Electron. Commun. Probab. **17** (2012), no. 13, 1–8.
DOI: 10.1214/ECP.v17-1865
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Laha-Lukacs properties of some free processes*

Wiktor Ejsmont†

Abstract

We study the Laha-Lukacs property of the free Meixner laws (processes). We prove that some families of free Meixner distribution have the linear regression function. We also show that this families have the property of quadratic conditional variances.

Keywords: Free Meixner law; conditional expectation; free cumulants; Laha-Lukacs theorem; noncommutative quadratic regression; von Neumann algebras.
AMS MSC 2010: 46L54 , 46L53.
Submitted to ECP on July 28, 2011, final version accepted on March 7, 2012.

1 Introduction

The original motivation for this paper comes from a desire to understand the results on conditional expectation in the work of Bożejko and Bryc [7]. They proved, that if the first conditional moment is a linear regression and conditional variances are quadratic functions, then the corresponding variables have free Meixner type laws (theorem 3.2). An open problem in this area is a converse of their theorem. We will show that free Meixner variables satisfy the condition from theorem 3.2 of [7]. In particular, we will apply this result to describe characterization of free Lévy processes. It is natural to study relations between classical and free probability. We will present a theorem which is the free non-commutative analog of the classical result by Wesolowski [15, 16]. Let us mention that he followed the argument in [9]. Laha and Lukacs in [9] characterized all the (classical) Meixner distributions using a quadratic regression property. Wesołowski proved that in classical probability the quadratic conditional variance characterize a subclass of Lévy processes. Similar results have been obtained in boolean probability by Anshelevich [2]. He showed that in the boolean theory the Laha-Lukacs property characterizes only the Bernoulli distributions. It is worthwhile to mention the work of Bryc [8], where the Laha-Lukacs property for q-Gaussian processes was shown. Bryc proved that classical processes corresponding to operators which satisfy a q-commutation relations have linear regressions and quadratic conditional variances. For $q = 0$ we have the free case, so that his result is a special case of the free Wigner's semicircle elements, which we consider.

*Research partially supported by Polish MNiSW grant NN201 364436.
†Mathematical Institute University of Wrocław pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland.
E-mail: wiktor.ejsmont@math.uni.wroc.pl

2 Free Meixner laws, free cumulants and conditional expectation

Classical Meixner distributions first appeared in the theory of orthogonal polynomials in the work of Meixner [10]. In free probability the Meixner systems of polynomials were introduced by Anshelevich [1], Bożejko, Leinert, Speicher [6] and Saitoh and Yoshida [12]. They showed that free Meixner system can be classified into six types of laws: the Wigner semicircle, the free Poisson, the free Pascal (free negative binomial), the free Gamma, a law that we will call pure free Meixner and the free binomial law. We assume that our probability space is a von Neumann algebra $\mathcal{A}$ with a normal faithful tracial state $\tau : \mathcal{A} \to \mathbb{C}$ i.e., $\tau()$ is linear, weak*-continuous and $\tau(\mathbb{X}\mathbb{Y}) = \tau(\mathbb{Y}\mathbb{X})$, $\tau(\mathbb{I}) = 1$, $\tau(\mathbb{X}\mathbb{X}^*) \geq 0$ and $\tau(\mathbb{X}\mathbb{X}^*) = 0$ implies $\mathbb{X} = 0$ for all $\mathbb{X}, \mathbb{Y} \in \mathcal{A}$. A (noncommutative) random variable $\mathbb{X}$ is a self-adjoint ($\mathbb{X} = \mathbb{X}^*$) element of $\mathcal{A}$. We are interested in the two-parameter family of compactly supported probability measures (so that their moments grow at a geometric rate) $\{\mu_{a,b} : a \in \mathbb{R}, b \geq -1\}$ with the Cauchy-Stieltjes transform given by the formula

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-y}\mu_{a,b}(dy) = \frac{(1+2b)z + a - \sqrt{(z-a)^2 - 4(1+b)}}{2(bz^2 + az + 1)}, \tag{2.1}$$

where the branch of the analytic square root should be determined by the condition that $Im(z) > 0 \Rightarrow Im(G_\mu(z)) \leqslant 0$ (see [12]). Cauchy-Stieltjes transform of μ is a function G_μ defined on the upper half plane $\mathbb{C}^+ = \{s + ti | s, t \in \mathbb{R}, t > 0\}$ and takes values in the lower half plane $\mathbb{C}^- = \{s + ti | s, t \in \mathbb{R}, t \leq 0\}$.
Equation (2.1) describes the distribution with mean zero and variance one (see [12]). The moment generating function, which corresponds to the equation (2.1), has the form

$$M(z) = \frac{1}{z}G_\mu(\frac{1}{z}) = \frac{1 + 2b + az - \sqrt{(1-za)^2 - 4z^2(1+b)}}{2(z^2 + az + b)}, \tag{2.2}$$

for $|z|$ small enough. The $\mathcal{R}$-transform of a random variable $\mathbb{X}$ is $\mathcal{R}_{\mathbb{X}}(z) = \sum_{i=0}^{\infty} R_{i+1}(\mathbb{X})z^i$, where $R_i(\mathbb{X})$ is a sequences defined by (2.4) (see [4] for more details). For reader's convenience we recall that the $\mathcal{R}$-transform corresponding to $M(z)$ is equal to

$$\mathcal{R}_\mu(z) = \frac{2z}{1 - za + \sqrt{(1-za)^2 - 4z^2 b}}, \tag{2.3}$$

where the analytic square root is chosen so that $\lim_{z\to 0} \mathcal{R}_\mu(z) = 0$ (see [12]). If $\mathbb{X}$ has the distribution $\mu_{a,b}$, then sometimes we will write $\mathcal{R}_{\mathbb{X}}$ for the $\mathcal{R}$-transform of $\mathbb{X}$. For particular values of a, b the law of $\mathbb{X}$ is:

- the Wigner's semicircle law if $a = b = 0$;
- the free Poisson law if $b = 0$ and $a \neq 0$;
- the free Pascal (negative binomial) type law if $b > 0$ and $a^2 > 4b$;
- the free Gamma law if $b > 0$ and $a^2 = 4b$;
- the pure free Meixner law if $b > 0$ and $a^2 < 4b$;
- the free binomial law $-1 \leq b < 0$.

Given a seqence $\mathbb{X}_1, \mathbb{X}_2, \ldots$ let $\mathbb{C}\langle\mathbb{X}_1, \ldots, \mathbb{X}_n\rangle$ denote the non-commutative ring of polynomials in variables $\mathbb{X}_1, \ldots, \mathbb{X}_n$. The free (non-crossing) cumulants are the k-linear maps $R_k : \mathbb{C}\langle\mathbb{X}_1, \ldots, \mathbb{X}_k\rangle \to \mathbb{C}$ defined by the recursive formula (connecting them with mixed moments)

$$\tau(\mathbb{X}_1\mathbb{X}_2 \ldots \mathbb{X}_n) = \sum_{\nu \in NC(n)} R_\nu(\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_n), \tag{2.4}$$

where

$$R_\nu(\mathbb{X}_1, \mathbb{X}_2, \dots, \mathbb{X}_n) := \Pi_{B\in\nu} R_{|B|}(\mathbb{X}_i : i \in B) \tag{2.5}$$

and $NC(n)$ is the set of all non-crossing partitions of $\{1, 2, \dots, n\}$ (see [11, 13]). Sometimes we will write $R_k(\mathbb{X}) = R_k(\mathbb{X}, \dots, \mathbb{X})$.

Random variables $\mathbb{X}_1, \dots, \mathbb{X}_n$ are freely independent (free) if, for every $n \geq 1$ and every non-constant choice of $\mathbb{Y}_i \in \{\mathbb{X}_1, \dots, \mathbb{X}_n\}$, where $i \in \{1, \dots, k\}$ (for some positive integer k) we get $R_k(\mathbb{Y}_1, \dots, \mathbb{Y}_k) = 0$.

The following theorem shows connection between the Cauchy transform and the $\mathcal{R}$-transform. Part (B) describes additive free convolution. We shall apply this theorem without further comment (it can be found in Nica, Speicher [11]).

Theorem 2.1. *(A)The relation between the Cauchy transform $G_\mu(z)$ and the $\mathcal{R}_\mu(z)$-transform of a probability measure μ is given by*

$$G_\mu(\mathcal{R}_\mu(z) + 1/z) = z. \tag{2.6}$$

(B) The $\mathcal{R}$-transform linearizes the free convolution, i.e. if μ and ν are (compactly supported) probability measures on $\mathbf{R}$, then we have

$$\mathcal{R}_{\mu\boxplus\nu} = \mathcal{R}_\mu + \mathcal{R}_\nu, \tag{2.7}$$

where $\boxplus$ denotes the free convolution (the free convolution $\boxplus$ of measures μ, ν is the law of $\mathbb{X} + \mathbb{Y}$ where $\mathbb{X}, \mathbb{Y}$ are free and have laws μ, ν respectively).

Below we introduce Lemma 4.1 of [7], which we will use in the main theorem to calculate the moment generating function of free convolution.

Lemma 2.2. *Suppose $\mathbb{X}$,$\mathbb{Y}$ are free, self-adjoint and $\mathbb{X}/\sqrt{\alpha}$,$\mathbb{Y}/\sqrt{\beta}$ have the free Meixner laws $\mu_{a/\sqrt{\alpha},b/\alpha}$ and $\mu_{a/\sqrt{\beta},b/\beta}$ respectively, where $\alpha, \beta > 0$, $\alpha + \beta = 1$ and $a \in \mathbb{R}, b \geq -1$. Then the moment generating function $M(z)$ for $\mathbb{X} + \mathbb{Y}$ satisfies quadratic equation*

$$(z^2 + az + b)M^2(z) - (1 + az + 2b)M(z) + 1 + b = 0. \tag{2.8}$$

If $\mathcal{B} \subset \mathcal{A}$ is a von Neumann subalgebra and $\mathcal{A}$ has a trace τ, then there exists a unique conditional expectation from $\mathcal{A}$ to $\mathcal{B}$ with respect to τ, which we denote by $\tau(.|\mathcal{B})$. This map is a weakly continuous, completely positive, identity preserving, contraction and it is characterized by the property that, for any $\mathbb{X} \in \mathcal{A}$, $\tau(XY) = \tau(\tau(X|\mathcal{B})Y)$ for any $\mathbb{Y} \in \mathcal{B}$ (see [5, 14]). For fixed $\mathbb{X} \in \mathcal{A}$ by $\tau(.|\mathbb{X})$ we denote the conditional expectation corresponding to the von Neumann algebra $\mathcal{B}$ generated by $\mathbb{X}$. The conditional variance is defined as usual

$$Var(\mathbb{X}|\mathcal{B}) = \tau((\mathbb{X} - \tau(\mathbb{X}|\mathcal{B}))^2|\mathcal{B}). \tag{2.9}$$

The following lemma has been proven in [7].

Lemma 2.3. *Let W be a (self-adjoint) element of the von Neumann algebra $\mathcal{A}$, generated by a self-adjoint $V \in \mathcal{A}$. If, for all $n \geq 1$ we have $\tau(UV^n) = \tau(WV^n)$, then*

$$\tau(U|V) = W. \tag{2.10}$$

Now we introduce the following notation:

- $NC(n+2)$ is the set of all non-crossing partitions of $\{1, 2, \ldots, n+2\}$,
- $NC'(n+2)$ is the set of all non-crossing partitions of $\{1, 2, \ldots, n+2\}$ which separate 1 and 2,
- $NC''(n+2)$ is the set of all non-crossing partitions of $\{1, 2, \ldots, n+2\}$ with the first two elements in the same block.

Lemma 2.4. *Let $\mathbb{Z}$ be a (self-adjoint) element of the von Neumann $\mathcal{A}$. Then*

$$\sum_{\nu \in NC'(n+2)} R_\nu(\mathbb{Z}) = \sum_{i=1}^{n} m_i \sum_{\nu \in NC''(n+2-i)} R_\nu(\mathbb{Z}) + m_1 m_{n+1} \tag{2.11}$$

where $m_i := \tau(\mathbb{Z}^i)$.

Proof. At first, we will consider partitions with singleton 1, i.e. $\pi \in NC'(n+2)$ and $\pi = \{V_1, \ldots, V_k\}$ where $V_1 = \{1\}$. It is clear that the sum over all non-crossing partitions of this form corresponds to the term $m_1 m_{n+1}$. On the other hand, for such partitions $\nu \in NC'(n+2)$ let $k = k(\nu) \in \{3, 4, \ldots, n+2\}$ denote the most-left element of the block containing 1. This decomposes $NC'(n+2)$ into the n classes $NC'_j(n+2) = \{\nu \in NC'(n+2) : k(\nu) = j+2\}, j = 1, 2, \ldots, n$. The set $NC'_j(n+2)$ can be identified with the product $NC(j) \times NC''(n+2-j)$. Indeed, the blocks of $\nu \in NC'_j(n+2)$, which partition the elements $\{2, 3, 4, ..., j+1\}$, can be identified with an appropriate partition in $NC(j)$, and (under the additional constraint that the first two elements $1, j+2$ are in the same block) the remaining blocks, which partition the set $\{1, j+2, j+3, ..., n+2\}$, can be uniquely identified with a partition in $NC''(n+2-j)$. This gives the formula

$$\begin{aligned}\sum_{\nu \in NC'(n+2)} R_\nu(\mathbb{Z}) &= \sum_{i=1}^{n} \sum_{\nu \in NC(i)} R_\nu(\mathbb{Z}) \sum_{\nu \in NC''(n+2-i)} R_\nu(\mathbb{Z}) + m_1 m_{n+1} \\ &= \sum_{i=1}^{n} m_i \sum_{\nu \in NC''(n+2-i)} R_\nu(\mathbb{Z}) + m_1 m_{n+1},\end{aligned} \tag{2.12}$$

which proves the Lemma. □

3 The main result

The following is our main results of the paper.

Theorem 3.1. *Suppose $\mathbb{X}$,$\mathbb{Y}$ are free, self-adjoint and $\mathbb{X}/\sqrt{\alpha}$,$\mathbb{Y}/\sqrt{\beta}$ have the free Meixner laws $\mu_{a/\sqrt{\alpha},b/\alpha}$ and $\mu_{a/\sqrt{\beta},b/\beta}$ respectively, where $\alpha, \beta > 0$ and $a \in \mathbb{R}, b \geq -1$. Then*

$$\tau(\mathbb{X}|(\mathbb{X}+\mathbb{Y})) = \frac{\alpha}{\alpha+\beta}(\mathbb{X}+\mathbb{Y}) \tag{3.1}$$

$$Var(\mathbb{X}|\mathbb{X}+\mathbb{Y})$$

$$= \frac{\alpha\beta}{(b+(\alpha+\beta))(\alpha+\beta)^2}[(\alpha+\beta)^2\mathbb{I} + (\alpha+\beta)a(\mathbb{X}+\mathbb{Y}) + b(\mathbb{X}+\mathbb{Y})^2]. \tag{3.2}$$

Additionally, we assume that $b \geq max\{-\alpha, -\beta\}$ if $b < 0$ (free binomial case).

Proof. First we compute the law of $\mathbb{X}+\mathbb{Y}$. It is well-known that the $\mathcal{R}$-transform of the dilatation $D_\lambda(\mu)$ is $\lambda r_\mu(\lambda z)$ $(D_\lambda(\mu)(A) := \mu(A/\lambda))$. Multiplying variable $\mathbb{X}/\sqrt{\alpha}$ by $\sqrt{\alpha}$ we obtain

$$\mathcal{R}_{\mathbb{X}}(z) = \alpha \frac{2z}{1 - za + \sqrt{(1-za)^2 - 4z^2 b}}. \tag{3.3}$$

Similarly we compute $\mathcal{R}$-transform of the variable $\mathbb{Y}$. This allows us to find the $\mathcal{R}$-transform of $\mathbb{X}+\mathbb{Y}$ (assuming $b \neq 0$)

$$\begin{aligned}\mathcal{R}_{\mathbb{X}+\mathbb{Y}}(z) &= (\alpha+\beta)\frac{2z}{1-za+\sqrt{(1-za)^2-4z^2b}} \\ &= (\alpha+\beta)\frac{1-za-\sqrt{(1-za)^2-4z^2b}}{2zb}. \end{aligned} \tag{3.4}$$

From equations (3.3) and (3.4) it follows that

$$R_k(\mathbb{X}) = \frac{\alpha}{\alpha+\beta}R_k(\mathbb{X}+\mathbb{Y}). \tag{3.5}$$

Analogously we get

$$R_k(\mathbb{Y}) = \frac{\beta}{\alpha+\beta}R_k(\mathbb{X}+\mathbb{Y}). \tag{3.6}$$

This gives

$$\begin{aligned}\tau(\mathbb{X}(\mathbb{X}+\mathbb{Y})^n) &= \sum_{\nu\in NC(n+1)} R_\nu(\mathbb{X},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}) \\ &= \frac{\alpha}{\alpha+\beta}\sum_{\nu\in NC(n+1)} R_\nu(\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}) \\ &= \tau(\frac{\alpha}{\alpha+\beta}(\mathbb{X}+\mathbb{Y})(\mathbb{X}+\mathbb{Y})^n)\end{aligned} \tag{3.7}$$

which, by Lemma 2.3, implies that $\tau(\mathbb{X}|(\mathbb{X}+\mathbb{Y})) = \frac{\alpha}{\alpha+\beta}(\mathbb{X}+\mathbb{Y})$. Using Lemma 2.2 and simple parameters normalization (replacing α by $\frac{\alpha}{\alpha+\beta}$, β by $\frac{\beta}{\alpha+\beta}$, a by $\frac{a}{\sqrt{\alpha+\beta}}$, b by $\frac{b}{\alpha+\beta}$ and putting $z=\sqrt{\alpha+\beta}$) we obtain the moment generating series for $\mathbb{X}+\mathbb{Y}$

$$\begin{aligned}M^2(z)(b+za(\alpha+\beta)+(\alpha+\beta)^2z^2)+M(z)(-2b-(\alpha+\beta)-za(\alpha+\beta)) \\ +b+(\alpha+\beta)=0.\end{aligned} \tag{3.8}$$

Denote by $c_{n+2}=\tau((\beta\mathbb{X}-\alpha\mathbb{Y})^2(\mathbb{X}+\mathbb{Y})^n)$ $(n\geq 0)$ and $m_n=\tau((\mathbb{X}+\mathbb{Y})^n)$ $(n\geq 0)$. From equation (3.5),(3.6) we have $R_k(\beta\mathbb{X}-\alpha\mathbb{Y},\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y})=0$. From the last equality we get

$$\begin{aligned}c_{n+2} &= \sum_{\nu\in NC(n+2)} R_\nu(\beta\mathbb{X}-\alpha\mathbb{Y},\beta\mathbb{X}-\alpha\mathbb{Y},\underbrace{\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}}_{n\text{-times}}) \\ &= \sum_{\nu\in NC'(n+2)} R_\nu(\beta\mathbb{X}-\alpha\mathbb{Y},\beta\mathbb{X}-\alpha\mathbb{Y},\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}) \\ &+ \sum_{\nu\in NC''(n+2)} R_\nu(\beta\mathbb{X}-\alpha\mathbb{Y},\beta\mathbb{X}-\alpha\mathbb{Y},\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}) \\ &= \sum_{\nu\in NC''(n+2)} R_\nu(\beta\mathbb{X}-\alpha\mathbb{Y},\beta\mathbb{X}-\alpha\mathbb{Y},\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}).\end{aligned} \tag{3.9}$$

The fact that $\mathbb{X}$ and $\mathbb{Y}$ are freely independent implies that

$$\begin{aligned}&R_k(\beta\mathbb{X}-\alpha\mathbb{Y},\beta\mathbb{X}-\alpha\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}) \\ &\beta^2R_k(\mathbb{X},\mathbb{X},\mathbb{X},\dots,\mathbb{X})+\alpha^2R_k(\mathbb{Y},\mathbb{Y},\mathbb{Y},\dots,\mathbb{Y}) \\ &\overset{(3.5),(3.6)}{=} \alpha\beta R_k(\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\mathbb{X}+\mathbb{Y},\dots,\mathbb{X}+\mathbb{Y}).\end{aligned} \tag{3.10}$$

Using $m_1 = 0$ and Lemma 2.4, we get

$$\begin{aligned}
c_{n+2} = \\
\alpha\beta \sum_{\nu\in NC''(n+2)} R_\nu(\mathbb{X}+\mathbb{Y}) = \alpha\beta \sum_{\nu\in NC(n+2)\setminus NC'(n+2)} R_\nu(\mathbb{X}+\mathbb{Y}) \\
= \alpha\beta m_{n+2} - \alpha\beta \sum_{\nu\in NC'(n+2)} R_\nu(\mathbb{X}+\mathbb{Y}) \\
= \alpha\beta m_{n+2} - \alpha\beta \sum_{i=1}^{n} m_i \sum_{\nu\in NC''(n+2-i)} R_\nu(\mathbb{X}+\mathbb{Y}) \\
= \alpha\beta m_{n+2} - \sum_{i=1}^{n} m_i c_{n+2-i}.
\end{aligned} \tag{3.11}$$

We can thus compute the power series

$$\sum_{n=0}^{\infty} c_{n+2} z^{n+2} = \alpha\beta M(z) - \alpha\beta - (M(z)-1)\Big(\sum_{n=0}^{\infty} c_{n+2} z^{n+2}\Big). \tag{3.12}$$

If we denote $C(z) = \sum_{n=0}^{\infty} c_{n+2} z^{n+2}$, then this can be rewritten as

$$\alpha\beta = (\alpha\beta - C(z))M(z). \tag{3.13}$$

Thus $C(z)$ is an analytic function for small $|z|$. If in (3.8) we multiply both sides by $(\alpha\beta - C(z))$, we get

$$\begin{aligned}
\alpha\beta M(z)(b + za(\alpha+\beta) + (\alpha+\beta)^2 z^2) \\
+\alpha\beta(-2b - (\alpha+\beta) - za(\alpha+\beta)) + (b + (\alpha+\beta))(\alpha\beta - C(z)) = 0.
\end{aligned} \tag{3.14}$$

Expanding the above series $M(z) = 1 + \sum_{i=1}^{\infty} z^i m_i$, we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \alpha\beta(m_{n+2} b + (\alpha+\beta) a m_{n+1} + (\alpha+\beta)^2 m_n) z^{n+2} = \\
= \sum_{n=0}^{\infty} (b + (\alpha+\beta)) c_{n+2} z^{n+2}
\end{aligned} \tag{3.15}$$

because, by assumption, the coefficients at z and z^0 are equal. Therefore we get from equation (3.15) that

$$c_{n+2} = \alpha\beta(m_{n+2} b + (\alpha+\beta) a m_{n+1} + (\alpha+\beta)^2 m_n)/(b + (\alpha+\beta)) \tag{3.16}$$

for all $n \geq 0$ (using $b + (\alpha+\beta) > 0$). The equation (3.16) is equivalent to

$$\begin{aligned}
\tau((\beta\mathbb{X} - \alpha\mathbb{Y})^2(\mathbb{X}+\mathbb{Y})^n) \\
= \frac{\alpha\beta}{(b + (\alpha+\beta))} \tau([(\alpha+\beta)^2\mathbb{I} + (\alpha+\beta)a(\mathbb{X}+\mathbb{Y}) + b(\mathbb{X}+\mathbb{Y})^2](\mathbb{X}+\mathbb{Y})^n)
\end{aligned} \tag{3.17}$$

for all $n \geq 0$. Now we use the Lemma 2.3 which essentially shows that

$$\begin{aligned}
\tau((\beta\mathbb{X} - \alpha\mathbb{Y})^2 | \mathbb{X}+\mathbb{Y}) \\
= \frac{\alpha\beta}{(b + (\alpha+\beta))} [(\alpha+\beta)^2\mathbb{I} + (\alpha+\beta)a(\mathbb{X}+\mathbb{Y}) + b(\mathbb{X}+\mathbb{Y})^2]
\end{aligned} \tag{3.18}$$

The last equality implies, in particular, that

$$\begin{aligned} Var(\mathbb{X}|\mathbb{X}+\mathbb{Y}) &= \tau((\mathbb{X}-\tau(\mathbb{X}|\mathbb{X}+\mathbb{Y}))^2|\mathbb{X}+\mathbb{Y}) \\ &= \tau((\beta\mathbb{X}-\alpha\mathbb{Y})^2|\mathbb{X}+\mathbb{Y})/(\alpha+\beta)^2 \\ &= \frac{\alpha\beta}{(b+(\alpha+\beta))(\alpha+\beta)^2}[(\alpha+\beta)^2\mathbb{I}+(\alpha+\beta)a(\mathbb{X}+\mathbb{Y})+b(\mathbb{X}+\mathbb{Y})^2]. \end{aligned} \tag{3.19}$$

This proves the Theorem. □

The following proposition is a free version of the classical result of Wesołowski [15]. A non-commutative stochastic process (X_t) is a free Lévy process if it has free additive and stationary increments. For a more detailed discussion of free Lévy processes we refer to [3].

Proposition 3.2. *Suppose* $(\mathbb{X}_{t\geq 0})$ *is a free Lévy process such that the increments* $(\mathbb{X}_{t+s}-\mathbb{X}_t)/\sqrt{s}$ $(t,s>0)$ *have the free Meixner law* $\mu_{a/\sqrt{s},b/s}$ *(for some* $a,b>0$*). Then for all* $t<s$

$$\tau(\mathbb{X}_t|\mathbb{X}_s)=\frac{t}{s}\mathbb{X}_s \tag{3.20}$$

and

$$Var(\mathbb{X}_t|\mathbb{X}_s)=\frac{t(s-t)}{(b+s)s^2}[s^2\mathbb{I}+sa\mathbb{X}_s+b\mathbb{X}_s^2] \tag{3.21}$$

Remark 3.3. *The relation (3.20) is well-known from Bożejko, Bryc [7]. In the following calculation, we present different proof of it.*

Proof. Given $s>t>0$, let $\mathbb{X}=\mathbb{X}_t/\sqrt{t}$ and $\mathbb{Y}=(\mathbb{X}_s-\mathbb{X}_t)/\sqrt{s-t}$ be two random variables. Then $\mathbb{X},\mathbb{Y}$ are free, centered, and have distribution $\mu_{a/\sqrt{t},b/t}$ and $\mu_{a/\sqrt{s-t},b/(s-t)}$, respectively. By Theorem 3.1, we obtain equations (3.20) and (3.21) (because we have $\mathbb{X}_s=\mathbb{X}_s-\mathbb{X}_t+\mathbb{X}_t$). Thus the Proposition holds. □

Corollary 3.4. *Suppose* $(\mathbb{X}_{t\geq 0})$ *is a free Lévy process such that* $\tau(\mathbb{X}_t)=0$ *and* $\tau(\mathbb{X}_t^2)=t$*. Then* $(\mathbb{X}_{t+s}-\mathbb{X}_t)/\sqrt{s}$ $(t,s>0)$ *have the free Meixner law* $\mu_{a/\sqrt{s},b/s}$ *(for some* $a,b>0$*) if and only if*

$$Var(\mathbb{X}_t|\mathbb{X}_s)=\frac{t(s-t)}{(b+s)s^2}[s^2\mathbb{I}+sa\mathbb{X}_s+b\mathbb{X}_s^2] \tag{3.22}$$

for all $t<s$.

Proof. $\Rightarrow$: If we assume that the $(\mathbb{X}_{t+s}-\mathbb{X}_t)/\sqrt{s}$ $(t,s>0)$ have free Meixner law $\mu_{a/\sqrt{s},b/s}$, then, using Proposition 3.2, (3.22) follows. $\Leftarrow$: Assuming (3.22) and using Proposition 3.4 of [7] one can see that $(\mathbb{X}_{t+s}-\mathbb{X}_t)/\sqrt{s}$ $(t,s>0)$ have the free Meixner law $\mu_{a/\sqrt{s},b/s}$. □

Remark 3.5 (suggested by Marek Bożejko)**.** *In Proposition (and Theorem) of that paper we assume that random variables are bounded* $X_t\in\mathcal{A}$*. It will be interesting to show that the assumption can be replaced by* $X_t\in L^2(\mathcal{A},\tau)$*.*

References

[1] Anshelevich M.: Free martingale polynomials. Journal of Functional Analysis, 201, 228-261, (2003). MR-1986160

[2] Anshelevich M.: Appell polynomials and their relatives. II. Boolean theory, Indiana Univ.Math. J. 58(2), 929-968, (2009). MR-2514394

[3] Barndorff-Nielsen O.E., Thorbjørnsen S.: Self-decomposability and Lt'evy processes in free probability. Bernoulli, 8(3), 323-366, (2002).

[4] Benaych-Georges F.: Taylor expansions of $\mathcal{R}$-transforms, application to supports and moments, Indiana Univ. Math. J. 55 , 465-482, (2006). MR-2225442

[5] Biane P.: Processes with free increments. Math. Z., 227(1),143-174, (1998). MR-1605393

[6] Bożejko M., Leinert M., Speicher R.: Convolution and limit theorems for conditionally free random variables. Pacific J. Math., 175(2), 357-388, (1996). MR-1432836

[7] Bożejko M., Bryc W.: On a class of free Lévy laws related to a regression problem. J. Funct. Anal. 236, 59-77, (2006). MR-2227129

[8] Bryc W.: Classical Versions of q-Gaussian Processes:Conditional Moments and Bell's Inequality Comm. Math. Phys., 219(2), 259-270, (2001). MR-1833804

[9] Laha R. G., Lukacs E.: On a problem connected with quadratic regression. Biometrika, 47(300), 335-343, (1960). MR-0121922

[10] Meixner J.: Orthogonale Polynomsysteme mit einer besonderen Gestalt der erzeugenden Funktion. Journal of the London Mathematical Society, 9, 6-13, (1934).

[11] Nica A., Speicher R.: Lectures on the Combinatorics of Free Probability, London Mathematical Society Lecture Notes Series 365, Cambridge University Press, (2006). MR-2266879

[12] Saitoh N., Yoshida H.: The infinite divisibility and orthogonal polynomials with a constant recursion formula in free probability theory. Probab. Math. Statist., 21(1), 159-170, (2001). MR-1869728

[13] Speicher R.: Free probability theory and non-crossing partitions. Sém. Lothar. Combin., 39:Art. B39c, 38 pp. (electronic), (1997). MR-1490288

[14] Takesaki M. Conditional expectations in von Neumann algebras. J. Functional Analysis, 9, 306-321, (1972). MR-0303307

[15] Wesołowski J.: Characterizations of some processes by properties of conditional moments. Demonstratio Math., 22(2),537-556, (1989). MR-1037930

[16] Wesołowski J.: Stochastic processes with linear conditional expectation and quadratic conditional variance. Probab. Math. Statist., 14,33-44, (1993). MR-1267516

Acknowledgments. The author would like to thank M. Bożejko, W. Bryc, M. Anshelevich and J. Wysoczański for several discussions and helpful comments during preparation of this paper.