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UPPER BOUND ON THE EXPECTED SIZE OF THE INTRINSIC BALL

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Abstract
We give a short proof of Theorem 1.2(i) from [5]. We show that the expected size of the intrinsic ball of radius r is at most Cr if the susceptibility exponent γ is at most 1. In particular, this result follows if the so-called triangle condition holds.

Let $G = (V, E)$ be an infinite connected graph. We consider independent bond percolation on G. For $p \in [0, 1]$, each edge of G is open with probability p and closed with probability $1-p$ independently for distinct edges. The resulting product measure is denoted by $\mathbb{P}_p$. For two vertices $x, y \in V$ and an integer n, we write $x \longleftrightarrow y$ if there is an open path from x to y, and we write $x \xleftrightarrow{\le n} y$ if there is an open path of at most n edges from x to y. Let $C(x)$ be the set of all $y \in V$ such that $x \longleftrightarrow y$. For $x \in V$, the *intrinsic ball* of radius n at x is the set $B_I(x, n)$ of all $y \in V$ such that $x \xleftrightarrow{\le n} y$. Let $p_c = \inf\{p : \mathbb{P}_p(|C(x)| = \infty) > 0\}$ be the critical percolation probability. Note that p_c does not depend on a particular choice of $x \in V$, since G is a connected graph. For general background on Bernoulli percolation we refer the reader to [2].

In this note we give a short proof of Theorem 1.2(i) from [5]. Our proof is robust and does not require particular structure of the graph.

Theorem 1. *Let $x \in V$. If there exists a finite constant C_1 such that $\mathbb{E}_p|C(x)| \le C_1(p_c - p)^{-1}$ for all $p < p_c$, then there exists a finite constant C_2 such that for all n,*

$$\mathbb{E}_{p_c}|B_I(x, n)| \le C_2 n.$$

Before we proceed with the proof of this theorem, we discuss examples of graphs for which the assumption of Theorem 1 is known to hold. It is believed that as $p \nearrow p_c$, the expected size of $C(x)$ diverges like $(p_c - p)^{-\gamma}$. The assumption of Theorem 1 can be interpreted as the mean-field bound $\gamma \le 1$. It is well known that for vertex-transitive graphs this bound is satisfied if the triangle condition holds at p_c [1]: For $x \in V$,

$$\sum_{y,z \in V} \mathbb{P}_{p_c}(x \longleftrightarrow y)\mathbb{P}_{p_c}(y \longleftrightarrow z)\mathbb{P}_{p_c}(z \longleftrightarrow x) < \infty.$$

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This condition holds on certain Euclidean lattices [3, 4] including the nearest-neighbor lattice $\mathbb{Z}^d$ with $d \geq 19$ and sufficiently spread-out lattices with $d > 6$. It also holds for a rather general class of non-amenable transitive graphs [6, 8, 9, 10]. It has been shown in [7] that for vertex-transitive graphs, the triangle condition is equivalent to the so-called open triangle condition. The latter is often used instead of the triangle condition in studying the mean-field criticality.

Proof of Theorem 1. Let $p < p_c$. We consider the following coupling of percolation with parameter p and with parameter p_c. First delete edges independently with probability $1 - p_c$, then every present edge is deleted independently with probability $1 - (p/p_c)$. This construction implies that for $x, y \in V$, $p < p_c$, and an integer n,

$$\mathbb{P}_p(x \overset{\leq n}{\longleftrightarrow} y) \geq \left(\frac{p}{p_c}\right)^n \mathbb{P}_{p_c}(x \overset{\leq n}{\longleftrightarrow} y).$$

Summing over $y \in V$ and using the inequality $\mathbb{P}_p(x \overset{\leq n}{\longleftrightarrow} y) \leq \mathbb{P}_p(x \longleftrightarrow y)$, we obtain

$$\mathbb{E}_{p_c}|B_I(x,n)| \leq \left(\frac{p_c}{p}\right)^n \mathbb{E}_p|C(x)|.$$

The result follows by taking $p = p_c(1 - \frac{1}{2n})$. □

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References

[1] M. Aizenman and Ch. Newman. Tree graph inequalities and critical behavior in percolation models. *J. Statist. Phys.* **36**: 107-143, 1984. MR0762034

[2] G. Grimmett. *Percolation*. Springer-Verlag, Berlin, Second edition, 1999. MR1707339

[3] T. Hara and G. Slade. Mean-field critical behaviour for percolation in high dimensions. *Commun. Math. Phys.* **128**: 333-391, 1990. MR1043524

[4] M. Heydenreich, R. van der Hofstad and A. Sakai. Mean-field behavior for long- and finite range Ising model, percolation and self-avoiding walk. *J. Statist. Phys.* **132(6)**: 1001-1049, 2008. MR2430773

[5] G. Kozma and A. Nachmias. The Alexander-Orbach conjecture holds in high dimensions. *Invent. Math* **178(3)**: 635-654, 2009. MR2551766

[6] G. Kozma. Percolation on a product of two trees. arXiv:1003.5240.

[7] G. Kozma. The triangle and the open triangle. To appear in *Ann. Inst. Henri Poincaré Probab. Stat.*, 2010. arXiv:0907.1959.

[8] R. Schonmann. Multiplicity of phase transitions and mean-field criticality on highly non-amenable graphs. *Commun. Math. Phys.* **219(2)**: 271-322, 2001. MR1833805

[9] R. Schonmann. Mean-filed criticality for percolation on planar non-amenable graphs. *Commun. Math. Phys.* **225(3)**: 453-463, 2002. MR1888869

[10] C. Wu. Critical behavior of percolation and Markov fields on branching planes. *J. Appl. Probab.* **30(3)**: 538-547, 1993. MR1232733