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Hydrodynamic limit of zero range processes among random conductances on the supercritical percolation cluster

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Abstract

We consider i.i.d. random variables $\{\omega(b) : b \in \mathbb{E}_d\}$ parameterized by the family of bonds in $\mathbb{Z}^d$, $d \geqslant 2$. The random variable $\omega(b)$ is thought of as the conductance of bond b and it ranges in a finite interval $[0, c_0]$. Assuming the probability m of the event $\{\omega(b) > 0\}$ to be supercritical and denoting by $\mathscr{C}(\omega)$ the unique infinite cluster associated to the bonds with positive conductance, we study the zero range process on $\mathscr{C}(\omega)$ with $\omega(b)$–proportional probability rate of jumps along bond b. For almost all realizations of the environment we prove that the hydrodynamic behavior of the zero range process is governed by the nonlinear heat equation $\partial_t \rho = m\nabla \cdot (\mathscr{D}\nabla\phi(\rho/m))$, where the matrix $\mathscr{D}$ and the function ϕ are ω–independent. As byproduct of the above result and the blocking effect of the finite clusters, we discuss the bulk behavior of the zero range process on $\mathbb{Z}^d$ with conductance field ω. We do not require any ellipticity condition.

Key words: disordered system, bond percolation, zero range process, hydrodynamic limit, homogenization, stochastic domination.

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1 Introduction

Percolation provides a simple and, at the same time, very rich model of disordered medium [G], [Ke]. The motion of a random walker on percolation clusters has been deeply investigated in Physics (see [BH] and reference therein) and also numerous rigorous results are now available. In the last years, for the supercritical percolation cluster it has been possible to prove the convergence of the diffusively rescaled random walk to the Brownian motion for almost all realizations of the percolation [SS], [BB], [MP], improving the annealed invariance principle obtained in [DFGW]. We address here our attention to interacting random walkers, moving on the supercritical Bernoulli bond percolation cluster with additional environmental disorder given by random conductances (for recent results on random walks among random conductances see [BP], [M], [F] and reference therein).

Particle interactions can be of different kind. An example is given by site exclusion, the hydrodynamic behavior of the resulting exclusion process has been studied in [F]. Another basic example, considered here, is the zero range interaction: particles lie on the sites of the infinite cluster without any constraint, while the probability rate of the jump of a particle from site x to a neighboring site y is given by $g(\eta(x))\omega(x,y)$, where g is a suitable function on $\mathbb{N}$, $\eta(x)$ is the number of particles at site x and $\omega(x,y)$ is the conductance of the bond $\{x,y\}$. We suppose that the concuctances are i.i.d. random variables taking value in $[0,c_0]$.

The above exclusion and zero range processes are non–gradient systems, since due to the disorder the algebraic local current cannot be written as spatial gradient of some local function. Nevertheless, thanks to the independence of the conductances from any bond orientation, one can study the hydrodynamic behavior avoiding the heavy machinery of non–gradient particle systems [V], [KL][Chapter VII]. Indeed, in the case of exclusion processes, due to the above symmetry of the conductance field the infinitesimal variation of the occupancy number $\eta(x)$ is a linear combination of occupancy numbers. This degree conservation strongly simplifies the analysis of the limiting behavior of the random empirical measure with respect to genuinely non–gradient disordered models as in [Q1], [FM], [Q2], and can be reduced to an homogenization problem [F]. In the case of zero–range processes, this degree conservation is broken. Nevertheless, due to the symmetry of the conductance field, adapting the method of the corrected empirical measure [GJ1] to the present contest one can reduce the proof of the hydrodynamic limit to an homogenization problem plus the proof of the Replacement Lemma. The resulting diffusive hydrodynamic equation does not depend on the environment and keeps memory on the particle interaction.

The homogenization problem has been solved in [F] also for more general random conductance fields. The core of the problem here is the proof of the Replacement Lemma. This technical lemma compares the particle density on microscopic boxes with the particle density on macroscopic boxes and it is a key tool in order to go from the microscopic scale to the macroscopic one. This comparison is usually made by moving particles along macroscopic paths by microscopic steps and then summing the local variations at each step. The resulting method corresponds to the so called Moving Particle Lemma and becomes efficient if the chosen macroscopic paths allow a spread–out particle flux, without any concentration in some special bond. While for a.a. ω any two points x,y in a box Λ_N of side N centered at the origin can be connected inside the infinite cluster by a path $\gamma_{x,y}$ of length at most $O(N)$ [AP], it is very hard (maybe impossible) to exhibit such a family of paths $\{\gamma_{x,y}\}_{x,y\in\Lambda_N}$ with a reasonable upper bound of the number of paths going through a given bond b, uniformly in b. Due to this obstacle, we will prove the Moving Particle Lemma not in its standard form, but in a weaker form, allowing anyway to complete the proof of the Replacement Lemma. We point out that in this step we use some technical results of [AP], where the chemical distance inside the supercritical Bernoulli bond percolation cluster is studied. It is only here that we need the hypothesis of i.i.d. conductances. Extending part of the results of [AP], one would get the hydrodynamic limit of zero range processes among random conductances on infinite clusters of more general conductance fields as in [F].

We comment another technical problem we had to handle with. The discussion in [GJ1] refers to the zero range process on a finite toroidal grid with conductances bounded above and below by some positive constants, and some steps cannot work here due to the presence of infinite particles. A particular care has to be devoted to the control of phenomena of particle concentration and slightly stronger homogenization results are required.

Finally, in the Appendix we discuss the bulk behavior of the zero range process on $\mathbb{Z}^d$ with i.i.d. random conductances in $[0, c_0]$, in the case of initial distributions with slowly varying parameter. Due to the blocking effect of the clusters with finite size, the bulk behavior is not described by a nonlinear heat equation.

We recall that the problem of density fluctuations for the zero range process on the supercritical Bernoulli bond percolation cluster with constant conductances has been studied in [GJ2]. Recently, the hydrodynamic limit of other interacting particle systems on $\mathbb{Z}^d$, or fractal spaces, with random conductances has been proved (cf. [F1], [FJL], [JL], [LF], [Val]). We point out the pioneering paper [Fr], where J. Fritz has proved the hydrodynamic behaviour of a one-dimensional Ginzburg-Landau model with conservation law in the presence of random conductances.

2 Models and results

2.1 The environment

The environment ω modeling the disordered medium is given by i.i.d. random variables $\big(\omega(b) : b \in \mathbb{E}_d\big)$, parameterized by the set $\mathbb{E}_d$ of non–oriented bonds in $\mathbb{Z}^d$, $d \geqslant 2$. ω and $\omega(b)$ are thought of as the conductance field and the conductance at bond b, respectively. We call $\mathbb{Q}$ the law of the field ω and we assume that $\omega(b) \in [0, c_0]$ for $\mathbb{Q}$–a.a. ω, for some fixed positive constant c_0. Hence, without loss of generality, we can suppose that $\mathbb{Q}$ is a probability measure on the product space $\Omega := [0, c_0]^{\mathbb{E}_d}$. Moreover, in order to simplify the notation, we write $\omega(x, y)$ for the conductance $\omega(b)$ if $b = \{x, y\}$. Note that $\omega(x, y) = \omega(y, x)$.

Consider the random graph $G(\omega) = \big(V(\omega), E(\omega)\big)$ with vertex set $V(\omega)$ and bond set $E(\omega)$ defined as

$$
\begin{aligned}
E(\omega) &:= \big\{ b \in \mathbb{E}_d \,:\, \omega(b) > 0 \big\}, \\
V(\omega) &:= \big\{ x \in \mathbb{Z}^d \,:\, x \in b \text{ for some } b \in E(\omega) \big\}.
\end{aligned}
$$

Assuming the probability $\mathbb{Q}(\omega(b) > 0)$ to be supercritical, the translation invariant Borel subset $\Omega_0 \subset \Omega$ given by the configurations ω for which the graph $G(\omega)$ has a unique infinite connected component (cluster) $\mathscr{C}(\omega) \subset V(\omega)$ has $\mathbb{Q}$–probability 1 [G]. Below, we denote by $\mathscr{E}(\omega)$ the bonds in $E(\omega)$ connecting points of $\mathscr{C}(\omega)$ and we will often understand the fact that $\omega \in \Omega_0$.

For later use, given $c > 0$ we define the random field $\hat{\omega}_c = \big(\hat{\omega}_c(b) \,:\, b \in \mathbb{E}_d\big)$ as

$$\hat{\omega}_c(b) = \begin{cases} 1 & \text{if } \omega(b) > c\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.1}$$

For $c = 0$ we simply set $\hat{\omega} := \hat{\omega}_0$.

2.2 The zero range process on the infinite cluster $\mathscr{C}(\omega)$

We fix a nondecreasing function $g : \mathbb{N} \to [0, \infty)$ such that $g(0) = 0$, $g(k) > 0$ for all $k > 0$ and

$$g^* := \sup_{k \in \mathbb{N}} |g(k+1) - g(k)| < \infty\,. \tag{2.2}$$

Given a realization ω of the environment, we consider the zero range process η_t on the graph $\mathscr{G}(\omega) = \big(\mathscr{C}(\omega), \mathscr{E}(\omega)\big)$ where a particle jumps from x to y with rate $g(\eta(x))\omega(x,y)$. This is the Markov process with paths $\eta(t)$ in the Skohorod space $D\big([0,\infty), \mathbb{N}^{\mathscr{C}(\omega)}\big)$ whose Markov generator $\mathscr{L}$ acts on local functions as

$$\mathscr{L} f(\eta) = \sum_{e \in \mathscr{B}} \sum_{\substack{x \in \mathscr{C}(\omega): \\ x+e \in \mathscr{C}(\omega)}} g(\eta(x))\omega(x, x+e) \left(f(\eta^{x,x+e}) - f(\eta) \right)\,, \tag{2.3}$$

where $\mathscr{B} = \{\pm e_1, \pm e_2, \dots, \pm e_d\}$, $e_1, \dots, e_d$ being the canonical basis of $\mathbb{Z}^d$, and where in general

$$\eta^{x,y}(z) = \begin{cases} \eta(x) - 1, & \text{if } z = x\,, \\ \eta(y) + 1, & \text{if } z = y\,, \\ \eta(z), & \text{if } z \neq x, y\,. \end{cases}$$

We recall that a function f is called local if $f(\eta)$ depends only on $\eta(x)$ for a finite number of sites x. Since $\mathscr{C}(\omega)$ is infinite, the above process is well defined only for suitable initial distribution. As discussed in [A], the process is well defined when the initial distribution has support on configurations η such that $\|\eta\| := \sum_{x \in \mathscr{C}(\omega)} \eta(x) a(x) < \infty$, $a(\cdot)$ being a strictly positive real valued function on $\mathscr{C}(\omega)$ such that

$$\sum_{x \in \mathscr{C}(\omega)} \sum_{e \in \mathscr{B}} \omega(x, x+e) a(x+e) \leqslant M a(x)$$

for some positive constant M.

Given $\varphi \geqslant 0$, set $Z(\varphi) := \sum_{k \geqslant 0} \varphi^k / g(k)!$ where $g(0)! = 1$, $g(k)! = g(1)g(2)\cdots g(k)$ for $k \geqslant 1$. Since $Z(\varphi)$ is an increasing function and $g(k)! \geqslant g(1)^k$, there exists a critical value $\varphi_c \in (0, \infty]$ such that $Z(\varphi) < \infty$ if $\varphi < \varphi_c$ and $Z(\varphi) = \infty$ if $\varphi > \varphi_c$. Then, for $0 \leqslant \varphi < \varphi_c$ we define $\bar{\nu}_\varphi$ as the product probability measure on $\mathbb{N}^{\mathscr{C}(\omega)}$ such that

$$\bar{\nu}_\varphi(\eta(x) = k) = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g(k)!}\,, \qquad k \in \mathbb{N}\,, \qquad x \in \mathscr{C}(\omega)\,.$$

Taking for example $\alpha(x) = e^{-|x|}$ in the definition of $\|\eta\|$ one obtains that $\bar{\nu}_\varphi(\|\eta\|) < \infty$, thus implying that the zero range process is well defined whenever the initial distribution is given by $\bar{\nu}_\varphi$

or by a probability measure μ stochastically dominated by $\bar{\nu}_\varphi$. In this last case, as proven in [A], by the monotonicity of g one obtains that the zero range process η_t starting from μ is stochastically dominated by the zero range process ζ_t starting from $\bar{\nu}_\varphi$, i.e. one can construct on an enlarged probability space both processes η_t and ζ_t s.t. $\eta_t(x) \leqslant \zeta_t(x)$ almost surely. Finally, we recall that all measures $\bar{\nu}_\varphi$ are reversible for the zero range process and that $\bar{\nu}_\varphi(e^{\theta\eta(x)}) < \infty$ for some $\theta > 0$, thus implying that $\bar{\nu}_\varphi(\eta(x)^k) < \infty$ for all $k \geqslant 0$ (cf. Section 2.3 of [KL]).

As usually done, we assume that $\lim_{\varphi\uparrow\varphi_c} Z(\varphi) = \infty$. Then, cf. Section 2.3 in [KL], the function $R(\varphi) := \bar{\nu}_\varphi(\eta(0))$ is strictly increasing and gives a bijection from $[0, \varphi_c)$ to $[0, \infty)$. Given $\rho \in [0, \infty)$ we will write $\varphi(\rho)$ for the unique value such that $R(\varphi) = \rho$. Then we set

$$\nu_\rho := \bar{\nu}_{\varphi(\rho)}\,, \qquad \phi(\rho) := \nu_\rho(g(\eta_x)) \qquad x \in \mathscr{C}(\omega)\,. \tag{2.4}$$

As proven in Section 2.3 of [KL], ϕ is Lipschitz with constant g^*.

2.3 The hydrodynamic limit

Given an integer $N \geqslant 1$ and a probability measure μ^N on $\mathbb{N}^{\mathscr{C}(\omega)}$, we denote by $\mathbb{P}_{\omega,\mu^N}$ the law of the zero range process with generator $N^2\mathscr{L}$ (see (2.3)) and with initial distribution μ^N (assuming this dynamics to be admissible). We denote by $\mathbb{E}_{\omega,\mu^N}$ the associated expectation. In order to state the hydrodynamic limit, we define $B(\Omega)$ as the family of bounded Borel functions on Ω and let $\mathscr{D}$ be the $d \times d$ symmetric matrix characterized by the variational formula

$$(a, \mathscr{D}a) = \frac{1}{m} \inf_{\psi\in B(\Omega)} \left\{ \sum_{e\in\mathscr{B}_*} \int_\Omega \omega(0,e)(a_e + \psi(\tau_e\omega) - \psi(\omega))^2 \mathbb{I}_{0,e\in\mathscr{C}(\omega)} \mathbb{Q}(d\omega) \right\}\,, \quad \forall a \in \mathbb{R}^d\,, \tag{2.5}$$

where $\mathscr{B}_*$ denotes the canonical basis of $\mathbb{Z}^d$,

$$m := \mathbb{Q}(0 \in \mathscr{C}(\omega)) \tag{2.6}$$

and the translated environment $\tau_e\omega$ is defined as $\tau_e\omega(x,y) = \omega(x+e, y+e)$ for all bonds $\{x,y\}$ in $\mathbb{E}_d$. In general, $\mathbb{I}_A$ denotes the characteristic function of A.

The above matrix $\mathscr{D}$ is the diffusion matrix of the random walk among random conductances on the supercritical percolation cluster and it equals the identity matrix multiplied by a positive constant (see the discussion in [F] and references therein).

Theorem 2.1. *For $\mathbb{Q}$–almost all environments ω the following holds. Let $\rho_0 : \mathbb{R}^d \to [0,\infty)$ be a bounded Borel function and let $\{\mu_N\}_{N\geqslant 1}$ be a sequence of probability measures on $\mathbb{N}^{\mathscr{C}(\omega)}$ such that for all $\delta > 0$ and all continuous functions G on $\mathbb{R}^d$ with compact support (shortly $G \in C_c(\mathbb{R}^d)$), it holds*

$$\lim_{N\uparrow\infty} \mu^N\left(\left| N^{-d} \sum_{x\in\mathscr{C}(\omega)} G(x/N)\,\eta(x) - \int_{\mathbb{R}^d} G(x)\rho_0(x)dx \right| > \delta \right) = 0\,. \tag{2.7}$$

Moreover, suppose that there exist $\rho_0, \rho_*, C_0 > 0$ such that μ^N is stochastically dominated by ν_{ρ_0} and the entropy $H(\mu^N|\nu_{\rho_*})$ is bounded by C_0N^d.

Then, for all $t > 0$, $G \in C_c(\mathbb{R}^d)$ and $\delta > 0$, it holds

$$\lim_{N\uparrow\infty} \mathbb{P}_{\omega,\mu^N}\Big(\Big|N^{-d}\sum_{x\in\mathscr{C}(\omega)} G(x/N)\,\eta_t(x) - \int_{\mathbb{R}^d} G(x)\rho(x,t)dx\Big| > \delta\Big) = 0\,, \tag{2.8}$$

where $\rho : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ is assumed to be the unique weak solution of the heat equation

$$\partial_t \rho = m\nabla \cdot (\mathscr{D}\nabla\phi(\rho/m)) \tag{2.9}$$

with boundary condition ρ_0 at $t = 0$.

We define the empirical measure $\pi^N(\eta)$ associated to the particle configuration η as

$$\pi^N(\eta) := \frac{1}{N^d}\sum_{x\in\mathscr{C}(\omega)} \eta(x)\delta_{x/N} \in \mathscr{M}(\mathbb{R}^d)\,,$$

where $\mathscr{M}(\mathbb{R}^d)$ denotes the Polish space of non–negative Radon measures on $\mathbb{R}^d$ endowed with the vague topology (namely, $\nu_n \to \nu$ in $\mathscr{M}$ if and only if $\nu_n(f) \to \nu(f)$ for each $f \in C_c(\mathbb{R}^d)$). We refer to the Appendix of [S] for a detailed discussion about the space $\mathscr{M}$ endowed of the vague topology. We write π^N_t for the empirical measure $\pi^N(\eta_t)$, η_t being the zero range process with generator $N^2\mathscr{L}$. Then condition (2.7) simply means that under μ^N the random measure π^N converges in probability to $\rho_0(x)dx$, while under $\mathbb{P}_{\omega,\mu^N}$ the random measure π^N_t converges in probability to $\rho(x,t)dx$, for each fixed $t \geqslant 0$. In order to prove the conclusion of Theorem 2.1 one only needs to show that the law of the random path $\pi^N_\cdot \in D([0,T],\mathscr{M})$ weakly converges to the delta distribution concentrated on the path $[0,T] \ni t \to \rho(x,t)dx \in \mathscr{M}$ (see [KL][Chapter 5]). It is this stronger result that we prove here.

Let us give some comments on our assumptions. We have restricted to increasing functions g in order to assure attractiveness and therefore that the dynamics is well defined whenever the initial distributions are stochastically dominated by some invariant measure ν_{ρ_0}. This simplifies also some technical estimates. One could remove the monotone assumption on g and choose other conditions assuring a well defined dynamics and some basic technical estimates involved in the proof, which would be similar to the ones appearing in [KL][Chapter 5].

The entropy bound $H(\mu^N|\nu_{\rho_*}) \leqslant C_0 N^d$ is rather restrictive. Indeed, given a locally Riemann integrable bounded profile $\rho_0 : \mathbb{R}^d \to [0,\infty)$, let μ^N be the product measure on $\mathbb{N}^{\mathscr{C}(\omega)}$ with slowly varying parameter associated to the profile ρ_0/m at scale N. Namely, μ^N is the product measure on $\mathbb{N}^{\mathscr{C}(\omega)}$ such that

$$\mu^N(\eta(x) = k) = \nu_{\rho_0(x/N)/m}(\eta(x) = k)\,.$$

Due to the ergodicity of $\mathbb{Q}$ condition (2.7) is fulfilled and, setting $\rho' := \sup_x \rho_0(x)$, μ^N is stochastically dominated by $\nu_{\rho'/m}$. On the other hand, the entropy $H(\mu^N|\nu_{\rho_*})$ is given by

$$\frac{1}{N^d}H(\mu^N|\nu_{\rho_*}) = \frac{1}{N^d}\sum_{x\in\mathscr{C}(\omega)} \frac{\rho_0(x/N)}{m}\Big[\log\varphi\big(\frac{\rho_0(x/N)}{m}\big) - \log\varphi\left(\rho_*\right)\Big]\Big\} + \\ \frac{1}{N^d}\sum_{x\in\mathscr{C}(\omega)}\Big\{\log Z\left(\varphi\left(\rho_*\right)\right) - \log Z\big(\varphi\big(\frac{\rho_0(x/N)}{m}\big)\big) \tag{2.10}$$

Hence, $H(\mu^N|\mu_{\rho_*}) \leqslant C_0 N^d$ only if ρ_0 approaches sufficiently fast the constant $m\rho_*$ at infinity. All these technical problems are due to the infinite space. In order to weaken the entropic assumption one should proceed as in [LM]. Since here we want to concentrate on the Moving Particle Lemma, which is the real new problem, we keep our assumptions.

Finally, we have assumed uniqueness of the solution of differential equation (2.9) with initial condition ρ_0. Results on uniqueness can be found in [BC], [KL][Chapter 5] and [Va]. Proceeding as in [KL][Section 5.7] and using the ideas developed below, one can prove that the limit points of the sequence $\{\pi_t^N\}_{t\in[0,T]}$ are concentrated on paths $t \to \rho(x,t)dx$ satisfying an energy estimate.

3 Tightness of $\{\pi_t^N\}_{t\in[0,T]}$

As already mentioned, in order to reduce the proof of Theorem 2.1 to the Replacement Lemma one has to adapt the method of the corrected empirical measure developed in [GJ1] and after that invoke some homogenization properties proved in [F]. The discussion in [GJ1] refers to the zero range process on a finite toroidal grid and has to be modified in order to solve technical problems due to the presence of infinite particles.

Given $N \in \mathbb{N}_+$, we define $\mathbb{L}_N$ as the generator of the random walk among random conductances ω on the supercritical percolation cluster, after diffusive rescaling. More precisely, we define $\mathscr{C}_N(\omega) = \{x/N \,:\, x \in \mathscr{C}(\omega)\}$ and set

$$\mathbb{L}_N f(x/N) = N^2 \sum_{\substack{e\in\mathscr{B}:\\ x+e\in\mathscr{C}(\omega)}} \omega(x,x+e)\Big\{f((x+e)/N) - f(x/N)\Big\} \tag{3.1}$$

for all $x \in \mathscr{C}(\omega)$ and $f : \mathscr{C}_N(\omega) \to \mathbb{R}$. We denote by ν_ω^N the uniform measure on $\mathscr{C}_N(\omega)$ given by $\nu_\omega^N = N^{-d}\sum_{x\in\mathscr{C}_N(\omega)} \delta_x$. Below we will think of the operator $\mathbb{L}_N$ as acting on $L^2(\nu_\omega^N)$. We write $(\cdot,\cdot)_{\nu_\omega^N}$ and $\|\cdot\|_{L^2(\nu_\omega^N)}$ for the scalar product and the norm in $L^2(\nu_\omega^N)$, respectively. Note that $\mathbb{L}_N$ is a symmetric operator, such that $(f, -\mathbb{L}_N f)_{\nu_\omega^N} > 0$ for each nonzero function $f \in L^2(\nu_\omega^N)$. In particular, $\lambda\mathbb{I} - \mathbb{L}_N$ is invertible for each $\lambda > 0$. Moreover, it holds

$$(f, -\mathbb{L}_N g)_{\nu_\omega^N} = \frac{1}{2}\sum_{x\in\mathscr{C}(\omega)} \sum_{\substack{e\in\mathscr{B}:\\ x+e\in\mathscr{C}(\omega)}} \omega(x,x+e)\big[f(x+e)-f(x)\big]\cdot\big[g(x+e)-g(x)\big]$$

for all functions $f, g \in L^2(\nu_\omega^N)$. Given $\lambda > 0$, $G \in C_c^\infty(\mathbb{R}^d)$ and $N \in \mathbb{N}_+$ we define G_N^λ as the unique element of $L^2(\nu_\omega^N)$ such that

$$\lambda G_N^\lambda - \mathbb{L}_N G_N^\lambda = G^\lambda, \tag{3.2}$$

where G^λ is defined as the restriction to $\mathscr{C}_N(\omega)$ of the function $\lambda G - \nabla\cdot\mathscr{D}\nabla G \in C_c^\infty(\mathbb{R}^d)$.

Let us collect some useful facts on the function G_N^λ:

Lemma 3.1. *Fix $\lambda > 0$. Then, for each $G \in C_c^\infty(\mathbb{R}^d)$ it holds*

$$(G_N^\lambda, -\mathbb{L}_N G_N^\lambda)_{\nu_\omega^N} \leqslant c(G,\lambda), \tag{3.3}$$

$$\|G_N^\lambda\|_{L^1(\nu_\omega^N)},\ \|G_N^\lambda\|_{L^2(\nu_\omega^N)} \leqslant c(\lambda, G), \tag{3.4}$$

$$\|\mathbb{L}_N G_N^\lambda\|_{L^1(\nu_\omega^N)},\ \|\mathbb{L}_N G_N^\lambda\|_{L^2(\nu_\omega^N)} \leqslant c(\lambda, G), \tag{3.5}$$

for a suitable positive constant $c(\lambda, G)$ depending on λ and G, but not from N. Moreover, for $\mathbb{Q}$–a.s. conductance fields ω it holds

$$\lim_{N\uparrow\infty} \|G_N^\lambda - G\|_{L^2(\nu_\omega^N)} = 0\,, \qquad \forall G \in C_c^\infty(\mathbb{R}^d)\,, \tag{3.6}$$

$$\lim_{N\uparrow\infty} \|G_N^\lambda - G\|_{L^1(\nu_\omega^N)} = 0\,, \qquad \forall G \in C_c^\infty(\mathbb{R}^d)\,. \tag{3.7}$$

Proof. By taking the scalar product with G_N^λ in (3.2) one obtains that

$$\lambda \|G_N^\lambda\|^2_{L^2(\nu_\omega^N)} + (G_N^\lambda, -\mathbb{L}_N G_N^\lambda)_{\nu_\omega^N} = (G_N^\lambda, G^\lambda)_{\nu_\omega^N} \leqslant \|G_N^\lambda\|_{L^2(\nu_\omega^N)} \|G^\lambda\|_{L^2(\nu_\omega^N)}\,.$$

Using that $\sup_N \|G^\lambda\|_{L^2(\nu_\omega^N)} < \infty$, from the above expression one easily obtains the uniform upper bounds on $(G_N^\lambda, -\mathbb{L}_N G_N^\lambda)_{\nu_\omega^N}$ and $\|G_N^\lambda\|_{L^2(\nu_\omega^N)}$. Since $\sup_{N\geqslant 1} \|G^\lambda\|_{L^2(\nu_\omega^N)} < \infty$, by difference one obtains the uniform upper bound on $\|\mathbb{L}_N G_N^\lambda\|_{L^2(\nu_\omega^N)}$.

In order to estimate $\|G_N^\lambda\|_{L^1(\nu_\omega^N)}$ let us write $p_t^N(x,y)$ for the probability that the random walk on $\mathscr{C}_N(\omega)$ with generator $\mathbb{L}_N$ and starting point x is at site y at time t. Then, since the jump rates depend on the unoriented bonds, $p_t^N(x,y) = p_t^N(y,x)$. Since

$$G_N^\lambda(x) = \sum_{y\in\mathscr{C}_N(\omega)} \int_0^\infty e^{-\lambda t} p_t^N(x,y) G^\lambda(y)\,, \tag{3.8}$$

for all $x \in \mathscr{C}_N(\omega)$, the above symmetry allows to conclude that

$$\|G_N^\lambda\|_{L^1(\nu_\omega^N)} \leqslant \frac{1}{N^d} \sum_{x,y\in\mathscr{C}_N(\omega)} \int_0^\infty e^{-\lambda t} p_t^N(x,y) |G^\lambda(x)| =$$

$$\frac{1}{\lambda N^d} \sum_{x\in\mathscr{C}_N(\omega)} |G^\lambda(x)| \to \frac{1}{\lambda}\int_{\mathbb{R}^d} |G^\lambda(u)| du < \infty\,. \tag{3.9}$$

Again, since $\sup_{N\geqslant 1} \|G^\lambda\|_{L^1(\nu_\omega^N)} < \infty$, by difference one obtains the uniform upper bound on $\|\mathbb{L}_N G_N^\lambda\|_{L^1(\nu_\omega^N)}$.

The homogenization result (3.6) follows from Theorem 2.4 (iii) in [F]. Finally, let us consider (3.7). Given $\ell > 0$, using Schwarz inequality, one can bound

$$\|G_N^\lambda - G\|_{L^1(\nu_\omega^N)} \leqslant$$

$$\|G_N^\lambda(u)\mathbb{I}(|u| > \ell)\|_{L^1(\nu_\omega^N)} + \|G(u)\mathbb{I}(|u| > \ell)\|_{L^1(\nu_\omega^N)} + c\ell^{d/2} \|G_N^\lambda - G\|^{1/2}_{L^2(\nu_\omega^N)}\,.$$

Since $G \in C_c^\infty(\mathbb{R}^d)$ the second term in the r.h.s. is zero for ℓ large enough. The last term in the r.h.s. goes to zero due to (3.6). In order to conclude we need to show that

$$\overline{\lim_{\ell\uparrow\infty}}\, \overline{\lim_{N\uparrow\infty}} \|G_N^\lambda(u)\mathbb{I}(|u| > \ell)\|_{L^1(\nu_\omega^N)} = 0\,. \tag{3.10}$$

Since $G^\lambda \in C_c^\infty(\mathbb{R}^d)$ we can fix nonnegative functions $F, f \in C_c^\infty(\mathbb{R}^d)$ such that $-f \leqslant G^\lambda \leqslant F$. We call F_N^λ, f_N^λ the solutions in $L^2(\nu_\omega^N)$ of the equations

$$\lambda F_N^\lambda - \mathbb{L}_N F_N^\lambda = F\,,$$

$$\lambda f_N^\lambda - \mathbb{L}_N f_N^\lambda = f\,,$$

respectively. From the integral representation (3.8) we derive that F_N^λ, f_N^λ are nonnegative, and that $-f_N^\lambda \leqslant G_N^\lambda \leqslant F_N^\lambda$ on $\mathscr{C}_N(\omega)$. In particular, in order to prove (3.10) it is enough to prove the same claim with F_N^λ, f_N^λ instead of G_N^λ. We give the proof for F_N^λ, the other case is completely similar. Let us define $H \in C^\infty(\mathbb{R}^d)$ as the unique solution in $L^2(dx)$ of the equation

$$\lambda H - \nabla \cdot \mathscr{D} \nabla H = F\,. \tag{3.11}$$

Again, by a suitable integral representation, we get that H is nonnegative. Applying Schwarz inequality, we can estimate

$$\begin{aligned}\|F_N^\lambda(u)\mathbb{I}(|u|>\ell)\|_{L^1(\nu_\omega^N)} = \|F_N^\lambda\|_{L^1(\nu_\omega^N)} - \|F_N^\lambda(u)\mathbb{I}(|u|\leqslant \ell)\|_{L^1(\nu_\omega^N)} \leqslant \\ \|F_N^\lambda\|_{L^1(\nu_\omega^N)} - \|H(u)\mathbb{I}(|u|\leqslant \ell)\|_{L^1(\nu_\omega^N)} + \|(H(u)-F_N^\lambda)\mathbb{I}(|u|\leqslant \ell)\|_{L^1(\nu_\omega^N)} \leqslant \\ \|F_N^\lambda\|_{L^1(\nu_\omega^N)} - \|H(u)\mathbb{I}(|u|\leqslant \ell)\|_{L^1(\nu_\omega^N)} + c\ell^{d/2}\|F_N^\lambda - H\|_{L^2(\nu_\omega^N)}^{1/2}\,.\end{aligned} \tag{3.12}$$

Since F_N^λ and F are nonnegative functions, when repeating the steps in (3.9) with F_N^λ, F instead of G_N^λ, G^λ respectively, we get the the inequality is an equality and therefore

$$\|F_N^\lambda\|_{L^1(\nu_\omega^N)} \to \lambda^{-1}\|F\|_{L^1(dx)} = \|H\|_{L^1(dx)}\,.$$

This observation, the above bound (3.12) and Theorem 2.4 (iii) in [F] imply that

$$\varlimsup_{N\uparrow\infty} \|F_N^\lambda(u)\mathbb{I}(|u|>\ell)\|_{L^1(\nu_\omega^N)} \leqslant \|H(u)\mathbb{I}(|u|>\ell)\|_{L^1(\nu_\omega^N)}$$

At this point it is trivial to derive (3.10) for F_N^λ. □

In the rest of this section, we will assume that ω is a good conductance field, i.e. the infinite cluster $\mathscr{C}(\omega)$ is well–defined and ω satisfies Lemma 3.1. We recall that these properties hold $\mathbb{Q}$–a.s.

The first step in proving the hydrodynamic limit consists in showing that the sequence of processes $\{\pi_t^N\}_{t\in[0,T]}$ is tight in the Skohorod space $D([0,T],\mathscr{M})$. By adapting the proof of Proposition IV.1.7 in [KL] to the vague convergence, one obtains that it is enough to show that the sequence of processes $\{\pi_t^N[G]\}_{t\in[0,T]}$ is tight in the Skohorod space $D([0,T],\mathbb{R})$ for all $G\in C_c^\infty(\mathbb{R})$. A key relation between the zero range process and the random walk among random conductances is given by

$$N^2\mathscr{L}\left(\pi^N(\eta)[G]\right) = \frac{1}{N^d}\sum_{x\in\mathscr{C}(\omega)} g(\eta(x))(\mathbb{L}_N G)(x/N)\,. \tag{3.13}$$

The check of (3.13) is trivial and based on integration by parts. At this point, due to the disorder given by the conductance field ω, a second integration by parts as usually done for gradient systems (cf. [KL][Chapter 5]) would be useless since the resulting object would remain wild. A way to overcome this technical problem is given by the method of the corrected empirical measure: as explained below, the sequence of processes $\{\pi_t^N[G]\}_{t\in[0,T]}$ behaves asymptotically as $\{\pi_t^N[G_N^\lambda]\}_{t\in[0,T]}$, thus the tightness of the former follows from the tightness of the latter. We need some care since the total number of particles can be infinite, hence it is not trivial that the process $\{\pi_t^N[G_N^\lambda]\}_{t\in[0,T]}$ is well defined.

We start with a technical lemma which will be frequently used:

Lemma 3.2. *Let H be a nonnegative function on $\mathscr{C}_N(\omega)$ belonging to $L^1(\nu_\omega^N)\cap L^2(\nu_\omega^N)$ and let $k\geqslant 0$. Then*

$$\mathbb{P}_{\omega,\mu^N}\Big(\sup_{0\leqslant t\leqslant T}\frac{1}{N^d}\sum_{x\in\mathscr{C}(\omega)}\eta_t(x)^k H(x/N)>A\Big)\leqslant A^{-1}c(k,\rho_0)\sqrt{\|H\|^2_{L^1(\nu_\omega^N)}+\|H\|^2_{L^2(\nu_\omega^N)}} \tag{3.14}$$

for all $A>0$.

Proof. We use a maximal inequality for reversible Markov processes due to Kipnis and Varadhan [KV] (cf. Theorem 11.1 in Appendix 1 of [KL]). Let us set

$$F(\eta)=\frac{1}{N^d}\sum_{x\in\mathscr{C}(\omega)}\eta(x)^k H(x/N), \tag{3.15}$$

supposing first that H has bounded support. Note that $F(\eta)\leqslant F(\eta')$ if $\eta(x)\leqslant\eta'(x)$ for all $x\in\mathscr{C}(\omega)$. Hence by the stochastic domination assumption, it is enough to prove (3.14) with $\mathbb{P}_{\omega,\nu_{\rho_0}}$ (always referred to the diffusively accelerated process) instead of $\mathbb{P}_{\omega,\mu^N}$. We recall that ν_{ρ_0} is reversible w.r.t. the the zero range process. Moreover

$$\nu_{\rho_0}(F^2)=\frac{1}{N^{2d}}\sum_{x,y\in\mathscr{C}(\omega)}H(x/N)H(y/N)\nu_{\rho_0}(\eta(x)^k\eta(y)^k)\leqslant c(k,\rho_0)\|H\|^2_{L^1(\nu_\omega^N)}, \tag{3.16}$$

while

$$\nu_{\rho_0}(F,-N^2\mathscr{L}F)=\frac{N^2}{N^{2d}}\sum_{x,y\in\mathscr{C}(\omega)}H(x/N)H(y/N)\nu_{\rho_0}(\eta(x)^k,-\mathscr{L}\eta(y)^k)\leqslant$$
$$c(k,\rho_0)\frac{N^2}{N^{2d}}\sum_{x\in\mathscr{C}(\omega)}\sum_{\substack{y\in\mathscr{C}(\omega):\\|x-y|=1}}H(x/N)H(y/N).$$

Using the bound $H(x/N)H(y/N)\leqslant H(x/N)^2+H(y/N)^2$ and the fact that $d\geqslant 2$, we conclude that

$$\nu_{\rho_0}(F,-N^2\mathscr{L}F)\leqslant c(k,\rho_0)\|H\|^2_{L^2(\nu_\omega^N)}. \tag{3.17}$$

By the result of Kipnis and Varadhan it holds

$$\mathbb{P}_{\omega,\nu_{\rho_0}}\Big(\sup_{0\leqslant t\leqslant T}\frac{1}{N^d}\sum_{x\in\mathscr{C}(\omega)}\eta_t(x)^k H(x/N)>A\Big)\leqslant\frac{e}{A}\sqrt{\nu_{\rho_0}(F^2)+T\nu_{\rho_0}(F,-N^2\mathscr{L}F)}. \tag{3.18}$$

At this point the thesis follows from the above bounds (3.16) and (3.17). In order to remove the assumption that H is local, it is enough to apply the result to the sequence of functions $H_n(x):=H(x)\chi(|x|\leqslant n)$ and then apply the Monotone Convergence Theorem as $n\uparrow\infty$. □

Remark 1. *We observe that the arguments used in the proof of Lemma 4.3 in [CLO] imply that, given a function H of bounded support and defining F as in* (3.15)*, it holds*

$$\mathbb{E}_{\omega,\nu_{\rho_0}}\Big(\sup_{0\leqslant t\leqslant T}\big(F(\eta_t)-F(\eta_0)\big)^2\Big)\leqslant cT\nu_{\rho_0}(F,-N^2\mathscr{L}F).$$

In particular, it holds

$$\mathbb{E}_{\omega,\nu_{\rho_0}}\Big(\sup_{0\leqslant t\leqslant T} F(\eta_t)^2\Big) \leqslant c\nu_{\rho_0}(F^2) + cT\nu_{\rho_0}(F, -N^2\mathscr{L}F)\,.$$

Using the bounds (3.16) *and* (3.17)*, the domination assumption and the Monotone Convergence Theorem, under the same assumption of Lemma 3.2 one obtains*

$$\mathbb{E}_{\omega,\mu^N}\Big(\sup_{0\leqslant t\leqslant T}\Big[\frac{1}{N^d}\sum_{x\in\mathscr{C}(\omega)}\eta_t(x)^k H(x/N)\Big]^2\Big) \leqslant c(k,\rho_0)\Big[\|H\|^2_{L^1(\nu^N_\omega)} + \|H\|^2_{L^2(\nu^N_\omega)}\Big]\,. \tag{3.19}$$

Using afterwards the Markov inequality, one concludes that

$$\mathbb{P}_{\omega,\mu^N}\Big(\sup_{0\leqslant t\leqslant T}\frac{1}{N^d}\sum_{x\in\mathscr{C}(\omega)}\eta_t(x)^k H(x/N) > A\Big) \leqslant c(k,\rho_0)A^{-2}\Big[\|H\|^2_{L^1(\nu^N_\omega)} + \|H\|^2_{L^2(\nu^N_\omega)}\Big] \tag{3.20}$$

for all $A > 0$*. The use of* (3.14) *or* (3.20) *in the rest of the discussion is completely equivalent.*

Due to Lemma 3.2 and Lemma 3.1 the process $\{\pi^N_t[G^\lambda_N]\}_{t\in[0,T]}$ is well defined w.r.t $\mathbb{E}_{\omega,\mu^N}$. Let us explain why this process is a good approximation of the process $\{\pi^N_t[G]\}_{t\in[0,T]}$:

Lemma 3.3. *Let* $G\in C^\infty_c(\mathbb{R}^d)$*. Then, given* $\delta>0$*, it holds*

$$\lim_{N\uparrow\infty}\mathbb{P}_{\omega,\mu^N}\Big(\sup_{0\leqslant t\leqslant T}\Big|\pi^N_t[G^\lambda_N]-\pi^N_t[G]\Big| > \delta\Big) = 0\,. \tag{3.21}$$

Proof. By Lemma 3.2 we can bound the above probability by

$$c(\rho_0)\delta^{-1}\sqrt{\|G^\lambda_N - G\|^2_{L^1(\nu^N_\omega)} + \|G^\lambda_N - G\|^2_{L^2(\nu^N_\omega)}}\,.$$

At this point the thesis follows from Lemma 3.1. □

Due to the above Lemma, in order to prove the tightness of $\{\pi^N_t[G]\}_{t\in[0,T]}$ it is enough to prove the tightness of $\{\pi^N_t[G^\lambda_N]\}_{t\in[0,T]}$. Now we can go on with the standard method based on martingales and Aldous criterion for tightness (cf. [KL][Chapter 5]), but again we need to handle with care our objects due to the risk of explosion. We fix a good realization ω of the conductance field. Due to Lemma 3.1, Lemma 3.2 and the bound $g(k)\leqslant g^*k$, we conclude that the process $\{M^N_t\}_{0\leqslant t\leqslant T}$ where

$$M^N_t(G) := \pi^N_t(G^\lambda_N) - \pi^N_0(G^\lambda_N) - \int_0^t \frac{1}{N^d}\sum_{x\in\mathscr{C}(\omega)} g(\eta_s(x))\mathbb{L}_N G^\lambda_N(x/N)ds\,, \tag{3.22}$$

is well defined $\mathbb{P}_{\omega,\mu^N}$–a.s.

Lemma 3.4. *Given* $\delta>0$*,*

$$\lim_{N\uparrow\infty}\mathbb{P}_{\omega,\mu^N}\Big(\sup_{0\leqslant t\leqslant T}|M^N_t(G)|\geqslant\delta\Big) = 0\,. \tag{3.23}$$

Proof. Given $n \geqslant 1$, we define the cut–off function $G^\lambda_{N,n} : \mathscr{C}_N(\omega) \to \mathbb{R}$ as $G^\lambda_{N,n}(x) = G^\lambda_N(x)\mathbb{I}(|x| \leqslant n)$. Then $G^\lambda_{N,n}$ is a local function and by the results of [A] (together with the stochastic domination assumption) we know that

$$M_t^{N,n}(G) := \pi_t^N(G^\lambda_{N,n}) - \pi_0^N(G^\lambda_{N,n}) - \int_0^t \frac{1}{N^d} \sum_{x \in \mathscr{C}(\omega)} g(\eta_s(x)) \mathbb{L}_N G^\lambda_{N,n}(x/N) ds$$

is an L^2–martingale of quadratic variation

$$< M_t^{N,n}(G) >= \int_0^t \frac{N^2}{N^{2d}} \sum_{x \in \mathscr{C}(\omega)} \sum_{\substack{y \in \mathscr{C}(\omega): \\ |x-y|=1}} g(\eta_s(x)) \omega(x,y) \left[G^\lambda_{N,n}(y/N) - G^\lambda_{N,n}(x/N) \right]^2 ds\,.$$

Note that, by the stochastic domination assumption and the bound $g(k) \leqslant g^* k$,

$$\begin{aligned} &\mathbb{E}_{\omega,\mu^N}\left(< M_t^{N,n}(G) > \right) \leqslant \\ &\qquad g^* \int_0^t \frac{N^2}{N^{2d}} \sum_{x \in \mathscr{C}(\omega)} \sum_{\substack{y \in \mathscr{C}(\omega): \\ |x-y|=1}} \mathbb{E}_{\omega,\nu_{\rho_0}}\left[\eta_s(x) \right] \omega(x,y) \left[G^\lambda_{N,n}(y/N) - G^\lambda_{N,n}(x/N) \right]^2 ds \\ &\qquad\qquad = g^* \rho_0 t N^{-d} (G^\lambda_{N,n}, -\mathbb{L}_N G^\lambda_{N,n})_{\nu^N_\omega}\,. \end{aligned} \tag{3.24}$$

By Doob's inequality and (3.24), we conclude that

$$\begin{aligned} &\mathbb{P}_{\omega,\mu^N}\Big(\sup_{0 \leqslant t \leqslant T} |M_t^{N,n}(G)| \geqslant \delta \Big) \leqslant \frac{c}{\delta^2} \mathbb{E}_{\omega,\mu^N}\left(|M_T^{N,n}(G)|^2 \right) \leqslant \\ &\qquad \frac{c g^* T \rho_0}{\delta^2 N^d} (G^\lambda_{N,n}, -\mathbb{L}_N G^\lambda_{N,n})_{\nu^N_\omega} \leqslant \frac{c' g^* T \rho_0}{\delta^2 N^d}\,. \end{aligned} \tag{3.25}$$

Above we have used that $\lim_{n\uparrow\infty} (G^\lambda_{N,n}, -\mathbb{L}_N G^\lambda_{N,n})_{\nu^N_\omega} = (G^\lambda_N, -\mathbb{L}_N G^\lambda_N)_{\nu^N_\omega} \leqslant c(\lambda)$ (see Lemma 3.1).

The above process $\{M_t^{N,n}(G)\}_{t\in[0,T]}$ is a good approximation of $\{M_t^N(G)\}_{t\in[0,T]}$ as $n \uparrow \infty$. Indeed, it holds

$$\lim_{n\uparrow\infty} \mathbb{P}_{\omega,\mu^N}\Big(\sup_{0 \leqslant t \leqslant T} |M_t^{N,n}(G) - M_t^N(G)| > \delta \Big) = 0\,, \qquad \delta > 0\,. \tag{3.26}$$

Indeed, since $\|G^\lambda_{N,n} - G^\lambda_N\|_{L^1(\nu^N_\omega), L^2(\nu^N_\omega)}$ and $\|\mathbb{L}_N G^\lambda_{N,n} - \mathbb{L}_N G^\lambda_N\|_{L^1(\nu^N_\omega), L^2(\nu^N_\omega)}$ converge to zero as $N \uparrow \infty$ and since $g(k) \leqslant g^* k$, the above claim follows from Lemma 3.2.

At this point, (3.26) and (3.25) imply (3.23). □

Let us prove the tightness of $\{\pi_t^N[G^\lambda_N]\}_{t\in[0,T]}$ using Aldous criterion (cf. Proposition 1.2 and Proposition 1.6 in Section 4 of [KL]):

Lemma 3.5. *Given $G \in C_c^\infty(\mathbb{R}^d)$, the sequence of processes $\{\pi_t^N[G^\lambda_N]\}_{t\in[0,T]}$ is tight in $D([0,T],\mathbb{R})$.*

Proof. Fix $\theta > 0$ and uppose that τ is a stopping time w.r.t. the canonical filtration bounded by T. With some abuse of notation we write $\tau + \theta$ for the quantity $\min\{\tau + \theta, T\}$. Then, given $\varepsilon > 0$, by Lemmata 3.1 and 3.2,

$$\mathbb{P}_{\omega,\mu^N}\Big(\Big|\int_\tau^{\tau+\theta} \frac{1}{N^d} \sum_{x\in\mathscr{C}(\omega)} g(\eta_s(x))\mathbb{L}_N G_N^\lambda(x/N)ds\Big| > \varepsilon\Big) \leqslant \mathbb{P}_{\omega,\mu^N}\Big(\theta g^* \sup_{s\in[0,T]} \frac{1}{N^d} \sum_{x\in\mathscr{C}(\omega)} \eta_s(x)|\mathbb{L}_N G_N^\lambda(x/N)|ds > \varepsilon\Big) \leqslant C g^*\theta/\varepsilon\,. \quad (3.27)$$

In particular,

$$\lim_{\gamma\downarrow 0}\limsup_{N\uparrow\infty} \sup_{\tau,\theta\in[0,\gamma]} \mathbb{P}_{\omega,\mu^N}\Big(\Big|\int_\tau^{\tau+\theta} \frac{1}{N^d} \sum_{x\in\mathscr{C}(\omega)} g(\eta_s(x))\mathbb{L}_N G_N^\lambda(x/N)ds\Big| > \varepsilon\Big) = 0\,. \quad (3.28)$$

An estimate similar to (3.27) implies that

$$\lim_{\varepsilon\uparrow\infty} \sup_{N\geqslant 1} \mathbb{P}_{\omega,\mu^N}\Big(\Big|\int_0^t \frac{1}{N^d} \sum_{x\in\mathscr{C}(\omega)} g(\eta_s(x))\mathbb{L}_N G_N^\lambda(x/N)ds\Big| > \varepsilon\Big) = 0\,. \quad (3.29)$$

Let us now come back to Lemma 3.4. Let τ, θ as above. Then

$$\limsup_{N\uparrow\infty} \mathbb{P}_{\omega,\mu^N}\big(|M^N_{\tau+\theta} - M^N_\tau| > \varepsilon\big) \leqslant \limsup_{N\uparrow\infty} \mathbb{P}_{\omega,\mu^N}\big(\sup_{0\leqslant s\leqslant T} |M^N_s| > \varepsilon/2\big) = 0 \quad (3.30)$$

Collecting (3.28), (3.29) and (3.30), together with Lemma 3.4, we conclude that

$$\lim_{\gamma\downarrow 0}\limsup_{N\uparrow\infty} \sup_{\tau,\theta\in[0,\gamma]} \mathbb{P}_{\omega,\mu^N}\Big(\big|\pi^N_{\tau+\theta}[G_N^\lambda] - \pi^N_\tau[G_N^\lambda]\big| > \varepsilon\Big) = 0\,, \quad (3.31)$$

$$\lim_{\varepsilon\uparrow\infty} \sup_{N\geqslant 1} \mathbb{P}_{\omega,\mu^N}\big(|\pi^N_t[G_N^\lambda] > \varepsilon\big) = 0\,. \quad (3.32)$$

Aldous criterion for tightness allows to derive the thesis from (3.31) and (3.32).

□

Let us come back to (3.22) and investigate the integral term there. The following holds:

Lemma 3.6. *Let* $I(t) := \int_0^t N^{-d} \sum_{x\in\mathscr{C}(\omega)} g(\eta_s(x))\big(G_N^\lambda(x/N) - G(x/N)\big)ds$. *Then, for all* $\delta > 0$,

$$\lim_{N\uparrow\infty} \mathbb{P}_{\omega,\mu^N}\big(\sup_{0\leqslant t\leqslant T} |I(t)| > \delta\big) = 0\,. \quad (3.33)$$

Proof. Since $g(k) \leqslant g^* k$ and by Schwarz inequality we can bound

$$I(t) \leqslant J := T g^* \|G_N^\lambda - G\|^{1/2}_{L^1(\nu^N_\omega)} \sup_{0\leqslant s\leqslant T} \{N^{-d} \sum_{x\in\mathscr{C}(\omega)} \eta_s(x)^2 \big|G_N^\lambda(x/N) - G(x/N)\big|\}^{1/2}\,.$$

Using the stochastic domination assumption and applying Lemma 3.2 we obtain

$$\mathbb{P}_{\omega,\mu^N}\big(\sup_{0\leqslant t\leqslant T} |I(t)| > \delta\big) \leqslant \mathbb{P}_{\omega,\nu_{\rho_0}}(J > \delta) \leqslant (1/\delta)^2 T^2 (g^*)^2 \|G_N^\lambda - G\|_{L^1(\nu^N_\omega)} \sqrt{\|G_N^\lambda - G\|^2_{L^1(\nu^N_\omega)} + \|G_N^\lambda - G\|^2_{L^2(\nu^N_\omega)}}\,. \quad (3.34)$$

The thesis now follows by applying Lemma 3.1. □

We are finally arrived at the conclusion. Indeed, due to Lemma 3.3 and Lemma 3.5 we know that the sequence of processes $\{\pi_t^N\}_{t\in[0,T]}$ is tight in the Skohorod space $D([0,T],\mathscr{M})$. Moreover, starting from the identity (3.22), applying Lemma 3.4, using the identity (3.2) which equivalent to

$$\mathbb{L}_N G_N^\lambda = \lambda G_N^\lambda - G^\lambda = \lambda(G_N^\lambda - G) + \nabla\cdot\mathscr{D}\nabla G\,,$$

and finally invoking Lemma 3.6 we conclude that, fixed a good conductance field ω, for any $G\in C_c^\infty(\mathbb{R}^d)$ and for any $\delta>0$

$$\lim_{N\uparrow\infty}\mathbb{P}_{\omega,\mu^N}\Big(\sup_{0\leqslant t\leqslant T}\Big|\pi_t^N(G)-\pi_0^N(G)-\int_0^t\frac{1}{N^d}\sum_{x\in\mathscr{C}(\omega)}g(\eta_s(x))\nabla\cdot\mathscr{D}\nabla G(x/N)ds\Big|>\delta\Big)=0\,. \quad (3.35)$$

Using the stochastic domination assumption it is trivial to prove that any limit point of the sequence $\{\pi_t^N\}_{t\in[0,T]}$ is concentrate on trajectories $\{\pi_t\}_{t\in[0,T]}$ such that π_t is absolutely continuous w.r.t. to the Lebesgue measure. Moreover, in order to characterize the limit points as solution of the differential equation (2.9) one would need non only (3.35). Indeed, it is necessary to prove that, given ω good, for each function $G\in C_c^\infty\big([0,T]\times\mathbb{R}^d\big)$ it holds

$$\begin{aligned}\lim_{N\uparrow\infty}\mathbb{P}_{\omega,\mu^N}\Big(\sup_{0\leqslant t\leqslant T}\Big|&\pi_t^N(G_t)-\pi_0^N(G_0)-\\ &\int_0^t\frac{1}{N^d}\sum_{x\in\mathscr{C}(\omega)}g(\eta_s(x))\nabla\cdot\mathscr{D}\nabla G_s(x/N)ds-\int_0^t\pi_s^N(\partial_s G_s)ds\Big|>\delta\Big)=0\,, \quad (3.36)\end{aligned}$$

where $G_s(x):=G(s,x)$. One can easily recover (3.36) from the same estimates used to get (3.35) and suitable approximations of G which are piecewise linear in t as in the final part of Section 3 in [GJ1]. In order to avoid heavy notation will continue the investigation of (3.35) only.

4 The Replacement Lemma

As consequence of the discussion in the previous section, in order to prove the hydrodynamical limit stated in Theorem 2.1 we only need to control the term

$$\int_0^t\frac{1}{N^d}\sum_{x\in\mathscr{C}(\omega)}g(\eta_s(x))\nabla\cdot\mathscr{D}\nabla G(x/N)ds\,. \quad (4.1)$$

To this aim we first introduce some notation. Given a family of parameters $\alpha_1,\alpha_2,\dots,\alpha_n$, we will write

$$\limsup_{\alpha_1\to a_1,\alpha_2\to a_2,\dots,\alpha_n\to a_n}$$

instead of

$$\limsup_{\alpha_n\to a_n}\ \limsup_{\alpha_{n-1}\to a_{n-1}}\cdots\limsup_{\alpha_1\to a_1}\,.$$

Below, given $x\in\mathbb{Z}^d$ and $k\in\mathbb{N}$, we write $\Lambda_{x,k}$ for the box

$$\Lambda_{x,k}:=x+[-k,k]^d\cap\mathbb{Z}^d\,,$$

and we write $\eta^k(x)$ for the density

$$\eta^k(x) := \frac{1}{(2k+1)^d} \sum_{y \in \Lambda_{x,k} \cap \mathscr{C}(\omega)} \eta(y) .$$

If $x = 0$ we simply write Λ_k instead of $\Lambda_{0,k}$.

Then, we claim that for $\mathbb{Q}$–a.a. ω, given $G \in C_c(\mathbb{R}^d)$, $\delta > 0$ and a sequence μ^N of probability measures on $\mathbb{N}^{\mathscr{C}(\omega)}$ stochastically dominated by some ν_{ρ_0} and such that $H(\mu^N | \nu_{\rho_*}) \leqslant C_0 N^d$, it holds

$$\limsup_{N \uparrow \infty, \varepsilon \downarrow 0} P_{\omega,\mu^N} \Big(\Big| \int_0^t \frac{1}{N^d} \sum_{x \in \mathscr{C}(\omega)} g\left(\eta_s(x)\right) G(x/N) ds - \int_0^t \frac{m}{N^d} \sum_{x \in \mathbb{Z}^d} \phi\left(\eta_s^{\varepsilon N}(x)/m\right) G(x/N) ds \Big| > \delta \Big) = 0 . \quad (4.2)$$

Let us first assume the above claim and explain how to conclude, supposing for simplicity of notation that $\varepsilon N \in \mathbb{N}$.

Given $u \in \mathbb{R}^d$ and $\varepsilon > 0$, define $\iota_{u,\varepsilon} := (2\varepsilon)^{-d} \mathbb{I}\{u \in [-\varepsilon, \varepsilon]^d\}$. Then the integral $\pi^N\left[\iota_{x/N,\varepsilon}\right]$, $x \in \mathbb{Z}^d$, can be written as

$$\pi^N\left[\iota_{x/N,\varepsilon}\right] = \frac{(2\varepsilon N + 1)^d}{(2\varepsilon N)^d} \eta^{\varepsilon N}(x) . \quad (4.3)$$

Let us define

$$\int_0^t \frac{m}{N^d} \sum_{x \in \mathbb{Z}^d} \phi(\pi_s^N[\iota_{x/N,\varepsilon}/m]) \nabla \cdot \mathscr{D} \nabla G(x/N) ds . \quad (4.4)$$

Then, due to (4.3) and since ϕ is Lipschitz with constant g^*, we can estimate from above the difference between (4.4) and the second integral term in (4.2) with G substituted by $\nabla \cdot \mathscr{D} \nabla G$ as

$$\frac{(2\varepsilon N + 1)^d - (2\varepsilon N)^d}{(2\varepsilon N)^d} \int_0^t \frac{1}{N^d} \sum_x \eta_s^{\varepsilon N}(x) \left| \nabla \cdot \mathscr{D} \nabla G(x/N) \right| ds \quad (4.5)$$

Since the integral term in (4.5) has finite expectation w.r.t $\mathbb{P}_{\omega,\nu_{\rho_0}}$ and therefore also w.r.t. $\mathbb{P}_{\omega,\mu^N}$, we conclude that the above difference goes to zero in probability w.r.t. $\mathbb{P}_{\omega,\mu^N}$. At this point the conclusion of the proof of Theorem 2.1 can be obtained by the same arguments used in [KL][pages 78,79].

Let us come back to our claim. Since

$$\sum_{x \in \mathscr{C}(\omega)} g(\eta_s(x)) G(x/N) = \sum_{x \in \mathbb{Z}^d} g(\eta_s(x)) \mathbb{I}(x \in \mathscr{C}(\omega)) G(x/N) ,$$

by a standard integration by parts argument and using that $G \in C_c^\infty(\mathbb{R}^d)$ one can replace the first integral in (4.2) by

$$\int_0^t \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \Big[\frac{1}{(2\ell+1)^d} \sum_{y : y \in \Lambda_{x,\ell} \cap \mathscr{C}(\omega)} g\left(\eta_s(y)\right) \Big] G(x/N) .$$

Then the claim (4.2) follows from

Lemma 4.1. *(***Replacement Lemma***) For $\mathbb{Q}$–a.a. ω, given $\delta > 0$, $M \in \mathbb{N}$ and given a sequence of probability measures μ^N on $\mathbb{N}^{\mathscr{C}(\omega)}$ stochastically dominated by some ν_{ρ_0} and such that $H(\mu^N|\nu_{\rho_*}) \leqslant C_0 N^d$, it holds*

$$\limsup_{N\uparrow\infty,\varepsilon\downarrow 0} \mathbb{P}_{\omega,\mu^N}\Big[\int_0^t \frac{1}{N^d}\sum_{x\in\Lambda_{MN}} V_{\varepsilon N}(\tau_x\eta_s,\tau_x\omega)ds > \delta\Big] = 0\,, \tag{4.6}$$

where

$$V_\ell(\eta,\omega) = \Big|\frac{1}{(2\ell+1)^d}\sum_{y:y\in\Lambda_\ell\cap\mathscr{C}(\omega)} g\left(\eta(y)\right) - m\phi\left(\eta^\ell(0)/m\right)\Big|.$$

Let us define $\Upsilon_{C_0,N}$ as the set of measurable functions $f:\mathbb{N}^{\mathscr{C}(\omega)}\to[0,\infty)$ such that i) $\nu_{\rho_*}(f)=1$, (ii) $\mathscr{D}(f):=\nu_{\rho_*}(\sqrt{f},-\mathscr{L}\sqrt{f})\leqslant C_0N^{d-2}$ and (iii) $f\,d\nu_{\rho_*}$ is stochastically dominated by ν_{ρ_0} (shortly, $f\,d\nu_{\rho_*}\prec d\nu_{\rho_0}$). Using the assumption $H(\mu^N|\nu_{\rho_*})\leqslant C_0N^d$ and entropy production arguments as in [KL][Chapter 5], in order to prove the Replacement Lemma it is enough to show that for $\mathbb{Q}$–a.a. ω, given $\rho_0,\rho_*,C_0,M>0$, it holds

$$\limsup_{N\uparrow\infty,\varepsilon\downarrow 0}\ \sup_{f\in\Upsilon_{C_0,N}}\int\frac{1}{N^d}\sum_{x\in\Lambda_{MN}} V_{\varepsilon N}(\tau_x\eta,\tau_x\omega)f(\eta)\nu_{\rho_*}(d\eta)=0\,. \tag{4.7}$$

Trivially, since ν_{ρ_1} stochastically dominates ν_{ρ_2} if $\rho_1>\rho_2$, it is enough to prove that, given $\rho_0,\rho_*,C_0,M>0$, for $\mathbb{Q}$–a.a. ω (4.7) is verified. We claim that the above result follows from the the One Block and the Two Blocks estimates:

Lemma 4.2. *(***One block estimate***) Fix $\rho_0,\rho_*,C_0,M>0$. Then, for $\mathbb{Q}$–a.a. ω it holds*

$$\limsup_{N\uparrow\infty,\ell\uparrow\infty}\ \sup_{f\in\Upsilon_{C_0,N}}\int\frac{1}{N^d}\sum_{x\in\Lambda_{MN}} V_\ell(\tau_x\eta,\tau_x\omega)f(\eta)\nu_{\rho_*}(d\eta)=0\,. \tag{4.8}$$

Lemma 4.3. *(***Two blocks estimate)** *Fix $\rho_0,\rho_*,C_0,M>0$. Then, for $\mathbb{Q}$–a.a. ω it holds*

$$\limsup_{N\uparrow\infty,\varepsilon\downarrow 0,\ell\uparrow\infty}\ \sup_{f\in\Upsilon_{C_0,N}}$$

$$\int\frac{1}{N^d}\sum_{x\in\Lambda_{MN}}\Big[\frac{1}{(2\varepsilon N+1)^d}\sum_{y\in\Lambda_{x,\varepsilon N}}\left|\eta^\ell(y)-\eta^{\varepsilon N}(x)\right|\Big]f(\eta)\nu_{\rho_*}(d\eta)=0\,. \tag{4.9}$$

We point out that the form of the Two Blocks Estimate is slightly weaker from the one in [KL][Chapter 5], nevertheless it is strong enough to imply, together with the One Block Estimate, equation (4.7). Indeed, let us define $a(y):=\mathbb{I}(y\in\mathscr{C}(\omega))$ and

$$I_1(\eta):=\frac{1}{N^d}\sum_{x\in\Lambda_{MN}}\left|Av_{y\in\Lambda_{x,\varepsilon N}}g(\eta(y))a(y)-Av_{y\in\Lambda_{x,\varepsilon N}}Av_{z\in\Lambda_{y,\ell}}g(\eta(z))a(z)\right|\,, \tag{4.10}$$

$$I_2(\eta):=\frac{1}{N^d}\sum_{x\in\Lambda_{MN}}\left|Av_{y\in\Lambda_{x,\varepsilon N}}\left(Av_{z\in\Lambda_{y,\ell}}g(\eta(z))a(z)-m\phi(\eta^\ell(y)/m)\right)\right|\,, \tag{4.11}$$

$$I_3(\eta):=\frac{1}{N^d}\sum_{x\in\Lambda_{MN}}\left|Av_{y\in\Lambda_{x,\varepsilon N}}m\phi(\eta^\ell(y)/m)-m\phi(\eta^{\varepsilon N}(x)/m)\right)\,, \tag{4.12}$$

where Av denotes the standard average. Then

$$\frac{1}{N^d}\sum_{x\in\Lambda_{MN}} V_{\varepsilon N}(\tau_x\eta,\tau_x\omega) = \\ \frac{1}{N^d}\sum_{x\in\Lambda_{MN}}\Big|Av_{y\in\Lambda_{x,\varepsilon N}} g(\eta(y))a(y) - m\phi(\eta^{\varepsilon N}(x)/m)\Big| \leqslant (I_1+I_2+I_3)(\eta)\,. \quad (4.13)$$

Let us explain a simple bound that will be frequently used below, often without any mention. Consider a family of numbers $b(x)$, $x\in\mathbb{Z}^d$. Then, taking $L,\ell>0$ we can write

$$Av_{x\in\Lambda_L} b(x) - Av_{x\in\Lambda_L} Av_{u\in\Lambda_{x,\ell}} b(u) = \\ \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_{L+\ell}} b(x)\big(\mathbb{I}(x\in\Lambda_L) - \frac{1}{|\Lambda_\ell|}\sharp\{u\in\Lambda_L : |x-u|_\infty\leqslant\ell\}\big)\,.$$

In particular, it holds

$$\Big|Av_{x\in\Lambda_L} b(x) - Av_{x\in\Lambda_L} Av_{u\in\Lambda_{x,\ell}} b(u)\Big| \leqslant \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_{L+\ell}\setminus\Lambda_{L-\ell}} |b(x)|\,. \quad (4.14)$$

Due to the above bound we conclude that

$$I_1(\eta) \leqslant \frac{g^*}{N^d(2\varepsilon N+1)^d}\sum_{x\in\Lambda_{MN}}\sum_{u\in\Lambda_{x,\varepsilon N+\ell}\setminus\Lambda_{x,\varepsilon N-\ell}} \eta(u)a(u)\,.$$

In particular, using that $f\,d\nu_{\rho_*}\prec d\nu_{\rho_0}$, we conclude that $\int I_1(\eta)f(\eta)\nu_{\rho_*}(d\eta)\leqslant c\ell/(\varepsilon N)$. The second term $I_2(\eta)$ can be estimated for $\varepsilon\leqslant 1$ as

$$I_2(\eta) \leqslant \frac{1}{N^d}\sum_{x\in\Lambda_{MN}} Av_{y\in\Lambda_{x,\varepsilon N}} V_\ell(\tau_z\eta,\tau_z\omega) \leqslant \frac{1}{N^d}\sum_{x\in\Lambda_{(M+1)N}} V_\ell(\tau_x\eta,\tau_x\omega)\,.$$

Due to the One block estimate one gets that

$$\limsup_{N\uparrow\infty,\varepsilon\downarrow 0,\ell\uparrow\infty}\ \sup_{f\in\Upsilon_{C_0,N}} \int I_2(\eta)f(\eta)\nu_{\rho_*}(d\eta) = 0\,.$$

The same result holds also for $I_3(\eta)$ due to the Lipschitz property of ϕ and the Two Blocks estimate. The above observations together with (4.13) imply (4.7).

5 Proof of the Two Blocks Estimate

For simplicity of notation we set $\ell_* = (2\ell+1)^d$ and we take $M=1$ (the general case can be treated similarly). Moreover, given $\Delta\subset\mathbb{Z}^d$, we write $\mathcal{N}(\Delta)$ for the number of particles in the region Δ, namely $\mathcal{N}(\Delta) := \sum_{x\in\Delta\cap\mathscr{C}(\omega)}\eta(x)$.

Let us set

$$A(\eta) := \frac{1}{N^d} \sum_{x\in\Lambda_N} Av_{y\in\Lambda_{x,\varepsilon N}} \left|\eta^\ell(y) - \eta^{\varepsilon N}(x)\right|, \tag{5.1}$$

$$B(\eta) := \frac{1}{N^d} \sum_{x\in\Lambda_N} Av_{y\in\Lambda_{x,\varepsilon N}} Av_{z\in\Lambda_{x,\varepsilon N}} \left|\eta^\ell(y) - \eta^\ell(z)\right|, \tag{5.2}$$

$$C(\eta) := \frac{1}{N^d|\Lambda_{\varepsilon N}|} \sum_{x\in\Lambda_N} Av_{y\in\Lambda_{x,\varepsilon N}} \sum_{\substack{z\in\Lambda_{x,\varepsilon N}:\\ |z-y|_\infty \geqslant 2\ell}} \left|\eta^\ell(y) - \eta^\ell(z)\right|. \tag{5.3}$$

Since due to (4.14)

$$\eta^{\varepsilon N}(x) = Av_{z\in\Lambda_{x,\varepsilon N}} \eta^\ell(z) + \mathscr{E}, \qquad |\mathscr{E}| \leqslant c|\Lambda_{\varepsilon N}|^{-1} \sum_{u\in\Lambda_{x,\varepsilon N+\ell}\setminus\Lambda_{x,\varepsilon N-\ell}} \eta(u),$$

using that $f\,d\nu_{\rho_*} \prec d\nu_{\rho_0}$, in order to prove the Two Blocks estimate we only need to show that

$$\limsup_{N\uparrow\infty,\varepsilon\downarrow 0,\ell\uparrow\infty} \sup_{f\in\Upsilon_{C_0,N}} \int B(\eta) f(\eta) \nu_{\rho_*}(d\eta) = 0.$$

Since

$$B(\eta) \leqslant C(\eta) + \frac{1}{N^d|\Lambda_{\varepsilon N}|} \sum_{x\in\Lambda_N} Av_{y\in\Lambda_{x,\varepsilon N}} \sum_{\substack{z\in\Lambda_{x,\varepsilon N}:\\ |z-y|_\infty \leqslant 2\ell}} \left(\eta^\ell(y) + \eta^\ell(z)\right),$$

using again that $f\,d\nu_{\rho_*} \prec d\nu_{\rho_0}$, in order to prove the Two Blocks Estimate we only need to show that

$$\limsup_{N\uparrow\infty,\varepsilon\downarrow 0,\ell\uparrow\infty} \sup_{f\in\Upsilon_{C_0,N}} \int \frac{1}{N^d(\varepsilon N)^{2d}\ell^d} \sum_{x\in\Lambda_N} \sum_{y\in\Lambda_{x,\varepsilon N}} \sum_{\substack{z\in\Lambda_{x,\varepsilon N}:\\ |z-y|_\infty \geqslant 2\ell}} \left|\mathscr{N}(\Lambda_{y,\ell}) - \mathscr{N}(\Lambda_{z,\ell})\right| f(\eta) \nu_{\rho_*}(d\eta) = 0. \tag{5.4}$$

Let us now make an observation that will be frequently used below. Let $X \subset \mathbb{Z}^d$ be a subset possibly depending on ω and on some parameters (for simplicity, we consider a real–value parameter $L \in \mathbb{N}$). Suppose that for $\mathbb{Q}$–a.a. ω it holds

$$\limsup_{N\uparrow\infty,L\uparrow\infty} \frac{|X\cap\Lambda_N|}{N^d} = 0.$$

Then, in order to prove (5.4) we only need to show that

$$\limsup_{N\uparrow\infty,\varepsilon\downarrow 0,\ell\uparrow\infty,L\uparrow\infty} \sup_{f\in\Upsilon_{C_0,N}} \int \frac{1}{N^d(\varepsilon N)^{2d}\ell^d} \sum_{x\in\Lambda_N} \sum_{y\in\Lambda_{x,\varepsilon N}} \sum_{\substack{z\in\Lambda_{x,\varepsilon N}:\\ |z-y|_\infty \geqslant 2\ell}} \left|\mathscr{N}(\Lambda_{y,\ell}\setminus X) - \mathscr{N}(\Lambda_{z,\ell}\setminus X)\right| f(\eta) \nu_{\rho_*}(d\eta) = 0. \tag{5.5}$$

We know that there exists $\alpha_0 > 0$ such that for each $\alpha \in (0, \alpha_0]$ the random field $\hat{\omega}_\alpha$ defined in (2.1) is a supercritical Bernoulli bond percolation. Let us write $\mathscr{C}_\alpha(\omega)$ for the associated infinite cluster. By ergodicity,

$$\limsup_{N\uparrow\infty,\alpha\downarrow 0} |(\mathscr{C} \setminus \mathscr{C}_\alpha) \cap \Lambda_N|/|\Lambda_N| = \lim_{\alpha\downarrow 0} \mathbb{Q}(0 \in \mathscr{C} \setminus \mathscr{C}_\alpha) = 0\,. \tag{5.6}$$

Hence, due to the above considerations, (5.4) is proven if we show that, for each $\alpha \in (0, \alpha_0]$, it holds (5.4) with $\mathscr{C}$ replaced by $\mathscr{C}_\alpha$. Moreover, since $f\,d\nu_{\rho_*} \prec \nu_{\rho_0}$, using Chebyshev inequality it is simple to prove that

$$\limsup_{N\uparrow\infty,\varepsilon\downarrow 0,\ell\uparrow\infty,A\uparrow\infty} \int \frac{1}{N^d(\varepsilon N)^{2d}} \sum_{x\in\Lambda_N} \sum_{y\in\Lambda_{x,\varepsilon N}} \sum_{\substack{z\in\Lambda_{x,\varepsilon N}:\\ |z-y|_\infty \geqslant 2\ell}} \mathbb{I}(\mathscr{N}(\Lambda_{y,\ell} \cup \Lambda_{z,\ell}) > A\ell_*^d) f(\eta)\nu_{\rho_*}(d\eta) = 0\,.$$

At this point, we only need to prove the following: Fixed $\alpha \in (0, \alpha_0]$ and $A > 0$, for $\mathbb{Q}$–a.a. ω it holds

$$\limsup_{N\uparrow\infty,\varepsilon\downarrow 0,\ell\uparrow\infty} \sup_{f\in\Upsilon^*_{C_0,N}} \int \frac{1}{N^d(\varepsilon N)^{2d}\ell^d} \sum_{x\in\Lambda_N} \sum_{y\in\Lambda_{x,\varepsilon N}} \sum_{\substack{z\in\Lambda_{x,\varepsilon N}:\\ |z-y|_\infty \geqslant 2\ell}} \Big|\mathscr{N}(\mathscr{C}_\alpha \cap \Lambda_{y,\ell}) - \mathscr{N}(\mathscr{C}_\alpha \cap \Lambda_{z,\ell})\Big| \mathbb{I}(\mathscr{N}(\Lambda_{y,\ell} \cup \Lambda_{z,\ell}) \leqslant A\ell_*^d) f(\eta)\nu_{\rho_*}(d\eta) = 0\,, \tag{5.7}$$

where $\Upsilon^*_{C_0,N}$ is the family of measurable functions $f : \mathbb{N}^{\mathscr{C}(\omega)} \to [0, \infty)$ such that $\nu_{\rho_*}(f) = 1$ and $\mathscr{D}(f) \leqslant C_0 N^{d-2}$.

We now use the results of [AP] about the chemical distance in the supercritical Bernoulli bond percolation $\hat{\omega}_\alpha$, for some fixed $\alpha \in (0, \alpha_0]$. We fix a positive integer K (this corresponds to the parameter N in [AP], which is fixed large enough once for all). Given $\mathbf{a} \in \mathbb{Z}^d$ and $s > 0$, we set $\Delta_{\mathbf{a},s} := \Lambda_{(2K+1)\mathbf{a},s}$. As in [AP], we call $\hat{\omega}_\alpha$ the microscopic random field. The macroscopic one $\sigma = \{\sigma(\mathbf{a}) : \mathbf{a} \in \mathbb{Z}^d\} \in \{0,1\}^{\mathbb{Z}^d}$ is defined in [AP] stating that $\sigma(\mathbf{a}) = 1$ if and only if the microscopic field $\hat{\omega}_\alpha$ satisfies certain geometric properties inside the box $\Delta_{\mathbf{a},5K/4}$. These properties are described on page 1038 in [AP], but their content is not relevant here, hence we do not recall them. What is relevant for us is that there exists a function $\bar{p} : \mathbb{N} \to [0, 1)$ with $\lim_{K\uparrow\infty} \bar{p}(K) = 1$, such that σ stochastically dominates a Bernoulli site percolation of parameter $\bar{p}(K)$ (see Proposition 2.1 in [AP]). Below we denote by $\mathbb{P}_{\bar{p}(K)}$ the law of this last random field, taking K large enough such that $\bar{p}(K)$ is supercritical. As in [AP] we call a point $\mathbf{a} \in \mathbb{Z}^d$ white or black if $\sigma(\mathbf{a}) = 1$ or 0 respectively, and we write in boldface the sites referred to the macroscopic field. Recall that a subset of $\mathbb{Z}^d$ is $*$–connected if it is connected with respect to the adjacency relation

$$x \overset{*}{\sim} y \Leftrightarrow |x - y|_\infty = 1\,.$$

$\mathscr{C}^*$ is defined as the set of all $*$–connected macroscopic black clusters. Given $\mathbf{a} \in \mathbb{Z}^d$, $\mathbf{C}^*_{\mathbf{a}}$ denotes the element of $\mathscr{C}^*$ containing $\mathbf{a}$ (with the convention that $\mathbf{C}^*_{\mathbf{a}} = \emptyset$ if $\mathbf{a}$ is white), while $\bar{\mathbf{C}}^*_{\mathbf{a}}$ is defined as $\bar{\mathbf{C}}^*_{\mathbf{a}} = \mathbf{C}^*_{\mathbf{a}} \cup \partial^{out}\mathbf{C}^*_{\mathbf{a}}$. We recall that, given a finite subset $\Lambda \subset \mathbb{Z}^d$, its outer boundary is defined as

$$\partial^{out}\Lambda := \big\{x \in \Lambda^c : \exists y \in \Lambda,\ \{x, y\} \in \mathbb{E}_d\big\}\,.$$

We use the convention that $\partial^{out}\mathbf{C}^*_{\mathbf{a}} = \{\mathbf{a}\}$ for a white site $\mathbf{a} \in \mathbb{Z}^d$. Hence, for $\mathbf{a}$ white it holds $\bar{\mathbf{C}}^*_{\mathbf{a}} = \{\mathbf{a}\}$.

Let us recall the first part of Proposition 3.1 in [AP]. To this aim, given $x, y \in \mathbb{Z}^d$, we write $\mathbf{a}(x)$ and $\mathbf{a}(y)$ for the unique sites in $\mathbb{Z}^d$ such that $x \in \Delta_{\mathbf{a},K}$ and $y \in \Delta_{\mathbf{a},K}$. We set $n := |\mathbf{a}(x) - \mathbf{a}(y)|_1$ and choose a macroscopic path $\mathbf{A}_{x,y} = (\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n)$ with $\mathbf{a}_0 = \mathbf{a}(x)$ and $\mathbf{a}_n = \mathbf{a}(y)$ (in particular, we require that $|\mathbf{a}_i - \mathbf{a}_{i+1}|_\infty = 1$). We build the path $\mathbf{A}_{x,y}$ in the following way: we start in $\mathbf{a}(x)$, then we move by unitary steps along the line $\mathbf{a}(x) + \mathbb{Z}e_1$ until reaching the point $\mathbf{a}'$ having the same first coordinate as $\mathbf{a}(y)$, then we move by unitary steps along the line $\mathbf{a}' + \mathbb{Z}e_2$ until reaching the point having the same first two coordinates as $\mathbf{a}(y)$ and so on. Then, Proposition 3.1 in [AP] implies (for K large enough, as we assume) that given any points $x, y \in \mathscr{C}_\alpha$ there exists a path $\gamma_{x,y}$ joining x to y inside $\mathscr{C}_\alpha$ such that $\gamma_{x,y}$ is contained in

$$W_{x,y} := \cup_{\mathbf{a}\in\mathbf{A}_{x,y}} \left(\cup_{\mathbf{w}\in\bar{C}^*_{\mathbf{a}}} \Delta_{\mathbf{w},5K/4} \right) . \tag{5.8}$$

These are the main results of [AP] that we will use below. Note that, since the sets $\bar{\mathbf{C}}^*_{\mathbf{a}}$ can be arbitrarily large, the information that $\gamma_{x,y} \subset W_{x,y}$ is not strong enough to allow to repeat the usual arguments in order to prove the Moving Particle Lemma, and therefore the Two Blocks Estimate. Hence, one needs some new ideas, that now we present.

First, we isolate a set of bad points as follows. We fix a parameter $L > 0$ and we define the subsets $\mathbf{B}(L), B(L) \subset \mathbb{Z}^d$ as

$$\mathbf{B}(L) := \{\mathbf{a} \in \mathbb{Z}^d \,:\, |\mathbf{C}^*_{\mathbf{a}}| > L\}\,, \tag{5.9}$$

$$B(L) := \cup_{\mathbf{a}\in\mathbf{B}(L)} \Delta_{\mathbf{a},10K}\,. \tag{5.10}$$

Lemma 5.1. *Given α in $(0, \alpha_0]$, for $\mathbb{Q}$–a.a. ω it holds*

$$\limsup_{N\uparrow\infty, L\uparrow\infty} \frac{|B(L) \cap \Lambda_N|}{|\Lambda_N|} = 0\,. \tag{5.11}$$

Proof. Since $|B(L) \cap \Lambda_N| \leqslant c(K)|\mathbf{B}(L) \cap \Lambda_N|$, we only need to prove the thesis with $B(L)$ replaced by $\mathbf{B}(L)$. We introduce the nondecreasing function $\rho_L : \mathbb{N} \to [0, \infty)$ defined as $\rho_L(n) := \mathbb{I}(n > L)n$. Then we can bound

$$|\mathbf{B}(L) \cap \Lambda_N| \leqslant \sum_{\substack{\mathbf{C}^*\in\mathscr{C}^*:\\ \mathbf{C}^*\cap\Lambda_N\neq\emptyset}} \rho_L(\mathbf{C}^*)\,.$$

Since σ stochastically dominates the Bernoulli site percolation with law $\mathbb{P}_{\bar{p}(K)}$ and due to Lemma 2.3 in [DP], we conclude that

$$\mathbb{Q}(|\mathbf{B}(L) \cap \Lambda_N| \geqslant a|\Lambda_N|) \leqslant \mathbb{Q}\Big(\sum_{\substack{\mathbf{C}^*\in\mathscr{C}^*:\\ \mathbf{C}\cap\Lambda_N\neq\emptyset}} \rho_L(\mathbf{C}^*) \geqslant a|\Lambda_N| \Big) \leqslant$$

$$\mathbb{P}_{\bar{p}(K)}\Big(\sum_{\substack{\mathbf{C}^*\in\mathscr{C}^*:\\ \mathbf{C}\cap\Lambda_N\neq\emptyset}} \rho_L(\mathbf{C}^*) \geqslant a|\Lambda_N| \Big) \leqslant P\Big(\sum_{\mathbf{a}\in\Lambda_N} \rho_L(\tilde{\mathbf{C}}^*_{\mathbf{a}}) \geqslant a|\Lambda_N| \Big)\,, \tag{5.12}$$

where the random variables $\tilde{\mathbf{C}}^*_{\mathbf{a}}$ (called *pre–clusters*) are i.i.d. and have the same law of $\mathbf{C}^*_{\mathbf{0}}$ under $\mathbb{P}_{\bar{p}(K)}$. Their construction is due to Fontes and Newman [FN1], [FN2]. Due to formula (4.47) of [AP], $E_{\mathbb{P}_{\bar{p}(K)}}(|\mathbf{C}^*_{\mathbf{0}}|)$ is finite for K large, in particular

$$\lim_{L\uparrow\infty} E(\rho_L(\tilde{\mathbf{C}}^*_{\mathbf{0}})) = 0\,. \tag{5.13}$$

By applying Cramér's theorem, we deduce that

$$P\Big(\sum_{\mathbf{a}\in\Lambda_N}\rho_L(\tilde{\mathbf{C}}^*_{\mathbf{a}}) \geqslant 2E(\rho_L(\tilde{\mathbf{C}}^*_{\mathbf{0}}))|\Lambda_N|\Big) \leqslant e^{-c(L)N^d}\,,$$

for some positive constant $c(L)$ and for all $N \geqslant 1$. Hence, due to (5.12) and Borel–Cantelli lemma, we can conclude that for $\mathbb{Q}$–a.a. ω it holds

$$|\mathbf{B}(L)\cap\Lambda_N|/|\Lambda_N| \leqslant 2E(\rho_L(\tilde{\mathbf{C}}^*_{\mathbf{0}}))\,, \qquad \forall N \geqslant N_0(L,\omega)\,.$$

At this point, the thesis follows from (5.13). $\square$

At this point, due to the arguments leading to (5.5), we only need to prove the following: given $\alpha\in(0,\alpha_0]$ and $A>0$, for $\mathbb{Q}$–a.a. ω it holds

$$\limsup_{N\uparrow\infty,\varepsilon\downarrow 0,\ell\uparrow\infty,L\uparrow\infty}\ \sup_{f\in\Upsilon^*_{C_0,N}} \int \frac{1}{N^d(\varepsilon N)^{2d}\ell^d}\sum_{x\in\Lambda_N}\sum_{y\in\Lambda_{x,\varepsilon N}}\sum_{\substack{z\in\Lambda_{x,\varepsilon N}:\\ |z-y|_\infty\geqslant 2\ell}}$$
$$\Big|\mathcal{N}(\Gamma_{y,\ell,\alpha})-\mathcal{N}(\Gamma_{z,\ell,\alpha})\Big|\mathbb{I}(\mathcal{N}(\Gamma_{y,\ell,\alpha}\cup\Gamma_{z,\ell,\alpha})\leqslant A\ell_*^d)f(\eta)\nu_{\rho_*}(d\eta)=0 \tag{5.14}$$

where

$$\Gamma_{u,\ell,\alpha} = \big(\Lambda_{u,\ell}\cap\mathscr{C}_\alpha\big)\setminus B(L)\,, \qquad u\in\mathbb{Z}^d\,. \tag{5.15}$$

Above we have used also that

$$\mathbb{I}(\mathcal{N}(\Lambda_{y,\ell}\cup\Lambda_{z,\ell})\leqslant A\ell_*^d)\leqslant\mathbb{I}(\mathcal{N}(\Gamma_{y,\ell,\alpha}\cup\Gamma_{z,\ell,\alpha})\leqslant A\ell_*^d)\,.$$

Note that in the integral of (5.14), the function f multiplies an $\mathscr{F}_N$–measurable function, where $\mathscr{F}_N$ is the σ–algebra generated by the random variables $\{\eta(x) : x\in G_N\}$ and G_N is the set of *good points* define as

$$G_N := (\Lambda_{N+1}\cap\mathscr{C}_\alpha)\setminus B(L)\,. \tag{5.16}$$

Since $\mathscr{D}(\cdot)$ is a convex functional (see Corollary 10.3 in Appendix 1 of [KL]), it must be

$$\mathscr{D}(\nu_{\rho_*}(f|\mathscr{F}_N))\leqslant\mathscr{D}(f)\leqslant C_0N^{d-2}\,.$$

Hence, by taking the conditional expectation w.r.t. $\mathscr{F}_N$ in (5.14), we conclude that we only need to prove (5.14) by substituting $\Upsilon^*_{C_0,N}$ with $\Upsilon^\sharp_{C_0,N}$ defined as the family of $\mathscr{F}_N$–measurable functions $f:\mathbb{N}^{\mathscr{C}(\omega)}\to[0,\infty)$ such that $\nu(f)=1$ and $\mathscr{D}(f)\leqslant C_0N^{d-2}$.

Recall the definition of the function $\varphi(\cdot)$ given before (2.4). By the change of variable $\eta\to\eta-\delta_x$ one easily proves the identity

$$\nu_{\rho_*}\Big[g(\eta_x)\big(\sqrt{f}(\eta^{x,y})-\sqrt{f}(\eta)\big)^2\Big]=\varphi(\rho_*)\nu_{\rho_*}\Big[\big(\sqrt{f}(\eta^{x,+})-\sqrt{f}(\eta^{y,+})\big)^2\Big]\,, \tag{5.17}$$

where in general $\eta^{z,+}$ denotes the configuration obtained from η by adding a particle at site z, i.e. $\eta^{z,+} = \eta + \delta_z$. Let us write $\nabla_{x,y}$ for the operator

$$\nabla_{x,y} h(\eta) := h(\eta^{x,+}) - h(\eta^{y,+}) .$$

We can finally state our weak version of the Moving Particle Lemma:

Lemma 5.2. *For $\mathbb{Q}$–a.a. ω the following holds. Fixed $\alpha \in (0, \alpha_0]$ and $L > 0$, there exists a positive constant $\kappa = \kappa(L, \alpha)$ such that*

$$\frac{\varepsilon^{-2}\varphi(\rho_*)}{N^d(\varepsilon N)^{2d}\ell_*^{2d}} \sum_{x\in\Lambda_N} \sum_{y\in\Lambda_{x,\varepsilon N}} \sum_{\substack{z\in\Lambda_{x,\varepsilon N}:\\ |z-y|_\infty \geqslant 2\ell}} \sum_{u\in\Gamma_{y,\ell,\alpha}} \sum_{v\in\Gamma_{z,\ell,\alpha}} \nu_{\rho_*}\big((\nabla_{u,v}\sqrt{f})^2\big)$$
$$\leqslant N^{2-d}\mathscr{D}(f)/\kappa \leqslant C_0/\kappa\,, \quad (5.18)$$

for any function $f \in \Upsilon^{\sharp}_{C_0,N}$ and for any N, ℓ, C_0.

Proof. Recall the definition of the path $\gamma_{x,y}$ given for $x, y \in \mathscr{C}_\alpha$ in the discussion before (5.8). Given a bond b non intersecting G_N, since f is $\mathscr{F}_N$–measurable it holds $\nabla_b\sqrt{f} = 0$. Using this simple observation, by a standard telescoping argument together with Schwarz inequality, we obtain that

$$\nu_{\rho_*}\big((\nabla_{u,v}\sqrt{f})^2\big) \leqslant \Big\{\sum_{i=0}^{n-1}\mathbb{I}(\{u_i,u_{i+1}\}\cap G_N \neq \emptyset)\Big\}\cdot\Big\{\sum_{i=0}^{n-1}\nu_{\rho_*}\big((\nabla_{u_i,u_{i+1}}\sqrt{f})^2\big)\Big\}\,, \quad (5.19)$$

where the path $\gamma_{u,v}$ is written as $(u = u_0, u_1, \ldots, u_n = v)$. Recall that if $\nu_{\rho_*}\big((\nabla_{u_i,u_{i+1}}\sqrt{f})^2\big) \neq 0$ then the set $\{u_i, u_{i+1}\}$ must intersect the set of good points G_N defined in (5.16).

If b is a bond of $\gamma_{u,v}$, then b must be contained in the set $W_{u,v}$ defined in (5.8). In particular, there exists $\mathbf{a} \in \mathbf{A}_{u,v}$ and $\mathbf{w} \in \bar{\mathbf{C}}^*_{\mathbf{a}}$ such that b is contained in $\Delta_{\mathbf{w},5K/4}$. Denoting $d_\infty(\cdot,\cdot)$ the distance between subsets of $\mathbb{Z}^d$ induced by the uniform norm $|\cdot|_\infty$, we can write

$$d_\infty(b, (2K+1)\mathbf{w}) \leqslant 5K/4\,. \quad (5.20)$$

We claim that, if b intersects G_N, then $|\mathbf{C}^*_{\mathbf{a}}| \leqslant L$. If $\mathbf{a}$ is white then $|\mathbf{C}^*_{\mathbf{a}}| = 1$ and the claim is trivally true. Let us suppose that $\mathbf{a}$ is black and that $|\mathbf{C}^*_{\mathbf{a}}| > L$. By definition of $\bar{\mathbf{C}}^*_{\mathbf{a}}$, there exists some point $\mathbf{a}' \in \mathbf{C}^*_{\mathbf{a}}$ such that $|\mathbf{w} - \mathbf{a}'|_\infty \leqslant 1$, i.e.

$$d_\infty\big((2K+1)\mathbf{w}, (2K+1)\mathbf{a}'\big) \leqslant 2K+1\,. \quad (5.21)$$

Due to (5.20) and (5.21) we conclude that $b \subset \Delta_{\mathbf{a}',10K} = \Lambda_{(2K+1)\mathbf{a}',10K}$. On the other hand, $\mathbf{C}^*_{\mathbf{a}'} = \mathbf{C}^*_{\mathbf{a}}$ and by definition of the set of bad points we get that $\Delta_{\mathbf{a}',10K} \subset B(L)$. The above observations imply that $b \subset B(L)$ in contradiction with the fact that b intersects the set of good points G_N. This concludes the proof of our claim: $|\mathbf{C}^*_{\mathbf{a}}| \leqslant L$.

We define $|\gamma_{u,v}|_* := \sum_{i=0}^{n-1}\mathbb{I}(\{u_i,u_{i+1}\}\cap G_N \neq \emptyset)$. We claim that for almost all conductance field ω there exists a constant $c(K,L)$ depending only on K and L such that, for all (x,y,z,u,v) as in (5.18), $|\gamma_{u,v}|_* \leqslant c\varepsilon N$. Indeed, by the above claim, we get that

$$|\gamma_{u,v}|_* \leqslant |\Lambda_{5K/4}| \sum_{\mathbf{a}\in\mathbf{A}_{u,v}} |\bar{C}^*_{\mathbf{a}}|\mathbb{I}(|\mathbf{C}^*_{\mathbf{a}}| \leqslant L) \leqslant c(d)L|\Lambda_{5K/4}|\,|\mathbf{A}_{u,v}| \leqslant C(K,L)\varepsilon N\,.$$

Given a bond $b \in \mathbb{E}_d$ let us estimate the cardinality of the set $\mathbb{X}(b)$, given by the strings (x, y, z, u, v) with x, y, z, u, v as in the l.h.s. of (5.18), such that b is a bond of the path $\gamma_{u,v}$ and b intersects G_N. Up to now we know that there exist $\mathbf{w}, \mathbf{a}$ such that b intersects $\Delta_{\mathbf{w},5K/4}$, $\mathbf{a} \in \mathbf{A}_{u,v}$, $\mathbf{w} \in \bar{\mathbf{C}}^*_{\mathbf{a}}$ and $|\mathbf{C}^*_{\mathbf{a}}| \leqslant L$. In particular, it must be

$$d_\infty(b, (2K+1)\mathbf{A}_{u,v}) \leqslant \\ d_\infty(b, (2K+1)\mathbf{w}) + (2K+1)d_\infty(\mathbf{w}, \mathbf{C}_{\mathbf{a}}) + (2K+1)\mathrm{diam}(\mathbf{C}^*_{\mathbf{a}}) \leqslant \\ 5K/4 + (2K+1)(1+L)\,. \quad (5.22)$$

Hence, if $(x, y, z, u, v) \in \mathbb{X}(b)$, then the distance between b and $(2K+1)\mathbf{A}_{u,v}$ is bounded by some constant depending only on K and L. Note that the macroscopic path $\mathbf{A}_{u,v}$ has length bounded by $c\varepsilon N/K$. Let us consider now the set $\mathbb{Y}(b)$ of macroscopic path $(\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_k)$ such that

$$k \leqslant c\varepsilon N/K\,, \quad (5.23)$$

$$d_\infty\left(b, (2K+1)\{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_k\}\right) \leqslant 5K/4 + (2K+1)(1+L)\,. \quad (5.24)$$

By the same computations used in the proof of the standard Moving Particle Lemma, one easily obtains that $|\mathbb{Y}(b)| \leqslant c(K, L)(\varepsilon N)^{d+1}$. Fixed a macroscopic path in $\mathbb{Y}(b)$ there are at most $c(K)$ ways to choose microscopic points u, v such that the macroscopic path equals $\mathbf{A}_{u,v}$. Fixed also u and v, we have at most ℓ_*^{2d} ways to choose (y, z) as in (5.18). Fixed (y, z) we have at most $c(\varepsilon N)^d$ ways to choose x as in (5.18). Hence $|\mathbb{X}(b)| \leqslant c(K, L)(\varepsilon N)^{2d+1}\ell_*^{2d}$. Finally, recall that the length of $\gamma_{u,v}$ is bounded by $c\varepsilon N$. Since moreover all paths $\gamma_{u,v}$ are in $\mathscr{C}_\alpha$, from the above observations and from (5.17) we derive that

$$\text{l.h.s. } of\,(5.18) \leqslant \frac{(c\varepsilon N)\varepsilon^{-2}\varphi(\rho_*)}{N^d(\varepsilon N)^{2d}\ell_*^{2d}} \sum_{b \in \mathbb{E}_d} |\mathbb{X}(b)| \nu_{\rho_*}((\nabla_b \sqrt{f})^2) \leqslant \\ \frac{c(K, L)(\varepsilon N)^{2d+2}\varepsilon^{-2}\ell_*^{2d}}{N^d(\varepsilon N)^{2d}\ell_*^{2d}\alpha} \mathscr{D}(f) \leqslant C(K, L, \alpha)N^{2-d}\mathscr{D}(f) \leqslant C(K, L, \alpha)C_0\,. \quad (5.25)$$

□

Since α is fixed, we will stress only the dependence of the constant $\kappa(L, \alpha)$ in Lemma 5.2 on L writing simply $\kappa(L)$. We introduce the function

$$\mathscr{X}(\eta) := \frac{1}{N^d(\varepsilon N)^{2d}} \sum_{x \in \Lambda_N} \sum_{y \in \Lambda_{x,\varepsilon N}} \sum_{\substack{z \in \Lambda_{x,\varepsilon N}: \\ |z-y|_\infty \geqslant 2\ell}} \\ \left| \frac{\mathscr{N}(\Gamma_{y,\ell,\alpha})}{\ell_*^d} - \frac{\mathscr{N}(\Gamma_{z,\ell,\alpha})}{\ell_*^d} \right| \mathbb{I}(\mathscr{N}(\Gamma_{y,\ell,\alpha} \cup \Gamma_{z,\ell,\alpha}) \leqslant A\ell_*^d)\,.$$

Recall that we need to prove (5.14) where $\Upsilon^*_{C_0,N}$ is substituted by the family $\Upsilon^\sharp_{C_0,N}$ defined before (5.17). Due to the above Lemma, we can substitute $\Upsilon^\sharp_{C_0,N}$ with the family $\Upsilon^\natural_{C_0/\kappa(L),N}$ of measurable functions $f : \mathbb{N}^{\mathscr{C}(\omega)} \to [0,\infty)$ such that $\nu(f) = 1$ and such that the l.h.s. of (5.18) is bounded by $C_0/\kappa(L)$. Namely, we need to prove that, given $\alpha \in (0, \alpha_0]$ and $A > 0$, for $\mathbb{Q}$–a.a. ω it holds

$$\limsup_{N\uparrow\infty,\varepsilon\downarrow 0,\ell\uparrow\infty,L\uparrow\infty} \sup_{f \in \Upsilon^{\natural}_{C_0/\kappa(L),N}} \nu_{\rho_*}\big(\mathscr{X}(\eta) f(\eta)\big) = 0\,. \tag{5.26}$$

Since by Lemma 5.2

$$\sup_{f\in\Upsilon^{\natural}_{C_0/\kappa(L),N}} \nu_{\rho_*}(\sqrt{f}, -\frac{\varepsilon^{-2}\kappa(L)}{N^d(\varepsilon N)^{2d}\ell_*^{2d}} \sum_{x\in\Lambda_N}\sum_{y\in\Lambda_{x,\varepsilon N}} \sum_{\substack{z\in\Lambda_{x,\varepsilon N}:\\ |z-y|_\infty\geqslant 2\ell}} \sum_{u\in\Gamma_{y,\ell,\alpha}}\sum_{v\in\Gamma_{z,\ell,\alpha}} \nabla_{u,v}\sqrt{f}) \leqslant C\,,$$

we only need to prove that, given $\alpha \in (0, \alpha_0]$ and $A, \gamma > 0$, for $\mathbb{Q}$–a.a. ω it holds

$$\limsup_{N\uparrow\infty,\epsilon\downarrow 0,\ell\uparrow\infty,L\uparrow\infty} \sup spec_{L^2(\nu_{\rho_*})}\Big\{\mathscr{X}(\eta) + \frac{\gamma\kappa(L)}{\varepsilon^2\ell^{2d}} \sum_{u\in\Gamma_{y,\ell,\alpha}}\sum_{v\in\Gamma_{z,\ell,\alpha}} \nabla_{u,v}\Big]\Big\} \leqslant 0\,, \tag{5.27}$$

where $\sup spec_{L^2(\nu_{\rho_*})}(\cdot)$ denotes the supremum of the spectrum in $L^2(\nu_{\rho_*})$ of the given operator. Now we use the subadditivity property

$$\sup spec_{L^2(\nu_{\rho_*})}(\sum_i X_i) \leqslant \sum_i \sup spec_{L^2(\nu_{\rho_*})}(X_i)$$

to bound the l.h.s. of (5.27) by

$$\begin{aligned}\limsup_{N\uparrow\infty,\varepsilon\downarrow 0,\ell\uparrow\infty,L\uparrow\infty} \frac{1}{N^d(\varepsilon N)^{2d}} \sum_{x\in\Lambda_N}\sum_{y\in\Lambda_{x,\varepsilon N}} \sum_{\substack{z\in\Lambda_{x,\varepsilon N}:\\ |z-y|_\infty\geqslant 2\ell}} \sup spec_{L^2(\nu_{\rho_*})}\Big\{\Big|\frac{\mathscr{N}(\Gamma_{y,\ell,\alpha})}{\ell_*^d} - \frac{\mathscr{N}(\Gamma_{z,\ell,\alpha})}{\ell_*^d}\Big|\cdot \\ \mathbb{I}(\mathscr{N}(\Gamma_{y,\ell,\alpha}\cup\Gamma_{z,\ell,\alpha}) \leqslant A\ell_*^d) + \frac{\gamma\kappa(L)}{\varepsilon^2\ell_*^{2d}} \sum_{u\in\Gamma_{y,\ell,\alpha}}\sum_{v\in\Gamma_{z,\ell,\alpha}} \nabla_{u,v}\Big\}.\end{aligned} \tag{5.28}$$

We observe that the operator inside the $\{\cdot\}$–brackets depends only on η restricted to $\Gamma_{y,z} := \Gamma_{y,\ell,\alpha} \cup \Gamma_{z,\ell,\alpha}$. By calling $\nu_{k,y,z}$ the canonical measure on $\mathscr{S}_{k,y,z} := \{\eta \in \mathbb{N}^{\Gamma_{y,z}} : \mathscr{N}(\Gamma_{y,z}) = k\}$ obtained by conditioning the marginal of ν_{ρ_*} on $\mathbb{N}^{\Gamma_{y,z}}$ to the event $\{\mathscr{N}(\Gamma_{y,z}) = k\}$, we can bound (5.28) by

$$\begin{aligned}\limsup_{N\uparrow\infty,\varepsilon\downarrow 0,\ell\uparrow\infty,L\uparrow\infty} \frac{1}{N^d(\varepsilon N)^{2d}} \sum_{x\in\Lambda_N}\sum_{y\in\Lambda_{x,\varepsilon N}} \sum_{\substack{z\in\Lambda_{x,\varepsilon N}:\\ |z-y|_\infty\geqslant 2\ell_*}} \sup_{k\in\{0,1,\dots,A\ell_*^d\}} \\ \sup spec_{L^2(\nu_{k,y,z})}\Big\{\Big|\frac{\mathscr{N}(\Gamma_{y,\ell,\alpha})}{\ell_*^d} - \frac{\mathscr{N}(\Gamma_{z,\ell,\alpha})}{\ell_*^d}\Big| + \frac{\gamma\kappa(L)}{\varepsilon^2\ell_*^{2d}} \sum_{u\in\Gamma_{y,\ell,\alpha}}\sum_{v\in\Gamma_{z,\ell,\alpha}} \nabla_{u,v}\Big\}.\end{aligned} \tag{5.29}$$

Given integers $k, n_1, n_2 \in \mathbb{N}$, define for $j = 1, 2$ the set $\Gamma_j := \{1, 2, \dots, n_j\}$, with the convention that $\Gamma_j = \emptyset$ if $n_j = 0$. Then define the space

$$\mathscr{S}_{k,n_1,n_2} := \{(\zeta_1,\zeta_2) \in \mathbb{N}^{\Gamma_1}\times\mathbb{N}^{\Gamma_2} : \sum_{a_1\in\Gamma_1}\zeta_1(a_1) + \sum_{a_2\in\Gamma_2}\zeta_2(a_2) = k\}$$

and set $\mathcal{N}(\zeta_i) = \sum_{a_i \in \Gamma_i} \zeta_i(a_i)$. Finally, call ν_{k,n_1,n_2} the probability measure on $\mathcal{S}_{k,n_1,n_2}$ obtained by first taking the product measure on $\mathbb{N}^{\Gamma_1} \times \mathbb{N}^{\Gamma_2}$ with the same marginals as ν_{ρ_*}, and afterwards by conditioning this product measure to the event that the total number of particles is k. Finally, define

$$F(k,n_1,n_2,\varepsilon,\ell,L) := \sup spec_{L^2(\nu_{k,n_1,n_2})}\Big\{\ell_*^{-d}\Big|\mathcal{N}(\zeta_1) - \mathcal{N}(\zeta_2)\Big| + \frac{\gamma\kappa(L)}{\varepsilon^2 \ell_*^{2d}} \sum_{u\in\Gamma_1}\sum_{v\in\Gamma_2} \nabla_{u,v}\Big\}. \tag{5.30}$$

Note that the operator $\gamma\kappa(L)\ell_*^{-2d}\sum_{u\in\Gamma_1}\sum_{v\in\Gamma_2}\nabla_{u,v}$ is the Markov generator of a process on $\mathcal{S}_{k,n_1,n_2}$ such that the measure ν_{k,n_1,n_2} is reversible and ergodic. In particular, 0 is a simple eigenvalue for this process. Fixed ℓ, we will vary the triple (k,n_1,n_2) in a finite set, more precisely we will take $n_1, n_2 \leqslant \ell_*^d$ and $0 \leqslant k \leqslant A\ell_*^d$. Then, applying Perturbation Theory (see Corollary 1.2 in Appendix 3.1 of [KL]), we conclude that

$$\limsup_{\varepsilon\downarrow 0} \sup_{k,n_1,n_2} \Big|F(k,n_1,n_2,\varepsilon,\ell,L) - G(k,\ell,n_1,n_2)\Big| = 0 \tag{5.31}$$

where

$$G(k,\ell,n_1,n_2) = \nu_{k,n_1,n_2}\big(\ell_*^{-d}\Big|\mathcal{N}(\zeta_1) - \mathcal{N}(\zeta_2)\Big|\big)\,. \tag{5.32}$$

The above result implies that in order to prove that (5.29) is nonnegative we only need to show that

$$\limsup_{N\uparrow\infty,\varepsilon\downarrow 0,\ell\uparrow\infty,L\uparrow\infty} \frac{1}{N^d(\varepsilon N)^{2d}} \sum_{x\in\Lambda_N}\sum_{y\in\Lambda_{x,\varepsilon N}}\sum_{z\in\Lambda_{x,\varepsilon N}} \sup_{k\in\{0,1,\dots,A\ell_*^d\}} G(k,\ell,n_y,n_z) \leqslant 0\,, \tag{5.33}$$

where

$$n_y = |\Gamma_{y,\ell,\alpha}|\,, \qquad n_z = |\Gamma_{z,\ell,\alpha}|\,.$$

Lemma 5.3. *Given $\delta > 0$, for $\mathbb{Q}$–a.a. ω it holds*

$$\limsup_{N\uparrow\infty,\varepsilon\downarrow 0,\ell\uparrow\infty,L\uparrow\infty} \frac{1}{N^d(\varepsilon N)^{2d}} \sum_{x\in\Lambda_N}\sum_{y\in\Lambda_{x,\varepsilon N}}\sum_{z\in\Lambda_{x,\varepsilon N}} \mathbb{I}\big(|n_y/\ell_*^d - m_\alpha| > \delta \text{ or } |n_z/\ell_*^d - m_\alpha| > \delta\big) = 0\,, \tag{5.34}$$

where $m_\alpha = \mathbb{Q}(0 \in \mathscr{C}_\alpha)$.

Proof. Recall definition (5.15) and set $N_y = |\mathscr{C}_\alpha \cap \Lambda_{y,\ell}|$, $N_z = |\mathscr{C}_\alpha \cap \Lambda_{z,\ell}|$. Then

$$\frac{1}{N^d(\varepsilon N)^{2d}} \sum_{x\in\Lambda_N}\sum_{y\in\Lambda_{x,\varepsilon N}}\sum_{z\in\Lambda_{x,\varepsilon N}} \mathbb{I}\big(|n_y - N_y| > \delta\ell_*^d\big) \leqslant \frac{c}{N^d(\varepsilon N)^d} \sum_{x\in\Lambda_N}\sum_{y\in\Lambda_{x,\varepsilon N}} \mathbb{I}(|B(L)\cap\Lambda_{y,\ell}| > \delta\ell_*^d) \leqslant \frac{c}{\delta N^d(\varepsilon N)^d \ell_*^d} \sum_{x\in\Lambda_N}\sum_{y\in\Lambda_{x,\varepsilon N}} |B(L)\cap\Lambda_{y,\ell}| \leqslant \frac{C}{\delta N^d}|B(L)\cap\Lambda_{N+\varepsilon N}|\,.$$

Due to the above bound, the same bound for n_z and N_z, Lemma 5.1 and finally a $\delta/2$–argument, we conclude that it is enough to prove (5.34) substituting n_y and n_z with N_y and N_z respectively. Moreover, due to ergodicity, for $\mathbb{Q}$–a.a. ω it holds

$$\limsup_{N\uparrow\infty,\varepsilon\downarrow 0,\ell\uparrow\infty} \frac{1}{N^d(\varepsilon N)^d}\sum_{x\in\Lambda_N}\sum_{y\in\Lambda_{x,\varepsilon N}} \mathbb{I}\big(|N_y/\ell_*^d-m_\alpha|>\delta\big)$$
$$\leqslant c\limsup_{N\uparrow\infty,\ell\uparrow\infty}\frac{1}{N^d}\sum_{y\in\Lambda_{2N}}\mathbb{I}\big(|N_y/\ell_*^d-m_\alpha|>\delta\big)=c\limsup_{\ell\uparrow\infty}\mathbb{Q}\big(|N_y/\ell_*^d-m_\alpha|>\delta\big)=0\,. \quad (5.35)$$

A similar bound can be obtained for z instead of y, thus implying (5.34). □

Since $G(k,\ell,n_1,n_2)\leqslant A$ and since

$$G(k,\ell,n_1,n_2)\leqslant \nu_{k,n_1,n_2}\Big(\Big|\frac{\mathcal{N}(\zeta_1)}{\ell_*^d}-\frac{k}{n_1+n_2}\frac{n_1}{\ell_*^d}\Big|\Big)+ \nu_{k,n_1,n_2}\Big(\Big|\frac{\mathcal{N}(\zeta_2)}{\ell_*^d}-\frac{k}{n_1+n_2}\frac{n_2}{\ell_*^d}\Big|\Big)+A\frac{|n_1-n_2|}{n_1+n_2}\,, \quad (5.36)$$

by the above Lemma in order to prove (5.33) we only need to prove for $j=1,2$ that

$$\limsup_{\ell\uparrow\infty,\delta\downarrow 0}\sup_{(k,n_1,n_2)\in\mathcal{I}}\nu_{k,n_1,n_2}\Big(\Big|\frac{\mathcal{N}(\zeta_j)}{\ell_*^d}-\frac{k}{n_y+n_2}\frac{n_j}{\ell_*^d}\Big|\Big)=0\,, \quad (5.37)$$

where

$$\mathcal{I}=\Big\{(k,n_1,n_2)\in\mathbb{N}^3:\ \frac{k}{\ell_*^d}\leqslant A,\ \frac{n_1}{\ell_*^d}\in[m_\alpha-\delta,(m_\alpha+\delta)\wedge 1]\,,\ \frac{n_2}{\ell_*^d}\in[m_\alpha-\delta,(m_\alpha+\delta)\wedge 1]\Big\}\,. \quad (5.38)$$

At this point (5.33) derives from the local central limit theorem as in in [KL][page 89, Step 6].

6 Proof of the One Block Estimate

We use here several arguments developed in the previous section. In order to avoid repetitions, we will only sketch the proof. As before, for simplicity of notation we take $M=1$.

Let us define $m_\alpha:=\mathbb{Q}(0\in\mathcal{C}_\alpha)$. Note that $m=\lim_{\alpha\downarrow 0}m_\alpha$. Setting $a=\eta^\ell(y)$, $b=\mathcal{N}(\mathcal{C}_\alpha\cap\Lambda_{y,\ell})/\ell_*^d$, since ϕ is Lipschitz we can bound

$$\Big|m\phi(\frac{a}{m})-m_\alpha\phi(\frac{b}{m_\alpha})\Big|\leqslant|m-m_\alpha|\phi(\frac{a}{m})+m_\alpha\Big|\phi(\frac{a}{m})-\phi(\frac{b}{m})\Big|+m_\alpha\Big|\phi(\frac{b}{m})-\phi(\frac{b}{m_\alpha})\Big|\leqslant$$
$$\frac{|m-m_\alpha|}{m}g^*a+g^*\frac{m_\alpha}{m}|a-b|+g^*bm_\alpha\Big|\frac{1}{m}-\frac{1}{m_\alpha}\Big|\leqslant$$
$$\frac{|m-m_\alpha|}{m}g^*\eta^\ell(y)+g^*\frac{m_\alpha}{m\ell_*^d}\mathcal{N}(\Lambda_{y,\ell}\setminus\mathcal{C}_\alpha)+g^*\mathcal{N}(\mathcal{C}_\alpha\cap\Lambda_{y,\ell})\ell_*^{-d}m_\alpha\Big|\frac{1}{m}-\frac{1}{m_\alpha}\Big|=:\mathcal{G}(\eta,\omega)\,.$$

Note that $\mathscr{G}(\eta)$ is increasing in η. Hence, using that $f(\eta)\nu_{\rho_*}(d\eta) \prec f(\eta)\nu_{\rho_0}(d\eta)$, we easily obtain that

$$\limsup_{N\uparrow\infty,\ell\uparrow\infty,\alpha\downarrow 0} \sup_{f\in\Upsilon_{C_0,N}} \int \frac{1}{N^d}\sum_{x\in\Lambda_N} \mathscr{G}(\tau_x\eta,\tau_x\omega) f(\eta)\nu_{\rho_*}(d\eta) = 0\,. \tag{6.1}$$

Due to the above result and reasoning as in the derivation of (5.14) where $\Upsilon^*_{C_0,N}$ can be replaced by $\Upsilon^\sharp_{C_0,N}$ (see the discussion after (5.14)), we only need to prove that given $C_0 > 0$, $A > 0$ and $\alpha \in (0,\alpha_0]$, for $\mathbb{Q}$–a.a. ω it holds

$$\limsup_{N\uparrow\infty,\ell\uparrow\infty,L\uparrow\infty} \sup_{f\in\Upsilon^\sharp_{C_0,N}}$$
$$\int Av_{x\in\Lambda_N}\Big|\ell_*^{-d}\sum_{y\in\Gamma_{x,\ell,\alpha}} g(\eta(y)) - m_\alpha\phi\big(\mathcal{N}(\Gamma_{x,\ell,\alpha})/m_\alpha\ell_*^d\big)\Big|\mathbb{I}\big(\mathcal{N}(\Gamma_{x,\ell,\alpha})\leqslant A\ell_*^d\big)f(\eta)\nu_{\rho_*}(\eta) = 0\,. \tag{6.2}$$

At this point by the same arguments of Lemma 5.2, one can prove that

Lemma 6.1. *For $\mathbb{Q}$–a.a. ω the following holds. Fixed $\alpha \in (0,\alpha_0]$ and $L > 0$, there exists a positive constant $\kappa = \kappa(\alpha, L)$ such that*

$$\varphi(\rho_*)N^2\ell_*^{-2d-2}Av_{x\in\Lambda_N}\sum_{u\in\Gamma_{x,\ell,\alpha}}\sum_{v\in\Gamma_{x,\ell,\alpha}}\nu_{\rho_*}\big((\nabla_{u,v}\sqrt{f})^2\big) \quad\leqslant\quad N^{-d+2}\mathscr{D}(f)/\kappa \quad\leqslant\quad C_0/\kappa\,, \tag{6.3}$$

for any function $f \in \Upsilon^\sharp_{C_0,N}$ and for any N,ℓ,C_0.

Due to the above lemma, as in the derivation of (5.27) we only need to prove that given $\alpha \in (0,\alpha_0]$, $A,\gamma > 0$, for $\mathbb{Q}$–a.a. ω it holds:

$$\limsup_{N\uparrow\infty,\ell\uparrow\infty,L\uparrow\infty} \sup spec_{L^2(\nu_{\rho_*})}$$
$$\Big\{Av_{x\in\Lambda_N}\Big|\ell_*^{-d}\sum_{y\in\Gamma_{x,\ell,\alpha}} g(\eta(y)) - m_\alpha\phi\big(\mathcal{N}(\Gamma_{x,\ell,\alpha})/m_\alpha\ell_*^d\big)\Big|\mathbb{I}\big(\mathcal{N}(\Gamma_{x,\ell,\alpha})\leqslant A\ell_*^d\big)$$
$$+\frac{\gamma\kappa(L)N^2}{\ell^{2d+2}}\sum_{u\in\Gamma_{x,\ell,\alpha}}\sum_{v\in\Gamma_{x,\ell,\alpha}}\nabla_{u,v}\Big]\Big\}\leqslant 0\,. \tag{6.4}$$

Using subadditivity as in the derivation of (5.28) we can bound the above l.h.s. by

$$\limsup_{N\uparrow\infty,\ell\uparrow\infty,L\uparrow\infty} Av_{x\in\Lambda_N}\sup spec_{L^2(\nu_{\rho_*})}$$
$$\Big\{\Big|\ell_*^{-d}\sum_{y\in\Gamma_{x,\ell,\alpha}} g(\eta(y)) - m_\alpha\phi\big(\mathcal{N}(\Gamma_{x,\ell,\alpha})/m_\alpha\ell_*^d\big)\Big|\mathbb{I}\big(\mathcal{N}(\Gamma_{x,\ell,\alpha})\leqslant A\ell_*^d\big)$$
$$+\frac{\gamma\kappa(L)N^2}{\ell^{2d+2}}\sum_{u\in\Gamma_{x,\ell,\alpha}}\sum_{v\in\Gamma_{x,\ell,\alpha}}\nabla_{u,v}\Big]\Big\}. \tag{6.5}$$

Again by conditioning on the number of particles in $\Gamma_{x,\ell,\alpha}$ and afterwards applying perturbation theory (see (5.29), (5.31) and (5.33)), one only needs to show that

$$\limsup_{N\uparrow\infty,\ell\uparrow\infty,L\uparrow\infty} Av_{x\in\Lambda_N} \sup_{k\in\{0,1,\dots,A\ell_*^d\}} G(k,\ell,n_x) = 0\,, \tag{6.6}$$

where $n_x := |\Gamma_{x,\alpha,\ell}|$, $\nu_{k,n}$ is the measure on $\{\zeta \in \mathbb{N}^n : \sum_{i=1}^n \zeta(i) = k\}$ obtained by taking the product measure with the same marginals as ν_{ρ_*} and then conditioning on the event that the total number of particles equals k, and where

$$G(k,\ell,n) := \nu_{k,n}\Big[\Big|\ell_*^{-d}\sum_{i=1}^n g(\zeta(i)) - m_\alpha \phi(k/m_\alpha \ell_*^d)\Big|\Big]\,.$$

One can prove that for $\mathbb{Q}$–a.a. ω it holds

$$\limsup_{N\uparrow\infty,\ell\uparrow\infty,L\uparrow\infty} Av_{x\in\Lambda_N}\mathbb{I}(|n_x/\ell_*^d - m_\alpha| > \delta) = 0\,, \tag{6.7}$$

for each positive constant $\delta > 0$. As in [KL][Chapter 5], one has that

$$\lim_{\ell\uparrow\infty}\sup_{(k,n)\in\mathcal{J}} \nu_{k,n}\Big[\Big|n^{-1}\sum_{i=1}^n g(\zeta(i)) - \phi(k/n)\Big|\Big] = 0\,. \tag{6.8}$$

where

$$\mathcal{J} := \Big\{(k,n)\in\mathbb{N}^2 \,:\, \frac{k}{\ell_*^d} \leqslant A,\ \frac{n}{\ell_*^d} \in (m_\alpha - \delta, (m_\alpha+\delta)\wedge 1)\Big\}\,.$$

At this point (6.6) follows from (6.7) and (6.8).

A Zero range process on $\mathbb{Z}^d$ with random conductances

Recall that the enviroment $\omega = \big(\omega(b) \,:\, b \in \mathbb{E}_d\big)$ is given by a family of i.i.d. random variables parameterized by the set $\mathbb{E}_d$ of non–orientied bonds in $\mathbb{Z}^d$, $d \geqslant 2$. We denote by $\mathbb{Q}$ the law of ω, we assume that $\mathbb{Q}(\omega(b) \in [0, c_0]) = 1$ and that $\mathbb{Q}(\omega(b) > 0)$ is supercritical. We fix a function $g : \mathbb{N} \to [0,\infty)$ as in Subsection 2.2. Given a realization of ω, we consider the zero range process $\eta(t)$ on $\mathbb{Z}^d$ whose Markov generator $N^2\mathcal{L}$ acts on local functions as

$$N^2\mathcal{L}f(\eta) = N^2\sum_{e\in\mathcal{B}}\sum_{x\in\mathbb{Z}^d} g(\eta(x))\omega(x,x+e)\big(f(\eta^{x,x+e}) - f(\eta)\big)\,, \tag{A.1}$$

where $\mathcal{B} = \{\pm e_1, \pm e_2, \dots, \pm e_d\}$, $e_1,\dots,e_d$ being the canonical basis of $\mathbb{Z}^d$. Given an *admissible* initial distribution $\bar\mu^N$ on $\{0,1\}^{\mathbb{Z}^d}$ (i.e. such that the corresponding zero range process is well defined), we denote by $\mathbb{P}_{\omega,\bar\mu^N}$ the law of $(\eta_t \,:\, t \geqslant 0)$. Trivially, the zero range process behaves independently on the different clusters of the conductance field.

If $\mathbb{Q}(\omega(b) > 0) = 1$, then $\mathbb{Q}$–a.s. the infinite cluster $\mathcal{C}(\omega)$ coincides with $\mathbb{Z}^d$ and the hydrodynamic behavior of the zero range process on $\mathbb{Z}^d$ is described by Theorem 2.1. If $\mathbb{Q}(\omega(b) > 0) < 1$ the bulk behavior of the zero range process on $\mathbb{Z}^d$ is different due to the presence of finite clusters acting as traps, as we now explain.

First, we observe that the finite clusters cannot be too big. Indeed, as byproduct of Borel–Cantelli Lemma and Theorems (8.18) and (8.21) in [G], there exists a positive constant $\gamma > 0$ such that for $\mathbb{Q}$–a.a. ω the following property (P1) holds:

(P1) for each $N \geqslant 1$ and each finite cluster C intersecting the box $[-N,N]^d$, the diameter of C is bounded by $\gamma \ln(1+N)$.

Lemma A.1. *Suppose that ω has a unique infinite cluster $\mathscr{C}(\omega)$ and that ω satisfies the above property (P1). Let $G \in C_c(\mathbb{R}^d)$ and $\eta \in \mathbb{N}^{\mathbb{Z}^d}$ be such that $\eta(x) = 0$ for all $x \in \mathscr{C}(\omega)$. Call Δ_G the support of G and call $\bar\Delta_G$ the set of points $z \in \mathbb{R}^d$ having distance from Δ_G at most 1. Then there exist positive constants $N_0(G,\gamma)$, $C(G,\gamma)$ depending only on G and γ such that the zero range process on $\mathbb{Z}^d$ with initial configuration η satisfies a.s. the following properties: $\eta_t(x) = 0$ for all $x \in \mathscr{C}(\omega)$ and, for $N \geqslant N_0(G,\gamma)$,*

$$\Big|N^{-d}\sum_{x\in\mathbb{Z}^d} G(x/N)\eta(x) - N^{-d}\sum_{x\in\mathbb{Z}^d} G(x/N)\eta_t(x)\Big| \leqslant C(G,\gamma)\frac{\ln(1+N)}{N^{d+1}}\sum_{x\in\mathbb{Z}^d : x/N\in\bar\Delta_G}\eta(x)\,. \tag{A.2}$$

We point out that the zero range process is well defined when starting in η, indeed the dynamics reduces to a family of independent zero range processes on the finite clusters, while the infinite cluster $\mathscr{C}(\omega)$ remains empty.

Proof. The fact that $\eta_t(x) = 0$ with $x \notin \mathscr{C}(\omega)$ is trivial. Let us prove (A.2). Without loss of generality we can suppose that G has support in $[-1,1]^d$ (the general case is treated similarly). Let us write $C_1^N, C_2^N, \dots, C_{k_N}^N$ for the family of finite clusters intersecting the box $[-N,N]^d$. For each cluster C_i^N we fix a point $x_i^N \in C_i^N$. Since by the property (P1) each C_i^N has diameter at most $\gamma\ln(1+N)$, we have

$$|G(x/N) - G(x_i^N/N)| \leqslant C(G)\gamma\ln(1+N)/N\,, \qquad \forall i : 1 \leqslant i \leqslant k_N\,,\ \forall x \in C_i^N\,.$$

The above estimate implies that

$$\Big|N^{-d}\sum_{x\in\mathbb{Z}^d} G(x/N)\eta_t(x) - N^{-d}\sum_{i=1}^{k_N} G(x_i^N/N)\sum_{x\in C_i^N}\eta_t(x)\Big| \leqslant$$
$$C(G)\gamma N^{-d-1}\ln(1+N)\sum_{i=1}^{k_N}\sum_{x\in C_i^N}\eta_t(x)\,.$$

Using now that the number of particles in each cluster is time–independent and that for N large enough $C_i^N \subset [-2N, 2N]$ for all $i = 1, \dots, k_N$, we get the thesis. □

As a consequence of Lemma A.1, if ω has a unique infinite cluster and if ω satisfies property (P1), then for any admissible initial configuration η_0 (i.e. such that the zero range process on $\mathbb{Z}^d$ is well defined when starting in η_0) and any $G \in C_c(\mathbb{R}^d)$, it holds

$$N^{-d}\sum_{x\in\mathbb{Z}^d} G(x/N)\eta_t(x) = N^{-d}\sum_{x\in\mathscr{C}(\omega)} G(x/N)\eta_t(x) + N^{-d}\sum_{x\notin\mathscr{C}(\omega)} G(x/N)\eta_0(x) + o(1)\,, \tag{A.3}$$

where $|o(1)| \leqslant C(G,\gamma)\frac{\ln(1+N)}{N^{d+1}}\sum_{x\in\mathbb{Z}^d : x/N\in\tilde{\Delta}_G}\eta(x)$. At this point, denoting by $\bar{\mu}^N$ the initial distribution of the zero range process η_t on $\mathbb{Z}^d$, one can derive the hydrodynamic limit of η_t if the marginals of $\bar{\mu}^N$ on $\mathscr{C}(\omega)$ and $\mathbb{Z}^d \setminus \mathscr{C}(\omega)$, respectively, are associated to suitable macroscopic profiles. In what follows, we discuss a special case where this last property is satisfied.

We fix a smooth, bounded nonnegative function $\rho_0 : \mathbb{R}^d \to [0,\infty)$ and for each N we define $\bar{\mu}^N$ as the product probability measure on $\mathbb{N}^{\mathbb{Z}^d}$ such that for all $x \in \mathbb{Z}^d$ it holds

$$\bar{\mu}^N\left(\eta(x)=k\right) = \nu_{\rho_0(x/N)}(\eta(x)=k)\,, \tag{A.4}$$

where ν_ρ is defined as in Subsection 2.2 with the difference that now it is referred to all $\mathbb{Z}^d$ and not only to $\mathscr{C}(\omega)$.

We call μ^N the marginal of $\bar{\mu}^N$ on $\mathscr{C}(\omega)$: μ^N is a product probability measure on $\mathbb{N}^{\mathscr{C}(\omega)}$ satisfying (A.4) for all $x \in \mathscr{C}(\omega)$ (note that μ^N depends on ω). Similarly, we call $\nu_{\rho,\mathscr{C}(\omega)}$ the marginal of ν_ρ on $\mathscr{C}(\omega)$. By the discussion at the end of Section 2, if the smooth profile ρ_0 converges sufficiently fast at infinity to a positive constant ρ_*, it holds

$$\limsup_{N\uparrow\infty} N^{-d}H(\mu^N|\nu_{\rho_*,\mathscr{C}(\omega)}) < \infty \qquad \mathbb{Q}\text{–a.s.} \tag{A.5}$$

Theorem A.2. *Suppose that the bounded smooth profile $\rho_0 : \mathbb{R}^d \to [0,\infty)$ satisfies* (A.5). *Then for all $t>0$, $G \in C_c(\mathbb{R}^d)$ and $\delta > 0$, for $\mathbb{Q}$–a.a. ω it holds*

$$\lim_{N\uparrow\infty}\mathbb{P}_{\omega,\bar{\mu}^N}\left(\left|N^{-d}\sum_{x\in\mathbb{Z}^d}G(x/N)\,\eta_t(x)-\int_{\mathbb{R}^d}G(x)\rho(x,t)dx\right|>\delta\right)=0 \tag{A.6}$$

where, setting $m=\mathbb{Q}(0\in\mathscr{C}(\omega))$,

$$\rho(x,t) = m\tilde{\rho}(x,t)+(1-m)\rho_0(x) \tag{A.7}$$

and $\tilde{\rho} : \mathbb{R}^d\times[0,\infty)\to\mathbb{R}$ is the unique weak solution of the heat equation

$$\partial_t\tilde{\rho} = \nabla\cdot(\mathscr{D}\nabla\phi(\tilde{\rho})) \tag{A.8}$$

with boundary condition $\tilde{\rho}_0 = \rho$ at $t=0$.

Proof. Since $\bar{\mu}^N$ is a product measure and the dynamics on different clusters is independent, the process restricted to $\mathscr{C}(\omega)$ has law $\mathbb{P}_{\omega,\mu^N}$. As discussed at the end of Section 2, to this last process we can apply Theorem 2.1. Since, for $\mathbb{Q}$-a.a. ω the initial distributions μ^N are associated to the macroscopic profile $m\rho_0$, we conclude that for $\mathbb{Q}$-a.a. ω it holds

$$\lim_{N\uparrow\infty}\mathbb{P}_{\omega,\bar{\mu}^N}\left(\left|N^{-d}\sum_{x\in\mathscr{C}(\omega)}G(x/N)\,\eta_t(x)-m\int_{\mathbb{R}^d}G(x)\tilde{\rho}(x,t)dx\right|>\delta\right)=0\,. \tag{A.9}$$

Let us now consider the evolution outside the infinite cluster. Let us write

$$N^{-d}\sum_{x\notin\mathscr{C}(\omega)}G(x/N)\,\eta_0(x) = N^{-d}\sum_{x\in\mathbb{Z}^d}G(x/N)\,\eta_0(x)-N^{-d}\sum_{x\in\mathscr{C}(\omega)}G(x/N)\,\eta_0(x)\,.$$

We know that, when η is sampled with distribution $\bar{\mu}^N$, the addenda in the r.h.s. converge in probability to $\int G(x)\rho_0(x)$ and $m\int G(x)\rho_0(dx)$, for $\mathbb{Q}$–a.a. ω. As a consequence the l.h.s. converges in probability to $(1-m)\int G(x)\rho_0(x)$ for $\mathbb{Q}$–a.a. ω. In addition,

$$\sup_{N\geqslant 1}\left\{\int \bar{\mu}^N(d\eta)N^{-d}\sum_{x:x/N\in\bar{\Delta}_G}\eta(x)\right\}<\infty\,.$$

The above observations and Lemma A.1 (cf. (A.3)) imply that

$$\lim_{N\uparrow\infty}\mathbb{P}_{\omega,\bar{\mu}^N}\left(\left|N^{-d}\sum_{x\notin\mathscr{C}(\omega)}G(x/N)\,\eta_t(x)-(1-m)\int_{\mathbb{R}^d}G(x)\rho_0(x)dx\right|>\delta\right)=0\,. \tag{A.10}$$

The thesis then follows from (A.9) and (A.10). □

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