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On the size of the largest cluster in $2D$ critical percolation

J. van den Berg* R. Conijn†

Abstract

We consider (near-)critical percolation on the square lattice. Let $\mathcal{M}_n$ be the size of the largest open cluster contained in the box $[-n,n]^2$, and let $\pi(n)$ be the probability that there is an open path from O to the boundary of the box. It is well-known (see [2]) that for all $0 < a < b$ the probability that $\mathcal{M}_n$ is smaller than $an^2\pi(n)$ and the probability that $\mathcal{M}_n$ is larger than $bn^2\pi(n)$ are bounded away from 0 as $n \to \infty$. It is a natural question, which arises for instance in the study of so-called frozen-percolation processes, if a similar result holds for the probability that $\mathcal{M}_n$ is *between* $an^2\pi(n)$ and $bn^2\pi(n)$. By a suitable partition of the box, and a careful construction involving the building blocks, we show that the answer to this question is affirmative. The 'sublinearity' of $1/\pi(n)$ appears to be essential for the argument.

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1 Introduction and main result

Consider bond percolation on $\mathbb{Z}^2$ with parameter p. (See [3] for a general introduction to percolation theory.) Let $\Lambda_n = [-n,n]^2 \cap \mathbb{Z}^2$ and let, for $v \in \Lambda_n$, $\mathcal{C}_n(v)$ denote the size of the open cluster of v inside the box Λ_n:

$$\mathcal{C}_n(v) = |\{w \in \Lambda_n : v \leftrightarrow w, \quad \text{inside } \Lambda_n\}|,$$

where we use the standard notation $v \leftrightarrow w$ for the existence of an open path from v to w, and where the addition 'inside Λ_n' means that we require the existence of such a path which is located entirely in Λ_n. For a set $A \subset \mathbb{Z}^2$ we denote by ∂A the (internal) boundary of A:

$$\partial A = \{v \in A : \exists w \notin A : (v,w) \text{ is an edge }\}.$$

The remaining part of A will be called the interior of A. Let $\pi_p(n)$ be the probability $\mathbb{P}_p(O \leftrightarrow \partial\Lambda_n)$. For simplicity we write $\pi(n)$ for $\pi_{\frac{1}{2}}(n)$.

We are interested in the size of 'large' open clusters in Λ_n for the case where p is equal (or close) to the critical value $1/2$. It is known in the literature that, informally speaking, the size of the largest open cluster is typically of order $n^2\pi(n)$: For any $c > 0$, there is a 'reasonable' probability that it is larger (smaller) than $cn^2\pi(n)$, and this probability goes to 0 uniformly in n as $c \to \infty$ ($c \to 0$). (See [1], [2]; see also [4] Section 3.) However, the question whether for all $0 < a < b$ there is a 'reasonable' probability

*CWI and VU University Amsterdam, The Netherlands. E-mail: J.van.den.Berg@cwi.nl
†VU University Amsterdam, The Netherlands. E-mail: R.P.Conijn@vu.nl

that there is an open cluster with size *between* $an^2\pi(n)$ and $bn^2\pi(n)$ has not been investigated in the literature.

This question, which is also natural by itself, arises e.g. in the study of finite-parameter frozen-percolation models. In these models each edge is closed at time 0 and 'tries' to become open at some random time, independently of the other edges. However, an open cluster stops growing as soon as its size has reached a certain (large) value M, the parameter of the model. (See [9] where this was studied for the case where the 'size' of a cluster is defined as its diameter instead of its volume.) The investigation of such processes leads to the question how two open clusters which both have size of order M but smaller than M, merge to a cluster of size bigger than M, which in turn leads to the question at the end of the previous paragraph. To state our main result, an affirmative answer to that question, we first need a few more definitions.

For $k, l \in \mathbb{N}$, we denote by $HC(k,l)$ the event that there is an open horizontal crossing in the box $[0,k] \times [0,l]$. (This is an open path from the left side to the right side of the box, of which all vertices, except the starting and end point, are in the interior of the box). Let the "characteristic length" be as defined in e.g. [6] and [7]: For a fixed $\epsilon \in (0, \frac{1}{2})$:

$$L(p) = L_\epsilon(p) = \begin{cases} \min\{n \in \mathbb{N} : \mathbb{P}_p(HC(n,n)) \leq \epsilon\} & \text{if } p < \frac{1}{2}, \\ \min\{n \in \mathbb{N} : \mathbb{P}_p(HC(n,n)) > 1-\epsilon\} & \text{if } p > \frac{1}{2}, \end{cases} \tag{1.1}$$

and $L(\frac{1}{2}) = \infty$. The precise value of ϵ is not essential. Throughout this paper we will consider it as being fixed, and therefore we omit it from our notation.

As said before, our main question concerns the existence of *some* open cluster in Λ_n with size in some specific interval. The proof we obtained gives, with only a tiny bit of extra work, something stronger; it shows that with 'reasonable' probability the *maximal* open cluster has this property. Therefore we state our main result in this stronger form (and remark that we do not know an essentially simpler proof of the original weaker form):

Denote by $\mathcal{M}_n$ the size of the maximal open cluster in Λ_n. More precisely,

$$\mathcal{M}_n = \max_{v \in \Lambda_n} \mathcal{C}_n(v).$$

Theorem 1.1. *Let $0 < a < b$. There exist $\delta > 0$ and $N \in \mathbb{N}$ such that, for all $n \geq N$ and all p with $L(p) \geq n$,*

$$\mathbb{P}_p\left(\mathcal{M}_n \in (an^2\pi(n), bn^2\pi(n))\right) > \delta.$$

The proof is given in Section 3. Section 2 will list the main ingredients used in the proof. The proof involves a suitable partition of Λ_n in smaller boxes and annuli. A brief and informal summary is given in the beginning of Section 3, after the description of these objects.

2 Ingredients for the proof of Theorem 1.1

We will make amply use of standard RSW results of the following form: For all $l > 0$ there exists $\delta(l) > 0$ such that, for all k and all p with $L(p) \geq k$, $\mathbb{P}_p(HC(k, \lceil lk \rceil)) \geq \delta(l)$. For a set W of vertices define

$$\tilde{\mathcal{C}}(W) = |\{v \in W : v \leftrightarrow \partial W\}|. \tag{2.1}$$

For $k, n \in \mathbb{N}$, we use the notation $\Lambda_{k,n}$ for the rectangle $[-k,k] \times [-n,n]$. We will use the following properties of $\pi_p(n)$ from the literature.

Theorem 2.1. *There exist $\alpha, C_1, \cdots, C_6 > 0$ such that:*

(i) For all $m \le n$:

$$C_1(\frac{n}{m})^{\alpha} \le \frac{\pi(m)}{\pi(n)} \le C_2(\frac{n}{m})^{\frac{1}{2}}.$$

(ii) For all $n \in \mathbb{N}$:

$$\sum_{k=0}^{n} \pi(k) \le C_3 \cdot n\pi(n).$$

(iii) For all $p \in (0,1)$ and all $n \le L(p)$,

$$C_4\pi(n) \le \pi_p(n) \le C_5\pi(n).$$

(iv) For all $k, n \in \mathbb{N}$ and p with $L(p) \ge k \wedge n$,

$$\mathbb{E}_p[\tilde{\mathcal{C}}(\Lambda_{k,n})] \le C_6 kn\, \pi(k \wedge n).$$

Proof. The inequalities in (i) are well-known. (The first follows easily from RSW arguments, and the second goes back to [8]; see also for example [2]). Part (ii) follows from (7) in [5]. Part (iii) is Theorem 1 in [6]. Part (iv), of which versions are explicitly in the literature (see e.g. [5], [6] and [7]), is proved as follows (where we assume that $k \le n$):

$$\mathbb{E}_p[\tilde{\mathcal{C}}(\Lambda_{k,n})] = \sum_{v \in \Lambda_{k,n}} \mathbb{P}_p(v \leftrightarrow \partial\Lambda_{k,n})$$

$$\le \sum_{v \in \Lambda_{k,n}} \pi_p(d(v, \partial\Lambda_{k,n})) \ \le\ 8n \sum_{l=0}^{k} \pi_p(l) \le C_6 nk\pi(k),$$

where the last inequality uses part (ii) and (iii). □

Define

$$Y(m) = |\{v \in \Lambda_m : v \leftrightarrow \partial\Lambda_{2m}\}|.$$

We need the following result for the distribution of $Y(m)$, which is essentially in [1] and (for the special case $p = 1/2$) [5].

Theorem 2.2. *There exist $\delta_1, C_7 > 0$ such that, for all $p \in (0,1)$ and all $m \le L(p)$:*

$$\mathbb{P}_p(Y(m) \ge C_7 m^2 \pi(m)) \ge \delta_1. \tag{2.2}$$

Proof. By Lemma 6.1 in [1] there exists $C_8 > 0$ such that for all $p \in (0,1)$ and $m \le L(p)$, $\mathbb{E}_p[(Y(m))^2] \le C_8(m^2\pi(m))^2$. Further, by the definition of $\pi(n)$ and parts (i) and (iii) of Theorem 2.1, there exists $C_9 > 0$ such that $\mathbb{E}_p[Y(m)] \ge C_9 m^2 \pi(m)$. These two inequalities, and the one-sided Chebyshev's inequality, give Theorem 2.2. □

It was shown in [5] (and extended/generalized in [1] and [2]) that $\mathcal{M}_n$, the size of the largest open cluster in Λ_n, is typically of order $n^2\pi(n)$. In particular, its expectation has an upper and a lower bound which are linear in $n^2\pi(n)$. In the proof of Theorem 1.1 we use the following result from [2].

Theorem 2.3. ([2] Thm. 3.1 (i), Thm. 3.3 (ii))
Let p_n be a sequence, such that $n \le L(p_n)$ for all n. Then for all $K > 0$,

$$\liminf_{n\to\infty} \mathbb{P}_{p_n}\left(\frac{\mathcal{M}_n}{n^2\pi(n)} < K\right) > 0.$$

Finally, to streamline the arguments in Section 3.5 at the end of the proof of Theorem 1.1, we state here the following fact about 'steering' the outcome of the sum of independent random variables. It is a simple observation rather than a lemma, and versions of it have without doubt been used in the probability literature in various contexts.

Lemma 2.4. *Let $0 < \alpha < \beta$, and let $k \in \mathbb{N}$ be such that $\alpha/k < (\beta - \alpha)/2$. Further, let $\eta_1, \eta_2 > 0$ and let $(X_i)_{1 \leq i \leq k}$ be independent random variables, (not necesarrily identically distributed) which satisfy the following:*

$$\begin{aligned} \mathbb{P}\left(X_i \in (\frac{\alpha}{k}, \frac{\beta-\alpha}{2})\right) &\geq \eta_1; \\ \mathbb{P}\left(X_i \leq \frac{\beta-\alpha}{2k}\right) &\geq \eta_2. \end{aligned}$$

Then

$$\mathbb{P}\left(\sum_{i=1}^{k} X_i \in (\alpha, \beta)\right) \geq (\eta_1 \wedge \eta_2)^k.$$

Proof. For $1 \leq i \leq k$ we say that 'step i is proper' if

$$X_i \begin{cases} \in (\frac{\alpha}{k}, \frac{\beta-\alpha}{2}) & \text{if } \sum_{j=1}^{i-1} X_j < \alpha \\ \leq \frac{\beta-\alpha}{2k} & \text{otherwise.} \end{cases}$$

It is clear that if all steps $i = 1, \cdots, k$ are proper, then $\sum_{i=1}^{k} X_i \in (\alpha, \beta)$. It is also easy to see that, for each i, the conditional probability that step i is proper, given that all steps $1, \cdots, i-1$ are proper, is at least $\min(\eta_1, \eta_2)$. □

3 Proof of Theorem 1.1

We first give a proof for the special case $p = 1/2$ and therefore drop the subscript p from the notation $\mathbb{P}_p$ and $\mathbb{E}_p$. At the end of Section 3.5 we point out that (due to the 'uniformity' of the ingredients stated in Section 2) the proof for the general case is essentially the same.

3.1 More definitions, and brief outline of the proof

Let $s, t \in \mathbb{N}$ with $t \leq \frac{1}{3}s$. The proof involves a construction using the following boxes and annuli.

$$\begin{aligned} B_{0,0} &= \Lambda_s. \\ A^I_{0,0} &= \Lambda_s \setminus \Lambda_{s-t}; \quad A^{II}_{0,0} = \Lambda_{s-t} \setminus \Lambda_{s-2t}; \quad A^{III}_{0,0} = \Lambda_{s-2t} \setminus \Lambda_{s-3t}. \\ A'_{0,0} &= A^I_{0,0} \cup A^{II}_{0,0} \cup A^{III}_{0,0}. \\ H_{0,0} &= ([0, 4t] \times [0, t] + (s - 2t, 0)) \cap \mathbb{Z}^2. \\ V_{0,0} &= ([0, t] \times [0, 4t] + (0, s - 2t)) \cap \mathbb{Z}^2. \end{aligned}$$

More generally, for all $i, j \in \mathbb{Z}$ we define $B_{i,j} = B_{0,0} + (2is, 2js)$, $A^I_{i,j} = A^I_{0,0} + (2is, 2js)$, etcetera.

Before we go on, we give a very brief and informal summary of the proof (see Figure 1): The box Λ_n in the statement of the theorem will be (roughly) partitioned in m^2 boxes $B_{i,j}$ defined above, where the s (and hence m) and t will be chosen appropriately, depending on n, a and b. (For elegance/symmetry we take the number m odd). We will 'construct' an open cluster of which the 'skeleton' consists of circuits in the annuli $A^{II}_{i,j}$, 'glued' together by connections in the 'corridors' $V_{i,j}$ and $H_{i,j}$. (The other annuli

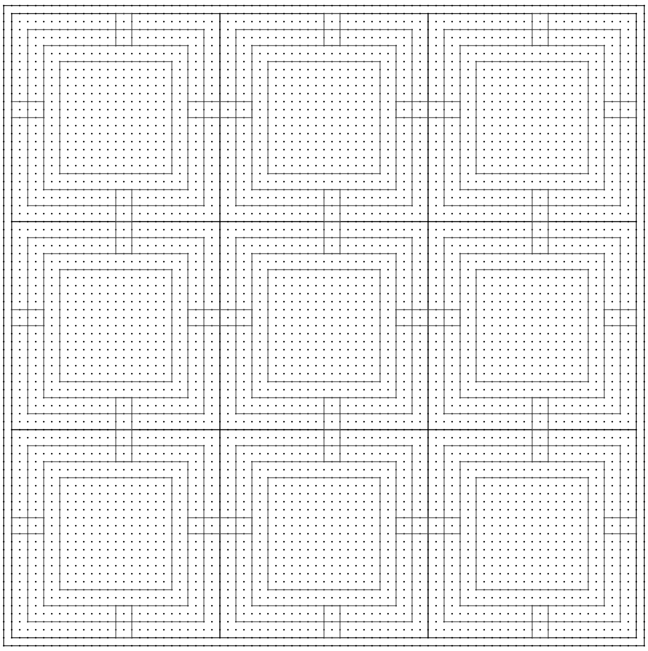

Figure 1: Partition of the box Λ_n. Here $n = 40, m = 3, s = 13, t = 2$.

defined above will be used for technical reasons in the proof). The setup is such that the contributions from the different $B_{i,j}$'s to the total cluster size are roughly independent, and that these contributions can be 'steered' to get the total sum inside the desired interval. In some sense this replaces the original problem for the box Λ_n by a similar problem, but now for the smaller boxes $B_{i,j}$. Apart from the technicalities involving the control of local dependencies, there is a subtle aspect in the proof related to the asymptotic behaviour of $\pi(n)$: Although the precise power-law behaviour of $\pi(n)$ is not important, it seems to be essential for the arguments that the exponent in a power-law upper bound is strictly smaller than 1 (see the note at the end of the proof of Lemma 3.6)).

Now we continue with the precise constructions mentioned above. First we give some more notation and definitions. Let E denote the set of edges of $\mathbb{Z}^2$ and $\Omega = \{0,1\}^E$. For $\omega \in \Omega$ and $F \subset E$ we will write $\omega_F \in \{0,1\}^F$ for the 'restriction' $(\omega_e, e \in F)$ of ω to F. Let $A \subset \Omega$ and $W \subset \mathbb{Z}^2$. We write $E(W)$ for the set of all edges of which both endpoints are in W. Informally, we use the notation $A(W)$ for the set of all configurations $\omega \in \Omega$ that belong to A or can be turned to an element of A by modifying ω outside $E(W)$. More precisely,

$$A(W) = \{\omega_{E(W)} : \omega \in A\} \times \{0,1\}^{E \setminus E(W)}. \tag{3.1}$$

We denote by $\tilde{O}^{s,t}$ the event that (i) - (iii) below occur (see Figure 2):

(i) $\forall i, j \in \mathbb{Z}$: the annulus $A^{II}_{i,j}$ contains an open circuit;

(ii) $\forall i, j \in \mathbb{Z}$: $H_{i,j}$ contains an open connection between the two widest open circuits in the annuli $A^{II}_{i,j}$ and $A^{II}_{i+1,j}$;

(iii) $\forall i, j \in \mathbb{Z}$: $V_{i,j}$ contains an open connection between the two widest open circuits in the annuli $A^{II}_{i,j}$ and $A^{II}_{i,j+1}$.

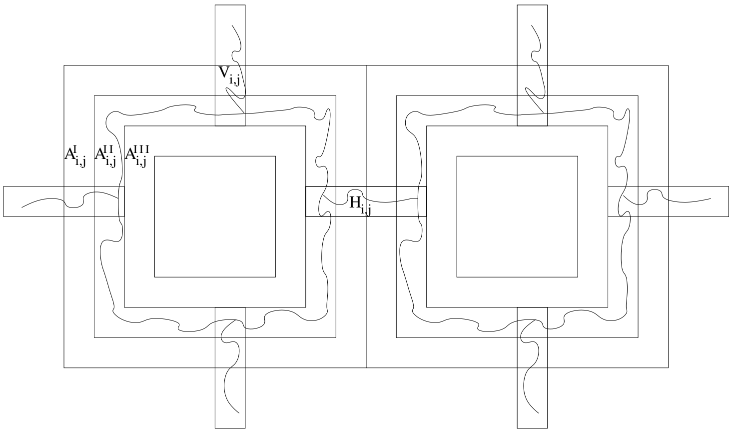

Figure 2: Illustration of the event $\tilde{O}^{s,t}$.

The introduction of this event looks meaningless since it has probability 0. It will only be used to give a 'compact' description of the following events (which do play a key role in the proof).

Definition 3.1. *Let $m, s, t \in \mathbb{N}$, with $t \leq \frac{1}{3}s$ and m odd. Let $i, j \in \mathbb{Z}$. We define, using notation* (3.1)*, the following events:*

$$\begin{aligned} O^{m,s,t} &= \tilde{O}^{s,t}(\Lambda_{ms}). \\ O^{s,t}_{i,j} &= \tilde{O}^{s,t}(B_{i,j}). \end{aligned}$$

Remark: From now on, for given m, s, t, the indices i, j under consideration will always be assumed to be in the set $\{-\frac{1}{2}(m-1), \cdots, 0, \cdots, \frac{1}{2}(m-1)\}$.

3.2 Expected cluster size in a narrow annulus

For a circuit γ in $\mathbb{Z}^2$ we denote by $\mathrm{Int}(\gamma)$ the bounded connected component of $\mathbb{Z}^2 \setminus \gamma$, and define

$$\mathcal{C}^{\gamma} = |\{v \in \mathrm{Int}(\gamma) : v \leftrightarrow \gamma\}|. \tag{3.2}$$

Further, for all i, j, let $\gamma_{i,j}$ denote the widest open circuit in the annulus $A^{II}_{i,j}$, and define, for $W \subset \Lambda_n$,

$$\mathcal{C}_{i,j}(W) = |\{v \in W : v \leftrightarrow \gamma_{i,j}\}|. \tag{3.3}$$

If there is no open circuit in $A^{II}_{i,j}$, then $\mathcal{C}_{i,j}(W) = 0$.

Recall the definition of $\tilde{\mathcal{C}}$ in (2.1).

Lemma 3.2. *There exists a constant $C_{10} > 0$ such that for all $s \in \mathbb{N}$, $t \leq \frac{1}{3}s$ and all i, j:*

$$\mathbb{E}[\mathcal{C}_{i,j}(A'_{i,j}) | O^{s,t}_{i,j}] \leq \mathbb{E}[\tilde{\mathcal{C}}(A'_{i,j}) | O^{s,t}_{i,j}] \leq C_{10} s t \pi(t).$$

Proof. The first inequality follows immediately from the fact that, on the event $O^{s,t}_{i,j}$, the circuit $\gamma_{i,j}$ is connected to the boundary of $B_{i,j}$ and hence $\mathcal{C}_{i,j}(A'_{i,j})$ is smaller than or equal to $\tilde{\mathcal{C}}(A'_{i,j})$. We prove the second inequality. Without loss of generality we take $i = j = 0$. We subdivide the annulus $A' = A'_{0,0} = \bigcup_{l=1}^{8} A'_l$, where A'_1, A'_2, A'_3, A'_4 are the $(3t \times 3t)$-squares in the four corners, and A'_5, A'_6, A'_7, A'_8 the remaining rectangles (see Figure 3). Note that $\tilde{\mathcal{C}}(A') \leq \sum_{l=1}^{8} \tilde{\mathcal{C}}(A'_l)$. Hence it is sufficient to show that there is a constant C_{11} such that for each $l = 1, \cdots, 8$,

$$\mathbb{E}[\tilde{\mathcal{C}}(A'_l) | O^{s,t}_{0,0}] \leq C_{11} s t \pi(t). \tag{3.4}$$

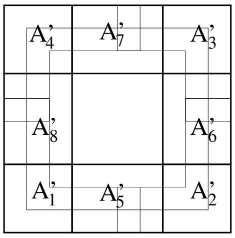

Figure 3: The subdivision of $A'_{0,0}$ in $A'_1, \cdots, A'_8$.

By symmetry we only have to handle the cases $l = 1$ and $l = 5$. For each l the l.h.s. of (3.4) is

$$\mathbb{E}[\tilde{\mathcal{C}}(A'_l)|O^{s,t}_{0,0}] = \frac{1}{\mathbb{P}(O^{s,t}_{0,0})} \sum_{v \in A'_l} \mathbb{P}(v \leftrightarrow \partial A'_l; O^{s,t}_{0,0}). \tag{3.5}$$

Recall the notation (3.1). For each $v \in A'_1$, obviously,

$$\mathbb{P}(v \leftrightarrow \partial A'_1; O^{s,t}_{0,0}) \leq \mathbb{P}(v \leftrightarrow \partial A'_1)\mathbb{P}(O^{s,t}_{0,0}(A' \setminus A'_1)). \tag{3.6}$$

Further, informally speaking, the event $O^{s,t}_{0,0}(A' \setminus A'_1)$ can, with a 'local surgery involving a bounded cost in terms of probability', be turned into the event $O^{s,t}_{0,0}$. More precisely, if $O^{s,t}_{0,0}(A' \setminus A'_1)$ holds, and there is a horizontal open crossing of the rectangle $[-s, -s + 6t] \times [-s + t, -s + 2t]$ and of the square $[-s, -s + 3t] \times [-s + 3t, -s + 6t]$, and a vertical open crossing of the rectangle $[-s + t, -s + 2t] \times [-s, -s + 6t]$ and of the square $[-s + 3t, -s + 6t] \times [-s, -s + 3t]$, then the event $O^{s,t}_{0,0}$ holds. Hence, by RSW (and FKG) we have a positive constant C_{12} such that $\mathbb{P}(O^{s,t}_{0,0}(A' \setminus A'_1)) \leq C_{12}\mathbb{P}(O^{s,t}_{0,0})$. Combining this with (3.5) and (3.6) gives

$$\mathbb{E}[\tilde{\mathcal{C}}(A'_1)|O^{s,t}_{0,0}] \leq C_{12} \sum_{v \in A'_1} \mathbb{P}(v \leftrightarrow \partial A'_1). \tag{3.7}$$

For the case $l = 5$ let, for $v \in A'_5$, $R = ([v_1 - t, v_1 + t] \times [-s, -s + 3t]) \cup V_{0,-1}$ and let $G(v)$ be the event that there are vertical open crossings in $[v_1 - t, v_1 - \frac{1}{2}t] \times [-s, -s + 3t]$ and $[v_1 + \frac{1}{2}t, v_1 + t] \times [-s, -s + 3t]$. By RSW (and FKG) arguments we have a positive constant C_{13} such that

$$\mathbb{P}(v \leftrightarrow \partial A'_5; O^{s,t}_{0,0}) \leq C_{13}\mathbb{P}(v \leftrightarrow \partial A'_5; O^{s,t}_{0,0}; G(v)) \leq C_{13}\mathbb{P}(v \leftrightarrow \partial A'_5)\mathbb{P}(O^{s,t}_{0,0}(R)). \tag{3.8}$$

Again some 'local surgery' gives a constant C'_{13} such that $\mathbb{P}(O^{s,t}_{0,0}(R)) \leq C'_{13}\mathbb{P}(O^{s,t}_{0,0})$. This, combined with (3.5) and (3.8), gives

$$\mathbb{E}[\tilde{\mathcal{C}}(A'_5)|O^{s,t}_{0,0}] \leq C_{13}C'_{13} \sum_{v \in A'_5} \mathbb{P}(v \leftrightarrow \partial A'_5). \tag{3.9}$$

Application of part (iv) of Theorem 2.1 to the right-hand sides of (3.7) and (3.9) gives (3.4). □

3.3 Properties of nice circuits

Let m, s, t be as in Definition 3.1, and recall the Remark about the values of the indices i, j at the end of Section 3.1. Let, for each i, j, $\gamma_{i,j}$ be as in the beginning of Section 3.2, and let $\tilde{\gamma}_{i,j}$ be a deterministic circuit in the annulus $A^{II}_{i,j}$. Further we will denote the collection of all $\gamma_{i,j}$'s by (γ), and the collection of all $\tilde{\gamma}_{i,j}$'s by $(\tilde{\gamma})$.

Definition 3.3. *We say that $\tilde{\gamma}_{i,j}$ is* (s,t)-nice *if*

$$\mathbb{E}[\tilde{\mathcal{C}}(A'_{i,j}) \mid O^{s,t}_{i,j}; \gamma_{i,j} = \tilde{\gamma}_{i,j}] \le 2C_{10} st\pi(t), \tag{3.10}$$

with C_{10} as in Lemma 3.2. Further, the collection $(\tilde{\gamma})$ is called (m,s,t)-nice if each circuit in the collection is (s,t)-nice.

We define $\Gamma^{s,t}_{i,j}$ as the event that $\gamma_{i,j}$ is (s,t)-nice, and define

$$\Gamma^{m,s,t} = \bigcap_{i,j} \Gamma^{s,t}_{i,j}. \tag{3.11}$$

Recall the notation (3.3). We will study the open cluster $\mathcal{C}_{0,0}(Q)$, where

$$Q = (\Lambda_n \setminus \Lambda_{ms}) \cup (\bigcup_{i,j} A'_{i,j}).$$

Lemma 3.4. *There exist positive constants C_{14} and C_{15} such that, for all m, s and t,*

(i)

$$\mathbb{P}(\Gamma^{m,s,t} | O^{m,s,t}) \ge (C_{14})^{m^2}.$$

(ii) For all (m,s,t)-nice $(\tilde{\gamma})$ and all n with $n - ms \le t$,

$$\mathbb{P}\left(\mathcal{C}_{0,0}(Q) \le C_{15} m^2 st\pi(t) \mid O^{m,s,t}; (\gamma) = (\tilde{\gamma})\right) \ge \frac{1}{2}.$$

Proof. We claim that there is a constant $C_{16} > 0$ such that for all i, j:

$$\mathbb{P}(O^{m,s,t}; (\gamma) = (\tilde{\gamma})) \ge C_{16} \mathbb{P}(D) \mathbb{P}(O^{s,t}_{i,j}; \gamma_{i,j} = \tilde{\gamma}_{i,j}), \tag{3.12}$$

where (with the notation (3.1))

$$D = O^{m,s,t}(\Lambda_n \setminus B_{i,j}) \cap \bigcap_{\tilde{i},\tilde{j}:(\tilde{i},\tilde{j}) \ne (i,j)} \{\gamma_{\tilde{i},\tilde{j}} = \tilde{\gamma}_{\tilde{i},\tilde{j}}\}.$$

To prove this claim we write

$$G = B_{i,j} \setminus A^I_{i,j}, \qquad J = A^I_{i,j} \cup \bigcup_{(\tilde{i},\tilde{j}) \in M_{i,j}} A^I_{\tilde{i},\tilde{j}}, \qquad K = \Lambda_n \setminus (G \cup J),$$

where $M_{i,j} = \{(i-1,j), (i,j-1), (i+1,j), (i,j+1)\} \cap \{-(m-1)/2, \cdots, (m-1)/2\}^2$. Let $D_1 = O^{s,t}_{i,j} \cap \{\gamma_{i,j} = \tilde{\gamma}_{i,j}\}$. We also need an event D_2 which, informally speaking, connects the structures in the definition of D_1 with those in D. More precisely,

$$D_2 = \bigcap_{(\tilde{i},\tilde{j}) \in M_{i,j}} D_2^{\tilde{i},\tilde{j}},$$

where $D_2^{i+1,j}$ is the event that (i) $H_{i,j} \cap J$ contains a horizontal crossing and (ii) $H_{i,j} \cap A^I_{i,j}$ and $H_{i,j} \cap A^I_{i+1,j}$ both contain a vertical crossing. The other $D_2^{\tilde{i},\tilde{j}}$'s are defined similarly. By RSW (and FKG) there is a positive constant C_{16} such that $\mathbb{P}(D_2) > C_{16}$. Note that D_2 is increasing and (with the notation in Section 3.1), the event D is increasing with respect to the edges outside $E(K)$, and D_1 is increasing with respect to the edges outside $E(G)$. We get

$$\begin{aligned}
&\mathbb{P}(O^{m,s,t};(\gamma)=(\tilde{\gamma}))\geq\mathbb{P}(D\cap D_1\cap D_2)\\
&=\sum_{\omega_1\in\{0,1\}^{E(K)}}\mathbb{P}(\omega_1)\sum_{\omega_2\in\{0,1\}^{E(G)}}\mathbb{P}(D\cap D_1\cap D_2|\omega_1,\omega_2)\mathbb{P}(\omega_2)\\
&\geq\sum_{\omega_1\in\{0,1\}^{E(K)}}\mathbb{P}(\omega_1)\sum_{\omega_2\in\{0,1\}^{E(G)}}\mathbb{P}(D|\omega_1,\omega_2)\mathbb{P}(D_1|\omega_1,\omega_2)\mathbb{P}(D_2|\omega_1,\omega_2)\mathbb{P}(\omega_2)\\
&=\sum_{\omega_1\in\{0,1\}^{E(K)}}\mathbb{P}(\omega_1)\mathbb{P}(D|\omega_1)\sum_{\omega_2\in\{0,1\}^{E(G)}}\mathbb{P}(D_1|\omega_2)\mathbb{P}(D_2)\mathbb{P}(\omega_2)\\
&\geq C_{16}\mathbb{P}(D)\mathbb{P}(D_1),
\end{aligned}$$

where we used FKG in the third line, and in the fourth line we used that D doesn't depend on the configuration on G, D_1 doesn't depend on the configuration on K, and D_2 doesn't depend on the configuration on $G\cup K$. This proves the claim.

By repeating the same arguments for each $B_{i,j}$, we eventually get the following 'extension' of (3.12):

$$\mathbb{P}\left((\gamma)=(\tilde{\gamma});O^{m,s,t}\right)\geq C_{16}^{m^2}\prod_{i,j}\mathbb{P}(\gamma_{i,j}=\tilde{\gamma}_{i,j};O^{s,t}_{i,j}). \tag{3.13}$$

Now we are ready to prove part (i):

$$\begin{aligned}
\mathbb{P}(\Gamma^{m,s,t}|O^{m,s,t}) &= \frac{1}{\mathbb{P}(O^{m,s,t})}\sum_{\tilde{\gamma}:(m,s,t)\text{-nice}}\mathbb{P}\left((\gamma)=(\tilde{\gamma});O^{m,s,t}\right)\\
&\geq C_{16}^{m^2}\prod_{i,j}\mathbb{P}(\Gamma^{s,t}_{i,j}|O^{s,t}_{i,j}),
\end{aligned}\tag{3.14}$$

where the inequality follows from (3.13) and the obvious inequality $\mathbb{P}(O^{m,s,t})\leq\prod_{i,j}\mathbb{P}(O^{s,t}_{i,j})$. This gives part (i) of the lemma because for each factor in the product of the last expression in (3.14) we have, by Definition 3.3, Markov's inequality and Lemma 3.2,

$$\begin{aligned}
\mathbb{P}(\Gamma^{s,t}_{i,j}|O^{s,t}_{i,j}) &= \mathbb{P}\left(\mathbb{E}[\tilde{\mathcal{C}}(A'_{i,j})\mid O^{s,t}_{i,j};\gamma_{i,j}]\leq 2C_{10}st\pi(t)\mid O^{s,t}_{i,j}\right)\\
&\geq 1-\frac{\mathbb{E}[\tilde{\mathcal{C}}(A'_{i,j})\mid O^{s,t}_{i,j}]}{2C_{10}st\pi(t)}\geq\frac{1}{2}.
\end{aligned}$$

To prove part (ii) first note that

$$\begin{aligned}
\mathbb{E}\left[\tilde{\mathcal{C}}(A'_{i,j})|O^{m,s,t};(\gamma)=(\tilde{\gamma})\right] &= \frac{\sum_{v\in A'_{i,j}}\mathbb{P}\left(v\leftrightarrow\partial A'_{i,j};O^{m,s,t};(\gamma)=(\tilde{\gamma})\right)}{\mathbb{P}(O^{m,s,t};(\gamma)=(\tilde{\gamma}))}\\
&\leq \frac{\sum_{v\in A'_{i,j}}\mathbb{P}(v\leftrightarrow\partial A'_{i,j};O^{s,t}_{i,j};\gamma_{i,j}=\tilde{\gamma}_{i,j})\mathbb{P}(D)}{C_{16}\mathbb{P}(D)\mathbb{P}(O^{s,t}_{i,j};\gamma_{i,j}=\tilde{\gamma}_{i,j})},
\end{aligned}$$

where we used (3.12) in the denominator. Hence

$$\mathbb{E}\left[\tilde{\mathcal{C}}(A'_{i,j})|O^{m,s,t};(\gamma)=(\tilde{\gamma})\right]\leq\frac{1}{C_{16}}\mathbb{E}[\tilde{\mathcal{C}}(A'_{i,j})|O^{s,t}_{i,j};\gamma_{i,j}=\tilde{\gamma}_{i,j}]\leq\frac{2C_{10}}{C_{16}}st\pi(t), \tag{3.15}$$

where the last inequality is just the 'niceness' property (Definition 3.3) of $(\tilde{\gamma})$. To finish the proof of part (ii), note that, for each $K>0$,

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{C}_{0,0}(Q)\leq Km^2st\pi(t)|O^{m,s,t};(\gamma)=(\tilde{\gamma})\right)\geq 1-\frac{\mathbb{E}\left[\mathcal{C}_{0,0}(Q)|O^{m,s,t};(\gamma)=(\tilde{\gamma})\right]}{Km^2st\pi(t)}\\
&\geq 1-\frac{\mathbb{E}\left[\tilde{\mathcal{C}}(\Lambda_n\setminus\Lambda_{ms})\right]+\sum_{i,j}\mathbb{E}\left[\tilde{\mathcal{C}}(A'_{i,j})|O^{m,s,t};(\gamma)=(\tilde{\gamma})\right]}{Km^2st\pi(t)}.
\end{aligned}$$

Applying part (iv) of Theorem 2.1 to the first expectation in the r.h.s. of the last expression, and (3.15) to each of the other expectations gives, by choosing K sufficiently large, the desired result. This completes the proof of part (ii) of Lemma 3.4. □

3.4 Cluster-size contributions inside the circuits

In this section we write the value t (the width of the relevant annuli and 'corridors' in the construction) as $\lfloor \varepsilon s \rfloor$. A suitable value for ε (depending on the values of a and b in the statement of Theorem 1.1) will be determined in the next section. The main result in the current section concerns the contribution from the interior of a nice circuit to the cluster of that circuit. Recall the notation (3.2).

Lemma 3.5. *There exist constants $C_{17}, C_{18}, \delta_2 > 0$, and for every $\varepsilon < \frac{1}{12}$ there exists $\delta_3(\varepsilon) > 0$, such that for all $s \in \mathbb{N}$ and all $(s, \lfloor \varepsilon s \rfloor)$-nice circuits $\tilde{\gamma}_{0,0}$ in $A^{II}_{0,0}$,*

(i)

$$\mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{0,0}} \in (C_{17} s^2 \pi(s), C_{18} s^2 \pi(s)) \mid \gamma_{0,0} = \tilde{\gamma}_{0,0}\right) \geq \delta_2. \tag{3.16}$$

(ii)

$$\mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{0,0}} < 4C_{10} s \lfloor \varepsilon s \rfloor \pi(\lfloor \varepsilon s \rfloor) \mid \gamma_{0,0} = \tilde{\gamma}_{0,0}\right) \geq \delta_3(\varepsilon). \tag{3.17}$$

Proof. Let $B'_{0,0} = B_{0,0} \setminus A'_{0,0}$. Let $\hat{Y} = |\{v \in B'_{0,0} : v \leftrightarrow \partial B_{0,0}\}|$. Clearly,

$$\mathbb{P}(\mathcal{C}^{\tilde{\gamma}_{0,0}} \geq C_{17} s^2 \pi(s) \mid \gamma_{0,0} = \tilde{\gamma}_{0,0}) \geq \mathbb{P}(\hat{Y} \geq C_{17} s^2 \pi(s)), \tag{3.18}$$

which (for a suitable choice of C_{17}) by Theorem 2.2 is at least a positive constant, which we write as $2\delta_2$. To complete the proof we need to find a $C_{18} > 0$ such that

$$\mathbb{P}(\mathcal{C}^{\tilde{\gamma}_{0,0}} \geq C_{18} s^2 \pi(s) \mid \gamma_{0,0} = \tilde{\gamma}_{0,0}) \leq \delta_2. \tag{3.19}$$

To do this we look for an upper bound for $\mathbb{E}[\mathcal{C}^{\tilde{\gamma}_{0,0}} \mid \gamma_{0,0} = \tilde{\gamma}_{0,0}]$. We have

$$\begin{aligned} \mathbb{E}[\mathcal{C}^{\tilde{\gamma}_{0,0}} \mid \gamma_{0,0} = \tilde{\gamma}_{0,0}] &= \mathbb{E}[\mathcal{C}^{\tilde{\gamma}_{0,0}} \mid O^{s,\lfloor \varepsilon s \rfloor}_{0,0}; \gamma_{0,0} = \tilde{\gamma}_{0,0}] \\ &\leq \mathbb{E}[\tilde{\mathcal{C}}(B'_{0,0})] + \mathbb{E}[\tilde{\mathcal{C}}(A'_{0,0}) \mid O^{s,\lfloor \varepsilon s \rfloor}_{0,0}; \gamma_{0,0} = \tilde{\gamma}_{0,0}] \end{aligned} \tag{3.20}$$

Applying part (iv) of Theorem 2.1 to the first expectation in the last line, and the niceness property of $\tilde{\gamma}_{0,0}$ to the other expectation, shows that the l.h.s. of (3.20) is at most $C_{19} s^2 \pi(s)$. Finally, Markov's inequality gives (3.19) with $C_{18} = \frac{C_{19}}{\delta_2}$. This completes the proof of part (i).

Now we prove part (ii). Let G be the event that there is a closed dual circuit in $A^{III}_{0,0}$. On this event, let $\beta_{0,0}$ denote the innermost of such circuits. Observe that, conditioned on $\beta_{0,0}$, the configuration outside $\beta_{0,0}$ is independent of the configuration inside. Also observe that, on the event G, all vertices in the interior of $\gamma_{0,0}$ that are connected to $\gamma_{0,0}$ are in $A'_{0,0}$. By these and related simple observations we have that

$$\begin{aligned} &\mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{0,0}} < 4C_{10} s \lfloor \varepsilon s \rfloor \pi(\lfloor \varepsilon s \rfloor) \mid \gamma_{0,0} = \tilde{\gamma}_{0,0}; \beta_{0,0} = \tilde{\beta}\right) \\ &= \mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{0,0}} < 4C_{10} s \lfloor \varepsilon s \rfloor \pi(\lfloor \varepsilon s \rfloor) \mid O^{s,\lfloor \varepsilon s \rfloor}_{0,0}; \gamma_{0,0} = \tilde{\gamma}_{0,0}; \beta_{0,0} = \tilde{\beta}\right) \\ &\geq \mathbb{P}\left(\tilde{\mathcal{C}}(A'_{0,0}) < 4C_{10} s \lfloor \varepsilon s \rfloor \pi(\lfloor \varepsilon s \rfloor) \mid O^{s,\lfloor \varepsilon s \rfloor}_{0,0}; \gamma_{0,0} = \tilde{\gamma}_{0,0}\right), \end{aligned}$$

which, by Markov's inequality and because $\tilde{\gamma}_{0,0}$ is nice, is at least $1/2$. Hence, the l.h.s. of (3.17) is at least $(1/2)\mathbb{P}(G)$, which by RSW is larger than some positive constant which depends only on ε.

□

3.5 Completion of the proof of Theorem 1.1

We are now ready to prove Theorem 1.1. First we still restrict to the case $p = 1/2$. Let $0 < a < b$ be given. See the brief outline in Section 3.1. The lengths of the building blocks $B_{i,j}$ and the widths of the annuli and 'corridors' in the partition of Λ_n, will be taken proportional to n, say (roughly) xn and εxn respectively, with suitably chosen x and ε. For this purpose we will use the following lemma:

Lemma 3.6. *There exist $x > 0, \varepsilon \in (0, \frac{1}{12})$ and $N \in \mathbb{N}$, with $\frac{1}{x}$ an odd integer, such that for all $n \geq N$ the following inequalities hold:*

$$C_{17} \lfloor xn \rfloor^2 \pi(\lfloor xn \rfloor) \cdot (\frac{1}{x})^2 \quad \geq \quad an^2 \pi(n); \tag{3.21}$$

$$C_{18} \lfloor xn \rfloor^2 \pi(\lfloor xn \rfloor) \quad \leq \quad \frac{1}{3}(b-a) n^2 \pi(n); \tag{3.22}$$

$$(4C_{10} \vee C_{15})(\frac{1}{x})^2 \lfloor \varepsilon \lfloor xn \rfloor \rfloor \lfloor xn \rfloor \pi(\lfloor \varepsilon \lfloor xn \rfloor \rfloor) \quad \leq \quad \frac{1}{3}(b-a) n^2 \pi(n). \tag{3.23}$$

Proof. It is easy to see (a weak form of the lower bound in part (i) of Theorem 2.1 suffices) that if x is sufficiently small (depending on a), then (3.21) holds for all sufficiently large n. It also easily follows (now from the upper bound in the same Theorem) that if x is sufficiently small (depending on $b - a$), (3.22) holds for all sufficiently large n. Finally, for x fixed, it follows (again from the upper bound in part (i) of Theorem 2.1) that if ε is sufficiently small (depending on $b - a$ and x), then (3.23) holds for all sufficiently large n.
Note that for this last step it is essential that the exponent ($1/2$) in part (i) of Theorem 2.1 is strictly smaller than 1.
This completes the proof of Lemma 3.6. □

Now let x, ε and N be as in Lemma 3.6. Moreover we assume (which we may, because we can enlarge N if necessary) that $n - \frac{1}{x}\lfloor xn \rfloor \leq \lfloor \varepsilon \lfloor xn \rfloor \rfloor$ for all $n \geq N$. Denote by D_n the event

$$D_n = \{\exists v \in \Lambda_n : \mathcal{C}_n(v) \in (an^2\pi(n), bn^2\pi(n))\}.$$

Let $n \geq N$ and let $m = \frac{1}{x}, s = \lfloor xn \rfloor$ and $t = \lfloor \varepsilon \lfloor xn \rfloor \rfloor$. By straightforward RSW and FKG arguments, there is a $\delta_4(x, \varepsilon) > 0$ such that $\mathbb{P}(O^{m,s,t}) > \delta_4(x, \varepsilon)$. Hence

$$\mathbb{P}(D_n) \geq \delta_4(x, \varepsilon) \mathbb{P}(D_n | O^{m,s,t}). \tag{3.24}$$

From Lemma 3.4 (i) it follows that

$$\mathbb{P}(D_n | O^{m,s,t}) \geq (C_{14})^{\frac{1}{x^2}} \mathbb{P}(D_n | O^{m,s,t}; \Gamma^{m,s,t}). \tag{3.25}$$

The next step is conditioning on the widest open circuits.

$$\mathbb{P}(D_n | O^{m,s,t}; \Gamma^{m,s,t}) = \sum_{(\tilde{\gamma}):(m,s,t)\text{-nice}} \mathbb{P}(D_n | (\gamma) = (\tilde{\gamma}); O^{m,s,t}) \, \mathbb{P}((\gamma) = (\tilde{\gamma}) | O^{m,s,t}; \Gamma^{m,s,t}). \tag{3.26}$$

For each $(\tilde{\gamma})$ we denote by $\mathcal{C}^{in}_{(\tilde{\gamma})}$ the number of all vertices that are in the interior of a circuit in the collection $(\tilde{\gamma})$ and connected to that circuit, and by $\mathcal{C}^{out}_{(\tilde{\gamma})}$ the number of vertices outside these circuits that are connected to one or more of these circuits, plus the number of vertices on these circuits. We have

$$\begin{aligned}
&\mathbb{P}(D_n|(\gamma)=(\tilde{\gamma});O^{m,s,t}) \qquad (3.27)\\
\geq\ & \mathbb{P}\left(\frac{\mathcal{C}^{in}_{(\tilde{\gamma})}}{n^2\pi(n)}\in(a,b-\frac{1}{3}(b-a))\mid\frac{\mathcal{C}^{out}_{(\tilde{\gamma})}}{n^2\pi(n)}\leq\frac{1}{3}(b-a);(\gamma)=(\tilde{\gamma});O^{m,s,t}\right)\\
&\cdot\mathbb{P}\left(\mathcal{C}^{out}_{(\tilde{\gamma})}\leq\frac{1}{3}(b-a)n^2\pi(n)\mid(\gamma)=(\tilde{\gamma});O^{m,s,t}\right)\\
\geq\ & \frac{1}{2}\mathbb{P}\left(\frac{\mathcal{C}^{in}_{(\tilde{\gamma})}}{n^2\pi(n)}\in(a,\frac{1}{3}a+\frac{2}{3}b)\right),
\end{aligned}$$

where the last inequality holds by (3.23) and Lemma 3.4 (ii), and because the configurations in the interiors of the $\tilde{\gamma}_{i,j}$'s are obviously independent of the event conditioned on in the expression in the r.h.s. of the first inequality. Note that $\mathcal{C}^{in}_{(\tilde{\gamma})}=\sum_{i,j}\mathcal{C}^{\tilde{\gamma}_{i,j}}$. The $\mathcal{C}^{\tilde{\gamma}_{i,j}}$'s are independent and for each i,j we have

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{i,j}}\in(ax^2n^2\pi(n),\frac{1}{3}(b-a)n^2\pi(n))\right)\\
&\overset{(3.21),(3.22)}{\geq}\ \mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{i,j}}\in\left(C_{17}\lfloor xn\rfloor^2\pi(\lfloor xn\rfloor),C_{18}\lfloor xn\rfloor^2\pi(\lfloor xn\rfloor)\right)\right)\overset{\text{Lem.}3.5(i)}{\geq}\delta_2,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{i,j}}\leq x^2\frac{1}{3}(b-a)n^2\pi(n)\right)\\
&\overset{(3.23)}{\geq}\ \mathbb{P}\left(\mathcal{C}^{\tilde{\gamma}_{i,j}}\leq 4C_{10}\lfloor\varepsilon\lfloor xn\rfloor\rfloor\lfloor xn\rfloor\pi(\lfloor\varepsilon\lfloor xn\rfloor\rfloor)\right)\overset{\text{Lem.}3.5(ii)}{\geq}\delta_3(\varepsilon).
\end{aligned}$$

Hence the conditions of Lemma 2.4 (with $k=(1/x)^2$, $\alpha=an^2\pi(n)$, $\beta=(\frac{1}{3}a+\frac{2}{3}b)n^2\pi(n)$) are satisfied. Hence, by that lemma the l.h.s. of (3.27) is at least $\frac{1}{2}(\delta_2\wedge\delta_3(\varepsilon))^{(1/x)^2}$. Together with (3.24) - (3.26) this shows that

$$\mathbb{P}(D_n)>\delta_5, \qquad (3.28)$$

with δ_5 a positive constant which depends only on a and b.

Now we will show that, by the way we 'constructed' the open cluster, a similar result holds for the *maximal* open cluster in Λ_n. First note that the 'constructed' cluster has the property that it contains an open horizontal and an open vertical crossing of the box Λ_{ms}. Also note that there is at most one open cluster with this property. Given the exact location of the (unique) open cluster with this property, the conditional probability that it is the maximal cluster in Λ_n is, if its size is larger than $an^2\pi(n)$, clearly larger than or equal to the probability that the remaining part of Λ_n contains no open cluster of size larger than $an^2\pi(n)$. By obvious monotonicity this probability is at least $\mathbb{P}(\mathcal{M}_n\leq an^2\pi(n))$, which by Theorem 2.3 is at least some positive constant δ_2 (which depends only on a). This argument gives

$$\mathbb{P}\left(\mathcal{M}_n\in(an^2\pi(n),bn^2\pi(n))\right)\geq\delta_2\delta_5,$$

which completes the proof of Theorem 1.1 for $p=1/2$.

Now let, more generally, p be such that $L(p)\geq n$. It is straightforward to check that (due to the 'uniformity' in p of the results in Section 2) each step in the proof remains essentially valid. For instance, it is easy to see from the arguments used that Lemma 3.2 (now with $\mathbb{P}$ replaced by $\mathbb{P}_p$) remains valid as long as $L(p)\geq s$. Since we take $s\leq n$ (application of) this lemma (and, similarly, the other lemma's) can be carried out as before. This completes the proof of Theorem 1.1.

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