

A Jump type SDE approach to positive self-Similar Markov processes*

Leif Döring[†]

Mátyás Barczy[‡]

Abstract

We present a new approach to positive self-similar Markov processes (pssMps) by reformulating Lamperti's transformation via jump type SDEs. As applications, we give direct constructions of pssMps (re)started continuously at zero if the Lamperti transformed Lévy process is spectrally negative. Our paper can be seen as a continuation of similar studies for continuous state branching processes.

Keywords: Lévy process, self-similar Markov process, Lamperti's transformation, jump type SDE.

AMS MSC 2010: Primary 60G18, 60H15; Secondary 60G51.

Submitted to EJP on November 15, 2011, final version accepted on October 27, 2012.

1 Introduction

Positive self-similar Markov processes (pssMps) play an important role in various branches of probability theory such as branching processes and fragmentation theory. In recent years two problems concerning the behavior of pssMps at zero have attracted a lot of attention. Solutions were heavily based on Lévy process theory. The main objective of this paper is to derive a jump type stochastic differential equation (SDE) for pssMps and prove well-posedness. As application of this approach we derive a new representation of pssMps (re)started continuously at zero by applying stochastic calculus.

1.1 Lamperti's transformation and "problems at zero"

We will use the following canonical notations. Let \mathbb{R}_+ denote the set of non-negative real numbers and let \mathbb{D} be the space of càdlàg functions $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ (right continuous

*L. Döring acknowledges support from Fondation Science Mathématique de Paris. M. Barczy acknowledges support from Marie Curie action NKTH-OTKA-EU FP7 co-funded 'MOBILITY' Grant No. OMF0610/2010, and by the Hungarian Scientific Research Fund under Grant No. OTKA T-079128.

[†]Leif Döring, Laboratoire de Probabilités et Modèles Aléatoires, Université Paris 6, France.
E-mail: leif.doering@googlemail.com

[‡]Mátyás Barczy, University of Debrecen, Faculty of Informatics, Debrecen, Hungary.
E-mail: barczy.matyas@inf.unideb.hu

with left limits) endowed with the Borel sigma-field \mathcal{D} generated by Skorokhod's J_1 topology. Let us consider a family of probability measures $\{\mathbb{P}_z, z \geq 0\}$ on $(\mathbb{D}, \mathcal{D})$ under which the coordinate process $Z_t(\omega) := \omega(t)$, $t \geq 0$, is a strong Markov process starting from z and fulfills the following scaling property: There exists a constant $a > 0$, called the index of self-similarity, such that

$$\text{the law of } (c^{-a}Z_{ct})_{t \geq 0} \text{ under } \mathbb{P}_z \text{ is } \mathbb{P}_{c^{-a}z} \tag{1.1}$$

for all $c > 0$ and $z \geq 0$. These laws are called non-negative self-similar Markov distributions and simultaneously the canonical process Z on the probability space $(\mathbb{D}, \mathcal{D}, \mathbb{P}_z)$ is called a non-negative self-similar Markov process started from z . Let us denote the corresponding process absorbed at the (possibly infinite) first hitting time T_0 of zero by

$$Z_t^\dagger := Z_t \mathbf{1}_{\{t \leq T_0\}}, \quad t \geq 0, \quad \text{with } T_0 := \inf\{t \geq 0 : Z_t = 0\}.$$

The laws of the process $(Z_t^\dagger)_{t \geq 0}$ under $\{\mathbb{P}_z, z \geq 0\}$, denoted by $\{\mathbb{P}_z^\dagger, z \geq 0\}$, are non-negative self-similar Markov distributions with the same index of self-similarity. They will be called positive self-similar Markov distributions and simultaneously, Z^\dagger is called a positive self-similar Markov process (pssMp). Note that \mathbb{P}_0^\dagger is the law of the process identical to 0, and that 0 is a trap for Z^\dagger , i.e. the corresponding laws $\{\mathbb{P}_z^\dagger, z \geq 0\}$ are carried by $\{\omega \in \mathbb{D} : \omega(t) = 0, \forall t \geq T_0(\omega)\}$. Alternatively, by a family of positive self-similar Markov distributions we mean a self-similar strong Markov family carried by $\{\omega \in \mathbb{D} : \omega(t) = 0, \forall t \geq T_0(\omega)\}$. It is important to note that, by our convention, the difference between positive and non-negative self-similar Markov processes is only that pssMps always have 0 as a trap. The formulation of pssMps might seem inconvenient but allows us for a more natural formulation of the results.

The systematic study of pssMps goes back to the 1960s, in particular to Lamperti's seminal article [24]. Lamperti [24, Lemmas 2.5 and 3.2] showed that the following trichotomy holds:

1. either $\mathbb{P}_z^\dagger(T_0 = \infty) = 1$ for all $z > 0$,
2. or $\mathbb{P}_z^\dagger(T_0 < \infty \text{ and } Z_{T_0-} = 0) = 1$ for all $z > 0$,
3. or $\mathbb{P}_z^\dagger(T_0 < \infty \text{ and } Z_{T_0-} > 0) = 1$ for all $z > 0$.

Let us next recall what is typically referred to as Lamperti's transformation (or Lamperti's representation). If Z is a pssMp of self-similarity index $a > 0$ starting from $z > 0$, then there is a possibly killed Lévy process $(\xi_t)_{t \geq 0}$ (whose law does not depend on z) such that the canonical process Z can be written as

$$Z_t = z \exp(\xi_{\tau(tz^{-1/a})}), \quad 0 \leq t < T_0, \tag{1.2}$$

where

$$\tau(t) := \inf\{s \geq 0 : I_s \geq t\} \quad \text{and} \quad I_t := \int_0^t \exp\left(\frac{1}{a}\xi_s\right) ds, \quad t \geq 0.$$

Since the transformation is invertible it yields a bijection between the class of pssMps and that of Lévy processes and furthermore it implies

- (1) $\iff \xi$ has infinite lifetime and $\mathbb{P}_z^\dagger(\limsup_{t \rightarrow \infty} \xi_t = \infty) = 1, \forall z > 0$,
- (2) $\iff \xi$ has infinite lifetime and $\mathbb{P}_z^\dagger(\lim_{t \rightarrow \infty} \xi_t = -\infty) = 1, \forall z > 0$,
- (3) $\iff \xi$ has finite lifetime, i.e. is killed at rate $q > 0$ with $-\infty$ as a cemetery.

Thus, with our definition of pssMps, (1.2) holds equally for $t \geq 0$ with the convention $\tau(tz^{-1/a}) = \infty$ for $t \geq T_0$.

Lamperti's representation has two drawbacks: first, it has no counterpart for non-negative self-similar Markov processes that do not have 0 as a trap since the infinite time-horizon $[0, \infty)$ is compressed via τ to the possibly finite time-horizon $[0, T_0)$ and for such a non-negative self-similar Markov process (1.2) is valid only up to T_0 . Hence, the entire path of ξ is already used to describe Z until T_0 . Secondly, in all cases Lamperti's representation holds for strictly positive initial conditions $z > 0$ but not for $z = 0$. Natural questions are then how to describe self-similar extensions of $(Z_t)_{t \in [0, T_0)}$ after T_0 and how to describe self-similar Markov processes started from zero. In the sequel we shall refer to these two problems as the "problems at zero". The problems at zero have been resolved in recent years based on criteria involving the Lamperti transformed Lévy process ξ .

Apparently, the first problem only occurs if ξ drifts to $-\infty$ - which includes being killed in an almost surely finite time - as otherwise $T_0 = \infty$ almost surely. In the special case when Z is a Brownian motion killed at 0, the problem was already solved by Lamperti [24, Theorem 5.1] and then the general case subsequently was tackled by Rivero [28] and Vuolle-Apiala [33]. Finally, it has been completely solved by Fitzsimmons [17, Theorem 1] and Rivero [29, Theorem 2] who have independently proved that the existence of a unique recurrent self-similar extension that leaves 0 continuously is equivalent to the following Cramér type condition

$$\text{there is a } 0 < \theta < \frac{1}{a} \text{ such that } \mathbb{E}(e^{\theta \xi_1}; \zeta > 1) = 1, \tag{1.3}$$

where ζ denotes the possibly infinite killing time of ξ .

Since the law of ξ does not depend on $z > 0$, in condition (1.3) we simply wrote \mathbb{E} instead of \mathbb{E}_z^\dagger (omitting also \dagger). We recall that a recurrent self-similar extension of a pssMp $\{\mathbb{P}_z^\dagger, z \geq 0\}$ is a non-negative self-similar Markov process $\{\mathbb{P}_z, z \geq 0\}$ on $(\mathbb{D}, \mathcal{D})$ of same index of self-similarity such that the law of the process $(Z_t^\dagger)_{t \geq 0}$ under $\{\mathbb{P}_z, z \geq 0\}$ equals the law of $(Z_t)_{t \geq 0}$ under $\{\mathbb{P}_z^\dagger, z \geq 0\}$ and 0 is not a trap for $\{\mathbb{P}_z, z \geq 0\}$. The recurrent self-similar extension $\{\mathbb{P}_z, z \geq 0\}$ is said to leave zero continuously if additionally

$$\mathbb{P}_0(Z_s = 0 \text{ for all } s \in G) = 1,$$

where G denotes the set of strictly positive left endpoints of the maximal intervals in the complement of the closure of the zero set $\{t \geq 0 : Z_t = 0\}$. For more information, in particular on the uniqueness of the extensions, see Fitzsimmons [17] and Rivero [28], [29]. In both articles, the recurrent extensions were characterized via their excursion measures.

The second problem can be formalized as follows: Does weak convergence $w\text{-}\lim_{z \downarrow 0} \mathbb{P}_z^\dagger = \mathbb{P}_0$ hold for a non-degenerate weak limit \mathbb{P}_0 and, if so, how can \mathbb{P}_0 be characterized? Both tasks were solved subsequently by Bertoin and Caballero [3], Bertoin and Yor [5], Caballero and Chaumont [8] and Chaumont et al. [10]. We should mention that in cases (2) and (3) the first problem at zero contains the second problem at zero, since the first hitting time T_0 of zero tends to zero almost surely for initial condition z tending to zero. Hence, in cases (2) and (3) the solution was already given by Fitzsimmons [17] and Rivero [28], [29] so that case (1) can be assumed. The main result is due to Caballero and Chaumont [8, Theorem 2]: the weak limit $w\text{-}\lim_{z \downarrow 0} \mathbb{P}_z^\dagger = \mathbb{P}_0$ exists if and only if the overshoot process

$$\xi_{T_x} - x, \quad x \geq 0, \quad \text{with} \quad T_x := \inf\{t \geq 0 : \xi_t \geq x\},$$

converges, as x tends to $+\infty$, weakly towards the law of a finite random variable and a further technical condition on ξ is satisfied. It was later shown in Chaumont et al. [10] that the additional assumption is superfluous. The convergence of the overshoot process has an equivalent analytic reformulation. According to Doney and Maller [15, Theorem 8], see also the discussion on page 1014 of Caballero and Chaumont [8], it is equivalent to

$$\xi \text{ is not arithmetic and } \begin{cases} \text{either } 0 < \mathbb{E}(\xi_1) \leq \mathbb{E}(|\xi_1|) < \infty, \\ \text{or } \mathbb{E}(|\xi_1|) < \infty, \mathbb{E}(\xi_1) = 0 \text{ and } J < \infty, \end{cases}$$

where

$$J := \int_1^\infty \frac{x\Pi((x, \infty))}{1 + \int_0^x \int_y^\infty \Pi((-\infty, -z)) dz dy} dx,$$

and Π denotes the Lévy measure of ξ . Finally, let us mention that the construction of \mathbb{P}_0 of Caballero and Chaumont [8] is based on involved Lévy process theory and not easily accessible. A simpler construction of \mathbb{P}_0 was given in Bertoin and Savov [4] via Lévy processes indexed by \mathbb{R} .

We will show below how to use jump type SDEs to construct the extensions in all cases. This comes directly from Lamperti’s representation which, surprisingly, offers more insight if it is reformulated via jump type SDEs. Furthermore, it seems reasonable that more applications, in particular for real-valued self-similar Markov processes, can be deduced via the powerful methods of stochastic analysis.

1.2 A motivating example

Let us start with a simple motivation by considering the only pssMp of self-similarity index 1 with continuous sample paths. If ξ is a Brownian motion with drift, say $\xi_t = (\delta - 2)t + 2B_t$, $t \geq 0$, $\delta \in \mathbb{R}$, then the Lamperti transformed pssMp is Z^\dagger , where Z is the non-negative self-similar Markov process of index 1 defined by

$$dZ_t = \delta dt + 2\sqrt{Z_t} dB_t. \tag{1.4}$$

For classical results on squared Bessel processes we refer to Chapter XI of Revuz and Yor [27]; recall in particular that for $\delta \geq 0$ and initial condition $z \geq 0$, (1.4) has a unique strong solution defined for $t \geq 0$ and that Z hits zero in finite time if and only if $\delta < 2$. In case $\delta < 0$ solutions exist only up to first hitting zero and are then extended by 0. The starting point of our analysis is the simple identity $\log \mathbb{E}(e^{\xi_1}) = \delta$, so that (1.4) can be written as

$$dZ_t = \log \mathbb{E}(e^{\xi_1}) dt + 2\sqrt{Z_t} dB_t. \tag{1.5}$$

Thus, precisely if $\log \mathbb{E}(e^{\xi_1}) > 0$ it has unique strong solutions, defined for all $t \geq 0$ and for all non-negative initial conditions, which do not have 0 as a trap. This gives the following explanation for the two problems in case of leaving zero continuously:

- If Z^\dagger hits zero in finite time and $\log \mathbb{E}(e^{\xi_1}) > 0$, then the unique recurrent self-similar extension of Z^\dagger that leaves zero continuously is nothing else but the unique strong solution Z of (1.5) for $t \geq 0$.
- If Z^\dagger does not hit zero in finite time (equivalently $\log \mathbb{E}(e^{\xi_1}) \geq 2$), then $w\text{-}\lim_{z \downarrow 0} \mathbb{P}_z = \mathbb{P}_0$ and \mathbb{P}_0 is nothing else but the unique strong solution of the SDE (1.5) started at zero (see for instance Stroock and Varadhan [32, Corollary 11.1.5] for the convergence).

For the particular example of the squared Bessel process we found naturally the condition $\log \mathbb{E}(e^{\xi_1}) > 0$ but not Condition (1.3). This can be seen here from the convexity of the Laplace exponent defined by $\psi(\lambda) := \log \mathbb{E}(e^{\lambda \xi_1})$ for $\lambda \in \mathbb{R}$: recalling that

$$Z^\dagger \text{ hits zero in finite time} \iff \xi \text{ drifts to } -\infty \iff \psi'(0+) = \mathbb{E}(\xi_1) < 0, \quad (1.6)$$

the convexity of ψ implies the equivalence

$$\psi(1) = \log \mathbb{E}(e^{\xi_1}) > 0 \iff \text{Condition (1.3) holds with } a = 1 \text{ and } q = 0, \quad (1.7)$$

if ξ drifts to $-\infty$. Note that (1.6) and (1.7) hold more generally for spectrally negative Lévy processes possibly with killing, see, e.g. Kyprianou [22, page 81].

1.3 A brief reminder of jump type SDEs for branching processes

To motivate our approach let us recall a development in the theory of continuous state branching processes (CSBPs), i.e. \mathbb{R}_+ -valued càdlàg strong Markov processes that satisfy the branching property $\mathbb{P}_x * \mathbb{P}_y = \mathbb{P}_{x+y}$, $x, y \geq 0$. Lamperti [23] proved that any CSBP Y can be represented as

$$Y_t = \xi_{\bar{\tau}(t)}, \quad t \geq 0, \quad \text{where } \bar{\tau}(t) := \inf\{t \geq 0 : \bar{I}_s \geq t\} \quad \text{and} \quad \bar{I}_t := \int_0^{\min(t, \bar{T}_0)} \frac{1}{\xi_s} ds,$$

where ξ is a Lévy process with no negative jumps started at $y \geq 0$ and \bar{T}_0 is the first time that ξ hits 0. By Volkonskii's formula (see, e.g. Williams [34, Section III.38]) this time-change representation is equivalent to the statement that the infinitesimal generator of Y has the form $\mathcal{A}_Y f(y) = y \mathcal{A}_\xi f(y)$ for sufficiently smooth functions f , where

$$\mathcal{A}_\xi f(y) = \gamma f'(y) + \frac{\sigma^2}{2} f''(y) + \int_0^\infty (f(y+u) - f(y) - f'(y)u \mathbf{1}_{\{u \leq 1\}}) \Pi(du),$$

is the infinitesimal generator of the Lévy process ξ with Lévy triplet (γ, σ^2, Π) . Recall that $\gamma \in \mathbb{R}$, $\sigma > 0$ and Π is a deterministic measure on \mathbb{R} satisfying $\int_{\mathbb{R}} \min(1, u^2) \Pi(du) < \infty$. From the generator \mathcal{A}_Y it is not hard to guess that Y is a (weak) solution to the jump type SDE

$$\begin{aligned} Y_t = & y + \gamma \int_0^t Y_s ds + \sigma \int_0^t \sqrt{Y_s} dB_s \\ & + \int_0^t \int_0^{Y_{s-}} \int_0^1 u (\mathcal{N} - \mathcal{N}')(ds, dr, du) + \int_0^t \int_0^{Y_{s-}} \int_1^\infty u \mathcal{N}(ds, dr, du), \quad t \geq 0, \end{aligned} \quad (1.8)$$

where \mathcal{N} is a Poisson random measure independent of the Brownian motion B having intensity measure $\mathcal{N}'(ds, dr, du) = ds \otimes dr \otimes \Pi(du)$. Interestingly, Y is not only a weak solution to (1.8) but actually a strong solution since pathwise uniqueness holds. This non-trivial additional fact, due to Dawson and Li [11, Section 5], suggests that the jump type SDE (1.8) might be a strong tool to study CSBPs. For further reading on Lamperti's transformation for CSBPs we refer to the overview article Caballero et al. [9].

2 Results

2.1 A jump type SDE approach to positive self-similar Markov processes

Mirroring the jump type SDE approach to CSBPs we now derive a jump type SDE for pssMps of self-similarity index 1. This can be seen as natural generalization of (1.5). According to the discussion for CSBPs, using Volkonskii's formula and Lamperti's

transformation (1.2) with $a = 1$, one gets the infinitesimal generator identity $\mathcal{A}_Z f(z) = \frac{1}{z} \mathcal{A}_{e^\xi} f(z)$ for sufficiently smooth functions f . In order to derive from this a jump type SDE we proceed in two steps. Recall that the Lévy-Itô representation of ξ takes the form

$$\xi_t = \gamma t + \sigma B_t + \int_0^t \int_{|u| \leq 1} u (\mathcal{N}_0 - \mathcal{N}'_0)(ds, du) + \int_0^t \int_{|u| > 1} u \mathcal{N}_0(ds, du), \quad t \geq 0, \quad (2.1)$$

where (γ, σ^2, Π) is the Lévy triplet of ξ , B is a standard Wiener process and \mathcal{N}_0 is an independent Poisson random measure on $(0, \infty) \times \mathbb{R}$ with intensity measure $\mathcal{N}'_0(ds, du) = ds \otimes \Pi(du)$. First, applying Itô's formula to $z e^{\xi_t}$ and afterwards including the generator correction $1/z$ similarly as the correction y in $\mathcal{A}_Y f(y) = y \mathcal{A}_\xi f(y)$ is added to (1.8), our Ansatz - which is verified below in Proposition 3.14 - is the jump type SDE

$$\begin{aligned} Z_t = z &+ \left(\gamma + \frac{\sigma^2}{2} + \int_{|u| \leq 1} (e^u - 1 - u) \Pi(du) \right) t + \sigma \int_0^t \sqrt{Z_s} dB_s \\ &+ \int_0^t \int_0^\infty \int_{|u| \leq 1} \mathbf{1}_{\{r Z_{s-} \leq 1\}} Z_{s-} (e^u - 1) (\mathcal{N} - \mathcal{N}')(ds, dr, du) \\ &+ \int_0^t \int_0^\infty \int_{|u| > 1} \mathbf{1}_{\{r Z_{s-} \leq 1\}} Z_{s-} (e^u - 1) \mathcal{N}(ds, dr, du) \end{aligned} \quad (2.2)$$

which, as in Lamperti's transformation (1.2), is a priori restricted to $t \leq T_0$, since the constant drift might be negative. Here, B is a standard Wiener process and \mathcal{N} is an independent Poisson random measure on $(0, \infty) \times (0, \infty) \times \mathbb{R}$ with intensity measure $\mathcal{N}'(ds, dr, du) = ds \otimes dr \otimes \Pi(du)$. On first view there is not too much to learn from this reformulation of Lamperti's transformation (1.2). However, if we additionally assume that $E(e^{\xi_1}) < \infty$, then one can add and subtract the compensation of large jumps to (2.2), and we get

$$\begin{aligned} Z_t = z &+ \left(\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^u - 1 - u \mathbf{1}_{\{|u| \leq 1\}}) \Pi(du) \right) t + \sigma \int_0^t \sqrt{Z_s} dB_s \\ &+ \int_0^t \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{r Z_{s-} \leq 1\}} Z_{s-} [e^u - 1] (\mathcal{N} - \mathcal{N}')(ds, dr, du) \end{aligned}$$

for $t \leq T_0$. Simplifying via the Lévy-Khintchin formula (more precisely the extension in Theorem 25.17 of Sato [30]) we obtain

$$\begin{aligned} Z_t = z &+ (\log \mathbb{E}(e^{\xi_1})) t + \sigma \int_0^t \sqrt{Z_s} dB_s \\ &+ \int_0^t \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{r Z_{s-} \leq 1\}} Z_{s-} [e^u - 1] (\mathcal{N} - \mathcal{N}')(ds, dr, du) \end{aligned} \quad (2.3)$$

for $t \leq T_0$. Let us now explain how to modify the Ansatz if we allow ξ to be killed, i.e. set ξ to the cemetery state $-\infty$ after an independent exponential lifetime ζ . After taking the exponential for the first step of the derivation, this is equivalent to including jumps to zero at an independent exponential rate. As before, the time-change is included via an extra integral so that Equation (2.3) transforms into

$$\begin{aligned} Z_t = z &+ (\log \mathbb{E}(e^{\xi_1})) t + \sigma \int_0^t \sqrt{Z_s} dB_s - \int_0^t \int_0^\infty \mathbf{1}_{\{r Z_{s-} \leq 1\}} Z_{s-} \mathcal{M}(ds, dr) \\ &+ \int_0^t \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{r Z_{s-} \leq 1\}} Z_{s-} [e^u - 1] (\mathcal{N} - \mathcal{N}')(ds, dr, du) \end{aligned}$$

for $t \leq T_0$, where \mathcal{M} is an independent Poisson random measure on $(0, \infty) \times (0, \infty)$ with intensity measure $\mathcal{M}'(ds, dr) = q ds \otimes dr$ and $q \geq 0$ is the killing rate (i.e. $\mathbb{P}(\zeta > 1) = e^{-q}$). Adding and subtracting the compensation gives

$$Z_t = z + (\log \mathbb{E}(e^{\xi_1}) - q)t + \sigma \int_0^t \sqrt{Z_s} dB_s - \int_0^t \int_0^\infty \mathbf{1}_{\{rZ_{s-} \leq 1\}} Z_{s-} (\mathcal{M} - \mathcal{M}') (ds, dr) + \int_0^t \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{rZ_{s-} \leq 1\}} Z_{s-} [e^u - 1] (\mathcal{N} - \mathcal{N}') (ds, dr, du)$$

which is simplified to

$$Z_t = z + (\log \mathbb{E}(e^{\xi_1}; \zeta > 1))t + \sigma \int_0^t \sqrt{Z_s} dB_s - \int_0^t \int_0^\infty \mathbf{1}_{\{rZ_{s-} \leq 1\}} Z_{s-} (\mathcal{M} - \mathcal{M}') (ds, dr) + \int_0^t \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{rZ_{s-} \leq 1\}} Z_{s-} [e^u - 1] (\mathcal{N} - \mathcal{N}') (ds, dr, du) \tag{2.4}$$

for $t \leq T_0$. This is our generalization of (1.5) which has a surprising advantage compared to Lamperti's original representation (1.2):

Remark 2.1. *The Lamperti transformation (1.2) only works for $t \leq T_0$, since, due to the time-change, the infinite time-horizon $[0, \infty)$ of ξ is transformed into the (possibly finite) time-horizon $[0, T_0)$. Nonetheless, if $0 < \log \mathbb{E}(e^{\xi_1}; \zeta > 1) < \infty$ then, due to the positive constant drift, the jump type SDE (2.4) is not restricted to $t \leq T_0$ but can be defined for all $t \geq 0$. This is not surprising since we already noted in Section 1.2 that ξ drifting to $-\infty$ implies the equivalence*

$$\log \mathbb{E}(e^{\xi_1}; \zeta > 1) > 0 \iff (1.3) \text{ holds with } a = 1 \tag{2.5}$$

if the occurring quantities are finite. Hence, if we can show that (2.4) is a "good" SDE, then the problems at zero for the corresponding pssMp can be explained as simple as for the squared Bessel process.

To formulate our results, let us start with a rigorous definition of solutions that are constructed on a stochastic basis $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, P)$ satisfying the usual conditions. We will use *weak solutions*, i.e. $(\mathcal{G}_t)_{t \geq 0}$ adapted stochastic processes $(Z_t)_{t \geq 0}$ with almost surely càdlàg paths that satisfy the SDE (2.4) almost surely with some (\mathcal{G}_t) -standard Wiener process B and some independent (\mathcal{G}_t) -Poisson random measure \mathcal{N} on $(0, \infty) \times (0, \infty) \times \mathbb{R}$ that has intensity measure $\mathcal{N}'(ds, dr, du) = ds \otimes dr \otimes \Pi(du)$. If additionally Z is adapted to the augmented filtration generated by B and \mathcal{N} , then Z is said to be a *strong solution*. We say that *pathwise uniqueness* holds for the SDE (2.4) if for any two weak solutions Z^1 and Z^2 defined on the same probability space with the same standard Wiener process and Poisson random measure, they are indistinguishable. Weak solutions for the SDE (2.4) *up to a first hitting time* are defined via the localized version of the corresponding martingale problem as in Chapter 4.6 of Ethier and Kurtz [16].

Theorem 2.2. *Suppose that ξ is a spectrally negative Lévy process with Lévy triplet (γ, σ^2, Π) killed at rate $q \geq 0$. If $\log \mathbb{E}(e^{\xi_1}; \zeta > 1) > 0$, then*

- (a) *for any initial condition $z \geq 0$ there is a pathwise unique non-negative strong solution $(Z_t^{(z)})_{t \geq 0}$ of the SDE (2.4) associated to ξ killed at rate q ,*
- (b) *the solutions $\{Z^{(z)}, z \geq 0\}$ define a non-negative self-similar Markov process of self-similarity index 1,*

(c) the solutions $\{(Z^{(z)})^\dagger, z \geq 0\}$ define a positive self-similar Markov process of index 1 and its Lamperti transformed Lévy process has Lévy triplet (γ, σ^2, Π) and is killed at rate q .

Recently, in Fu and Li [18], Li and Mytnik [25] and Dawson and Li [12], the pathwise uniqueness problem for Poisson driven jump type SDEs has been studied. Our SDE (2.4) does not fit precisely into those frameworks because of missing monotonicity properties and the explosive jump rate at zero. However, we can adapt some of their arguments in our special case. At the beginning of Section 3 we briefly explain why we assume that ξ is spectrally negative and we explain why we conjecture that the minimal assumption $\mathbb{E}(e^{\xi_1}) < \infty$ should be enough for deriving the SDE (2.4).

2.2 Applications to the problems at zero

Before stating our contributions to the problems at zero let us discuss how to reduce pssMps of index $a > 0$ to those of index 1. For this sake we use the following fact which follows readily from the definition of Lamperti’s transformation (see for instance Bertoin and Yor [6, page 203]):

$$\begin{aligned} Z^\dagger \text{ is a pssMp of index } a \text{ with associated Lévy process } \xi \\ \iff X^\dagger := (Z^\dagger)^{1/a} \text{ is a pssMp of index 1 with associated Lévy process } \frac{1}{a}\xi. \end{aligned} \tag{2.6}$$

Hence, in order to study recurrent self-similar extensions $(Z_t)_{t \geq 0}$ of Z^\dagger it suffices to find an extension $(X_t)_{t \geq 0}$ of X^\dagger , since then $Z := X^a$ is an extension of Z^\dagger . Since the Lamperti transformed Lévy process for X^\dagger is explicitly given by $\frac{1}{a}\xi$, this can be done via Theorem 2.2 provided that the Lamperti transformed Lévy process ξ is known for Z^\dagger .

We can now state a new characterization of the unique recurrent self-similar extension of a pssMp that leaves zero continuously in the spirit of Fitzsimmons [17] and Rivero [29].

Theorem 2.3. *Suppose that ξ is a spectrally negative Lévy process killed at rate $q \geq 0$ and the Cramér type condition (1.3) holds. Further, let $(X_t)_{t \geq 0}$ be the unique strong solution of (2.4) associated to $\frac{1}{a}\xi$ killed at rate q with initial condition $z^{1/a} \geq 0$. If $Z^{(z)} := X^a$, then*

- (a) $\{(Z^{(z)})^\dagger, z \geq 0\}$ is a pssMp of self-similarity index a with Lamperti transformed Lévy process ξ killed at rate q ,
- (b) $\{Z^{(z)}, z \geq 0\}$ is the unique recurrent self-similar extension of $\{(Z^{(z)})^\dagger, z \geq 0\}$ that leaves zero continuously.

Note that under the assumptions of Theorem 2.3, the Lévy process ξ drifts to $-\infty$ so that $\log \mathbb{E}(e^{\frac{1}{a}\xi_1}; \zeta > 1) > 0$ by (2.5). Hence, the SDE (2.4) associated to $\frac{1}{a}\xi$ killed at rate q has a strictly positive constant drift and the result is a consequence of (2.6) and Theorem 2.2. As usual in the study of pssMps, Theorem 2.3 is only applicable for pssMps for which ξ is known. For some examples of pssMps for which the Lévy triplet of the Lamperti transformed Lévy process ξ can be calculated explicitly, we refer to Caballero and Chaumont [7].

Finally, we utilize the fact that the SDE (2.4) can be issued from zero once the constant drift is strictly positive. This gives a characterization and convergence statement for pssMps started from zero via stochastic calculus.

Theorem 2.4. *Let $a > 0$ and suppose that ξ is a spectrally negative Lévy process killed at rate $q \geq 0$ and that either*

- ξ does not drift to $-\infty$,
- or ξ drifts to $-\infty$ and Condition (1.3) holds.

If $\{\mathbb{P}_z^\dagger, z \geq 0\}$ denotes the law of the pssMp of self-similarity index a with Lamperti transformed Lévy process ξ killed at rate q , then

$$w\text{-}\lim_{z \downarrow 0} \mathbb{P}_z^\dagger = \mathbb{P}_0 \quad \text{in the Skorokhod topology,}$$

where \mathbb{P}_0 is the law of $Z^{(0)} := X^a$ and X is the unique strong solution of the SDE (2.4) associated to $\frac{1}{a}\xi$ killed at rate q with initial condition $X_0 = 0$.

Theorems 2.3 and 2.4 were motivated by results on CSBPs with state-dependent immigration in Berestycki et al. [2]. In their context the pssMp was already defined by a jump type SDE and the extensions could be constructed more directly.

Remark 2.5. In the setting of Theorem 2.2 with $q = 0$, applying Itô's formula to the SDE (2.4), one can check that the infinitesimal generator of the pssMp of index 1 associated to ξ takes the form

$$\begin{aligned} \mathcal{A}f(z) &= (\log \mathbb{E}(e^{\xi_1}))f'(z) + \frac{\sigma^2}{2}zf''(z) + \frac{1}{z} \int_{\mathbb{R}} (f(ze^u) - f(z) - zf'(z)(e^u - 1))\Pi(du) \\ &= \left(\gamma + \frac{\sigma^2}{2}\right)f'(z) + \frac{\sigma^2}{2}zf''(z) + \frac{1}{z} \int_{\mathbb{R}} (f(ze^u) - f(z) - zf'(z)u\mathbf{1}_{\{|u| \leq 1\}})\Pi(du) \end{aligned}$$

for $z > 0$ and for appropriate smooth test-functions f . The drift coefficient is $\gamma + \frac{\sigma^2}{2}$ (and not γ) which is a misprint in formula (6.5) of Lamperti [24, Theorem 6.1].

3 Proofs

From now on we will work on a stochastic basis $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, P)$ satisfying the usual conditions and we will denote the underlying probability measure and the expectation with respect to it by P and E , respectively, instead of \mathbb{P} and \mathbb{E} (in contrast to the introduction, but without confusion). Further, to simplify the notations (in contrast to the formulation of Theorems 2.2, 2.3 and 2.4) we will write Z instead of $Z^{(z)}$, i.e. we omit denoting the initial value $z \geq 0$.

Before turning to the proofs we briefly discuss the difficulties of the SDE (2.4). The pathwise uniqueness proof is based on a Yamada-Watanabe type argument. This argument has been reinvented for jump-type SDEs by Fu and Li [18] motivating our approach as well even though the coefficients of the SDE (2.4) do not satisfy the standard requirements. To understand the additional difficulty of proving pathwise uniqueness for (2.4) we stress two issues:

For the first, one has to consider carefully the large jumps, i.e. those jumps which come from an atom (s, r, u) of \mathcal{N} with $|u|$ larger than some fixed $M > 0$. In the setting of Fu and Li [18], it was possible to truncate the jump measure at infinity. Pathwise uniqueness for their jump type SDE with truncated intensity measure could be proved via second moment arguments and large jumps were added by interlacing since their occurrence does not accumulate. Unfortunately, the large jumps accumulate for our SDE (2.4) when solutions approach zero. Hence, the interlacement procedure cannot be fully used in our case so that estimates need to be carried out on the full Lévy measure.

The second issue directly occurs in the estimates of the Yamada-Watanabe argument (or local time argument) for the pathwise uniqueness proof. The adaptation of the Yamada-Watanabe argument is relatively easy if for two solutions X and Y

$$(X_{s-} - Y_{s-})\Delta(X_s - Y_s) \geq 0, \quad s > 0, \tag{3.1}$$

i.e. the larger solution has the larger jumps which equivalently says that the integrand of the Poissonian integral is non-decreasing. This monotonicity property is for instance verified for the CSBPs discussed in Section 1.3 and is a main assumption in Fu and Li [18] and Li and Mytnik [25]. Modifying some arguments, uniqueness can still be proved if the smaller solution does not exceed the larger solution by a jump, i.e. after a jump the difference $X - Y$ does not change sign. This is the key observation allowing to prove pathwise uniqueness for generalized Fleming-Viot processes in Dawson and Li [12]. In fact, this is the precise origin of our hypothesis ξ being spectrally negative. The monotonicity property (3.1) always fails due to the jump rate $1/Z_{s-}$ in Equation (2.4) but a change of sign for the difference could occur only for jumps corresponding to $u > 0$.

Remark 3.1. *Apart from the proof of pathwise uniqueness given in Proposition 3.4, all of our arguments hold equally if $E(e^{2\xi_1}) < \infty$ which is trivially fulfilled if ξ is spectrally negative. To simplify a possible later extension, we kept the proofs in this larger generality, but behind always supposing that ξ is spectrally negative.*

Remark 3.2. *In the meantime the pathwise uniqueness has been generalized by Li and Pu [26] to more general jump-type SDEs containing the SDE (2.4).*

We now start with the proof of Theorem 2.2 for which we first prove the pathwise uniqueness statement and then construct a strong solution via approximation. To prove pathwise uniqueness for the SDE (2.4) the non-Lipschitz integrand $x \mapsto \sqrt{x}$ of the Brownian part forces us to use Yamada-Watanabe type arguments going back to [35] and [36]. Let us start with some notations. Suppose $a_0 = 1$ and $0 < \dots < a_k < a_{k-1} < \dots \leq a_0$, $k \in \mathbb{N}$, are such that $\int_{a_k}^{a_{k-1}} \frac{1}{z} dz = k$ for all $k \geq 1$. For completeness, we note that $a_k = e^{-k} a_{k-1}$, which yields that

$$a_k = e^{-\frac{k(k+1)}{2}}, \quad k \in \mathbb{N},$$

and hence $\lim_{k \rightarrow \infty} a_k = 0$. In what follows we will not use the explicit form of $(a_k)_{k \in \mathbb{N}}$. For all $k \in \mathbb{N}$, let $\psi_k : \mathbb{R} \rightarrow \mathbb{R}_+$ be a non-negative continuous function with support in (a_k, a_{k-1}) satisfying $\int_{a_k}^{a_{k-1}} \psi_k(x) dx = 1$ and $\psi_k(x) \leq \frac{2}{kx}$, $x > 0$. Next, let us define

$$\phi_k(z) := \int_0^{|z|} \int_0^y \psi_k(x) dx dy, \quad z \in \mathbb{R}.$$

Then it is apparent that ϕ_k is even, twice continuously differentiable,

$$\lim_{k \rightarrow \infty} \phi_k(z) = |z|, \quad z \in \mathbb{R}, \tag{3.2}$$

and the sequence $(\phi_k)_{k \in \mathbb{N}}$ is non-decreasing. Furthermore, for fixed $k \in \mathbb{N}$ and $x, y \in \mathbb{R}_+$, we have

$$\phi_k''(x - y)[\sqrt{x} - \sqrt{y}]^2 = \psi_k(|x - y|)[\sqrt{x} - \sqrt{y}]^2 \leq \frac{2[\sqrt{x} - \sqrt{y}]^2}{k|x - y|} \leq \frac{2}{k}. \tag{3.3}$$

Supposing that there are two non-negative weak solutions X and Y of the SDE (2.4) on the same stochastic basis $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, P)$ with the same standard Wiener process and Poisson random measures such that $X_0 = Y_0$, our aim is to estimate $E(\phi_k(X_t - Y_t))$ so that via Fatou's lemma and (3.2) an estimate for $E(|X_t - Y_t|)$ can be derived. In the following we denote by

$$\Delta_h f(x) := f(x + h) - f(x) \quad \text{and} \quad D_h f(x) := \Delta_h f(x) - f'(x)h$$

if the right-hand sides are meaningful and additionally abbreviate

$$\begin{aligned} \mathbf{g}(x, r, u) &= \mathbf{1}_{\{rx \leq 1\}} x(e^u - 1), & x \geq 0, r \geq 0, u \in \mathbb{R}, \\ \mathbf{h}(x, r) &= -\mathbf{1}_{\{rx \leq 1\}} x, & x \geq 0, r \geq 0. \end{aligned}$$

From now on we will also abbreviate $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$ for $a, b \in \mathbb{R}$. Let us start with a simple lemma about convergence to infinity of a suitable sequence of stopping times that are used to ensure that the appearing stopped local martingales are martingales.

Lemma 3.3. *Suppose that Z is a non-negative weak solution of the SDE (2.4) and let $\tau_m := \inf\{t \geq 0 : Z_t \geq m\}$ for $m \in \mathbb{N}$. Then the sequence $(\tau_m)_{m \in \mathbb{N}}$ is increasing and tends to infinity almost surely.*

Proof. First let us assume that $q = 0$. Note that τ_m is increasing in m so that $\lim_{m \rightarrow \infty} \tau_m$ exists almost surely. By Proposition 2.3 in Fu and Li [18], we find that τ_m indeed converges to infinity almost surely as $m \rightarrow \infty$. To check that we are allowed to use this proposition, their conditions (2.a) and (2.b) need to be verified. In order to do so, let

$$b(x) := \begin{cases} \log E(e^{\xi_1}) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \quad \text{and} \quad \sigma(x) := \begin{cases} \sigma\sqrt{x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$\mu_1 := 0, g_1 := 0, U_0 := (0, \infty) \times \mathbb{R}, \mu_0(dr, du) := dr \otimes \Pi(du)$ and $g_0(x, r, u) := \mathbf{g}(x, r, u)$. Then condition (2.a) in Fu and Li [18] is satisfied with $K := \log E(e^{\xi_1})$, and, for all $x > 0$,

$$\begin{aligned} &\sigma(x)^2 + \int_0^\infty \int_{\mathbb{R}} \left[|g_0(x, r, u)| \wedge g_0(x, r, u)^2 \right] \mu_0(dr, du) \\ &\leq \sigma^2 x + \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{xr \leq 1\}} x^2 (e^u - 1)^2 dr \Pi(du) \\ &\leq \sigma^2 x + x \int_{\mathbb{R}} (e^u - 1)^2 dr \Pi(du), \end{aligned}$$

which implies that condition (2b) in Fu and Li [18] is also satisfied. Since we have to suppose that ξ is spectrally negative, the proof can be extended easily to $q > 0$ if the Poisson random measure \mathcal{M} is chosen to be $\mathcal{M}(ds, dr) = \mathcal{N}(ds, dr, \{1\})$, with an additional atom of weight q for Π at $\{1\}$, and \mathbf{g} is extended as $\mathbf{g}(x, r, 1) = -\mathbf{1}_{\{rx \leq 1\}} x$ for $x, r \geq 0$. \square

We can now proceed to prove the pathwise uniqueness. The proof is a combination of the Yamada-Watanabe argument with the classical Tanaka's formula approach for pathwise uniqueness. To avoid the local time argument we use a dominated convergence argument.

Proposition 3.4. *Suppose that ξ is as in Theorem 2.2, then pathwise uniqueness for non-negative solutions holds for the SDE (2.4) for $t \geq 0$.*

Proof. Suppose that $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are two non-negative weak solutions of the SDE (2.4) on the same stochastic basis with the same standard Wiener process and Poisson random measures such that $X_0 = Y_0 = z \geq 0$. In the sequel we are going to show that X and Y are indistinguishable. First, by Itô's formula (see, e.g. Di Nunno et al. [13, Theorem 9.5] or Ikeda and Watanabe [19, Chapter II, Theorem 5.1]) taking the

difference of these solutions, the drifts cancel each other and it remains

$$\begin{aligned} X_t - Y_t &= \sigma \int_0^t [\sqrt{X_s} - \sqrt{Y_s}] dB_s \\ &+ \int_0^t \int_0^\infty \int_{\mathbb{R}} [\mathbf{g}(X_{s-}, r, u) - \mathbf{g}(Y_{s-}, r, u)] (\mathcal{N} - \mathcal{N}') (ds, dr, du) \\ &+ \int_0^t \int_0^\infty [\mathbf{h}(X_{s-}, r) - \mathbf{h}(Y_{s-}, r)] (\mathcal{M} - \mathcal{M}') (ds, dr), \quad t \geq 0, \end{aligned}$$

since the noises are independent. Next, for all $m \in \mathbb{N}$, let $\tau_m := \inf \{t \geq 0 : X_t \geq m \text{ or } Y_t \geq m\}$ which tends to infinity as $m \rightarrow \infty$ almost surely due to Lemma 3.3. Again by Itô's formula, now applied to the semi-martingale $Z_t := X_t - Y_t, t \geq 0$, we obtain

$$\begin{aligned} \phi_k(Z_{t \wedge \tau_m}) &= \frac{\sigma^2}{2} \int_0^{t \wedge \tau_m} \phi_k''(Z_s) [\sqrt{X_s} - \sqrt{Y_s}]^2 ds \\ &+ \int_0^{t \wedge \tau_m} \int_0^\infty \int_{\mathbb{R}} D_{\mathbf{g}(X_s, r, u) - \mathbf{g}(Y_s, r, u)} \phi_k(Z_s) ds dr \Pi(du) \\ &+ q \int_0^{t \wedge \tau_m} \int_0^\infty D_{\mathbf{h}(X_s, r) - \mathbf{h}(Y_s, r)} \phi_k(Z_s) ds dr \\ &+ \sigma \int_0^{t \wedge \tau_m} \phi_k'(Z_s) [\sqrt{X_s} - \sqrt{Y_s}] dB_s \\ &+ \int_0^{t \wedge \tau_m} \int_0^\infty \int_{\mathbb{R}} \Delta_{\mathbf{g}(X_{s-}, r, u) - \mathbf{g}(Y_{s-}, r, u)} \phi_k(Z_{s-}) (\mathcal{N} - \mathcal{N}') (ds, dr, du) \\ &+ \int_0^{t \wedge \tau_m} \int_0^\infty \Delta_{\mathbf{h}(X_{s-}, r) - \mathbf{h}(Y_{s-}, r)} \phi_k(Z_{s-}) (\mathcal{M} - \mathcal{M}') (ds, dr) \end{aligned} \tag{3.4}$$

for $k, m \in \mathbb{N}$ and $t \geq 0$. Let us first show that the last three integrals are proper martingales in t with respect to the filtration (\mathcal{G}_t) . By Ikeda and Watanabe [19, Chapter II, Proposition 2.4 and page 62], for this it is enough to check that

$$E \left(\int_0^{t \wedge \tau_m} (\phi_k'(Z_s))^2 (\sqrt{X_s} - \sqrt{Y_s})^2 ds \right) < \infty, \tag{3.5}$$

$$E \left(\int_0^{t \wedge \tau_m} \int_0^\infty \int_{\mathbb{R}} (\Delta_{\mathbf{g}(X_s, r, u) - \mathbf{g}(Y_s, r, u)} \phi_k(Z_s))^2 ds dr \Pi(du) \right) < \infty, \tag{3.6}$$

$$E \left(q \int_0^{t \wedge \tau_m} \int_0^\infty (\Delta_{\mathbf{h}(X_s, r) - \mathbf{h}(Y_s, r)} \phi_k(Z_s))^2 ds dr \right) < \infty, \tag{3.7}$$

for $k, m \in \mathbb{N}$ and $t \geq 0$. Using that $X_t < m$ and $Y_t < m$ for $0 \leq t < \tau_m$, and that $(\phi_k'(z))^2 \leq 1, z \in \mathbb{R}$, (3.5) follows. For (3.6), using the estimate $(\Delta_h \phi_k(x))^2 \leq h^2, h \in \mathbb{R}, x \in \mathbb{R}$, we get the upper bound

$$\begin{aligned} &E \left(\int_0^{t \wedge \tau_m} \int_0^\infty \int_{\mathbb{R}} (\mathbf{g}(X_s, r, u) - \mathbf{g}(Y_s, r, u))^2 ds dr \Pi(du) \right) \\ &\leq E \left(\int_0^{t \wedge \tau_m} |Z_s| ds \right) \int_{\mathbb{R}} (e^u - 1)^2 \Pi(du) < \infty, \end{aligned}$$

since for all $x, y \in \mathbb{R}_+$,

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}} (\mathbf{1}_{\{rx \leq 1\}}x - \mathbf{1}_{\{ry \leq 1\}}y)^2 (e^u - 1)^2 dr \Pi(du) \\ &= \left(x + y - 2 \int_0^\infty \mathbf{1}_{\{ry \leq 1\}} \mathbf{1}_{\{rx \leq 1\}} xy dr \right) \int_{\mathbb{R}} (e^u - 1)^2 \Pi(du) \\ &= (x + y - 2 \min(x, y)) \int_{\mathbb{R}} (e^u - 1)^2 \Pi(du) \\ &= |x - y| \int_{\mathbb{R}} (e^u - 1)^2 \Pi(du). \end{aligned}$$

This shows (3.6) since $|Z_s| \leq 2m$ for $0 \leq s < \tau_m$. Almost the identical estimate shows (3.7). Note also that the above calculations show that we had the right to use Itô's formula for deriving (3.4). Now we are in a position to carry out the Yamada-Watanabe argument for (3.4). The martingales vanish if we take expectations and the first integral can be estimated via (3.3) having limit 0 as $k \rightarrow \infty$ for every fixed $t \geq 0$ and $m \in \mathbb{N}$. If we suppose additionally that, for any $t \geq 0$ and $m \in \mathbb{N}$,

$$\lim_{k \rightarrow \infty} E \left(\int_0^{t \wedge \tau_m} \int_0^\infty \int_{\mathbb{R}} D_{\mathbf{g}(X_s, r, u) - \mathbf{g}(Y_s, r, u)} \phi_k(X_s - Y_s) ds dr \Pi(du) \right) = 0, \tag{3.8}$$

$$\lim_{k \rightarrow \infty} E \left(q \int_0^{t \wedge \tau_m} \int_0^\infty D_{\mathbf{h}(X_s, r) - \mathbf{h}(Y_s, r)} \phi_k(X_s - Y_s) ds dr \right) = 0 \tag{3.9}$$

then (3.2), (3.4) and Fatou's lemma lead us to

$$0 \leq E(|Z_{t \wedge \tau_m}|) \leq \lim_{k \rightarrow \infty} E(\phi_k(Z_{t \wedge \tau_m})) = 0$$

for every $t \geq 0$ and $m \in \mathbb{N}$. Using that τ_m tends to ∞ almost surely as $m \rightarrow \infty$, again by Fatou's lemma, we have

$$0 \leq E(|Z_t|) \leq \lim_{m \rightarrow \infty} E(|Z_{t \wedge \tau_m}|) = 0, \quad t \geq 0.$$

Hence, the processes $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ are modifications of one another. Since both processes have right continuous sample paths they are also indistinguishable (see, e.g., Karatzas and Shreve [21, Problem 1.1.5]).

To finish the proof we still have to verify (3.8) and (3.9). Let us first define $\phi(z) := |z|$, $z \in \mathbb{R}$, and note that, with $\phi'(z) = \text{sign}(z)$ for $z \neq 0$,

$$\begin{aligned} & \int_0^\infty D_{\mathbf{g}(x, r, u) - \mathbf{g}(y, r, u)} \phi(x - y) dr \\ &= \mathbf{1}_{\{x > y\}} \int_0^{1/x} (|(x - y) + (x - y)(e^u - 1)| - |x - y| - (x - y)(e^u - 1)) dr \\ & \quad + \mathbf{1}_{\{x > y\}} \int_{1/x}^{1/y} (|(x - y) - y(e^u - 1)| - |x - y| + y(e^u - 1)) dr \\ & \quad + \mathbf{1}_{\{x \leq y\}} \int_0^{1/y} (|(x - y) + (x - y)(e^u - 1)| - |x - y| + (x - y)(e^u - 1)) dr \\ & \quad + \mathbf{1}_{\{x \leq y\}} \int_{1/y}^{1/x} (|(x - y) + x(e^u - 1)| - |x - y| + x(e^u - 1)) dr \\ &= 0 \end{aligned} \tag{3.10}$$

for any $x, y \geq 0$ and any $u \leq 0$. In fact, each of the four integrands is constant zero. By Fubini's theorem (justified by the trivial fact that $|0| = 0$) this shows that

$$E \left(\int_0^{t \wedge \tau_m} \int_0^\infty \int_{\mathbb{R}} D_{\mathbf{g}(X_s, r, u) - \mathbf{g}(Y_s, r, u)} \phi(X_s - Y_s) \, ds \, dr \, \Pi(du) \right) = 0$$

so that (3.8) follows if we can show that

$$\begin{aligned} & \lim_{k \rightarrow \infty} E \left(\int_0^{t \wedge \tau_m} \int_0^\infty \int_{\mathbb{R}} D_{\mathbf{g}(X_s, r, u) - \mathbf{g}(Y_s, r, u)} \phi_k(X_s - Y_s) \, ds \, dr \, \Pi(du) \right) \\ &= E \left(\int_0^{t \wedge \tau_m} \int_0^\infty \int_{\mathbb{R}} D_{\mathbf{g}(X_s, r, u) - \mathbf{g}(Y_s, r, u)} \phi(X_s - Y_s) \, ds \, dr \, \Pi(du) \right). \end{aligned}$$

By Fubini's theorem, separating the four cases as before in (3.10) and integrating out r we see that we can equally show that for any $t \geq 0$ (the corresponding two cases for $X_s \leq Y_s$ are similar)

$$\begin{aligned} & \lim_{k \rightarrow \infty} E \left(\int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{X_s > Y_s\}} \frac{1}{X_s} \left(\phi_k(X_s - Y_s + (X_s - Y_s)(e^u - 1)) \right. \right. \\ & \quad \left. \left. - \phi_k(X_s - Y_s) - \phi'_k(X_s - Y_s)(X_s - Y_s)(e^u - 1) \right) ds \, \Pi(du) \right) \\ &= E \left(\int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{X_s > Y_s\}} \frac{1}{X_s} \left(|X_s - Y_s + (X_s - Y_s)(e^u - 1)| \right. \right. \\ & \quad \left. \left. - |X_s - Y_s| - (X_s - Y_s)(e^u - 1) \right) ds \, \Pi(du) \right) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} & \lim_{k \rightarrow \infty} E \left(\int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{X_s > Y_s\}} \left(\frac{1}{Y_s} - \frac{1}{X_s} \right) \left(\phi_k(X_s - Y_s - Y_s(e^u - 1)) \right. \right. \\ & \quad \left. \left. - \phi_k(X_s - Y_s) + \phi'_k(X_s - Y_s)Y_s(e^u - 1) \right) ds \, \Pi(du) \right) \\ &= E \left(\int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{X_s > Y_s\}} \left(\frac{1}{Y_s} - \frac{1}{X_s} \right) \left(|X_s - Y_s - Y_s(e^u - 1)| \right. \right. \\ & \quad \left. \left. - |X_s - Y_s| + Y_s(e^u - 1) \right) ds \, \Pi(du) \right). \end{aligned} \tag{3.12}$$

Indeed, one can apply Fubini's theorem for (3.11) and (3.12), since in all cases the integrands are non-negative which follows by the mean-value theorem representation for the remainder of Taylor's theorem and $\phi''_k(x) = \psi_k(|x|) \geq 0$, $x \in \mathbb{R}$. Due to the pointwise convergences $\lim_{k \rightarrow \infty} \phi_k(x) = |x|$, $x \in \mathbb{R}$, and $\lim_{k \rightarrow \infty} \phi'_k(x) = \text{sign}(x)$ for $x \neq 0$ it suffices to verify dominated convergence for both integrals. To prove (3.11), we split the large and small negative jumps to obtain

$$\begin{aligned} & \mathbf{1}_{\{X_s > Y_s\}} \frac{1}{X_s} \left(\phi_k(X_s - Y_s + (X_s - Y_s)(e^u - 1)) \right. \\ & \quad \left. - \phi_k(X_s - Y_s) - \phi'_k(X_s - Y_s)(X_s - Y_s)(e^u - 1) \right) \\ & \leq \mathbf{1}_{\{X_s > Y_s\}} \mathbf{1}_{\{u < -1\}} \frac{1}{X_s} \phi'_k(X_s - Y_s)(X_s - Y_s)(1 - e^u) \\ & \quad + \mathbf{1}_{\{X_s > Y_s\}} \mathbf{1}_{\{-1 \leq u \leq 0\}} \frac{1}{X_s} \frac{1}{2} \phi''_k(\varrho_s)(X_s - Y_s)^2(e^u - 1)^2 \end{aligned}$$

for some $\varrho_s \in ((X_s - Y_s)e^u, X_s - Y_s)$, where we used again the mean-value theorem representation for the remainder of Taylor's theorem and that $\phi_k(x)$ is increasing in

$x \geq 0$ and $\phi'_k(x) \geq 0, x \in \mathbb{R}$. Using that $\phi'_k(x) \leq 1$ and $\phi''_k(x) \leq \frac{2}{kx}$ for $x > 0$ this gives the uniform in k upper bound

$$\mathbf{1}_{\{u < -1\}} + \mathbf{1}_{\{-1 \leq u \leq 0\}} e^{-u} (e^u - 1)^2$$

which is integrable with respect to $P \otimes \mathbf{1}_{\{s \leq t\}} ds \otimes \Pi(du)$. Hence, (3.11) follows from the dominated convergence theorem.

Now we turn to prove (3.12) which is more delicate as (3.12) corresponds to the jumps possibly contradicting (3.1) if u was positive, i.e. if the Lévy process ξ was not spectrally negative. We only use Taylor's theorem and the mean-value theorem to estimate

$$\begin{aligned} & \mathbf{1}_{\{X_s > Y_s\}} \left(\frac{1}{Y_s} - \frac{1}{X_s} \right) (\phi_k(X_s - Y_s - Y_s(e^u - 1)) - \phi_k(X_s - Y_s) + \phi'_k(X_s - Y_s) Y_s (e^u - 1)) \\ & \leq \mathbf{1}_{\{X_s > Y_s\}} \frac{1}{2} \phi''_k(\aleph_s) \frac{X_s - Y_s}{X_s Y_s} Y_s^2 (e^u - 1)^2, \end{aligned}$$

where $\aleph_s \in (X_s - Y_s, X_s - Y_s + Y_s(1 - e^u))$. The right-end of the interval is larger than the left-end due to the assumption $u \leq 0$. If u was positive, then $X_s - Y_s + Y_s(1 - e^u)$ might even become negative. In this case we should estimate ϕ''_k uniformly in k in an interval containing zero but, by definition, ϕ''_k explodes as k tends to infinity around zero. Using again that $\phi''_k(x) \leq \frac{2}{kx}$ for $x > 0$, we obtain the $P \otimes \mathbf{1}_{\{s \leq t\}} ds \otimes \Pi(du)$ -integrable upper bound $(e^u - 1)^2$ so that (3.12) and thus (3.8) is verified.

To verify (3.9), we proceed similarly. First note that for $x, y \geq 0$

$$\begin{aligned} & \int_0^\infty D_{\mathbf{h}(x,r) - \mathbf{h}(y,r)} \phi(x - y) dr \\ & = \mathbf{1}_{\{x > y\}} \int_0^{1/x} (|x - y + (-x + y)| - |x - y| - (-x + y)) dr \\ & \quad + \mathbf{1}_{\{x > y\}} \int_{1/x}^{1/y} (|x - y + y| - |x - y| - y) dr \\ & \quad + \mathbf{1}_{\{x \leq y\}} \int_0^{1/y} (|x - y + (-x + y)| - |x - y| + (-x + y)) dr \\ & \quad + \mathbf{1}_{\{x \leq y\}} \int_{1/y}^{1/x} (|x - y + (-x)| - |x - y| + (-x)) dr \\ & = 0 \end{aligned}$$

since all integrands are constant zero. In particular, this shows that

$$E \left(q \int_0^{t \wedge \tau_m} \int_0^\infty D_{\mathbf{h}(X_s,r) - \mathbf{h}(Y_s,r)} \phi(X_s - Y_s) ds dr \right) = 0$$

so that (3.9) follows if we can show that

$$\begin{aligned} & \lim_{k \rightarrow \infty} E \left(q \int_0^{t \wedge \tau_m} \int_0^\infty D_{\mathbf{h}(X_s,r) - \mathbf{h}(Y_s,r)} \phi_k(X_s - Y_s) ds dr \right) \\ & = E \left(q \int_0^{t \wedge \tau_m} \int_0^\infty D_{\mathbf{h}(X_s,r) - \mathbf{h}(Y_s,r)} \phi(X_s - Y_s) ds dr \right). \end{aligned}$$

As before, separating the four cases and integrating out r we see that we can equally show that for any $t \geq 0$ (the corresponding two cases for $X_s \leq Y_s$ are similar)

$$\begin{aligned} & \lim_{k \rightarrow \infty} E \left(q \int_0^t \mathbf{1}_{\{X_s > Y_s\}} \frac{1}{X_s} \left(-\phi_k(X_s - Y_s) + \phi'_k(X_s - Y_s)(X_s - Y_s) \right) ds \right) \\ & = E \left(q \int_0^t \mathbf{1}_{\{X_s > Y_s\}} \frac{1}{X_s} \left(-|X_s - Y_s| + X_s - Y_s \right) ds \right) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} & \lim_{k \rightarrow \infty} E \left(q \int_0^t \mathbf{1}_{\{X_s > Y_s\}} \left(\frac{1}{Y_s} - \frac{1}{X_s} \right) \left(\phi_k(X_s) - \phi_k(X_s - Y_s) - \phi'_k(X_s - Y_s) Y_s \right) ds \right) \\ &= E \left(q \int_0^t \mathbf{1}_{\{X_s > Y_s\}} \left(\frac{1}{Y_s} - \frac{1}{X_s} \right) \left(|X_s| - |X_s - Y_s| - Y_s \right) ds \right). \end{aligned} \tag{3.14}$$

Here we note that, as earlier, all the integrands are non-negative. The convergence of (3.13) follows by dominated convergence, since, using that $\phi_k(x) \geq 0$, $x \in \mathbb{R}$, and $\phi'_k(x) \leq 1$, $x \geq 0$, we have the $P \otimes \mathbf{1}_{\{s \leq t\}}$ -integrable upper bound

$$\mathbf{1}_{\{X_s > Y_s\}} \frac{1}{X_s} \left(-\phi_k(X_s - Y_s) + \phi'_k(X_s - Y_s)(X_s - Y_s) \right) \leq \mathbf{1}_{\{X_s > Y_s\}} \frac{X_s - Y_s}{X_s} \leq 1$$

for the integrand. For (3.14) we use dominated convergence based on

$$\begin{aligned} & \mathbf{1}_{\{X_s > Y_s\}} \left(\frac{1}{Y_s} - \frac{1}{X_s} \right) \left(\phi_k((X_s - Y_s) + Y_s) - \phi_k(X_s - Y_s) - \phi'_k(X_s - Y_s) Y_s \right) \\ &= \mathbf{1}_{\{X_s > Y_s\}} \left(\frac{X_s - Y_s}{X_s Y_s} \right) \frac{1}{2} \phi''_k(\aleph_s) Y_s^2 \end{aligned}$$

for some $\aleph_s \in (X_s - Y_s, X_s - Y_s + Y_s)$, so that $\phi''_k(x) \leq \frac{2}{kx}$, $x > 0$, gives the $P \otimes \mathbf{1}_{\{s \leq t\}}$ -integrable upper bound $\frac{1}{kX_s Y_s} Y_s^2 \leq 1$, $k \in \mathbb{N}$. This completes the proof. \square

We now turn our attention to the existence of solutions; the proof is given via a sequence of lemmas utilizing ideas of Fu and Li [18]. The strategy is to construct solutions by first considering the compensated equation suppressing all jumps, then adding jumps carefully (with truncations that allow to do this) and finish with a weak convergence argument.

We start with proving existence of a unique strong solution for the (ordinary) SDE obtained from (2.4) by truncations and omitting the jumps. Namely, for all $z \geq 0$ and $0 < \varepsilon < m$, let us consider the SDE

$$\begin{aligned} Z_t &= z + (\log E(e^{\xi_1}; \zeta > 1))t + \sigma \int_0^t \sqrt{(Z_s \wedge m)} dB_s \\ &+ q \int_0^t \int_0^{1/\varepsilon} \mathbf{1}_{\{rZ_s \leq 1\}} (Z_s \wedge m) ds dr \\ &- \int_0^t \int_0^{1/\varepsilon} \int_{|u| \geq \varepsilon} \mathbf{1}_{\{rZ_s \leq 1\}} (Z_s \wedge m) (e^u - 1) ds dr \Pi(du). \end{aligned} \tag{3.15}$$

An easy calculation shows that this is the same as the SDE

$$\begin{aligned} Z_t &= z + (\log E(e^{\xi_1}; \zeta > 1))t + \sigma \int_0^t \sqrt{Z_s \wedge m} dB_s \\ &- \left(\int_{|u| \geq \varepsilon} (e^u - 1) \Pi(du) - q \right) \int_0^t b^{\varepsilon, m}(Z_s) ds, \end{aligned} \tag{3.16}$$

where

$$b^{\varepsilon, m}(x) := \mathbf{1}_{\{x > 0\}} \frac{x \wedge m}{x \vee \varepsilon} = \begin{cases} \frac{m}{x} & \text{if } x \geq m, \\ 1 & \text{if } \varepsilon \leq x \leq m, \\ \frac{x}{\varepsilon} & \text{if } 0 \leq x \leq \varepsilon, \\ 0 & \text{if } x < 0. \end{cases}$$

Lemma 3.5. *Suppose that $\log E(e^{\xi_1}; \zeta > 1) > 0$ and $z \geq 0$. Then, for all $0 < \varepsilon < m$, there is a pathwise unique non-negative strong solution to the (ordinary) SDE (3.16) for $t \geq 0$.*

Proof. First let us consider the modified equation

$$Z_t = z + (\log E(e^{\xi_1}; \zeta > 1))t - \left(\int_{|u| \geq \varepsilon} (e^u - 1) \Pi(du) - q \right) \int_0^t b^{\varepsilon, m}(Z_s) ds + \sigma \int_0^t \sqrt{(Z_s \wedge m) \vee 0} dB_s, \quad t \geq 0, \tag{3.17}$$

so that the integrand of the Brownian integral is a priori well-defined. The drift is globally Lipschitz (since it is piecewise continuously differentiable with bounded derivatives) and further $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$, $x, y \geq 0$, so that the results of Yamada and Watanabe (see, e.g. Ikeda and Watanabe [19, Theorems IV.2.3 and IV.3.2]) ensure that the SDE (3.17) has a pathwise unique strong solution for all non-negative initial values $z \geq 0$. The non-negativity of the solution follows directly from the positivity of the drift close to zero. To give a precise argument we utilize Proposition 2.1 of Fu and Li [18]: Indeed, the functions

$$\mathbb{R} \ni x \mapsto \log E(e^{\xi_1}; \zeta > 1) - \left(\int_{|u| \geq \varepsilon} (e^u - 1) \Pi(du) - q \right) b^{\varepsilon, m}(x)$$

and

$$\mathbb{R} \ni x \mapsto \sqrt{(x \wedge m) \vee 0}$$

are continuous, $\sqrt{(x \wedge m) \vee 0} = 0$ if $x \leq 0$ and

$$\log E(e^{\xi_1}; \zeta > 1) - \left(\int_{|u| \geq \varepsilon} (e^u - 1) \Pi(du) - q \right) b^{\varepsilon, m}(x) > 0, \quad x \leq 0.$$

Since solutions are non-negative we can drop the additional maximum with 0 in the integrand for the Brownian integral so that there is a pathwise unique non-negative strong solution to (3.16). \square

Next, we add jumps to the SDE (3.16) (or equivalently to (3.15)). By the choice of the truncation, jumps come with finite rate so that they can be added merely "by hands" (the procedure is called interlacing).

Lemma 3.6. *Suppose that $\log E(e^{\xi_1}; \zeta > 1) > 0$ and $z \geq 0$. Then, for all $0 < \varepsilon < m$, there is a pathwise unique non-negative strong solution to*

$$Z_t = z + (\log E(e^{\xi_1}; \zeta > 1))t + \sigma \int_0^t \sqrt{(Z_s \wedge m)} dB_s - \int_0^t \int_0^{1/\varepsilon} \mathbf{1}_{\{rZ_{s-} \leq 1\}} (Z_s \wedge m) (\mathcal{M} - \mathcal{M}') (ds, dr) + \int_0^t \int_0^{1/\varepsilon} \int_{|u| \geq \varepsilon} \mathbf{1}_{\{rZ_{s-} \leq 1\}} (Z_{s-} \wedge m) (e^u - 1) (\mathcal{N} - \mathcal{N}') (ds, dr, du) \tag{3.18}$$

for $t \geq 0$.

Proof. First note that the existence of a pathwise unique non-negative strong solution to the SDE (3.15) for $t \geq 0$ follows by Lemma 3.5. Next, we add the jumps that are chosen according to independent Poisson random measures \mathcal{N} on $(0, \infty) \times (0, \infty) \times \mathbb{R}$

with intensity measure $\mathcal{N}'(ds, dr, du) = ds \otimes dr \otimes \Pi(du)$ and \mathcal{M} on $(0, \infty) \times (0, \infty)$ with intensity measure $\mathcal{M}'(ds, dr) = q ds \otimes dr$. Note that the truncated SDE (3.16) was chosen in a way that it only contains the compensation on sets with finite intensity. Hence, only finitely many jumps have to be added in finite time intervals. Those can be added via interlacing, i.e. using the pathwise unique strong solution of the SDE (3.16), as in the proof of Proposition 2.2 in Fu and Li [18]. To keep the proof shorter we assume $q = 0$ since the additional integral corresponding to killing can be dealt with likewise. Let $\{S_k : k \in \mathbb{N}\}$ be the set of jump times of the Poisson process

$$\mathbb{R} \ni t \mapsto \int_0^t \int_0^{1/\varepsilon} \int_{|u| \geq \varepsilon} 1 \mathcal{N}(ds, dr, du) = \mathcal{N}((0, t) \times (0, 1/\varepsilon) \times \{u \in \mathbb{R} : |u| \geq \varepsilon\}).$$

Clearly, S_k is a sum of independent exponentially distributed random variables with parameter $\frac{1}{\varepsilon} \Pi(|u| \geq \varepsilon)$ and hence we have $S_k \rightarrow \infty$ almost surely as $k \rightarrow \infty$. For $0 \leq t < S_1$, let $Z_t^{\varepsilon, m}$ be the solution of (3.15) with $q = 0$ given by Lemma 3.5. To set up an induction, suppose that $Z_t^{\varepsilon, m}$ has been defined for $0 \leq t < S_k$, and let

$$\xi := Z_{S_k}^{\varepsilon, m} + \int_{\{S_k\}} \int_0^{1/\varepsilon} \int_{|u| \geq \varepsilon} \mathbf{1}_{\{r Z_{s-}^{\varepsilon, m} \leq 1\}} (Z_{s-}^{\varepsilon, m} \wedge m) (e^u - 1) \mathcal{N}(ds, dr, du). \quad (3.19)$$

Note that ξ is non-negative which follows by the simple fact that $x + \mathbf{1}_{\{x \leq \varepsilon\}}(x \wedge m)(e^u - 1) \geq 0$, $x \geq 0$, $u \in \mathbb{R}$. Since the SDE (3.16) has a pathwise unique non-negative strong solution, there is also a pathwise unique non-negative strong solution, say $(X_k(t))_{t \geq 0}$, to the SDE

$$\begin{aligned} Z_t &= \xi + (\log E(e^{\xi_1}))t + \sigma \int_0^t \sqrt{(Z_s \wedge m)} dB_{S_k+s} \\ &\quad - \int_{|u| \geq \varepsilon} (e^u - 1) \Pi(du) \int_0^t b^{\varepsilon, m}(Z_s) ds, \quad t \geq 0. \end{aligned} \quad (3.20)$$

Here we call the attention to the fact that the SDE (3.16) (and hence the SDE (3.20), too) has a pathwise unique non-negative strong solution for all non-negative random initial condition (indeed, Yamada and Watanabe's theorems and Proposition 2.1 in Fu and Li [18] are valid for non-negative random initial conditions, too). We also note that the strong Markov property of B and the independence of B and \mathcal{N} yield that $(B_{S_k+t} - B_{S_k})_{t \geq 0}$ is a standard Brownian motion with respect to its natural filtration and it is independent of \mathcal{G}_{S_k+} , see, e.g. Karatzas and Shreve [21, Theorem 2.6.16]. For $S_k \leq t < S_{k+1}$, we set $Z_t^{\varepsilon, m} := X_k(t - S_k)$. Then, for $S_k \leq t < S_{k+1}$, we have

$$\begin{aligned} Z_t^{\varepsilon, m} &= X_k(t - S_k) = \xi + (\log E(e^{\xi_1}))(t - S_k) + \sigma \int_0^{t-S_k} \sqrt{(X_k(s) \wedge m)} dB_{S_k+s} \\ &\quad - \int_{|u| \geq \varepsilon} (e^u - 1) \Pi(du) \int_0^{t-S_k} b^{\varepsilon, m}(X_k(s)) ds \\ &= Z_{S_k}^{\varepsilon, m} + \int_{\{S_k\}} \int_0^{1/\varepsilon} \int_{|u| \geq \varepsilon} \mathbf{1}_{\{r Z_{s-}^{\varepsilon, m} \leq 1\}} (Z_{s-}^{\varepsilon, m} \wedge m) (e^u - 1) \mathcal{N}(ds, dr, du) \\ &\quad + (\log E(e^{\xi_1}))(t - S_k) + \sigma \int_{S_k}^t \sqrt{(Z_s^{\varepsilon, m} \wedge m)} dB_s \\ &\quad - \int_{|u| \geq \varepsilon} (e^u - 1) \Pi(du) \int_{S_k}^t b^{\varepsilon, m}(Z_s^{\varepsilon, m}) ds. \end{aligned}$$

By the induction hypothesis, we obtain

$$\begin{aligned} Z_{S_k}^{\varepsilon,m} &= z + (\log E(e^{\xi_1}))S_k + \sigma \int_0^{S_k} \sqrt{(Z_s^{\varepsilon,m} \wedge m)} dB_s \\ &\quad - \int_0^{S_k} \int_0^{1/\varepsilon} \int_{|u| \geq \varepsilon} \mathbf{1}_{\{rZ_s^{\varepsilon,m} \leq 1\}} (Z_s^{\varepsilon,m} \wedge m)(e^u - 1) ds dr \Pi(du) \\ &\quad + \sum_{\ell=1}^{k-1} \int_{\{S_\ell\}} \int_0^{1/\varepsilon} \int_{|u| \geq \varepsilon} \mathbf{1}_{\{rZ_{s-}^{\varepsilon,m} \leq 1\}} (Z_{s-}^{\varepsilon,m} \wedge m)(e^u - 1) \mathcal{N}(ds, dr, du), \end{aligned}$$

so that, for $S_k \leq t < S_{k+1}$, $Z_t^{\varepsilon,m}$ satisfies the SDE

$$\begin{aligned} Z_t &= z + (\log E(e^{\xi_1}))t + \sigma \int_0^t \sqrt{(Z_s \wedge m)} dB_s - \int_{|u| \geq \varepsilon} (e^u - 1) \Pi(du) \int_0^t b^{\varepsilon,m}(Z_s) ds \\ &\quad + \int_0^t \int_0^{1/\varepsilon} \int_{|u| \geq \varepsilon} \mathbf{1}_{\{rZ_{s-} \leq 1\}} (Z_{s-} \wedge m)(e^u - 1) \mathcal{N}(ds, dr, du). \end{aligned} \tag{3.21}$$

Hence, by induction, this defines a process $(Z_t^{\varepsilon,m})_{t \geq 0}$ which is a non-negative strong solution to the SDE (3.21) for all $t \geq 0$. By calculating the compensated integral in the SDE (3.18) we get that the SDE (3.21) is the same as the SDE (3.18) yielding that $Z^{\varepsilon,m}$ is a non-negative strong solution of (3.18). The non-negativity of $Z^{\varepsilon,m}$ follows by the construction. Finally, pathwise uniqueness for the SDE (3.18) follows from that of (3.16) and (3.20). Indeed, since jumps come with finite rate, uniqueness holds between the jumps so that also the jumps themselves are uniquely determined. \square

So far, we got around the problem of having infinitely many small jumps by cutting the Lévy measure Π at ε . Note also that infinitely many large jumps were avoided by cutting the state-dependent jump intensity by $1/\varepsilon$. This allowed us to construct solutions by standard SDE theory and "by hands" (interlacing procedure). Next, we get rid of these two restrictions via martingale problem arguments. The additional truncation by m remains in order to ensure tightness when ε goes to zero.

Lemma 3.7. *Suppose that $\log E(e^{\xi_1}; \zeta > 0) > 0$, $z \geq 0$ and for all $0 < \varepsilon < m$, let $Z^{\varepsilon,m}$ be the pathwise unique non-negative strong solution to (3.18). Then, for every $m > 0$ and every sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ tending to zero, the family $\{Z^{\varepsilon_n,m} : n \in \mathbb{N}\}$ is tight in Skorokhod's J_1 topology on \mathbb{D} .*

Proof. For the proof we apply Aldous's tightness criterion (see Aldous [1] or Chapter 3.8 of Ethier and Kurtz [16]). According to this, to prove that $\{Z^{\varepsilon_n,m} : n \in \mathbb{N}\}$ is tight in \mathbb{D} it is enough to show that

- (i) for every fixed $t \geq 0$, the set of random variables $\{Z_t^{\varepsilon_n,m} : n \in \mathbb{N}\}$ is tight,
- (ii) for every sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ (with respect to the filtration $(\mathcal{G}_t)_{t \geq 0}$) bounded above by $T > 0$ and for every sequence of positive real numbers $(\delta_n)_{n \in \mathbb{N}}$ converging to 0, $Z_{\tau_n + \delta_n}^{\varepsilon_n,m} - Z_{\tau_n}^{\varepsilon_n,m} \rightarrow 0$ in probability as $n \rightarrow \infty$.

To prove (i), by Markov's inequality it is enough to check that, for every fixed $t \geq 0$,

$$\sup_{n \in \mathbb{N}} E \left[(Z_t^{\varepsilon_n,m})^2 \right] < \infty. \tag{3.22}$$

Using that $(a + b + c + d + e)^2 \leq 5(a^2 + b^2 + c^2 + d^2 + e^2)$, $a, b, c, d, e \in \mathbb{R}$, we obtain that $E\left[(Z_t^{\varepsilon_n, m})^2\right]$ can be bounded by

$$\begin{aligned} & 5z^2 + 5(\log E(e^{\xi_1}; \zeta > 1))^2 t^2 + 5\sigma^2 E\left(\left(\int_0^t \sqrt{(Z_s^{\varepsilon_n, m} \wedge m)} dB_s\right)^2\right) \\ & + 5E\left(\left(\int_0^t \int_0^{1/\varepsilon_n} \mathbf{1}_{\{rZ_{s-}^{\varepsilon_n, m} \leq 1\}} (Z_{s-}^{\varepsilon_n, m} \wedge m)(\mathcal{M} - \mathcal{M}')(\mathrm{d}s, \mathrm{d}r)\right)^2\right) \\ & + 5E\left(\left(\int_0^t \int_0^{1/\varepsilon_n} \int_{|u| \geq \varepsilon_n} \mathbf{1}_{\{rZ_{s-}^{\varepsilon_n, m} \leq 1\}} (Z_{s-}^{\varepsilon_n, m} \wedge m)(e^u - 1)(\mathcal{N} - \mathcal{N}')(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u)\right)^2\right), \end{aligned}$$

which, by Itô's isometry and page 62 in Ikeda and Watanabe [19], can be bounded from above by

$$\begin{aligned} & 5z^2 + 5(\log E(e^{\xi_1}; \zeta > 1))^2 t^2 + 5\sigma^2 E\left(\int_0^t (Z_s^{\varepsilon_n, m} \wedge m) \mathrm{d}s\right) \\ & + 5E\left(q \int_0^t \int_0^{1/\varepsilon_n} \mathbf{1}_{\{rZ_{s-}^{\varepsilon_n, m} \leq 1\}} (Z_{s-}^{\varepsilon_n, m} \wedge m)^2 \mathrm{d}s \mathrm{d}r\right) \\ & + 5E\left(\int_0^t \int_0^{1/\varepsilon_n} \int_{|u| \geq \varepsilon_n} \mathbf{1}_{\{rZ_{s-}^{\varepsilon_n, m} \leq 1\}} (Z_{s-}^{\varepsilon_n, m} \wedge m)^2 (e^u - 1)^2 \mathrm{d}s \mathrm{d}r \Pi(\mathrm{d}u)\right) \\ & \leq 5z^2 + 5(\log E(e^{\xi_1}; \zeta > 1))^2 t^2 + 5\sigma^2 mt + 5E\left(q \int_0^t (Z_s^{\varepsilon_n, m} \wedge m) \mathrm{d}s\right) \\ & + 5E\left(\int_0^t \int_{|u| \geq \varepsilon_n} (Z_s^{\varepsilon_n, m} \wedge m)(e^u - 1)^2 \mathrm{d}s \Pi(\mathrm{d}u)\right) \\ & \leq 5z^2 + 5(\log E(e^{\xi_1}; \zeta > 1))^2 t^2 + 5mt \left(\sigma^2 + q + \int_{|u| \geq \varepsilon_n} (e^u - 1)^2 \Pi(\mathrm{d}u)\right), \quad t \geq 0, \end{aligned}$$

which is uniformly bounded in $n \in \mathbb{N}$ since $E(e^{2\xi_1}) < \infty$ and Π is a Lévy measure.

Now we turn to (ii) proving the stronger statement that $Z_{\tau_n + \delta_n}^{\varepsilon_n, m} - Z_{\tau_n}^{\varepsilon_n, m}$ converges to 0 in L^2 as $n \rightarrow \infty$. Namely, by the SDE (3.18) and using that $(a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)$, we have

$$\begin{aligned} & E\left(|Z_{\tau_n + \delta_n}^{\varepsilon_n, m} - Z_{\tau_n}^{\varepsilon_n, m}|^2\right) \\ & = E\left[\left|(\log E(e^{\xi_1}; \zeta > 1))\delta_n + \sigma \int_{\tau_n}^{\tau_n + \delta_n} \sqrt{(Z_s^{\varepsilon_n, m} \wedge m)} \mathrm{d}B_s\right.\right. \\ & \quad - \int_{\tau_n}^{\tau_n + \delta_n} \int_0^{1/\varepsilon_n} \mathbf{1}_{\{rZ_{s-}^{\varepsilon_n, m} \leq 1\}} (Z_{s-}^{\varepsilon_n, m} \wedge m)(\mathcal{M} - \mathcal{M}')(\mathrm{d}s, \mathrm{d}r) \\ & \quad \left.\left. + \int_{\tau_n}^{\tau_n + \delta_n} \int_0^{1/\varepsilon_n} \int_{|u| \geq \varepsilon_n} \mathbf{1}_{\{rZ_{s-}^{\varepsilon_n, m} \leq 1\}} (Z_{s-}^{\varepsilon_n, m} \wedge m)(e^u - 1)(\mathcal{N} - \mathcal{N}')(\mathrm{d}s, \mathrm{d}r, \mathrm{d}u)\right|^2\right] \end{aligned}$$

$$\begin{aligned} &\leq 4(\log E(e^{\xi_1}; \zeta > 1))^2 \delta_n^2 + 4\sigma^2 E \left(\int_{\tau_n}^{\tau_n + \delta_n} \sqrt{(Z_s^{\varepsilon_n, m} \wedge m)} dB_s \right)^2 \\ &+ 4E \left(\int_{\tau_n}^{\tau_n + \delta_n} \int_0^{1/\varepsilon_n} \mathbf{1}_{\{rZ_{s-}^{\varepsilon_n, m} \leq 1\}} (Z_{s-}^{\varepsilon_n, m} \wedge m) (\mathcal{M} - \mathcal{M}') (ds, dr) \right)^2 \\ &+ 4E \left(\int_{\tau_n}^{\tau_n + \delta_n} \int_0^{1/\varepsilon_n} \int_{|u| \geq \varepsilon_n} \mathbf{1}_{\{rZ_{s-}^{\varepsilon_n, m} \leq 1\}} (Z_{s-}^{\varepsilon_n, m} \wedge m) (e^u - 1) (\mathcal{N} - \mathcal{N}') (ds, dr, du) \right)^2. \end{aligned}$$

By Proposition 3.2.10 in Karatzas and Shreve [21],

$$E \left(\int_{\tau_n}^{\tau_n + \delta_n} \sqrt{(Z_s^{\varepsilon_n, m} \wedge m)} dB_s \right)^2 = E \left(\int_{\tau_n}^{\tau_n + \delta_n} (Z_s^{\varepsilon_n, m} \wedge m) ds \right),$$

and, by Theorem II.1.33 in Jacod and Shiryaev [20], the optimal stopping theorem, using the same arguments as in the proof of (3.2.22) in Karatzas and Shreve [21], we get

$$\begin{aligned} &E \left(\int_{\tau_n}^{\tau_n + \delta_n} \int_0^{1/\varepsilon_n} \mathbf{1}_{\{rZ_{s-}^{\varepsilon_n, m} \leq 1\}} (Z_{s-}^{\varepsilon_n, m} \wedge m) (\mathcal{M} - \mathcal{M}') (ds, dr) \right)^2 \\ &= E \left(q \int_{\tau_n}^{\tau_n + \delta_n} \int_0^{1/\varepsilon_n} \mathbf{1}_{\{rZ_{s-}^{\varepsilon_n, m} \leq 1\}} (Z_{s-}^{\varepsilon_n, m} \wedge m)^2 ds dr \right) \end{aligned}$$

and

$$\begin{aligned} &E \left(\int_{\tau_n}^{\tau_n + \delta_n} \int_0^{1/\varepsilon_n} \int_{|u| \geq \varepsilon_n} \mathbf{1}_{\{rZ_{s-}^{\varepsilon_n, m} \leq 1\}} (Z_{s-}^{\varepsilon_n, m} \wedge m) (e^u - 1) (\mathcal{N} - \mathcal{N}') (ds, dr, du) \right)^2 \\ &= E \left(\int_{\tau_n}^{\tau_n + \delta_n} \int_0^{1/\varepsilon_n} \int_{|u| \geq \varepsilon_n} \mathbf{1}_{\{rZ_{s-}^{\varepsilon_n, m} \leq 1\}} (Z_{s-}^{\varepsilon_n, m} \wedge m)^2 (e^u - 1)^2 ds dr \Pi(du) \right). \end{aligned}$$

Hence,

$$\begin{aligned} &E (|Z_{\tau_n + \delta_n}^{\varepsilon_n, m} - Z_{\tau_n}^{\varepsilon_n, m}|^2) \\ &\leq 4(\log E(e^{\xi_1}; \zeta > 1))^2 \delta_n^2 + 4m\delta_n \left(\sigma^2 + q + \int_{|u| \geq \varepsilon_n} (e^u - 1)^2 \Pi(du) \right), \end{aligned}$$

which tends to zero as n tends to infinity. This proves (ii) and finishes the proof of the lemma. \square

Having proved the tightness we identify the limit points as weak solutions to

$$\begin{aligned} Z_t &= z + (\log E(e^{\xi_1}; \zeta > 1))t + \sigma \int_0^t \sqrt{(Z_s \wedge m)} dB_s \\ &- \int_0^t \int_0^\infty (Z_{s-} \wedge m) (\mathcal{M} - \mathcal{M}') (ds, dr) \\ &+ \int_0^t \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{rZ_{s-} \leq 1\}} (Z_{s-} \wedge m) (e^u - 1) (\mathcal{N} - \mathcal{N}') (ds, dr, du). \end{aligned} \tag{3.23}$$

This is achieved via martingale problem characterizations of jump type SDEs.

Lemma 3.8. *Let $m > 0$ and Z^m be any limiting point of the tight family $\{Z^{\varepsilon_k, m} : k \in \mathbb{N}\}$ (given in Lemma 3.7) as $\varepsilon_k \downarrow 0$. Then Z^m is a weak solution to (3.23) for $t \geq 0$.*

Proof. First note that, due to Itô's formula, $Z^{\varepsilon_k, m}$ solves the following martingale problem:

$$M_t^{\varepsilon_k, m} := f(Z_t^{\varepsilon_k, m}) - f(z) - \int_0^t (\mathcal{A}^{\varepsilon_k, m} f)(Z_s^{\varepsilon_k, m}) ds, \quad t \geq 0,$$

is a martingale for all $f \in C_b^2(\mathbb{R}_+)$, where

$$\begin{aligned} (\mathcal{A}^{\varepsilon_k, m} f)(x) &:= \frac{1}{2} \sigma^2(x \wedge m) f''(x) + (\log E(e^{\xi_1}; \zeta > 1)) f'(x) \\ &+ q \int_0^{1/\varepsilon_k} \left(f(x - \mathbf{1}_{\{rx \leq 1\}}(x \wedge m)) - f(x) + f'(x) \mathbf{1}_{\{rx \leq 1\}}(x \wedge m) \right) dr \\ &+ \int_0^{1/\varepsilon_k} \int_{|u| \geq \varepsilon_k} \left(f(x + \mathbf{1}_{\{rx \leq 1\}}(x \wedge m)(e^u - 1)) - f(x) \right. \\ &\quad \left. - f'(x) \mathbf{1}_{\{rx \leq 1\}}(x \wedge m)(e^u - 1) \right) dr \Pi(du), \quad x \geq 0, \end{aligned}$$

and $C_b^2(\mathbb{R}_+)$ denotes the set of bounded twice continuously differentiable functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ of which the first and second derivatives are also bounded. Without loss of generality we can assume that $(\varepsilon_k)_{k \in \mathbb{N}}$ realizes the weak convergence of $Z^{\varepsilon_k, m}$ to Z^m in \mathbb{D} as $k \rightarrow \infty$. By Skorokhod's representation theorem and Riesz's theorem we may assume that $Z^{\varepsilon_k, m}$ converges to Z^m almost surely in \mathbb{D} (possibly on a different probability space and changing also the subsequence $(\varepsilon_k)_{k \in \mathbb{N}}$). By Ethier and Kurtz [16, page 118], this yields that $P(\bar{\Omega}) = 1$, where

$$\bar{\Omega} := \left\{ \omega \in \Omega : \lim_{k \rightarrow \infty} Z_t^{\varepsilon_k, m}(\omega) = Z_t^m(\omega) \text{ for } t \geq 0 \text{ at which } (Z_u^m(\omega))_{u \geq 0} \text{ is continuous} \right\}.$$

Next we show that for all $f \in C_b^2(\mathbb{R}_+)$,

$$M_t^m := f(Z_t^m) - f(z) - \int_0^t (\mathcal{A}^m f)(Z_s^m) ds, \quad t \geq 0, \tag{3.24}$$

is a martingale with respect to its natural filtration, where

$$\begin{aligned} (\mathcal{A}^m f)(x) &:= \frac{1}{2} \sigma^2(x \wedge m) f''(x) + (\log E(e^{\xi_1}; \zeta > 1)) f'(x) \\ &+ q \int_0^\infty \left(f(x - \mathbf{1}_{\{rx \leq 1\}}(x \wedge m)) - f(x) + f'(x) \mathbf{1}_{\{rx \leq 1\}}(x \wedge m) \right) dr \\ &+ \int_0^\infty \int_{\mathbb{R}} \left(f(x + \mathbf{1}_{\{rx \leq 1\}}(x \wedge m)(e^u - 1)) - f(x) \right. \\ &\quad \left. - f'(x) \mathbf{1}_{\{rx \leq 1\}}(x \wedge m)(e^u - 1) \right) dr \Pi(du), \quad x \geq 0. \end{aligned}$$

In what follows let $\omega \in \bar{\Omega}$ be fixed. First, using the dominated convergence theorem, we show that

$$\lim_{k \rightarrow \infty} \int_0^t (\mathcal{A}^{\varepsilon_k, m} f)(Z_s^{\varepsilon_k, m})(\omega) ds = \int_0^t (\mathcal{A}^m f)(Z_s^m)(\omega) ds, \quad t \geq 0. \tag{3.25}$$

Let us introduce the notation

$$D_m(\omega) := \{t \geq 0 : (Z_u^m(\omega))_{u \geq 0} \text{ is continuous at } t\}, \quad m > 0.$$

Then $\lim_{k \rightarrow \infty} Z_t^{\varepsilon_k, m}(\omega) = Z_t^m(\omega)$ for all $t \in D_m(\omega)$, and, by Ethier and Kurtz [16, page 114], $[0, \infty) \setminus D_m(\omega)$ is at most countable. Using the boundedness and continuity of f , f' and f'' , we check that for the integrands of the left-hand side of (3.25)

$$\lim_{k \rightarrow \infty} (\mathcal{A}^{\varepsilon_k, m} f)(Z_s^{\varepsilon_k, m})(\omega) = (\mathcal{A}^m f)(Z_s^m)(\omega) \quad \text{for all } s \in D_m(\omega). \tag{3.26}$$

Since $s \in D_m(\omega)$ and f', f'' are continuous, the first two summands in $(\mathcal{A}^{\varepsilon_k, m} f)(Z_s^{\varepsilon_k, m})(\omega)$ trivially converge as $k \rightarrow \infty$ to that of $(\mathcal{A}^m f)(Z_s^m)(\omega)$ so that it suffices to show the convergence of the integrals in $(\mathcal{A}^{\varepsilon_k, m} f)(Z_s^{\varepsilon_k, m})(\omega)$. In the following we show the convergence for the second integral, the arguments for the integral corresponding to killing are similar (replacing $(e^u - 1)^2$ by 1 and skipping the integration with respect to u).

On the one hand, for all $m > 0$ and $s \in D_m(\omega)$, the integrand (as a function of r and u)

$$\begin{aligned} & \mathbf{1}_{\{r \leq 1/\varepsilon_k\}} \mathbf{1}_{\{|u| \geq \varepsilon_k\}} f(Z_s^{\varepsilon_k, m}(\omega) + \mathbf{1}_{\{r Z_s^{\varepsilon_k, m}(\omega) \leq 1\}} (Z_s^{\varepsilon_k, m}(\omega) \wedge m)(e^u - 1)) - f(Z_s^{\varepsilon_k, m}(\omega)) \\ & - f'(Z_s^{\varepsilon_k, m}(\omega)) \mathbf{1}_{\{r Z_s^{\varepsilon_k, m}(\omega) \leq 1\}} (Z_s^{\varepsilon_k, m}(\omega) \wedge m)(e^u - 1) \end{aligned}$$

converges, as $k \rightarrow \infty$, pointwise to

$$\begin{aligned} & f(Z_s^m(\omega) + \mathbf{1}_{\{r Z_s^m(\omega) \leq 1\}} (Z_s^m(\omega) \wedge m)(e^u - 1)) - f(Z_s^m(\omega)) \\ & - f'(Z_s^m(\omega)) \mathbf{1}_{\{r Z_s^m(\omega) \leq 1\}} (Z_s^m(\omega) \wedge m)(e^u - 1) \end{aligned}$$

possibly except at the point $r = Z_s^m(\omega)$ (it can only occur if $Z_s^m(\omega) > 0$). On the other hand, by Taylor expansion of second order, we can derive the upper bound

$$\begin{aligned} & \sup_{k \in \mathbb{N}} \mathbf{1}_{\{r \leq 1/\varepsilon_k\}} \mathbf{1}_{\{|u| \geq \varepsilon_k\}} \left| f(Z_s^{\varepsilon_k, m}(\omega) + \mathbf{1}_{\{r Z_s^{\varepsilon_k, m}(\omega) \leq 1\}} (Z_s^{\varepsilon_k, m}(\omega) \wedge m)(e^u - 1)) - f(Z_s^{\varepsilon_k, m}(\omega)) \right. \\ & \quad \left. - f'(Z_s^{\varepsilon_k, m}(\omega)) \mathbf{1}_{\{r Z_s^{\varepsilon_k, m}(\omega) \leq 1\}} (Z_s^{\varepsilon_k, m}(\omega) \wedge m)(e^u - 1) \right| \\ & \leq \frac{1}{2} \sup_{x \in \mathbb{R}_+} |f''(x)| \sup_{k \in \mathbb{N}} \mathbf{1}_{\{r Z_s^{\varepsilon_k, m}(\omega) \leq 1\}} (Z_s^{\varepsilon_k, m}(\omega) \wedge m)^2 (e^u - 1)^2. \end{aligned}$$

The upper bound is integrable with respect to $dr \otimes \Pi(du)$, since an easy calculation shows that, for all $m > 0$ and $s \in D_m(\omega)$,

$$\mathbf{1}_{\{r Z_s^{\varepsilon_k, m}(\omega) \leq 1\}} (Z_s^{\varepsilon_k, m}(\omega) \wedge m)^2 \leq \left(\frac{1}{r} \wedge m \right)^2, \quad k \in \mathbb{N},$$

and hence the integral in $(\mathcal{A}^{\varepsilon_k, m} f)(Z_s^{\varepsilon_k, m})(\omega)$ is bounded above by

$$\begin{aligned} & \frac{1}{2} \sup_{x \in \mathbb{R}_+} |f''(x)| \int_0^\infty \left(\frac{1}{r} \wedge m \right)^2 dr \int_{\mathbb{R}} (e^u - 1)^2 \Pi(du) \\ & = \frac{1}{2} \sup_{x \in \mathbb{R}_+} |f''(x)| (2m) \int_{\mathbb{R}} (e^u - 1)^2 \Pi(du). \end{aligned}$$

Using dominated convergence theorem we have (3.26). Next, using again dominated convergence theorem, we check (3.25). By (3.26), using also that $[0, \infty) \setminus D_m(\omega)$ is at most countable, for all $t \geq 0$,

$$\lim_{k \rightarrow \infty} (\mathcal{A}^{\varepsilon_k, m} f)(Z_s^{\varepsilon_k, m})(\omega) = (\mathcal{A}^m f)(Z_s^m)(\omega) \quad \text{Lebesgue a.e. } s \in [0, t].$$

Further, for all $s \in [0, t]$,

$$\begin{aligned} \sup_{k \in \mathbb{N}} (\mathcal{A}^{\varepsilon_k, m} f)(Z_s^{\varepsilon_k, m})(\omega) & \leq \frac{\sigma^2 m}{2} \sup_{x \in \mathbb{R}_+} |f''(x)| + (\log E(e^{\zeta^1}); \zeta > 1) \sup_{x \in \mathbb{R}_+} |f'(x)| \\ & \quad + qm \sup_{x \in \mathbb{R}_+} |f''(x)| + m \sup_{x \in \mathbb{R}_+} |f''(x)| \int_{\mathbb{R}} (e^u - 1)^2 \Pi(du), \end{aligned} \tag{3.27}$$

which is trivially integrable over $[0, t]$ with respect to s . Hence (3.25) follows by dominated convergence theorem. Note also that, since $P(\bar{\Omega}) = 1$ and almost sure convergence implies weak convergence, we have

$$\left(\int_0^t (\mathcal{A}^{\varepsilon_k, m} f)(Z_s^{\varepsilon_k, m}) ds \right)_{t \geq 0} \text{ converges weakly to } \left(\int_0^t (\mathcal{A}^m f)(Z_s^m) ds \right)_{t \geq 0} \quad (3.28)$$

in \mathbb{D} as $k \rightarrow \infty$.

The martingale property of M^m with respect to its own filtration can be checked as follows. First, by Jacod and Shiryaev [20, Corollary IX.1.19], we show that $(M_t^m)_{t \geq 0}$ is a local martingale. To check the conditions of this corollary we have to show that $M^{\varepsilon_k, m}$ converges weakly in \mathbb{D} as $k \rightarrow \infty$ to M^m and that there is some $b \geq 0$ such that $|M_t^{\varepsilon_k, m} - M_{t-}^{\varepsilon_k, m}| \leq b$ for all $t > 0, k, m \in \mathbb{N}$, almost surely. Using that $Z^{\varepsilon_k, m}$ converges weakly in \mathbb{D} to Z^m as $k \rightarrow \infty$ and f is continuous and bounded, we have $f(Z^{\varepsilon_k, m})$ converges weakly in \mathbb{D} to $f(Z^m)$ as $k \rightarrow \infty$. Since the limit process in (3.28) is continuous, by Jacod and Shiryaev [20, Proposition VI.1.23], we obtain that $M^{\varepsilon_k, m}$ converges weakly in \mathbb{D} as $k \rightarrow \infty$ to M^m . Further, almost surely for all $t \geq 0$,

$$|M_t^{\varepsilon_k, m} - M_{t-}^{\varepsilon_k, m}| = |f(Z_t^{\varepsilon_k, m}) - f(Z_{t-}^{\varepsilon_k, m})| \leq 2 \sup_{x \in \mathbb{R}_+} |f(x)| < \infty.$$

The martingale property of the local martingale M^m follows by Jacod and Shiryaev [20, Proposition I.1.47], since, by the very same arguments as before, for all $T > 0$,

$$E \left(\sup_{t \in [0, T]} |M_t^m| \right) \leq 2 \sup_{x \in \mathbb{R}_+} |f(x)| + T \left(\frac{\sigma^2 m}{2} \sup_{x \in \mathbb{R}_+} |f''(x)| + (\log E(e^{\xi^1}; \zeta > 1)) \sup_{x \in \mathbb{R}_+} |f'(x)| + qm \sup_{x \in \mathbb{R}_+} |f''(x)| + m \sup_{x \in \mathbb{R}_+} |f''(x)| \int_{\mathbb{R}} (e^u - 1)^2 \Pi(du) \right) < \infty.$$

To finish the proof, we can resort to Proposition 4.2 of Fu and Li [18] (see also Theorem II.2.42 of Jacod and Shiryaev [20]). Since Z^m is a solution to the martingale problem (3.24), it is then a weak solution of the SDE (3.23). \square

The last step for the existence proof is to get rid of the additional truncation by m which was only included to validate Aldous's criterion in Lemma 3.7. We start with a simple observation.

Lemma 3.9. *Under the conditions of Theorem 2.2 pathwise uniqueness for non-negative solutions holds for the truncated SDE (3.23) for $t \geq 0$.*

Proof. This follows along the lines of the proof of Proposition 3.4 for the modified SDE (3.23). \square

Combining Lemma 3.9 with Lemma 3.8, we can construct a sequence $(Z^m)_{m \in \mathbb{N}}$ of strong solutions to the jump type SDEs (3.23) on the same probability space. Indeed, under pathwise uniqueness weak solutions are automatically strong solutions, see, e.g. Situ [31, page 104], and they can be defined on the same probability space, since, by the adaptedness, strong solutions are deterministic functions of the driving noises B, \mathcal{M} and \mathcal{N} . Let us define by

$$\tau_m := \inf\{t \geq 0 : Z_t^m \geq m\}$$

the sequence of exceeding times of level m .

Lemma 3.10. *The sequence $(\tau_m)_{m \in \mathbb{N}}$ is increasing and tends to infinity almost surely.*

Proof. As in Lemma 3.3 we can assume $q = 0$ as the killing integral does not influence the explosion in finite time. As long as $Z_t^m \leq m$ taking minimum with m is superfluous so that Z^m satisfies the SDE (2.4) up to time τ_m . Hence, by Proposition 3.4, for all $m \leq m', m, m' \in \mathbb{N}$, two solutions Z^m and $Z^{m'}$ are almost surely identical until τ_m . This implies that $\tau_m \leq \tau_{m'}$ almost surely for all $m \leq m'$ yielding that the limit $\lim_{m \rightarrow \infty} \tau_m$ exists almost surely.

To show that τ_m indeed converges to infinity we use an estimate of Fu and Li [18]. By the proof of their Proposition 2.3 we want to use the upper bound

$$(1 + m)P(\tau_m < t) \leq (1 + z) \exp((\log E(e^{\xi_1}))t) = (1 + z)(E(e^{\xi_1}))^t, \quad t \geq 0. \quad (3.29)$$

To check that we are allowed to work with (3.29), conditions (2.a) and (2.b) of Fu and Li [18] need to be verified. This can be done almost identical to the proof of Lemma 3.3 modifying the definitions of the functions σ and g_0 given there to $\sigma\sqrt{x \wedge m}$, $x \geq 0$, and to $\mathbf{1}_{\{xr \leq 1\}}(x \wedge m)(e^u - 1)$, $x \in \mathbb{R}$, $r \geq 0$, $u \in \mathbb{R}$, respectively.

Let us finally check that (3.29) implies $P(\lim_{m \rightarrow \infty} \tau_m = \infty) = 1$. Suppose that $\tau := \lim_{m \rightarrow \infty} \tau_m$ (this limit exists with probability one since the sequence $(\tau_m)_{m \in \mathbb{N}}$ is monotone increasing), then, by (3.29), for all continuity points $t \geq 0$ of the distribution function of τ , we get

$$P(\tau < t) \leq \liminf_{m \rightarrow \infty} \frac{(1 + z)(E(e^{\xi_1}))^t}{1 + m} = 0.$$

Since the distribution function of τ has only at most countably many discontinuity points, this yields that $P(\tau = \infty) = 1$. \square

We can now combine the lemmas to prove strong existence of solutions to the SDE (2.4).

Proposition 3.11. *Under the conditions of Theorem 2.2 there is a strong solution to the SDE (2.4).*

Proof. To construct a strong solution to the jump type SDE (2.4) we use the sequence of strong solutions $(Z^m)_{m \in \mathbb{N}}$ to (3.23) and stopping times $(\tau_m)_{m \in \mathbb{N}}$ constructed and defined above. For $t \in [0, \tau_1)$, we set $Z_t := Z_t^1$ and inductively set $Z_t := Z_t^m$ for $t \in [\tau_{m-1}, \tau_m)$, $m \geq 2$. This definition is consistent, since Z^m satisfies the SDE (2.4) up to time τ_m and, due to the pathwise uniqueness for the SDE (2.4), if $m' > m$, then the solutions Z^m and $Z^{m'}$ coincide up to τ_m almost surely. The construction clearly shows that Z is a strong solution to the SDE (2.4) for all $t \geq 0$. \square

The pathwise uniqueness serves us mainly as a tool to prove the self-similarity of solutions.

Proposition 3.12. *Under the conditions of Theorem 2.2 the unique strong solutions of the SDE (2.4) are self-similar of index 1.*

Proof. Let $c > 0$ be fixed and define $\bar{Z}_t := \frac{1}{c}Z_{ct}$, $t \geq 0$, where $(Z_t)_{t \geq 0}$ is the unique strong solution of the SDE (2.4) with initial condition $z \geq 0$. Then we obtain

$$\begin{aligned} \bar{Z}_t &= \frac{z}{c} + (\log E(e^{\xi_1}; \zeta > 1))t + \frac{\sigma}{c} \int_0^{ct} \sqrt{Z_s} dB_s \\ &\quad - \frac{1}{c} \int_0^{ct} \int_0^\infty \mathbf{1}_{\{rZ_{s-} \leq 1\}} Z_{s-} (\mathcal{M} - \mathcal{M}')(ds, dr) \\ &\quad + \frac{1}{c} \int_0^{ct} \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{rZ_{s-} \leq 1\}} Z_{s-} [e^u - 1] (\mathcal{N} - \mathcal{N}')(ds, dr, du), \quad t \geq 0. \end{aligned}$$

By Revuz and Yor [27, Proposition V.1.5] we have the almost sure identity

$$\frac{\sigma}{c} \int_0^{ct} \sqrt{Z_s} dB_s = \sigma \int_0^t \sqrt{\frac{1}{c} Z_{cs}} d\left(\frac{1}{\sqrt{c}} B_{cs}\right), \quad t \geq 0,$$

since Z has càdlàg paths so that $P(\int_0^t Z_s ds < \infty) = 1$ for all $t \geq 0$. Next, we also use the analogue almost sure identities

$$\begin{aligned} & \int_0^{ct} \int_0^\infty \mathbf{1}_{\{rZ_{s-} \leq 1\}} Z_{s-} (\mathcal{M} - \mathcal{M}')(ds, dr) \\ &= \int_0^t \int_0^\infty \mathbf{1}_{\{r\frac{1}{c}Z_{(cs)-} \leq 1\}} Z_{(cs)-} (\mathcal{M} - \mathcal{M}') (c ds, c^{-1} dr), \quad t \geq 0, \end{aligned}$$

and

$$\begin{aligned} & \int_0^{ct} \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{rZ_{s-} \leq 1\}} Z_{s-} [e^u - 1] (\mathcal{N} - \mathcal{N}') (ds, dr, du) \\ &= \int_0^t \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{r\frac{1}{c}Z_{(cs)-} \leq 1\}} Z_{(cs)-} [e^u - 1] (\mathcal{N} - \mathcal{N}') (c ds, c^{-1} dr, du), \quad t \geq 0. \end{aligned}$$

We do not give the justifications for these simple facts and refer to the arguments of the proof of Proposition 3.14, Steps 4c), 4d) in a more delicate context. Motivated by the above three identities, we define the Wiener process $\bar{B}_t := \frac{1}{\sqrt{c}} B_{ct}$, $t \geq 0$, and the independent Poisson random measures $\bar{\mathcal{N}}$ on $(0, \infty) \times (0, \infty) \times \mathbb{R}$ by $\bar{\mathcal{N}}(ds, dr, du) := \mathcal{N}(c ds, c^{-1} dr, du)$ and $\bar{\mathcal{M}}$ on $(0, \infty) \times (0, \infty)$ by $\bar{\mathcal{M}}(ds, dr) := \mathcal{M}(c ds, c^{-1} dr)$.

It follows directly from the definition of a Poisson random measure that $\bar{\mathcal{N}}$ is a Poisson random measure with the same intensity measure as \mathcal{N} and $\bar{\mathcal{M}}$ is a Poisson random measure with the same intensity measure as \mathcal{M} . With these definitions, the above calculation leads to

$$\begin{aligned} \bar{Z}_t &= \frac{z}{c} + (\log E(e^{\xi^1}; \zeta > 1)) t + \sigma \int_0^t \sqrt{\bar{Z}_s} d\bar{B}_s - \int_0^t \int_0^\infty \mathbf{1}_{\{r\bar{Z}_{s-} \leq 1\}} \bar{Z}_{s-} (\bar{\mathcal{M}} - \bar{\mathcal{M}}') (ds, dr) \\ &+ \int_0^t \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{r\bar{Z}_{s-} \leq 1\}} \bar{Z}_{s-} [e^u - 1] (\bar{\mathcal{N}} - \bar{\mathcal{N}}') (ds, dr, du), \quad t \geq 0. \end{aligned}$$

Hence, \bar{Z} is a strong solution to (2.4) with initial condition $\frac{z}{c}$ so that pathwise uniqueness (which implies weak uniqueness) for (2.4) proved in Proposition 3.4 implies the claim. \square

Finally, we prove the strong Markov property of solutions to (2.4).

Proposition 3.13. *Under the conditions of Theorem 2.2 the unique strong solution of the jump type SDE (2.4) is a strong Markov process.*

Proof. In the following for all $0 < \varepsilon < m$ and $z \geq 0$, let us denote by $\mathbb{P}_z^{\varepsilon, m}$, \mathbb{P}_z^m , \mathbb{P}_z the laws of the unique solutions to (3.18), (3.23), (2.4) started in $z \in \mathbb{R}_+$ on $(\mathbb{D}, \mathcal{D})$.

Step 1 (measurability of $z \mapsto \mathbb{P}_z$): We first prove that the martingale problem corresponding to the infinitesimal generator of the strong solutions $Z^{\varepsilon, m}$ to (3.18) is well-posed. The well-posedness for deterministic initial conditions $z \geq 0$ follows from part (a) of Theorem 2.2, and can be extended to general initial distributions (i.e. probability measures on \mathbb{R}_+): by Problem 49 on page 273 of Ethier and Kurtz [16] it suffices to verify the conditions of their Lemma 4.5.13. Namely, we have to check that the domain of the infinitesimal generator of $\mathcal{A}^{\varepsilon, m}$ contains an algebra that separates points and vanishes nowhere (i.e. not all of its elements simultaneously vanish at a point) and that

for each compact set $K \subseteq \mathbb{R}_+$ and $\eta > 0$ there exist a sequence of compact subsets $K_n \subseteq \mathbb{R}_+$, $K \subseteq K_n$, $n \in \mathbb{N}$, and smooth functions $f_n : \mathbb{R}_+ \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, with compact support such that for $F_n := \{z \in \mathbb{R}_+ : \inf_{x \in K_n} |x - z| \leq \eta\}$ we have

$$\beta_{n,\eta} := \inf_{y \in K} f_n(y) - \sup_{y \in \mathbb{R}_+ \setminus F_n} f_n(y) > 0, \quad n \in \mathbb{N}, \tag{3.30}$$

$$\lim_{n \rightarrow \infty} \beta_{n,\eta}^{-1} \sup_{y \in F_n} (\mathcal{A}^{\varepsilon,m} f_n)^-(y) = 0, \tag{3.31}$$

$$\lim_{n \rightarrow \infty} \beta_{n,\eta}^{-1} \left(\sup_{x \in \mathbb{R}_+} |f_n(x)| - \inf_{y \in K} f_n(y) \right) = 0, \tag{3.32}$$

where $(\mathcal{A}^{\varepsilon,m} f)^-$ denotes the negative part of $\mathcal{A}^{\varepsilon,m} f$. First note that the set of bounded twice continuously differentiable functions on \mathbb{R}_+ with compact support and bounded first and second derivatives serves as an appropriate choice for such an algebra. Next let $k \in \mathbb{N}$ be such that $K \subseteq [0, k]$, $K_n := [0, k + n]$, $n \in \mathbb{N}$, and let us choose the functions f_n , $n \in \mathbb{N}$, as in Example 4.5.14 in Ethier and Kurtz [16], namely $f_n : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a smooth function with compact support such that

- $f_n(x) = 1 + \log(1 + (k + n + \eta)^2) - \log(1 + x^2)$ if $0 \leq x \leq k + n + \eta$,
- $f_n(x) \in [0, 1]$ if $x > k + n + \eta$,
- $\sup_{x \in \mathbb{R}_+ \setminus F_n} |f'_n(x)| < \infty$ and $\sup_{x \in \mathbb{R}_+ \setminus F_n} |f''_n(x)| < \infty$.

By Example 4.5.14 in Ethier and Kurtz [16], (3.30) and (3.32) hold. By (3.27) we have

$$\sup_{y \in F_n} (\mathcal{A}^{\varepsilon,m} f_n)^-(y) \leq \sup_{y \in \mathbb{R}_+} (\mathcal{A}^{\varepsilon,m} f_n)(y) < \infty, \quad n \in \mathbb{N},$$

further, by the definition of f_n ,

$$\beta_{n,\eta} \geq \log(1 + (k + n + \eta)^2) \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

which yields (3.32). Then Theorem 4.4.6 of Ethier and Kurtz [16] implies measurability of $z \mapsto \mathbb{P}_z^{\varepsilon,m}$ mapping $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ to $(\mathcal{P}(\mathbb{D}), \rho)$, where $\mathcal{P}(\mathbb{D})$ is the set of all probability measures on $(\mathbb{D}, \mathcal{D})$ equipped with the Prokhorov metric ρ .

Since for all $z \geq 0$, $\mathbb{P}_z^{\varepsilon,m}$ converges to \mathbb{P}_z^m with respect to the Prokhorov metric ρ as $\varepsilon \downarrow 0$, the map $z \mapsto \mathbb{P}_z^m$ is measurable as pointwise limit of a sequence of measurable maps. Finally, notice that our pathwise construction of Z via Z^m given in the proof of Proposition 3.11 implies in particular that Z^m converges weakly to Z as $m \rightarrow \infty$ in the Skorokhod's topology of \mathbb{D} . Indeed, by the construction, if $m' \geq m$, then $Z_t^{m'} = Z_t^m$ for $t \in [0, \tau_m)$, and, by Lemma 3.10, $(\tau_m)_{m \in \mathbb{N}}$ is increasing and tends to infinity almost surely. Hence, Z^m converges to Z as $m \rightarrow \infty$ in the local uniform topology on \mathbb{D} . Since this topology is weaker than the Skorokhod's topology (see, e.g. Jacod and Shiryaev [20, Proposition VI.1.17]), we find that Z^m converges weakly to Z as $m \rightarrow \infty$ in the Skorokhod's topology of \mathbb{D} . Using again that the pointwise limit of a sequence of measurable mappings is measurable we have the measurability of the map $z \mapsto \mathbb{P}_z$.

Step 2 (uniqueness for general initial distributions): Well-posedness for the martingale problem corresponding to the infinitesimal generator of the strong solutions Z to (2.4) with general non-negative initial distributions follows then from well-posedness for deterministic initial conditions $z \in \mathbb{R}_+$ combined with the measurability of the first step (see for instance Lemma IX.4.4 of Jacod and Shiryaev [20]).

Step 3 (strong Markov property): Finally, the strong Markov property for the unique strong solutions to the SDE (3.18) follows from the first two steps for instance by Theorem 4.4.2 of Ethier and Kurtz [16]. □

We now come to the arguments needed to turn the heuristic derivation of the jump type SDE (2.4) given in the introduction into mathematics. In order to determine the Lamperti transformed Lévy process corresponding to the unique strong solution Z absorbed at zero, we show that applying Lamperti's transformation to the Lévy process ξ associated to the SDE (2.4) leads to a weak solution of (2.4) up to the first hitting time of zero.

Proposition 3.14. *Under the conditions of Theorem 2.2 let Z be the pathwise unique strong solution to (2.4). Then the Lamperti transformed Lévy process ξ of the pssMp Z^\dagger of index 1 has Lévy triplet (γ, σ^2, Π) and is killed at rate $q \geq 0$.*

Proof. By Proposition 3.12, the unique strong solution to (2.4) is self-similar of self-similarity index 1 so that the absorbed process Z^\dagger is a pssMp of self-similarity index 1. Now let ξ be a Lévy process with Lévy triplet (γ, σ^2, Π) and killing rate q and, for some $z > 0$, define

$$Y_t := z \exp(\xi_{\tau(tz^{-1})}), \quad t \geq 0, \tag{3.33}$$

where $(\tau(t))_{t \geq 0}$ is the (extended real-valued) continuous, increasing inverse of $I_t = \int_0^t e^{\xi_s} ds$, $t \geq 0$, with the convention $Y_t := 0$ for $t \geq S_0$, where $S_0 := \inf\{t \geq 0 : Y_t = 0\}$. To prove the statement of the proposition, by pathwise uniqueness (which implies well-posedness of the corresponding martingale problem), it is then enough to check that $(Y_t)_{t \in [0, S_0)}$ is a weak solution of the SDE (2.4) with initial condition z up to the first hitting time of 0.

Since the killing only occurs for the first jump of Y down to zero at S_0 , which - as we explained in the derivation of (2.4) - resulted in the killing integral of (2.4) we assume from now on $q = 0$.

We will use throughout the proof that $S_0 < \infty$ almost surely if and only if ξ drifts to $-\infty$ so that in this case the infinite interval $[0, \infty)$ is compressed via τ to the finite interval $[0, S_0)$. More formally, this says that almost surely $\tau(tz^{-1}) < \infty$ for $t < S_0$ with $\lim_{t \uparrow S_0} \tau(tz^{-1}) = \infty$.

Step 1 (integrals are well-defined): First we check that with the definition (3.33), Y can be a solution to (2.4) in the sense that the integrals in the SDE (2.4) can be defined as local martingales.

Step 1a (localization sequence): We show that there exists a sequence of stopping times $(\delta_m)_{m \in \mathbb{N}}$ defined via Y such that $P(\lim_{m \rightarrow \infty} \delta_m = \infty) = 1$. Let us define the increasing sequence of stopping times $\delta_m := \inf\{t \geq 0 : Y_t \geq m\}$, $m \in \mathbb{N}$. Since δ_m increases there exists a random variable C such that $\lim_{m \rightarrow \infty} \delta_m = C$ almost surely. We aim at proving that $P(C = \infty) = 1$. First note that due to the aforementioned properties of τ , $(Y_t)_{t \geq 0}$ has almost surely càdlàg paths. In what follows we distinguish the cases according to ξ drifting to $-\infty$, $+\infty$ or oscillating.

If ξ drifts to $+\infty$ or oscillates, then $S_0 = \infty$ almost surely and hence $\lim_{t \rightarrow \infty} \tau(tz^{-1}) = \infty$ so that $(\xi_{\tau(tz^{-1})})_{t \geq 0}$ drifts to $+\infty$ or oscillates. Let us suppose that $P(C < \infty) > 0$ and let $\omega \in \Omega$ be such that $C(\omega) < \infty$. Using that $\delta_m(\omega) \leq C(\omega)$, $m \in \mathbb{N}$, by the definition of an infimum, there exists a sequence $(t_m)_{m \in \mathbb{N}}$ (depending on ω) such that $Y_{t_m}(\omega) \geq m$, $m \in \mathbb{N}$, and $t_m < C(\omega)$, $m \in \mathbb{N}$ (in the case of $\delta_m(\omega) < C(\omega)$, $m \in \mathbb{N}$), or $\lim_{m \rightarrow \infty} t_m(\omega) = C(\omega)$ (in the case of $\delta_{m_0}(\omega) = C(\omega)$ for some $m_0 \in \mathbb{N}$). Hence, since τ is increasing and continuous, $\tau(t_m z^{-1})(\omega) \leq \tau(Cz^{-1})(\omega)$, $m \in \mathbb{N}$, or $\lim_{m \rightarrow \infty} \tau(t_m z^{-1})(\omega) = \tau(Cz^{-1})(\omega)$. Further, since $Y_{t_m}(\omega) \geq m$, $m \in \mathbb{N}$, we have $\lim_{m \rightarrow \infty} Y_{t_m}(\omega) = \infty$ and then $\lim_{m \rightarrow \infty} \xi_{\tau(t_m z^{-1})}(\omega) = \infty$. Then it would imply that ξ would not be bounded on the interval $[0, \tau(Cz^{-1})(\omega) + 1]$. This is a contradiction, since, by $C(\omega) < S_0(\omega) = \infty$, we have $\tau(Cz^{-1})(\omega) < \infty$ implying that the interval $[0, \tau(Cz^{-1})(\omega) + 1]$ is finite. Hence $P(C = \infty) = 1$ if ξ drifts to $+\infty$ or it oscillates.

If ξ drifts to $-\infty$, then S_0 is almost surely finite and $\lim_{t \uparrow S_0} Y_t = \lim_{t \uparrow S_0} z e^{\xi_{\tau(tz^{-1})}} = 0$ since $\lim_{t \uparrow S_0} \tau(tz^{-1}) = \infty$. Recalling also that, by convention, $Y_t = 0$ for $t \geq S_0$ and Y has càdlàg paths, we find that the trajectories of $(Y_t)_{t \geq 0}$ are bounded from above implying that δ_m tends to infinity as $m \rightarrow \infty$. Hence $P(C = \infty) = 1$ also in the case of ξ drifting to $-\infty$.

Step 1b (the integral with respect to B): To define the integral $\int_0^t \sqrt{Y_s} dB_s$ we use the localized estimate

$$E \left(\int_0^{t \wedge \delta_m} Y_s ds \right) \leq mE(t \wedge \delta_m) \leq tm < \infty, \quad m \in \mathbb{N}, t \geq 0, \tag{3.34}$$

and Definition 2.4 in Ikeda and Watanabe [19, page 57].

Step 1c (the integral with respect to $\mathcal{N} - \mathcal{N}'$): To define the Poissonian integral

$$\int_0^t \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{rY_{s-} \leq 1\}} Y_{s-} (e^u - 1) (\mathcal{N} - \mathcal{N}') (ds, dr, du),$$

we use

$$\begin{aligned} & E \left(\int_0^{t \wedge \delta_m} \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{rY_{s-} \leq 1\}} (Y_s)^2 (e^u - 1)^2 ds dr \Pi(du) \right) \\ &= E \left(\int_0^{t \wedge \delta_m} Y_s ds \right) \int_{\mathbb{R}} (e^u - 1)^2 \Pi(du) < \infty, \end{aligned}$$

which follows by (3.34) and that $E(e^{2\xi_1}) < \infty$ (which holds since ξ is spectrally negative). Hence, the localized definition of Ikeda and Watanabe [19, page 62] can be applied.

Step 2 (SDE for $z \exp(\xi)$): First let us define $\eta_t := z \exp(\xi_t)$, $t \geq 0$. Applying Itô's formula (see, e.g. Ikeda and Watanabe [19, Chapter II, Theorem 5.1]) to the Lévy-Itô representation (2.1) of ξ , we have

$$\begin{aligned} \eta_t &= z + \gamma \int_0^t \eta_s ds + \sigma \int_0^t \eta_s dB_s + \frac{\sigma^2}{2} \int_0^t \eta_s ds \\ &\quad + \int_0^t \int_{|u| \leq 1} (ze^{\xi_{s-} + u} - ze^{\xi_{s-}}) (\mathcal{N}_0 - \mathcal{N}'_0) (ds, du) \\ &\quad + \int_0^t \int_{|u| > 1} (ze^{\xi_{s-} + u} - ze^{\xi_{s-}}) \mathcal{N}_0 (ds, du) \\ &\quad + \int_0^t \int_{|u| \leq 1} (ze^{\xi_s + u} - ze^{\xi_s} - uze^{\xi_s}) ds \Pi(du) \tag{3.35} \\ &= z + \left(\gamma + \frac{\sigma^2}{2} + \int_{|u| \leq 1} (e^u - 1 - u) \Pi(du) \right) \int_0^t \eta_s ds + \sigma \int_0^t \eta_s dB_s \\ &\quad + \int_0^t \int_{|u| \leq 1} \eta_{s-} (e^u - 1) (\mathcal{N}_0 - \mathcal{N}'_0) (ds, du) \\ &\quad + \int_0^t \int_{|u| > 1} \eta_{s-} (e^u - 1) \mathcal{N}_0 (ds, du), \quad t \geq 0. \end{aligned}$$

Step 3 (some preparations): In order to deduce from (3.35) that the time-changed process $Y_t = \eta_{\tau(tz^{-1})}$, $t \in [0, S_0)$, is a solution of (2.4) with $q = 0$ up to first hitting zero we need some preparations.

Step 3a (the local martingale W and the Poisson random measure \mathcal{P}): We denote by $(t_n, \Delta_n)_{n \in \mathbb{N}}$ an arbitrary labeling of the pairs associated to jump times and jump sizes of the time-changed Lévy process $(\xi_{\tau(tz^{-1})})_{t \in [0, S_0]}$. By an enlarging procedure, we can assume that we are given a Wiener process $(\bar{B}_t)_{t \geq 0}$ and an independent Poisson random measure \mathcal{Q} on $(0, \infty) \times (0, \infty) \times \mathbb{R}$ with intensity measure $ds \otimes dr \otimes \Pi(du)$ both defined independently of $(B_{\tau(tz^{-1})})_{t \in [0, S_0]}$ and $(t_n, \Delta_n)_{n \in \mathbb{N}}$ on a common filtered probability space $(\Omega', \mathcal{H}, (\mathcal{H}_t)_{t \geq 0}, P')$. Note that $\mathcal{G}_{\tau(tz^{-1})} \subseteq \mathcal{H}_t, t \leq 0 < S_0$. To keep the notation simple, we still use P and E for the (extended) probability measure and the expectation with respect to it, respectively. Additionally, we choose an independent sequence of random variables $(R_n)_{n \in \mathbb{N}}$ uniformly distributed on $(0, 1)$ such that R_n is \mathcal{H}_{t_n} -measurable and independent of \mathcal{H}_{t_n-} (the existence of such a sequence can be assumed by the above mentioned enlargement of filtration). Since $(B_{\tau(tz^{-1})})_{t \in [0, S_0]}$ is a continuous local martingale (not necessarily a Brownian motion) with respect to the filtration $(\mathcal{H}_t)_{t \in [0, S_0]}$, we can define

$$W_t := \int_0^t \sqrt{Y_s} \mathbf{1}_{\{Y_s \neq 0\}} dB_{\tau(sz^{-1})} + \int_0^t \mathbf{1}_{\{Y_s = 0\}} d\bar{B}_s, \quad t \geq 0.$$

Note that if ξ drifts to $-\infty$, then $S_0 < \infty$ almost surely and $\tau(sz^{-1}) = \infty$ for $s \geq S_0$. Hence $B_{\tau(sz^{-1})}$ is not defined in this case, however, by convention, $Y_s = 0$ for $s \geq S_0$ and then in this case the first integral in the definition of W is understood to be 0. We also define a new point measure by

$$\begin{aligned} \mathcal{P}(A_1 \times A_2 \times A_3) &:= \sum_{n=1}^{\infty} \mathbf{1}_{A_1 \times A_2 \times A_3}((t_n, R_n/Y_{t_n-}, \Delta_n)) \\ &\quad + \int_{A_1} \int_{A_2} \int_{A_3} (\mathbf{1}_{\{rY_{s-} > 1\}} + \mathbf{1}_{\{Y_{s-} = 0\}}) \mathcal{Q}(ds, dr, du) \end{aligned}$$

for all $A_1, A_2 \in \mathcal{B}((0, \infty))$ and $A_3 \in \mathcal{B}(\mathbb{R})$. Here we note that we never divide by zero ($Y_{t_n-} \neq 0, n \in \mathbb{N}$), since $t_n < S_0, n \in \mathbb{N}$, and hence \mathcal{P} is well-defined.

Step 3b (W is a Brownian motion): First observe that $(W_t)_{t \geq 0}$ is a continuous local martingale with respect to the filtration $(\mathcal{H}_t)_{t \geq 0}$, since by localizing with the sequence $(\delta_m)_{m \in \mathbb{N}}$ defined in Step 1a

$$\begin{aligned} &E \left(\int_0^{t \wedge \delta_m} Y_s \mathbf{1}_{\{Y_s \neq 0\}} d\tau(sz^{-1}) + \int_0^{t \wedge \delta_m} \mathbf{1}_{\{Y_s = 0\}} ds \right) \\ &= E \left(\int_0^{t \wedge \delta_m} \mathbf{1}_{\{Y_s \neq 0\}} ds + \int_0^{t \wedge \delta_m} \mathbf{1}_{\{Y_s = 0\}} ds \right) \leq t, \quad t \geq 0, \quad m \in \mathbb{N}, \end{aligned}$$

so that Definition 2.4 in Ikeda and Watanabe [19, page 57] can be used. The equality above is justified by

$$\tau'(t) = \frac{z}{Y_{tz}}, \quad t \in [0, S_0/z], \tag{3.36}$$

which easily follows, since, by the definition of the time-change τ ,

$$\int_0^{\tau(t)} \exp(\xi_s) ds = t, \quad t \in [0, S_0/z],$$

and differentiating both sides with respect to t gives

$$\exp(\xi_{\tau(t)}) \tau'(t) = 1, \quad t \in [0, S_0/z].$$

Furthermore, using the bilinearity of quadratic variation and the independence of the processes $(\bar{B}_t)_{t \geq 0}$ and $(B_{\tau(tz^{-1})})_{t \in [0, S_0]}$, the quadratic variation process of the (\mathcal{H}_t) -local martingale W is

$$\langle W \rangle_t = \int_0^t Y_s \mathbf{1}_{\{Y_s \neq 0\}} d\tau(sz^{-1}) + \int_0^t \mathbf{1}_{\{Y_s = 0\}} ds = \int_0^t \mathbf{1}_{\{Y_s \neq 0\}} ds + \int_0^t \mathbf{1}_{\{Y_s = 0\}} ds = t$$

for $t \geq 0$. Here we call the attention to the fact that $\int_0^t \sqrt{Y_s} \mathbf{1}_{\{Y_s \neq 0\}} dB_{\tau(sz^{-1})}$, $t \in [0, S_0]$, is not a square-integrable martingale in general, only a continuous local martingale, however, its quadratic variation process equals $\int_0^t Y_s \mathbf{1}_{\{Y_s \neq 0\}} d\tau(sz^{-1})$, $t \in [0, S_0]$, see, e.g., Karatzas and Shreve [21, page 147]. Finally, by Lévy's theorem, this shows that $(W_t)_{t \geq 0}$ is a (\mathcal{H}_t) -Wiener process.

Step 3c (\mathcal{P} is a Poisson random measure): We show that \mathcal{P} is a Poisson random measure on $(0, \infty) \times (0, \infty) \times \mathbb{R}$ with intensity measure $ds \otimes dr \otimes \Pi(du)$. Since \mathcal{P} was defined to be an optional random measure, by Jacod and Shiryaev [20, Theorems II.1.8 and II.4.8], it is enough to show that

$$E \left(\int_0^\infty \int_0^\infty \int_{\mathbb{R}} H(s, r, u) \mathcal{P}(ds, dr, du) \right) = E \left(\int_0^\infty \int_0^\infty \int_{\mathbb{R}} H(s, r, u) ds dr \Pi(du) \right) \quad (3.37)$$

for every non-negative predictable function H on $\Omega \times (0, \infty) \times (0, \infty) \times \mathbb{R}$. Here predictability means measurability with respect to the σ -algebra $\mathcal{A} \otimes \mathcal{B}((0, \infty) \times \mathbb{R})$, where \mathcal{A} is the σ -algebra on $\Omega \times (0, \infty)$ generated by all left-continuous adapted processes (considered as mappings on $\Omega \times (0, \infty)$). Note also that (3.37) includes the case when H is not integrable with respect to \mathcal{P} meaning that both sides of (3.37) are infinite in this case. By the definition of \mathcal{P} we can write

$$\begin{aligned} & E \left(\int_0^\infty \int_0^\infty \int_{\mathbb{R}} H(s, r, u) \mathcal{P}(ds, dr, du) \right) \\ &= E \left(\sum_{n=1}^\infty H(t_n, R_n/Y_{t_n-}, \Delta_n) \right) \\ &+ E \left(\int_0^\infty \int_0^\infty \int_{\mathbb{R}} H(s, r, u) (\mathbf{1}_{\{rY_{s-} > 1\}} + \mathbf{1}_{\{Y_{s-} = 0\}}) \mathcal{Q}(ds, dr, du) \right). \end{aligned} \quad (3.38)$$

To express the first summand we apply Theorem II.1.8 of Jacod and Shiryaev [20] to the non-negative predictable function

$$\tilde{H}(s, r, u) := H(s, r/Y_{s-}, u), \quad s > 0, r > 0, u \in \mathbb{R},$$

and the Poisson random measure on $(0, \infty) \times (0, \infty) \times \mathbb{R}$ defined by

$$\tilde{\mathcal{P}}(A_1 \times A_2 \times A_3) := \sum_{n=1}^\infty \mathbf{1}_{A_1 \times A_2 \times A_3}((t_n, R_n, \Delta_n)), \quad A_1, A_2 \in \mathcal{B}((0, \infty)), A_3 \in \mathcal{B}(\mathbb{R}),$$

to obtain

$$\begin{aligned} E \left(\sum_{n=1}^\infty H(t_n, R_n/Y_{t_n-}, \Delta_n) \right) &= E \left(\int_0^\infty \int_0^\infty \int_{\mathbb{R}} \tilde{H}(s, r, u) \tilde{\mathcal{P}}(ds, dr, du) \right) \\ &= E \left(\int_0^{S_0} \int_0^1 \int_{\mathbb{R}} H(s, r/Y_{s-}, u) d\tau(sz^{-1}) dr \Pi(du) \right), \end{aligned}$$

where the second equality holds since, by construction, the compensator measure of $\tilde{\mathcal{P}}$ is

$$\mathbf{1}_{(0, S_0)}(s) \mathbf{1}_{(0, 1)}(r) d\tau(sz^{-1}) dr \Pi(du).$$

Utilizing (3.36) and a change of variable in the second coordinate of H , we can further simplify the right-hand side to

$$\begin{aligned} & E \left(\int_0^{S_0} \int_0^1 \int_{\mathbb{R}} \frac{1}{Y_{s-}} H(s, r/Y_{s-}, u) \, ds \, dr \, \Pi(du) \right) \\ &= E \left(\int_0^{S_0} \int_0^{1/Y_{s-}} \int_{\mathbb{R}} H(s, r, u) \, ds \, dr \, \Pi(du) \right). \end{aligned}$$

Similarly, applying Theorem II.1.8 of Jacod and Shiryaev [20] to \mathcal{Q} , the second summand of the right hand side of (3.38) equals

$$\begin{aligned} & E \left(\int_0^\infty \int_0^\infty \int_{\mathbb{R}} H(s, r, u) (\mathbf{1}_{\{rY_{s-} > 1\}} + \mathbf{1}_{\{Y_{s-} = 0\}}) \, \mathcal{Q}(ds, dr, du) \right) \\ &= E \left(\int_0^{S_0} \int_{1/Y_{s-}}^\infty \int_{\mathbb{R}} H(s, r, u) \, ds \, dr \, \Pi(du) \right) + E \left(\int_{S_0}^\infty \int_0^\infty \int_{\mathbb{R}} H(s, r, u) \, ds \, dr \, \Pi(du) \right). \end{aligned}$$

Adding the right hand sides of the above two equalities, by (3.38), we have (3.37).

Step 3d (W and P are independent): This follows from Ikeda and Watanabe [19, Chapter II, Theorem 6.3], since the compensator measure of \mathcal{P} , by Step 3c, is deterministic.

Step 4 (deriving the SDE (2.4) - plugging-in): We first observe that, by (3.35), the time-changed process $Y_t = \eta_{\tau(tz^{-1})}$, $t \in [0, S_0)$, satisfies the integral equation

$$\begin{aligned} Y_t &= z + \left(\gamma + \frac{\sigma^2}{2} + \int_{|u| \leq 1} (e^u - 1 - u) \Pi(du) \right) \int_0^{\tau(tz^{-1})} \eta_s \, ds + \sigma \int_0^{\tau(tz^{-1})} \eta_s \, dB_s \\ &\quad + \int_0^{\tau(tz^{-1})} \int_{|u| \leq 1} \eta_{s-} (e^u - 1) (\mathcal{N}_0 - \mathcal{N}'_0)(ds, du) \\ &\quad + \int_0^{\tau(tz^{-1})} \int_{|u| > 1} \eta_{s-} (e^u - 1) \mathcal{N}_0(ds, du) \end{aligned} \tag{3.39}$$

for $t < S_0$. In the following we replace the last four summands on an event having probability one in such a way that (3.39) becomes (2.2) for $t < S_0$ driven by W and \mathcal{P} (i.e., replacing B and \mathcal{N}_0 by W and \mathcal{P} , respectively).

Step 4a (transforming the drift): Using (3.36), the first summand simplifies to the desired form, since by a change of variable, we have

$$\int_0^{\tau(tz^{-1})} \eta_s \, ds = \int_0^{\tau(tz^{-1})} z \exp(\xi_s) \, ds = t, \quad t < S_0. \tag{3.40}$$

Step 4b (transforming the Brownian integral): First we check that almost surely

$$\int_0^{\tau(tz^{-1})} \eta_s \, dB_s = \int_0^t \eta_{\tau(uz^{-1})} \, dB_{\tau(uz^{-1})} = \int_0^t Y_u \, dB_{\tau(uz^{-1})}, \quad t \in [0, S_0). \tag{3.41}$$

Note that, for all $t \geq 0$,

$$\int_0^t \eta_s^2 \, d\langle B \rangle_s = \int_0^t z^2 e^{2\xi_s} \, ds < +\infty \quad \text{a.s.}, \tag{3.42}$$

since the Lévy process ξ does not explode in finite time. We distinguish the cases according to ξ drifting to $+\infty$, $-\infty$ or oscillating. If ξ drifts to $+\infty$ or oscillates, then

$S_0 = \infty$ almost surely and hence $\tau(tz^{-1})$ is finite almost surely for all $t \geq 0$. By (3.42), we can use Revuz and Yor [27, Proposition V.1.5] with the following choices

$$C_t := \tau(tz^{-1}), \quad H_t := \eta_t, \quad X_t := B_t,$$

to infer (3.41). If ξ drifts to $-\infty$, then $S_0 < \infty$ almost surely and $C_t = \infty$ almost surely for $t \geq S_0$, which yield that one can not use directly Revuz and Yor [27, Proposition V.1.5]. However, we can proceed as follows: For all $\varepsilon > 0$, let us define $S_\varepsilon := \inf\{t \geq 0 : Y_t < \varepsilon\}$. Since we assumed that ξ drifts to $-\infty$ we may use that $P(Y_{S_0-} = 0) = 1$ to deduce that $P(\lim_{\varepsilon \downarrow 0} S_\varepsilon = S_0) = 1$. Next, we define the modified time-change $\tau_\varepsilon(t) := \tau((t \wedge S_\varepsilon)z^{-1})$ for any $t \geq 0$. Then τ_ε is a stopping time in the sense of Revuz and Yor [27, Definition V.1.2] and $\tau_\varepsilon(t) < \infty$ for all $t \geq 0$, since it remains constant for $t \geq S_\varepsilon$. By (3.42), we can use Revuz and Yor [27, Proposition V.1.5] to infer that for all $\varepsilon > 0$

$$\int_0^{\tau_\varepsilon(t)} \eta_s dB_s = \int_0^t \eta_{\tau_\varepsilon(s)} dB_{\tau_\varepsilon(s)}, \quad t \geq 0.$$

Since $\tau_\varepsilon(t) = \tau(tz^{-1})$, $0 \leq t \leq S_\varepsilon$, we have

$$\int_0^{\tau(tz^{-1})} \eta_s dB_s = \int_0^t \eta_{\tau(sz^{-1})} dB_{\tau(sz^{-1})}, \quad 0 \leq t \leq S_\varepsilon.$$

Using that $P(\lim_{\varepsilon \downarrow 0} S_\varepsilon = S_0) = 1$, we have (3.41) also in the case of ξ drifting to $-\infty$. Then, by the definition of W and using that $Y_{s-} \neq 0$, $0 < s < S_0$, we have the almost sure identity

$$\int_0^{\tau(tz^{-1})} \eta_s dB_s = \int_0^t Y_s dB_{\tau(sz^{-1})} = \int_0^t \sqrt{Y_s} dW_s, \quad t \in [0, S_0).$$

This completes the arguments for the Brownian integral.

Step 4c (transforming the integral for large jumps): Next, we check that almost surely

$$\begin{aligned} & \int_0^{\tau(tz^{-1})} \int_{|u|>1} \eta_{s-}(e^u - 1) \mathcal{N}_0(ds, du) \\ &= \int_0^t \int_0^\infty \int_{|v|>1} \mathbf{1}_{\{rY_{s-} \leq 1\}} Y_{s-}(e^v - 1) \mathcal{P}(ds, dr, dv) \end{aligned} \tag{3.43}$$

for all $t < S_0$. By the definition of the integral with respect to the Poisson random measure \mathcal{N}_0 , the left-hand side of (3.43) equals

$$\sum_{\{n \in \mathbb{N} : \tilde{t}_n \leq \tau(tz^{-1})\}} z e^{\xi_{\tilde{t}_n}} (e^{\tilde{\Delta}_n} - 1) \mathbf{1}_{\{|\tilde{\Delta}_n| \geq 1\}},$$

where $(\tilde{t}_n, \tilde{\Delta}_n)$, $n \in \mathbb{N}$, is a labeling of the pairs associated to jump times and jump sizes of the Lévy process ξ such that $\tilde{t}_n = \tau(t_n z^{-1})$ and $\tilde{\Delta}_n = \Delta_n$ (for the definition of t_n and Δ_n , see Step 3a). This sum has only finitely many terms almost surely, since $\tau(tz^{-1}) < \infty$, $t \in [0, S_0)$, and on each interval a Lévy process has only finitely many jumps which are of magnitude greater than or equal to 1. By the definition of Y and \mathcal{P} , the right-hand side of (3.43) equals the following sum

$$\sum_{\{n \in \mathbb{N} : t_n \leq t\}} z e^{\xi_{\tau(t_n z^{-1})}} (e^{\Delta_n} - 1) \mathbf{1}_{\{|\Delta_n| \geq 1\}},$$

where we used that $P(R_n \leq 1) = 1$, $n \in \mathbb{N}$. The equality of the above two sums follows readily, since $\tilde{t}_n = \tau(t_n z^{-1})$ and $\tilde{\Delta}_n = \Delta_n$, $n \in \mathbb{N}$.

Note also that the additional Poisson random measure \mathcal{Q} in the definition of \mathcal{P} is only added in order to get the compensator measure $ds \otimes dr \otimes \Pi(du)$ (i.e., to make the rate fit); all jumps according to \mathcal{Q} do not influence the integral on the right-hand side of (3.43) as they are ruled out by the indicator.

Step 4d (transforming the integral for small jumps): It remains to check the almost sure identity

$$\begin{aligned} & \int_0^{\tau(tz^{-1})} \int_{|u| \leq 1} \eta_{s-} (e^u - 1) (\mathcal{N}_0 - \mathcal{N}'_0) (ds, du) \\ &= \int_0^t \int_0^\infty \int_{|v| \leq 1} \mathbf{1}_{\{rY_{s-} \leq 1\}} Y_{s-} (e^v - 1) (\mathcal{P} - \mathcal{P}') (ds, dr, dv), \quad 0 \leq t < S_0. \end{aligned} \tag{3.44}$$

The left- and right-hand sides of (3.44) are square-integrable local martingales in t on the time interval $[0, \infty)$. For the right-hand side this can be checked as in Step 1c, for the left-hand side one can argue as follows. Namely, by Step 1a, we have $(\delta_m)_{m \in \mathbb{N}}$ is a sequence of stopping times such that $P(\lim_{m \rightarrow \infty} \delta_m = \infty) = 1$ implying that $(\tau((\delta_m \wedge S_0)z^{-1}))_{m \in \mathbb{N}}$ is a sequence of stopping times such that $P(\lim_{m \rightarrow \infty} \tau((\delta_m \wedge S_0)z^{-1}) = \infty) = 1$ and hence it is enough to show that

$$E \left(\int_0^{\tau(tz^{-1}) \wedge \tau((\delta_m \wedge S_0)z^{-1})} \int_{|u| \leq 1} \eta_s^2 (e^u - 1)^2 ds \Pi(du) \right) < \infty, \quad m \in \mathbb{N}, t \geq 0,$$

which (as in Step 1c) follows from $\int_{|u| \leq 1} (e^u - 1)^2 \Pi(du) < \infty$ and

$$\begin{aligned} & E \left(\int_0^{\tau(tz^{-1}) \wedge \tau((\delta_m \wedge S_0)z^{-1})} \eta_s^2 ds \right) \\ &= E \left(\int_0^{\tau((t \wedge \delta_m \wedge S_0)z^{-1})} \eta_s^2 ds \right) = E \left(\int_0^{t \wedge \delta_m \wedge S_0} Y_u du \right) \leq tm, \end{aligned} \tag{3.45}$$

where the second equality follows by a change of variable using (3.36). This yields that, by definition of the localized Poissonian integrals (see, e.g. Ikeda and Watanabe [19, page 63]), for all $m \in \mathbb{N}$,

$$\int_0^{\tau(tz^{-1}) \wedge \tau((\delta_m \wedge S_0)z^{-1})} \int_{|u| \leq 1} \eta_{s-} (e^u - 1) (\mathcal{N}_0 - \mathcal{N}'_0) (ds, du) \tag{3.46}$$

is the L^2 -limit, as $\varepsilon \downarrow 0$, of

$$\int_0^{\tau(tz^{-1}) \wedge \tau((\delta_m \wedge S_0)z^{-1})} \int_{\varepsilon \leq |u| \leq 1} \eta_{s-} (e^u - 1) (\mathcal{N}_0 - \mathcal{N}'_0) (ds, du); \tag{3.47}$$

and

$$\int_0^{t \wedge \delta_m} \int_0^\infty \int_{|v| \leq 1} \mathbf{1}_{\{rY_{s-} \leq 1\}} Y_{s-} (e^v - 1) (\mathcal{P} - \mathcal{P}') (ds, dr, dv) \tag{3.48}$$

is the L^2 -limit, as $\varepsilon \downarrow 0$, of

$$\int_0^{t \wedge \delta_m} \int_0^\infty \int_{\varepsilon \leq |v| \leq 1} \mathbf{1}_{\{rY_{s-} \leq 1\}} Y_{s-} (e^v - 1) (\mathcal{P} - \mathcal{P}') (ds, dr, dv). \tag{3.49}$$

Note also that for all $m \in \mathbb{N}$ and $0 < \varepsilon < 1$, the integrals (3.47) and (3.49) can be written into two parts separating the integration with respect to the Poisson random measures and their intensities (see for instance Jacod and Shiryaev [20, Proposition II.1.28]) since for all $t \geq 0$, $m \in \mathbb{N}$, and $0 < \varepsilon < 1$, similarly to (3.45), we have

$$E \left(\int_0^{\tau(tz^{-1}) \wedge \tau((\delta_m \wedge S_0)z^{-1})} \int_{\varepsilon \leq |u| \leq 1} \eta_{s-} |e^u - 1| ds \Pi(du) \right) \leq t \int_{\varepsilon \leq |u| \leq 1} |e^u - 1| \Pi(du) < \infty$$

and

$$E \left(\int_0^{t \wedge \delta_m} \int_0^\infty \int_{\varepsilon \leq |v| \leq 1} \mathbf{1}_{\{rY_{s-} \leq 1\}} Y_{s-} |e^v - 1| ds dr \Pi(dv) \right) \leq t \int_{\varepsilon \leq |u| \leq 1} |e^u - 1| \Pi(du) < \infty.$$

Hence, by the same arguments as in Step 4c and using also (3.40), for all $m \in \mathbb{N}$, (3.46) equals the L^2 -limit, as $\varepsilon \downarrow 0$, of

$$\begin{aligned} & \sum_{\{n \in \mathbb{N} : \tilde{t}_n \leq \tau((t \wedge \delta_m \wedge S_0)z^{-1})\}} z e^{\xi \tilde{t}_n} - (e^{\tilde{\Delta}_n} - 1) \mathbf{1}_{\{\varepsilon \leq |\tilde{\Delta}_n| \leq 1\}} \\ & - \int_0^{\tau((t \wedge \delta_m \wedge S_0)z^{-1})} \eta_s ds \int_{\varepsilon \leq |u| \leq 1} (e^u - 1) \Pi(du) \\ = & \sum_{\{n \in \mathbb{N} : \tilde{t}_n \leq \tau((t \wedge \delta_m \wedge S_0)z^{-1})\}} z e^{\xi \tilde{t}_n} - (e^{\tilde{\Delta}_n} - 1) \mathbf{1}_{\{\varepsilon \leq |\tilde{\Delta}_n| \leq 1\}} \\ & - (t \wedge \delta_m \wedge S_0) \int_{\varepsilon \leq |u| \leq 1} (e^u - 1) \Pi(du). \end{aligned}$$

Similarly, using that $\mathcal{P}'(ds, dr, dv) = ds \otimes dr \otimes \Pi(dv)$, we find that for all $m \in \mathbb{N}$, (3.48) equals the L^2 -limit, as $\varepsilon \downarrow 0$, of

$$\begin{aligned} & \int_0^{t \wedge \delta_m} \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{rY_{s-} \leq 1\}} \mathbf{1}_{\{\varepsilon \leq |v| \leq 1\}} Y_{s-} (e^v - 1) \mathcal{P}(ds, dr, dv) \\ & - \int_0^{t \wedge \delta_m} \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{rY_{s-} \leq 1\}} \mathbf{1}_{\{\varepsilon \leq |v| \leq 1\}} Y_{s-} (e^v - 1) ds dr \Pi(dv) \\ = & \sum_{\{n \in \mathbb{N} : t_n \leq t \wedge \delta_m\}} \mathbf{1}_{\{\varepsilon \leq |\Delta_n| \leq 1\}} Y_{t_n} - (e^{\Delta_n} - 1) \\ & + \int_0^{t \wedge \delta_m} \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{\{rY_{s-} \leq 1\}} \mathbf{1}_{\{\varepsilon \leq |v| \leq 1\}} Y_{s-} (e^v - 1) (\mathbf{1}_{\{rY_{s-} > 1\}} + \mathbf{1}_{\{Y_{s-} = 0\}}) \mathcal{Q}(ds, dr, dv) \\ & - (t \wedge \delta_m) \int_{\varepsilon \leq |v| \leq 1} (e^v - 1) \Pi(dv) \\ = & \sum_{\{n \in \mathbb{N} : t_n \leq t \wedge \delta_m\}} \mathbf{1}_{\{\varepsilon \leq |\Delta_n| \leq 1\}} Y_{t_n} - (e^{\Delta_n} - 1) - (t \wedge \delta_m) \int_{\varepsilon \leq |v| \leq 1} (e^v - 1) \Pi(dv), \end{aligned}$$

where (for the sum) we used that $R_n/Y_{t_n} \leq 1/Y_{t_n}$, $n \in \mathbb{N}$, almost surely. Using that $\tilde{t}_n = \tau(t_n z^{-1})$, $n \in \mathbb{N}$, and $\tilde{\Delta}_n = \Delta_n$, $n \in \mathbb{N}$, by the uniqueness part of the definition of a square-integrable local martingale, we have (3.44).

Step 4e (putting the pieces together): Using Steps 4a,b,c,d and that for all $0 \leq t < S_0$,

$$\int_0^t \int_0^\infty \int_{|v| > 1} \mathbf{1}_{\{rY_{s-} \leq 1\}} Y_{s-} (e^v - 1) \mathcal{P}'(ds, dr, dv) = t \int_{|v| > 1} (e^v - 1) \Pi(dv) < \infty,$$

which holds since $E(e^{\xi_1}) < \infty$, we have

$$Y_t = z + \left(\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^u - 1 - u\mathbf{1}_{\{|u|\leq 1\}}) \Pi(du) \right) t + \sigma \int_0^t \sqrt{Y_s} dW_s + \int_0^t \int_0^\infty \int_{|v|\leq 1} \mathbf{1}_{\{rY_{s-} \leq 1\}} Y_{s-} (e^v - 1) (\mathcal{P} - \mathcal{P}') (ds, dr, dv), \quad 0 \leq t < S_0.$$

By Sato [30, Theorem 25.17] the drift coefficient takes the desired form $\log E(e^{\xi_1})$, and hence we have the SDE (2.4) with $q = 0$.

Step 5 (end of proof): In Step 4 we proved that the last four integrals of (3.39) can be replaced on an event of full probability by the desired integrals. Hence, Y is a weak solution of (2.4) with $q = 0$ up to the first hitting time S_0 of zero. Since the pathwise uniqueness of (2.4) implies well-posedness of the corresponding martingale problem (see Step 2 in the proof of Proposition 3.13), by Theorem 4.6.1 of Ethier and Kurtz [16] also the localized martingale problem is well-posed. Hence, Z^\dagger and Y (extended by 0 after S_0) are equal in law. \square

Proof of Theorem 2.2. The proof is a combination of Propositions 3.4, 3.11, 3.12, 3.13 and 3.14. \square

Proof of Theorem 2.3. Under the conditions of the theorem, by (2.5) the drift coefficient $\log E(e^{\frac{1}{a}\xi_1}; \zeta > 1)$ for the SDE (2.4) associated to $\frac{1}{a}\xi$ killed at rate q is positive. Part (a) follows from Theorem 2.2 and Proposition 3.14 combined with the equivalence of (2.6). For part (b), taking into account the discussion before the theorem, we only have to prove that the solution Z constructed in Theorem 2.2 leaves zero continuously. The uniqueness of the extension follows from Theorem 2 in Rivero [28]. If $q = 0$, solutions hit zero continuously, so that by SDE (2.4), since the integrands of both stochastic integrals vanish if $Z_{s-} = 0$, the positive constant drift pushes the solution into the interior of \mathbb{R}_+ . This implies that solutions leave zero continuously. If $q > 0$, then zero is only hit by a negative jump due to the killing integral. Since in this case $Z_{s-} > 0$, one might think that the integral driven by \mathcal{N} jumps away from zero. But since \mathcal{M} and \mathcal{N} are two independent Poisson random measures, the Poissonian integrals corresponding to them do not jump together almost surely. Hence, the integral driven by \mathcal{N} does not jump if $Z_{s-} > 0$ and $Z_s = 0$, and solutions leave zero again continuously due to the positive drift. \square

Proof of Theorem 2.4. First of all, note that if ξ does not drift to $-\infty$, then the convexity of the Laplace exponent of ξ trivially implies that $\log E(e^{\frac{1}{a}\xi_1}; \zeta > 1) > 0$ since it is null at zero. Further, due to (2.5), the same is true under the assumption of the theorem if ξ drifts to $-\infty$. Hence, the drift coefficient $\log E(e^{\frac{1}{a}\xi_1}; \zeta > 1)$ for the SDE (2.4) associated to $\frac{1}{a}\xi$ killed at rate q is strictly positive.

As for the continuous case of Section 1.2, we refer to a result from stochastic calculus such as Theorem IX.4.8 of Jacod and Shiryaev [20] applied in a rather trivial fashion, since we only change the initial conditions. In their notation (compare also their Section III.2.c), in particular Theorem III.2.26 and Remark III.2.29), the sequence of martingale problems corresponding to the jump type SDE (2.4) has initial conditions $\eta^z = z, z \geq 0$, and time-homogeneous coefficients

$$K^z(y, A) = \int_0^\infty \int_{\mathbb{R}} \mathbf{1}_{A \setminus \{0\}} (\mathbf{1}_{\{ry \leq 1\}} y (e^u - 1)) dr \Pi^a(du) + q \int_0^\infty \mathbf{1}_{A \setminus \{0\}} (-\mathbf{1}_{\{ry \leq 1\}} y) dr,$$

$A \in \mathcal{B}(\mathbb{R})$, $y \geq 0$, and

$$b^z(y) = \log E(e^{\frac{1}{a}\xi_1}; \zeta > 1) - \int_{|v|>1} v K^z(y, dv),$$

$$c^z(y) = y,$$

$y \geq 0$. By Π^a we denote the Lévy measure of the Lévy process $\frac{1}{a}\xi$. For completeness, we extend c^z, b^z and K^z by zero for $y < 0$. In what follows we check the conditions of Theorem IX.4.8 of Jacod and Shiryaev [20]. First, we note that for all Borel measurable and non-negative functions $f : \mathbb{R} \rightarrow \mathbb{R}_+$ with $f(0) = 0$, we have

$$\int_{\mathbb{R}} f(v) K^z(y, dv) = \int_0^\infty \int_{\mathbb{R}} f(\mathbf{1}_{\{ry \leq 1\}} y(e^u - 1)) dr \Pi^a(du) + q \int_0^\infty f(-\mathbf{1}_{\{ry \leq 1\}} y) dr \tag{3.50}$$

for $y, z \geq 0$, in the sense that the left and right hand sides are both finite or infinite at the same time. Indeed, if f is a linear combination of indicator functions of Borel subsets of \mathbb{R} not containing 0, then it follows by the definition of $K^z(y, \cdot)$, if f is Borel measurable, non-negative and bounded, then it is a consequence of dominated convergence theorem, while for a general f follows by monotone convergence. The coefficients do not depend on z , η^z converges trivially to η^0 as $z \downarrow 0$ and, b^0 is continuous. By Theorem 2.2 and Step 1 in the proof of Proposition 3.13, we have condition (IX.4.3) of Jacod and Shiryaev [20] is satisfied for (b^0, c^0, K^0) . Next, the modification \tilde{c}^z is given by

$$\begin{aligned} \tilde{c}^z(y) &= c^z(y) + \int_{\mathbb{R}} v^2 \mathbf{1}_{\{|v| \leq 1\}} K^z(y, dv) \\ &= y + \int_{\mathbb{R}} y(e^u - 1)^2 \mathbf{1}_{\{|y(e^u - 1)| \leq 1\}} \Pi^a(du) + qy \mathbf{1}_{\{|y| \leq 1\}} \end{aligned}$$

for $y \geq 0$, where the second equality follows by (3.50) integrating out with respect to r . Hence \tilde{c}^z is continuous due to dominated convergence and our assumption that ξ is spectrally negative, which implies $\int_{\mathbb{R}} (e^u - 1)^2 \Pi^a(du) < \infty$. Similarly, for all $z \geq 0$ and continuous bounded non-negative test-functions g vanishing around zero, by dominated convergence, the function

$$(0, \infty) \ni y \mapsto \int_{\mathbb{R}} g(v) K^z(y, dv) = \frac{1}{y} \int_{\mathbb{R}} g(y(e^u - 1)) \Pi^a(du) + q \frac{g(-y)}{y}$$

is continuous having limit 0 as $y \downarrow 0$. To check condition (IX.4.9) in Jacod and Shiryaev [20], we use

$$K^0(y, \{v : |v| > b\}) = \frac{1}{y} \int_{\mathbb{R}} \mathbf{1}_{\{|e^u - 1| > b/y\}} \Pi^a(du) + q \frac{\mathbf{1}_{\{b < y\}}}{y}, \quad y > 0, b > 0,$$

which, uniformly in $y \in [0, N]$ for any $N > 0$, goes to zero as $b \rightarrow \infty$. Indeed, the second summand trivially tends to zero and for the first summand we obtain

$$\frac{1}{y} \int_{\mathbb{R}} \mathbf{1}_{\{|e^u - 1| > b/y\}} \Pi^a(du) \leq \frac{1}{y} \int_{\mathbb{R}} \frac{y^2(e^u - 1)^2}{b^2} \Pi^a(du) = \frac{y}{b^2} \int_{\mathbb{R}} (e^u - 1)^2 \Pi^a(du)$$

which tends to zero as $b \rightarrow \infty$. □

References

- [1] D. Aldous: *Stopping times and tightness. II*. Ann. Probab. **17(2)** 586–595, (1989). MR-0985380

- [2] J. Berestycki, L. Döring, L. Mytnik and L. Zambotti: Hitting properties and non-uniqueness for SDE driven by stable processes. arXiv:1111.4388, (2011).
- [3] J. Bertoin and M.-E. Caballero: Entrance from $0+$ for increasing semi-stable Markov processes. *Bernoulli* **8** 195–205, (2002). MR-1895890
- [4] J. Bertoin and M. Savov: Some applications of duality for Lévy processes in a half-line. *Bull. Lond. Math. Soc.* **43** 97–110, (2011). MR-2765554
- [5] J. Bertoin and M. Yor: The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Anal.* **17** 389–400, (2002). MR-1918243
- [6] J. Bertoin and M. Yor: Exponential functionals of Lévy processes. *Probab. Surv.* **2** 191–212, (2005). MR-2178044
- [7] M.-E. Caballero and L. Chaumont: Conditioned stable Lévy processes and Lamperti representation. *J. Appl. Prob.* **43**, 967–983, (2006). MR-2274630
- [8] M.-E. Caballero and L. Chaumont: Weak convergence of positive self-similar Markov processes and overshoots of Lévy processes. *Ann. Probab.* **34** 1012–1034, (2006). MR-2243877
- [9] M.-E. Caballero, A. Lambert and G. Uribe Bravo: Proof(s) of the Lamperti representation of continuous-state branching processes. *Probab. Surv.* **6** 62–89, (2009). MR-2592395
- [10] L. Chaumont, A. Kyprianou, J.C. Pardo and V. Rivero: Fluctuation theory and exit systems for positive self-similar Markov processes. *Ann. Probab.* **40(1)** 245–279, (2012). MR-2917773
- [11] D.A. Dawson and Z.H. Li: Skew convolution semigroups and affine Markov processes. *Ann. Probab.* **34(3)** 1103–1142, (2006). MR-2243880
- [12] D.A. Dawson and Z.H. Li: Stochastic equations, flows and measure-valued processes. *Ann. Probab.* **40(2)** 813–857, (2012). MR-2952093
- [13] G. Di Nunno, B. Øksendal and F. Proske: *Malliavin calculus for Lévy processes with applications to finance. Corrected second printing.* Springer, 2009. MR-2460554
- [14] R.A. Doney: *Fluctuation theory for Lévy processes. Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005. Edited and with a foreword by Jean Picard. Lecture Notes in Mathematics, 1897.* Springer-Verlag, Berlin, 2007. MR-2320889
- [15] R.A. Doney and R.A. Maller: Stability of the overshoot for Lévy processes. *Ann. Appl. Probab.* **30(1)** 188–212, (2002). MR-1894105
- [16] S.N. Ethier and T.G. Kurtz: *Markov processes: characterization and convergence.* Wiley, 1986. MR-1567615
- [17] P. Fitzsimmons: On the existence of recurrent extensions of self-similar Markov processes. *Electron. Comm. Probab.* **11** 230–241, (2006). MR-2266714
- [18] Z. Fu and Z.H. Li: Stochastic equations of non-negative processes with jumps. *Stochastic Process. Appl.* **120** 306–330, (2010). MR-2584896
- [19] N. Ikeda and S. Watanabe: *Stochastic differential equations and diffusion processes.* North-Holland Publishing Company, 1981. MR-1011252
- [20] J. Jacod and A.N. Shiryaev: *Limit theorems for stochastic processes. Second edition.* Springer-Verlag, Berlin, 2003. MR-1943877
- [21] I. Karatzas and S.E. Shreve: *Brownian motion and stochastic calculus. Second edition.* Springer-Verlag, New York, 1991. MR-1121940
- [22] A.E. Kyprianou: *Introductory lectures on fluctuations of Lévy process with applications.* Universitext, Springer, 2006. MR-2250061
- [23] J. Lamperti: Continuous state branching processes. *Bull. Amer. Math. Soc.* **73** 382–386, (1967). MR-0208685
- [24] J. Lamperti: Semi-stable Markov processes. I. *Z. Wahr. und Verw. Gebiete* **22** 205–225, (1972). MR-0307358
- [25] Z.H. Li and L. Mytnik: Strong solutions for stochastic differential equations with jumps. *Ann. Inst. H. Poincaré Probab. Statist.* **47(4)** 1055–1067, (2011). MR-2884224
- [26] Z.H. Li and F. Pu: Strong solutions of jump-type stochastic equations *Electron. Commun. Probab.* **17**, Article 33, 1–13, (2012).

- [27] D. Revuz and M. Yor: *Continuous martingales and Brownian motion. Third edition.* Springer-Verlag Berlin Heidelberg, 1999. MR-1725357
- [28] V. Rivero: Recurrent extensions of self-similar Markov processes and Cramér's condition. *Bernoulli* **11**(3) 471–509 (2005). MR-2146891
- [29] V. Rivero: Recurrent extensions of self-similar Markov processes and Cramér's condition. II. *Bernoulli* **13**(4) 1053–1070 (2007). MR-2364226
- [30] K.-I. Sato: *Lévy processes and infinitely divisible distributions.* Cambridge University Press, Cambridge, 1999. MR-1739520
- [31] R. Situ: *Theory of stochastic differential equations with jumps and applications.* Springer, Berlin, 2005. MR-2160585
- [32] D.W. Stroock and S.R.S. Varadhan: *Multidimensional diffusion processes.* Springer, Berlin, 1979. MR-0532498
- [33] J. Vuolle-Apiala: Itô excursion theory for self-similar Markov processes. *Ann. Probab.* **22** 546–565, (1994). MR-1288123
- [34] D. Williams: *Diffusions, Markov Processes, and Martingales, Volume 1: Foundations.* John Wiley & Sons, Chichester - New York, 1979. MR-0531031
- [35] S. Yamada and T. Watanabe: On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.* **11** 155–167, (1971). MR-0278420
- [36] S. Yamada and T. Watanabe: On the uniqueness of solutions of stochastic differential equations. II. *J. Math. Kyoto Univ.* **11** 553–563, (1971). MR-0288876

Acknowledgments. The authors would like to thank to Jean Bertoin, Zenghu Li, Pierre-Henri Cumenge and Lorenzo Zambotti for valuable discussions on the topic. Special thanks to Mladen Savov for discussions on Proposition 3.4. We are grateful to the referee and one of the associate editors for valuable comments that have led to an improvement of the manuscript.