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CENTRAL LIMIT THEOREM FOR THE THIRD MOMENT IN SPACE OF THE BROWNIAN LOCAL TIME INCREMENTS

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Abstract
The purpose of this note is to prove a central limit theorem for the third integrated moment of the Brownian local time increments using techniques of stochastic analysis. The main ingredients of the proof are an asymptotic version of Knight's theorem and the Clark-Ocone formula for the third integrated moment of the Brownian local time increments.

1 Introduction

Let $\{B_t, t \geq 0\}$ be a standard one-dimensional Brownian motion. We denote by $\{L_t^x, t \geq 0, x \in \mathbb{R}\}$ a continuous version of its local time. The following central limit theorem for the L^2 modulus of continuity of the local time has been recently proved:

$$h^{-\frac{3}{2}} \left(\int_{\mathbb{R}} (L_t^{x+h} - L_t^x)^2 dx - 4th \right) \xrightarrow{\mathscr{L}} \frac{8}{\sqrt{3}} \left(\int_{\mathbb{R}} (L_t^x)^2 dx \right)^{\frac{1}{2}} \eta, \tag{1}$$

where η is a $N(0,1)$ random variable independent of B and $\mathscr{L}$ denotes the convergence in law. This result has been first proved in [3] by using the method of moments. In [4] we gave a simple proof based on Clark-Ocone formula and an asymptotic version of Knight's theorem (see Revuz and Yor [9], Theorem (2.3), page 524). Another simple proof of this result with the techniques of stochastic analysis has been given in [11].
The following extension of this result to the case of the third integrated moment has been proved recently by Rosen in [12] using the method of moments.

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Theorem 1. *For each fixed $t > 0$*

$$\frac{1}{h^2}\int_{\mathbb{R}} (L_t^{x+h} - L_t^x)^3 dx \xrightarrow{\mathscr{L}} 8\sqrt{3}\left(\int_{\mathbb{R}} (L_t^x)^3 dx\right)^{\frac{1}{2}} \eta$$

as h tends to zero, where η is a normal random variable with mean zero and variance one that is independent of B.

The purpose of this paper is to provide a proof of Theorem 1 using the same ideas as in [4]. The main ingredient is to use Clark-Ocone stochastic integral representation formula which allows us to express the random variable

$$F_t^h = \int_{\mathbb{R}} (L_t^{x+h} - L_t^x)^3 dx \tag{2}$$

as a stochastic integral. In comparison with the L^2 modulus of continuity, the situation is here more complicated and we require some new and different techniques. First, there are four different terms (instead of two) in the stochastic integral representation, and two of them are martingales. Surprisingly, some of the terms of this representation converge in $L^2(\Omega)$ to the derivative of the self-intersection local time and the limits cancel out. Finally, there is a remaining martingale term to which we can apply the asymptotic version of Knight's theorem. As in the proof of (1), to show the convergence of the quadratic variation of this martingale and other asymptotic results we make use of Tanaka's formula for the time-reversed Brownian motion and backward Itô stochastic integrals.

We believe that a similar result could be established for the integrated moment of order p for an integer $p \geq 4$ using Clark-Ocone representation formula, but the proof would be much more involved.

These results are related to the behavior of the Brownian local time in the space variable. It was proved by Perkins [7] that for any fixed $t > 0$, $\{L_t^x, x \in \mathbb{R}\}$ is a semimartingale with quadratic variation $\langle L_t \rangle_b - \langle L_t \rangle_a = 4\int_a^b L_t^x dx$. This property provides an heuristic explanation of the central limit theorems presented above. The proof of these theorems, however, requires more complicated tools.

The paper is organized as follows. In the next section we recall some preliminaries on Malliavin calculus. In Section 3 we establish a stochastic integral representation for the derivative of the self-intersection local time, which has its own interest, and for the random variable F_t^h defined in (2). Section 4 is devoted to the proof of Theorem 1, and the Appendix contains two technical lemmas.

2 Preliminaries on Malliavin Calculus

Let us recall some basic facts on the Malliavin calculus with respect the Brownian motion $B = \{B_t, t \geq 0\}$. We refer to [5] for a complete presentation of these notions. We assume that B is defined on a complete probability space $(\Omega, \mathscr{F}, P)$ such that $\mathscr{F}$ is generated by B. Consider the set $\mathscr{S}$ of smooth random variables of the form $F = f\left(B_{t_1}, \dots, B_{t_n}\right)$,, where $t_1, \dots, t_n \geq 0$, $n \in \mathbb{N}$ and f is bounded and infinitely differentiable with bounded derivatives of all orders. The derivative operator D on a smooth random variable of this is defined by

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(B_{t_1}, \dots, B_{t_n}\right) \mathbf{1}_{[0,t_i]}(t).$$

We denote by $\mathbb{D}^{1,2}$ the completion of $\mathscr{S}$ with respect to the norm $\|F\|_{1,2}$ given by

$$\|F\|_{1,2}^2 = E\left[F^2\right] + E\left(\int_0^\infty (D_t F)^2 dt\right).$$

The classical Itô representation theorem asserts that any square integrable random variable can be expressed as

$$F = E[F] + \int_0^\infty u_t dB_t,$$

where $u = \{u_t, t \geq 0\}$ is a unique adapted process such that $E\left(\int_0^\infty u_t^2 dt\right) < \infty$. If F belongs to $\mathbb{D}^{1,2}$, then $u_t = E[D_t F|\mathscr{F}_t]$, where $\{\mathscr{F}_t, t \geq 0\}$ is the filtration generated by B, and we obtain the Clark-Ocone formula (see [6])

$$F = E[F] + \int_0^\infty E[D_t F|\mathscr{F}_t] dB_t. \tag{3}$$

3 Stochastic integral representations

Consider the random variable γ_t defined rigorously as the limit in $L^2(\Omega)$

$$\gamma_t = \lim_{\varepsilon \to 0} \int_0^t \int_0^u p'_\varepsilon(B_u - B_s) ds du, \tag{4}$$

where $p_\varepsilon(x) = (2\pi\varepsilon)^{-\frac{1}{2}} \exp(-x^2/2\varepsilon)$. The process γ_t coincides with the derivative $-\frac{d}{dy}\alpha_t(y)|_{y=0}$ of the self-intersection local time

$$\alpha_t(y) = \int_0^t \int_0^u \delta_y(B_u - B_s) ds du.$$

The derivative of the self-intersection local time has been studied by Rogers and Walsh in [10] and by Rosen in [11]. We are going to use Clark-Ocone formula to show that the limit (4) exists and to provide an integral representation for this random variable.

Lemma 2. *Set $\gamma_t^\epsilon = \int_0^t \int_0^u p'_\varepsilon(B_u - B_s) ds du$. Then, γ_t^ϵ converges in $L^2(\Omega)$ as ε tends to zero to the random variable*

$$\gamma_t = 2\int_0^t \left(\int_0^r p_{t-r}(B_r - B_s) ds - L_r^{B_r}\right) dB_r.$$

Proof. By Clark-Ocone formula applied to γ_t^ϵ we obtain the integral representation

$$\gamma_t^\epsilon = \int_0^1 E(D_r \gamma_t^\epsilon | \mathscr{F}_r) dB_r,$$

where $\{\mathscr{F}_t, t \geq 0\}$ denotes the filtration generated by the Brownian motion. Then,

$$D_r \gamma_t^\epsilon = \int_0^t \int_0^u p''_\epsilon(B_u - B_s) \mathbf{1}_{[s,u]}(r) ds du,$$

and for any $r \le t$

$$\begin{aligned} E(D_r \gamma_t^{\epsilon} | \mathscr{F}_r) &= \int_r^t \int_0^r p''_{\epsilon+u-r}(B_r - B_s) ds du = 2 \int_r^t \int_0^r \frac{\partial p_{\epsilon+u-r}}{\partial u}(B_r - B_s) ds du \\ &= 2 \int_0^r (p_{\epsilon+t-r}(B_r - B_s) - p_{\epsilon}(B_r - B_s)) ds. \end{aligned}$$

As ε tends to zero this expression converges in $L^2(\Omega \times [0,t])$ to

$$2 \left(\int_0^r p_{t-r}(B_r - B_s) ds - L_r^{B_r} \right),$$

which completes the proof. □

Let us now obtain a stochastic integral representation for the third integrated moment $F_t^h = \int_{\mathbb{R}} (L_t^{x+h} - L_t^x)^3 dx$. Notice first that $E(F_t^h) = 0$ because F_t^h is an odd functional of the Brownian motion.

Proposition 3. *We have $F_t^h = \int_0^t \Phi_r dB_r$, where $\Phi_r = \sum_{i=1}^4 \Phi_r^{(i)}$, and*

$$\begin{aligned} \Phi_r^{(1)} &= 6 \int_{\mathbb{R}} \left(L_r^{z+h} - L_r^z \right)^2 \mathbf{1}_{[0,h]}(B_r - z) dz \\ \Phi_r^{(2)} &= -6 \int_{\mathbb{R}} \int_0^h \left(L_r^{z+h} - L_r^z \right)^2 p_{t-r}(B_r - z - y) dy dz \\ \Phi_r^{(3)} &= \frac{12h}{\sqrt{2\pi}} \int_0^r \int_{-h}^h \int_{\frac{h^2}{t-r}}^{\infty} p_{t-r-\frac{h^2}{z}}(B_r - B_s + y) z^{-\frac{3}{2}} (1 - e^{-\frac{z}{2}}) dz dy ds \\ \Phi_r^{(4)} &= -\frac{12h}{\sqrt{2\pi}} \int_0^r \mathbf{1}_{[-h,h]}(B_r - B_s) ds \int_{\frac{h^2}{t-r}}^{\infty} z^{-\frac{3}{2}} (1 - e^{-\frac{z}{2}}) dz. \end{aligned}$$

Proof. Let us write

$$\begin{aligned} F_t^h &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \left(\int_0^t \left[p_\varepsilon(B_s - x - h) - p_\varepsilon(B_s - x) \right] ds \right)^3 dx \\ &= 6 \lim_{\varepsilon \to 0} \int_D \int_{\mathbb{R}} \prod_{i=1}^3 \left[p_\varepsilon(B_{s_i} - x - h) - p_\varepsilon(B_{s_i} - x) \right] dx ds, \end{aligned}$$

where $D = \{(s_1, s_2, s_3) \in [0,t]^3 : s_1 < s_2 < s_3\}$. We can pass to the limit as ϵ tends to zero the first factor $p_\varepsilon(B_{s_1} - x - h) - p_\varepsilon(B_{s_1} - x)$, and we obtain

$$\begin{aligned} F_t^h &= \lim_{\varepsilon \to 0} 6 \int_D \Big\{ \left[p_\varepsilon(B_{s_2} - B_{s_1}) - p_\varepsilon(B_{s_2} - B_{s_1} + h) \right] \left[p_\varepsilon(B_{s_3} - B_{s_1}) - p_\varepsilon(B_{s_3} - B_{s_1} + h) \right] \\ &\quad - \left[p_\varepsilon(B_{s_2} - B_{s_1} - h) - p_\varepsilon(B_{s_2} - B_{s_1}) \right] \left[p_\varepsilon(B_{s_3} - B_{s_1} - h) - p_\varepsilon(B_{s_3} - B_{s_1}) \right] \Big\} ds \\ &= \lim_{\varepsilon \to 0} 6 \int_D \Phi_\varepsilon(s) ds. \end{aligned}$$

We are going to apply the Clark-Ocone formula to the random variable $\int_D \Phi_\varepsilon(s)ds$. Fix $r \in [0,t]$. We need to compute $\int_D E\left(D_r\left[\Phi_\varepsilon(s)\right]|\mathscr{F}_r\right)ds$. To do this we decompose the region D, up to a set of zero Lebesgue measure, as $D = D_0 \cup D_1 \cup D_2$, where

$$\begin{aligned} D_1 &= \{s : 0 \le s_1 < s_2 < r < s_3 \le t\}, \\ D_2 &= \{s : 0 \le s_1 < r < s_2 < s_3 \le t\}, \end{aligned}$$

and $D_0 = \{s : 0 \le r \le s_1\} \cup \{s : s_3 \le r \le t\}$. Notice that on D_0, $D_r\left[\Phi_\varepsilon(s)\right] = 0$.

Step 1 For the region D_1 we obtain

$$\begin{aligned} E\left(D_r\left[\Phi_\varepsilon(s)\right]|\mathscr{F}_r\right) &= \left[p_\varepsilon(B_{s_2} - B_{s_1}) - p_\varepsilon(B_{s_2} - B_{s_1} + h)\right] \\ &\quad \times \left[p'_{\varepsilon+s_3-r}(B_r - B_{s_1}) - p'_{\varepsilon+s_3-r}(B_r - B_{s_1} + h)\right] \\ &\quad - \left[p_\varepsilon(B_{s_2} - B_{s_1} - h) - p_\varepsilon(B_{s_2} - B_{s_1})\right] \\ &\quad \times \left[p'_{\varepsilon+s_3-r}(B_r - B_{s_1} - h) - p'_{\varepsilon+s_3-r}(B_r - B_{s_1})\right]. \end{aligned}$$

We can write this in the following form

$$\begin{aligned} E\left(D_r\left[\Phi_\varepsilon(s)\right]|\mathscr{F}_r\right) &= \int_0^h \left[p_\varepsilon(B_{s_2} - B_{s_1} + h) - p_\varepsilon(B_{s_2} - B_{s_1})\right] p''_{\varepsilon+s_3-r}(B_r - B_{s_1} + y)dy \\ &\quad - \int_0^h \left[p_\varepsilon(B_{s_2} - B_{s_1}) - p_\varepsilon(B_{s_2} - B_{s_1} - h)\right] p''_{\varepsilon+s_3-r}(B_r - B_{s_1} - y)dy \\ &= 2\int_0^h \left[p_\varepsilon(B_{s_2} - B_{s_1} + h) - p_\varepsilon(B_{s_2} - B_{s_1})\right] \frac{\partial p_{\varepsilon+s_3-r}}{\partial s_3}(B_r - B_{s_1} + y)dy \\ &\quad -2\int_0^h \left[p_\varepsilon(B_{s_2} - B_{s_1}) - p_\varepsilon(B_{s_2} - B_{s_1} - h)\right] \frac{\partial p_{\varepsilon+s_3-r}}{\partial s_3}(B_r - B_{s_1} - y)dy. \end{aligned}$$

Integrating with respect to the variable s_3 yields

$$\begin{aligned} \int_D E\left(D_r\left[\Phi_\varepsilon(s)\right]|\mathscr{F}_r\right)ds &= 2\int_{0\le s_1<s_2\le r}\int_0^h \left[p_\varepsilon(B_{s_2} - B_{s_1} + h) - p_\varepsilon(B_{s_2} - B_{s_1})\right] \\ &\quad \times \left[p_{\varepsilon+t-r}(B_r - B_{s_1} + y) - p_\varepsilon(B_r - B_{s_1} + y)\right] dy ds_1 ds_2 \\ &\quad -2\int_{0\le s_1<s_2\le r}\int_0^h \left[p_\varepsilon(B_{s_2} - B_{s_1}) - p_\varepsilon(B_{s_2} - B_{s_1} - h)\right] \\ &\quad \times \left[p_{\varepsilon+t-r}(B_r - B_{s_1} - y) - p_\varepsilon(B_r - B_{s_1} - y)\right] dy ds_1 ds_2. \end{aligned}$$

This expression can be written in terms of the local time:

$$\begin{aligned} \int_D E\left(D_r\left[\Phi_\varepsilon(s)\right]|\mathscr{F}_r\right)ds &= 2\int_{\mathbb{R}^2}\int_0^h\int_0^r \left[p_\varepsilon(x - z + h) - p_\varepsilon(x - z)\right](L_r^x - L_s^x)L_{ds}^z \\ &\quad \times \left[p_{\varepsilon+t-r}(B_r - z + y) - p_\varepsilon(B_r - z + y)\right] dy dx dz \\ &\quad -2\int_{\mathbb{R}^2}\int_0^h\int_0^r \left[p_\varepsilon(x - z) - p_\varepsilon(x - z - h)\right](L_r^x - L_s^x)L_{ds}^z \\ &\quad \times \left[p_{\varepsilon+t-r}(B_r - z - y) - p_\varepsilon(B_r - z - y)\right] dy dx dz. \end{aligned}$$

We make the change of variables $y \to h-y$ and $z \to z+h$ in the last integral and we obtain

$$\begin{aligned}\int_D E\left(D_r\left[\Phi_\varepsilon(s)\right]|\mathscr{F}_r\right)ds &= 2\int_{\mathbb{R}^2}\int_0^h\int_0^r \left[p_\varepsilon(x-z+h)-p_\varepsilon(x-z)\right](L_r^x-L_s^x)\left(L_{ds}^z-L_{ds}^{z-h}\right)\\ &\quad\times\left[p_{\varepsilon+t-r}(B_r-z+y)-p_\varepsilon(B_r-z+y)\right]dydxdz.\end{aligned}$$

Taking the limit as ε tends to zero in $L^2(\Omega\times[0,t])$ and integrating in x yields

$$\begin{aligned}\lim_{\epsilon\to 0}\int_D E\left(D_r\left[\Phi_\varepsilon(s)\right]|\mathscr{F}_r\right)ds &= 2\int_{\mathbb{R}}\int_0^h\int_0^r \left(L_r^{z-h}-L_s^{z-h}-L_r^z+L_s^z\right)\left(L_{ds}^z-L_{ds}^{z-h}\right)\\ &\quad\times\left[p_{t-r}(B_r-z+y)-\delta_0(B_r-z+y)\right]dydz\\ &= \int_{\mathbb{R}}\left(L_r^{z+h}-L_r^z\right)^2\ \mathbf{1}_{[0,h]}(B_r-z)dz\\ &\quad-\int_{\mathbb{R}}\int_0^h\left(L_r^{z+h}-L_r^z\right)^2 p_{t-r}(B_r-z-y)dydz := \Psi_r^{(1)}.\end{aligned}$$

Step 2 For the region D_2, taking first the conditional expectation with respect to $\mathscr{F}_{s_2}$ which contains $\mathscr{F}_r$, and integrating with respect to the law of $B_{s_2}-B_r$ we obtain

$$\begin{aligned}E\left(D_r\left[\Phi_\varepsilon(s)\right]|\mathscr{F}_r\right) &= \int_{\mathbb{R}} dy\, p_{s_2-r}(y)\{\left[p'_\varepsilon(B_r-B_{s_1}+y)-p'_\varepsilon(B_r-B_{s_1}+y+h)\right]\\ &\quad\times\left[p_{\varepsilon+s_3-s_2}(B_r-B_{s_1}+y)-p_{\varepsilon+s_3-s_2}(B_r-B_{s_1}+h+y)\right]\\ &\quad+\left[p_\varepsilon(B_r-B_{s_1}+y)-p_\varepsilon(B_r-B_{s_1}+y+h)\right]\\ &\quad\times\left[p'_{\varepsilon+s_3-s_2}(B_r-B_{s_1}+y)-p'_{\varepsilon+s_3-s_2}(B_r-B_{s_1}+h+y)\right]\\ &\quad-\left[p'_\varepsilon(B_r-B_{s_1}+y-h)-p'_\varepsilon(B_r-B_{s_1}+y)\right]\\ &\quad\times\left[p_{\varepsilon+s_3-s_2}(B_r-B_{s_1}+y-h)-p_{\varepsilon+s_3-s_2}(B_r-B_{s_1}+y)\right]\\ &\quad-\left[p_\varepsilon(B_r-B_{s_1}+y-h)-p_\varepsilon(B_r-B_{s_1}+y)\right]\\ &\quad\times\left[p'_{\varepsilon+s_3-s_2}(B_r-B_{s_1}+y-h)-p'_{\varepsilon+s_3-s_2}(B_r-B_{s_1}+y)\right]\}.\end{aligned}$$

Integrating by parts yields

$$\begin{aligned}E\left(D_r\left[\Phi_\varepsilon(s)\right]|\mathscr{F}_r\right) &= -\int_{\mathbb{R}} dy\, p'_{s_2-r}(y)\{\left[p_\varepsilon(B_r-B_{s_1}+y)-p_\varepsilon(B_r-B_{s_1}+y+h)\right]\\ &\quad\times\left[p_{\varepsilon+s_3-s_2}(B_r-B_{s_1}+y)-p_{\varepsilon+s_3-s_2}(B_r-B_{s_1}+h+y)\right]\\ &\quad-\left[p_\varepsilon(B_r-B_{s_1}+y-h)-p_\varepsilon(B_r-B_{s_1}+y)\right]\\ &\quad\times\left[p_{\varepsilon+s_3-s_2}(B_r-B_{s_1}+y-h)-p_{\varepsilon+s_3-s_2}(B_r-B_{s_1}+y)\right]\}.\end{aligned}$$

Letting ε tend to zero we obtain

$$\begin{aligned}\lim_{\epsilon\to 0}E\left(D_r\left[\Phi_\varepsilon(s)\right]|\mathscr{F}_r\right) &= \left[p_{s_3-s_2}(0)-p_{s_3-s_2}(h)\right]\left[p'_{s_2-r}(B_r-B_{s_1}+h)-p'_{s_2-r}(B_r-B_{s_1}-h)\right]\\ &= \left[p_{s_3-s_2}(0)-p_{s_3-s_2}(h)\right]\int_{-h}^h p''_{s_2-r}(B_r-B_{s_1}+y)dy.\end{aligned}$$

Hence,

$$\lim_{\epsilon\to 0}\int_{D_2} E\left(D_r\left[\Phi_\varepsilon(s)\right]|\mathscr{F}_r\right)ds = 2\int_0^r ds_1\int_r^t ds_2\left(\left(\int_{s_2}^t\left[p_{s_3-s_2}(0)-p_{s_3-s_2}(h)\right]ds_3\right)\right.$$
$$\times\int_{-h}^h \frac{\partial p_{s_2-r}}{\partial s_2}(B_r-B_{s_1}+y)dy := \Psi_r^{(2)}. \tag{5}$$

We have

$$\int_{s_2}^t\left[p_{s_3-s_2}(0)-p_{s_3-s_2}(h)\right]ds_3 = \frac{h}{\sqrt{2\pi}}\int_{\frac{h^2}{t-s_2}}^\infty z^{-\frac{3}{2}}(1-e^{-\frac{z}{2}})dz. \tag{6}$$

Substituting (6) into (5) yields

$$\Psi_r^{(2)} = \frac{2h}{\sqrt{2\pi}}\int_{-h}^h\int_{\frac{h^2}{t-r}}^\infty\int_r^{t-\frac{h^2}{z}}\int_0^r \frac{\partial p_{u-r}}{\partial u}(B_r-B_s+y)\times z^{-\frac{3}{2}}(1-e^{-\frac{z}{2}})dsdudzdy.$$

Now we integrate in the variable u and we obtain

$$\begin{aligned}\Psi_r^{(2)} &= \frac{2h}{\sqrt{2\pi}}\int_{-h}^h\int_{\frac{h^2}{t-r}}^\infty\int_0^r p_{t-r-\frac{h^2}{z}}(B_r-B_s+y)z^{-\frac{3}{2}}(1-e^{-\frac{z}{2}})dsdzdy\\ &\quad -\frac{2h}{\sqrt{2\pi}}\int_0^r \mathbf{1}_{[-h,h]}(B_r-B_s)ds\int_{\frac{h^2}{t-r}}^\infty z^{-\frac{3}{2}}(1-e^{-\frac{z}{2}})dz.\end{aligned}$$

Thus,

$$M_t = 6\int_0^t \Psi_r^{(1)}dB_r + 6\int_0^t \Psi_r^{(2)}dB_r,$$

which completes the proof. □

4 Proof of Theorem 1

The proof will be done in several steps. Along the proof we will denote by C a generic constant, which may be different from line to line.

Step 1 Notice first that by Lemma 2 and the equation

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty z^{-\frac{3}{2}}(1-e^{-\frac{z}{2}})dz = 1, \tag{7}$$

we obtain that $h^{-2}\int_0^t(\Phi_r^{(3)}+\Phi_r^{(4)})dB_r$ converges in $L^2(\Omega)$ to $12\gamma_t$.

Step 2 In order to handle the term $h^{-2}\int_0^t(\Phi_r^{(1)}+\Phi_r^{(2)})dB_r$ we consider the function

$$\phi_h(\xi) = \int_0^h p_{t-r}(\xi-y)dy - \mathbf{1}_{[0,h]}(\xi),$$

and we can write

$$\Phi_r^{(1)} + \Phi_r^{(2)} = -6\int_{\mathbb{R}} \left(L_r^{B_r-x+h} - L_r^{B_r-x}\right)^2 \phi_h(x)dx.$$

Applying Tanaka's formula to the time reversed Brownian motion $\{B_r - B_s, 0 \le s \le r\}$, we obtain

$$\begin{aligned}\frac{1}{2}\left(L_r^{B_r-x+h} - L_r^{B_r-x}\right) &= -(-x+h)^+ + (-x)^+ + (B_r - x + h)^+ - (B_r - x)^+ \\ &\quad - \int_0^r \mathbf{1}_{[-h,0]}(B_r - B_s - x)d\widehat{B}_s,\end{aligned}$$

where $d\widehat{B}_s$ denotes the backward Itô integral. Clearly, the only term that gives a nonzero contribution is

$$24\int_{\mathbb{R}} \left(\int_0^r \mathbf{1}_{[-h,0]}(B_r - B_s - x)d\widehat{B}_s\right)^2 \phi_h(x)dx.$$

We are going to use the following notation

$$\delta_{r,s}^h(x) = \mathbf{1}_{[-h,0]}(B_r - B_s - x), \tag{8}$$

$$A_{\sigma,r}^h(x) = \int_\sigma^r \delta_{r,s}^h(x)d\widehat{B}_s, \tag{9}$$

where $0 < \sigma < r$. With this notation we want to find the limit in distribution of

$$Y_h := -24h^{-2}\int_0^t \left(\int_{\mathbb{R}} (A_{0,r}^h(x))^2 \phi_h(x)dx\right) dB_r,$$

as h tends to zero. By Itô's formula,

$$(A_{0,r}^h(x))^2 = 2\int_0^r A_{\sigma,r}^h(x)\delta_{r,\sigma}^h(x)d\widehat{B}_\sigma + \int_0^r \delta_{r,s}^h(x)ds.$$

Therefore,

$$\begin{aligned} Y_h &= -48h^{-2}\int_0^t \left(\int_{\mathbb{R}} \left(\int_0^r A_{\sigma,r}^h(x)\delta_{r,\sigma}^h(x)d\widehat{B}_\sigma\right)\phi_h(x)dx\right) dB_r \\ &\quad -24h^{-2}\int_0^t \left(\int_{\mathbb{R}}\int_0^r \delta_{r,s}^h(x)\phi_h(x)dsdx\right) dB_r. \end{aligned} \tag{10}$$

Notice that, by Lemma 2

$$\begin{aligned} &-24h^{-2}\int_0^t \left(\int_{\mathbb{R}}\int_0^r \delta_{r,s}^h(x)\phi_h(x)dsdx\right) dB_r \\ &= -24h^{-2}\int_0^t \left(\int_{\mathbb{R}}\int_0^r \mathbf{1}_{[-h,0]}(B_r - B_s - x)\left(\int_0^h p_{t-r}(x-y)dy - \mathbf{1}_{[0,h]}(x)\right)dsdx\right) dB_r \end{aligned}$$

converges in $L^2(\Omega)$ to $-12\gamma_t$, which cancels with the limit obtained in Step 1.

To handle the first summand in the right-hand side of (10) we make the decomposition

$$\int_{\mathbb{R}} \left(\int_0^r A^h_{\sigma,r}(x) \delta^h_{r,\sigma}(x) d\widehat{B}_\sigma \right) \phi_h(x) dx = \Gamma_{r,h} - \Delta_{r,h},$$

where

$$\Gamma_{r,h} = \int_0^h \int_{\mathbb{R}} \left(\int_0^r A^h_{\sigma,r}(x) \delta^h_{r,\sigma}(x) d\widehat{B}_\sigma \right) p_{t-r}(x-y) dx dy, \tag{11}$$

and

$$\Delta_{r,h} = \int_0^h \left(\int_0^r A^h_{\sigma,r}(x) \delta^h_{r,\sigma}(x) d\widehat{B}_\sigma \right) dx, \tag{12}$$

Step 3 We claim that

$$\lim_{h\to 0} h^{-4} E(\Gamma^2_{r,h}) = 0, \tag{13}$$

which implies that $h^{-2}\int_0^t \Gamma_{r,h} dB_r$ converges to 0 in $L^2(\Omega)$ as h tends to zero. Let us prove (13). We can write

$$\begin{aligned} E(\Gamma^2_{r,h}) &= \int_0^r \left(\int_0^h \int_{\mathbb{R}} A^h_{\sigma,r}(x) \delta^h_{r,\sigma}(x) p_{t-r}(x-y) dx dy \right)^2 d\sigma \\ &\le E\left(\sup_{0\le\sigma\le r\le t} \sup_{x\in\mathbb{R}} |A^h_{\sigma,r}(x)|^2 \int_0^r \left(\int_0^h \int_{\mathbb{R}} \delta^h_{r,\sigma}(x) p_{t-r}(x-y) dx dy \right)^2 d\sigma \right). \end{aligned}$$

Clearly, for any $p \ge 1$,

$$h^{-4} \int_0^r \left(\int_0^h \int_{\mathbb{R}} \delta^h_{r,\sigma}(x) p_{t-r}(x-y) dx dy \right)^2 d\sigma$$

converges in $L^p(\Omega)$ to $\int_0^r p_{t-r}(B_r - B_s)^2 ds$, and, on the other hand, by Lemma 4 in the Appendix, $\| \sup_{0\le\sigma\le r\le t} \sup_{x\in\mathbb{R}} |A^h_{\sigma,r}(x)| \|_p$ converges to zero as h tend to zero, for any $p \ge 2$. This completes the proof of (13).

Step 4 Finally, we will discuss the limit of the martingale

$$M^h_t = 48h^{-2} \int_0^t \Delta_{r,h} dB_r,$$

where $\Delta_{r,h}$ is defined in (12). Applying the asymptotic version of Knight's theorem (see Revuz and Yor [9], Theorem 2.3 page 524) as in [4], it suffices to show the following convergences in probability as h tends to zero

$$\langle M^h, B \rangle_t \to 0, \tag{14}$$

uniformly in compact sets, and

$$\langle M^h \rangle_t \to 192 \int_{\mathbb{R}} (L^x_t)^3 dx. \tag{15}$$

Exchanging the order of integration we can write $\Delta_{r,h}$ as

$$\Delta_{r,h} = \int_0^r \left(\int_0^h A^h_{\sigma,r}(x) \delta^h_{r,\sigma}(x) dx \right) d\widehat{B}_\sigma = \int_0^r \Psi^h_{r,\sigma} d\widehat{B}_\sigma, \tag{16}$$

where

$$\Psi^h_{r,\sigma} = \int_0^h A^h_{\sigma,r}(x)\delta^h_{r,\sigma}(x)dx. \tag{17}$$

Step 5 Let us prove (14). For any $p \geq 2$ we have, by Burkholder's inequality

$$\begin{aligned} E\left|\langle M^h, B\rangle_t - \langle M^h, B\rangle_s\right|^p &\leq c_p h^{-2p} E\left|\int_0^s \left(\int_s^t \int_0^h A^h_{\sigma,r}(x)\delta^h_{r,\sigma}(x)dxdr\right)^2 d\sigma\right|^{\frac{p}{2}} \\ &\quad + c_p h^{-2p} E\left|\int_s^t \left(\int_\sigma^t \int_0^h A^h_{\sigma,r}(x)\delta^h_{r,\sigma}(x)dxdr\right)^2 d\sigma\right|^{\frac{p}{2}} \\ &= c_p h^{-2p}(B_1 + B_2). \end{aligned}$$

For the term B_1 we can write

$$B_1 \leq E\left(\sup_{x\in\mathbb{R}} \sup_{0\leq\sigma\leq s\leq r\leq t} |A^h_{\sigma,r}(x)|^p \int_0^s \left|\int_s^t \int_0^h \delta^h_{r,\sigma}(x)dxdr\right|^p d\sigma\right).$$

Applying Cauchy-Schwarz inequality and lemmas 4 and 5 in the Appendix we obtain

$$B_1 \leq C_{p,t,\epsilon}(t-s)^{\frac{p}{2}} h^{2p+\frac{p}{2}-\epsilon}.$$

A similar estimate can be deduced for the term B_2. Finally, an application of the Garsia-Rodemich-Rumsey lemma allows us to conclude that the convergence in (14) holds in probability, uniformly in compact sets.

Step 6 Let us prove (15). We have, by Itô's formula and in view of (16) and (17)

$$\begin{aligned} \langle M^h\rangle_t &= 48^2 h^{-4}\int_0^t \Delta^2_{r,h}dr = 48^2 h^{-4}\int_0^t \int_0^r (\Psi^h_{r,\sigma})^2 d\sigma dr \\ &\quad + 48^2 \times 2h^{-4}\int_0^t \int_0^r \Psi^h_{r,\sigma}\left(\int_\sigma^r \Psi^h_{r,s} d\widehat{B}_s\right) d\widehat{B}_\sigma dr \\ &:= 48^2 h^{-4}\left(R^1_{t,h} + 2R^2_{t,h}\right). \end{aligned} \tag{18}$$

We are going to see that only the first summand in the above expression gives a nonzero contribution to the limit. Consider first the term $R^1_{t,h}$. We can express $(\Psi^h_{r,\sigma})^2$ as

$$(\Psi^h_{r,\sigma})^2 = \int_0^h \int_0^h A^h_{\sigma,r}(x)A^h_{\sigma,r}(y)\delta^h_{r,\sigma}(x)\delta^h_{r,\sigma}(y)dxdy,$$

and, by Itô's formula

$$A^h_{\sigma,r}(x)A^h_{\sigma,r}(y) = \int_\sigma^r \delta^h_{r,s}(x)\delta^h_{r,s}(y)ds + \int_\sigma^r A^h_{s,r}(x)\delta^h_{r,s}(y)d\widehat{B}_s + \int_\sigma^r A^h_{s,r}(y)\delta^h_{r,s}(x)d\widehat{B}_s.$$

Substituting the above equality in the expression of $(\Psi^h_{r,\sigma})^2$ yields

$$\begin{aligned} R^1_{t,h} &= \int_0^t \int_0^r \int_\sigma^r \int_0^h \int_0^h \delta^h_{r,s}(x)\delta^h_{r,s}(y)\delta^h_{r,\sigma}(x)\delta^h_{r,\sigma}(y)dxdydsd\sigma dr \\ &+ \int_0^t \int_0^r \int_0^h \int_0^h \left(\int_\sigma^r A^h_{s,r}(x)\delta^h_{r,s}(y)d\widehat{B}_s \right) \delta^h_{r,\sigma}(x)\delta^h_{r,\sigma}(y)dxdyd\sigma dr \\ &+ \int_0^t \int_0^r \int_0^h \int_0^h \left(\int_\sigma^r A^h_{s,r}(y)\delta^h_{r,s}(x)d\widehat{B}_s \right) \delta^h_{r,\sigma}(x)\delta^h_{r,\sigma}(y))dxdyd\sigma dr \\ = & \sum_{i=1}^3 A^{i,h}_t. \end{aligned}$$

Only the first term in the above sum will give a nonzero contribution to the limit. Let us consider first this term. We have

$$\int_0^h \delta^h_{r,s}(x)\delta^h_{r,\sigma}(x)dx = g_h(B_r - B_s, B_r - B_\sigma),$$

where

$$g_h(x,y) = (h - |x| - |y|)_+ \mathbf{1}_{\{xy<0\}} + [(h-|x|)_+ \wedge (h-|y|)_+]\mathbf{1}_{\{xy\geq 0\}}.$$

As a consequence,

$$\begin{aligned} 48^2 h^{-4} A^{1,h}_t &= 48^2 h^{-4} \int_0^t \int_0^r \int_\sigma^r g_h(B_r - B_s, B_r - B_\sigma)^2 \, dsd\sigma dr \\ &= \frac{1}{2} 48^2 h^{-4} \int_0^t \int_0^r \int_0^r g_h(B_r - B_s, B_r - B_\sigma)^2 \, dsd\sigma dr \\ &= \frac{1}{2} 48^2 h^{-4} \int_0^t \left(\int_{\mathbb{R}^2} g_h(B_r - x, B_r - y)^2 \, L^x_r L^y_r dxdy \right) dr. \end{aligned}$$

As h tends to zero this converges to $\frac{1}{4} 48^2 \int_0^t (L^{B_r}_r)^2 dr = 192 \int_{\mathbb{R}} (L^x_t)^3 dx$. This follows form the fact that

$$\int_{\mathbb{R}^2} g_h(x,y)^2 dxdy = \frac{1}{2} h^4.$$

Let us show that the other terms $h^{-4}A^{2,h}_t$ and $h^{-4}A^{3,h}_t$ converge to zero in $L^1(\Omega)$ as h tends to zero.

Using Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$ yields

$$\begin{aligned} h^{-4}\left|A_t^{2,h}\right| &\leq h^{-4}\int_0^t\int_0^r\left(\int_0^h\int_0^h \delta_{r,\sigma}^h(x)\delta_{r,\sigma}^h(y)dxdy\right)^{\frac{1}{p}} \\ &\quad\times\left(\int_0^h\int_0^h\left|\int_\sigma^r A_{s,r}^h(x)\delta_{r,s}^h(y)d\widehat{B}_s\right|^q dxdy\right)^{\frac{1}{q}} d\sigma dr \\ &\leq h^{-4}\int_0^t\int_0^r (h-|B_r-B_\sigma|)_+^{\frac{2}{p}}\left(\int_0^h\int_0^h\left|\int_\sigma^r A_{s,r}^h(x)\delta_{r,s}^h(y)d\widehat{B}_s\right|^q dxdy\right)^{\frac{1}{q}} d\sigma dr \\ &\leq h^{-4}\sup_r\left(\int_0^r (h-|B_r-B_\sigma|)_+^{\frac{2}{p}}d\sigma\right) \\ &\quad\times\int_0^t\left(\sup_\sigma\int_0^h\int_0^h\left|\int_\sigma^r A_{s,r}^h(x)\delta_{r,s}^h(y)d\widehat{B}_s\right|^q dxdy\right)^{\frac{1}{q}} dr \\ &= h^{-4}B_t^{1,h}B_t^{2,h}. \end{aligned}$$

The term $B_t^{1,h}$ can be estimated as follows

$$\int_0^r (h-|B_r-B_\sigma|)_+^{\frac{2}{p}}d\sigma=\int_{\mathbb{R}}(h-|B_r-x|)_+^{\frac{2}{p}}L_r^x dx\leq\frac{2p}{p+2}\left(\sup_{r\leq t,x\in\mathbb{R}}L_r^x\right)h^{\frac{2}{p}+1}.$$

Furthermore, for the term $B_t^{2,h}$ we can write

$$\begin{aligned} \left\|B_t^{2,h}\right\|_q &\leq C\sup_{r\leq t}\left(\int_0^h\int_0^h E\left|\int_0^r A_{s,r}^h(x)\delta_{r,s}^h(y)d\widehat{B}_s\right|^q dxdy\right)^{\frac{1}{q}} \\ &\leq C\sup_{r\leq t}\left(\int_0^h\int_0^h E\left|\int_0^r A_{s,r}^h(x)^2\delta_{r,s}^h(y)ds\right|^{\frac{q}{2}} dxdy\right)^{\frac{1}{q}} \\ &\leq Ch^{\frac{1}{q}}\sup_{r\leq t}\left(E\left(\sup_{x\in\mathbb{R}}\sup_{0\leq s\leq r\leq t}\left|A_{s,r}^h(x)\right|^q\int_0^r\int_0^h\delta_{r,s}^h(y)dsdy\right)\right)^{\frac{1}{q}} \\ &\leq Ch^{\frac{3}{q}}\left(E\left(\sup_{x\in\mathbb{R}}\sup_{0\leq s\leq r\leq t}\left|A_{s,r}^h(x)\right|^q\times\sup_{x\in\mathbb{R}}\sup_{0\leq r\leq t}L_r^x\right)\right)^{\frac{1}{q}}. \end{aligned}$$

Using Lemma 4 in the Appendix with the exponent q yields

$$\left\|B_t^{2,h}\right\|_q\leq Ch^{\frac{3}{q}+\frac{1}{2}-\frac{\epsilon}{q}}.$$

As a consequence,

$$h^{-4}E\left|A_t^{2,h}\right|\leq Ch^{-4}h^{\frac{2}{p}+1+\frac{2}{q}+\frac{1}{q}+\frac{1}{2}-\frac{\epsilon}{q}}=Ch^{\frac{1}{q}-\frac{1}{2}-\frac{\epsilon}{q}},$$

and for $0<q<2(1-\epsilon)$ this converges to zero. In the same way we can show that $h^{-4}E\left|A_t^{3,h}\right|$ tends to zero as h tends to zero.

It only remains to show that the term $h^{-4}R^2_{t,h}$ in the right-hand side of (18) converges to zero. Using Fubini's theorem we can write

$$R^2_{t,h} = \int_0^t \int_0^r \Psi^h_{r,\sigma} \left(\int_\sigma^r \Psi^h_{r,s} d\widehat{B}_s \right) d\widehat{B}_\sigma dr = \int_0^t \left(\int_\sigma^t \Psi^h_{r,\sigma} \left(\int_\sigma^r \Psi^h_{r,s} d\widehat{B}_s \right) dr \right) d\widehat{B}_\sigma,$$

hence,

$$\begin{aligned} E(R^2_{t,h})^2 &= E\int_0^t \left(\int_\sigma^t \Psi^h_{r,\sigma} \left(\int_\sigma^r \Psi^h_{r,s} d\widehat{B}_s \right) dr \right)^2 d\sigma \\ &= E\int_0^t \int_\sigma^t \int_\sigma^t \Psi^h_{r,\sigma}\Psi_{\rho,\sigma} \left(\int_\sigma^r \Psi^h_{r,s} d\widehat{B}_s \right) \left(\int_\sigma^\rho \Psi^h_{\rho,s} d\widehat{B}_s \right) dr d\rho d\sigma \\ &\le 2E\int_0^t \int_\sigma^t \int_\sigma^t \left| \Psi_{r,\sigma}\Psi^h_{\rho,\sigma} \right| \left(\int_\sigma^r \Psi^h_{r,s} d\widehat{B}_s \right)^2 dr d\rho d\sigma \\ &\le 2 \left\| \sup_r \left(\int_0^r \left| \Psi^h_{r,\sigma} \right| d\sigma \right) \right\|_a \left\| \sup_\sigma \left(\int_\sigma^t \left| \Psi^h_{\rho,\sigma} \right| d\rho \right) \right\|_b \\ &\quad \times \left\| \left(\int_0^t \sup_{\sigma \le r} \left(\int_\sigma^r \Psi^h_{r,s} d\widehat{B}_s \right)^2 dr \right) \right\|_c \end{aligned}$$

with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Using lemmas 4 and 5 in the Appendix we can show that the first two factors are bounded by a constant times $h^{\frac{5}{2}-\epsilon}$ for some arbitrarily small $\epsilon > 0$. Using Doob's maximal inequality, the third factor can be estimated by

$$\sup_{0 \le r \le t} \left\| \int_0^r (\Psi^h_{r,s})^2 ds \right\|_{\frac{c}{2}},$$

and

$$\int_0^r (\Psi^h_{r,s})^2 ds \le \sup_{x\in\mathbb{R}} \sup_{0\le s\le r\le t} |A^h_{s,r}|^2 \int_0^r (h - |B_r - B_s|)^2_+ ds \le Ch^3 \sup_{x\in\mathbb{R}} \sup_{0\le s\le r\le t} |A^h_{s,r}|^2 \sup_{x\in\mathbb{R}} L^x_r.$$

Finally, applying Lemma 4, we obtain

$$h^{-8} E(R^2_{t,h})^2 \le Ch^{1-\delta}$$

for some arbitrary small δ. This completes the proof of Theorem 1.

5 Appendix

In this section we prove two technical results used in the paper.

Lemma 4. *Consider the random variable $A^h_{\sigma,r}$ introduced in (9). Then, for any $p \ge 2$ and $\epsilon \in (0, \frac{p}{2})$ there exists a constant $C_{t,p,\epsilon}$ such that*

$$E \sup_{x\in\mathbb{R}} \sup_{0\le\sigma\le r\le t} |A^h_{\sigma,r}(x)|^p \le C_{t,p,\epsilon} h^{\frac{p}{2}-\epsilon}.$$

Proof. By Tanaka's formula applied to the time reversed Brownian motion we can write

$$\begin{aligned} A_{\sigma,r}(x) &= -(B_r - B_\sigma + h)^+ + (B_r - B_\sigma)^+ + (B_r - x + h)^+ - (B_r - x)^+ \\ &\quad + \frac{1}{2}\left(L_\sigma^{B_r - x + h} - L_\sigma^{B_r - x} - L_r^{B_r - x + h} + L_r^{B_r - x}\right). \end{aligned}$$

Therefore,

$$|A_{r,\sigma}(x)| \le 2h + \sup_{x\in\mathbb{R}} \sup_{0\le r\le t} |L_r^{x+h} - L_r^x|. \tag{19}$$

Finally, the result follows from the inequalities for the local time proved by Barlow and Yor in [1]. □

Lemma 5. *Let $\delta^h_{r,\sigma}(x)$ be the random variable defined in (9). Then, for any $p \ge 2$ there exists a constant $C_{t,p}$ such that for all $0 \le s \le t$*

$$E \sup_{0\le\sigma\le s} \left| \int_s^t \int_0^h \delta^h_{r,\sigma}(x)\,dx\,dr \right|^p \le h^{2p} C_{t,p} (t-s)^{\frac{p}{2}}.$$

Proof. We can write

$$\begin{aligned} \int_s^t \int_0^h \delta^h_{r,\sigma}(x)\,dx\,dr &= \int_s^t (h - |B_r - B_\sigma|)^+ dr = \int_{\mathbb{R}} (L_t^x - L_s^x)(h - |x - B_\sigma|)^+ dx \\ &\le h^2 \sup_x (L_t^x - L_s^x). \end{aligned} \tag{20}$$

Finally,

$$\sup_x (L_t^x - L_s^x) \le \sup_x \int_s^t \delta_x(B_u - B_s)du,$$

and $\int_s^t \delta_x(B_u - B_s)du$ has the same distribution as L^x_{t-s}, or, by the scaling properties of the local time, as $\sqrt{t-s}L_1^{x/\sqrt{t-s}}$, so

$$E \sup_x (L_t^x - L_s^x)^p \le (t-s)^{\frac{p}{2}} E \sup_x (L_1^x)^p.$$

□

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