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Large deviations for self-intersection local times in subcritical dimensions

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Abstract

Let $(X_t, t \geq 0)$ be a simple symmetric random walk on $\mathbb{Z}^d$ and for any $x \in \mathbb{Z}^d$, let $l_t(x)$ be its local time at site x. For any $p > 1$, we denote by $I_t = \sum_{x\in\mathbb{Z}^d} l_t(x)^p$ the p-fold self-intersection local times (SILT). Becker and König [6] recently proved a large deviations principle for I_t for all $p > 1$ such that $p(d-2/p) < 2$. We extend these results to a broader scale of deviations and to the whole subcritical domain $p(d-2) < d$. Moreover, we unify the proofs of the large deviations principle using a method introduced by Castell [9] for the critical case $p(d-2) = d$.

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1 Introduction and main results

Let $(X_t, t \geq 0)$ be a simple random walk in $\mathbb{Z}^d$ started from the origin. We denote by Δ its generator given by

$$\Delta f(x) = \sum_{y\sim x} (f(y) - f(x)),$$

where the sum is over the nearest neighbors of x. Let P be the underlying probability measure and E the corresponding expectation.

For any $x \in \mathbb{Z}^d$, we denote by $l_t(x)$ the local time at state x of the random walk:

$$\forall x \in \mathbb{Z}^d, \forall t > 0,\ l_t(x) = \int_0^t \delta_x(X_s)ds,$$

where δ_x is the Kronecker symbol. In this article we are interested in the self-intersection local times (SILT):

$$\forall p > 1, \forall t > 0,\ I_t = \sum_{x\in\mathbb{Z}^d} l_t(x)^p.$$

When p is an integer it is easy to see that the SILT measures how much the random walk intersects itself since the SILT can be rewritten as

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$$\begin{aligned} I_t &= \sum_{x \in \mathbb{Z}^d} l_t^p(x) = \sum_{x \in \mathbb{Z}^d} \left(\int_0^t \delta_x(X_s) ds \right)^p \\ &= \int_0^t \cdots \int_0^t \mathbb{1}\left\{ X_{s_1} = \cdots = X_{s_p} \right\} ds_1 \cdots ds_p. \end{aligned}$$

1.1 Motivations

Processes' intersections are studied since the fifties with works of Erdős, Dvoretzki and Kakutani on various quantities characterizing processes' intersections like the range of a random walk [19] or the multiple points of a Brownian motion [20]. After 1975 and the major work of Donsker and Varadhan [18] on the Wiener sausage, these questions have raised constant interest. During the eighties, Le Gall accomplished a great amount of studies on intersections of random walks [27] and intersections of Brownian motions [26].

Questions about processes' intersections are motivated by physical models and mathematical questions. In 1965, Edwards [21] introduced the following continuous polymer model. A polymer is modeled by a continuous process as a Brownian motion $B(t)$. The idea is to penalize the polymer's intersections which are physically forbidden, and to introduce a measure ν whose density with respect to the Wiener measure μ is given for any $a > 0$ by

$$\frac{d\nu}{d\mu} = \frac{\exp\left(-a\beta(1)\right)}{E\left[\exp\left(-a\beta(1)\right)\right]},$$

where $\beta(1)$ is the SILT of a Brownian motion up to time 1. As we will see later, this quantity is not always defined but let's write it for now. The discrete analogous model is the so-called Domb-Joyce [17] model where the Brownian motion is replaced by a random walk. Interested readers can refer to the survey of van der Hofstad and König [33].

The SILT is also linked with mathematical questions. In 1979, Kesten and Spitzer [24] and Borodin [7], [8] simultaneously introduced the model of random walks in random sceneries. Their goal was to build some new self-similar processes. The model is the following. We consider our symmetric random walk $(X_t, t \geq 0)$ moving on a random field $(Y_z, z \in \mathbb{Z}^d)$ independent of the random walk. We consider the process

$$Z_t = \int_0^t Y(X_s) ds = \sum_{z \in \mathbb{Z}^d} Y(z) l_t(z).$$

The SILT (for $p = 2$) is the variance of the conditional law of $(Z_t, t \geq 0)$ given the random walk. Interested lectors can find in [3] a short survey of large deviations questions for this model. Note that this model was also considered by geophysicists Matheron and de Marsily in [30] for studying anomalous dispersion in layered random flows.

1.2 About the SILT

We present now what is known about the SILT of random walks. For simplicity, let us consider a simple symmetric random walk. In dimension one the random walk is recurrent and it intersects itself a lot. Its typical value is $t^{(p+1)/2}$ and we have the following limit theorem:

$$\frac{1}{t^{(p+1)/2}} \sum_{x \in \mathbb{Z}} l_t(x)^p \xrightarrow{d} \beta(1),$$

where

$$\beta(1) = \int_{\mathbb{R}} \mathcal{L}_1(x)^p dx$$

where $\mathcal{L}_1(x)$ is the local time of the Brownian motion at state x up to time 1. This theorem can be seen as a consequence of results of Jain and Pruitt [23], Le Gall [27] and Chen and Li [14].

In dimension 2, the random walk is still recurrent but the typical value of the SILT is of order $t(\log t)^{p+1}$, which is smaller than in dimension 1. Černy proved in [10] a strong Law of Large Numbers. We have also the following Central Limit Theorem:

$$\frac{1}{t}\left(I_t - E[I_t]\right) \xrightarrow{d} \beta(1),$$

where $\beta(1)$ is the renormalized SILT of the Brownian motion (see the work of Varadhan [34]). This theorem summarizes the results obtained in different kind of generality by Le Gall [27] and Rosen [31].

For dimensions larger than 3, the random walk is transient, it typically spends one unit of time in each visited site, thus $I_t \sim t$. Becker and König recently proved a strong Law of Large Numbers [5] to which we can add the following Central Limit Theorem: for $d = 3$ and $p = 2$,

$$\frac{1}{\sqrt{t \log t}}\left(I_t - E[I_t]\right) \xrightarrow{d} \lambda_1 U,$$

for $d \geq 4$ and $p = 2$,

$$\frac{1}{\sqrt{t}}\left(I_t - E[I_t]\right) \xrightarrow{d} \lambda_2 U,$$

where U is a standard Gaussian variable and λ_1 and λ_2 some constants. Refer to Le Gall [27], Chen [12] and Asselah [2].

1.3 Large deviations

Once the Law of Large Numbers and Central Limit Theorem are established, it is natural to be interested in the large deviations for the SILT. In this article, we wonder how I_t can exceed its mean, i.e. we want to compute the asymptotics of the probability $P(I_t \geq t^p r_t^p)$ where $t^p r_t^p \gg E[I_t]$. We consider three types of deviations, very large deviations ($t^p r_t^p \gg E[I_t]$), large deviations ($t^p r_t^p \sim E[I_t]$) and moderate deviations ($P(I_t - E[I_t] \geq t^p r_t^p)$ for $t^p r_t^p \geq \sqrt{\mathrm{Var}(I_t)}$). Heuristically, it is interesting to ask how the random walk can realize this kind of atypical events for a minimal cost. We propose to the random walk some strategies to realize large deviations of its SILT.

To increase the intersections of the random walk, a solution is to localize it in a ball of radius R up to time $\tau \leq t$. On one hand, the walk goes out of the ball in R^2 units of time, and the probability of this localization is of order $\exp(-\frac{\tau}{R^2})$. On the other hand, the random walk spends about $\frac{\tau}{R^d}$ units of time on each site of the ball, so I_t increases to $\left(\frac{\tau}{R^d}\right)^p R^d = \tau^p R^{d(1-p)}$. We recall that we want $I_t = t^p r_t^p$, which gives the value of τ : $\tau = t r_t R^{\frac{d(p-1)}{p}}$. Thus, the probability of the localization is of order

$$\exp\left(-\frac{\tau}{R^2}\right) = \exp\left(-t r_t R^{\frac{d(p-1)}{p}-2}\right).$$

To optimize the probability of the localization, we maximize this quantity over R which gives us three cases:

1. $\frac{d(p-1)}{p} - 2 > 0 \Leftrightarrow p(d-2) > d$ (supercritical case): in this case, the optimal choice for R is 1. A good strategy to realize large deviations is to spend a time of order $t r_t$ in a ball of radius 1, thus:

$$P(I_t \geq t^t r_t^p) \sim \exp(-t r_t).$$

2. $\frac{d(p-1)}{p} - 2 = 0 \Leftrightarrow p(d-2) = d$ (critical case): here the choice of R doesn't matter. Every strategy consisting in spending a time of order $tr_t R^2$ in a ball of radius R such that $1 \leq R \ll 1/\sqrt{r_t}$ could be a good strategy, and

$$P(I_t \geq t^p r_t^p) \sim \exp(-tr_t).$$

3. $\frac{d(p-1)}{p} - 2 < 0 \Leftrightarrow p(d-2) < d$ (subcritical case): a good strategy is to stay up to time t in a ball of maximal radius, i.e. $r_t^{-\frac{p}{d(p-1)}}$, thus

$$P(I_t \geq tr_t) \sim \exp\left(-tr_t^{\frac{2p}{d(p-1)}}\right).$$

The question of large deviations for the SILT has been studied in detail during the last decade. Let's have a brief review of the results obtained so far.

1. Let us consider first the subcritical case $p(d-2) < d$ which is the most studied. First, Chen and Li in [14] proved a very large deviations principle for all $p > 1$, but only in the one dimensional case. They use the Central Limit Theorem given previously to obtain the large deviations principle for the SILT of the random walk from the large deviations principle for the SILT of the limit process.

   Then, Chen in [11] obtained the same kind of result for the mutual intersections of p independent random walks. There are also two moderate deviations results. The first one is due to Bass, Chen and Rosen [4] who proved a moderate deviations principle in dimension 2. Then, Chen in [13] proved the same result in dimension 3. These results are obtained for all the scales of deviations, but they only cover the case of the double intersections ($p = 2$). In their proofs, they successfully use the triangular decomposition introduced by Varadhan in [34] to renormalize the SILT of the Brownian motion in dimension 2.

   Recently, Becker and König [6] obtained a very large deviations principle in the subcritical case with two restrictions. On one hand, they obtain the principle only for $p(d-2/p) < d$, and on the other hand, the principle is obtained for scales of deviations exceeding the mean by a polynomial factor. As we will see later, the final constant is expressed in term of the best constant in a Gagliardo-Nirenberg inequality.
2. In the critical case $p(d-2) = d$, Castell [9] proved a very large deviations principle using a version of Dynkin isomorphism theorem settled by Eisenbaum (theorem 2.3), which links the law of the local time with the law of a Gaussian process. The constant of large deviations is expressed in terms of the best constant in a Sobolev inequality.
3. In the supercritical case $p(d-2) > d$, Chen and Mörters [16] proved a large deviations principle for integer value of p, computing large moments of the SILT. Asselah [1] obtained a large deviations principle but only for $p = 2$. Finally, we proved in [25] a large deviations principle for all $p > 1$ and for α-stable random walk with $p(d-\alpha) \geq d$, using Dynkin isomorphism theorem.

A recent monograph of Chen [13] summarizes these results. Interested readers can refer to it for an exhaustive treatment of the subject.

In this paper, we extend the result of Chen and Li [14] and improve the results of Becker and König [6] in two directions. We prove a very large deviations principle and we cancel their condition $p(d-2/p) < d$. Furthermore, we unify the proofs of the large deviations principle in the three different cases, showing that the method based on the Dynkin isomorphism theorem also works in the subcritical case.

1.4 Main results

Let us introduce some notations to state our results. We denote by ∇ and by $\|\cdot\|_p$ the gradient and the L^p-norm of functions defined on $\mathbb{R}^d$, by $H^1(\mathbb{R}^d)$ the classical Sobolev space and by q the conjugate of p.

Theorem 1.1. *Let*

$$\chi_{d,p} := \inf \left\{ \frac{1}{2} \|\nabla g\|_2^2 \, , g \in H^1(\mathbb{R}^d) \text{ such that } \|g\|_2 = \|g\|_{2p} = 1 \right\}.$$

Assume that $p(d-2) < d$ and that
- in dimension $d = 1$, $\frac{1}{t^{1/2q}} \ll r_t \ll 1$
- in dimension $d = 2$, $\left(\frac{\log t}{t}\right)^{1/q} \ll r_t \ll 1$
- in dimension $d \geq 3$, $\frac{1}{t^{1/q}} \ll r_t \ll 1$
then we have

$$\lim_{t \to +\infty} \frac{1}{t r_t^{2q/d}} \log P\left(I_t \geq t^p r_t^p\right) = -\chi_{d,p}.$$

Remark 1.2 (About scales of deviations)**.** *Note that in Theorem 1.1, our conditions on r_t are equivalent to $t^p r_t^p \gg E[I_t]$, thus we didn't succeed to reach the order of the mean. In our approach, the SILT is represented by the norm of a Gaussian process Z. To obtain the right constant of deviations, we introduce a median of the Gaussian process. The control of this median reduces our results to scales of very large deviations. A better understanding of the behavior of this median could allow us to improve scales in our results.*

Note that the question of moderate deviations is still open with the exception of dimensions 2 and 3 for $p = 2$ in the subcritical case $p(d-2) < d$ (see the monograph of Chen [13]). In the case where the dimension is larger than 5 and for $p = 2$ (supercritical case $p(d-2) > d$), Asselah [1] succeeded to obtain the constant of deviations up to the scale of the mean.

Remark 1.3 ($\chi_{d,p}$ is non degenerate)**.** *We prove that constant $\chi_{d,p}$ is non degenerate, linking it to the best constant in the Gagliardo-Nirenberg inequality. We recall that Gagliardo-Niremberg constant $K_{d,p}$ is defined by*

$$K_{d,p} = \sup_{g \in H^1(\mathbb{R}^d)} \left\{ \frac{\|g\|_{2p}}{\|\nabla g\|_2^{d/2q} \|g\|_2^{1-d/2q}} \right\},$$

and is a non degenerate constant in the subcritical case $p(d-2) < d$ (see for example Lemma 2 in [15]). This expression being invariant under the transformation $g_\beta(\cdot) = \beta^{d/2p} g(\beta \cdot)$, we take the supremum over $\|g\|_2 = 1$ to obtain

$$K_{d,p} = \sup_{g \in H^1(\mathbb{R}^d)} \left\{ \frac{\|g\|_{2p}}{\|\nabla g\|_2^{d/2q}}, \|g\|_2 = 1 \right\}.$$

Again, we remark that this expression is invariant under the transformation $g_\beta(\cdot) = \beta^{d/2} g(\beta \cdot)$. So we can take the supremum over $\|g\|_{2p} = 1$ then

$$K_{d,p} = \sup_{g \in H^1(\mathbb{R}^d)} \left\{ \|\nabla g\|_2^{-d/2q}, \|g\|_2 = \|g\|_{2p} = 1 \right\}.$$

So $\chi_{d,p} = \frac{1}{2} K_{d,p}^{-4q/d}$.

1.5 Sketch of proof

The proof of the lower bound of the large deviations principle (Section 3) is quite classical. Let

$$L_t = \frac{\alpha_t^d}{t} l_t(\lfloor \alpha_t x \rfloor)$$

be a rescaled version of l_t. Gantert, König and Shi (Lemma 3.1 in [22]) proved that for any $R > 0$, under the sub-probability measure $P(\cdot, supp(L_t) \subset [-R, R]^d)$, L_t satisfies a large deviations principle on

$$\mathcal{F} = \left\{ \mu \in \mathcal{M}_1(\mathbb{R}^d) \text{ such that } d\mu = \psi^2 dx \text{ and } supp(\psi) \subset [-R, R]^d \right\}$$

endowed with the weak topology, with speed $t\alpha_t^{-2}$, whose rate function is given by

$$\begin{aligned} \mathcal{I} : \mathcal{M}_1(\mathbb{R}^d) &\to \mathbb{R} \\ \mu &\to \begin{cases} \frac{1}{2} \|\nabla \psi\|_2^2, \text{ if } d\mu = \psi^2 dx, \\ +\infty \text{ else.} \end{cases} \end{aligned}$$

Let $\|\cdot\|_{p,R}$ be the L^p-norm of functions defined on $[-R, R]^d$. The function

$$\mu \in \mathcal{F} \to \left\| \frac{d\mu}{dx} \right\|_{p,R} = \sup \left\{ \int_{[-R,R]^d} \phi(x) d\mu(x), \|\phi\|_{q,R} = 1 \right\}$$

being lower semi-continuous in the weak topology, we can apply a contraction principle to transfer the large deviations lower bound from L_t to the SILT. Taking $\alpha_t = r_t^{-q/d}$ we have the desired result.

For the upper bound, we cannot proceed in the same way as I_t is only a lower semi-continuous function of L_t. Various methods have been developed to overcome this difficulty. In this paper, we use the same method as Castell [9] and Laurent [25] using Dynkin-Eisenbaum isomorphism theorem (Theorem 2.3). Let us describe this method. We first compare the SILT of the random walk with the SILT of the random walk projected on the discrete torus $\mathbb{T}_{R\alpha_t}$ of radius $R\alpha_t$, and stopped at an exponential time τ of parameter λ_t independent of the random walk (Lemma 2.2). On one hand, the projection of the random walk leads to an increase of the SILT which is not a problem for the upper bound, but on the other hand, we have to make a good choice of λ_t to compensate the stopping of the random walk.

Then, we apply Dynkin-Eisenbaum's theorem (Theorem 2.3). Roughly speaking, this theorem says that the law of the SILT is the same as the law of the $2p$-norm of a centered Gaussian process $(Z_x, x \in \mathbb{T}_{R\alpha_t})$ whose covariance is given by

$$G_{R\alpha_t, \lambda_t}(x, y) = E_x \left[\int_0^\tau \delta_x(X_s^{R\alpha_t}) ds \right]$$

(Lemma 2.4) where $(X_s^{R\alpha_t}, s \geq 0)$ is the random walk projected on $\mathbb{T}_{R\alpha_t}$.

Gaussian concentration inequalities lead to a first upper bound $\rho_1(a, R, t)$ given by a variational formula expressed in a discrete space:

$$\rho_1(a, R, t) = \inf \left\{ \lambda_t N_{2,R\alpha_t}^2(h) + \frac{1}{2} N_{2,R\alpha_t}^2(\tilde{\nabla} h), h \in L^{2p}(\mathbb{T}_{R\alpha_t}) \text{ such that } N_{2p,R\alpha_t}(h) = 1 \right\}, \tag{1.1}$$

where $N_p(\cdot)$ is the l^p-norm of functions defined on $\mathbb{Z}^d$, $N_{p,A}(\cdot)$ the l^p-norm of A-periodic functions defined on $\mathbb{Z}^d$, and $\tilde{\nabla}$ is the discrete gradient defined by

$$\forall x \in \mathbb{Z}^d, \forall i \in \{1, ..., d\}, \tilde{\nabla}_i f(x) = f(x + e_i) - f(x),$$

where $(e_1, ..., e_d)$ is the canonical base of $\mathbb{R}^d$.

The main work is then to take the limit in (1.1) over time and space and to prove the following proposition:

Proposition 1.4. *Let*

$$\rho(a) = \inf\left\{a\left\|h\right\|_2^2 + \frac{1}{2}\left\|\nabla h\right\|_2^2, h \in H^1(\mathbb{R}^d) \text{ such that } \left\|h\right\|_{2p} = 1\right\}.$$

Assume that $\alpha_t = r_t^{-q/d}$ *and* $\lambda_t = a\alpha_t^{-2} = ar_t^{2q/d}$*. If* $p(d-2) < d$*, then*

$$\liminf_{R\to+\infty}\liminf_{t\to+\infty} r_t^{1-2q/d}\rho_1(a,R,t) \geq \rho(a).$$

The proof of Proposition 1.4 is inspired by the proof of Lemma 2.1 of Becker and König [6]. The main difficulty is to pass from the variational formula $\rho_1(a,R,t)$ expressed in a discrete space, to the variational formula $\rho(a)$ expressed in a continuous space. First, we take a sequence h_n of functions defined on $\mathbb{Z}^d$ that approaches the infimum in the definition of $\rho_1(a,R,t)$. Then, we extend these functions on $\mathbb{R}^d$ to build a sequence g_n of continuous functions defined on $\mathbb{R}^d$. The sequence g_n is our candidate to approach the infimum in the definition of $\rho(a)$. Functions g_n are built as following. We split $\mathbb{Z}^d$ into unit cubes and we split each cubes in $d!$ tetrahedra. Functions h_n are defined on each vertex of each cubes. We extend them linearly on each tetrahedra. Thus, we have a linear interpolation of h_n.

We finally prove that the upper and the lower bound are equal with the following proposition:

Proposition 1.5.

$$\inf\left\{a - \rho(a), a > 0\right\} = -\chi_{d,p}.$$

2 Proof of the upper bound of Theorem 1.1

Let us begin with a lemma. We denote by $p_s^{R\alpha_t}(\cdot,\cdot)$ the probability transition of the random walk $(X_s^{R\alpha_t}, s \geq 0)$.

Lemma 2.1. *Behavior of* $G_{R\alpha_t,\lambda_t}(0,0)$.
For any $a > 0$ *we set* $\lambda_t = a\alpha_t^{-2}$*. If* $\alpha_t \to +\infty$*, then for any* $R > 0$*,*

1. *for* $d = 1$*,* $G_{R\alpha_t,\lambda_t}(0,0) = O(\alpha_t)$*.*
2. *for* $d = 2$*,* $G_{R\alpha_t,\lambda_t}(0,0) = O(\log\alpha_t)$*.*
3. *for* $d \geq 3$*,* $G_{R\alpha_t,\lambda_t}(0,0) = O(1)$*.*

Proof. Applying theorems 3.3.15 and 2.3.1 in [32], we know by Nash inequalities that

$$\exists C > 0 \text{ such that } \forall s > 0,\ \left|p_s^{R\alpha_t}(0,0) - \frac{1}{(R\alpha_t)^d}\right| \leq \frac{C}{s^{d/2}}.$$

So

$$\begin{aligned}
&G_{R\alpha_t,\lambda_t}(0,0)\\
=&\int_0^{+\infty}\exp(-s\lambda_t)p_s^{R\alpha_t}(0,0)ds\\
\leq&1 + \int_1^{+\infty}\frac{\exp(-s\lambda_t)}{(R\alpha_t)^d}ds + \int_1^{+\infty}\exp(-s\lambda_t)\frac{C}{s^{d/2}}ds\\
\leq&1 + \frac{1}{\lambda_t(R\alpha_t)^d} + \int_1^{1/\lambda_t}\frac{C}{s^{d/2}}ds + \int_{1/\lambda_t}^{+\infty}C\lambda_t^{d/2}\exp(-s\lambda_t)ds\\
\leq&1 + \frac{1}{\lambda_t(R\alpha_t)^d} + \int_1^{1/\lambda_t}\frac{C}{s^{d/2}}ds + C\lambda_t^{d/2-1}.
\end{aligned}$$

Remember that $\lambda_t = a\alpha_t^{-2}$ and that $\alpha_t \to +\infty$, then we have the result in the three cases. □

2.1 Step 1: comparison with the SILT of the random walk on the torus stopped at an exponential time

Lemma 2.2. *Let τ be an exponential time of parameter λ_t independent of the random walk $(X_t, t \geq 0)$ and*

$$l_{R\alpha_t,\tau}(x) = \int_0^\tau \delta_x(X_s^{R\alpha_t})\, ds.$$

Then for all $R, \alpha_t > 0$:

$$P\left[N_p(l_t) \geq tr_t\right] \leq e^{t\lambda_t} P\left[N_{p,R\alpha_t}(l_{R\alpha_t,\tau}) \geq tr_t\right].$$

Proof. It follows by convexity that

$$\begin{aligned} N_p^p(l_t) &= \sum_{x\in\mathbb{Z}^d} l_t^p(x) = \sum_{x\in\mathbb{T}_{R\alpha_t}} \sum_{k\in\mathbb{Z}^d} l_t^p(x + kR\alpha_t) \\ &\leq \sum_{x\in\mathbb{T}_{R\alpha_t}} \left(\sum_{k\in\mathbb{Z}^d} l_t(x + kR\alpha_t)\right)^p = \sum_{x\in\mathbb{T}_{R\alpha_t}} l_{R\alpha_t,t}^p(x) = N_{p,R\alpha_t}^p(l_{R\alpha_t,t}). \end{aligned}$$

Then, using the fact that $\tau \sim \mathcal{E}(\lambda_t)$ is independent of $(X_s, s \geq 0)$ we get:

$$\begin{aligned} P\left[N_p(l_t) \geq tr_t\right] \exp\left(-t\lambda_t\right) &\leq P\left[N_{p,R\alpha_t}(l_{R\alpha_t,t}) \geq tr_t\right] P(\tau \geq t) \\ &= P\left[N_{p,R\alpha_t}(l_{R\alpha_t,t}) \geq tr_t, \tau \geq t\right] \\ &\leq P\left[N_{p,R\alpha_t}(l_{R\alpha_t,\tau}) \geq tr_t\right]. \end{aligned}$$

Finally, $P\left[N_p(l_t) \geq tr_t\right] \leq e^{t\lambda_t} P\left[N_{p,R\alpha_t}(l_{R\alpha_t,\tau}) \geq tr_t\right]$.

□

2.2 Step 2: the Eisenbaum isomorphism theorem

Theorem 2.3. *(Eisenbaum, see for instance Corollary 8.1.2 page 364 in [29]). Let τ be as in Lemma 2.2 and let $(Z_x, x \in \mathbb{T}_{R\alpha_t})$ be a centered Gaussian process with covariance matrix*

$$G_{R\alpha_t,\lambda_t} = E_x\left[\int_0^\tau \delta_x(X_s^{R\alpha_t}) ds\right]$$

independent of τ and of the random walk $(X_s, s \geq 0)$. For $s \neq 0$, consider the process

$$S_x := l_{R\alpha_t,\tau}(x) + \frac{1}{2}(Z_x + s)^2,$$

then for all measurable and bounded function $F : \mathbb{R}^{\mathbb{T}_{R\alpha_t}} \mapsto \mathbb{R}$:

$$E\left[F((S_x; x \in \mathbb{T}_{R\alpha_t}))\right] = E\left[F\left((\frac{1}{2}(Z_x + s)^2; x \in \mathbb{T}_{R\alpha_t})\right)\left(1 + \frac{Z_0}{s}\right)\right].$$

2.3 Step 3: Comparison between $N_{p,R\alpha_t}(l_{R\alpha_t,\tau})$ and $N_{2p,R\alpha_t}(Z)$

Lemma 2.4. *Let τ and $(Z_x, x \in \mathbb{T}_{R\alpha_t})$ be defined as in Theorem 2.3. For all $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that for all $a, R, \alpha_t, r_t > 0$:*

$$P\left(N_{p,R\alpha_t}(l_{R\alpha_t,\tau}) \geq tr_t\right) \leq C(\epsilon)\left(1 + \frac{(R\alpha_t)^{d/2p}}{\epsilon\sqrt{2\epsilon tr_t\lambda_t}}\right) \frac{P\left(N_{2p,R\alpha_t}(Z) \geq \sqrt{2tr_t}(1 + \circ(\epsilon))\right)^{1/(1+\epsilon)}}{P(N_{2p,R\alpha_t}(Z) \geq (1+\epsilon)\sqrt{2tr_t\epsilon})}.$$

Proof.

$$S_x := l_{R\alpha_t,\tau}(x) + \frac{1}{2}(Z_x+s)^2 \Rightarrow S_x^p \geq l^p_{R\alpha_t,\tau}(x) + \left(\frac{1}{2}(Z_x+s)^2\right)^p$$
$$\Rightarrow N^p_{p,R\alpha_t}(S) \geq N^p_{p,R\alpha_t}(l_{R\alpha_t,\tau}) + \frac{1}{2^p}N^{2p}_{2p,R\alpha_t}(Z+s).$$

By independence of $(Z_x, x\in\mathbb{T}_{R\alpha_t})$ with the random walk $(X_s, s\geq 0)$ and the exponential time τ, we have for all $\epsilon>0$,

$$\begin{aligned}
&P\left(N^p_{p,R\alpha_t}(l_{R\alpha_t,\tau}) \geq t^p r_t^p\right) P\left(\frac{1}{2^p}N^{2p}_{2p,R\alpha_t}(Z+s)\geq t^p r_t^p\epsilon^p\right)\\
=&P\left(N^p_{p,R\alpha_t}(l_{R\alpha_t,\tau}) \geq t^p r_t^p, \frac{1}{2^p}N^{2p}_{2p,R\alpha_t}(Z+s)\geq t^p r_t^p\epsilon^p\right)\\
\leq&P\left(N^p_{p,R\alpha_t}(l_{R\alpha_t,\tau}) + \frac{1}{2^p}N^{2p}_{2p,R\alpha_t}(Z+s)\geq t^p r_t^p(1+\epsilon^p)\right)\\
\leq&P\left(N^p_{p,R\alpha_t}(S) \geq t^p r_t^p(1+\epsilon^p)\right)\\
=&E\left[\left(1+\frac{Z_0}{s}\right); \frac{1}{2^p}N^{2p}_{2p,R\alpha_t}(Z+s)\geq t^p r_t^p(1+\epsilon^p)\right],
\end{aligned} \tag{2.1}$$

where the last equality comes from Theorem 2.3. Moreover, by Hölder's inequality, for all $\epsilon>0$,

$$\begin{aligned}
&E\left[\left(1+\frac{Z_0}{s}\right); \frac{1}{2^p}N^{2p}_{2p,R\alpha_t}(Z+s)\geq t^p r_t^p(1+\epsilon^p)\right]\\
\leq&E\left[\left|1+\frac{Z_0}{s}\right|^{1+1/\epsilon}\right]^{\epsilon/(1+\epsilon)} P\left(N^{2p}_{2p,R\alpha_t}(Z+s)\geq 2^p t^p r_t^p(1+\epsilon^p)\right)^{1/(1+\epsilon)}.
\end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2) we obtain that for all $a,\epsilon>0$,

$$\begin{aligned}
&P\left(N_{p,R\alpha_t}(l_{R\alpha_t,\tau})\geq tr_t\right)\\
\leq&E\left[\left|1+\frac{Z_0}{s}\right|^{1+1/\epsilon}\right]^{\epsilon/(1+\epsilon)} \frac{P\left(N_{2p,R\alpha_t}(Z+s)\geq\sqrt{2tr_t}(1+\circ(\epsilon))\right)^{1/(1+\epsilon)}}{P(N_{2p,R\alpha_t}(Z+s)\geq\sqrt{2tr_t\epsilon})}.
\end{aligned} \tag{2.3}$$

Then using the fact that $Var(Z_0)=G_{R\alpha_t,\lambda_t}(0,0)\leq E[\tau]=\frac{1}{\lambda_t}$,

$$\begin{aligned}
&P\left(N_{p,R\alpha_t}(l_{R\alpha_t,\tau})\geq tr_t\right)\\
\leq&C(\epsilon)\left(1+\frac{1}{s\sqrt{\lambda_t}}\right) \frac{P\left(N_{2p,R\alpha_t}(Z+s)\geq\sqrt{2tr_t}(1+\circ(\epsilon))\right)^{1/(1+\epsilon)}}{P(N_{2p,R\alpha_t}(Z+s)\geq\sqrt{2tr_t\epsilon})}.
\end{aligned} \tag{2.4}$$

Choosing

$$s=\frac{\epsilon\sqrt{2tr_t\epsilon}}{(R\alpha_t)^{\frac{d}{2p}}},$$

using triangle inequality and the fact that $N_{2p,R\alpha_t}(s)=s(R\alpha_t)^{\frac{d}{2p}}$, we have:

$$P\left(N_{p,R\alpha_t}(l_{R\alpha_t,\tau})\geq tr_t\right)\leq C(\epsilon)\left(1+\frac{(R\alpha_t)^{d/2p}}{\epsilon\sqrt{2\epsilon tr_t\lambda_t}}\right)\frac{P\left(N_{2p,R\alpha_t}(Z)\geq\sqrt{2tr_t}(1+\circ(\epsilon))\right)^{1/(1+\epsilon)}}{P(N_{2p,R\alpha_t}(Z)\geq(1+\epsilon)\sqrt{2tr_t\epsilon})}.$$

□

2.4 Step 4: Large deviations for $N_{2p,R}(Z)$

Lemma 2.5. *Let τ and $(Z_x, x \in \mathbb{T}_{R\alpha_t})$ be defined as in Theorem 2.3, and $\rho_1(a,R,t)$ be defined by (1.1). Under assumptions of Proposition 1.4 and Theorem 1.1,*

1. $\forall a, R, t > 0$, $\lambda_t \leq \rho_1(a,R,t) \leq aR^{d/q}\alpha_t^{d/q-2}$.
2. $\forall a, \epsilon, R, t > 0$,

$$P\left[N_{2p,R\alpha_t}(Z) \geq \sqrt{tr_t\epsilon}\right] \geq \frac{1}{\sqrt{2\pi tr_t\epsilon\rho_1(a,R,t)}}\left(1 - \frac{1}{tr_t\epsilon\rho_1(a,R,t)}\right)\exp\left(-\frac{1}{2}tr_t\epsilon\rho_1(a,R,t)\right).$$

3. $\forall a, \epsilon, R, t > 0$,

$$\begin{aligned}&P\left(N_{2p,R\alpha_t}(Z) \geq \sqrt{2tr_t}(1+\circ(\epsilon))\right)\\ \leq&\frac{\sqrt{2}}{(\sqrt{2tr_t}(1+\circ(\epsilon))+\circ(\sqrt{tr_t}))\sqrt{\pi\rho_1(a,R,t)}}\exp\left(-\frac{\rho_1(a,R,t)\left(\sqrt{2tr_t}(1+\circ(\epsilon))+\circ(\sqrt{tr_t})\right)^2}{2}\right).\end{aligned}$$

Proof. 1. For the upper bound, it suffices to take $f = (R\alpha_t)^{-d/2p}$ to obtain the result. For the lower bound, we remark that $N_{2p,R\alpha_t}(h) = 1$ implies that for all $x \in \mathbb{T}_{R\alpha_t}$, $|h(x)| \leq 1$, and then $N^{2p}_{2p,R\alpha_t}(h) \leq N^2_{2,R\alpha_t}(h)$. Therefore $\rho_1(a,R,t) \geq \lambda_t$.

2. By Hölder's inequality, for any f such that $\|f\|_{(2p)',R\alpha_t} = 1$,

$$P\left[N_{2p,R\alpha_t}(Z) \geq \sqrt{tr_t\epsilon}\right] \geq P\left[\sum_{x\in\mathbb{T}_{R\alpha_t}} f_x Z_x \geq \sqrt{tr_t\epsilon}\right].$$

Since $\sum_{x\in\mathbb{T}_{R\alpha_t}} f_x Z_x$ is a real centered Gaussian variable with variance

$$\sigma^2_{a,R,t}(f) = \sum_{x,y\in\mathbb{T}_{R\alpha_t}} G_{R\alpha_t,\lambda_t}(x,y) f_x f_y\,,$$

we have:

$$\begin{aligned}P\left[\|Z\|_{2p,R\alpha_t} \geq \sqrt{tr_t\epsilon}\right] &\geq \frac{\sigma_{a,R,t}(f)}{\sqrt{2\pi}\sqrt{tr_t\epsilon}}\left(1 - \frac{\sigma^2_{a,R,t}(f)}{tr_t\epsilon}\right)\exp\left(-\frac{tr_t\epsilon}{2\sigma^2_{a,R,t}(f)}\right)\\ &\geq \frac{\sigma_{a,R,t}(f)}{\sqrt{2\pi}\sqrt{tr_t\epsilon}}\left(1 - \frac{\rho_2(a,R,t)}{tr_t\epsilon}\right)\exp\left(-\frac{tr_t\epsilon}{2\sigma^2_{a,R,t}(f)}\right),\end{aligned}$$

where

$$\rho_2(a,R,t) = \sup\left\{\sigma^2_{a,R,t}(f), N_{(2p)',R\alpha_t}(f) = 1\right\}.$$

Taking the supremum over f we obtain that $\forall a, R, t, \epsilon > 0$,

$$P\left[N_{2p,R\alpha_t}(Z) \geq \sqrt{tr_t\epsilon}\right] \geq \frac{\sqrt{\rho_2(a,R,t)}}{\sqrt{2\pi tr_t\epsilon}}\left(1 - \frac{\rho_2(a,R,t)}{tr_t\epsilon}\right)\exp\left(-\frac{tr_t\epsilon}{2\rho_2(a,R,t)}\right).$$

Then it suffices to prove that $\rho_2(a,R,t) = \frac{1}{\rho_1(a,R,t)}$ to obtain the result.

We denote by $< \cdot,\cdot >_{R\alpha_t}$ the scalar product on $l^2(\mathbb{T}_{R\alpha_t})$. On one hand, by Hölder inequality,

$$< f, G_{R\alpha_t,\lambda_t} f >_{R\alpha_t} \leq N_{2p,R\alpha_t}(G_{R\alpha_t,\lambda_t} f),\ \forall f \text{ such that } N_{(2p)',R\alpha_t}(f) = 1.$$

Since $G^{-1}_{R\alpha_t,\lambda_t} = \lambda_t - \Delta$,

$$\begin{aligned}
< f, G_{R\alpha_t,\lambda_t} f >_{R\alpha_t} &=< G^{-1}_{R\alpha_t,\lambda_t} G_{R\alpha_t,\lambda_t} f, G_{R\alpha_t,\lambda_t} f >_{R\alpha_t} \\
&= \lambda_t N^2_{2,R\alpha_t}(G_{R\alpha_t,\lambda_t} f) + \frac{1}{2} N^2_{2,R\alpha_t}(\nabla G_{R\alpha_t,\lambda_t} f) \\
&\geq \rho_1(a,R,t) N^2_{2p,R\alpha_t}(G_{R\alpha_t,\lambda_t} f).
\end{aligned}$$

Therefore, for all f such that $N_{(2p)',R\alpha_t}(f) = 1$,

$$< f, G_{R\alpha_t,\lambda_t} f >^2_{R\alpha_t} \leq \frac{< f, G_{R\alpha_t,\lambda_t} f >_{R\alpha_t}}{\rho_1(a,R,t)}.$$

Then, taking the supremum over f, $\rho_2(a,R,t) \leq 1/\rho_1(a,R,t)$.

On the other hand, let f_0 achieving the infimum in the definition of $\rho_1(a,R,t)$.

$$\begin{aligned}
\rho_2(a,R,t) &= \sup_{N_{(2p)',R\alpha_t}(f)=1} \{< f, G_{R\alpha_t,\lambda_t} f >_{R\alpha_t}\} \\
&\geq \frac{< G^{-1}_{R\alpha_t,\lambda_t} f_0, f_0 >_{R\alpha_t}}{N^2_{(2p)',R\alpha_t}(G^{-1}_{R\alpha_t,\lambda_t} f_0)} = \frac{\rho_1(a,R,t)}{N_{(2p)',R\alpha_t}(G^{-1}_{R\alpha_t,\lambda_t} f_0)}.
\end{aligned}$$

Furthermore, using the Lagrange multipliers method, we know that

$$N^2_{(2p)',R\alpha_t}(G^{-1}_{R\alpha_t,\lambda_t} h_0) = \rho_1(a,R,t).$$

Hence $\rho_2(a,R,t) \geq 1/\rho_1(a,R,t)$, and then $\rho_2(a,R,t) = 1/\rho_1(a,R,t)$.

3. Let M be a median of $N_{2p,R\alpha_t}(Z)$. We can easily see that

$$\begin{aligned}
&P\left(N_{2p,R\alpha_t}(Z) \geq \sqrt{2tr_t}(1 + \circ(\epsilon))\right) \\
\leq &P\left(|N_{2p,R\alpha_t}(Z) - M| \geq \sqrt{2tr_t}(1 + \circ(\epsilon)) - M\right).
\end{aligned} \tag{2.5}$$

Using concentration inequalities for norms of Gaussian processes (see for instance Lemma 3.1 in [28]), $\forall u > 0$,
$P\left[|N_{2p,R\alpha_t}(Z) - M| \geq \sqrt{u}\right] \leq 2P(Y \geq \sqrt{\frac{u}{\rho_2(a,R,t)}})$ where $Y \sim \mathcal{N}(0,1)$. Then for $tr_t \gg M^2$,

$$\begin{aligned}
&P\left(|N_{2p,R\alpha_t}(Z) - M| \geq \sqrt{2tr_t}(1 + \circ(\epsilon)) - M\right) \\
\leq &2P\left(Y \geq \frac{\sqrt{2tr_t}(1 + \circ(\epsilon)) - M}{\sqrt{\rho_2(a,R,t)}}\right) \\
\leq &\frac{2\sqrt{\rho_2(a,R,t)}}{(\sqrt{2tr_t}(1 + \circ(\epsilon)) - M)\sqrt{2\pi}} \exp\left(-\frac{\left(\sqrt{2tr_t}(1 + \circ(\epsilon)) - M\right)^2}{2\rho_2(a,R,t)}\right).
\end{aligned} \tag{2.6}$$

Let us now prove that under our assumptions we have $tr_t \gg M^2$. Since $M = (\text{median}(\sum_{x\in\mathbb{T}_{R\alpha_t}} Z_x^{2p}))^{1/2p}$ and that for any random variable $X \geq 0$, $\text{median}(X) \leq 2E[X]$, we get:

$$\begin{aligned}
M^2 &= (\text{median}(\sum_{x\in\mathbb{T}_{R\alpha_t}} Z_x^{2p}))^{1/p} \\
&\leq (2E[\sum_{x\in\mathbb{T}_{R\alpha_t}} Z_x^{2p}])^{1/p} \\
&\leq C(p)(\sum_{x\in\mathbb{T}_{R\alpha_t}} G_{R\alpha_t,\lambda_t}(0,0)^p E[Y^{2p}])^{1/p}, \text{ where } Y \sim \mathcal{N}(0,1) \\
&\leq C(p)(R\alpha_t)^{d/p} G_{R\alpha_t,\lambda_t}(0,0)(E[Y^{2p}])^{1/p} \\
&\leq C(p)(R\alpha_t)^{d/p} G_{R\alpha_t,\lambda_t}(0,0).
\end{aligned}$$

Recall that we have $\lambda_t = a\alpha_t^{-2}$ and $\alpha_t = r_t^{-q/d}$.
For $d=1$, by Lemma 2.1, $M^2 = O\left(\alpha_t^{1+1/p}\right) = O\left(r_t^{-\frac{p+1}{p-1}}\right)$. Then as $r_t \gg \frac{1}{t^{1/2q}}$ we have $M^2 \ll tr_t$.
For $d=2$, by Lemma 2.1, $M^2 = O\left(\alpha_t^{2/p}\log\alpha_t\right) = O\left(r_t^{-1/(p-1)}\log\frac{1}{r_t}\right)$. Then as $r_t \gg \left(\frac{\log t}{t}\right)^{1/q}$ we have $M^2 \ll tr_t$.
For $d\geq 3$, by Lemma 2.1, $M^2 \leq C\alpha_t^{d/p} = Cr_t^{-1/(p-1)}$. Then as $r_t \gg t^{-1/q}$, we have $M^2 \ll tr_t$.

□

2.5 End of proof of the upper bound in theorem 1.1

Combining Lemma 2.2 and Lemma 2.4 we have proved that: $\forall \epsilon, a, R, t > 0$,

$$\begin{aligned}&P\left(N_p(l_t)\geq tr_t\right)\\ \leq& C(\epsilon)\exp(t\lambda_t)\left(1+\frac{(R\alpha_t)^{\frac{d}{2p}}}{\epsilon\sqrt{2\epsilon tr_t\lambda_t}}\right)\frac{P\left(N_{2p,R\alpha_t}(Z)\geq\sqrt{2tr_t}(1+\circ(\epsilon))\right)^{1/(1+\epsilon)}}{P\left(N_{2p,R}(Z)\geq 2\sqrt{2tr_t\epsilon}\right)}.\end{aligned} \tag{2.7}$$

First we look for an upper bound for the numerator in (2.7). By 1 and 3 of Lemma 2.5, we have that

$$\begin{aligned}&\limsup_t \frac{1}{tr_t^{2q/d}}\log P\left(N_{2p,R\alpha_t}(Z)\geq\sqrt{2tr_t}(1+\circ(\epsilon))\right)^{1/(1+\epsilon)}\\ &\leq -\liminf_{t\to+\infty} r_t^{1-2q/d}\rho_1(a,R,t)(1+\circ(\epsilon)).\end{aligned} \tag{2.8}$$

Now we work on the denominator in (2.7). Using 1 and 2 of Lemma 2.5, we obtain:

$$\begin{aligned}&P\left(N_{2p,R}(Z)\geq 2\sqrt{2tr_t\epsilon}\right)\\ \geq&\frac{1}{\sqrt{16\pi tr_t\epsilon\rho_1(a,R,t)}}\left(1-\frac{1}{8tr_t\epsilon\rho_1(a,R,t)}\right)\exp\left(-4tr_t\epsilon\rho_1(a,R,t)\right)\\ \geq&\frac{1}{\sqrt{16\pi ta\epsilon R^{d/q}r_t^{2q/d}}}\left(1-\frac{1}{8tr_t\epsilon\lambda_t}\right)\exp\left(-4\epsilon atr_t^{2q/d}R^{d/q}\right)\end{aligned}$$

Therefore,

$$\liminf_t \frac{1}{tr_t^{2q/d}}\log P\left(N_{2p,R}(Z)\geq 2\sqrt{2tr_t\epsilon}\right)\geq -4a\epsilon R^{d/q}. \tag{2.9}$$

Now we combine (2.7),(2.8),(2.9) to have:

$$\limsup_t \frac{1}{tr_t^{2q/d}}\log P\left(N_p(l_t)\geq tr_t\right)\leq a-(1+\circ(\epsilon))\liminf_t r_t^{1-2q/d}\rho_1(a,R,t)+4a\epsilon R^{d/q-2}.$$

Then we let $\epsilon\to 0$:

$$\limsup_t \frac{1}{tr_t^{2q/d}}\log P\left(N_p(l_t)\geq tr_t\right)\leq a-\liminf_t r_t^{1-2q/d}\rho_1(a,R,t).$$

Then we take the limit over R using Proposition 1.4:

$$\limsup_t \frac{1}{tr_t^{2q/d}}\log P\left(N_p(l_t)\geq tr_t\right)\leq a-\rho(a).$$

We finish the proof taking the infimum over $a>0$ and using Proposition 1.5.

3 Proof of the lower bound of Theorem 1.1

Proof. Let for all x in $\mathbb{R}^d$,

$$L_t = \frac{\alpha_t^d}{t} l_t(\lfloor \alpha_t x \rfloor)$$

be the rescaled version of l_t. Thanks to the work of Gantert, König and Shi (Lemma 3.1 in [22]) we know that for $R > 0$, under the sub-probability measure $P(\cdot, supp(L_t) \subset [-R, R]^d)$, L_t satisfies a large deviations principle on

$$\mathcal{F} = \left\{ \mu \in \mathcal{M}_1(\mathbb{R}^d) \text{ such that } d\mu = \psi^2 dx \text{ and } supp(\psi) \subset [-R, R]^d \right\}$$

endowed with the weak topology, with speed $t\alpha_t^{-2}$, whose rate function is given by

$$\begin{aligned} \mathcal{I} : \mathcal{M}_1(\mathbb{R}^d) &\to \mathbb{R} \\ \mu &\to \begin{cases} \frac{1}{2} \|\nabla \psi\|_2^2, \text{ if } d\mu = \psi^2 dx, \\ +\infty \text{ else.} \end{cases} \end{aligned}$$

So, for $r_t = \alpha_t^{-d/q}$,

$$\begin{aligned} P(N_p(l_t) \geq tr_t) &= P(\|L_t\|_p \geq 1) \\ &\geq P(\|L_t\|_p > 1, supp(L_t) \subset [-R, R]^d). \end{aligned}$$

Then, as

$$\mu \to \|\mu\|_p = \sup \left\{ \int_{\mathbb{R}^d} f(x) d\mu(x), \|f\|_q = 1 \right\}$$

is a lower semi-continuous function in the weak topology, we have:

$$\liminf_{t \to +\infty} \frac{1}{tr_t^{2q/d}} \log P(N_p(l_t) \geq tr_t) \geq - \inf \left\{ \frac{1}{2} \|\nabla \psi\|_2^2, \|\psi\|_2 = 1, \|\psi\|_{2p} > 1, supp(\psi) \subset [-R, R]^d \right\}.$$

Let $R \to +\infty$,

$$\begin{aligned} \liminf_{t \to +\infty} \frac{1}{tr_t^{2q/d}} \log P(N_p(l_t) \geq tr_t) &\geq - \inf \left\{ \frac{1}{2} \|\nabla \psi\|_2^2, \|\psi\|_2 = 1, \|\psi\|_{2p} > 1 \right\} \\ &= - \inf \left\{ \frac{1}{2} \|\nabla \psi\|_2^2, \|\psi\|_2 = \|\psi\|_{2p} = 1 \right\}. \end{aligned}$$

□

4 Proofs of Propositions 1.4 and 1.5

We denote by $\mathfrak{S}_d$ the set of the permutations on $\{1, ..., d\}$, by $\lfloor \cdot \rfloor$ the integer part, by $B(s)$ the ball of radius s and by $Vol(B(s))$ its volume.

Proof of Proposition 1.4:
Let choose a sequence (R_n, t_n, h_n) such that $R_n \to +\infty$, $t_n \to +\infty$, $N_{2p, R\alpha_{t_n}}(h_n) = 1$ and such that

$$\begin{aligned} &\liminf_{R \to +\infty} \liminf_{t \to +\infty} \inf \left\{ \frac{a}{\alpha_t^{d/q}} N_{2, R\alpha_t}^2(h) + \frac{1}{2} \alpha_t^{2-d/q} N_{2, R\alpha_t}^2(\tilde{\nabla} h), N_{2p, R\alpha_t}(h) = 1 \right\} \\ \geq &\frac{a}{\alpha_{t_n}^{d/q}} N_{2, R_n \alpha_{t_n}}^2(h_n) + \frac{1}{2} \alpha_{t_n}^{2-d/q} N_{2, R_n \alpha_{t_n}}^2(\tilde{\nabla} h_n) - \frac{1}{n}. \end{aligned}$$

h_n is a sequence of functions defined on $\mathbb{T}_{R_n\alpha_{t_n}}$. Note that we can assume without loss of generality that h_n is non-negative. We want to extend these functions to $\mathbb{R}^d$. In this perspective, we split $\mathbb{Z}^d$ into cubes

$$C(k) = \bigotimes_{i=1}^{d} [k_i, k_i + 1]$$

for any $k \in \mathbb{Z}^d$. Then, each cube $C(k)$ is again splitted into $d!$ tetrahedra $T_\sigma(k)$, where for any $\sigma \in \mathfrak{S}_d$, $T_\sigma(k)$ is the convex hull of $k, k + e_{\sigma(1)}, ..., k + e_{\sigma(1)} + \cdots + e_{\sigma(d)}$. For any $y \in \mathbb{R}^d$ we denote by $\sigma(y)$ a permutation such that $y \in T_{\sigma(y)}(\lfloor y \rfloor)$. Note that this permutation is not unique when y is at the boarder of two or more tetrahedra, but it won't be a problem. Set for any $x \in \mathbb{R}^d$,

$$g_n(x) = \alpha_{t_n}^{d/2p} h_n(\lfloor \alpha_{t_n} x \rfloor) + \alpha_{t_n}^{d/2p} \sum_{i=1}^{d} f_{n,\sigma(\alpha_{t_n} x),i}(\alpha_{t_n} x), \tag{4.1}$$

where $\forall y \in \mathbb{R}^d$, $\forall i \in \{1, ..., d\}$, $\forall \sigma \in \mathfrak{S}^d$,

$$f_{n,\sigma,i}(y) = \big(h_n(\lfloor y \rfloor + e_{\sigma(1)} + ... + e_{\sigma(i)}) - h_n(\lfloor y \rfloor + e_{\sigma(1)} + ... + e_{\sigma(i-1)})\big)\,(y_{\sigma(i)} - \lfloor y_{\sigma(i)} \rfloor).$$

Note that g_n is well defined for any y in $\mathbb{R}^d$, even for y at the boarder of two or more tetrahedra because the value of g_n is the same if we consider y living in one tetrahedra or the other. Following the work of Becker and König, it can be proved that g_n is continuous and R_n-periodic.

Now, for $\epsilon \in]0, 1[$, we set Ψ_{R_n} a truncation function that verify

$$\Psi_{R_n} = \bigotimes_{i=1}^{d} \psi_{R_n} : \mathbb{R}^d \to [0, 1],$$

where

$$\psi_{R_n} = \begin{cases} 0 \text{ outside } [-R_n, R_n], \\ \text{linear in } [-R_n, -R_n + R_n^\epsilon] \text{ and in } [R_n - R_n^\epsilon, R_n], \\ 1 \text{ in } [-R_n + R_n^\epsilon, R_n - R_n^\epsilon]. \end{cases}$$

The function

$$\frac{g_n \Psi_{R_n}}{\|g_n \Psi_{R_n}\|_{2p}}$$

is our candidate to realize the infimum in the definition of $\rho(a)$. Therefore, we have to bound from below $N^2_{2,R_n\alpha_{t_n}}(h_n)$ by $\|(g_n\Psi_n)\|_2^2$ and $N^2_{2,R_n\alpha_{t_n}}(\tilde{\nabla} h_n)$ by $\|\nabla(g_n\Psi_n)\|_2^2$.
First we prove the following:

$$\forall \delta > 0,\ \alpha_{t_n}^{-d/q} N^2_{2,R_n\alpha_{t_n}}(h_n) \geq \frac{\delta}{\delta + 1} \|g_n \Psi_{R_n}\|_2^2 - \delta d \alpha_{t_n}^{-d/q} N^2_{2,R_n\alpha_{t_n}}(\tilde{\nabla} h_n). \tag{4.2}$$

Using the definition (4.1) of g_n and the triangle inequality, we have that

$$\|g_n\|_{2,R_n} \leq \alpha_{t_n}^{-d/2q} N_{2,R_n\alpha_{t_n}}(h_n) + \alpha_{t_n}^{-d/2q} \left\| \sum_{i=1}^{d} f_{n,\sigma(\cdot),i}(\cdot) \right\|_{2,R_n\alpha_{t_n}}. \tag{4.3}$$

We bound from above now the norm of $f_{n,\sigma(y),i}(y)$ for any $y \in T_\sigma(k)$:

$$|\sum_{i=1}^{d} f_{n,\sigma(y),i}(y)|^2 \leq d \sum_{i=1}^{d} |h_n(\lfloor y \rfloor + e_{\sigma(1)} + \cdots + e_{\sigma(i)}) - h_n(\lfloor y \rfloor + e_{\sigma(1)} + \cdots + e_{\sigma(i-1)})|^2.$$

Then,

$$\left\|\sum_{i=1}^{d} f_{n,\sigma,i}\right\|_{2,R\alpha_{t_n}}^2 \leq d \sum_{k\in B(R_n\alpha_{t_n})} \sum_{\sigma\in\mathfrak{S}(d)} \int_{y\in T_\sigma(k)} \sum_{i=1}^{d} |\tilde{\nabla}_{\sigma(i)} h_n(k+e_{\sigma(1)}+\cdots+e_{\sigma(i-1)})|^2 dy$$
$$= \frac{d}{d!} \sum_{\sigma\in\mathfrak{S}(d)} \sum_{i=1}^{d} \sum_{k\in B(R_n\alpha_{t_n})} |\tilde{\nabla}_{\sigma(i)} h_n(k+e_{\sigma(1)}+\cdots+e_{\sigma(i-1)})|^2$$
$$= dN^2_{2,R_n\alpha_{t_n}}(\tilde{\nabla}(h_n)).$$

Moreover, thanks to the definition (4.1) of g_n, it is easy to see that

$$\|\nabla g_n\|^2_{2,R_n} = \alpha_{t_n}^{2-d/q} N^2_{2,R_n\alpha_{t_n}}(\tilde{\nabla} h_n). \tag{4.4}$$

Thus:

$$\left\|\sum_{i=1}^{d} f_{n,\sigma(\cdot),i}(\cdot)\right\|_{2,R_n\alpha_{t_n}} \leq \sqrt{d} N_{2,R_n\alpha_{t_n}}(\tilde{\nabla} h_n) = \frac{\sqrt{d}}{\alpha_{t_n}^{1-d/2q}} \|\nabla g_n\|_{2,R_n}. \tag{4.5}$$

Combining (4.3) and (4.5) we have:

$$\|g_n\|_{2,R_n} \leq \alpha_{t_n}^{-d/2q} N_{2,R_n\alpha_{t_n}}(h_n) + \sqrt{d}\alpha_{t_n}^{-1} \|\nabla g_n\|_{2,R_n}. \tag{4.6}$$

Taking the square in (4.6), we have that $\forall \delta > 0$,

$$\|g_n\|^2_{2,R_n} \leq (1+\frac{1}{\delta})\alpha_{t_n}^{-d/q} N^2_{2,R_n\alpha_{t_n}}(h_n) + (1+\delta)d\alpha_{t_n}^{-2} \|\nabla g_n\|^2_{2,R_n}.$$

So,

$$\alpha_{t_n}^{-d/q} N^2_{2,R_n\alpha_{t_n}}(h_n) \geq \frac{\delta}{\delta+1} \|g_n\|^2_{2,R_n} - \delta d\alpha_{t_n}^{-2} \|\nabla g_n\|^2_{2,R_n}$$
$$\geq \frac{\delta}{\delta+1} \|g_n\Psi_{R_n}\|^2_2 - \delta d\alpha_{t_n}^{-2} \|\nabla g_n\|^2_{2,R_n}.$$

Therefore, we succeeded to prove (4.2):

$$\forall \delta > 0,\ \alpha_{t_n}^{-d/q} N^2_{2,R_n\alpha_{t_n}}(h_n) \geq \frac{\delta}{\delta+1} \|g_n\Psi_{R_n}\|^2_2 - \delta d\alpha_{t_n}^{-d/q} N^2_2(\tilde{\nabla} h_n).$$

Now we want to prove that there exists $C > 0$ such that for n large enough

$$\alpha_{t_n}^{2-d/q} N^2_{2,R_n\alpha_{t_n}}(\tilde{\nabla} h_n) \geq \left(1-\frac{C}{R_n^\epsilon}\right) \|\nabla(g_n\Psi_{R_n})\|^2_2 - \frac{C}{\alpha_{t_n}^{d/q} R_n^\epsilon} N^2_{2,R_n\alpha_{t_n}}(h_n). \tag{4.7}$$

By the work of Becker and König, we know that

$$\|\nabla(g_n\Psi_{R_n})\|^2_2 \leq \left(1+\frac{1}{R_n^\epsilon}\right) \|\nabla g_n\|^2_{2,R_n} + \frac{2}{R_n^\epsilon} \|g_n\|^2_{2,R_n}. \tag{4.8}$$

Putting together (4.4),(4.8) and (4.6), we have:

$$\begin{aligned}\|\nabla g_n\|_{2,R_n}^2 \geq & \frac{R_n^\epsilon}{1+R_n^\epsilon}\|\nabla(g_n\Psi_{R_n})\|_2^2 - \frac{2}{1+R_n^\epsilon}\left(\frac{1}{\alpha_{t_n}^{d/2q}}N_{2,R_n\alpha_{t_n}}(h_n) + \frac{\sqrt{d}}{\alpha_{t_n}}\|\nabla g_n\|_{2,R_n}\right)^2 \\ \geq & \frac{R_n^\epsilon}{1+R_n^\epsilon}\|\nabla(g_n\Psi_{R_n})\|_2^2 - \frac{2}{1+R_n^\epsilon}\left(\frac{1}{\alpha_{t_n}^{d/q}}N_{2,R_n\alpha_{t_n}}^2(h_n) + \frac{d}{\alpha_{t_n}^2}\|\nabla g_n\|_{2,R_n}^2\right. \\ & \left. + \frac{2\sqrt{d}}{\alpha_{t_n}^{1+d/2q}}N_{2,R_n\alpha_{t_n}}(h_n)\|\nabla g_n\|_{2,R_n}\right) \\ \geq & \frac{R_n^\epsilon}{1+R_n^\epsilon}\|\nabla(g_n\Psi_{R_n})\|_2^2 - \frac{2}{1+R_n^\epsilon}\left(\frac{1}{\alpha_{t_n}^{d/q}}N_{2,R_n\alpha_{t_n}}^2(h_n) + \frac{d}{\alpha_{t_n}^2}\|\nabla g_n\|_2^2\right. \\ & \left. + \frac{\sqrt{d}}{\alpha_{t_n}^{1+d/2q}}N_{2,R_n\alpha_{t_n}}^2(h_n) + \frac{\sqrt{d}}{\alpha_{t_n}^{1+d/2q}}\|\nabla g_n\|_{2,R_n}^2\right).\end{aligned}$$

So,

$$\begin{aligned}&\|\nabla g_n\|_{2,R_n}^2\left(1+\frac{2}{1+R_n^\epsilon}\left(\frac{d}{\alpha_{t_n}^2}+\frac{\sqrt{d}}{\alpha_{t_n}^{1+d/q}}\right)\right) \\ \geq & \frac{R_n^\epsilon}{1+R_n^\epsilon}\|\nabla(g_n\Psi_{R_n})\|_2^2 - \frac{2}{1+R_n^\epsilon}\left(\frac{1}{\alpha_{t_n}^{d/q}}+\frac{\sqrt{d}}{\alpha_{t_n}^{1+d/2q}}\right)N_{2,R_n\alpha_{t_n}}^2(h_n).\end{aligned}$$

Then, using (4.4), the fact that $\alpha_{t_n} \to +\infty$ and that $1 + d/2q > d/q$, we obtain:

$$\begin{aligned}\alpha_{t_n}^{2-d/q}N_{2,R_n\alpha_{t_n}}^2(\tilde{\nabla}h_n) \geq & \frac{R_n^\epsilon}{1+R_n^\epsilon+2\left(\frac{d}{\alpha_{t_n}^2}+\frac{\sqrt{d}}{\alpha_{t_n}^{1+d/q}}\right)}\|\nabla(g_n\Psi_{R_n})\|_2^2 \\ & - \frac{2}{1+R_n^\epsilon+2\left(\frac{d^2}{\alpha_{t_n}^2}+\frac{\sqrt{d}}{\alpha_{t_n}^{1+d/q}}\right)}\left(\frac{1}{\alpha_{t_n}^{d/q}}+\frac{\sqrt{d}}{\alpha_{t_n}^{1+d/2q}}\right)N_{2,R_n\alpha_{t_n}}^2(h_n) \\ \geq & \left(1-\frac{C}{R_n^\epsilon}\right)\|\nabla(g_n\Psi_{R_n})\|_2^2 - \frac{C}{\alpha_{t_n}^{d/q}R_n^\epsilon}N_{2,R_n\alpha_{t_n}}^2(h_n).\end{aligned}$$

Thus, we succeeded to prove (4.7).
At this point of the proof, we have bounded from below $N_2^2(\tilde{\nabla}h_n)$ by $\|\nabla(g_n\Psi_{R_n})\|_2^2$ (4.7), and $N_2^2(h_n)$ by $\|(g_n\Psi_{R_n})\|_2^2$ (4.2). Therefore, combining these two results, we have for n large enough:

$$\begin{aligned}&\frac{a}{\alpha_{t_n}^{d/q}}N_{2,R_n\alpha_{t_n}}^2(h_n) + \frac{1}{2}\alpha_{t_n}^{2-d/q}N_{2,R_n\alpha_{t_n}}^2(\tilde{\nabla}h_n) \\ \geq & a\left(\frac{\delta}{\delta+1}\|g_n\Psi_{R_n}\|_2^2 - \delta d\alpha_{t_n}^{-d/q}N_{2,R_n\alpha_{t_n}}^2(\tilde{\nabla}h_n)\right) + \frac{1}{2}\left(1-\frac{C}{R_n^\epsilon}\right)\|\nabla(g_n\Psi_{R_n})\|_2^2 \\ & - \frac{C}{\alpha_{t_n}^{d/q}R_n^\epsilon}N_{2,R_n\alpha_{t_n}}^2(h_n) \\ \geq & \min\left(\frac{\delta}{\delta+1}, 1-\frac{C}{R_n^\epsilon}\right)\left(a\|g_n\Psi_{R_n}\|_2^2 + \frac{1}{2}\|\nabla(g_n\Psi_{R_n})\|_2^2\right) - a\delta d\alpha_{t_n}^{-d/q}N_{2,R_n\alpha_{t_n}}^2(\tilde{\nabla}h_n) \\ & - C\alpha_{t_n}^{-d/q}R_n^{-\epsilon}N_{2,R_n\alpha_{t_n}}^2(h_n) \\ \geq & \min\left(\frac{\delta}{\delta+1}, 1-\frac{C}{R_n^\epsilon}\right)\rho(a)\|g_n\Psi_{R_n}\|_{2p}^2 - a\delta d\alpha_{t_n}^{-d/q}N_{2,R_n\alpha_{t_n}}^2(\tilde{\nabla}h_n) \\ & - C\alpha_{t_n}^{-d/q}R_n^{-\epsilon}N_{2,R_n\alpha_{t_n}}^2(h_n). \qquad (4.9)\end{aligned}$$

It remains now to prove that $\|g_n\Psi_{R_n}\|_{2p}^2$ is close to 1. First, we prove that without loss of generality, we can assume that there exists $C>0$ such that

$$\left(1-CR_n^{\epsilon-1}\right)\|g_n\|_{2p,R_n}^2 \le \|g_n\Psi_{R_n}\|_{2p}^2 . \tag{4.10}$$

Indeed, for any $a\in B_{R_n}$ let $g_{n,a}(x)=g_n(x-a)$. By periodicity of g_n, on one side we have

$$\begin{aligned}\int\limits_{B_{R_n}}\int\limits_{B_{R_n}\setminus B_{R_n-R_n^\epsilon}} g_n^{2p}(x-a)dx\,da &= \int\limits_{B_{R_n}\setminus B_{R_n-R_n^\epsilon}}\int\limits_{B_{R_n}} g_n^{2p}(x-a)dx\,da\\ &= \int\limits_{B_{R_n}\setminus B_{R_n-R_n^\epsilon}}\int\limits_{B_{R_n}} g_n^{2p}(x)dx\,da\\ &\le CR_n^{d-1+\epsilon}\|g_n\|_{2p,R_n}^{2p},\end{aligned}$$

and on the opposite side we have

$$\int\limits_{B_{R_n}}\int\limits_{B_{R_n}\setminus B_{R_n-R_n^\epsilon}} g_n^{2p}(x-a)dx\,da \ge Vol(B(1))R_n^d \inf_{a\in B_{R_n}}\left\{\int\limits_{B_{R_n}\setminus B_{R_n-R_n^\epsilon}} g_n^{2p}(x-a)dx\,da\right\}.$$

Therefore

$$\inf_{a\in B_{R_n}}\left\{\int\limits_{B_{R_n}\setminus B_{R_n^\epsilon}} g_n^{2p}(x-a)dx\,da\right\}\le CR_n^{\epsilon-1}\|g_n\|_{2p,R_n}^{2p}.$$

Remark that g_n being periodic, for any $a\in B_{R_n}$, $\|g_n\|_2=\|g_{n,a}\|$ and $\|\nabla g_n\|_2=\|\nabla g_{n,a}\|_2$. Hence, combining (4.9) and (4.10) we obtain that: $\exists C>0$ such that $\forall\delta>0$,

$$\begin{aligned}&\frac{a}{\alpha_{t_n}^{d/q}}N_{2,R_n\alpha_{t_n}}^2(h_n)+\frac{1}{2}\alpha_{t_n}^{2-d/q}N_{2,R_n\alpha_{t_n}}^2(\tilde\nabla h_n)\\ &\ge\min\left(\frac{\delta}{\delta+1},1-\frac{C}{R_n^\epsilon}\right)\left(1-CR_n^{\epsilon-1}\right)\rho(a)\|g_n\|_{2p,R_n}^2-a\delta d\alpha_{t_n}^{-d/q}N_{2,R_n\alpha_{t_n}}^2(\tilde\nabla h_n)\\ &\qquad -C\alpha_{t_n}^{-d/q}R_n^{-\epsilon}N_{2,R_n\alpha_{t_n}}^2(h_n).\end{aligned}$$

It remains now to prove that $\|g_n\|_{2p,R_n}^2$ is close to 1. Using the definition (4.1) of g_n and the triangle inequality, we can prove that

$$\|g_n\|_{2p,R_n}\ge 1-C\alpha_{t_n}^{-1}\|\nabla g_n\|_{2p,R_n}.$$

Therefore, for all $\gamma>0$,

$$(1+\gamma)\|g_n\|_{2p,R_n}^2+\frac{1+\gamma}{\gamma}C\alpha_{t_n}^{-2}\|\nabla g_n\|_{2p,R_n}^2\ge 1.$$

Finally, there exists $C>0$ such that for all $\delta,\gamma>0$,

$$\begin{aligned}&\frac{a}{\alpha_{t_n}^{d/q}}N_{2,R_n\alpha_{t_n}}^2(h_n)+\frac{1}{2}\alpha_{t_n}^{2-d/q}N_{2,R_n\alpha_{t_n}}^2(\tilde\nabla h_n)\\ &\ge\min\left(\frac{\delta}{\delta+1},1-\frac{C}{R_n^\epsilon}\right)\left(1-CR_n^{\epsilon-1}\right)\left(\frac{1}{1+\gamma}-\frac{C}{\gamma\alpha_{t_n}^2}\|\nabla g_n\|_{2p,R_n}^2\right)\rho(a)\\ &\quad -a\delta d\alpha_{t_n}^{-d/q}N_{2,R_n\alpha_{t_n}}^2(\tilde\nabla h_n)-C\alpha_{t_n}^{-d/q}R_n^{-\epsilon}N_{2,R_n\alpha_{t_n}}^2(h_n).\end{aligned}$$

Remember that we have assumed that h is non-negative, thus, as $\|h_n\|_\infty \leq 1$ we have that

$$\|\nabla g_n\|_{2p,R_n} = \alpha_{t_n} N_{2p,R_n\alpha_{t_n}}(\tilde{\nabla} h_n) \leq \alpha_{t_n} N^{1/p}_{2,R_n\alpha_{t_n}}(\tilde{\nabla} h_n).$$

It follows that there exists $C > 0$ such that for all $\delta, \gamma > 0$,

$$\begin{aligned}
&\frac{a}{\alpha_{t_n}^{d/q}} N^2_{2,R_n\alpha_{t_n}}(h_n) + \frac{1}{2}\alpha_{t_n}^{2-d/q} N_{2,R_n\alpha_{t_n}}(\tilde{\nabla} h_n)\\
&\geq \min\left(\frac{\delta}{\delta+1}, 1 - \frac{C}{R_n^\epsilon}\right)\left(1 - CR_n^{\epsilon-1}\right)\left(\frac{1}{1+\gamma} - \frac{C}{\gamma} N^{2/p}_{2,R_n\alpha_{t_n}}(\tilde{\nabla} h_n)\right)\rho(a)\\
&\quad - a\delta d \alpha_{t_n}^{-d/2q} N^2_{2,R_n\alpha_{t_n}}(\tilde{\nabla} h_n) - C\alpha_{t_n}^{-d/q} R_n^{-\epsilon} N^2_{2,R_n\alpha_{t_n}}(h_n).
\end{aligned}$$

We want now let n going to infinity. It's easy to see that if $\alpha_{t_n}^{2-d/q} N_{2,R_n\alpha_{t_n}}(\tilde{\nabla} h_n) \to +\infty$ or $\alpha_{t_n}^{-d/q} N^2_{2,R_n\alpha_{t_n}}(h_n) \to +\infty$, then the result is obvious. Therefore, we can assume that

$$\liminf_{n\to+\infty} \frac{a}{\alpha_{t_n}^{d/q}} N^2_{2,R_n\alpha_{t_n}}(h_n) + \frac{1}{2}\alpha_{t_n}^{2-d/q} N_{2,R_n\alpha_{t_n}}(\tilde{\nabla} h_n) < +\infty.$$

Thus, $N_{2,R_n\alpha_{t_n}}(\tilde{\nabla} h_n) \to 0$ because $2 - d/q > 0$ and $\alpha_{t_n}^{-d/q} R_n^{-\epsilon} N^2_{2,R_n\alpha_{t_n}}(h_n) \to 0$. Therefore, we have:

$$\liminf_n \frac{a}{\alpha_{t_n}^{d/q}} N^2_{2,R_n\alpha_{t_n}}(h_n) + \frac{1}{2}\alpha_{t_n}^{2-d/q} N^2_{2,R_n\alpha_{t_n}}(\tilde{\nabla} h_n) \geq \min\left(\frac{\delta}{\delta+1}, 1\right)\frac{1}{1+\gamma}\rho(a).$$

Then we let $\delta \to +\infty$ and $\gamma \to 0$ to obtain the result.

Proof of Proposition 1.5:
We recall that

$$\rho(a) = \inf\left\{a\|h\|_2^2 + \frac{1}{2}\|\nabla h\|_2^2, \|h\|_{2p} = 1\right\}.$$

Set $h_\beta(\cdot) = \beta^{d/2p} h(\beta\cdot)$. We remark that $\|h_\beta\|_{2p} = 1$, $\|h_\beta\|_2 = \beta^{-d/2q}\|h\|_2$ and $\|\nabla h_\beta\|_2 = \beta^{1-d/2q}\|\nabla h\|_2$. Then we minimize over β the function:

$$\Phi_h(\beta) = a\beta^{-d/q}\|h\|_2^2 + \frac{1}{2}\beta^{2-d/q}\|\nabla h\|_2^2.$$

Picking the optimal value

$$\beta^* = \sqrt{\frac{2ad}{2q-d}}\frac{\|h\|_2}{\|\nabla h\|_2}$$

we have that

$$\rho(a) = a^{1-d/2q}\left(\frac{2q}{2q-d}\right)\left(\frac{2q-d}{2d}\right)^{d/2q} \inf\left\{\|h\|_2^{2-d/q}\|\nabla h\|_2^{d/q}, \|h\|_{2p} = 1\right\}.$$

Then optimizing over $a > 0$ the expression $\inf\{a - \rho(a), a > 0\}$ with the optimal value

$$a^* = \frac{2q-d}{2d}\inf\left\{\|h\|_2^{4q/d-2}\|\nabla h\|_2^2, \|h\|_{2p} = 1\right\}$$

we have that

$$\inf\{a - \rho(a), a > 0\} = -\inf\left\{\frac{1}{2}\|h\|_2^{4q/d-2}\|\nabla h\|_2^2, \|h\|_{2p} = 1\right\}.$$

Note that the expression is invariant under the transformation $h_\beta(\cdot) = \beta^{d/2p} h(\beta\cdot)$, therefore we can freely add the condition $\|h\|_2 = 1$. Thus,

$$\inf\{a - \rho(a), a > 0\} = -\inf\left\{\frac{1}{2}\|\nabla h\|_2^2, \|h\|_{2p} = \|h\|_2 = 1\right\} := -\chi_{d,p}.$$

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