

On the existence of solutions of a class of SDEs with discontinuous drift and singular diffusion*

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Abstract

The classical result by Itô on the existence of strong solutions of stochastic differential equations (SDEs) with Lipschitz coefficients can be extended to the case where the drift is only measurable and bounded. These generalizations are based on techniques presented by Zvonkin and Veretennikov, which rely on the uniform ellipticity of the diffusion coefficient.

In this paper we study the case of degenerate ellipticity and give sufficient conditions for the existence of a solution. The conditions on the diffusion coefficient are more general than previous results and we gain fundamental insight into the geometric properties of the discontinuity of the drift on the one hand and the diffusion vector field on the other hand.

Besides presenting existence results for the degenerate elliptic situation, we give an example illustrating the difficulties in obtaining more general results than those given.

The particular types of SDEs considered arise naturally in the framework of combined optimal control and filtering problems.

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1 Introduction

In this article we are going to consider a d -dimensional time-homogeneous stochastic differential equation (SDE) of the form

$$dX_t = \mu(X_t) dt + \sigma(X_t) dW_t. \quad (1.1)$$

Itô's well-known existence and uniqueness theorem for SDEs states that for (locally) Lipschitz μ, σ there exists a unique (maximal local) strong solution, see, e.g., [7]. However, in this article, we are interested in the case where μ is not Lipschitz.

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Zvonkin [16] (for the one-dimensional case) and Veretennikov [13] (for the multi-dimensional case) prove that if σ is bounded and Lipschitz and the infinitesimal generator of the SDE is *uniformly elliptic*, i.e., if there exists a constant $\lambda > 0$ such that for all $x \in \mathbb{R}^d$ and all $v \in \mathbb{R}^d$ we have $v^\top \sigma(x) \sigma(x)^\top v \geq \lambda v^\top v$, then there still exists a strong solution, even if μ is only measurable and bounded. Veretennikov [14] generalizes the result by requiring that uniform ellipticity needs to hold only for those components in which the drift is non-Lipschitz.

Zvonkin's method is extended by Zhang [15] to a locally integrable drift function and non-degenerate diffusion. Beyond the aforementioned classical results one can find several approaches for dealing with discontinuous drift coefficients in the literature. A very natural way is to use smooth approximations for discontinuous coefficients and analyze the corresponding limiting process. Such a procedure is presented in a general form by Krylov and Liptser [5], but still the presence of ellipticity is crucial. This technique is also related to the question of stability of SDEs, see Protter [10, Chapter V.4], for which results commonly ask for the a-priori existence of the limiting process. In Meyer-Brandis and Proske [9] the existence question for the situation of a measurable drift function and non-degenerate diffusion coefficient is dealt with using techniques from Malliavin calculus.

Another method, different in spirit, is introduced by Halidias and Kloeden [3] who prove existence of a solution to an SDE with a drift that is increasing in every coordinate via a construction using sub- and super solutions. For our result, no monotonicity of the drift is needed and it is therefore a more general existence and uniqueness result for the degenerate setup.

That ellipticity plays a crucial role can be illustrated by the following example for which uniform ellipticity fails: consider the 2-dimensional SDE

$$\begin{aligned} dX_t^1 &= \left(\frac{1}{2} - \operatorname{sgn}(X_t^1 + X_t^2) \right) dt + dW_t, \\ dX_t^2 &= -dW_t, \end{aligned} \tag{1.2}$$

with $X_0 = (0, 0)$. If (1.2) had a strong solution then we see from adding the two equations that there would exist a one-dimensional adapted process $\tilde{X} = X^1 + X^2$ satisfying

$$\tilde{X}_t = \int_0^t \left(\frac{1}{2} - \operatorname{sgn}(\tilde{X}_s) \right) ds. \tag{1.3}$$

For the sake of completeness we show in Section 4 that such a process \tilde{X} cannot exist.

The aim of this article is to give sufficient conditions on μ, σ such that the SDE has a solution for the case where $\sigma \sigma^\top$ fails to be uniformly elliptic and where μ is allowed to be discontinuous and unbounded. This question is motivated by an example from Leobacher et al. [6], where a SDE appears that clearly violates the classical conditions of the theorem by Itô and does not necessarily include an increasing drift like the example in Halidias and Kloeden [3]. In fact, we are especially interested in cases where the drift is decreasing. We will present the example from Leobacher et al. [6] as an application of our results in Section 4.

The first main result of our contribution is Theorem 3.1 which states that there exists a unique maximal local solution of (1.1), if σ is sufficiently smooth, $(\sigma \sigma^\top)_{11} \geq c > 0$, and if μ is discontinuous in $\{x_1 = 0\}$, but sufficiently smooth everywhere else. The result can be generalized by a transformation of the domain, to allow for a discontinuity along a

hypersurface, but what remains is a certain dependence of the result on the geometry of the discontinuity of the drift coefficient.

The contribution of this article is two-fold: firstly, it closes a gap in combined filtering-control problems, namely that of admissibility of threshold and band strategies, which are very common types of optimal strategies. Secondly, it contributes to the theory of SDEs, not only by giving one of the most general existence and uniqueness results for the degenerate-elliptic case, but also by highlighting intriguing connections between the geometry of a discontinuity of the drift on the one hand and the diffusion vector field on the other hand.

The paper is organized as follows. In Section 2 we first fix notations and present the classical notions of strong, local, and maximal local solutions. Towards the statement of the main theorem a technical lemma and a particular form of Itô’s formula are proven. In Section 3 we prove the main results of this paper, and in Section 4 we apply our results to a concrete problem coming from mathematical finance.

2 Definitions and first results

In the whole paper we work with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where the filtration satisfies the usual conditions. Furthermore, we consider a d -dimensional standard Brownian motion $W = (W_t)_{t \geq 0}$ on that space. By $\|\cdot\|$ we always denote the d -dimensional Euclidean norm. When using the notion “solution” we always refer to “strong solution”.

First, let us recall the definitions of local, maximal local, and global solutions of SDEs. Consider an SDE of the form

$$X_t = x_0 + \int_0^t \mu(X_s) ds + \int_0^t \sigma(X_s) dW_s, \tag{2.1}$$

with initial data $X_0 = x_0 \in \mathcal{L}^2_{\mathcal{F}_0}(\Omega)$, where

$$\mu : \mathbb{R}^d \rightarrow \mathbb{R}^d \text{ and } \sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}.$$

Definition 2.1 ([13]). *An \mathbb{R}^d -valued stochastic process $(X_t)_{0 \leq t \leq T}$ is called a solution of (2.1), if it is continuous, progressively measurable w.r.t. $(\mathcal{F}_t)_{t \geq 0}$, and if it fulfills (2.1) for all $t \in [0, T]$ a.s.*

A solution $(X_t)_{0 \leq t \leq T}$ is said to be unique, if any other solution $(\bar{X}_t)_{0 \leq t \leq T}$ is indistinguishable from $(X_t)_{0 \leq t \leq T}$.

Definition 2.2 ([8, Definition 3.14]). *Let ζ be an $(\mathcal{F}_t)_{t \geq 0}$ -stopping time such that $0 \leq \zeta \leq T$ a.s. An \mathbb{R}^d -valued $(\mathcal{F}_t)_{t \geq 0}$ -adapted continuous stochastic process $(X_t)_{0 \leq t < \zeta}$ is called a local solution of equation (2.1), if $X_0 = x_0$, and, moreover, there is a non-decreasing sequence $\{\zeta_k\}_{k \geq 1}$ of $(\mathcal{F}_t)_{t \geq 0}$ -stopping times such that $0 \leq \zeta_k \uparrow \zeta$ a.s. and*

$$X_t = x_0 + \int_0^{t \wedge \zeta_k} \mu(X_s) ds + \int_{t_0}^{t \wedge \zeta_k} \sigma(X_s) dW_s$$

holds for any $t \in [0, T)$ a.s.

If, furthermore,

$$\limsup_{t \rightarrow \zeta} \|X_t\| = \infty \text{ whenever } \zeta < T,$$

then it is called a maximal local solution and ζ is called the explosion time.

A maximal local solution $(X_t)_{0 \leq t < \zeta}$ is said to be unique, if any other maximal local solution $(\bar{X}_t)_{0 \leq t < \bar{\zeta}}$ is indistinguishable from it, namely $\zeta = \bar{\zeta}$ and $X_t = \bar{X}_t$ for $0 \leq t < \zeta$ a.s.

Definition 2.3 ([8, p. 94]). *If the assumptions of an existence and uniqueness theorem hold on every finite subinterval $[0, T]$ of $[0, \infty)$, then (2.1) has a unique solution $(X_t)_{t \geq 0}$ on the entire interval $[0, \infty)$. Such a solution is called a global solution.*

Furthermore, recall that a function ϕ on \mathbb{R}^d is locally Lipschitz, if for all $n \in \mathbb{N}$ there is a constant $L_n > 0$ such that for those $x_1, x_2 \in \mathbb{R}^d$ with $\max\{\|x_1\|, \|x_2\|\} \leq n$ the following condition holds:

$$\|\phi(x_1) - \phi(x_2)\| \leq L_n \|x_1 - x_2\|.$$

Consider the following system of SDEs on \mathbb{R}^d :

$$dX_t = \mu(X_t)dt + \sigma(X_t) dW_t, \tag{2.2}$$

$$X_0 = x.$$

Assumption 2.4. *We assume the following for the coefficients of (2.2).*

- (i) σ_{ij} , for $i = 2, \dots, d, j = 1, \dots, d$ are locally Lipschitz,
- (ii) $\sigma_{1j} \in C^{1,3}(\mathbb{R} \times \mathbb{R}^{d-1})$, for $j = 1, \dots, d$,
- (iii) $(\sigma\sigma^\top)_{11} \geq c > 0$ for some constant c and for all $x \in \mathbb{R}^d$.

The function $\mu : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is allowed to be discontinuous. However, the form of the discontinuity needs to be a special one.

Assumption 2.5. *We assume the following for μ : there exist functions $\mu^+, \mu^- \in C^{1,3}(\mathbb{R} \times \mathbb{R}^{d-1})$ such that*

$$\mu(x_1, \dots, x_d) = \begin{cases} \mu^+(x_1, x_2, \dots, x_d) & \text{if } x_1 > 0 \\ \mu^-(x_1, x_2, \dots, x_d) & \text{if } x_1 < 0 \end{cases}$$

Remark 2.6. The value of μ for $x_1 = 0$ is of no significance for us since, due to Assumption 2.4 (iii) a solution X to (2.2) does not spend a positive amount of time in the hyperplane $\{x_1 = 0\}$.

Now, we are going to study the existence of a solution to system (2.2). Suppose first that a solution to (2.2) exists and define

$$Z_t^1 = g_1(X_t)$$

for some suitable function $g_1 : \mathbb{R}^d \rightarrow \mathbb{R}$. For the sake of readability we are going to skip the arguments in the following.

Heuristically using Itô's formula we calculate

$$\begin{aligned} dZ_t^1 &= \sum_{i=1}^d \frac{\partial g_1}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 g_1}{\partial x_i \partial x_j} d[X^i, X^j]_t \\ &= \left[\mu_1 \frac{\partial g_1}{\partial x_1} + \frac{1}{2} (\sigma\sigma^\top)_{11} \frac{\partial^2 g_1}{\partial x_1^2} \right] dt + \sum_{i=2}^d \mu_i \frac{\partial g_1}{\partial x_i} dt \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d (1 - \delta_{11}(i,j)) (\sigma\sigma^\top)_{ij} \frac{\partial^2 g_1}{\partial x_i \partial x_j} dt + \sum_{i,j=1}^d \sigma_{ij} \frac{\partial g_1}{\partial x_i} dW_t^j. \end{aligned}$$

Now we would like to choose $g_1 \neq 0$ such that

$$\mu_1 \frac{\partial g_1}{\partial x_1} + \frac{1}{2}(\sigma\sigma^\top)_{11} \frac{\partial^2 g_1}{(\partial x_1)^2} = 0 \tag{2.3}$$

to eliminate the problematic term μ_1 . This extends the idea of Zvonkin [16]. A solution g_1 can be obtained as follows.

$$\begin{aligned} \frac{\frac{\partial^2 g_1}{(\partial x_1)^2}}{\frac{\partial g_1}{\partial x_1}} &= -\frac{2\mu_1}{(\sigma\sigma^\top)_{11}} = \frac{\partial}{\partial x_1} \left(\log \left(\frac{\partial g_1}{\partial x_1} \right) \right) \\ \Leftrightarrow g_1 &= C_0(x_2, \dots, x_d) \int \exp \left(-\int \frac{2\mu_1}{(\sigma\sigma^\top)_{11}} dx_1 \right) dx_1 + C_1(x_2, \dots, x_d). \end{aligned}$$

We are free to choose $C_1(x_2, \dots, x_d) \equiv 0$ and $C_0(x_2, \dots, x_d) \equiv 1$, such that

$$g_1(x) = \int_0^{x_1} \exp \left(-\int_0^\xi \frac{2\mu_1(t, x_2, \dots, x_d)}{(\sigma\sigma^\top)_{11}(t, x_2, \dots, x_d)} dt \right) d\xi.$$

Note that $\frac{\partial g_1}{\partial x_1}(0, x_2, \dots, x_d) > 0$ for all $(x_2, \dots, x_d) \in \mathbb{R}^{d-1}$. Note further that for $i \neq 1$ the terms of the form $\mu_i \frac{\partial g_1}{\partial x_i}$ are locally Lipschitz since μ_i is locally bounded and $\frac{\partial g_1}{\partial x_i}$ is zero on $\{x_1 = 0\}$.

Using similar considerations we can find $g_k : \mathbb{R}^d \rightarrow \mathbb{R}$ such that for $Z^k = X^k + g_k(X)$, $k = 2, \dots, d$, the drift coefficient is Lipschitz. The corresponding ordinary differential equation is

$$\mu_k + \mu_1 \frac{\partial g_k}{\partial x_1} + \frac{1}{2}(\sigma\sigma^\top)_{11} \frac{\partial^2 g_k}{(\partial x_1)^2} = 0 \tag{2.4}$$

for which we get a special solution

$$g_k(x_1, \dots, x_d) := \int_0^{x_1} C_k(\xi, x_2, \dots, x_d) \exp \left(-\int_0^\xi \frac{2\mu_1(t, x_2, \dots, x_d)}{(\sigma\sigma^\top)_{11}(t, x_2, \dots, x_d)} dt \right) d\xi,$$

where

$$C_k(\xi, x_2, \dots, x_d) := -\int_0^\xi \frac{2\mu_k(\eta, x_2, \dots, x_d)}{(\sigma\sigma^\top)_{11}(\eta, x_2, \dots, x_d)} \exp \left(\int_0^\eta \frac{2\mu_1(t, x_2, \dots, x_d)}{(\sigma\sigma^\top)_{11}(t, x_2, \dots, x_d)} dt \right) d\eta$$

by the method of variation of constants. Note that for all $k \neq 1$ and all $i = 1, \dots, d$ we have $\frac{\partial g_k}{\partial x_i}(0, x_2, \dots, x_d) = 0$ for all $(x_2, \dots, x_d) \in \mathbb{R}^{d-1}$.

Lemma 2.7. Consider a function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ of the form

$$g(x_1, \dots, x_d) = \int_0^{x_1} C(\xi, x_2, \dots, x_d) \exp \left(\int_0^\xi a(t, x_2, \dots, x_d) dt \right) d\xi,$$

where C is of the form

$$C(\xi, x_2, \dots, x_d) = \int_0^\xi c_1(\eta, x_2, \dots, x_d) d\eta + c_0$$

for some $c_0 \in \mathbb{R}$ and a, c_1 are such that

$$\begin{aligned} a(x_1, \dots, x_d) &= \begin{cases} a^+(x_1, x_2, \dots, x_d) & \text{if } x_1 > 0 \\ a^-(x_1, x_2, \dots, x_d) & \text{if } x_1 < 0 \end{cases} \\ c_1(x_1, \dots, x_d) &= \begin{cases} c_1^+(x_1, x_2, \dots, x_d) & \text{if } x_1 > 0 \\ c_1^-(x_1, x_2, \dots, x_d) & \text{if } x_1 < 0 \end{cases} \end{aligned}$$

for some functions $a^+, a^-, c_1^+, c_1^- \in C^{1,3}(\mathbb{R} \times \mathbb{R}^{d-1})$.

Then Dg is locally Lipschitz for $D \in \{1, \frac{\partial}{\partial x_i}, \frac{\partial^2}{\partial x_i \partial x_j} \mid i, j = 1, \dots, d\} \setminus \{\frac{\partial^2}{\partial x_1^2}\}$.

Proof. For $i \neq 1$ we may exchange differentiation w.r.t. x_i and integration (see, e.g., [11, Theorem 9.42]). Furthermore, we use that a, c_1 have continuous third derivatives w.r.t. (x_2, \dots, x_d) , as warranted by our assumptions on the functions a, c_1 . Thus we get that $g, \frac{\partial g}{\partial x_i}$ for $i = 1, \dots, d$ are locally Lipschitz and $\frac{\partial^2 g}{\partial x_i \partial x_j}$ is locally Lipschitz for $i, j \neq 1$.

It remains to show that $\frac{\partial^2 g}{\partial x_j \partial x_1}$ is locally Lipschitz for all $j > 1$. Let $r > 0$ and let $x, y \in B_r(0)$. $\frac{\partial^3 g}{\partial x_j^2 \partial x_1}, \frac{\partial^3 g}{\partial x_j \partial x_1^2}$ exist and are bounded on $B_r(0)$, say by K . Consider first the case where $x_1, y_1 > 0$ (i.e., both points lie on the same side of the hyperplane $\{x_1 = 0\}$):

$$\left| \frac{\partial^2 g}{\partial x_j \partial x_1}(y) - \frac{\partial^2 g}{\partial x_j \partial x_1}(x) \right| \leq \sup_{\|z\| \leq r} \left\| \left(\frac{\partial^3 g}{\partial x_j^2 \partial x_1}(z), \frac{\partial^3 g}{\partial x_j \partial x_1^2}(z) \right) \right\| \|y - x\| \leq \sqrt{2}K \|y - x\|.$$

If exactly one of the x_1, y_1 is zero, the same estimate holds.

If $x_1 < 0 < y_1$, let z be the intersection of the hyperplane $\{x_1 = 0\}$ with the line connecting x and y . Now make the same estimate as above twice.

Finally, let $x_1 = y_1 = 0$. Let $z_1 = \|y - x\|/2, (z_2, \dots, z_d) = \frac{1}{2}((x_2, \dots, x_d) + (y_2, \dots, y_d))$. Then

$$\begin{aligned} \left| \frac{\partial^2 g}{\partial x_j \partial x_1}(y) - \frac{\partial^2 g}{\partial x_j \partial x_1}(x) \right| &\leq \left| \frac{\partial^2 g}{\partial x_j \partial x_1}(y) - \frac{\partial^2 g}{\partial x_j \partial x_1}(z) \right| + \left| \frac{\partial^2 g}{\partial x_j \partial x_1}(z) - \frac{\partial^2 g}{\partial x_j \partial x_1}(x) \right| \\ &\leq \sqrt{2}K(\|y - z\| + \|z - x\|) = 2K\|y - z\|. \end{aligned}$$

□

From Lemma 2.7 together with Assumptions 2.4 (ii) and 2.5 it follows that for all $k = 1, \dots, d$ the function Dg_k is locally Lipschitz for $D \in \{1, \frac{\partial}{\partial x_i}, \frac{\partial^2}{\partial x_i \partial x_j} \mid i, j = 1, \dots, d\} \setminus \{\frac{\partial^2}{\partial x_1^2}\}$.

Define a function $G : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by

$$G(x_1, \dots, x_d) := (g_1(x), x_2 + g_2(x), \dots, x_d + g_d(x)).$$

Then $\frac{\partial G_k}{\partial x_i}(0, x_2, \dots, x_d) = \delta_{i,k}$. In particular, due to the inverse function theorem [11, Theorem 9.24], G is locally invertible, i.e., for every $x_0 \in \mathbb{R}^d$ there exist $r > 0$ and $H : G(B_r(x_0)) \rightarrow \mathbb{R}^d$ such that

$$H \circ G = id_{B_r(x_0)} \text{ and } G \circ H = id_{G(B_r(x_0))}.$$

Note that H inherits the smoothness from G .

Before proceeding we need to clarify the notion *local* in our context. Below we are going to use the locally defined function H for establishing the existence of a solution to (2.2). Naturally, for $X_0 = x_0$ the – still to be constructed – solution X exists on $B_r(x_0)$. Setting $\zeta^k = \zeta = \inf\{t > 0 \mid X_t \notin B_r(x_0)\}$ for $k \in \mathbb{N}$ in Definition 2.2 we will have that $(X_{t \wedge \zeta^k})_{t \geq 0}$ fulfills (2.2). Therefore it is a local solution in the sense of Definition 2.2 but on the restricted domain $B_r(x_0)$ we will have existence of a *unique* solution to (2.2) for every initial point $X_0 \in B_r(x_0)$. It is maximal in the sense that there will be no explosion before reaching the boundary of $B_r(x_0)$.

Now, let us assume for the moment that a solution X to the SDE (2.2) exists. Let $Z_t := G(X_t)$. The functions g_k for $k = 2, \dots, d$ are defined in a way to guarantee that the drift of Z^k is locally Lipschitz, i.e., the discontinuities are removed from the drift.

Now, let us consider the following ‘transformed’ SDE:

$$dZ_t = (\nabla G(X_t)\mu(X_t) + \text{tr}(\sigma(X_t)^\top \nabla^2 G(X_t)\sigma(X_t))) dt + \nabla G(X_t)\sigma(X_t)dW_t$$

That is, X solves $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$ iff $Z_t = G(X_t)$ solves

$$\left. \begin{aligned} dZ_t &= \tilde{\mu}(Z_t)dt + \tilde{\sigma}(Z_t)dW_t && \text{with} \\ \tilde{\mu}(z) &= (\nabla G)(H(z))\mu(H(z)) + \text{tr}(\sigma(H(z))^\top (\nabla^2 G)(H(z))\sigma(H(z))) && \\ \tilde{\sigma}(z) &= (\nabla G)(H(z))\sigma(H(z)). \end{aligned} \right\} \quad (2.5)$$

Lemma 2.8. *Let Assumptions 2.4 and 2.5 be fulfilled. Then system (2.5) has a unique local solution Z .*

Proof. The drift and diffusion coefficients of (2.5) are locally Lipschitz by Lemma 2.7. Thus, from [8, Theorem 3.15] we get that (2.5) has a unique local solution Z . \square

For proving our main result we need an Itô-type formula for the function H and the solution Z . There is a great number of extensions of the classical Itô formula for functions that are not necessarily C^2 , e.g., Russo and Vallois [12], Föllmer and Protter [2], Eisenbaum [1], but usually they rely on non-degeneracy of the diffusion coefficient of the argument process, either explicitly, or implicitly, by having Brownian motion as the argument.

Theorem 2.9 (Itô's formula). *Let $\mathcal{D} \subseteq \mathbb{R}^d$ be an open set and $f : \mathcal{D} \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$, $f \in C^1(\mathcal{D})$, $\frac{\partial^2}{\partial x_i \partial x_j} f(x) \in C(\mathcal{D})$ for all $i = 1, \dots, d, j = 2, \dots, d$, $\frac{\partial^2}{\partial x_1^2} f(x) \in C((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{d-1}) \cap \mathcal{D}$, $\sup_{\{x | x_1 \neq 0\}} |\frac{\partial^2}{\partial x_1^2} f(x)| < \infty$. Furthermore, let X be an Itô process and $\zeta = \inf\{t > 0 | X_t \notin \mathcal{D}\}$.*

Then for all $t \geq 0$

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^{t \wedge \zeta} \frac{\partial}{\partial x_i} f(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^{t \wedge \zeta} \frac{\partial^2}{\partial x_i \partial x_j} f(X_s) d[X^i, X^j]_s.$$

Proof. Let

$$\psi(t) := \begin{cases} 0, & t < -1 \\ \frac{15}{16} \int_{-1}^t (\tau + 1)^2 (\tau - 1)^2 d\tau, & -1 \leq t \leq 1 \\ 1, & t > 1, \end{cases}$$

$\psi \in C^2$, $0 \leq \psi(t) \leq 1$ for all $t \in \mathbb{R}$. For every $n \in \mathbb{N}$ and $t \in \mathbb{R}$ we have that

$$\psi(2n(-t + 1)) + \psi(2n(t - 1))\psi(2n(-t - 1)) + \psi(2n(t + 1)) = 1.$$

Furthermore, let for every $n \in \mathbb{N}$, \tilde{f}_n be a C^2 function with $\lim_{n \rightarrow 0} \|D(f - \tilde{f}_n)\|_\infty = 0$ for every $D \in \{1, \frac{\partial}{\partial x_i}, \frac{\partial^2}{\partial x_i \partial x_j} | i, j = 1, \dots, d\} \setminus \{\frac{\partial^2}{\partial x_1^2}\}$ and such that for all n we have $\|\frac{\partial^2}{\partial x_1^2} \tilde{f}_n\|_\infty \leq K$ for some constant K . (E.g. \tilde{f}_n could be the convolution of f with a suitable C^2 function.)

Now define

$$f_n(x) := \psi(2n(-x_1 + 1))f(x) + \psi(2n(x_1 - 1))\psi(2n(-x_1 - 1))\tilde{f}_n(x) + \psi(2n(x_1 + 1))f(x).$$

Since $f_n \in C^2$ we may apply Itô's formula:

$$f_n(X_t) = f_n(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} f_n(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f_n(X_s) d[X^i, X^j]_s. \quad (2.6)$$

Let us consider the term with the second derivative w.r.t. x_1 :

$$\int_0^t \frac{\partial^2}{\partial x_1^2} f_n(X_s) d[X^1]_s = \int_0^t 1_{\{|X_s^1| > \frac{1}{n}\}} \frac{\partial^2}{\partial x_1^2} f(X_s) d[X^1]_s + \int_0^t 1_{\{|X_s^1| \leq \frac{1}{n}\}} \frac{\partial^2}{\partial x_1^2} f_n(X_s) d[X^1]_s.$$

Since

$$\left| \int_0^t 1_{\{|X_s^1| \leq \frac{1}{n}\}} \frac{\partial^2}{\partial x_1^2} f_n(X_s) d[X^1]_s \right| \leq K \int_0^t 1_{\{|X_s^1| \leq \frac{1}{n}\}} d[X^1]_s$$

we have for the limit

$$\left| \lim_{n \rightarrow \infty} \int_0^t \frac{\partial^2}{\partial x_1^2} f_n(X_s) d[X^1]_s - \int_0^t 1_{\{|X_s^1| \neq 0\}} \frac{\partial^2}{\partial x_1^2} f(X_s) d[X^1]_s \right| \leq K \int_0^t 1_{\{|X_s^1| = 0\}} d[X^1]_s$$

by bounded convergence.

From [4, Chapter 3, Theorem 7.1] we know that

$$\int_0^t 1_{\{|X_s^1| = 0\}} d[X^1]_s = 2 \int_{\mathbb{R}} 1_{\{0\}}(a) \Lambda_t(a) da = 0,$$

where $\Lambda_t(a)$ denotes the local time of X^1 in a up to time t .

Thus, for $n \rightarrow \infty$ all terms in (2.6) converge to the corresponding term with f_n replaced by f . □

3 Main result

Theorem 3.1. *Let Assumptions 2.4 and 2.5 be fulfilled. Let Z be the unique local solution of (2.5). Then*

$$X = H(Z)$$

is the unique local solution to (2.2).

Proof. We have that $\nabla G(0, x_2, \dots, x_d)$ is the $(d \times d)$ -identity matrix, and if $x_1 = 0$, then also $z_1 = 0$. Furthermore, the components of H fulfill the assumptions of Theorem 2.9. Therefore, Itô's formula still holds. Its application to $X = H(Z)$ yields

$$dX_t = \mu(X_t) dt + \sigma(X_t) dW_t,$$

which proves the claim. □

From Theorem 3.1 we know that a unique local solution of system (2.2) exists. Now it remains to prove that there is a unique maximal local solution.

Theorem 3.2. *Let Assumptions 2.4 and 2.5 be fulfilled. Then system (2.2) has a unique maximal local solution.*

Proof. The proof consists of three steps.

Step 1: For each $x \in \mathbb{R}^d$ there is a ball $B_{\varepsilon_x}(x)$ with radius $\varepsilon_x > 0$ such that (2.2) with $X_0 = x$ has a unique local solution due to Theorem 3.1.

Step 2: Let $\mathcal{D}_n, n \in \mathbb{N}$ be compact subsets of \mathbb{R}^d with $\mathcal{D}_n \uparrow \mathbb{R}^d$, i.e., $\mathbb{R}^d = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$. Furthermore, for all $x \in \mathbb{R}^d$ there is an $n_0 \in \mathbb{N}$ such that $x \in \mathcal{D}_n$ for all $n \geq n_0$.

We know from above that for all $x \in \mathcal{D}_n$ there is a radius $\varepsilon_x > 0$ such that (2.2) with $X_0 = x$ has a unique local solution on $B_{\varepsilon_x}(x)$. Clearly, $\mathcal{D}_n \subseteq \bigcup_{x \in \mathcal{D}_n} B_{\varepsilon_x}(x)$ and since \mathcal{D}_n is compact, there exists $m < \infty$ such that $\mathcal{D}_n \subseteq \bigcup_{k=1}^m B_{\varepsilon_{x_k}}(x_k)$, i.e., the covering is finite. Now, consider some fixed $x \in \mathcal{D}_n, x \in B_{\varepsilon_{x_{k_1}}}(x_{k_1}), \dots, B_{\varepsilon_{x_{k_{\bar{m}}}}}(x_{k_{\bar{m}}})$ for $\{k_1, \dots, k_{\bar{m}}\} \subseteq \{1, \dots, m\}$ and $\bar{m} \leq m$. Since we have uniqueness of the solution on every ball, we have uniqueness on any finite intersection $\bigcap_j B_{\varepsilon_{x_{k_j}}}(x_{k_j})$. Thus, (2.2) with $X_0 = x$ has a unique

local solution (Θ^x, ζ^x) on $\bigcap_{j=1}^{\bar{m}} B_{\varepsilon_{x_{k_j}}}(x_{k_j})$, where $\Theta = X$.

In detail, this means that there exist mappings

$$\begin{aligned} \zeta &: \Omega \times \mathcal{D}_n \rightarrow [0, \infty], \\ \Theta &: \Omega \times \mathcal{D}_n \times [0, \zeta) \rightarrow \mathcal{D}_n, \end{aligned}$$

such that

- Θ and ζ are measurable (cf. [10, Chapter V.6, Theorem 31] and due to a continuous transformation of a measurable function),
- $\zeta^x : \Omega \rightarrow [0, \infty]$, $\zeta^x = \inf\{t > 0 \mid \Theta_t^x \notin \bigcap_{j=1}^{\bar{m}} B_{\varepsilon_{x_{k_j}}}(x_{k_j})\}$ is a stopping time w.r.t. our filtration, $\zeta^x > 0$ a.s.
- $\Theta^x : \Omega \times [0, \zeta^x] \rightarrow \mathcal{D}_n$ locally solves (2.2) with $X_0 = x$.

We can now use this to construct a solution on \mathcal{D}_n . Let $\omega \in \Omega$. Define

$$\begin{aligned} \zeta_1(\omega) &:= \zeta^x(\omega), \\ X_t^{x,n}(\omega) &:= \Theta_t^x(\omega), \quad \text{for } t \in [0, \zeta_1(\omega)], \end{aligned}$$

and

$$\begin{aligned} \zeta_{i+1}(\omega) &:= \zeta^{X_{\zeta_i(\omega)}^{x,n}}(\omega) + \zeta_i(\omega), \\ X_t^{x,n}(\omega) &:= \Theta_{t-\zeta_i(\omega)}^{X_{\zeta_i(\omega)}^{x,n}}(\omega), \quad \text{for } t \in [\zeta_i(\omega), \zeta_{i+1}(\omega)]. \end{aligned}$$

Then, due to the strong Markov property (cf. [10, Chapter V.6, Theorem 32]), $X^{x,n}$ stopped at $\zeta^{\mathcal{D}_n} := \inf\{t \geq 0 \mid X_t^{x,n} \notin \mathcal{D}_n\}$ defines a unique local solution to (2.2) on \mathcal{D}_n .

Step 3: It remains to extend the solution to the whole domain \mathbb{R}^d . We already know that for each $x \in \mathbb{R}^d$ there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $x \in \mathcal{D}_n$. The sequence of stopping times $\zeta^{\mathcal{D}_n} = \inf\{t \geq 0 \mid X_t^{x,n} \notin \mathcal{D}_n\}$ is increasing in n and thus we may define $\hat{\zeta} := \lim_{n \rightarrow \infty} \zeta^{\mathcal{D}_n}$. From Step 2 we know that for all such $n \geq n_0$ a unique local solution

$$X^{x,n} : \Omega \times [0, \zeta^{\mathcal{D}_n}) \rightarrow \mathbb{R}^d$$

exists. Clearly, for $n_1, n_2 \geq 0$, $n_1 \neq n_2$ we have

$$X_t^{x,n_1} = X_t^{x,n_2} \quad \forall t \in [0, \zeta^{\mathcal{D}_{n_1}} \wedge \zeta^{\mathcal{D}_{n_2}}] \text{ a.s.}$$

Therefore, define

$$X_t^x(\omega) := X_t^{x,n}(\omega) \quad \text{for } t \leq \zeta^{\mathcal{D}_n},$$

which is our unique maximal local solution. □

We proved existence and uniqueness of a unique maximal local solution of system (2.2) under Assumptions 2.4 and 2.5. Naturally, by imposing stronger conditions on the coefficients we can show the corresponding global result.

Theorem 3.3. *Let Assumptions 2.4 and 2.5 be fulfilled. Additionally, let σ be globally Lipschitz and let μ satisfy a linear growth condition, i.e., $\|\mu(x)\| \leq D_1 + D_2\|x\|$ for constants $D_1, D_2 > 0$.*

Then there exists a unique global solution to (2.2).

Proof. For the unique maximal local solution X we need to show that

$$\mathbb{P} \left(\limsup_{t \rightarrow \zeta} \|X_t\|^2 = \infty; \zeta < \infty \right) = 0$$

for all stopping times ζ . Now, let ζ be a stopping time with $\mathbb{P}(\zeta < \infty) > 0$. Furthermore, let $(T_n)_{n \geq 0}$ be the sequence of stopping times defined by

$$T_n := \inf \left\{ t \geq 0 : \|X_t\|^2 \geq n \right\},$$

and let

$$X_t^{T_n} = x_0 + \int_0^{t \wedge T_n} \mu(X_s^{T_n}) ds + \int_0^{t \wedge T_n} \sigma(X_s^{T_n}) dW_s.$$

For $T > 0$ large enough such that $\mathbb{P}(\zeta < T) > 0$ we consider

$$\begin{aligned} & \mathbb{E} \left(\sup_{t \leq T} \|X_t^{T_n}\|^2 \right) \\ & \leq \mathbb{E} \left(\|x_0\|^2 \right) + \mathbb{E} \left(\sup_{t \leq T} \left\| \int_0^{t \wedge T_n} \mu(X_s^{T_n}) ds \right\|^2 \right) + \mathbb{E} \left(\sup_{t \leq T} \left\| \int_0^{t \wedge T_n} \sigma(X_s^{T_n}) dW_s \right\|^2 \right) \\ & =: E_1 + E_2 + E_3. \end{aligned}$$

Let D_1, D_2 be as above.

$$\begin{aligned} E_2 & \leq \mathbb{E} \left(\sup_{t \leq T} (t \wedge T_n) \int_0^{t \wedge T_n} \|\mu(X_s^{T_n})\|^2 ds \right) \\ & \leq \mathbb{E} \left(\sup_{t \leq T} (t \wedge T_n) \int_0^{t \wedge T_n} (D_1 + D_2 \|X_s^{T_n}\|)^2 ds \right) \\ & \leq 2D_1^2 T^2 + 2TD_2^2 \int_0^T \mathbb{E} \left(\sup_{s \leq t} \|X_s^{T_n}\|^2 \right) dt. \end{aligned}$$

By Doob's \mathcal{L}^2 -inequality and Itô's isometry we get

$$\begin{aligned} E_3 & \leq 4\mathbb{E} \left(\sum_{i=1}^d \left(\int_0^T \sum_{j=1}^d \sigma_{ij}(X_s^{T_n}) dW_s^j \right)^2 \right) = 4 \sum_{i=1}^d \mathbb{E} \left(\int_0^T \left(\sum_{j=1}^d \sigma_{ij}(X_s^{T_n}) \right)^2 ds \right) \\ & \leq 4d \sum_{i=1}^d \mathbb{E} \left(\int_0^T \sum_{j=1}^d \sigma_{ij}(X_s^{T_n})^2 ds \right) \leq C_1 T + C_2 L^2 \int_0^T \mathbb{E} \left(\|X_t^{T_n}\|^2 \right) dt \\ & \leq C_1 T + C_2 L^2 \int_0^T \mathbb{E} \left(\sup_{s \leq t} \|X_s^{T_n}\|^2 \right) dt, \end{aligned}$$

where C_1, C_2 are constants independent of n and L is the maximum of all appearing Lipschitz constants. Thus, we have

$$\begin{aligned} E_1 + E_2 + E_3 & \leq \|x_0\|^2 + 2D_1^2 T^2 + C_1 T + (2TD_2^2 + C_2 L^2) \int_0^T \mathbb{E} \left(\sup_{s \leq t} \|X_s^{T_n}\|^2 \right) dt \\ & =: A(T) + B(T) \int_0^T \mathbb{E} \left(\sup_{s \leq t} \|X_s^{T_n}\|^2 \right) dt. \end{aligned}$$

Therefore,

$$\mathbb{E} \left(\sup_{t \leq T^*} \|X_t^{T_n}\|^2 \right) \leq A(T) + B(T) \int_0^{T^*} \mathbb{E} \left(\sup_{s \leq t} \|X_s^{T_n}\|^2 \right) dt$$

$\forall T^* \leq T$. We can directly apply Gronwall's inequality and get

$$\mathbb{E} \left(\sup_{t \leq T^*} \|X_t^{T_n}\|^2 \right) \leq A(T)e^{B(T)T^*} \quad \forall n \text{ and } \forall T^* \leq T.$$

Sending $n \rightarrow \infty$ we arrive at

$$\mathbb{E} \left(\sup_{t \leq T} \|X_t\|^2 \right) \leq C(T) < \infty.$$

Since the above expectation is finite for each T , we can conclude

$$\begin{aligned} \mathbb{P} \left(\limsup_{t \rightarrow \zeta} \|X_t\|^2 = \infty; \zeta < \infty \right) &= \lim_{T \rightarrow \infty} \mathbb{P} \left(\limsup_{t \rightarrow \zeta} \|X_t\|^2 = \infty; \zeta < T \right) \\ &= \lim_{T \rightarrow \infty} \mathbb{P} \left(\limsup_{t \rightarrow \zeta} \|X_t\|^2 = \infty \mid \zeta < T \right) \mathbb{P}(\zeta < T) = 0. \end{aligned}$$

□

As a generalization we would like to allow discontinuities not only along $\{x_1 = 0\}$, but also along some sufficiently regular hypersurface $\{x \in \mathbb{R}^d : f(x) = 0\}$. We consider the system

$$dX_t = \mu(f(X_t), X_t^2, \dots, X_t^d)dt + \sigma(X_t) dW_t, \tag{3.1}$$

$X_0 = x_0$.

Assumption 3.4. We assume the following for the coefficients of (3.1).

- (i) $\sigma_{ij} \in C^{1,3}(\mathbb{R} \times \mathbb{R}^{d-1})$, for $i, j = 1, \dots, d$,
- (ii) $f \in C^{3,5}(\mathbb{R} \times \mathbb{R}^{d-1})$, and $|\frac{\partial f}{\partial x_1}| > 0$,
- (iii) for some constant c and for all $x \in \mathbb{R}^d$

$$\|\nabla f(x) \cdot \sigma(x)\|^2 \geq c > 0.$$

Remark 3.5. One may notice that item (iii) replaces the item (iii) from Assumption 2.4. Item (iii) has a nice geometric interpretation: the diffusion component must not be parallel to the surface where the drift is discontinuous.

Now we have a look at the transformation $U_t = f(X_t)$. Due to our assumptions and [11, Theorem 9.24], for any x_0 there exists a function $e \in C^{3,5}(\mathbb{R} \times \mathbb{R}^{d-1})$ such that

$$\begin{aligned} e(f(x), x_2, \dots, x_d) &= x_1, \\ f(e(u, x_2, \dots, x_d), x_2, \dots, x_d) &= u. \end{aligned}$$

We define

$$\begin{aligned} \bar{\mu}_1(u, x_2, \dots, x_d) &= \sum_{i=1}^d \mu_i(u, x_2, \dots, x_d) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^\top)_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}, \\ \bar{\mu}_i(u, x_2, \dots, x_d) &= \mu_i(u, x_2, \dots, x_d), \quad i = 2, \dots, d, \\ \bar{\sigma}_{1j}(u, x_2, \dots, x_d) &= \sum_{i=1}^d \sigma_{ij} \frac{\partial f}{\partial x_i}, \quad j = 1, \dots, d, \\ \bar{\sigma}_{ij}(u, x_2, \dots, x_d) &= \sigma_{ij}, \quad i = 2, \dots, d; j = 1, \dots, d, \end{aligned}$$

where all missing arguments are $(e(u, x_2, \dots, x_d), x_2, \dots, x_d)$.

This leads to the system

$$(dU_t, dX_t^2, \dots, dX_t^d)^\top = \bar{\mu}(U_t, X_t^2, \dots, X_t^d)dt + \bar{\sigma}(U_t, X_t^2, \dots, X_t^d)dW_t, \quad (3.2)$$

$$(U_0, X_0^2, \dots, X_0^d) = (u, x_2, \dots, x_d) = (f(x), x_2, \dots, x_d).$$

Theorem 3.6. *Let system (3.1) fulfill Assumptions 2.5 and 3.4. Then (3.1) has a unique maximal local solution. If, in addition, $\bar{\sigma}$ is globally Lipschitz and $\bar{\mu}$ satisfies linear growth, then (3.1) has a unique global solution.*

Proof. Note that for the transformed system (3.2) our original assumptions on the coefficients are fulfilled. Therefore, $X = e(U, X^2, \dots, X^d)$ is the unique local solution to (3.2). Furthermore, we can apply Theorem 3.2 and get that system (3.2) has a unique maximal local solution. The remaining assertion follows immediately from Theorem 3.3. \square

Remark 3.7. While in Veretennikov [14] only those components of the drift are allowed to be discontinuous, the related part of the diffusion of which is uniformly elliptic, our result allows all drift components to be discontinuous along a sufficiently smooth hypersurface. Furthermore, in [14] all coefficients need to be bounded, which is not required herein. Instead however, we require more smoothness of the coefficients. The proof of the main result from [14] relies on highly evolved methods from the literature on PDEs. Concretely, in [14] first weak existence is proven and then pathwise uniqueness. In contrast to that our proof is more direct.

4 Examples

At this point we return to the initially cited problem from [6], which originates from stochastic optimal control in mathematical finance.

Example 4.1. In [6] the question arises whether the system

$$\begin{aligned} dX_t^1 &= \left(X_t^2 - \kappa 1_{\{X_t^1 \geq b(X_t^2)\}} \right) dt + \sigma dW_t^1, \\ dX_t^2 &= \frac{1}{\sigma} (\theta_2 - X_t^2)(X_t^2 - \theta_1) dW_t^1, \end{aligned} \quad (4.1)$$

where $\sigma > 0$, and θ_1, θ_2 are constants with $\theta_1 \leq X_t^2 \leq \theta_2$ for $t \geq 0$, has a solution. The motivation of system (4.1) is as follows. X^1 originally corresponds to a dividend paying firm value process of a company which is described by a Brownian motion with drift. Since the drift is assumed to be unobservable, it is replaced by its estimator, which by an application of filtering theory is given by the process X^2 . The dividend payments are governed by the threshold function b such that whenever $X_t^1 \geq b(X_t^2)$, dividends are paid at a constant rate κ .

If $f(x, y) = x - b(y)$ fulfills items (ii) and (iii) of Assumption 3.4, all conditions on the coefficients are fulfilled. Thus, applying Theorem 3.6 implies the existence of a unique global solution of (4.1).

Now we complete the counterexample mentioned in the introduction: if the crucial condition $(\sigma\sigma^\top)_{11}(x) \geq c > 0$ is violated, then there is no solution.

Example 4.2. There is no measurable function \tilde{X} satisfying the differential equation

$$\begin{aligned} \tilde{X}_t &= \int_0^t \left(\frac{1}{2} - \text{sgn}(\tilde{X}_s) \right) ds, \\ \tilde{X}_0 &= 0. \end{aligned}$$

Proof. Suppose such an \tilde{X} would indeed exist. Since $\text{sgn}(\tilde{X}_s)$ is bounded, $t \mapsto \int_0^t \text{sgn}(\tilde{X}_s) ds$ is continuous and therefore \tilde{X} is continuous.

Let $\varepsilon > 0$. \tilde{X} cannot be positive on $(0, \varepsilon)$, since then the integrand would equal $-\frac{1}{2}$ on $(0, \varepsilon)$ and thus $\tilde{X}_\varepsilon = \tilde{X}_t - \int_t^\varepsilon \frac{1}{2} ds = \tilde{X}_t - \frac{\varepsilon-t}{2}$. From this we get $\tilde{X}_t = \tilde{X}_\varepsilon + \frac{\varepsilon-t}{2} > \frac{\varepsilon-t}{2}$, thus contradicting the continuity of \tilde{X} in $t = 0$.

Now suppose there was $t_1 > 0$ with $\tilde{X}_{t_1} > 0$. Then the set $A = \{t \in [0, t_1] : \tilde{X}_t = 0\}$ is non-empty and bounded from above. Set $t_0 := \sup A$. From the continuity of \tilde{X} it follows that $\tilde{X}_{t_0} = 0$ and hence $t_0 < t_1$. Thus $\hat{X}_t := \tilde{X}_{t-t_0}$ defines a solution of (1.3) with $\hat{X}_0 = 0$ and $\hat{X}_t > 0$ on $(0, t_1 - t_0)$. But we have already ruled out the existence of such a solution.

Thus \tilde{X} can never be positive and, by analogous arguments, \tilde{X} can never be negative. But neither can we have for $\varepsilon > 0$ that $\tilde{X}_t = 0$ for all $t \in (0, \varepsilon)$, since that would imply $\tilde{X}_t = \int_0^t (\frac{1}{2} - \text{sgn}(0)) ds = \frac{t}{2} \neq 0$. \square

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