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EXPONENTIAL TAIL BOUNDS FOR MAX-RECURSIVE SEQUENCES

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Abstract
Exponential tail bounds are derived for solutions of max-recursive equations and for max-recursive random sequences, which typically arise as functionals of recursive structures, of random trees or in recursive algorithms. In particular they arise in the worst case analysis of divide and conquer algorithms, in parallel search algorithms or in the height of random tree models. For the proof we determine asymptotic bounds for the moments or for the Laplace transforms and apply a characterization of exponential tail bounds due to Kasahara (1978).

1 Introduction

Stochastic recursive equations of max-type arise in a great variety of problems with a recursive stochastic component as in the probabilistic analysis of algorithms or in combinatorial optimization problems. For a list of examples in this area see the survey paper of Aldous and Bandyopadhyay (2005). We consider in this paper random sequences (X_n), satisfying recurrences of the type

$$X_n \stackrel{d}{=} \bigvee_{r=1}^{K} \left(A_r(n) X^{(r)}_{I_r^{(n)}} + b_r(n) \right), \ n \geq n_0, \tag{1}$$

which fit with the general **divide and conquer** paradigma. The $I_r^{(n)} \in \{0,\dots,n-1\}$ are subgroup sizes of the K subproblems in which a problem of size n is split, $b_r(n)$ are random toll terms arising from the splitting process, $A_r(n)$ is a random weighting term for subproblem r

and $(X_n^{(r)})$ are independent copies of (X_n) describing the parameter of the r-th subproblem. It is assumed that $(X_n^{(1)}), \ldots, (X_n^{(K)}), (A(n) = (A_1(n), \ldots, A_K(n)), I^{(n)} = (I_1^{(n)}, \ldots, I_K^{(n)}), b^{(n)} = (b_1(n), \ldots, b_r(n)))$ are independent while the coefficients and subgroup sizes $A(n)$, $I^{(n)}$, $b^{(n)}$ may be dependent. $\stackrel{d}{=}$ denotes equality in distribution. X_n coincides with the maximum (worst case) of the weighted parameters of the subproblems $1, \ldots, K$ in distribution.
A general distributional limit theorem for this type of max-recurrences was given in Neininger and Rüschendorf (2005) , see also [9], [10] by means of the contraction method. The limit of X_n after normalization is characterized as unique solution of a fixpoint equation of the form

$$X \stackrel{d}{=} \bigvee_{r=1}^{K} (A_r X_r + b_r), \tag{2}$$

where (A_r, b) are limits in L^2 of the coefficients $(A_r(n), b_r(n))$ and X_r are independent copies of X.
In the present paper we study some conditions on the coefficients such that the normalized recursive sequences

$$Y_n = \frac{X_n - EX_n}{s_n} \tag{3}$$

for some scaling sequences s_n have exponential tails. We also give conditions which imply exponential tails of solutions of the fixpoint equation (2). In section 2 we consider the case of solutions X of the max-recursive equation in (2). For this case some existence and uniqueness results have been obtained in [8], [10]. Some results on tail bounds in particular for the worst case of FIND equation have been given in Grübel and Rösler (1996), Devroye (2001) and Janson (2004). We derive bounds for the moments of X and obtain by Kasahara's theorem (1978) exponential tail bounds for X. In section 3 we consider the case of max recursive sequences (X_n). We establish various conditions which imply bounds for the asymptotics of moments and Laplace transforms which again lead by Kasahara's theorem to exponential tail bounds for max-recurrences (X_n). As example we discuss the worst case of FIND sequence.

2 Exponential tail bounds for max-recursive equations

Exponential tail bounds are not easily directly accessible for max-recursive sequences (X_n) as in (1) or for solutions of max-recursive equations in (2). It will however turn out to be possible to get suitable bounds on the moments or on the Laplace transforms of (X_n) resp. X. These imply exponential tail bounds by the following lemma, which is a consequence of a more general theorem of Kasahara (1978).
We define for functions f, g on $\mathbb{R}_+ \to \mathbb{R}_+$

$$f(x) \leq_{as} g(x) \quad \text{if} \quad \overline{\lim_{x\to\infty}} \frac{f(x)}{g(x)} \leq 1 \tag{4}$$

and

$$f(x) \sim_{as} g(x) \quad \text{if} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1. \tag{5}$$

Lemma 2.1 (Kasahara (1978)) *Let X be a random variable $p, a > 0$*

and $b := \left(\frac{1}{pea}\right)^{1/p}$
1) For $X \geq 0$ *are equivalent:*
a)
$$-\ln P(X > x) \leq_{as} ax^p \tag{6}$$
and
b)
$$\|X\|_q \leq_{as} bq^{1/p} \quad \textit{for } q \to \infty,\ q \in 2\mathbb{N} \tag{7}$$
2) For general X *and* $p \geq 1$ *a) is further equivalent to:*
c)
$$\ln Ee^{tX} \leq_{as} ct^q \tag{8}$$
where $c = q^{-1}(pa)^{-(q-1)}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$,
3) The statements in 1),2) remain valid also if $\leq_{as}$ *is replaced by asymptotic equivalence* $\sim_{as}$.

Remark: In the paper of Kasahara (1978) the statement of Lemma 2.1 was given for the asymptotic equivalence case (as in part 3)). The method of proof in that paper however also allows to cover the $\leq_{as}$-bounds as in parts 1), 2) of Lemma 2.1. □

Theorem 2.2 *Consider the max recursive equation* (2) *and assume that* $E\sum_{r=1}^K A_r^q < 1$ *for* $q \geq q_0$. *If for some* $p > 0$

$$f(q) := \frac{\|\bigvee b_r\|_q}{1 - (E\sum_{r=1}^K A_r^q)^{1/q}} \leq_{as} bq^{1/p}, \tag{9}$$

then (2) *has a unique solution* X *in* M_{q_0}, *the class of all distributions with finite moments of order* q_0. *Further* X *has moments of any order and*

$$P(X > x) \leq_{as} e^{-ax^p}$$

where $a = \frac{1}{peb^p}$.

Proof: The existence and uniqueness of a solution X (in distribution) of the max recursive equation (2) follows from Neininger and Rüschendorf (2005), Theorem 5. For the proof of (9) we establish bounds for the moments of X:

$$\begin{aligned}
\|X\|_q \quad &= \|\bigvee_{r=1}^K (A_r X_r + b_r)\|_q \\
&\leq \|\bigvee_{r=1}^K A_r X_r\|_q + \|\bigvee_{r=1}^K b_r\|_q \\
&= \left(E\bigvee_{r=1}^K (A_r X_r)^q\right)^{1/q} + \|\bigvee_{r=1}^K b_r\|_q \\
&\leq \left(\sum_{r=1}^K EA_r^q EX_r^q\right)^{1/q} + \|\bigvee_{r=1}^K b_r\|_q \\
&\leq \left(\sum_{r=1}^K EA_r^q\right)^{1/q} \|X\|_q + \|\bigvee_{r=1}^K b_r\|_q.
\end{aligned}$$

This implies that

$$\|X\|_q \quad \leq \frac{\|\bigvee b_r\|_q}{1-\left(\sum_{r=1}^{K} EA_r^q\right)^{1/q}} = f(q) \leq_{as} bq^{1/p}. \tag{10}$$

Thus the tail estimate follows from an application of Lemma 2.1. □

Remark: Condition (9) can arise in various different ways. Two typical of these are

a) (9) holds if

$$\widetilde{b} := \sup_q \|\bigvee b_r\|_q = \|\bigvee b_r\|_\infty < \infty \tag{11}$$

and

$$\left(\sum_{r=1}^{k} EA_r^q\right)^{1/q} \quad \leq_{as} \left(1-\frac{\widetilde{b}}{b}\, q^{-1/p}\right). \tag{12}$$

Condition (12) implies in particular that $\sup_{r\leq K} \|A_r\|_\infty \leq 1$.

b) If

$$\delta := \lim\sup_{q\to\infty} \left(\sum_{r=1}^{k} EA_r^q\right)^{1/q} < 1 \tag{13}$$

and

$$\|\bigvee_{r=1}^{K} b_r\|_q \leq_{as} cq^{1/p} \tag{14}$$

then (10) holds with $b = \frac{c}{1-\delta}$ and $a = \left(\frac{1-\delta}{b}\right)^p \frac{1}{pe}$.

Condition b) allows more general toll terms b_r but puts a stronger condition on the weights A_r. □

Example 2.1 *Let X be the unique solution of the following max-recursive equation with finite moments of any order*

$$X \stackrel{d}{=} UX_1 \vee (1-U)X_2 \quad +1, \tag{15}$$

where $U \stackrel{d}{=} U[0,1]$ is uniformly distributed on $[0,1]$ (see [2], [4]). This fixpoint equation characterizes the limit of the worst case of FIND. Then

$$\|U\|_q = \|1-U\|_q = \left(\frac{1}{q+1}\right)^{1/q}$$

and thus by Theorem 2.2, (see (10))

$$\|X\|_q \leq f(q) = \frac{1}{1-2^{1/q}\left(\frac{1}{q+1}\right)^{1/q}} = \frac{1}{1-\exp\left(\frac{1}{q}\ln\frac{2}{q+1}\right)^{1/q}} \sim \frac{1}{1-\exp\left(-\frac{1}{q}\ln q\right)}.$$

Using $1-e^{-x} \geq (1-\frac{1}{e})x$ *for* $x \in (0,1)$ *and* $0 < \frac{1}{q}\ln q < 1$*, this implies for* $q \geq 3$

$$E\|X\|_q \leq \frac{q}{\ln q}\frac{e}{e{-}1}(1+o(1)) \leq q\frac{e}{e{-}1}(1+o(1)).$$

Therefore, Lemma 2.1 implies the tail estimate

$$P(X \geq x) \leq \exp(-ax(1+o(1)))$$

with $a = \frac{e-1}{e^2}$.

Remark: Exponential tail bounds for the limit X of the normalized worst case of FIND algorithm $X_n = \frac{T_n}{n}$ were established in Grübel and Rösler (1996) and in Devroye (2001) and in a similar way as above in Janson (2004). Theorem 2.2 generalizes these bounds to a general class of max recursive equations. The normalized worst case of FIND algorithm X_n is stochastically majorized by X i.e.

$$P\left(\frac{T_n}{n} \geq t\right) \leq P(X \geq t) \tag{16}$$

(see Devroye (2001)). Therefore, the exponential tail bounds hold also for the normalized worst case of FIND algorithm $X_n = \frac{T_n}{n}$ uniformly in $n \in \mathbb{N}$. □

3 Tail bounds for max-recursive sequences

In this section we establish tail bounds for max-recursive sequences (X_n) as in (1). A central limit theorem for these kind of recursions has been given in a recent paper of Neininger and Rüschendorf (2005). We assume in the following version that equation (1) is already in stabilized form.

Theorem 3.1 (Limit theorem for max-recursive sequences, see [8]) *Let* (X_n) *be a max-recursive sequence as in* (1) *with* $X_n \in L^s$, $\forall n$ *and assume the following conditions:*

1. *stabilization:*
   $(A_1(n),\dots,A_K(n),b_1(n),\dots,b_K(n)) \to (A_1,\dots,A_K,b_1,\dots,b_K)$ *in* L^s.
2. *contraction:* $E\sum_{r=1}^K |A_r|^s < 1$
3. *nondegeneracy:* $E1_{\{I_r^{(n)} \leq l\}}|A_r(n)|^s \to 0$ *for all* $l{\in}\mathbb{N}$, $r{=}1,\dots K$.

   Then (X_n) *converges in distribution to a limit* X^**. Further,* $l_s(X_n,X^*) \to 0$ *and* X^* *is the unique solution of the recursive equation* (2) *in* M_s*, the class of distributions with finite s-th moments.*

In the following theorem we supplement this limit theorem by giving tail bounds for (X_n). For the proof we establish uniformly in $n \in \mathbb{N}$ bounds on the asymptotics of the moments and then obtain exponential tail bounds by Lemma 2.1. Let $s_n = s(n)$ be a monotonically nondecreasing norming sequence of X_n and consider the normalized sequence

$$Y_n := \frac{X_n - EX_n}{s_n}. \tag{17}$$

Theorem 3.2 *Let (X_n) be a max-recursive sequence as in (2), let $p > 0$ and let $\|Y_i\|_r \leq_{as} cr^{1/p}$ as $r \to \infty$ for $i = 0, \dots, n_0 - 1$. Further we assume*

$$\text{a)} \qquad \left\| \frac{1}{s_n} \bigvee_{r=1}^{K} (b_r(n) - EX_n + A_r(n) EX_{I_r^{(n)}}) \right\|_r \leq_{as} br^{1/p} \tag{18}$$

and

$$\frac{1}{\eta} + \sum_{r=1}^{K} \frac{s(I_r^{(n)})}{s_n} |A_r(n)| \leq 1, \quad n \geq n_0, \text{ for some } \eta > 1 \tag{19}$$

$$\text{or b)} \qquad \left\| \bigvee_{r=1}^{K} (b_r(n) - EX_n + A_r(n) EX_{I_r^{(n)}}) \right\|_r \leq_{as} br^{1/p} \tag{20}$$

and

$$\frac{1}{s_n} + \sum_{r=1}^{K} \frac{s(I_r^{(n)})}{s_n} |A_r(n)| \leq 1, \quad n \geq n_0, \tag{21}$$

Then there exists some function $h(x)$, such that

$$h(x) \leq_{as} e^{-ax^p} \tag{22}$$

and

$$P(Y_n \leq x) \leq h(x), \quad \forall n \in \mathbb{N}_0. \tag{23}$$

where $a = \dfrac{1}{2e(\max\{\eta b, e\})^p}$ in case a) and $a = \dfrac{1}{2e(\max\{b, e\})^p}$ in case b).

Thus we obtain exponential tail bounds of order p uniformly for all Y_n.

Proof: We establish by induction uniformly in $n \in \mathbb{N}_0$ moment estimates for (Y_n) which by Lemma 2.1 correspond to the exponential tail bounds in (23). For $n = 0, \dots, n_0 - 1$ these bounds are given by assumption. The normalized sequence (Y_n) satisfies the modified recursive equation

$$Y_n \stackrel{d}{=} \bigvee_{r=1}^{K} \left(A_r(n) \frac{s(I_r^{(n)})}{s_n} Y_{I_r^{(n)}} + b_r^{(n)} \right) \tag{24}$$

where

$$b_r^{(n)} := \frac{1}{s_n} (b_r(n) - EX_n + A_r(n) EX_{I_r^{(n)}}). \tag{25}$$

We denote by Υ_n the distribution of $(A(n), I^{(n)}, b^{(n)})$ where

$$A(n) = (A_1(n), \dots, A_K(n)), I^{(n)} = (I_1^{(n)}, \dots, I_K^{(n)}) \text{ and } b^{(n)} = (b_1^{(n)}, \dots, b_K^{(n)}).$$

Then conditioning by the vector $(A(n), I^{(n)}, b^{(n)})$ we obtain in the induction step with $\beta := \bigvee_{r=1}^{K} b_r^{(n)}$

$$\begin{aligned}
E|Y_n|^q \quad &= E\left|\bigvee_{r=1}^{K}\left(A_r(n)\frac{s(I_r^{(n)})}{s_n}Y^{(r)}_{I_r^{(n)}} + b_r^{(n)}\right)\right|^q \\
&\leq E\left(\sum_{r=1}^{K}|A_r(n)|\frac{s(I_r^{(n)})}{s_n}\ |Y^{(r)}_{I_r^{(n)}}| + |b_r^{(n)}|\right)^q \\
&= \int\sum_{l=0}^{q}\binom{q}{l}E\left(\sum_{r=1}^{K}|a_r|\frac{s(i_r)}{s_n}\ |Y^{(r)}_{i_r}|\right)^l |\beta|^{q-l}d\Upsilon_n \\
(*) &\leq \int\sum_{i=0}^{q}\binom{q}{l}\left(cl^{\frac{1}{p}}(1+o(1))\right)^l\left(\sum_{r=1}^{K}\frac{s(i_r)}{s_n}|a_r|\right)^l |\beta|^{q-l}d\Upsilon_n
\end{aligned}$$

Further in case a) we obtain

$$\begin{aligned}
E|Y_n|^q \quad &\leq \left(\max\{\eta b, c\}q^{\frac{1}{p}}(1+o(1))\right)^q E\left(\tfrac{1}{\eta}+1-\tfrac{1}{\eta}\right)^q \\
&= \left(\max\{\eta b, c\}q^{\frac{1}{p}}(1+o(1))\right)^q. \qquad (26)
\end{aligned}$$

In case b) we obtain

$$\begin{aligned}
E|Y_n|^q \quad &\leq \left(\max\{b, c\}q^{\frac{1}{p}}(1+o(1))\right)^q E\left(\tfrac{1}{s_n}+1-\tfrac{1}{s_n}\right)^q \\
&= \left(\max\{b, c\}q^{\frac{1}{p}}(1+o(1))\right)^q. \qquad (27)
\end{aligned}$$

For the proof of $(*)$ we consider

$$\begin{aligned}
&E\left(\sum_{r=1}^{K}\frac{s(i_r)}{s_n}|a_r||Y^{(r)}_{i_r}|\right)^l \\
&= \sum_{j_1+\ldots+j_K=l}\binom{l}{j_1\ldots j_K}\prod_{r=1}^{K}\left(\frac{s(i_r)}{s_n}\ |a_r|\right)^{j_r} E\ |Y^{(r)}_{i_r}|^{j_r} \\
&\overset{i_r<n}{\leq} \sum_{j_1+\ldots+j_K=l}\binom{l}{j_1\ldots j_K}\prod_{r=1}^{K}\left(\frac{s(i_r)}{s_n}\ |a_r|\right)^{j_r}\left(cj_r^{\frac{1}{p}}(1+o(1))\right)^{j_r} \\
&\leq \left(cl^{\frac{1}{p}}(1+o(1))\right)^l\sum_{j_1+\ldots+j_K=l}\binom{l}{j_1\ldots j_K}\prod_{r=1}^{K}\left(\frac{s(i_r)}{s_n}|a_r|\right)^{j_r} \\
&= \left(cl^{\frac{1}{p}}(1+o(1))\right)^l\left(\sum_{r=1}^{K}\frac{s(i_r)}{s_n}\ |a_r|\right)^l.
\end{aligned}$$

The terms $cl^{\frac{1}{p}} + o(1)$ as $q \to \infty$ are independent of $n \in \mathbb{N}_o$ and thus we obtain from Lemma 2.1 a tail bound uniformly in $n \in \mathbb{N}$ as in (23). $\square$

Remark: To estimate the crucial term in (18) one can use the bound

$$\left|\bigvee_{r=1}^{K}\Big(b_r(n)-EX_n+A_r(n)EX_{I_r^{(n)}}\Big)\right|\leq\left|\bigvee_{r=1}^{K}b_r(n)\right|+\left|\sum_{r=1}^{K}|A_r(n)EX_{I_r^{(n)}}|-EX_n\right|. \tag{28}$$

Further, $|\bigvee b_r(n)|\leq\sum_r|b_r(n)|$, which however is only a good estimate if one of the $|b_r(n)|$ is big while all other $|b_r(n)|$ are small. □

Without toll terms the following improved conditions yield subgaussian tail bounds.

Theorem 3.3 *Let (X_n) be a max-recursive sequence as in (1) with zero toll terms, $b_r(n)=0, r=1,\dots,K$ and $X_n\geq 0, \quad \forall n$. We assume*

a)

$$-\ln P(X_i>x)=ax^2(1+o(1)) \text{ as } x\to\infty, \text{ for } i=0,\dots,n_0-1, \tag{29}$$

and

b)

$$\sup_{r\leq K,n\geq n_0}A_r(n)<1. \tag{30}$$

Then

$$-\ln P(X_i>x)\leq_{as}ax^2 \text{ uniformly in } i\in\mathbb{N}_0. \tag{31}$$

Proof: The proof is by induction. For $i\leq n_0-1$ (31) holds by assumption (29). For the induction step we obtain by conditioning as in section 2 for $x\geq 1$

$$\begin{aligned}
P\left(\left(\bigvee_{r=1}^{K}A_r(n)X_{I_r^{(n)}}^{(r)}\right)^{2q}\leq x\right) &= \int P\left(\left(\bigvee_{r=1}^{K}a_rX_{i_r}^{(r)}\right)^{2q}\leq x\right)dP^{(A(n),I^{(n)})}\\
&= \int\prod_{r=1}^{K}P\left(X_{i_r}^{(r)}\leq\frac{x^{\frac{1}{2q}}}{a_r}\right)dP^{(A(n),I^{(n)})}\\
&= \int\prod_{r=1}^{K}\left(1-e^{-a\left(\frac{x^{\frac{1}{2q}}}{a_r}\right)^2(1+o(1))}\right)dP^{(A(n),I^{(n)})}
\end{aligned}$$

With $\Upsilon_n=P^{(A(n),I^{(n)})}$ we thus obtain from majorized convergence

$$\begin{aligned}
EX_n^{2q} &= EX_n^{2q}1_{\{X_n<1\}}+EX_n^{2q}1_{\{X_n\geq1\}}\\
&= E1_{\{X_n\geq1\}}\left(\bigvee_{r=1}^{K}A_r(n)X_{I_r^{(n)}}^{(r)}\right)^{2q}+o(1)\\
&\leq \int\int_1^\infty\left(1-\prod_{r=1}^{K}\left(1-e^{-a\left(\frac{x^{\frac{1}{2q}}}{a_r}\right)^2(1+o(1))}\right)\right)dx\,d\Upsilon_n+o(1)+P(X_n\geq1)\\
&= \int\int_1^\infty\left(1-\prod_{r=1}^{K}\left(1-e^{-ax^2(1+o(1))}\right)\right)2q(a_rx)^{2q-1}dx\,d\Upsilon_n+O(1)\\
&= \int\int_1^\infty\left(\sum_{j=1}^{K}\binom{K}{j}(-1)^{j+1}e^{-ajx^2(1+o(1))}\right)2q(a_jx)^{2q-1}dx\,d\Upsilon_n+O(1)
\end{aligned}$$

By induction in $q \geq 1$ and partial integration we obtain

$$\int_0^\infty e^{-ajx^2} x^{2q-1} dx = \frac{(q-1)!}{2(aj)^q}, \qquad q \geq 1.$$

To deal with the $(1+o(1))$ term let

$$\begin{aligned} -\ln P(X_j > x) &= ax^2(1+o(1)) \\ =: ax^2(1+g_i(x), \quad &i \leq n-1, g_i(x) \to 0 \text{ as } x \to \infty \end{aligned}$$

and let $g(x) := \sup_{i \leq n-1} g_i(x)$. Choosing $K_0 \geq 1$ such that $g(x) \leq 1-c$ for $x \geq K_0$, $0 < c < 1$, we obtain by some calculation

$$\int_1^\infty e^{(1+g(x))} e^{-ajx^2} x^{2i-1} dx \qquad \leq C^j \int_0^\infty e^{-ajcx^2} x^{2i-1} dx$$

and thus we obtain

$$\begin{aligned} EX_n^{2q} &= \sum_{j=1}^K \tbinom{K}{j} (-1)^{j+1} E(A_j(n)^{2q-1}) 2q \textstyle\int_1^\infty e^{-ajx^2(1+o(1))} x^{2q-1} dx + O(1) \\ &\leq \sum_{j=1}^K \tbinom{K}{j} C^j |E(A_j(n)^{2q-1})| 2q \tfrac{(2q-2)(2q-4)\cdots 2}{(2acj)^q} + O(1) \\ &\leq \sup_{r \leq K, n \geq n_0} E(A_r(n)^{2q-1}) 2^q q! \frac{1}{(2ac)^q} \sum_{j=1}^K \tbinom{K}{j} C^j + O(1) \\ &\leq \sup_{r \leq K, n \geq n_0} |E(A_r(n)^{2q-1})| \left(\frac{e(2+\varepsilon)}{c} \right)^q q! \frac{1}{((2+\varepsilon)ae)^q} (1+C)^K + O(1) \\ &\leq \left(\frac{1}{((2+\varepsilon)ae)} 2q \right)^{\frac{2q}{2+\varepsilon}} \left(\left(\frac{1}{(2+\varepsilon)ae} 2q \right)^{\frac{2+\varepsilon}{2}} \left(\frac{(2+\varepsilon)e}{c2q} \right)^q \cdot \right. \\ &\qquad \left. \sup_{r \leq K, n \geq n_0} E(A_r(n)^{2q-1}) q! (1+C)^K \right) + O(1) \\ &\leq \left(\frac{1}{((2+\varepsilon)ae)} (2q) \right)^{\frac{2q}{2+\varepsilon}} (1+o(1)). \end{aligned}$$

Lemma 2.1 implies

$$\begin{aligned} P(X_n > x) &\leq e^{-ax^{2+\varepsilon}(1+o(1))} \\ &= e^{-ax^{2+\varepsilon}+o(x^{2+\varepsilon})} < e^{-ax^{2+\varepsilon}+|\tilde{c}|x^{2+\varepsilon}} \\ &= e^{-ax^2(x^\varepsilon(1+|\tilde{c}|))} < e^{-ax^2} \text{ for } x \geq 1. \end{aligned}$$

Thus at the induction step it is possible to choose the same constants c, C. □

The bounds in the following theorem are based on the Laplace transform and allow unbounded toll terms.

Theorem 3.4 *Let (X_n) be a max-recursive sequence as in (1) and let for some nondecreasing sequence $s_n \geq 1$ and $q > 1$*

$$EX_n = \mu s_n^q + r_n,\ r_n = o(s_n^q). \tag{32}$$

Let $Y_n = \frac{X_n - EX_n}{s_n}$ denote the normalized sequence and assume

a) $Ee^{\lambda Y_i} \leq e^{c\lambda^q}$ *for some* $c > 0, i = 0, \dots, n_0 - 1$ *as* $\lambda \geq \lambda_0$

b) $Ee^{\lambda \frac{b_r(n)}{s_n}} \leq e^{D_1 \lambda^q \overline{r}_n}, \quad \lambda \geq 1,\ n \geq n_0,\ \lambda \to \infty,\ \ \overline{r}_n = \max(r_n, 1)$

c) $A_r^q(n) s(I_r^{(n)})^q - s_n^q \leq -\delta\ s_n^q\ \overline{r}_n, \quad r \leq K,\ n \geq n_0,\ \delta > 0$

d) $Ee^{\frac{\lambda}{s_n}(A_r(n) EX^{(r)}_{I_r^{(n)}} - EX_n)} \leq e^{D_2 \lambda^q \overline{r}_n}, \quad \lambda \geq \lambda_1,\ n \geq n_0$

Then there exists some constants L, λ_2 such that for all n

$$E\exp(\lambda Y_n) \leq \exp(L\lambda^q(1 + o(1))),\ \textit{for } \lambda \geq \lambda_2,$$

and

$$P(Y_n > x) \leq_{as} e^{-ax^p}, \tag{33}$$

where $a = \frac{q-1}{q}\left(\frac{1}{Lq}\right)^{\frac{1}{q-1}}$ *and* $\frac{1}{q} + \frac{1}{p} = 1$.

Proof: The scaled sequence (Y_n) satisfies the recursive equation

$$Y_n \quad \stackrel{d}{=} \bigvee_{r=1}^{K} \left(A_r(n) \frac{s(I_r^{(n)})}{s_n} Y^{(r)}_{I_r^{(n)}} + b_r^{(n)} \right),$$

where $b_r^{(n)} = \frac{1}{s_n}(b_r(n) - EX_n + A_r(n) EX^{(r)}_{I_r^{(n)}})$. By assumption we have

$$E\exp(\lambda Y_j) \leq \exp(c\lambda^q(1 + o(1)), \quad \lambda \to \infty, \quad 1 \leq j \leq n_o - 1$$

for some $c > 0$. We will prove by induction, that for some $L \geq c$

$$E\exp(\lambda Y_j) \leq \exp L\lambda^q(1 + o(1)), \lambda \geq \lambda_2, \quad j \in \mathbb{N}.$$

For the induction step let Υ_n denote the distribution of $(I^{(n)}, b^{(n)}, A(n))$. By conditioning and using the induction hypothesis we obtain as in the proof of Theorem 3.2

$$\begin{aligned}
E\exp(\lambda Y_n) \quad &= \quad E\exp\left(\lambda \bigvee_{r=1}^{K} \left(\frac{s(I_r^{(n)})}{s_n} A_r(n) Y^{(r)}_{I_r^{(n)}} + b_r^{(n)} \right)\right) \\
&\leq \quad \sum_{r=1}^{K} \int E\exp\left(\lambda \frac{s(j_r)}{s_n} a_r Y^{(r)}_{j_r} + \lambda\beta\right) d\Upsilon_n(j, \beta, a) \\
&= \quad \exp(L\lambda^q(1 + o(1))) \sum_{r=1}^{K} E\exp\left(L\lambda^q \left(\left(\frac{s(I_r^{(n)})}{s_n} A_r(n) \right)^q - 1 \right) \right. \\
&\qquad \left. \cdot (1 + o(1)) + \lambda b_r^{(n)} \right).
\end{aligned}$$

Thus it remains to show that for $\lambda \geq \lambda_2$

$$A := \sup_{n \geq n_o} E \exp\left[\left(L\lambda^q\left(\left(\left(\frac{s(I_r^{(n)})}{s_n}\right)A_r(n)\right)^q - 1\right)(1+o(1)) + \lambda b_r^{(n)})\right] \leq \frac{1}{K}. \tag{34}$$

By assumption we obtain from the Cauchy-Schwarz inequality

$$\begin{aligned} E\exp(\lambda b_r^{(n)}) &= Ee^{\lambda \frac{b_r(n)}{s_n} + \frac{\lambda}{s_n}(A_r(n)EX^{(r)}_{I_r^{(n)}} - EX_n)} \\ &\leq \left(Ee^{2\lambda\frac{b_r(n)}{s_n}}\right)^{\frac{1}{2}}\left(Ee^{2\frac{\lambda}{s_n}(A_r(n)EX^{(r)}_{I_r^{(n)}} - EX_n)}\right)^{\frac{1}{2}} \\ &\leq e^{D_1\lambda^q\overline{r}_n + D_2\lambda^q\overline{r}_n} = e^{D\lambda^q\overline{r}_n}, \quad D := D_1 + D_2. \end{aligned}$$

This implies by assumption c)

$$\begin{aligned} A &\leq e^{-L\delta\lambda^q\overline{r}_n(1+o(1)) + D\lambda^q\overline{r}_n} \\ &\leq e^{-(L\delta - D)\lambda^q\overline{r}_n(1+o(1))} \\ &\leq e^{-(L\frac{\delta}{2}+D)} \\ &\leq \frac{1}{K} \quad \text{for } \lambda \geq \max(1,\lambda_1,\lambda_2) \end{aligned}$$

if $L \geq 2\frac{\ln K + D}{\delta}$ using $\overline{r}_n \geq 1$. The tail bound then follows from Lemma 2.1. □

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