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On the large deviations for Engel's, Sylvester's series and Cantor's products

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Abstract

In this paper, we study the large deviations for Engel's series, Sylvester's series and Cantor's products from number theory.

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For a real number, there are various ways to represent it as an expansion of digits. For example, perhaps the most well-known expansion is continued fractions. Statistical properties of the digits in the representations of real numbers have been well studied in the literature, see e.g. Kuzmin [5], Borel [2], Rényi [11], Raikov [7], Riesz [12], Ryll-Nardzewski [13], Rényi [8], Lévy [6], Rényi and Révész [9], Erdős et al. [4] and for a brief summary see e.g. Rényi [10].

One particular class of representations of a real number is Engel's series, and it is related to Sylvester's series and Cantor's products, see e.g. Erdős et al. [4] and Rényi [8]. The statistical properties, including the central limit theorem, law of iterated logarithms have been well understood for Engel's series, Sylvester's series and Cantor's products. In this article, we are interested to study the large deviations for these representations.

A sequence of probability measures $(P_n)_{n\in\mathbb{N}}$ on a topological space X satisfies a large deviation principle with speed n and rate function $I : X \to \mathbb{R}$ if I is non-negative, lower semicontinuous and for any measurable set A, we have

$$-\inf_{x\in A^o} I(x) \leq \liminf_{n\to\infty}\frac{1}{n}\log P_n(A) \leq \limsup_{n\to\infty}\frac{1}{n}\log P_n(A) \leq -\inf_{x\in\overline{A}} I(x). \tag{0.1}$$

For example, one can study the probability that the empirical mean of a sequence of random variables deviates away from its ergodic mean. Those probabilities are exponentially small in general and follow a large deviation principle. For an introduction to the theory of large deviations, we refer to Dembo and Zeitouni [3] and Varadhan [14].

We will obtain the large deviation principles for Engel's series in Section 1, Sylvester's series in Section 2 and Cantor's products in Section 3.

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1 Engel's Series

For any real number $0 < x < 1$, it can be represented in the form of Engel's series,

$$x = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \cdots + \frac{1}{q_1 q_2 \cdots q_n} + \cdots , \tag{1.1}$$

where $q_n \in \mathbb{N}$ and $q_{n+1} \geq q_n \geq 2$ for any $n \in \mathbb{N}$.

Borel [1] announced, without proof, that for a.e. x,

$$\lim_{n\to\infty} \sqrt[n]{q_n} = e. \tag{1.2}$$

Lévy [6] proved this and also pointed out that $\log(q_{n+1}/q_n)$ are in a certain sense asymptotically independent and identially distributed as exponential random variables. Moreover, Lévy [6] showed that

$$\lim_{n\to\infty} \mathbb{P}\left\{ x \in (0,1) : \frac{\log q_n - n}{\sqrt{n}} \leq t \right\} = \Phi(t), \tag{1.3}$$

where $\mathbb{P}$ is the Lebesgue measure and $\Phi(t) := \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$. The proof was simplified in Erdős et al. [4] by observing that the sequence q_n follows a homogeneous Markov chain with transition probabilities given by

$$\mathbb{P}(q_n = k | q_{n-1} = j) = \frac{j-1}{k(k-1)}, \qquad k \geq j. \tag{1.4}$$

Rényi and Révész [9] showed that (1.3) holds for any probability measure $\mathbb{Q}$,

$$\lim_{n\to\infty} \mathbb{Q}\left(\frac{\log q_n - n}{\sqrt{n}} \leq t \right) = \Phi(t), \tag{1.5}$$

if $\mathbb{Q}$ is absolutely continuous with respect to the Lebesgue measure.

Now, we can ask the question, what is the probability of the rate event that $\frac{\log q_n}{n}$ deviates away from its ergodic mean? Informally speaking, we are interested to estimate the small probability $\mathbb{Q}(\frac{\log q_n}{n} \simeq x)$, where $x \neq 1$. That leads to the studies of large deviations.

Before we go to the statement and proof of the large deviations result for Engel's series, let us first state and prove the following two lemmas.

Lemma 1.1. *Assume x is a uniformly distributed random variable on $(0,1)$ and let $(q_n)_{n=1}^{\infty}$ be the Engel's series defined in (1.1). Then,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[e^{\theta \log q_n}] = \begin{cases} -\log 2 & \text{if } \theta \leq -1 \\ \log\left(\frac{1}{1-\theta}\right) & \text{if } -1 < \theta < 1 \\ +\infty & \text{if } \theta \geq 1 \end{cases} . \tag{1.6}$$

Proof. Recall that q_n is a homogeneous Markov chain with transition probability $\mathbb{P}(q_n = k | q_{n-1} = j) = \frac{j-1}{k(k-1)}$, for $k \geq j$. Large deviations theory for homogeneous Markov chain for the compact state space is well known, see e.g. Dembo and Zeitouni [3]. However, in our case, $(q_n)_{n=1}^{\infty}$ is supported on $\mathbb{N} \backslash \{1\}$ and the general results do not apply directly. Some careful analysis is needed.

First, observe that for any $\theta \geq 1$,

$$\mathbb{E}[e^{\theta \log q_n}] \geq \mathbb{E}[q_n] \geq \mathbb{E}[q_1] = \sum_{k=2}^{\infty} \frac{k}{k(k-1)} = \infty. \tag{1.7}$$

Next, let us show that for any $-1 < \theta < 1$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[e^{\theta \log q_n}\right] = \log\left(\frac{1}{1-\theta}\right). \tag{1.8}$$

For any $\theta < 1$,

$$\begin{aligned}
\mathbb{E}[e^{\theta \log q_n}] &= \sum_{k=2}^{\infty} \mathbb{P}(q_n = k) k^{\theta} \\
&= \sum_{k=2}^{\infty} \sum_{j=2}^{k} \mathbb{P}(q_{n-1} = j) \frac{j-1}{k(k-1)} k^{\theta} \\
&= \sum_{j=2}^{\infty} \mathbb{P}(q_{n-1} = j) j^{\theta} \sum_{k=j}^{\infty} \frac{j-1}{k(k-1)} \left(\frac{k}{j}\right)^{\theta}.
\end{aligned} \tag{1.9}$$

For any $\epsilon > 0$,

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^{2-\theta}} dx &= \sum_{k=j}^{\infty} \int_{\frac{k}{j}}^{\frac{k+1}{j}} \frac{1}{x^{2-\theta}} dx \\
&\leq \sum_{k=j}^{\infty} \frac{1}{j} \left(\frac{k}{j}\right)^{\theta-2} \\
&\leq (1+\epsilon) \sum_{k=j}^{\infty} \frac{j-1}{k(k-1)} \left(\frac{k}{j}\right)^{\theta},
\end{aligned} \tag{1.10}$$

for $j \geq N$, where N is a sufficiently large positive integer depending on $\epsilon > 0$. Therefore, we have

$$\begin{aligned}
\sum_{k=N}^{\infty} \mathbb{P}(q_n = k) k^{\theta} &= \sum_{k=N}^{\infty} \sum_{j=2}^{k} \mathbb{P}(q_{n-1} = j) \frac{j-1}{k(k-1)} k^{\theta} \\
&\geq \sum_{k=N}^{\infty} \sum_{j=N}^{k} \mathbb{P}(q_{n-1} = j) \frac{j-1}{k(k-1)} k^{\theta} \\
&\geq \sum_{j=N}^{\infty} \mathbb{P}(q_{n-1} = j) j^{\theta} \sum_{k=j}^{\infty} \frac{j-1}{k(k-1)} \left(\frac{k}{j}\right)^{\theta} \\
&\geq \frac{1}{1+\epsilon} \int_1^{\infty} \frac{1}{x^{2-\theta}} dx \sum_{j=N}^{\infty} \mathbb{P}(q_{n-1} = j) j^{\theta} \\
&= \frac{1}{1+\epsilon} \frac{1}{1-\theta} \sum_{j=N}^{\infty} \mathbb{P}(q_{n-1} = j) j^{\theta}.
\end{aligned} \tag{1.11}$$

This implies that there exists some constant $M > 0$, so that

$$\sum_{k=N}^{\infty} \mathbb{P}(q_n = k) k^{\theta} \geq M \left(\frac{1}{1-\theta} \frac{1}{1+\epsilon}\right)^n. \tag{1.12}$$

On the other hand,

$$\mathbb{E}[e^{\theta \log q_n}] - \sum_{k=N}^{\infty} \mathbb{P}(q_n = k) k^{\theta} = \sum_{k=2}^{N-1} \mathbb{P}(q_n = k) k^{\theta} \leq \frac{C(N)}{2^n}, \tag{1.13}$$

for some positive constant $C(N)$ only depending on N. To see this, notice that

$$\sum_{k=2}^{N-1} \mathbb{P}(q_n = k)k^\theta \leq (N-1)^{|\theta|} \sum_{k=2}^{N-1} \mathbb{P}(q_n = k).$$

In Erdős et al. [4], it was proved that

$$\sum_{k=2}^{N-1} \mathbb{P}(q_n = k) = \frac{1}{(n-1)!} \int_0^\infty u^{n-1} e^{-u} [1 - (1 - e^{-u})^{N-1}] du. \tag{1.14}$$

Note that for any $0 < y < 1$ and $\alpha \geq 1$, $(1-y)^\alpha \geq 1 - \alpha y$. Thus, for $N \geq 2$,

$$\sum_{k=2}^{N-1} \mathbb{P}(q_n = k) \leq \frac{1}{(n-1)!} \int_0^\infty u^{n-1} e^{-u} (N-1) e^{-u} du = \frac{N-1}{2^n}. \tag{1.15}$$

Hence, by (1.12) and (1.13), for any $-1 < \theta < 1$, for $\epsilon > 0$ sufficiently small,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}[e^{\theta \log q_n}] \geq \log \left(\frac{1}{1-\theta} \frac{1}{1+\epsilon} \right). \tag{1.16}$$

Since it holds for any $\epsilon > 0$, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}[e^{\theta \log q_n}] \geq \log \left(\frac{1}{1-\theta} \right). \tag{1.17}$$

On the other hand, for any ϵ,

$$\begin{aligned} \int_1^\infty \frac{1}{x^{2-\theta}} dx &= \sum_{k=j}^\infty \int_{\frac{k}{j}}^{\frac{k+1}{j}} \frac{1}{x^{2-\theta}} dx \qquad (1.18) \\ &\geq \sum_{k=j}^\infty \frac{1}{j} \left(\frac{k+1}{j} \right)^{\theta-2} \\ &\geq (1-\epsilon) \sum_{k=j}^\infty \frac{j-1}{k(k-1)} \left(\frac{k}{j} \right)^\theta, \end{aligned}$$

for $j \geq N$, where N is sufficiently large that depends on ϵ. Therefore, for any $\theta < 1$,

$$\begin{aligned} \sum_{k=N}^\infty \mathbb{P}(q_n = k)k^\theta &= \sum_{j=N}^\infty \mathbb{P}(q_{n-1} = j) j^\theta \sum_{k=j}^\infty \frac{j-1}{k(k-1)} \left(\frac{k}{j} \right)^\theta \qquad (1.19) \\ &\leq \frac{1}{1-\epsilon} \frac{1}{1-\theta} \sum_{j=N+1}^\infty \mathbb{P}(q_{n-1} = j) j^\theta. \end{aligned}$$

Together with (1.13), for any $-1 < \theta < 1$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}[e^{\theta \log q_n}] \leq \log \left(\frac{1}{1-\theta} \right). \tag{1.20}$$

Next, let us show that for any $\theta < -1$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[e^{\theta \log q_n} \right] = -\log 2. \tag{1.21}$$

On the one hand,

$$\mathbb{E}\left[e^{\theta \log q_n} \right] = \sum_{k=2}^\infty \mathbb{P}(q_n = k)k^\theta \geq \mathbb{P}(q_n = 2)2^\theta, \tag{1.22}$$

and $\mathbb{P}(q_n = 2) = \frac{1}{2^n}$. Thus,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[e^{\theta \log q_n}\right] \geq -\log 2. \tag{1.23}$$

On the other hand, Erdős et al. [4] showed that for any $k \in \mathbb{N}$,

$$\mathbb{P}(q_n \leq k) = \frac{1}{(n-1)!} \int_0^\infty u^{n-1} e^{-u} [1 - (1-e^{-u})^{k-1}] du. \tag{1.24}$$

Therefore, for any $\theta < -1$,

$$\begin{aligned} \sum_{k=2}^\infty \mathbb{P}(q_n = k) k^\theta &= \frac{1}{(n-1)!} \int_0^\infty u^{n-1} e^{-u} \sum_{k=2}^\infty \left[(1-e^{-u})^{k-2} - (1-e^{-u})^{k-1}\right] k^\theta du \qquad (1.25) \\ &= \frac{1}{(n-1)!} \int_0^\infty u^{n-1} e^{-2u} \sum_{k=2}^\infty (1-e^{-u})^{k-2} k^\theta du \\ &\leq \frac{1}{(n-1)!} \int_0^\infty u^{n-1} e^{-2u} du \sum_{k=2}^\infty k^\theta \\ &= \frac{1}{2^n} \sum_{k=2}^\infty k^\theta. \end{aligned}$$

This gives us the desired upper bound. Hence, we showed (1.21) for any $\theta < -1$.

Finally, for $\theta = -1$, since $q_n \geq 2$, for any $\delta > 0$, we have

$$\mathbb{E}\left[e^{(-1-\delta)\log q_n}\right] \leq \mathbb{E}\left[e^{-\log q_n}\right] \leq \mathbb{E}\left[e^{(-1+\delta)\log q_n}\right]. \tag{1.26}$$

By (1.8), (1.21), and $\lim_{\theta \downarrow -1} \log\left(\frac{1}{1-\theta}\right) = -\log 2$, we get

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[e^{-\log q_n}\right] = -\log 2. \tag{1.27}$$

□

Theorem 1.2. *Let $\mathbb{Q}$ be any probability measure that is equivalent to $\mathbb{P}$ so that $\frac{d\mathbb{P}}{d\mathbb{Q}} \in L^p(\mathbb{Q})$ and $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^p(\mathbb{P})$ for any $1 < p < \infty$, where $\mathbb{P}$ is the Lebesgue measure on $(0,1)$. Then, $\mathbb{Q}(\frac{\log q_n}{n} \in \cdot)$ satisfies a large deviation principle with rate function*

$$I(x) = \begin{cases} x - 1 - \log x & \text{if } x > \frac{1}{2} \\ \log 2 - x & \text{if } 0 \leq x \leq \frac{1}{2} \\ +\infty & \text{otherwise} \end{cases}. \tag{1.28}$$

Proof. Let $\mathbb{P}$ be the Lebesgue measure on $(0,1)$. By Lemma 1.1 and Gärtner-Ellis theorem (see e.g. Dembo and Zeitouni [3]), $\mathbb{P}(\frac{\log q_n}{n} \in \cdot)$ satisfies a large deviation principle with rate function

$$I(x) = \sup_{\theta \in \mathbb{R}} \{\theta x - \Gamma(\theta)\}, \tag{1.29}$$

where $\Gamma(\theta) := \frac{1}{n} \log \mathbb{E}[e^{\theta \log q_n}]$ exists and is given by (1.6). It is not hard to compute that $\sup_{\theta \in \mathbb{R}}\{\theta x - \Gamma(\theta)\}$ is the same as (1.28).

Let $\mathbb{Q}$ be any probability measure that is equivalent to $\mathbb{P}$ so that $\frac{d\mathbb{P}}{d\mathbb{Q}} \in L^p(\mathbb{Q})$ and $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^p(\mathbb{P})$ for any $1 < p < \infty$. Then, for any Borel set A and positive numbers p, q so that $\frac{1}{p} + \frac{1}{q} = 1$, by Hölder's inequality,

$$\begin{aligned}\mathbb{P}\left(\frac{\log q_n}{n} \in A\right) &= \mathbb{E}^{\mathbb{Q}}\left[1_{\frac{\log q_n}{n} \in A} \frac{d\mathbb{P}}{d\mathbb{Q}}\right] \\ &\leq \mathbb{Q}\left(\frac{\log q_n}{n} \in A\right)^{\frac{1}{p}} \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)^q\right]^{\frac{1}{q}}.\end{aligned} \tag{1.30}$$

Therefore,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log q_n}{n} \in A\right) \leq \frac{1}{p} \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{Q}\left(\frac{\log q_n}{n} \in A\right). \tag{1.31}$$

Since it holds for any $p > 1$, we get

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log q_n}{n} \in A\right) \leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{Q}\left(\frac{\log q_n}{n} \in A\right). \tag{1.32}$$

Similarly, one can show that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{Q}\left(\frac{\log q_n}{n} \in A\right) \leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\log q_n}{n} \in A\right). \tag{1.33}$$

Therefore the large deviation principle also holds for $\mathbb{Q}(\frac{\log q_n}{n} \in \cdot)$ with the same rate function $I(x)$. □

Remark 1.3. *Prof. S. R. S. Varadhan pointed out to the author that if we consider the conditional probability measures $\mathbb{P}(\cdot|q_1 = q)$, $q \geq 2$, $q \in \mathbb{N}$, then, one can show that $\mathbb{P}(\frac{\log q_n}{n} \in \cdot|q_1 = q)$ satisfies a large deviation principle with rate function*

$$I_q(x) = \begin{cases} x - 1 - \log x & \text{if } x > \frac{1}{q} \\ \log q - (q-1)x & \text{if } 0 \leq x \leq \frac{1}{q} \\ +\infty & \text{otherwise} \end{cases}. \tag{1.34}$$

Remark 1.4. *Lévy [6] pointed out that for uniformly distributed x on $(0,1)$, as $n \to \infty$, $\log \frac{q_n(x)}{q_{n-1}(x)}$ are asymptotically close to i.i.d. exponential random variables with mean 1. However, as we have seen in Theorem 1.2, the rate function $I(x)$ coincides with the rate function for the empirical mean of i.i.d. exponentially distributed random variables only when $x \geq \frac{1}{2}$. That indicates that in the context of tail probabilities and large deviations, the approximation of $\frac{q_n(x)}{q_{n-1}(x)}$ by i.i.d. exponential random variables is not enough. What is interesting though, is that we will see later in Section 2 and Section 3 that the rate functions of the large deviations for Sylvester's series and Cantor's products coincide with the rate function of the exponentials.*

2 Sylvester's Series

For a real number $x \in (0,1)$, Sylverster's series, also known as Engel's series of the second kind, is defined via the expansion

$$x = \frac{1}{Q_1} + \frac{1}{Q_2} + \cdots + \frac{1}{Q_n} + \cdots, \tag{2.1}$$

where $Q_{n+1} \geq Q_n(Q_n - 1) + 1$, $Q_n \geq 2$, $Q_n \in \mathbb{N}$. Let $\mathbb{P}$ be the probability measure so that x is a uniformly distributed random variable on $(0,1)$. Then, we have, see e.g. Erdős et al. [4]

$$\mathbb{P}(Q_1 = k_1, Q_2 = k_2, \ldots, Q_n = k_n) = \frac{1}{k_n(k_n - 1)}, \tag{2.2}$$

provided that $k_1 \geq 2$ and $k_j \geq k_{j-1}(k_{j-1}-1)+1$ for any $2 \leq j \leq n$. Thus, Q_n is a homogenous Markov chain and

$$\mathbb{P}(Q_n = k | Q_{n-1} = j) = \frac{j(j-1)}{k(k-1)}, \tag{2.3}$$

for any $j \geq 2$ and $k \geq j(j-1)+1$. Erdős et al. [4] proved the following central limit theorem.

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log \frac{Q_n}{Q_1\cdots Q_{n-1}} - n}{\sqrt{n}} \leq t\right) = \Phi(t), \tag{2.4}$$

for any $t \in \mathbb{R}$, where $\Phi(t) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^t e^{-u^2/2}du$.

Lemma 2.1. *Assume x is a uniformly distributed random variable on $(0,1)$ and let $(Q_n)_{n=1}^\infty$ be the Sylvester's series defined in* (2.1)*. Then,*

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left[\left(\frac{Q_n}{Q_1\cdots Q_{n-1}}\right)^\theta\right] = \begin{cases} \log\left(\frac{1}{1-\theta}\right) & \text{if } \theta < 1 \\ +\infty & \text{if } \theta \geq 1 \end{cases}. \tag{2.5}$$

Proof. For any $\theta \geq 1$,

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{Q_n}{Q_1\cdots Q_{n-1}}\right)^\theta\right] &= \sum_{k_1,\ldots,k_n}\left(\frac{k_n}{k_1\cdots k_{n-1}}\right)^\theta \frac{1}{k_n(k_n-1)} \\
&\geq \sum_{k_1,\ldots,k_n}\frac{k_n}{k_1^\theta\cdots k_{n-1}^\theta}\frac{1}{k_n(k_n-1)} \\
&= \sum_{k_1,\ldots,k_{n-1}}\frac{1}{k_1^\theta\cdots k_{n-1}^\theta}\sum_{k_n}\frac{1}{k_n-1} \\
&= \infty,
\end{aligned} \tag{2.6}$$

since $\sum_{k_n\geq N}\frac{1}{k_n-1} = \infty$ for any $N \in \mathbb{N}$.

On the other hand, for any $\theta < 1$,

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{Q_n}{Q_1\cdots Q_{n-1}}\right)^\theta\right] \\
&= \sum_{k_1,\ldots,k_n}\left(\frac{k_n}{k_1\cdots k_{n-1}}\right)^\theta \frac{1}{k_n(k_n-1)} \\
&= \sum_{k_1,\ldots,k_{n-1}}\left(\frac{k_{n-1}}{k_1\cdots k_{n-2}}\right)^\theta \sum_{k_n\geq k_{n-1}(k_{n-1}-1)+1}\frac{k_n^\theta}{(k_{n-1}^2)^\theta}\frac{1}{k_n(k_n-1)}.
\end{aligned} \tag{2.7}$$

For any $\epsilon > 0$, there exists a sufficiently large $N \in \mathbb{N}$ so that for any $j \geq N$,

$$(1-\epsilon)\frac{1}{1-\theta} \leq \sum_{k\geq j(j-1)+1}\frac{k^\theta}{j^{2\theta}k(k-1)} \leq (1+\epsilon)\frac{1}{1-\theta}. \tag{2.8}$$

Since $k_n \geq 2$ for any $n \in \mathbb{N}$, we have $k_n \geq k_{n-1}(k_{n-1}-1)+1 \geq k_{n-1}+1$ and thus $k_n \geq n+1$ for any $n \in \mathbb{N}$.

Therefore, for any $n \geq N$, we have

$$\frac{1-\epsilon}{1-\theta}\mathbb{E}\left[\left(\frac{Q_{n-1}}{Q_1\cdots Q_{n-2}}\right)^\theta\right] \leq \mathbb{E}\left[\left(\frac{Q_n}{Q_1\cdots Q_{n-1}}\right)^\theta\right] \leq \frac{1+\epsilon}{1-\theta}\mathbb{E}\left[\left(\frac{Q_{n-1}}{Q_1\cdots Q_{n-2}}\right)^\theta\right]. \tag{2.9}$$

Hence,

$$\log\left(\frac{1-\epsilon}{1-\theta}\right) \leq \liminf_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left[\left(\frac{Q_n}{Q_1\cdots Q_{n-1}}\right)^{\theta}\right] \tag{2.10}$$
$$\leq \limsup_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left[\left(\frac{Q_n}{Q_1\cdots Q_{n-1}}\right)^{\theta}\right] \leq \left(\frac{1+\epsilon}{1-\theta}\right).$$

Since it is true for any $\epsilon > 0$, (2.5) holds. □

Theorem 2.2. *Let $\mathbb{Q}$ be any probability measure that is equivalent to $\mathbb{P}$ so that $\frac{d\mathbb{P}}{d\mathbb{Q}} \in L^p(\mathbb{Q})$ and $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^p(\mathbb{P})$ for any $1 < p < \infty$, where $\mathbb{P}$ is the Lebesgue measure on $(0,1)$. Then, $\mathbb{Q}(\frac{1}{n}\log(\frac{Q_n}{Q_1\cdots Q_{n-1}}) \in \cdot)$ satisfies a large deviation principle with rate function*

$$I(x) = \begin{cases} x - 1 + \log x & \text{if } x > 0 \\ +\infty & \text{otherwise} \end{cases}. \tag{2.11}$$

Proof. Following the last part of the proof of Theorem 1.2, it is sufficient to consider Lebesgue measure $\mathbb{P}$ on $(0,1)$. Lemma 2.1 and Gärtner-Ellis theorem imply that

$$\mathbb{P}\left(\frac{1}{n}\log(\frac{Q_n}{Q_1\cdots Q_{n-1}}) \in \cdot\right)$$

satisfies a large deviation principle with the rate function given by (2.11). □

3 Cantor's Products

Consider a real number $1 < x < 2$. Cantor's product is defined via, see e.g. Rényi [8]

$$x = \prod_{n=1}^{\infty}\left(1 + \frac{1}{S_n}\right), \tag{3.1}$$

where $S_n \in \mathbb{N}$ and $S_{n+1} \geq S_n^2$. Let $\mathbb{P}$ be the probability measure under which x is uniformly distributed on $(1,2)$. Rényi [8] showed that S_n is a Markov chain with transition probabilities

$$\mathbb{P}(S_{n+1} = k | S_n = j) = \frac{j^2 - 1}{k(k-1)}, \qquad k \geq j^2, j \geq 2, k, j \in \mathbb{N}. \tag{3.2}$$

Rényi showed the strong law of large numbers, $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty}\left(\frac{S_n}{S_1\cdots S_{n-1}}\right)^{\frac{1}{n}} = e, \tag{3.3}$$

and he also obtained central limit theorem,

$$\lim_{n\to\infty}\mathbb{P}\left(\frac{\log\frac{S_n}{S_1\cdots S_{n-1}} - n}{\sqrt{n}} \leq t\right) = \Phi(t), \tag{3.4}$$

for any $t \in \mathbb{R}$, where $\Phi(t) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-u^2/2}du$.

Similar to the proof of Theorem 2.2, we obtain the following large deviations. The proof is omitted.

Theorem 3.1. *Let $\mathbb{Q}$ be any probability measure that is equivalent to $\mathbb{P}$ so that $\frac{d\mathbb{P}}{d\mathbb{Q}} \in L^p(\mathbb{Q})$ and $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^p(\mathbb{P})$ for any $1 < p < \infty$, where $\mathbb{P}$ is the Lebesgue measure on $(1,2)$. Then, $\mathbb{P}(\frac{1}{n}\log(\frac{S_n}{S_1\cdots S_{n-1}}) \in \cdot)$ satisfies a large deviation principle with rate function*

$$I(x) = \begin{cases} x - 1 + \log x & \text{if } x > 0 \\ +\infty & \text{otherwise} \end{cases}. \tag{3.5}$$

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