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Chaoticity of the stationary distribution of rank-based interacting diffusions

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Abstract

We consider Brownian diffusions on the real line, interacting through rank-dependent drifts. It is known that in the mean-field limit, such particle systems behave like independent copies of a so-called nonlinear diffusion process. We prove a similar asymptotic behaviour at the level of stationary distributions. Our proof is based on explicit expressions for the Laplace transforms of the stationary distributions of both the particle system and the nonlinear diffusion process, and yields convergence of the marginal distributions in Wasserstein distances of all orders. We highlight the consequences of this result on the study of rank-based models of equity markets, such as the Atlas model.

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1 Introduction

1.1 Rank-based interacting diffusions

Let $b : [0,1] \to \mathbb{R}$ be a continuous function, and take a real number $\sigma \neq 0$. For all $n \geq 2$, consider the system of *rank-based interacting diffusions*, or *particles*, defined by

$$\forall i \in \{1,\ldots,n\}, \qquad \mathrm{d}X_i^n(t) = b_n\left(\sum_{j=1}^n \mathbb{1}_{\{X_j^n(t) \leq X_i^n(t)\}}\right)\mathrm{d}t + \sigma \mathrm{d}W_i(t), \tag{1.1}$$

where $(W_1(t),\ldots,W_n(t))_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}^n$, and for all $k \in \{1,\ldots,n\}$,

$$b_n(k) := n\int_{v=\frac{k-1}{n}}^{\frac{k}{n}} b(v)\mathrm{d}v = n\left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right)\right), \qquad B(u) := \int_{v=0}^{u} b(v)\mathrm{d}v.$$

By the Girsanov theorem, the stochastic differential equation (1.1) possesses a unique weak solution, and actually a unique strong solution [21]. It describes the evolution of n Brownian particles on the real line, such that the particle with k-th rank in the increasing order of positions has a drift $b_n(k)$. This is an instance of a system of *competing particles* [16, 1, 17, 18].

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Assume that the law of the vector of initial positions $(X_1^n(0), \ldots, X_n^n(0))$ is *chaotic*, that is to say that it behaves like independent copies of a scalar random variable $X(0)$ when n grows to infinity (see Definition 1.3 below). Then it is a classical result for such kind of particle systems that the chaoticity of the initial positions is *propagated* by the temporal evolution (1.1), in the sense that the processes $(X_n^1(t))_{t \geq 0}, \ldots, (X_n^n(t))_{t \geq 0}$ behave themselves like independent copies of a scalar diffusion process $(X(t))_{t \geq 0}$. The latter process can be explicitly identified, it is the unique weak solution to the nonlinear (in McKean's sense [14]) stochastic differential equation

$$\begin{cases} \mathrm{d}X(t) = b(F_t(X(t)))\mathrm{d}t + \sigma \mathrm{d}W(t), \\ F_t(x) = \mathbb{P}(X(t) \leq x). \end{cases} \tag{1.2}$$

The expression *nonlinearity in McKean's sense* refers to the fact that the drift coefficient of the stochastic differential equation depends on the law of $X(t)$, which is the effect of the nonlinearity of the associated Fokker-Planck equation. For this reason, $(X(t))_{t \geq 0}$ will be referred to as the *nonlinear diffusion process*. The propagation of chaos results for the particle system (1.1) were obtained in [8, 10]; we also refer to [9, 19, 11, 5] for nonconstant diffusion coefficients.

These propagation of chaos results are not uniform in time, and therefore do not provide any indication of the link between the long time behaviour of the particle system and the long time behaviour of the nonlinear diffusion process. The purpose of this article is to clarify this link by showing that the stationary distribution of a suitably modified version of the particle system, to which we shall refer as the *projected* particle system, is chaotic with respect to the stationary distribution of the nonlinear process, which completes the picture concerning the long time and large scale behaviour of the particle system detailed on Figure 1.

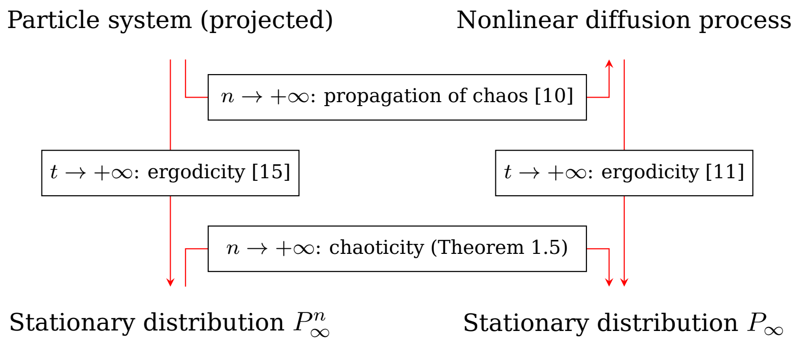

Figure 1: A summary of convergence results, in long time as well as for a large number of particles, for the projected particle system.

We first recall what is known on the long time behaviour of both the particle system and the nonlinear diffusion process in Subsection 1.2, then we state our main result in Subsection 1.3. We discuss some applications of our result and some possible extensions related to Stochastic Portfolio Theory in Subsection 1.4, and provide an outline of the article in Subsection 1.5.

1.2 Projected particle system and nonlinear diffusion process

As was remarked in [10], the solution to (1.1) cannot converge to an equilibrium, since its projection along the direction $(1, \ldots, 1)$ is a Brownian motion with constant drift. One can however address the long time behaviour of the projection onto the hyperplane

$$M_n := \{(z_1, \ldots, z_n) \in \mathbb{R}^n : z_1 + \cdots + z_n = 0\},$$

which is orthogonal to the singular direction $(1,\dots,1)$. The resulting process is called the *projected particle system*, it is the M_n-valued diffusion process solving

$$\mathrm{d}Z_i^n(t) = \left(b_n\left(\sum_{j=1}^n \mathbb{1}_{\{Z_j^n(t)\leq Z_i^n(t)\}}\right) - \bar{b}\right)\mathrm{d}t + \sigma\frac{n-1}{n}\mathrm{d}W_i(t) - \frac{\sigma}{n}\sum_{j\neq i}\mathrm{d}W_j(t), \tag{1.3}$$

where $\bar{b} := \frac{1}{n}\sum_{k=1}^n b_n(k) = B(1)$. Propagation of chaos for the projected particle system toward the nonlinear diffusion process (1.2) was established in [10].

In order to describe the long time behaviour of both the projected particle system and the nonlinear diffusion process, we introduce the following set of *equilibrium assumptions*:

(E1) $\bar{b} = B(1) = 0$,

(E2) $B(u) > (1-u)B(0) + uB(1)$ for all $u \in (0,1)$,

(E3) $b(0) > 0 > b(1)$.

Assumption (E1) concerns the average drift of the centre of mass of the particle system. Assumption (E2) is usually referred to as the *Oleinik E-entropy condition* in the literature of hyperbolic differential equations, and it ensures the existence of an equilibrium for both the projected particle system and the nonlinear diffusion process. Note that under Assumption (E1), it simply rewrites $B(u) > 0$ for all $u \in (0,1)$. Assumption (E3) is a nondegeneracy condition for the drift of extremal particles, it implies in particular exponential tails for the stationary distribution of the nonlinear diffusion process.

The following proposition concerning the long time behaviour of the projected particle system is due to Pal and Pitman [15, Theorem 8]. We use the notation $z_{(1)} \leq \cdots \leq z_{(n)}$ to refer to the order statistics of a vector $(z_1,\dots,z_n) \in \mathbb{R}^n$.

Proposition 1.1. *Under Assumption* (E2)*, for all* $n \geq 2$*,*

$$\mathcal{Z}_n := \int_{z\in M_n} \exp\left(\frac{2}{\sigma^2}\sum_{k=1}^n b_n(k)z_{(k)}\right)\mathrm{d}z < +\infty,$$

and the probability distribution with density

$$p_\infty^n(z) := \frac{1}{\mathcal{Z}_n}\exp\left(\frac{2}{\sigma^2}\sum_{k=1}^n b_n(k)z_{(k)}\right)$$

with respect to the surface measure $\mathrm{d}z$ *on* M_n *is the unique stationary distribution of the process* $(Z_1^n(t),\dots,Z_n^n(t))_{t\geq 0}$.

Let us remark that the density $p_\infty^n(z)$ only depends on the order statistics of z, and therefore is invariant under the permutations of the coordinates of z. As a consequence, the probability distribution $P_\infty^n := p_\infty^n(z)\mathrm{d}z$ is a symmetric probability distribution on $\mathbb{R}^n$, which gives full measure to M_n. Under the assumption that b be decreasing on $[0,1]$, which together with Assumption (E1) implies both Assumptions (E2) and (E3), Jourdain and Malrieu [10, Theorem 2.12] proved that P_∞^n satisfies a Poincaré inequality with a uniform constant with respect to n, and deduced uniform exponential convergence to equilibrium for the projected particle system.

On the other hand, the stationary distributions of the nonlinear diffusion process were described in [11]. This description relies on the function $\Phi : (0,1) \to \mathbb{R}$ defined under Assumptions (E1), (E2) and (E3) by

$$\forall u \in (0,1), \qquad \Phi(u) := \frac{\sigma^2}{2}\left(\int_{v=0}^u \frac{v}{B(v)}\mathrm{d}v - \int_{v=u}^1 \frac{1-v}{B(v)}\mathrm{d}v\right). \tag{1.4}$$

It is recalled in Lemma 2.1 below that Φ is the inverse function of the cumulative distribution function F_∞ of a certain probability distribution P_∞ on $\mathbb{R}$, which satisfies

$$\int_{x\in\mathbb{R}} |x| P_\infty(\mathrm{d}x) = \int_{u=0}^1 |\Phi(u)|\mathrm{d}u < +\infty, \qquad \int_{x\in\mathbb{R}} x P_\infty(\mathrm{d}x) = \int_{u=0}^1 \Phi(u)\mathrm{d}u = 0.$$

We can now describe the set of stationary distributions of the nonlinear process, which follows from [11, Proposition 4.1].

Proposition 1.2. *Under Assumptions* (E1)*,* (E2) *and* (E3)*, the stationary probability distributions for the nonlinear process $(X(t))_{t\geq 0}$ are the translations of the probability distribution P_∞; that is to say, the probability distributions with cumulative distribution function $x \mapsto F_\infty(x + \bar{x})$ for some $\bar{x} \in \mathbb{R}$.*

Ergodicity results for the nonlinear diffusion process were obtained in [10, 11]. Note that in [11], the stationary distributions are proven to be the translations of the function Ψ defined on $(0,1)$ by

$$\forall u \in (0,1), \qquad \Psi(u) := \frac{\sigma^2}{2}\int_{v=\frac{1}{2}}^u \frac{\mathrm{d}v}{B(v)}.$$

Since Φ and Ψ have the same derivative, it is clear that the set of translations of Φ^{-1} coincides with the set of translations of Ψ^{-1}.

As a consequence of Proposition 1.2, a stationary distribution for the nonlinear process is characterised by its expectation. In particular, P_∞ is the unique centered stationary distribution of the nonlinear process.

1.3 Main result

In order to state our main result, we first recall the definition of the notion of chaoticity [20, Definition 2.1, p. 177]. If P^n is a probability distribution on $\mathbb{R}^n$ and $k \in \{1, \dots, n\}$, we denote by $P^{k,n}$ the marginal distribution of the k first coordinates under P^n.

Definition 1.3. *For all $n \geq 1$, let P^n be a symmetric probability distribution on $\mathbb{R}^n$, and let P be a probability distribution on $\mathbb{R}$. The sequence $(P^n)_{n\geq 1}$ is said to be P-chaotic if, for all $k \geq 1$, $P^{k,n}$ converges weakly to the product measure $P^{\otimes k}$.*

Recall that we denote by P_∞^n the unique stationary distribution of the projected particle system; it is the probability distribution on $\mathbb{R}^n$ with density $p_\infty^n(z)$ with respect to the surface measure $\mathrm{d}z$ on M_n. On the other hand, P_∞ refers to the unique centered stationary distribution of the nonlinear diffusion process. Of course, our purpose is to establish the P_∞-chaoticity of the sequence $(P_\infty^n)_{n\geq 1}$. We shall actually prove the convergence of $P_\infty^{k,n}$ in a stronger sense than in Definition 1.3; namely, in Wasserstein distance [22].

Definition 1.4. *Let $k \geq 1$ and $q \in [1, +\infty)$. The* Wasserstein distance *of order q between two probability distributions μ and ν on $\mathbb{R}^k$ is defined by*

$$\mathrm{W}_q(\mu,\nu) := \inf_{(X,Y)\in\Pi(\mu,\nu)} \left(\mathbb{E}[|X-Y|^q]\right)^{1/q},$$

where $\Pi(\mu,\nu)$ refers to the set of pairs of random variables with marginal distributions μ and ν.

We can now state our main result.

Theorem 1.5. *Under Assumptions* (E1)*,* (E2) *and* (E3)*, for all $k \geq 1$, for all $q \in [1, +\infty)$,*

$$\lim_{n\to+\infty} \mathrm{W}_q\left(P_\infty^{k,n}, (P_\infty)^{\otimes k}\right) = 0.$$

In particular, the sequence of stationary distributions P^n_∞ of the projected particle system is P_∞-chaotic.

The proof of Theorem 1.5 relies on the explicit computation of the Laplace transform of $P^{2;n}_\infty$, see Subsection 1.5 below.

Remark 1.6. The definition of W_q depends on the choice of the norm $|\cdot|$ on $\mathbb{R}^k$. But since all norms are equivalent on $\mathbb{R}^k$, all the associated distances W_q are also equivalent. Therefore, convergence results for the W_q topology do not depend on the choice of the underlying norm. In this article, we take the convention that the Wasserstein distance of order q is defined with respect to the ℓ^q norm $|x| := (|x_1|^q + \cdots + |x_k|^q)^{1/q}$ on $\mathbb{R}^k$.

1.4 Discussion and comments

Figure 1 illustrates the fact that, when it makes sense, the interversion of the limits '$n \to +\infty$' and '$t \to +\infty$' is generically correct for functionals of systems of rank-based interacting particles.

This remark is of interest in the study of rank-based models of equity markets, such as the Atlas model introduced by Fernholz in the framework of Stochastic Portfolio Theory [6, 3, 7]. Indeed, in this context, relevant quantities such as capital distribution curves or growth rates of portfolios are expressed in terms of the stationary distribution P^n_∞ described in Proposition 1.1. The asymptotic behaviour of these quantities, when the size of the market grows to infinity, were investigated in [3, 4]. On the other hand, it was suggested in [12] to use the propagation of chaos results of [11] to obtain a functional description of an infinite market first, and then apply the available ergodicity results on the nonlinear diffusion process to derive closed formulas for these relevant quantities. Theorem 1.5 is a first step toward the validation of the equivalence of both approaches, and we refer to [12] for a detailed account.

The *gap process* $(Y^n_1(t), \ldots, Y^n_{n-1}(t))_{t\geq 0}$ is defined in terms of the particle system by

$$\forall k \in \{1, \ldots, n-1\}, \qquad Y^n_k(t) := X^n_{(k+1)}(t) - X^n_{(k)}(t),$$

where we recall that $x_{(1)} \leq \cdots \leq x_{(n)}$ refers to the order statistics of the vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Note that the gap process can also and indifferently be defined by replacing the particle system $(X^n_1(t), \ldots, X^n_n(t))_{t\geq 0}$ with the projected particle system $(Z^n_1(t), \ldots, Z^n_n(t))_{t\geq 0}$. This process is of particular importance in Stochastic Portfolio Theory [6, 3] and under Assumption (E2), its long time behaviour is known to be described by an inhomogeneous product of exponentially distributed random variables [15, Theorem 8]. In the present case, it is straightforward to check that the pushforward measure of P^n_∞ by the application

$$(z_1, \ldots, z_n) \in M_n \mapsto \left(z_{(2)} - z_{(1)}, \ldots, z_{(n)} - z_{(n-1)}\right) \in [0, +\infty)^{n-1}$$

is the product measure

$$q^n_\infty := \bigotimes_{k=1}^{n-1} \mathrm{Exp}\left(\frac{2n}{\sigma^2} B\left(\frac{k}{n}\right)\right).$$

In view of the discussion above, it is natural to wonder whether Theorem 1.5 has a natural transcription in terms of the gap process. It actually turns out that the description of the large scale behaviour of the stationary distribution of the gap process only follows from elementary observations, and does not rely on Theorem 1.5. First, the independence structure between the stationary gaps is obviously preserved when n varies. Second, precise asymptotics for the marginal distribution of each stationary gap are obtained by studying the behaviour of the quantity $\frac{2n}{\sigma^2}B(\frac{k}{n})$ when k varies in $\{1, \ldots, n-1\}$ and n grows to infinity. In particular, under Assumption (E3), the first and last gaps converge to (nondegenerate) exponentially distributed random variable with respective parameters $\frac{2}{\sigma^2}b(0)$ and $-\frac{2}{\sigma^2}b(1)$, while the k_n-th gap, with $k_n/n \to u \in (0,1)$, converges to 0 at rate $\frac{2n}{\sigma^2}B(u)$.

As a possible extension of the present work, it is very natural to address similar questions for models with a nonconstant diffusion coefficient σ^2. On the one hand, the long time behaviour of the associated nonlinear diffusion process was studied in [11] and the result is a straightforward extension of Proposition 1.2, up to replacing the definition of the function Φ with

$$\forall u \in (0,1), \qquad \Phi(u) := \int_{v=0}^{u} \frac{\sigma^2(v)v}{2B(v)}\mathrm{d}v - \int_{v=u}^{1} \frac{\sigma^2(v)(1-v)}{2B(v)}\mathrm{d}v.$$

On the other hand, the stationary distribution of the projected particle system is in general not explicit, with the notable exception of the case of an affine variance function $\sigma^2(u) = \sigma_0^2 + \rho u$, with $\sigma_0^2 > 0$, $\sigma_0^2 + \rho > 0$. Then [2, Theorem 2, p. 622] implies that Proposition 1.1 holds true with a stationary density $\hat{p}^n_\infty$ with respect to the surface measure on M_n, defined by

$$\hat{p}^n_\infty(z) := \frac{1}{\hat{\mathcal{Z}}_n} \exp\left(\frac{2}{\sigma_0^2}\sum_{k=1}^{n} \hat{b}_n(k) z_{(k)}\right), \qquad \hat{b}_n(k) := n\left(\frac{B(\frac{k}{n})}{1 + \frac{\rho}{\sigma_0^2}\frac{k+1/2}{n}} - \frac{B(\frac{k-1}{n})}{1 + \frac{\rho}{\sigma_0^2}\frac{k-1/2}{n}}\right).$$

Since $\hat{b}_n(k)$ cannot be written as the increment on $[\frac{k-1}{n}, \frac{k}{n}]$ of some function $\hat{B}$, the extension of Theorem 1.5 to this case is expected to require more involved algebra than in the present paper. We will therefore not address this extension, but leave it open for future research. Further generalisations of our results to more elaborated models could also include hybrid Atlas models [2], systems with asymmetric collisions [13], and general systems of competing particles [17, 18].

1.5 Outline of the article

In Section 2, we give an explicit expression of the Laplace transform of P_∞ and $P^{2,n}_\infty$. In Section 3, we use these expressions to prove the weak convergence of $P^{2,n}_\infty$ to the product measure $(P_\infty)^{\otimes 2}$, thereby obtaining chaoticity for pairs of particles. It is a well-known fact that this result is enough to ensure chaoticity for collections of k particles for arbitrary $k \geq 2$, and the strengthening of this result in Wasserstein distance leading to Theorem 1.5 is detailed in Section 4.

1.6 Notations

Throughout the article, we use the following notations: for all $s, t \in \mathbb{R}$, $s \wedge t := \min\{s,t\}$, $s \vee t := \max\{s,t\}$, $[s]^- := 0 \vee (-s)$ and $[s]^+ := 0 \vee s$. Besides, $\lfloor s \rfloor$ denotes the integer part of s.

2 Expression of the Laplace transforms

In this section, we obtain explicit expressions for the Laplace transforms of P_∞ and $P^{2,n}_\infty$. These expressions are respectively derived in Subsections 2.1 and 2.2.

2.1 Expression of $L_\infty(r)$

For all $r \in \mathbb{R}$, we denote by

$$L_\infty(r) := \int_{x\in\mathbb{R}} \exp(rx) P_\infty(\mathrm{d}x)$$

the Laplace transform of P_∞. Since the inverse of the cumulative distribution function F_∞ of P_∞ is Φ, $L_\infty(r)$ has a natural expression in terms of the function Φ, namely

$$L_\infty(r) = \int_{u=0}^1 \exp(r\Phi(u))\mathrm{d}u.$$

In order to study the domain on which L_∞ is finite, we first recall a few properties of Φ.

Lemma 2.1. *Under Assumptions* (E1), (E2) *and* (E3), *the function* Φ *is* C^2 *and increasing on* $(0,1)$, *and satisfies*

$$\Phi(u) \sim \frac{\sigma^2}{2b(0)}\log(u) \quad \textit{when } u \downarrow 0, \qquad \Phi(u) \sim \frac{\sigma^2}{2b(1)}\log(1-u) \quad \textit{when } u \uparrow 1. \tag{2.1}$$

Besides, it is integrable on $[0,1]$ *and such that*

$$\int_{u=0}^1 \Phi(u)\mathrm{d}u = 0. \tag{2.2}$$

Proof. Assumptions (E1) and (E3) together with the continuity of b imply that

- when $u \downarrow 0$, $B(u) \sim b(0)u$, with $b(0) > 0$,
- when $u \uparrow 1$, $B(u) \sim -b(1)(1-u)$, with $b(1) < 0$,

and combining these estimates with Assumption (E2), we deduce that the integrals in the right-hand side of (1.4) are finite, and the function Φ is C^2 and increasing on $(0,1)$, and satisfies (2.1). The integrability of Φ on $[0,1]$ follows from (2.1) and the continuity of Φ on $(0,1)$, and by the Fubini-Tonelli theorem,

$$\int_{u=0}^1 \int_{v=0}^u \frac{v}{B(v)}\mathrm{d}v\mathrm{d}u = \int_{v=0}^1 \frac{v(1-v)}{B(v)}\mathrm{d}v = \int_{u=0}^1 \int_{v=u}^1 \frac{1-v}{B(v)}\mathrm{d}v\mathrm{d}u,$$

whence (2.2). □

As a consequence, the point (2.1) ensures that, under the assumptions of Lemma , as soon as r is taken in the set

$$\mathcal{V} := \{r \in \mathbb{R} : -2b(0)/\sigma^2 < r < -2b(1)/\sigma^2\},$$

then $L_\infty(r) < +\infty$. We note that $\mathcal{V}$ is an open subset of $\mathbb{R}$ and contains 0.

2.2 Expression of $L_\infty^{2,n}(s,t)$

For all $(s,t) \in \mathbb{R}^2$, we denote by

$$L_\infty^{2,n}(s,t) := \int_{z \in M_n} \exp(sz_1 + tz_2) p_\infty^n(z)\mathrm{d}z$$

the Laplace transform of $P_\infty^{2,n}$. Its expression is given in the next proposition.

Proposition 2.2. *Under Assumptions* (E1), (E2) *and* (E3), *for all* (s,t) *taken in the set*

$$\mathcal{V}_2 := \{(s,t) \in \mathcal{V} \times \mathcal{V} : s + t \in \mathcal{V}\},$$

there exists $n_0 \geq 2$ *such that, for all* $n \geq n_0$, $L_\infty^{2,n}(s,t)$ *is finite and writes*

$$L_\infty^{2,n}(s,t) = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i} J_{i,j}^n(s,t),$$

where, for all $1 \leq i < j \leq n$*,* $J^n_{i,j}(s,t)$ *is defined by*

$$\prod_{k=1}^{i-1} \frac{1}{1-(s+t)\frac{\sigma^2}{2n}\frac{k/n}{B(k/n)}} \prod_{k=i}^{j-1} \frac{1}{1-t\frac{\sigma^2}{2n}\frac{k/n}{B(k/n)}+s\frac{\sigma^2}{2n}\frac{1-k/n}{B(k/n)}} \prod_{k=j}^{n-1} \frac{1}{1+(s+t)\frac{\sigma^2}{2n}\frac{1-k/n}{B(k/n)}},$$

while, for all $1 \leq j < i \leq n$*,* $J^n_{i,j}(s,t)$ *is defined by*

$$\prod_{k=1}^{j-1} \frac{1}{1-(s+t)\frac{\sigma^2}{2n}\frac{k/n}{B(k/n)}} \prod_{k=j}^{i-1} \frac{1}{1-s\frac{\sigma^2}{2n}\frac{k/n}{B(k/n)}+t\frac{\sigma^2}{2n}\frac{1-k/n}{B(k/n)}} \prod_{k=i}^{n-1} \frac{1}{1+(s+t)\frac{\sigma^2}{2n}\frac{1-k/n}{B(k/n)}}.$$

We note that $\mathcal{V}_2$ is an open subset of $\mathbb{R}^2$ and contains 0.

The proof of Proposition 2.2 is provided in §2.2.2. We first collect preliminary estimates in §2.2.1.

2.2.1 Preliminary estimates

Under the assumptions of Proposition 2.2, for all $r \in \mathcal{V}$, for all $n \geq 1$, for all $k \in \{1,\ldots,n-1\}$, let us define

$$f^+_{k,n}(r) := r\frac{\sigma^2}{2n}\frac{k/n}{B(k/n)}, \qquad f^-_{k,n}(r) := -r\frac{\sigma^2}{2n}\frac{1-k/n}{B(k/n)},$$

so that the quantities $J^n_{i,j}(s,t)$ introduced in Proposition 2.2 rewrite

$$J^n_{i,j}(s,t) := \prod_{k=1}^{i-1} \frac{1}{1-f^+_{k,n}(s+t)} \prod_{k=i}^{j-1} \frac{1}{1-f^+_{k,n}(t)-f^-_{k,n}(s)} \prod_{k=j}^{n-1} \frac{1}{1-f^-_{k,n}(s+t)} \qquad \text{if } i<j,$$

$$J^n_{i,j}(s,t) := \prod_{k=1}^{j-1} \frac{1}{1-f^+_{k,n}(s+t)} \prod_{k=j}^{i-1} \frac{1}{1-f^+_{k,n}(s)-f^-_{k,n}(t)} \prod_{k=i}^{n-1} \frac{1}{1-f^-_{k,n}(s+t)} \qquad \text{if } i>j.$$

In this paragraph, we exhibit upper bounds on the quantities $f^+_{k,n}(r)$ and $f^-_{k,n}(r)$, for $r \in \{s,t,s+t\}$, which ensure that the quantities $J^n_{i,j}(s,t)$ are well defined for n large enough. We roughly proceed as follows: when k/n is far from 1, then $\frac{k/n}{B(k/n)}$ remains bounded by above, so that $f^+_{k,n}(r)$ is arbitrarily small for n large enough. On the contrary, when k/n is close to 1, then

$$\frac{k/n}{B(k/n)} \simeq -\frac{1}{b(1)(1-k/n)},$$

so that

$$f^+_{k,n}(r) \simeq -r\frac{\sigma^2}{2b(1)}\frac{1}{n-k},$$

and the fact that $r \in \mathcal{V}$ provides natural bounds on the right-hand side. The same ideas allow to obtain similar bounds on $f^-_{k,n}(r)$.

We now give a rigorous formulation of these arguments. Under Assumption (E3), for all $\epsilon > 0$ such that $\epsilon < b(0) \wedge (-b(1))$, we introduce

$$\mathcal{V}^\epsilon := \{r \in \mathbb{R} : -2(b(0)-\epsilon)/\sigma^2 < r < 2(-b(1)-\epsilon)/\sigma^2\},$$
$$\mathcal{V}^\epsilon_2 := \{(s,t) \in \mathcal{V}^\epsilon \times \mathcal{V}^\epsilon : s+t \in \mathcal{V}^\epsilon\}.$$

Similary to $\mathcal{V}$ and $\mathcal{V}_2$, the sets $\mathcal{V}^\epsilon$ and $\mathcal{V}^\epsilon_2$ are open subsets of $\mathbb{R}$ and $\mathbb{R}^2$ respectively, and both contain 0. For all $r \in \mathcal{V}^\epsilon$, we define

$$\alpha_+(r) := \frac{[r]^+\sigma^2}{2(-b(1)-\epsilon)} \in [0,1), \qquad \alpha_-(r) := \frac{[r]^-\sigma^2}{2(b(0)-\epsilon)} \in [0,1).$$

For $\delta \in (0,1/2)$ small enough and depending on ϵ,

- for all $u \in [0,\delta]$, $B(u) \geq u(b(0)-\epsilon)$,
- for all $u \in [1-\delta,1]$, $B(u) \geq (1-u)(-b(1)-\epsilon)$.

Besides, by Assumptions (E1), (E2) and (E3), the functions $u \mapsto B(u)/u$ and $u \mapsto B(u)/(1-u)$ are positive on the respective intervals $(0,1-\delta]$ and $[\delta,1)$, and have a positive limit at the respective points 0 and 1. Therefore we have

$$m_-(\delta) := \inf_{u\in[0,1-\delta]} \frac{B(u)}{u} > 0, \qquad m_+(\delta) := \inf_{u\in[\delta,1]} \frac{B(u)}{1-u} > 0.$$

The heuristic arguments detailed at the beginning of the subsection translate into the following precise estimates: fixing $(s,t) \in \mathcal{V}_2^\epsilon$, we have, for all $r \in \{s,t,s+t\}$, for all $n \geq 1$, for all $k \in \{1,\dots,n-1\}$,

$$f^+_{k,n}(r) = r\frac{\sigma^2}{2n}\frac{k/n}{B(k/n)} \leq \begin{cases} \dfrac{[r]^+\sigma^2}{2nm_-(\delta)} & \text{if } k < n(1-\delta), \\ \dfrac{[r]^+\sigma^2}{2(n-k)}\dfrac{k/n}{-b(1)-\epsilon} \leq \dfrac{\alpha_+(r)}{n-k} & \text{if } k \geq n(1-\delta). \end{cases} \tag{2.3}$$

Similarly,

$$f^-_{k,n}(r) = -r\frac{\sigma^2}{2n}\frac{1-k/n}{B(k/n)} \leq \begin{cases} \dfrac{[r]^-\sigma^2}{2nm_+(\delta)} & \text{if } k > n\delta, \\ \dfrac{[r]^-\sigma^2}{2k}\dfrac{1-k/n}{b(0)-\epsilon} \leq \dfrac{\alpha_-(r)}{k} & \text{if } k \leq n\delta. \end{cases} \tag{2.4}$$

In particular, if n is chosen so that

$$\frac{[s+t]^+\sigma^2}{2nm_-(\delta)} \leq \frac{1}{2},$$

then we deduce from (2.3) that, for all $k \in \{1,\dots,n-1\}$,

$$f^+_{k,n}(s+t) \leq \begin{cases} 1/2 & \text{if } k < n(1-\delta), \\ \alpha_+(s+t) & \text{if } k \geq n(1-\delta). \end{cases}$$

Similarly, if n is chosen so that

$$\frac{[t]^+\sigma^2}{2nm_-(\delta)} \leq \frac{1-\alpha_-(s)}{2}, \quad \frac{[t]^+\sigma^2}{2nm_-(\delta)} + \frac{[s]^-\sigma^2}{2nm_+(\delta)} \leq \frac{1}{2}, \quad \frac{[s]^-\sigma^2}{2nm_+(\delta)} \leq \frac{1-\alpha_+(t)}{2},$$

then we deduce from (2.3) and (2.4) that, for all $k \in \{1,\dots,n-1\}$,

$$f^+_{k,n}(t) + f^-_{k,n}(s) \leq \begin{cases} (\alpha_-(s)+1)/2 & \text{if } k \leq n\delta, \\ 1/2 & \text{if } n\delta < k < n(1-\delta), \\ (\alpha_+(t)+1)/2 & \text{if } k \geq n(1-\delta). \end{cases}$$

These results are gathered together in the following lemma.

Lemma 2.3. *Let $(s,t) \in \mathcal{V}_2$. Under the assumptions of Proposition 2.2, there exists $\epsilon > 0$ such that $(s,t) \in \mathcal{V}_2^\epsilon$. Let $\delta \in (0,1/2)$ depending on ϵ and satisfying the conditions above. Let us define $\bar\alpha \in [1/2,1)$ by*

$$\bar\alpha := \max\left\{\frac{1}{2}, \alpha_+(s+t), \alpha_-(s+t), \frac{\alpha_+(s)+1}{2}, \frac{\alpha_-(s)+1}{2}, \frac{\alpha_+(t)+1}{2}, \frac{\alpha_-(t)+1}{2}\right\}.$$

Then, there exists $n_0 \geq 2$ such that, for all $n \geq n_0$, for all $k \in \{1,\dots,n-1\}$

$$\max\left\{f^+_{k,n}(s+t), f^+_{k,n}(t)+f^-_{k,n}(s), f^+_{k,n}(s)+f^-_{k,n}(t), f^-_{k,n}(s+t)\right\} \leq \bar\alpha.$$

2.2.2 Computation of $L_\infty^{2,n}(s,t)$

Let us first note that, since $p_\infty^n(z)\mathrm{d}z$ is a symmetric probability distribution on $\mathbb{R}^n$, then for all symmetric and nonnegative function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{z\in M_n} f(z_1,\dots,z_n)p_\infty^n(z_1,\dots,z_n)\mathrm{d}z = \int_{z\in M_n} f(z_1,\dots,z_n)\tilde{p}_\infty^n(z_1,\dots,z_n)\mathrm{d}z,$$

where, for all $z=(z_1,\dots,z_n)\in M_n$,

$$\tilde{p}_\infty^n(z_1,\dots,z_n) = n!\mathbb{1}_{\{z_1\le\cdots\le z_n\}}\frac{1}{\mathcal{Z}_n}\exp\left(\frac{2}{\sigma^2}\sum_{k=1}^n b_n(k)z_k\right).$$

Using the symmetry of $p_\infty^n(z)\mathrm{d}z$ again, we deduce that, for all $(s,t)\in\mathcal{V}_2$,

$$L_\infty^{2,n}(s,t) = \frac{1}{n(n-1)}\sum_{i=1}^n\sum_{j\neq i}\int_{z\in M_n}\exp(sz_i+tz_j)\tilde{p}_\infty^n(z)\mathrm{d}z.$$

Let us now fix $i\in\{1,\dots,n\}$ and $j\neq i$, and define

$$J_{i,j}^n(s,t) := \int_{z\in M_n}\exp(sz_i+tz_j)\tilde{p}_\infty^n(z)\mathrm{d}z.$$

Note that, at this stage, nothing prevents $J_{i,j}^n(s,t)$ from being infinite. We use the parametrisation of M_n by the $n-1$ coordinates

$$x_1 = z_1,\dots,x_{i-1}=z_{i-1},x_{i+1}=z_{i+1},\dots,x_n=z_n.$$

Then M_n is defined by $x_i = F(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ in $\mathbb{R}^n$, with

$$F(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) := -(x_1+\cdots+x_{i-1}+x_{i+1}+\cdots+x_n),$$

so that the surface measure writes

$$\begin{aligned}\mathrm{d}z &= \sqrt{1+|\nabla F(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)|^2}\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_n\\ &= \sqrt{n}\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_n,\end{aligned}$$

therefore we obtain

$$\begin{aligned}J_{i,j}^n(s,t) = \frac{n!\sqrt{n}}{\mathcal{Z}_n}&\int_{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)\in\mathbb{R}^{n-1}}\exp\left(tx_j+\sum_{k\neq i}\left(\frac{2}{\sigma^2}(b_n(k)-b_n(i))-s\right)x_k\right)\\ &\times\mathbb{1}_{\{x_1\le\cdots\le x_{i-1}\le-(x_1+\cdots+x_{i-1}+x_{i+1}+\cdots+x_n)\le x_{i+1}\le\cdots\le x_n\}}\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_n.\end{aligned}$$

We denote $S := x_1+\cdots+x_{i-1}+x_{i+1}+\cdots+x_n$, and let $y_k := x_k+S$, for all $k\neq i$, in the right-hand side above. Then we have

$$\mathbb{1}_{\{x_1\le\cdots\le x_{i-1}\le-(x_1+\cdots+x_{i-1}+x_{i+1}+\cdots+x_n)\le x_{i+1}\le\cdots\le x_n\}} = \mathbb{1}_{\{y_1\le\cdots\le y_{i-1}\le 0\le y_{i+1}\le\cdots\le y_n\}},$$

and

$$\begin{aligned}tx_j+\sum_{k\neq i}\left(\frac{2}{\sigma^2}(b_n(k)-b_n(i))-s\right)x_k &= ty_j+\sum_{k\neq i}\left(\frac{2}{\sigma^2}(b_n(k)-b_n(i))-s\right)y_k\\ &\quad - S\left(t+\sum_{k\neq i}\left(\frac{2}{\sigma^2}(b_n(k)-b_n(i))-s\right)\right).\end{aligned}$$

Note that $S = \sum_{k\neq i}(y_k - S)$, which implies $S = \frac{1}{n}\sum_{k\neq i} y_k$. Besides,

$$\sum_{k\neq i}\left(\frac{2}{\sigma^2}(b_n(k)-b_n(i)) - s\right) = \sum_{k\neq i}\frac{2}{\sigma^2}b_n(k) - (n-1)\frac{2}{\sigma^2}b_n(i) - (n-1)s$$
$$= -n\frac{2}{\sigma^2}b_n(i) - (n-1)s,$$

since $\sum_{k=1}^n b_n(k) = n\bar{b} = 0$ by Assumption (E1). As a consequence,

$$tx_j + \sum_{k\neq i}\left(\frac{2}{\sigma^2}(b_n(k)-b_n(i)) - s\right)x_k = \sum_{k\neq i}\gamma_{i,j}^{k,n}(s,t)y_k,$$

where

$$\gamma_{i,j}^{k,n}(s,t) := -\frac{s+t}{n} + \frac{2}{\sigma^2}b_n(k) + \mathbb{1}_{\{k=j\}}t.$$

As a conclusion,

$$\begin{aligned} J_{i,j}^n(s,t) &= \frac{n!\sqrt{n}}{n\mathcal{Z}_n}\int_{(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_n)\in\mathbb{R}^{n-1}} \exp\left(\sum_{k\neq i}\gamma_{i,j}^{k,n}(s,t)y_k\right) \\ &\quad \times \mathbb{1}_{\{y_1\leq\cdots\leq y_{i-1}\leq 0\leq y_{i+1}\leq\cdots\leq y_n\}}\mathrm{d}y_1\cdots\mathrm{d}y_{i-1}\mathrm{d}y_{i+1}\cdots\mathrm{d}y_n \\ &= \frac{n!\sqrt{n}}{n\mathcal{Z}_n}J_{i,j}^{-,n}(s,t)J_{i,j}^{+,n}(s,t), \end{aligned}$$

where $J_{i,j}^{-,n}(s,t)$ is defined by

$$\int_{y_{i-1}=-\infty}^{0}\int_{y_{i-2}=-\infty}^{y_{i-1}}\cdots\int_{y_1=-\infty}^{y_2}\exp(\gamma_{i,j}^{1,n}(s,t)y_1+\cdots+\gamma_{i,j}^{i-1,n}(s,t)y_{i-1})\mathrm{d}y_1\cdots\mathrm{d}y_{i-1},$$

and $J_{i,j}^{+,n}(s,t)$ is defined by

$$\int_{y_{i+1}=0}^{+\infty}\int_{y_{i+2}=y_{i+1}}^{+\infty}\cdots\int_{y_n=y_{n-1}}^{+\infty}\exp(\gamma_{i,j}^{n,n}(s,t)y_n+\cdots+\gamma_{i,j}^{i+1,n}(s,t)y_{i+1})\mathrm{d}y_n\cdots\mathrm{d}y_{i+1}.$$

Let $n_0 \geq 2$ and $\bar{\alpha} \in [1/2, 1)$ be given by Lemma 2.3. We deduce from the definition of $\gamma_{i,j}^{k,n}(s,t)$ that, if $n \geq n_0$, then, for all $i,j \in \{1,\ldots,n\}$ such that $i \neq j$, for all $k \in \{1,\ldots,i-1\}$,

$$\begin{aligned} \gamma_{i,j}^{1,n}(s,t)+\cdots+\gamma_{i,j}^{k,n}(s,t) &= -(s+t)\frac{k}{n} + \frac{2n}{\sigma^2}B\left(\frac{k}{n}\right) \\ &= \frac{2n}{\sigma^2}B\left(\frac{k}{n}\right)(1 - f_{k,n}^+(s+t)) \\ &\geq \frac{2n}{\sigma^2}B\left(\frac{k}{n}\right)(1-\bar{\alpha}) > 0, \end{aligned}$$

where Assumptions (E1) and (E2) yield $B(\frac{k}{n}) > 0$; similarly, for all $k \in \{i+1,\ldots,n\}$,

$$\begin{aligned} \gamma_{i,j}^{k,n}(s,t)+\cdots+\gamma_{i,j}^{n,n}(s,t) &= -(s+t)\frac{n-k+1}{n} + t - \frac{2n}{\sigma^2}B\left(\frac{k-1}{n}\right) \\ &= -\frac{2n}{\sigma^2}B\left(\frac{k-1}{n}\right)(1 - f_{k-1,n}^+(t) - f_{k-1,n}^-(s)) \\ &\leq -\frac{2n}{\sigma^2}B\left(\frac{k-1}{n}\right)(1-\bar{\alpha}) < 0, \end{aligned}$$

which ensures that $J^{-;n}_{i,j}(s,t)$ and $J^{+;n}_{i,j}(s,t)$ are finite. By successive integrations, we obtain

$$J^{-;n}_{i,j}(s,t) = \prod_{k=1}^{i-1} \frac{1}{\gamma^{1,n}_{i,j}(s,t) + \cdots + \gamma^{k,n}_{i,j}(s,t)}, \quad J^{+;n}_{i,j}(s,t) = \prod_{k=i+1}^{n} \frac{-1}{\gamma^{n,n}_{i,j}(s,t) + \cdots + \gamma^{k,n}_{i,j}(s,t)},$$

which finally gives the expression

$$\frac{n!\sqrt{n}}{n\mathcal{Z}_n} \prod_{k=1}^{i-1} \frac{1}{\frac{-(s+t)k}{n} + \frac{2n}{\sigma^2}B(\frac{k}{n})} \prod_{k=i}^{j-1} \frac{1}{\frac{-tk+s(n-k)}{n} + \frac{2n}{\sigma^2}B(\frac{k}{n})} \prod_{k=j}^{n-1} \frac{1}{\frac{(s+t)(n-k)}{n} + \frac{2n}{\sigma^2}B(\frac{k}{n})}$$

for $J^n_{i,j}(s,t)$ if $i < j$, and

$$\frac{n!\sqrt{n}}{n\mathcal{Z}_n} \prod_{k=1}^{j-1} \frac{1}{\frac{-(s+t)k}{n} + \frac{2n}{\sigma^2}B(\frac{k}{n})} \prod_{k=j}^{i-1} \frac{1}{\frac{-sk+t(n-k)}{n} + \frac{2n}{\sigma^2}B(\frac{k}{n})} \prod_{k=i}^{n-1} \frac{1}{\frac{(s+t)(n-k)}{n} + \frac{2n}{\sigma^2}B(\frac{k}{n})}$$

if $i > j$.

To complete the proof, we remark that

$$L^{2,n}_{\infty}(0,0) = 1 = \frac{n!\sqrt{n}}{n\mathcal{Z}_n} \prod_{k=1}^{n-1} \frac{1}{\frac{2n}{\sigma^2}B(\frac{k}{n})},$$

which allows us to get rid of the constant term $\frac{n!\sqrt{n}}{n\mathcal{Z}_n}$ and to obtain the expected expression of $J^n_{i,j}(s,t)$ in Proposition 2.2, for $(s,t) \in \mathcal{V}_2$.

3 Convergence of the Laplace transforms

This section is dedicated to the proof of the following result.

Proposition 3.1. *Under the assumptions of Proposition 2.2, for all* $(s,t) \in \mathcal{V}_2$*,*

$$\lim_{n\to+\infty} L^{2,n}_{\infty}(s,t) = L_{\infty}(s)L_{\infty}(t).$$

We proceed in two steps: first, we prove that, for all $t \in \mathcal{V}$, the Laplace transform $L^{1,n}_{\infty}(t) = L^{2,n}_{\infty}(t,0)$ of $P^{1,n}_{\infty}$ converges to $L_{\infty}(t)$. Second, we check that, for $(s,t) \in \mathcal{V}_2$, the difference between $L^{2,n}_{\infty}(s,t)$ and the product $L^{1,n}_{\infty}(s)L^{1,n}_{\infty}(t)$ vanishes. These two steps are addressed in the respective Subsections 3.2 and 3.3. The preliminary Subsection 3.1 gathers useful elementary results.

3.1 Elementary inequalities

We shall use the following inequalities, which are elementary consequences of the Taylor-Lagrange inequality.

(TL1) For all $\alpha \in [0,1)$, for all $x \in [-\alpha,+\infty)$, $|\log(1+x) - x| \leq \kappa(\alpha)x^2$, where $\kappa(\alpha) := \frac{1}{2(1-\alpha)^2}$.

(TL2) For all $x, y \in \mathbb{R}$, $|\exp(x) - \exp(y)| \leq \exp(y)\left(|x-y| + |R(x-y)|\right)$, where the function $R : z \mapsto \exp(z) - 1 - z$ is such that, for all $C \in [0,+\infty)$, for all $z \in [-C, C]$, $|R(z)| \leq \frac{1}{2}\exp(C)z^2$.

We also recall that, for all $C \in [0,+\infty)$,

$$\lim_{n\to+\infty} \left(1 - \frac{C}{n}\right)^{-n} = \exp(C). \tag{3.1}$$

3.2 Convergence of $L_\infty^{1,n}(t)$

Let us fix $t \in \mathcal{V}$. By the results of Section 2, there exists $n_0 \geq 2$ such that, for all $n \geq n_0$, the Laplace transform $L_\infty^{1,n}(t) = L_\infty^{2,n}(t,0)$ of $p_\infty^{1,n}$ is finite and writes

$$L_\infty^{1,n}(t) = \frac{1}{n}\sum_{i=1}^n I_i^n(t), \qquad I_i^n(t) := \prod_{k=1}^{i-1} \frac{1}{1 - t\frac{\sigma^2}{2n}\frac{k/n}{B(k/n)}} \prod_{k=i}^{n-1} \frac{1}{1 + t\frac{\sigma^2}{2n}\frac{1-k/n}{B(k/n)}}.$$

Let us briefly explain the heuristics of the proof. For $u \in (0,1)$ and $i \simeq nu$,

$$\begin{aligned}\log I_i^n(t) &= -\sum_{k=1}^{i-1}\log\left(1 - t\frac{\sigma^2}{2n}\frac{k/n}{B(k/n)}\right) - \sum_{k=i}^{n-1}\log\left(1 + t\frac{\sigma^2}{2n}\frac{1-k/n}{B(k/n)}\right)\\ &\simeq \frac{t\sigma^2}{2n}\sum_{k=1}^{n-1} \mathbb{1}_{\{k/n \leq u\}}\frac{k/n}{B(k/n)} + \mathbb{1}_{\{k/n \geq u\}}\frac{1-k/n}{B(k/n)},\end{aligned}$$

and by a Riemann sum argument, the right-hand side converges to $t\Phi(u)$. We deduce that

$$L_\infty^{1,n}(t) = \frac{1}{n}\sum_{i=1}^n I_i^n(t) \simeq \frac{1}{n}\sum_{i=1}^n \exp\left(t\Phi\left(\frac{i}{n}\right)\right),$$

which now converges to $L_\infty(t)$ thanks to a second Riemann sum argument.

To make the whole argument rigorous, we first write

$$|L_\infty^{1,n}(t) - L_\infty(t)| \leq \sum_{i=1}^n \int_{u=\frac{i-1}{n}}^{\frac{i}{n}} |I_i^n(t) - \exp(t\Phi(u))|\mathrm{d}u. \tag{3.2}$$

Let $\epsilon > 0$ and $\delta \in (0,1/2)$ be given by Lemma 2.3 for the pair $(t,0) \in \mathcal{V}_2$. We split the sum appearing in the right-hand side of (3.2) into *boundary terms*, corresponding to $i \leq n\delta$ and $i \geq n(1-\delta)$, and a *central term*, corresponding to $n\delta < i < n(1-\delta)$. These terms are addressed separately, in the respective §3.2.1 and §3.2.2.

3.2.1 Boundary terms

For all $n \geq n_0$,

$$\sum_{i \leq n\delta}\int_{u=\frac{i-1}{n}}^{\frac{i}{n}} |I_i^n(t) - \exp(t\Phi(u))|\mathrm{d}u \leq \frac{1}{n}\sum_{i=1}^{\lfloor n\delta\rfloor} I_i^n(t) + \int_{u=0}^{\delta}\exp(t\Phi(u))\mathrm{d}u.$$

It is an easy consequence of the result of Subsection 2.1 that the integral in the right-hand side above vanishes with δ. The purpose of this paragraph is to show that

$$\lim_{\delta\downarrow 0}\limsup_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{\lfloor n\delta\rfloor} I_i^n(t) = 0. \tag{3.3}$$

Let us first assume that $t \geq 0$. Then, for all $i \leq n\delta$,

$$\prod_{k=i}^{n-1}\frac{1}{1 + t\frac{\sigma^2}{2n}\frac{1-k/n}{B(k/n)}} \leq 1.$$

We now use the fact that, if $i \leq n\delta$, then for all $k \in \{1,\ldots,i-1\}$, $\frac{k/n}{B(k/n)} \leq \frac{1}{b(0)-\epsilon}$, to write

$$\prod_{k=1}^{i-1}\frac{1}{1 - t\frac{\sigma^2}{2n}\frac{k/n}{B(k/n)}} \leq \left(1 - \frac{t\sigma^2}{2n(b(0)-\epsilon)}\right)^{-(i-1)} \leq \left(1 - \frac{t\sigma^2}{2n(b(0)-\epsilon)}\right)^{-n},$$

as soon as n is large enough to ensure that $t\sigma^2/(2n(b(0)-\epsilon)) < 1$. Using (3.1), we deduce that

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{\lfloor n\delta\rfloor} I_i^n(t) \leq \delta \exp\left(\frac{t\sigma^2}{2(b(0)-\epsilon)}\right),$$

and (3.3) easily follows.

Let us now assume that $t<0$. Then, for $i \leq n\delta$, we still have the rough bound

$$\prod_{k=1}^{i-1} \frac{1}{1-t\frac{\sigma^2}{2n}\frac{k/n}{B(k/n)}} \leq 1,$$

but we need to be more careful as far as the product

$$\prod_{i=k}^{n-1} \frac{1}{1+t\frac{\sigma^2}{2n}\frac{1-k/n}{B(k/n)}} = \prod_{i=k}^{n-1} \frac{1}{1-f^-_{k,n}(t)}$$

is concerned. On the one hand,

$$\log\left(\prod_{k=\lfloor n\delta\rfloor+1}^{n-1} \frac{1}{1-f^-_{k,n}(t)}\right) = -\sum_{k=\lfloor n\delta\rfloor+1}^{n-1} \log\left(1-f^-_{k,n}(t)\right),$$

and combining Lemma 2.3 with the inequality (TL1) yields

$$-\sum_{k=\lfloor n\delta\rfloor+1}^{n-1} \log\left(1-f^-_{k,n}(t)\right) \leq \sum_{k=\lfloor n\delta\rfloor+1}^{n-1} \left(f^-_{k,n}(t) + \kappa(\bar{\alpha})\left(f^-_{k,n}(t)\right)^2\right).$$

Since the definition of $f^-_{k,n}(t)$ yields

$$\sum_{k=\lfloor n\delta\rfloor+1}^{n-1} f^-_{k,n}(t) = -\frac{t\sigma^2}{2n}\sum_{k=1}^{n-1} \mathbb{1}_{\{k/n>\delta\}}\frac{1-k/n}{B(k/n)}$$

while (2.4) implies

$$\sum_{k=\lfloor n\delta\rfloor+1}^{n-1} \left(f^-_{k,n}(t)\right)^2 \leq \frac{1}{n}\left(\frac{t\sigma^2}{2m_+(\delta)}\right)^2,$$

we deduce that

$$\lim_{n\to+\infty} \log\left(\prod_{k=\lfloor n\delta\rfloor+1}^{n-1} \frac{1}{1-f^-_{k,n}(t)}\right) = -\frac{t\sigma^2}{2}\int_{v=\delta}^{1} \frac{1-v}{B(v)}\mathrm{d}v.$$

On the other hand, (2.4) gives

$$\prod_{k=i}^{\lfloor n\delta\rfloor} \frac{1}{1-f^-_{k,n}(t)} \leq \prod_{k=i}^{\lfloor n\delta\rfloor} \frac{1}{1-\alpha/k},$$

where $\alpha := \alpha_-(t) < 1$. Using (TL1) again, we write

$$\begin{aligned}\log\left(\prod_{k=i}^{\lfloor n\delta\rfloor} \frac{1}{1-\alpha/k}\right) = -\sum_{k=i}^{\lfloor n\delta\rfloor} \log\left(1-\frac{\alpha}{k}\right) &\leq \sum_{k=i}^{\lfloor n\delta\rfloor} \left(\frac{\alpha}{k} + \kappa(\alpha)\frac{\alpha^2}{k^2}\right) \\ &\leq \alpha\sum_{k=i}^{\lfloor n\delta\rfloor} \frac{1}{k} + \kappa(\alpha)\alpha^2\frac{\pi^2}{6} \\ &\leq \alpha\left(1+\log(n\delta)-\log(i)\right) + \kappa(\alpha)\alpha^2\frac{\pi^2}{6},\end{aligned}$$

so that
$$\prod_{k=i}^{\lfloor n\delta \rfloor} \frac{1}{1-f^-_{k,n}(t)} \leq K(\alpha)\delta^\alpha \frac{1}{(i/n)^\alpha},$$
where $K(\alpha) := \exp(\alpha + \kappa(\alpha)\alpha^2\pi^2/6)$. Since
$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{\lfloor n\delta \rfloor} \frac{1}{(i/n)^\alpha} = \int_{v=0}^{\delta} \frac{\mathrm{d}v}{v^\alpha} = \frac{\delta^{1-\alpha}}{1-\alpha},$$
we conclude that
$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{\lfloor n\delta \rfloor} I^n_i(t) \leq \frac{K(\alpha)}{1-\alpha}\delta \exp\left(-\frac{t\sigma^2}{2}\int_{v=\delta}^1 \frac{1-v}{B(v)}\mathrm{d}v\right) =: M(\delta).$$
To obtain (3.3), we now have to check that $M(\delta)$ vanishes with δ. To this aim, we fix $0 < \eta < b(0) \wedge (-b(1))$ such that $t \in \mathcal{V}^\eta$. Since the diverging integral $\int_{v=\delta}^1 \frac{1-v}{B(v)}\mathrm{d}v$ is equivalent to $-\log(\delta)/b(0)$ when δ vanishes, we deduce that, for δ small enough, we have
$$\int_{v=\delta}^1 \frac{1-v}{B(v)}\mathrm{d}v \leq \frac{-\log\delta}{b(0)-\eta},$$
so that
$$\exp\left(-\frac{t\sigma^2}{2}\int_{v=\delta}^1 \frac{1-v}{B(v)}\mathrm{d}v\right) \leq \delta^{-\beta}, \qquad \text{with } \beta := \frac{-t\sigma^2}{2(b(0)-\eta)} \in (0,1).$$
As a conclusion, $M(\delta)$ is of order $\delta^{1-\beta}$ when δ is small, whence (3.3).

The boundary term corresponding to $i \geq n(1-\delta)$ can be handled by symmetric arguments.

3.2.2 Central term

We now prove that
$$\lim_{\delta\downarrow 0}\limsup_{n\to+\infty} \sum_{i=\lfloor n\delta\rfloor+1}^{\lfloor n(1-\delta)\rfloor-1} \int_{u=\frac{i-1}{n}}^{\frac{i}{n}} |I^n_i(t) - \exp(t\Phi(u))|\mathrm{d}u = 0. \tag{3.4}$$
To this aim we fix $i \in \{\lfloor n\delta\rfloor+1, \ldots, \lfloor n(1-\delta)\rfloor-1\}$ and $u \in [\frac{i-1}{n}, \frac{i}{n}]$. By (TL2),
$$|I^n_i(t) - \exp(t\Phi(u))| \leq \exp(t\Phi(u))(|\Delta| + |R(\Delta)|),$$
where $\Delta := \Delta_1 + \Delta_2$, with
$$\Delta_1 := -\sum_{k=1}^{i-1} \log\left(1 - \frac{t\sigma^2}{2n}\frac{k/n}{B(k/n)}\right) - \frac{t\sigma^2}{2}\int_{v=0}^u \frac{v}{B(v)}\mathrm{d}v,$$
$$\Delta_2 := -\sum_{k=i}^{n-1} \log\left(1 + \frac{t\sigma^2}{2n}\frac{1-k/n}{B(k/n)}\right) + \frac{t\sigma^2}{2}\int_{v=0}^u \frac{1-v}{B(v)}\mathrm{d}v.$$
For all $k \in \{1,\ldots,i-1\}$, we deduce from Lemma 2.3, the inequality (TL1) and the estimate (2.3) that
$$\begin{aligned}|\Delta_1| &\leq \left|\sum_{k=1}^{i-1} \frac{t\sigma^2}{2n}\frac{k/n}{B(k/n)} - \frac{t\sigma^2}{2}\int_{v=0}^u \frac{v}{B(v)}\mathrm{d}v\right| + \frac{\kappa(\bar\alpha)}{n}\left(\frac{t\sigma^2}{2m_-(\delta)}\right)^2 \\ &\leq \frac{|t|\sigma^2}{2}\sum_{k=1}^{i-1}\int_{v=\frac{k-1}{n}}^{\frac{k}{n}} \left|\frac{k/n}{B(k/n)} - \frac{v}{B(v)}\right|\mathrm{d}v + \frac{|t|\sigma^2}{2}\int_{v=\frac{i-1}{n}}^u \frac{v}{B(v)}\mathrm{d}v + \frac{\kappa(\bar\alpha)}{n}\left(\frac{t\sigma^2}{2m_-(\delta)}\right)^2.\end{aligned}$$

Using the uniform continuity of $v/B(v)$ on $[0,1-\delta]$, we deduce that for n large enough, for all $k \in \{1,\dots,\lfloor n(1-\delta)\rfloor - 1\}$,

$$\forall v \in \left[\frac{k-1}{n}, \frac{k}{n}\right], \qquad \left|\frac{k/n}{B(k/n)} - \frac{v}{B(v)}\right| \leq \delta.$$

As a consequence,

$$|\Delta_1| \leq \frac{|t|\sigma^2}{2}\delta + \frac{1}{n}\left(\frac{|t|\sigma^2}{2m_-(\delta)} + \kappa(\bar{\alpha})\left(\frac{t\sigma^2}{2m_-(\delta)}\right)^2\right) =: M_1(n,\delta),$$

and we note that $M_1(n,\delta)$ does not depend on $i \in \{\lfloor n\delta\rfloor + 1, \dots, \lfloor n(1-\delta)\rfloor - 1\}$ and satisfies

$$\lim_{\delta\downarrow 0}\limsup_{n\to+\infty} M_1(n,\delta) = 0.$$

We similarly construct $M_2(n,\delta)$ satisfying the same conditions as $M_1(n,\delta)$ and such that $|\Delta_2| \leq M_2(n,\delta)$ for n large enough. As a consequence, for $\delta > 0$ small enough and n large enough, we have $|\Delta_1| + |\Delta_2| \leq 1$, so that (TL2) yields

$$|I_i^n(t) - \exp(t\Phi(u))| \leq \exp(t\Phi(u))(M_1(n,\delta) + M_2(n,\delta) + \frac{\exp(1)}{2}(M_1(n,\delta) + M_2(n,\delta))^2),$$

and finally

$$\begin{aligned}\sum_{i=\lfloor n\delta\rfloor+1}^{\lfloor n(1-\delta)\rfloor-1} \int_{u=\frac{i-1}{n}}^{\frac{i}{n}} &|I_i^n(t) - \exp(t\Phi(u))|\mathrm{d}u \\ &\leq \left(M_1(n,\delta) + M_2(n,\delta) + \frac{\exp(1)}{2}(M_1(n,\delta) + M_2(n,\delta))^2\right)\int_{u=0}^{1}\exp(t\Phi(u))\mathrm{d}u,\end{aligned}$$

which completes the proof of (3.4).

3.3 Convergence of $L_\infty^{2,n}(s,t) - L_\infty^{1,n}(s)L_\infty^{1,n}(t)$

Let $n_0 \geq 2$ be given by Lemma 2.3. Then for all $n \geq n_0$,

$$\begin{aligned}|L_\infty^{2,n}(s,t) - L_\infty^{1,n}(s)L_\infty^{1,n}(t)| &= \left|\frac{1}{n(n-1)}\sum_{i=1}^n\sum_{j\neq i} J_{i,j}^n(s,t) - \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n I_i^n(s)I_j^n(t)\right| \\ &\leq \frac{1}{n(n-1)}\sum_{i=1}^n\sum_{j\neq i}|J_{i,j}^n(s,t) - I_i^n(s)I_j^n(t)| \\ &\quad + \frac{1}{n(n-1)}\sum_{i=1}^n I_i^n(s)I_i^n(t) + \frac{1}{n-1}L_\infty^{1,n}(s)L_\infty^{1,n}(t).\end{aligned}$$

By the results of Subsection 3.2, the last term in the right-hand side above vanishes when n grows to infinity. The diagonal term $\frac{1}{n(n-1)}\sum_{i=1}^n I_i^n(s)I_i^n(t)$ is addressed in §3.3.1, and the main term $\frac{1}{n(n-1)}\sum_{i=1}^n\sum_{j\neq i}|J_{i,j}^n(s,t) - I_i^n(s)I_j^n(t)|$ is addressed in §3.3.2.

3.3.1 Diagonal term

In this paragraph, we prove that

$$\lim_{n\to+\infty}\frac{1}{n(n-1)}\sum_{i=1}^n I_i^n(s)I_i^n(t) = 0. \tag{3.5}$$

To this aim, we write, for all $i \in \{1, \ldots, n\}$,

$$I_i^n(s)I_i^n(t) = \prod_{k=1}^{i-1} \frac{1}{1 - \frac{s\sigma^2}{2n}\frac{k/n}{B(k/n)}} \frac{1}{1 - \frac{t\sigma^2}{2n}\frac{k/n}{B(k/n)}} \prod_{k=i}^{n-1} \frac{1}{1 + \frac{s\sigma^2}{2n}\frac{1-k/n}{B(k/n)}} \frac{1}{1 + \frac{t\sigma^2}{2n}\frac{1-k/n}{B(k/n)}},$$

and note that if $st \geq 0$, then $I_i^n(s)I_i^n(t) \leq I_i^n(s+t)$, so that (3.5) follows from the results of Subsection 3.2. On the other hand, if $st < 0$, say $s < 0 < t$, then

$$I_i^n(s)I_i^n(t) \leq \prod_{k=1}^{i-1} \frac{1}{1 - \frac{t\sigma^2}{2n}\frac{k/n}{B(k/n)}} \prod_{k=i}^{n-1} \frac{1}{1 + \frac{s\sigma^2}{2n}\frac{1-k/n}{B(k/n)}}.$$

Let us fix $\epsilon > 0$ and $\delta \in (0, 1/2)$ as in §2.2.1. Arguing as in §3.2.1, we obtain

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{\lfloor n\delta \rfloor} \prod_{k=1}^{i-1} \frac{1}{1 - \frac{t\sigma^2}{2n}\frac{k/n}{B(k/n)}} \prod_{k=i}^{n-1} \frac{1}{1 + \frac{s\sigma^2}{2n}\frac{1-k/n}{B(k/n)}} \leq \exp\left(\frac{t\sigma^2}{2(b(0)-\epsilon)}\right) M(\delta) < +\infty,$$

and the same arguments apply to the sum for $i \geq n(1-\delta)$. On the other hand, combining the estimates (2.3), (2.4) with (3.1) yields

$$\limsup_{n\to+\infty} \sum_{i=\lfloor n\delta \rfloor+1}^{\lfloor n(1-\delta) \rfloor-1} \prod_{k=1}^{i-1} \frac{1}{1 - \frac{t\sigma^2}{2n}\frac{k/n}{B(k/n)}} \prod_{k=i}^{n-1} \frac{1}{1 + \frac{s\sigma^2}{2n}\frac{1-k/n}{B(k/n)}} \leq \exp\left(\frac{t\sigma^2}{2m_-(\delta)} - \frac{s\sigma^2}{2m_+(\delta)}\right).$$

We deduce that

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} I_i^n(s)I_i^n(t) < +\infty,$$

whence (3.5).

3.3.2 Main term

In this paragraph, we finally check that

$$\lim_{n\to+\infty} \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j\neq i} |J_{i,j}^n(s,t) - I_i^n(s)I_j^n(t)| = 0. \tag{3.6}$$

By (TL2), we have, for all $i, j \in \{1, \ldots, n\}$ such that $i \neq j$,

$$|J_{i,j}^n(s,t) - I_i^n(s)I_j^n(t)| \leq I_i^n(s)I_j^n(t) \left(|\rho_{i,j}^n(s,t)| + |R(\rho_{i,j}^n(s,t))|\right),$$

where

$$\rho_{i,j}^n(s,t) := \log(J_{i,j}^n(s,t)) - \log(I_i^n(s)I_j^n(t)).$$

Remark that $\rho_{i,j}^n(s,t)$ writes as a sum, for $k \in \{1, \ldots, n-1\}$, of terms of the form

$$\log(1 - f_{k,n}^{\pm}(s) - f_{k,n}^{\pm}(t)) - \log(1 - f_{k,n}^{\pm}(s)) - \log(1 - f_{k,n}^{\pm}(t)),$$

where Lemma 2.3 ensures that each term $f_{k,n}^{\pm}(s)$, $f_{k,n}^{\pm}(t)$ and $f_{k,n}^{\pm}(s) + f_{k,n}^{\pm}(t)$ is lower than $\bar{\alpha} < 1$. As a consequence, (TL1) yields

$$\begin{aligned}
&|\log(1 - f_{k,n}^{\pm}(s) - f_{k,n}^{\pm}(t)) - \log(1 - f_{k,n}^{\pm}(s)) - \log(1 - f_{k,n}^{\pm}(t))| \\
&\quad \leq \kappa(\bar{\alpha})((f_{k,n}^{\pm}(s) + f_{k,n}^{\pm}(t))^2 + (f_{k,n}^{\pm}(s))^2 + (f_{k,n}^{\pm}(t))^2) \\
&\quad \leq 3\kappa(\bar{\alpha})((f_{k,n}^{\pm}(s))^2 + (f_{k,n}^{\pm}(t))^2),
\end{aligned}$$

hence $|\rho^n_{i,j}(s,t)| \leq 3\kappa(\bar{\alpha})\{F^n_i(s) + F^n_j(t)\}$, where

$$F^n_i(s) := \sum_{k=1}^{i-1} \left(f^-_{k,n}(s)\right)^2 + \sum_{k=i}^{n-1} \left(f^+_{k,n}(s)\right)^2 = \sum_{k=1}^{i-1} \left(\frac{s\sigma^2}{2n}\frac{k/n}{B(k/n)}\right)^2 + \sum_{k=i}^{n-1} \left(\frac{s\sigma^2}{2n}\frac{1-k/n}{B(k/n)}\right)^2.$$

We deduce from the estimates (2.3) and (2.4) that

$$\begin{aligned}F^n_i(s) \leq \frac{1}{n}&\left(\left(\frac{s\sigma^2}{2m_-(\delta)^2}\right)^2 + \left(\frac{s\sigma^2}{2m_+(\delta)^2}\right)^2\right)\\ &+ \frac{\pi^2}{6}\left(\left(\frac{|s|\sigma^2}{2(b(0)-\epsilon)}\right)^2 \mathbb{1}_{\{i\leq n\delta\}} + \left(\frac{|s|\sigma^2}{2(-b(1)-\epsilon)}\right)^2 \mathbb{1}_{\{i\geq n(1-\delta)\}}\right),\end{aligned}$$

so that that there exist a nonnegative and finite constant $M(\delta)$, that depends on δ, and a nonnegative and finite constant C, that does not depend on δ, such that

$$|\rho^n_{i,j}(s,t)| \leq \frac{M(\delta)}{n} + C\left(\mathbb{1}_{\{i\leq n\delta\}} + \mathbb{1}_{\{i\geq n(1-\delta)\}} + \mathbb{1}_{\{j\leq n\delta\}} + \mathbb{1}_{\{j\geq n(1-\delta)\}}\right).$$

For n large enough, the right-hand side above is lower that $3C$, so that (TL2) yields

$$\begin{aligned}|R(\rho^n_{i,j}(s,t))| &\leq \frac{\exp(3C)}{2}\left\{\frac{M(\delta)}{n} + C\left(\mathbb{1}_{\{i\leq n\delta\}} + \mathbb{1}_{\{i\geq n(1-\delta)\}} + \mathbb{1}_{\{j\leq n\delta\}} + \mathbb{1}_{\{j\geq n(1-\delta)\}}\right)\right\}^2\\ &\leq \frac{3}{2}\exp(3C)\left\{\frac{M(\delta)^2}{n^2} + C^2\left(\mathbb{1}_{\{i\leq n\delta\}} + \mathbb{1}_{\{i\geq n(1-\delta)\}}\right)^2 + C^2\left(\mathbb{1}_{\{j\leq n\delta\}} + \mathbb{1}_{\{j\geq n(1-\delta)\}}\right)^2\right\}\\ &= \frac{3}{2}\exp(3C)\left\{\frac{M(\delta)^2}{n^2} + C^2\left(\mathbb{1}_{\{i\leq n\delta\}} + \mathbb{1}_{\{i\geq n(1-\delta)\}} + \mathbb{1}_{\{j\leq n\delta\}} + \mathbb{1}_{\{j\geq n(1-\delta)\}}\right)\right\}.\end{aligned}$$

As a consequence, there exist a nonnegative and finite constant $M'(\delta)$, that depends on δ, and a nonnegative and finite constant C', that does not depend on δ, such that, for n large enough, for all $i \neq j$ in $\{1,\ldots,n\}$,

$$\begin{aligned}&|J^n_{i,j}(s,t) - I^n_i(s)I^n_j(t)|\\ &\quad\leq I^n_i(s)I^n_j(t)\left\{\frac{M'(\delta)}{n} + C'\left(\mathbb{1}_{\{i\leq n\delta\}} + \mathbb{1}_{\{i\geq n(1-\delta)\}} + \mathbb{1}_{\{j\leq n\delta\}} + \mathbb{1}_{\{j\geq n(1-\delta)\}}\right)\right\}.\end{aligned} \tag{3.7}$$

To complete the proof of (3.6), we now check that

$$\lim_{\delta\downarrow 0}\limsup_{n\to+\infty}\frac{1}{n(n-1)}\sum_{i=1}^n\sum_{j\neq i} I^n_i(s)I^n_j(t)\{\cdots\} = 0,$$

where $\{\cdots\}$ refers to the braced term in the right-hand side of (3.7). Note that, on account of the results of §3.3.1, it is equivalent to show that

$$\lim_{\delta\downarrow 0}\limsup_{n\to+\infty}\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n I^n_i(s)I^n_j(t)\{\cdots\} = 0.$$

On the one hand,

$$\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n I^n_i(s)I^n_j(t)\frac{M'(\delta)}{n} = \frac{M'(\delta)}{n}L^{1,n}_\infty(s)L^{1,n}_\infty(t)$$

vanishes when n grows to infinity. On the other hand,

$$\limsup_{n\to+\infty}\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n I^n_i(s)I^n_j(t)C'\mathbb{1}_{\{i\leq n\delta\}} = C'L_\infty(t)\limsup_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{\lfloor n\delta\rfloor} I^n_i(s),$$

and it was proved in §3.2.1 that the last term in the right-hand side vanishes with δ. Addressing the other boundary terms similarly, we obtain (3.6) and thereby complete the proof of Proposition 3.1.

4 Proof of Theorem 1.5

This last section contains the proof of Theorem 1.5.

Proof of Theorem 1.5. Since $\mathcal{V}_2$ is an open subset of $\mathbb{R}^2$ containing $(0,0)$, Proposition 3.1 implies the weak convergence of $P_\infty^{2;n}$ to the product measure $(P_\infty)^{\otimes 2}$. According to the proof of [20, Proposition 2.2, p. 177], this is enough to ensure the P_∞-chaoticity of the sequence $(P_\infty^n)_{n\geq 1}$.

Let us now establish the convergence in Wasserstein distance and fix $k \geq 1$, $q \in [1, +\infty)$. Following Remark 1.6 and [22, Theorem 6.9], to prove that $P_\infty^{k;n}$ converges to $(P_\infty)^{\otimes k}$ in Wasserstein distance of order q, it suffices to check that

$$\lim_{n\to+\infty} \mathbb{E}[|X_1^n|^q + \cdots + |X_k^n|^q] = \mathbb{E}[|X_1|^q + \cdots + |X_k|^q],$$

where, for all $n \geq 1$, $(X_1^n, \ldots, X_n^n)$ is distributed according to $P_\infty^{k;n}$, while $(X_1, \ldots, X_n)$ is distributed according to $(P_\infty)^{\otimes k}$. Using the linearity of the expectation and the symmetry of P_∞^n, we deduce that it is enough to check this result for $k = 1$. Then we already know that X_1^n converges in distribution to X_1, and we now check that the sequence of random variables $(|X_1^n|^q)_{n\geq 1}$ is uniformly integrable, which implies the convergence of $\mathbb{E}[|X_1^n|^q]$ and completes the proof.

To check the uniform integrability of the sequence $(|X_1^n|^q)_{n\geq 1}$, we fix $r > q$ and prove that the sequence $(\mathbb{E}[|X_1^n|^r])_{n\geq 1}$ is bounded. To this aim, we fix $\rho > 0$ such that $-\rho \in \mathcal{V}$ and $\rho \in \mathcal{V}$. Then, there exists $M \geq 0$ such that, for all $x \in \mathbb{R}$, $|x|^r \leq M + (\exp(-\rho x) + \exp(\rho x))$, so that, for all $n \geq 1$,

$$\mathbb{E}[|X_1^n|^r] \leq M + L_\infty^{2;n}(-\rho, 0) + L_\infty^{2;n}(\rho, 0),$$

and we deduce from Proposition 3.1 that the right-hand side converges to a finite value when n grows to infinity, which implies that the left-hand side is uniformly bounded with respect to n. $\square$

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