

Electron. J. Probab. **28** (2023), article no. 8, 1–40.
 ISSN: 1083-6489 <https://doi.org/10.1214/22-EJP875>

Sparse matrices: convergence of the characteristic polynomial seen from infinity*

Simon Coste[†]

Abstract

We prove that the reverse characteristic polynomial $\det(I_n - zA_n)$ of a random $n \times n$ matrix A_n with iid Bernoulli(d/n) entries converges in distribution towards the random infinite product

$$\prod_{\ell=1}^{\infty} (1 - z^\ell)^{Y_\ell}$$

where Y_ℓ are independent Poisson(d^ℓ/ℓ) random variables. We show that this random function is a Poisson analog of more classical Gaussian objects such as the Gaussian holomorphic chaos. As a byproduct, we obtain new simple proofs of previous results on the asymptotic behaviour of extremal eigenvalues of sparse Erdős-Rényi digraphs: for every $d > 1$, the greatest eigenvalue of A_n is close to d and the second greatest is smaller than \sqrt{d} , a Ramanujan-like property for irregular digraphs. For $d < 1$, the only non-zero eigenvalues of A_n converge to a Poisson multipoint process on the unit circle.

Our results also extend to the semi-sparse regime where d is allowed to grow to ∞ with n , slower than $n^{o(1)}$. We show that the reverse characteristic polynomial converges towards a more classical object written in terms of the exponential of a log-correlated real Gaussian field. In the semi-sparse regime, the empirical spectral distribution of $A_n/\sqrt{d_n}$ converges to the circle distribution; as a consequence of our results, the second eigenvalue sticks to the edge of the circle.

Keywords: random directed graphs; random matrices; sparse matrices; eigenvalues.

MSC2020 subject classifications: 60B20.

Submitted to EJP on July 31, 2021, final version accepted on November 2, 2022.

Supersedes arXiv:2106.00593v2.

*The author is supported by ERC NEMO, under the European Union's Horizon 2020 research and innovation programme grant agreement number 788851.

[†]École Normale Supérieure, France. E-mail: simon.coste@ens.fr

1 Introduction

Let A_n be a square $n \times n$ matrix whose n^2 entries are independent Bernoulli(d_n/n) random variables. This non-Hermitian matrix arises, for example, as the adjacency matrix of a directed Erdős-Rényi graph with mean in-degree and mean out-degree d_n . Its *empirical spectral distribution* is the atomic measure defined by

$$\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i(A_n)} \quad (1.1)$$

where $|\lambda_i(A_n)| \geq \dots \geq |\lambda_n(A_n)|$ are the complex eigenvalues of A_n ordered by decreasing modulus. It is a striking result that when $d_n \rightarrow \infty$, the random measure μ_n suitably rescaled converges towards the *circular law*, a uniform distribution on a disk ([2, 30]). This phenomenon cannot hold when d_n is bounded independently of n , because in this case, any potential limit should have an atom at zero, as noted for example in [30]. In this $d_n = O(1)$ regime called *sparse*, the existence of a weak limit for (1.1) is not known. If this limit exists, there are no conjectures on its shape. This is in contrast with the same problem when A_n is the adjacency matrix of a random directed d -regular graph (when $d \geq 3$ is an integer); there, the limiting distribution is conjectured to have a closed-form expression, the oriented Kesten-McKay density. In the Erdős-Rényi model, there are reasons to think that no closed-form expressions will exist; more generally, the spectral behaviour of A_n in the sparse regime is still largely unknown and we refer to the physics-oriented survey [25] for insights on spectra of sparse, non-Hermitian matrices. Recently, [9] showed that when $d_n = d$, all the eigenvalues of A_n are asymptotically close to the disk $D(0, \sqrt{d})$ except one which is close to d — this is a Ramanujan-like property for sparse digraphs. It notably implies the tightness of the sequence of random measures (μ_n) , and the fact that all limit points are supported in $D(0, \sqrt{d})$, but the existence of a *unique* limit point is not guaranteed.

In this paper, the main result is that when $d_n = d > 0$, the sequence of random polynomials

$$q_n : z \mapsto \det(I_n - zA_n) = \prod_{i=1}^n (1 - z\lambda_i),$$

restricted to the disk $D(0, d^{-1/2})$, weakly converges towards an explicit random analytic function F . This is inspired by a recent advance in [7]. As a corollary, we draw a simple proof of the aforementioned result from [9]. The limiting function F seems to be new; it is a Poisson analog of the Gaussian holomorphic chaos [26], and has connections with the combinatorics of multiset-partitions. In this paper, we only sketch some of its elementary properties, but a deeper study might be of independent interest. Since this object arises as the limit of the polynomials q_n which are themselves linked with the λ_i , a better understanding of F might be useful for understanding the asymptotic spectral properties of A_n .

In the regime where $d_n \rightarrow \infty$ slower than $n^{o(1)}$, we prove a similar result for the rescaled polynomials $d_n^{-1/2} q_n(d_n^{-1/2} z)$. We show that when restricted to $D(0, 1)$, they converge towards $-z\sqrt{1-z^2}G(z)$, where G is the exponential of a log-correlated Gaussian field, and we draw consequences on the eigenvalues of A_n which show that the second eigenvalue of $A_n/\sqrt{d_n}$ is close to 1: it sticks to the edge of the circular law.

2 Results in the sparse regime

In all this section, we fix $d > 0$ and we consider a random $n \times n$ matrix A_n whose n^2 entries are independent zero-one random variables with Bernoulli(d/n) distribution.

2.1 Convergence of the secular polynomials and eigenvalue asymptotics

The main result of this paper is the convergence of the reverse characteristic polynomial of A_n , which is defined as

$$q_n(z) = \det(I - zA_n). \quad (2.1)$$

This is a sequence of random polynomials. They can explicitly be written in terms of the eigenvalues of A_n , namely $q_n(z) = \prod_{i=1}^n (1 - \lambda_i z)$, or alternatively they can be expressed through their coefficients, that is: $q_n(z) = 1 + \sum_{k=1}^n (-1)^k z^k \Delta_k(A)$, where

$$\Delta_k(A) = \sum_{\substack{I \subset [n] \\ |I|=k}} \det((a_{i,j})_{i,j \in I}).$$

These coefficients are known in the literature under the name *secular coefficients*, see for instance [16] in the context of circular β -ensembles, and sometimes we will refer to q_n as the *secular polynomials*. Since they are symmetric functions in the λ_i , they can be expressed through Newton's formulas as polynomials in the power sums $\lambda_1^k + \dots + \lambda_n^k = \text{tr}(A_n^k)$: more precisely, there is a polynomial P_k with degree k and real coefficients such that

$$\Delta_k(A_n) = (-1)^k \frac{P_k(\text{tr}(A_n^1), \dots, \text{tr}(A_n^k))}{k!},$$

this will be recalled in Subsection 6.1. The traces of A_n^k can be studied using classical methods in combinatorics, and their limit is identified by the following definition and the theorem after.

Definition 2.1. Let $d > 0$, and let $(Y_\ell : \ell \in \mathbb{N}^*)$ be a family of independent random variables, with $Y_\ell \sim \text{Poi}(d^\ell/\ell)$. We define a family of (non-independent) random variables by

$$X_k := \sum_{\ell|k} \ell Y_\ell \quad (k \in \mathbb{N}^*) \quad (2.2)$$

where $a|b$ means that b is a nonzero multiple of a .

Theorem 2.2 (trace asymptotics). For every integer k , the following joint weak convergence holds:

$$(\text{tr}(A_n^1), \dots, \text{tr}(A_n^k)) \xrightarrow[n \rightarrow \infty]{\text{law}} (X_1, \dots, X_k). \quad (2.3)$$

In particular, their joint convergence in distribution implies the convergence in distribution of any polynomial in the $\text{tr}(A_n^k)$, and in particular of the coefficients $\Delta_k(A_n)$ towards $(-1)^k P_k(X_1, \dots, X_k)/k!$. We define a (formal) series by

$$F(z) := 1 + \sum_{k=1}^{\infty} (-1)^k P_k(X_1, \dots, X_k) \frac{z^k}{k!}, \quad (2.4)$$

the convergence of which will be justified later. We just saw that for each k , the k -th coefficient of q_n converges when $n \rightarrow \infty$ towards the k -th coefficient of F : this is called *finite-dimensional convergence*. Generally, if f_n, f are formal series, we say that $f_n \rightarrow f$ in the finite-dimensional sense if for every integer k , the k -th coefficient of f_n converges as $n \rightarrow \infty$ towards the k -th coefficient of f . This is a very loose notion of convergence.

To upgrade this result to *functional* weak convergence, we need to introduce some tools. Let \mathbb{H}_r the space of analytic functions on the open disk $D(0, r)$, for $r > 0$. This set is endowed with the topology of uniform convergence on compact subsets, and with the corresponding Borel sigma-algebra (all the technical details regarding random analytic

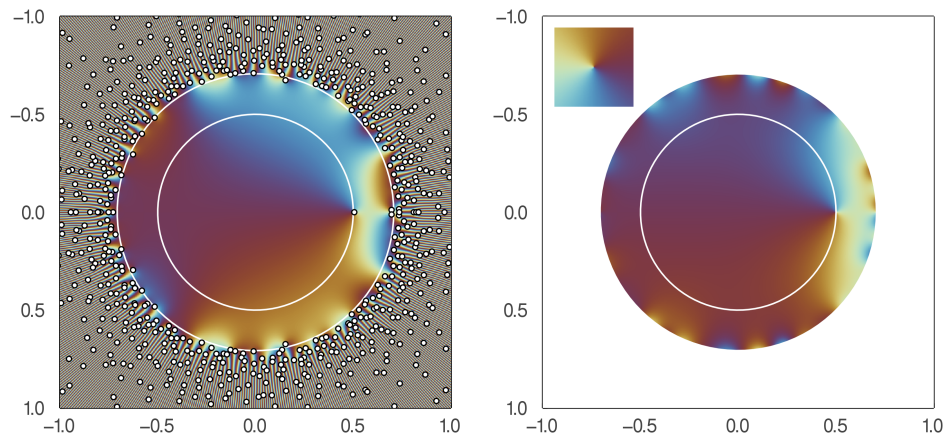


Figure 1: An illustration of Theorem 2.3 when $d > 1$. The color scheme used for these domain colourings is depicted in the small inset of the right picture. **Left** is the domain colouring of $z \mapsto \det(I - zA)$, where A is an $n \times n$ random matrix with independent entries equal to 1 with probability d/n and 0 otherwise ($n = 500$ and $d = 2$). The inverse eigenvalues of A are in white and the two circles have radius $1/d$ and $1/\sqrt{d}$. **Right** is the colouring of the random analytic function F in (2.4). What we see inside $D(0, 1/\sqrt{d})$ in the left picture converges in distribution towards what we see in the right picture.

functions will be recalled in Section 5). The sum representation in (2.4) is not very informative, and some effort will be deployed for getting more explicit representations of F in the next subsection, including infinite product representations in terms of the Y_ℓ . For the moment, we will only need that

$$\text{almost surely, } F \text{ is in } \mathbb{H}_{d^{-1/2}}. \quad (2.5)$$

We can now state our main result, where for completeness all the definitions are recalled.

Theorem 2.3 (weak convergence). *Let $d > 0$ and let A_n be a random $n \times n$ matrix whose n^2 entries are independent Bernoulli random variables with parameter d/n , and let $q_n(z) = \det(I_n - zA_n)$. Then,*

$$q_n \xrightarrow[n \rightarrow \infty]{\text{law}} F \quad (2.6)$$

where the convergence is the weak convergence of probability measures on $\mathbb{H}_{d^{-1/2}}$, and F is the random element in $\mathbb{H}_{d^{-1/2}}$ defined in (2.4).

2.2 Eigenvalue asymptotics

The zeros of analytic functions are continuous with respect to the uniform convergence of compact sets (Hurwitz's theorem). It is thus natural to study the zeroes of F . We introduce a random multiset, or equivalently a Radon measure with integer values. Let \mathbb{U}_ℓ be the set of ℓ -th roots of unity. If A, B are two multisets, their multiset union is noted $A \uplus B$; we note $A^m = A \uplus \dots \uplus A$ (m times), which means the multiset containing each element of A exactly m times.

Definition 2.4. *Let $(Y_\ell : \ell \geq 1)$ be the family of independent $\text{Poisson}(d^\ell/\ell)$ random variables in Definition 2.1. We set $\mathcal{Z}_d(\ell) = \mathbb{U}_\ell^{Y_\ell}$, and*

$$\mathcal{Z}_d = \biguplus_{\ell \geq 1} \mathcal{Z}_d(\ell). \quad (2.7)$$

In other words, each ℓ -th root of unity is put in \mathcal{Z}_d with multiplicity Y_ℓ . Note that the same root of unity can be added multiple times: in fact, if $x = e^{\frac{k2i\pi}{\ell}}$ with k, ℓ mutually primes, then x is a primitive ℓ -th root of unity, and the total multiplicity of x in \mathcal{Z}_d will be

$$M_\ell = \sum_{j=1}^{\infty} Y_{j\ell}.$$

When $d < 1$, this sum is almost surely finite, because $\mathbf{E}[M_\ell] = \sum_{j=1}^{\infty} d^{j\ell}/(j\ell) < \infty$. When $d = 1$, then $\mathbf{E}[M_\ell] = \infty$ and we will see that this sum is also almost surely infinite. We can now describe the random multiset of zeroes of F .

Proposition 2.5. *The following trichotomy holds.*

- When $d > 1$, the random analytic function F has exactly one zero in $D(0, d^{-1/2})$. It is located at $1/d$ and it is simple.
- When $d = 1$, the random analytic function F does not vanish in $D(0, 1)$.
- When $d < 1$, F is a polynomial and its zeroes have the same distribution as the random multiset described in Definition 2.4.

The proof is a straightforward consequence of the representations of F given in Theorem 2.7 below. Now, with this proposition and a simple probabilistic analysis of the weak convergence on $\mathbb{H}_{d-1/2}$ done in the inspiring papers [3, 7], we will almost effortlessly show the following result which was already partially proved in [9, 15] using a different, *ad hoc* method.

Theorem 2.6. *Let $|\lambda_1| \geq \dots \geq |\lambda_n|$ be the eigenvalues of the random matrix A_n defined in the preceding theorem. If $d > 1$, then for any $\varepsilon > 0$ the following holds:*

$$\lim_{n \rightarrow \infty} \mathbf{P}(|\lambda_1 - d| > \varepsilon) = 0 \quad \lim_{n \rightarrow \infty} \mathbf{P}(|\lambda_2| > \sqrt{d} + \varepsilon) = 0. \quad (2.8)$$

If $d = 1$, then

$$\lim_{n \rightarrow \infty} \mathbf{P}(|\lambda_1| > 1 + \varepsilon) = 0.$$

Finally, if $d < 1$, then the eigenvalues of A_n are either zero or roots of unity. The random multiset Φ_n of the non-zero eigenvalues of A_n satisfies

$$\Phi_n \xrightarrow[n \rightarrow \infty]{\text{law}} \mathcal{Z}_d \quad (2.9)$$

where \mathcal{Z}_d is the random multiset of zeroes of F and the convergence is the vague convergence of Radon measures on \mathbb{C} .

In the case $d > 1$, it is supposed that $|\lambda_2|$ actually converges in probability towards \sqrt{d} , but to the knowledge of the author this has not been proved yet; we refer to the related work section on this topic.

Incidentally, when $d < 1$ there is a positive probability that A_n has no non-zero eigenvalues, that is, A_n is nilpotent. We will see in the proof that this probability is asymptotically equal to $1-d$. As soon as $d \geq 1$, this is no longer the case and A_n must have at least one non-zero eigenvalue with probability going to 1 as $n \rightarrow \infty$. Regarding the $d = 1$ case, we could not extract further information from our proof, the only conclusion we have is that all the eigenvalues asymptotically have modulus smaller than $1 + o_{\mathbf{P}}(1)$, but some numerical experiments tend to show that there is a circle of eigenvalues with modulus close to 1 (and not equal to 1 as in the $d < 1$ case), as well as other smaller non-zero eigenvalues, see the middle picture in Figure 2.

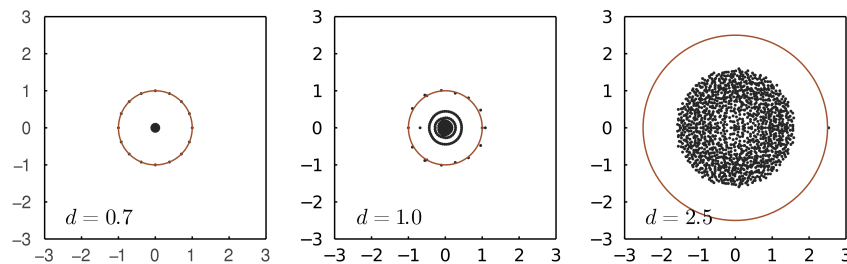


Figure 2: Complex eigenvalues of a realization A_n for $n = 2000$ and different values of d . The circles have radii $\max\{1, d\}$.

2.3 The Poisson multiplicative function and its secular coefficients

We now study the random analytic function F . The results in this section will notably imply (2.5) or Proposition 2.5, but they are also interesting on their own.

The log-generating function of the random variables X_k will play a central role, as well as its centered version:

$$f(z) = \sum_{k=1}^{\infty} X_k \frac{z^k}{k}, \quad \mathcal{P}(z) = \sum_{n=1}^{\infty} (\tau_k - X_k) \frac{z^k}{k} \quad \text{where} \quad \tau_k = \mathbf{E}[X_k] = \sum_{\ell|k} d^\ell. \quad (2.10)$$

Theorem 2.7 (limiting object). *Almost surely, for all z inside the disk of convergence of f , we have*

$$F(z) = e^{-f(z)} \quad (2.11)$$

and

$$F(z) = \prod_{\ell=1}^{\infty} (1 - z^\ell)^{Y_\ell}. \quad (2.12)$$

We now turn to the radius of convergence of f and F .

- (i) If $d < 1$, then the following alternative holds true. With probability $1 - d$, the function f is identically zero. Or, with probability d , the radius of convergence of f is equal to 1. In both cases F is a polynomial so its radius of convergence is ∞ .
- (ii) If $d = 1$, almost surely the radius of convergence of f is 1, hence the radius of convergence of F is not smaller than 1. Moreover, F is not a polynomial.
- (iii) If $d > 1$, almost surely the radius of convergence of f is $1/d$. The radius of convergence of \mathcal{P} is $d^{-1/2}$. For every $z \in D(0, d^{-1/2})$,

$$F(z) = e^{\mathcal{P}(z)} \times \prod_{\ell=1}^{\infty} (1 - dz^\ell)^{\frac{1}{\ell}} \quad (2.13)$$

where the infinite product is uniformly convergent on compact subsets $D(0, d^{-1/2})$, hence the radius of convergence of F is not smaller than $d^{-1/2}$.

There is not so much to say about F when $d < 1$: the expression (2.12) says that F is a polynomial with roots located on the unit circle. However, when $d > 1$ the situation is richer. Let c_n be the (random) coefficients of $F = e^{-f}$, so that $F(z) = \sum_{n=0}^{\infty} c_n z^n$. The c_n are also called the secular coefficients of F ; with the definition in (2.4) they are given by $c_n = P_k(X_1, \dots, X_k)/k!$ but this expression will not be very useful. The c_n are random variables, and since F is analytic inside $D(0, 1/\sqrt{d})$, Hadamard's formula

says that $\limsup |c_n|^{1/n} \leq \sqrt{d}$ almost surely, and it is natural to ask what happens at the border of this disk, a question similar to the construction of the Gaussian Multiplicative Chaos on the circle, see [29] for a survey (as a side note, we will also see that f is itself a log-correlated field).

Remark 2.8. The preceding theorem does not prove that the radius of convergence of F is equal to $d^{-1/2}$, but only greater than $d^{-1/2}$. We strongly suppose that it is indeed an equality.

Now, Cauchy's formula says that for any $m \in \mathbb{Z}$ and $r < d^{-1/2}$,

$$\frac{1}{2\pi} \int_0^{2\pi} F(re^{it}) r^{-m} e^{-imt} dt = \begin{cases} c_m & \text{if } m \geq 0 \\ 0 & \text{if } m < 0. \end{cases}.$$

Consequently, the limit

$$\lim_{r \rightarrow d^{-1/2}} \frac{1}{2\pi} \int_0^{2\pi} F(re^{it}) \varphi(re^{it}) dt$$

exists for every $\varphi(z) = \sum_{k=d_1}^{d_2} a_k z^k$ with $d_1, d_2 \in \mathbb{Z}$, ie for every trigonometric polynomial φ on the circle $\mathbb{T}_{d^{-1/2}} = \{|z| = d^{-1/2}\}$. This provides us with a simple construction of F extended to $\mathbb{T}_{d^{-1/2}}$: we see it as a random distribution, that is, a continuous linear function on the set of trigonometric polynomials on $\mathbb{T}_{d^{-1/2}}$.

Definition 2.9. The Poisson Holomorphic Chaos of index $d > 1$, noted PHC_d , is the random distribution on $\mathbb{T}_{d^{-1/2}}$ almost surely defined by

$$(\text{PHC}_d, \varphi) = \lim_{r \rightarrow d^{-1/2}} \frac{1}{2\pi} \int_0^{2\pi} F(re^{it}) \varphi(re^{it}) dt. \quad (2.14)$$

A distribution D on $\mathbb{T}_{d^{-1/2}}$ is entirely characterized by its Fourier coefficients $\hat{D}(m) := (D, e_m)$ where $e_m(t) = d^{-1/2} e^{-imt}$. Consequently, we can define the s -Sobolev norm ($s \in \mathbb{R}$) by

$$\|D\|_s^2 = \sum_{n \in \mathbb{Z}} (1 + n^2)^s |\hat{D}(n)|^2. \quad (2.15)$$

A distribution is s -Sobolev when the sum above is finite. Since the Fourier coefficients of the Poisson holomorphic chaos are given by $\widehat{\text{PHC}_d}(m) = d^{-m/2} c_m$ if $m \geq 0$ and 0 if $m < 0$, the Sobolev norm is simply given by $\sum_{m \in \mathbb{N}} (1 + m^2)^s d^{-m/2} |c_m|^2$.

Proposition 2.10. Let $d > 1$. Almost surely, the random distribution PHC_d is s -Sobolev for every $s < -1/2$.

Future work will be devoted to a wider analysis of the Sobolev-regularity of F . Studying the Sobolev norms of F requires a good understanding of the integrability properties of the secular coefficients c_n . We saw that these coefficients are polynomials in the X_k (hence of the Y_ℓ), but this expression is difficult to manipulate; however, we have access to their moments by means of a combinatorial analysis. For every integer $k > 0$, we note Odd_k the set of nonempty subsets of $[k] = \{1, \dots, k\}$ with an odd number of elements, and Even_k the set of nonempty subsets of $[k] = \{1, \dots, k\}$ with an even number of elements.

Theorem 2.11. For any z_1, \dots, z_k in the domain of definition of F , one has

$$\mathbf{E}[F(z_1) \cdots F(z_k)] = \frac{\prod_{S \in \text{Odd}_k} (1 - d \prod_{s \in S} z_s)}{\prod_{S \in \text{Even}_k} (1 - d \prod_{s \in S} z_s)}. \quad (2.16)$$

To give a few examples,

$$\begin{aligned}\mathbf{E}[F(z)] &= 1 - zd \\ \mathbf{E}[F(y)F(z)] &= \frac{(1 - dy)(1 - dz)}{1 - dyz} \\ \mathbf{E}[F(x)F(y)F(z)] &= \frac{(1 - dx)(1 - dy)(1 - dz)(1 - dxyz)}{(1 - dxy)(1 - dxz)(1 - dyz)}.\end{aligned}$$

In particular, we see the c_n are centered random variables except c_0 and c_1 , and the analytic series $(1 - z)^{-1} = \sum z^n$ gives $\mathbf{E}[|c_n|^2] = d^n(d + 1)$ — in particular $\mathbf{E}[|c_n|^2] = O(d^n)$.

The formula given above is our analog of the generating-function formula for the Gaussian Holomorphic Chaos in [26]. Therein, the combinatorial interpretation of the secular coefficients was easily linked with the enumeration of magic squares. A similar simple interpretation of the coefficients of (2.16) is not clear for the moment, but some thoughts are gathered in Subsection 8.6 at page 26.

3 Results in the denser regime where $d_n \rightarrow \infty$

The primary interest of this paper was to study random zero-one matrices with a constant mean degree $d_n = d > 0$ not depending on n , a regime which is more difficult than the ‘semi-sparse’ regime where d_n is allowed to grow to infinity. This is what we study now, and our results essentially have the same flavour as [7] in that the spectral limits of A_n are Gaussian. We emphasize that this method gives a unified point of view on the convergence of the characteristic polynomial in *all regimes of d* , sparse or not. Our results are stated and proved in the regime where d goes to infinity but not too fast; in fact, we will from now on suppose

$$\lim_{n \rightarrow \infty} d_n = +\infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\log(d_n)}{\log(n)} = 0. \quad (\text{H})$$

There is no doubt that the result will still hold for denser regimes such as $d = n^\alpha$, $\alpha < 1$; we did not pursue this route further.

Our first result is on the identification of traces of $A_n/\sqrt{d_n}$ when $d_n \rightarrow \infty$; in the sparse regime, everything was essentially Poisson; in this semi-sparse regime, everything is essentially Gaussian.

Theorem 3.1. *Let $(N_k : k \geq 1)$ be a family of iid standard real Gaussian random variables. Then, under (H), for any k ,*

$$\left(\frac{\text{tr}(A)}{\sqrt{d_n}} - \sqrt{d_n}, \dots, \frac{\text{tr}(A^k)}{\sqrt{d_n^k}} - \sqrt{d_n^k} \right) \xrightarrow[n \rightarrow \infty]{\text{law}} (N_1, \dots, \sqrt{k}N_k) + (0, 1, 0, 1, \dots, \mathbf{1}_{k \text{ is even}}). \quad (3.1)$$

The proof is at Section 10. It is well known that if Z_λ is a $\text{Poi}(\lambda)$ random variable, then $\lambda^{-1/2}(Z_\lambda - \lambda)$ converges in distribution towards a standard Gaussian. From this, it is almost immediate to see that if the X_k are the random variables arising in Definition 2.1 in the sparse regime, then $d^{-1/2}(X_1, \dots, X_k)$ converges in distribution as $d \rightarrow \infty$ to the RHS in (3.1).

We now define

$$G(z) = \exp \left\{ \sum_{k=1}^{\infty} N_k \frac{z^k}{\sqrt{k}} \right\}.$$

It is easily seen that the analytic function inside the exponential, say $g(z)$, almost surely has a radius of convergence equal to 1, hence G is in \mathbb{H}_1 (analytic functions in $D(0, 1)$). On a side note, it is not difficult to check that if g is a log-correlated Gaussian field, in that $\text{Cov}(g(z), g(w)) = -\log(1 - z\bar{w})$.

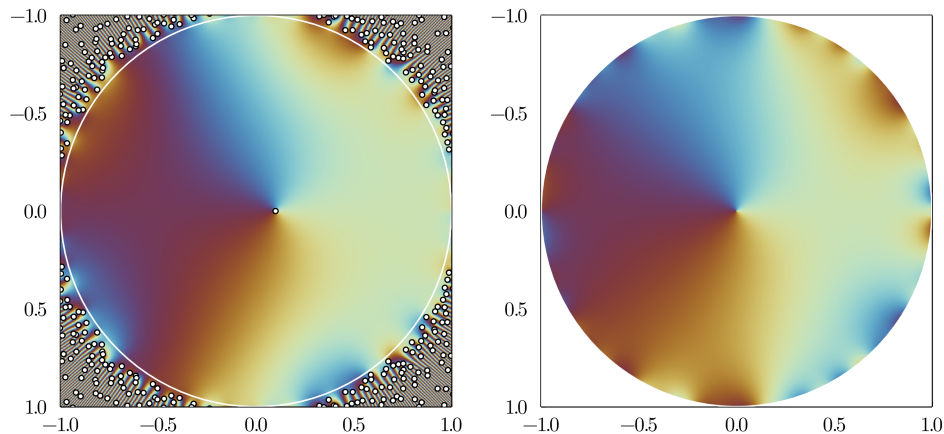


Figure 3: Same plot as in Figure 1, but for $\det(I - zA/\sqrt{d_n})$; here we took $n = 2000$ and $d_n = 100$. The function on the right is the phase portrait of $z \mapsto -z\sqrt{1 - z^2}G(z)$ as in Theorem 3.2.

Theorem 3.2 (weak convergence). *Let (d_n) be a sequence satisfying (H) and let A_n be a random $n \times n$ matrix whose n^2 entries are independent Bernoulli random variables with parameter d_n/n , and let $q_n(z) = \det(I_n - zA_n/\sqrt{d_n})$. Then,*

$$\frac{q_n(z)}{\sqrt{d_n}} \xrightarrow[n \rightarrow \infty]{\text{law}} -z\sqrt{1 - z^2}G(z) \quad (3.2)$$

weakly on \mathbb{H}_1 .

We note that the limit in (3.2) is nearly the same as the one in [7]. The main difference is in the presence of this z factor, and it is due to the fact that the entries in our matrix are not centered. If we replaced A with $A - \mathbf{E}[A]$, this term would not be present.

We now give the analog of Theorem 2.6 in the $d_n \rightarrow \infty$ setting.

Theorem 3.3. *Let $|\lambda_1| \geq \dots \geq |\lambda_n|$ be the eigenvalues of the random matrix A_n defined in the preceding theorem, under hypothesis (H). Then, $\lambda_1/\sqrt{d_n} \rightarrow \infty$ almost surely and for any $\varepsilon > 0$,*

$$\lim_{n \rightarrow \infty} \mathbf{P}(|\lambda_2/\sqrt{d_n}| - 1| > \varepsilon) = 0. \quad (3.3)$$

In any regime where $d_n \rightarrow \infty$, the circle law was proved in [2, 30]: the empirical spectral distribution of the $\lambda_i/\sqrt{d_n}$ converges weakly towards the uniform distribution on $D(0, 1)$. The convergence in (3.3) says that apart from the first eigenvalue, there are no outliers in the spectrum, ie the second eigenvalue cannot wander away from the boundary of the support $D(0, 1)$ of the limiting distribution.

Since the circular distribution puts a strictly positive mass $c(\varepsilon)$ on any domain $\mathbb{C} \setminus D(0, 1 - \varepsilon)$, then by the Portemanteau theorem, the fraction of eigenvalues of $A_n/\sqrt{d_n}$ which are bigger than $1 - \varepsilon$ is asymptotically $\geq c(\varepsilon)$, and in particular is strictly positive: not only is $|\lambda_2/\sqrt{d_n}|$ greater than $1 - \varepsilon$, but also a linearly growing number of $|\lambda_k/\sqrt{d_n}|$.

Plan of the paper

In Section 4, we give an overview of the origins of our method and various related work. We mention a set of questions and possible extensions. Section 5 is a summary of classical notions on random analytic functions; the specific properties of the Poisson

multiplicative function which are stated in Subsection 2.3 are proved in Section 8. The core results of the paper, namely Theorem 2.3 and 2.6, are proved in Section 6 as consequences of Theorem 2.2, which identifies the limits of the traces. This theorem is proved in Section 9. Section 10 deals with our results in the $d_n \rightarrow \infty$ regime.

4 Related work and comments

Convergence of the reverse characteristic polynomial Our work is inspired by the recent paper [7]. There, the authors prove the convergence of $\det(I_n - zX_n/\sqrt{n})$, where X_n is a random matrix whose entries are iid random variables, with mean 0 and variance 1, and whose law does not depend on n ; for instance, Ginibre matrices. The circular law phenomenon holds for this model; that is, the empirical spectral distribution of X_n/\sqrt{n} converges weakly towards the uniform distribution over $D(0, 1)$. The goal of [7] was to prove the ‘no outliers phenomenon’: that the greatest eigenvalue of X_n/\sqrt{n} , noted ρ_n , converges in probability towards 1, with no extra assumptions on the moments of the entries of X . We refer to the introduction of [6] and references therein for some history on this theorem. In our paper we closely follow the method introduced in [7]. This method is itself inspired by [3] and Appendix A therein. The crucial addition of [7] was a truncation procedure and the identification of a trace CLT, which allowed to identify the limiting distribution. However, the model studied in [3] is completely different (perturbations of banded matrices).

Eigenvalue asymptotics for sparse matrices and trace methods Theorem 2.6 was already proved in [9], following a long line of research initiated in [5, 10] and continued, for example, in [13, 8, 12]. These papers reach results like (2.8) by using a very sophisticated high-trace method, namely, they study the asymptotics of $\text{tr}(A_n^{k_n})$, with k_n allowed to grow to ∞ . This technique dates back at least to [20]. In our method, (2.8) is a direct consequence of Theorem 2.3, which is itself proved using the classical trace method with k fixed as in Theorem 2.2. It is considerably simpler. Trace asymptotics are standard in random matrix theory, even for non-Hermitian random matrices; however, we could not find Theorem 2.2 in the literature and to the knowledge of the authors, there are very few similar results for other models of sparse matrices. The closest result can be seen in [18] (or in its bipartite version [33]) for traces of non-backtracking matrices on regular graphs: see the definition of CNBW_k^∞ at page 15 in [18].

Sparse models: extension to regular graphs Our paper only treats the case of Erdős-Rényi directed graphs, whose edges are independent. The other main model for sparse non-Hermitian matrices is the *random regular digraph model* mentioned in the introduction. For an integer $d \geq 3$, one samples A uniformly from the set of adjacency matrices of d -regular directed graphs. A statement similar to Theorem 2.6 for this model was proven in [13] using the high-trace method, but the method in our paper should also provide a simpler proof. In fact, the limits of the traces as in Theorem 2.2 is thought to be the same for random regular, with a similar proof; but the main difficulty lies in the tension of the polynomials q_n , where our method and Proposition 6.2 are not easily generalized. This exploration was recently done in the preprint [14].

Similarly, one can be interested in applying this model to random undirected regular graphs. The asymptotics of the second eigenvalue, in this case, is given by the Alon-Friedman theorem and the proofs relied on delicate subgraphs asymptotics [19], or the use of the non-backtracking matrix B , see [5]. The asymptotics of the traces of B are known, see [18], but here again the tension of the family of polynomials $\det(I - zB)$ on $\mathbb{H}_{1/2\sqrt{d-1}}$ is not proved for the moment.

Sparse models: extensions to inhomogeneous graphs The inhomogeneous random digraph is defined as follows: instead of appearing with probability $p = d/n$, an edge (i, j) appears with probability $p_{i,j}$ depending on i and j . When the mean matrix P has low rank and has delocalized entries, a theorem similar to Theorem 2.6 was proved in [15]. Since this model encompasses popular generative models in network science (such as the directed Stochastic Block-Model, its weighted, labelled or degree-corrected variants, or the rank-1 inhomogeneous random graph), our method could provide an interesting alternative to [10] and the high-trace methods; however, as it is now our method cannot provide information on the eigenvectors of A_n , and it could be interesting to see if any refinement can achieve this.

Alon-Boppana for non-normal matrices In the theory of sparse non-Hermitian matrices, there is still a crucial lack of lower bounds for the second eigenvalue. For instance, our result in Theorem 2.6 states that when $d > 1$ the second eigenvalue of Erdős-Rényi graphs satisfies $|\lambda_2| \leq \sqrt{d} + o_{\mathbf{P}}(1)$. However, the matching lower bound, namely that $|\lambda_2| \geq \sqrt{d} + o_{\mathbf{P}}(1)$, is only conjectured; in fact, it is generally believed that for any fixed k , $|\lambda_k| \geq \sqrt{d} + o_{\mathbf{P}}(1)$. We call this kind of upper bounds ‘Alon-Boppana bounds’, in reference of the famous lower bound for λ_2 in [27] in the context of regular graphs. For Hermitian or normal matrices, they are easily reachable thanks to the min-max characterizations of eigenvalues, but this tool is not available for non-normal matrices such as adjacency matrices of random digraphs. In many models of sparse random matrices such as [15], Alon-Boppana bounds are conjectured, but at the moment they are not proved.

One possible way to prove these bounds is to directly prove that the ESD of A converges towards a limiting measure supported in $D(0, \sqrt{d})$, but as mentioned in the introduction this seems very difficult in sparse settings.

The method developed in [2, 7] and this paper suggests another strategy. The limiting function F is conjectured to have a radius of convergence of $1/\sqrt{d}$. If we had $\limsup |\lambda_2| < c$ for some $c < \sqrt{d}$ on an event with positive probability, then on this event all the roots of q_n except $1/\lambda_1$ would be outside the circle of radius $1/c$. It seems reasonable to say that this should imply that the radius of convergence of F would then be greater than $1/c$ on this event, a contradiction. We could not formalize this idea.

The Poisson analog of the GHC Exponentials of Gaussian fields have a very rich history, see the survey [29]. Of special interest is the Gaussian holomorphic chaos, ie

$$C(z) = \exp \left(\sum_{k=0}^{\infty} N_k \frac{z^k}{\sqrt{k}} \right) \quad (4.1)$$

where the N_k are iid standard complex Gaussian variables. This random analytic function on $D(0, 1)$ was proved to be the limit of the secular polynomials in the circular β -ensemble for $\beta = 2$ in [17], and then for every $\beta > 0$ in [23]. This link between characteristic polynomials and Gaussian functions was also central in a series of conjectures in [21] and subsequent work. The remarkable paper [26] thoroughly studies more refined properties of the GHC linked with the circular ensembles; therein, the very clear combinatorial interpretation of the moments of the GHC already seen in [16] is proven by elementary means ([26, Theorem 1.6]). For the moment, we could not reach such an elegant description for our Poisson analog, see Theorem 2.7.

Subsequent work should be devoted to a less shallow study of the Poisson multiplicative function F and its extension to the circle $\mathbb{T}_{d-1/2}$ as in Definition (2.1). The following questions arise.

1. We strongly suppose that the radius of convergence of F is $d^{-1/2}$, we did only prove that it is $\geq d^{-1/2}$.
2. Proposition 2.10 only gives a tiny information on the regularity of PHC. Can we fully characterize the Sobolev regularity of this object? As noted in Subsection 8.4, the answer could depend on the limit of $\int |F(re^{it})|^2 dt$ — the total mass of a putative random measure on $\{|z| = d^{-1/2}\}$ defined by $\lim_{r \rightarrow d^{-1/2}} \int e^{2\operatorname{Re}(f(re^{it}))} dt$.
3. Is there a simple combinatorial interpretation of the generating functions for the moments of PHC_d in (2.16)?
4. What can be said about the convergence of the secular coefficients $\Delta_k(A_n)$ when k is allowed to go to infinity with n ?

5 General facts about random analytic functions

In this section, we recall some basic facts on random analytic functions.

Let \mathbb{H}_r be the set of analytic functions on the open disk $D(0, r)$. A textbook treatment of the properties of \mathbb{H}_r is in [32, vol. 2a, ch. 6]. A classical consequence of Cauchy's formula is that the elements in \mathbb{H}_r can be represented as power series

$$\sum_{k=0}^{\infty} a_k z^k \quad (5.1)$$

with $\limsup |a_k|^{1/k} \leq 1/r$ (Hadamard's formula). The space \mathbb{H}_r is endowed with the compact-convergence topology: we say that a sequence f_n converges to f if, for every compact set $K \subset D(0, r)$,

$$\|f_n - f\|_K = \sup_{z \in K} |f_n(z) - f(z)| \rightarrow 0.$$

Endowed with this topology, $\mathcal{H}(D)$ is *topolish* — it is separable, complete and there is a metric distance generating the topology.

We now turn to random variables in \mathbb{H}_r . We endow this space with the Borel sigma-algebra. Random variables in \mathbb{H}_r are random analytic functions; equivalently, they are series as in (5.1), where the a_i are random variables satisfying Hadamard's limsup condition. The laws of two random functions $\sum a_k z^k$ and $\sum b_k z^k$ are equal if and only if the finite-dimensional distributions of (a_i) and (b_i) agree, that is, if $(a_0, \dots, a_k) \stackrel{\text{law}}{=} (b_0, \dots, b_k)$ for every k . The classical text on random analytic functions is [24], especially Chapter 3.

We endow the space of probability distributions on \mathbb{H}_r with the topology of weak convergence of measures, that is: the law of f_n converges weakly towards the law of f if and only if $\mathbb{E}[\Phi(f_n)] \rightarrow \mathbb{E}[\Phi(f)]$ for every continuous bounded function $\Phi : \mathbb{H}_r \rightarrow [0, \infty[$. With a common abuse of language, when we say that random variables converge in law, we mean that their distributions converge in law as described above. The following theorem summarizes the results in [31]. See also [7, Lemma 3.2].

Theorem 5.1. *Let f, f_1, f_2, \dots be random variables in \mathbb{H}_r . Then, f_n converges weakly towards f if and only if*

- (i) *The sequence of random variables $\|f_n\|_{K_i}$ is tight for every i , where K_i is a sequence of compacts exhausting $D(0, r)$.*
- (ii) *The finite-dimensional laws of f_n converge in law towards those of f .*

Proving the tightness of $\|f_n\|_K$ when f_n is a random sequence of analytic function can be simplified by the following device, a statement close to [31], Lemma 2.6 and the remark just after.

Lemma 5.2 (Hardy-type criterion). *Let $f_n(z) = \sum a_{n,k} z^k$ be a sequence of random analytic functions on $D(0, r)$. If there is a sequence (a_k) such that $\sup_n \mathbf{E}[|a_{n,k}|^2] \leq a_k$ for every k and such that $\limsup |a_k|^{1/2k} \leq 1/r$, then (f_n) is tight in \mathbb{H}_r .*

Proof. Let $K \subset D(0, r)$ be a compact set and let $s < r$ be the radius of an open disk containing it. By Cauchy's formula, for every $z \in K$,

$$\begin{aligned} |f_n(z)|^2 &= \left| \frac{1}{2i\pi} \int_{|w|=s} \frac{f_n(w)}{w-z} dw \right|^2 = \frac{1}{(2\pi)^2} \left| \int_0^{2\pi} \frac{f_n(se^{it}) se^{it}}{se^{it}-z} dt \right|^2 \\ &\leq \frac{s^2}{2\pi\delta^2} \int_0^{2\pi} |f_n(se^{it})|^2 dt \end{aligned}$$

where we used Cauchy-Schwarz's inequality and we set $\delta = s - \max_{x \in K} |x| > 0$. Using Parseval's formula, we obtain

$$\forall z \in K, \quad |f_n(z)|^2 \leq \frac{s^2}{\delta^2} \sum_{k=0}^{\infty} |a_{n,k}|^2 s^{2k}.$$

The expectation of the RHS is by hypothesis smaller than $(s^2/\delta^2) \sum_k a_k s^{2k}$, a continuous function in $s \in [0, r)$ because we supposed that $\limsup |a_k|^{1/2k} \leq 1/r$. Consequently, $\mathbf{E}[\sup_{z \in K} |f_n(z)|^2] \leq c_K$ for some constant c_K depending on K . It readily implies the tightness of the sequence. \square

Remark 5.3. The 2-Hardy norm on a disk $D(0, s)$ is $(\sum |a_k|^2 s^{2k})^{1/2}$. The preceding statement says that if (f_n) has a uniformly bounded 2-Hardy norm in every $D(0, s)$ for $s < r$ then the sequence is tight. This is slightly more specific than the criterion in [31], in which one directly proves that $\mathbf{E}[|f_n(z)|^2]$ is bounded by a continuous function. The reason why we do this is the following: if we directly want to study $\mathbf{E}[|f_n(z)|^2]$, we obtain

$$\mathbf{E}[|f_n(z)|^2] = \sum_{k,\ell} z^k \bar{z}^\ell \mathbf{E}[a_{n,k} \bar{a}_{n,\ell}]$$

and in most study cases, the random variables $a_{n,\ell}$ and $a_{n,k}$ are independent or decorrelated for $k \neq \ell$, as in [7], but this is not the case for us, see the examples in Subsection 6.3.

6 Proof of Theorem 2.3

6.1 Proof of the finite-dimensional convergence

To prove Theorem 2.3, namely that q_n converges weakly towards F , we use Theorem 5.1. The key for the finite-dimensional convergence is the identification of the distributional limits of the traces of A^k in Theorem 2.2. Given this theorem which will be proved in Section 9, we can classically link the traces of A^k with the secular coefficients, the coefficients of $\det(I - zA)$.

To do this we introduce a family of polynomials P_k , each P_k having k variables. We note \mathfrak{S}_k the group of permutations of $[k]$; each permutation σ can be uniquely written as a composition of ℓ cycles with disjoint support, say $\sigma = c_1 \circ \dots \circ c_\ell$; the length of a cycle c is noted $|c|$. Then, we define P_k as

$$P_k(z_1, \dots, z_k) = \sum_{\sigma \in \mathfrak{S}_k} (-1)^\ell z_{|c_1|} \dots z_{|c_\ell|}. \quad (6.1)$$

Proposition 6.1. For any $f(z) = \sum a_n z^n$ analytic in some disk $D(0, r)$,

$$e^{-\sum_{k=1}^{\infty} a_k \frac{z^k}{k}} = 1 + \sum_{k=1}^{\infty} P_k(a_1, \dots, a_k) \frac{z^k}{k!}. \quad (6.2)$$

Moreover, for any $n \times n$ matrix B , for any z ,

$$\det(I - zB) = 1 + \sum_{k=1}^n P_k(\text{tr}(B), \dots, \text{tr}(B^k)) \frac{z^k}{k!}. \quad (6.3)$$

Proof. Part (6.2) can be found in [1, Corollary 3.6]. For the second statement, we note $C(z)$ the matrix logarithm of $I - zB$, which exists if $|z| < 1/\|B\|$ and is defined by

$$C(z) = -\sum_{k=1}^{\infty} \frac{B^k}{k} z^k.$$

Then $\det(I - zB) = \det(e^{C(z)}) = e^{\text{tr}(C(z))} = e^{-\sum_{k=1}^{\infty} \frac{\text{tr}(B^k)}{k} z^k}$. This proves (6.3) (we can truncate the sum at n since it is a polynomial of degree n) for $|z| < 1/\|B\|$, and it extends analytically for all z . \square

The coefficients of q_n are $\Delta_k(A_n) = P_k(\text{tr}(A), \dots, \text{tr}(A^k))/k!$ and polynomials are continuous with respect to weak convergence, so Theorem 2.2 implies that

$$(\Delta_1(A_n), \dots, \Delta_k(A_n)) \xrightarrow{\text{law}} (P_1(X_1), \dots, P_k(X_1, \dots, X_k)/k!).$$

Thanks to (6.2) and the definition of F in (2.4), this is exactly the finite-dimensional convergence of q_n towards $F = e^{-f}$, ie the second point in Theorem 5.1. To complete the proof, we need to check the first point of this theorem.

6.2 Proof of the tightness

We now have to prove the first point on Theorem 5.1, that is, the tightness of (q_n) , and to do this we use the Hardy-type criterion in Lemma 5.2. We recall that $q_n(z) = \det(I - zA) = 1 + \sum_{k=1}^n (-1)^k z^k \Delta_k(A)$ where the $\Delta_k(A)$ are the secular coefficients; from now on, we'll simply note them Δ_k , and they are given by

$$\Delta_k = \sum_{\substack{I \subset [n] \\ |I|=k}} \det(A(I)) \quad (6.4)$$

with $A(I) = (a_{i,j})_{i,j \in I}$. Our goal for Lemma 5.2 is to give an upper bound for

$$\mathbf{E}[|\Delta_k|^2] = \sum_{\substack{|I|=k \\ |J|=k}} \mathbf{E}[\det(A(I)) \det(A(J))] \quad (6.5)$$

which does not depend on n . For any finite set E , we note $\mathfrak{S}(E)$ the group of its bijections and $\varepsilon : \mathfrak{S}(E) \rightarrow \{-1, +1\}$ the signature (the unique nonconstant group morphism). If I, J are fixed, then

$$\mathbf{E}[\det(A(I)) \det(A(J))] = \sum_{\substack{\sigma \in \mathfrak{S}(I) \\ \tau \in \mathfrak{S}(J)}} \varepsilon(\sigma) \varepsilon(\tau) \mathbf{E} \left[\prod_{\substack{i \in I \\ j \in J}} a_{i, \sigma(i)} a_{j, \tau(j)} \right] \quad (6.6)$$

and when the entries $a_{i,j}$ are centered and have a common variance, the former expression is easy to study: if $I \neq J$, the expectation is the sum is always zero by independence,

so the whole expression is nonzero iff $I = J$ and the only nonzero contribution in the sum comes from $\sigma = \tau$. This is the situation studied in [3] or [9]. It is not the case for us since we deal with non-centered entries. For instance, let us take $n = 2$, $I = \{1\}$ and $J = \{1, 2\}$, so that

$$\mathbf{E}[\det(A(I)) \det(A(J))] = \mathbf{E}[a_{1,1}^2 a_{2,2} - a_{1,1} a_{1,2} a_{2,1}].$$

If the $a_{i,j}$ are standard real Gaussian random variables, this is zero. But if they are $\text{Ber}(p)$, this is $p^2 - p^3$. Fortunately, the computations are accessible.

Proposition 6.2. *Set $p = d/n$. With the preceding notations,*

$$\mathbf{E}[|\Delta_k|^2] = (n)_k p^k (1-p)^{k-1} (1-p+np). \quad (6.7)$$

The proof will be in Section 6.3. We did not succeed in finding a simpler proof. Now, since $p \leq 1$ and $np = d$, we can bound $\mathbf{E}[|\Delta_k|^2]$ by $a_k := d^k(1+d)$, which does not depend on n ; we also have

$$\limsup |a_k|^{1/2k} = \sqrt{d}.$$

Lemma 5.2 applies and shows that (q_n) is a tight sequence in $\mathbb{H}_{d^{-1/2}}$.

6.3 Proof of Proposition 6.2

For any two nonempty subsets $I, J \subset [n]$, we will note

$$\delta(I, J) = \mathbf{E}[\det(A(I)) \det(A(J))]. \quad (6.8)$$

Proposition 6.2 is a direct consequence of the following theorem which could be of independent interest.

Theorem 6.3. *If I has k elements and J has h elements (wlog, $k \leq h$), then the followings holds for any matrix A with independent $\text{Ber}(p)$ entries.*

(i) *If $J \setminus I$ or $I \setminus J$ has more than two elements, then $\delta(I, J) = 0$.*

(ii) *If $I = J$, then*

$$\delta(I, J) = k! p^k (1-p)^{k-1} (1-p+kp). \quad (6.9)$$

(iii) *If $I \subset J$ and $J \setminus I$ has only one element, then $\delta(I, J) = k! p^{k+1} (1-p)^k$.*

(iv) *If $|I| = |J|$ and if $|I \cap J| = k-1$, then*

$$\delta(I, J) = (k-1)! p^{k+1} (1-p)^{k-1}.$$

We now prove Proposition 6.2, and we will only need the cases (i), (ii), (iv) — case (iii) is only included for completeness. We have

$$\mathbf{E}[|\Delta_k|^2] = \sum_{|I|=|J|=k} \delta(I, J)$$

and in this sum, the only non-zero contributions come from couples $I = J$ and couples I, J with $|I \cap J| = k-1$, so

$$\begin{aligned} \mathbf{E}[|\Delta_k|^2] &= \sum_{\substack{|I|=|J|=k \\ I=J}} \delta(I, J) + \sum_{\substack{|I|=|J|=k \\ |I \cap J|=k-1}} \delta(I, J) \\ &= \binom{n}{k} k! p^k (1-p)^{k-1} (1-p+kp) + \binom{n}{k} (n-k) k (k-1)! p^{k+1} (1-p)^{k-1} \\ &= (n)_k p^k (1-p)^{k-1} (1-p+kp) + (n)_k (n-k) p^{k+1} (1-p)^{k-1} \\ &= (n)_k p^k (1-p)^{k-1} (1-p+kp+p(n-k)) \\ &= (n)_k p^k (1-p)^{k-1} (1-p+np). \end{aligned}$$

Before jumping to the proof of Theorem 6.3, let us illustrate the 3×3 case for which computations can be checked by hand.

Example 6.4. We study

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}.$$

Case (i): let us take for instance $I = \{1\}$ and $J = \{1, 2, 3\}$, so

$$\begin{aligned} \delta(I, J) &= \mathbf{E}[a_{1,1} \det(A)] \\ &= \mathbf{E}[a_{1,1}(a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1})] \\ &= \mathbf{E}[a_{1,1}a_{2,2}a_{3,3} + a_{1,1}a_{1,2}a_{2,3}a_{3,1} \\ &\quad + a_{1,1}a_{1,3}a_{2,1}a_{3,2} - a_{1,1}a_{2,3}a_{3,2} - a_{1,1}a_{1,2}a_{2,1}a_{3,3} - a_{1,1}a_{1,3}a_{2,2}a_{3,1}] \\ &= p^3 + p^4 + p^4 - p^3 - p^4 - p^4 \\ &= 0. \end{aligned}$$

For case (ii) let us take $I = J = \{1, 2\}$, so that

$$\begin{aligned} \delta(I, J) &= \mathbf{E}[(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})^2] \\ &= \mathbf{E}[a_{1,1}^2a_{2,2}^2 + a_{1,2}^2a_{2,1}^2 - 2a_{1,1}a_{2,2}a_{1,2}a_{2,1}] \\ &= 2p^2 - 2p^4. \end{aligned}$$

For case (iii) we take $I = \{1, 2\}$ and $J = \{1, 2, 3\}$, we obtain

$$\begin{aligned} \delta(I, J) &= \mathbf{E}[(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) \times \\ &\quad (a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1})] \end{aligned}$$

and this is equal to $2p^5 - 4p^4 + 2p^3$.

Finally for case (iv) we take $I = \{1, 2\}$ and $J = \{2, 3\}$ and we obtain

$$\begin{aligned} \delta(I, J) &= \mathbf{E}[(a_{1,1}a_{2,2} - a_{1,2}a_{2,1})(a_{2,2}a_{3,3} - a_{2,3}a_{3,2})] \\ &= \mathbf{E}[a_{1,1}a_{2,2}a_{3,3} - a_{1,1}a_{2,2}a_{3,3}a_{2,2} - a_{1,2}a_{2,1}a_{2,2}a_{3,3} + a_{2,1}a_{1,2}a_{2,3}a_{3,2}] \\ &= p^3 - p^4. \end{aligned}$$

A tedious (but still doable) computation shows that case (iv), when $k = 3$ (that is, two 3×3 matrices overlapping on a 2×2 matrix), leads to the expression $2p^4 - 4p^5 + 2p^6$.

For the proof, we will prove case (i), then cases (ii)-(iii) together, then case (iv).

Proof of case (i). For simplicity, suppose that there are two elements, say k and ℓ , which are in I but not in J . Let A' be the matrix A , where the lines corresponding to k and ℓ have been exchanged. This operation does not change the distribution of A , that is, A and A' have the same distribution (because the rows of A are exchangeable), and in particular

$$\det(A(I)) \det(A(J)) \stackrel{\text{law}}{=} \det(A'(I)) \det(A'(J)). \quad (6.10)$$

Additionally, this operation leaves $A(J)$ completely unchanged, ie $A'(J) = A(J)$, hence $\det(A'(J)) = \det(A(J))$, but since it swaps two rows of $A(I)$, we have $\det(A(I)) = -\det(A'(I))$. Consequently,

$$\mathbf{E}[\det(A(I)) \det(A(J))] = \mathbf{E}[\det(A'(I)) \det(A'(J))] = -\mathbf{E}[\det(A(I)) \det(A(J))].$$

We deduce that this is equal to zero. On a sidenote which will be used later, this is true for any expression like $\mathbf{E}[\det X \det Y]$ where X, Y are sub-matrices of A who differ from at least two rows (ie, there are two rows of X which do not intersect Y , or the reverse). \square

Proof of cases (ii)-(iii). We consider the situation where $I \subset J$. Case (ii) corresponds to $I = J$ while case (iii) corresponds to $J \setminus I$ is only one element. In both cases, $I \cap J = I$. Let us fix some notations. For any discrete set $D \subset [n]$, the set of permutations of D is noted $\mathfrak{S}(D)$. If $\sigma \in \mathfrak{S}(I)$ and $\tau \in \mathfrak{S}(J)$, we introduce

$$\begin{aligned} \text{cp}(\sigma, \tau) &= \{i \in I : \sigma(i) = \tau(i)\} \\ \text{fp}(\tau) &= \{i \in I : \tau(i) = i\} \end{aligned}$$

the sets of (respectively) *common points* of σ, τ in I , and *fixed points* of τ in I . We will need the following observation. For completeness, if I is strictly included in J , we extend σ from I to J by naturally defining $\sigma(j) = j$ where j is the element of J not in I .

Lemma 6.5. $|\text{fp}(\tau \circ \sigma^{-1})| = |\text{cp}(\sigma, \tau)|$ (if $I \subset J$).

Proof. We fix $i \in I$. If i is a common point of τ and σ , ie $\tau(i) = \sigma(i)$, then $\tau \circ \sigma^{-1} \circ \sigma(i) = \sigma(i)$. Consequently, $\sigma(i)$ (which is an element of I) is a fixed point of $\tau \circ \sigma^{-1}$, in short $\sigma(\text{cp}) \subset \text{fp}$ and consequently $|\text{fp}(\tau \circ \sigma^{-1})| \geq |\text{cp}(\sigma, \tau)|$. Conversely, if i is a fixed point of $\tau \circ \sigma^{-1}$, then for $j = \sigma^{-1}(i) \in I$ one has

$$\tau(j) = \tau(\sigma^{-1}(\sigma(j))) = \tau(\sigma^{-1}(i)) = i = \sigma(j)$$

so j is a common point of τ, σ , in short $\sigma^{-1}(\text{fp}) \subset \text{cp}$ and $|\text{fp}(\tau \circ \sigma^{-1})| \leq |\text{cp}(\sigma, \tau)|$. \square

Equipped with this lemma, we prove (ii)-(iii). By the definition of the determinant,

$$\delta(I, J) = \mathbf{E}[\det(A(I)) \det(A(J))] = \sum_{\substack{\sigma \in \mathfrak{S}(I) \\ \tau \in \mathfrak{S}(J)}} \varepsilon(\sigma) \varepsilon(\tau) \mathbf{E} \left[\prod_{i \in I} a_{i, \sigma(i)} \prod_{j \in J} a_{j, \tau(j)} \right]. \quad (6.11)$$

In (6.11), we note that for fixed σ, τ , the expectation inside the sums is equal to $p^{k+h-|\text{cp}(\sigma, \tau)|}$. Since $\varepsilon(\tau \circ \sigma^{-1}) = \varepsilon(\tau) \varepsilon(\sigma^{-1}) = \varepsilon(\tau) \varepsilon(\sigma)$ and $\tau \rightarrow \tau \circ \sigma^{-1}$ is a bijection of $\mathfrak{S}(J)$, we obtain

$$\begin{aligned} \mathbf{E}[\det(A(I)) \det(A(J))] &= p^{k+h} \sum_{\sigma \in \mathfrak{S}(I)} \sum_{\tau \in \mathfrak{S}(J)} \varepsilon(\sigma) \varepsilon(\tau) p^{-|\text{fp}(\tau \circ \sigma^{-1})|} \\ &= p^{k+h} \sum_{\sigma \in \mathfrak{S}(I)} \sum_{\tau \in \mathfrak{S}(J)} \varepsilon(\tau \circ \sigma^{-1}) p^{-|\text{fp}(\tau \circ \sigma^{-1})|} \\ &= p^{k+h} \sum_{\sigma \in \mathfrak{S}(I)} \sum_{\tau \in \mathfrak{S}(J)} \varepsilon(\tau) p^{-|\text{fp}(\tau)|} \\ &= p^{k+h} k! \sum_{\tau \in \mathfrak{S}(J)} \varepsilon(\tau) p^{-|\text{fp}(\tau)|}. \end{aligned}$$

Let $M = M_{J,I}(p)$ be the matrix whose rows and columns are indexed by J , and with all entries equal to 1 except the diagonal entries $m_{i,i}$ with $i \in I$, which are equal to p^{-1} . Then,

$$\sum_{\tau \in \mathfrak{S}(J)} \varepsilon(\tau) p^{-|\text{fp}(\tau)|} = \det M_{J,I}(p). \quad (6.12)$$

Thanks to this representation we can finish the proof of cases (ii)-(iii). For this, we introduce some final notations. We note E_ℓ the $\ell \times \ell$ matrix with ones everywhere, and

we denote by χ_ℓ its characteristic polynomial $\chi_\ell(z) = \det(E_\ell - z)$. Since the eigenvalues of E_ℓ are ℓ and $\ell - 1$ zeroes, we have $\chi_\ell(z) = (\ell - z)(-z)^{\ell-1}$. Finally, we set $w = 1 - p^{-1}$.

Case (ii). Here $I = J$, $k = h$, and by definition, (6.12) is nothing but the characteristic polynomial of E_k evaluated at w , so

$$\delta(I, J) = p^{2k} k! \chi_k(w) = p^{2k} k! (k - 1 + p^{-1})(p^{-1} - 1)^{k-1}$$

which simplifies to the expression in the theorem.

Cases (iii). Here $J \setminus I$ only has one element, say i , hence $k = |I| = |J| - 1 = h - 1$; without loss of generality we can suppose that i is the last element in J , so that we can do the following manipulations:

$$\begin{aligned} \det M_{J,I}(p) &= \det \begin{bmatrix} p^{-1} & 1 & \dots & \dots & 1 \\ 1 & p^{-1} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & p^{-1} & 1 \\ 1 & \dots & \dots & 1 & 1 \end{bmatrix} \\ &= \det \begin{bmatrix} p^{-1} & 1 & \dots & \dots & 1 \\ 1 & p^{-1} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & p^{-1} & 1 \\ 1 & \dots & \dots & 1 & p^{-1} + w \end{bmatrix} \\ &= \det \begin{bmatrix} p^{-1} & 1 & \dots & \dots & 1 \\ 1 & p^{-1} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & p^{-1} & 1 \\ 1 & \dots & \dots & 1 & p^{-1} \end{bmatrix} + \det \begin{bmatrix} p^{-1} & 1 & \dots & 1 & 0 \\ 1 & p^{-1} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots \\ \vdots & \dots & \ddots & p^{-1} & 0 \\ 1 & \dots & \dots & 1 & w \end{bmatrix} \\ &= \det \begin{bmatrix} p^{-1} & 1 & \dots & \dots & 1 \\ 1 & p^{-1} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & p^{-1} & 1 \\ 1 & \dots & \dots & 1 & p^{-1} \end{bmatrix} + (-1)^{i+i} w \det \begin{bmatrix} p^{-1} & 1 & \dots & 1 \\ 1 & p^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & p^{-1} \end{bmatrix} \end{aligned}$$

and this is equal to $\chi_h(w) + w\chi_{h-1}(w) = (-1)^{h-1}w^{h-1} = (-1)^k w^k$, since in case (iii) we have $h = k + 1$. Overall, we obtain that in case (iii), $\delta(I, J) = k! p^{k+1} (1 - p)^k$. \square

Proof of case (iv). First of all, our proof will be valid only for $k > 2$, but the formula is still valid for $k = 2$ and yields $p^3 - p^4$ as checked in the end of Example 6.4. When $k > 2$, case (iv) actually reduces to the preceding cases. For simplicity, we will only treat the case where $I = \{1, \dots, k\}$ and $J = \{2, \dots, k + 1\}$. For notational simplicity, we note $J_{a,b,c,\dots} = J \setminus \{a, b, c, \dots\}$ and we recall that $A[X, Y]$ is the sub-matrix of A with rows indices in X and column indices in Y , and that $A(X) = A[X, X]$. Let us develop the determinant of $A(J)$ according to its last column:

$$\det(A(J)) = a_{k+1,k+1} \det A(J_{k+1}) + \sum_{\ell=2}^k a_{\ell,k+1} (-1)^{\ell-1+k} \det(A[J_\ell, J_{k+1}]).$$

Multiplying by $\det(A(I))$ and taking the expectation,

$$\delta(I, J) = \mathbf{E}[a_{k+1, k+1} \det(A(I)) \det(J_{k+1})] + \sum_{\ell=2}^k (-1)^{\ell-1+k} \mathbf{E}[a_{\ell, k+1} \det(A(I)) \det(A[J_\ell, J_{k+1}])]. \quad (6.13)$$

Let us consider each element in the sum. First, $a_{\ell, k+1}$ is independent of the couple $A(I), A(J_\ell, J_{k+1})$, hence

$$\mathbf{E}[a_{\ell, k+1} \det(A(I)) \det(A[J_\ell, J_{k+1}])] = p \mathbf{E}[\det(A(I)) \det(A[J_\ell, J_{k+1}])].$$

Second, if $\ell \in \{2, \dots, k\}$, there are two rows of $A(I)$ which are not in $A[J_\ell, J_{k+1}]$: these are the first and ℓ -th rows of A . Consequently, we can swap them exactly as in the proof of case (i) without changing the distribution of $\det(A(I)) \det(A[J_\ell, J_{k+1}])$ but still changing its sign, resulting in $\mathbf{E}[\det(A(I)) \det(A[J_\ell, J_{k+1}])] = 0$. The sum in the RHS of (6.13) thus vanishes, and only the first term remains. Now we get

$$\delta(I, J) = \mathbf{E}[a_{k+1, k+1} \det(A(I)) \det(J_{k+1})] = p \delta(I, J_{k+1})$$

which falls under the analysis of case (iii), since $J_{k+1} = \{2, \dots, k\} \subset I = \{1, \dots, k\}$, and is therefore equal to

$$p \times (k-1)! p^k (1-p)^{k-1}. \quad \square$$

7 Proof of Theorem 2.6

We identify multisets with integer-valued Radon measures as in [31] and endow the space of multisets with the topology of vague convergence, and the space of random multisets with the topology of weak convergence with respect to the vague topology.

Proposition 7.1. *Let f_n be a sequence of random elements in \mathbb{H}_r converging in law towards f which is supposed to be nonzero, and let Φ_n, Φ be the random multisets of the zeroes of f_n, f . Then, Φ_n converges in law towards Φ . Additionnaly, if Φ is almost surely equal to $\{\rho\}$ for some deterministic $\rho \in D(0, r)$, then for every $\varepsilon > 0$, for n is large enough depending on ε , f_n has a unique zero ρ_n in $D(0, r - \varepsilon)$, it is simple, and $|\rho_n - \rho| < \varepsilon$.*

Proof. The first statement is exactly [31], Proposition 2.3. For the second statement we detail the proof: by Skorokhod's representation theorem, on a possibly enlarged probability space, we can find random analytic functions g, g_1, g_2, \dots such that g has the same law as f and g_n has the same law as f_n for every n , and such that $g_n \rightarrow g$ in \mathbb{H}_r almost surely on an event Ω_0 with probability 1. On Ω_0 , we can apply Hurwitz's continuity theorem as in [32], Theorem 6.4.1, cases (b)-(c): for every $\omega \in \Omega_0$, for every $\varepsilon > 0$, there is an N_ω such that $\forall n > N_\omega$, the function g_n has exactly one zero in $D(\rho, \varepsilon)$, and has no other zeroes in $D(0, r - \varepsilon)$. \square

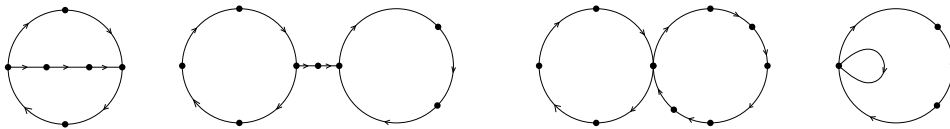
Proof of Theorem 2.6, case $d \geq 1$. The statements in Theorem 2.6 about the largest eigenvalues are direct consequences of Propositions 2.5 and 7.1. \square

Proof of Theorem 2.6, case $d < 1$. In this case we must additionnally prove that all the eigenvalues of A are either zero or roots of unity, which will entirely close the proof of the theorem. Let G be the digraph associated with A . We say that a directed graph is strongly connected if for any two vertices i, j , there is a directed path from i to j and a directed path from j to i . Let g_1, \dots, g_r be the maximal strongly connected subgraphs (MSCS) of G — we also say that vertices with either no out-going edge or no in-going edge or both are MSCS of their own (they only have one vertex), and we call them *trivial*.

It is always possible to label the n vertices of G so that A is a block matrix with diagonal blocks given by the adjacency matrices of the g_i , and lower-diagonal blocks are only filled with zeroes. Consequently, the eigenvalues of A are the eigenvalues of all the g_i . The eigenvalues of trivial MSCS are zero, so non-zero eigenvalues in the spectrum of A must be eigenvalues of non-trivial MSCS.

Lemma 7.2. *If $d < 1$, with high probability, G does not contain any non-trivial strongly connected subgraph other than cycles.*

Proof of the lemma. The argument is similar to the one for undirected graphs, see [22, Theorem 5.5]. The only strongly connected subgraphs with exactly as many edges as vertices are the directed cycles. Now, if a strongly connected subgraph contains strictly more edges than vertices, then it must contain a subgraph which is either a directed cycle with an extra inner directed path, or two directed cycles joined by a directed path. Essentially, these subgraphs look like one of the following ones:



Let D_k be the number of such digraphs with k vertices and let X_k be the number of subgraphs of G which are isomorphic to one of these. We have

$$\mathbf{E}[X_k] = \frac{(n)_k}{k!} \#(D_k) \left(\frac{d}{n}\right)^{k+1}$$

and it is easily seen that $\#(D_k) \leq ck!k^2$ for some constant c , so we have $\mathbf{E}[X_k] \leq ck^2 d^{k+1}/nk \leq ckd^k/n$. Consequently, noting $X = \sum_{k \geq 2} X_k$ and using $d < 1$, we obtain that $\mathbf{E}[X] = O(1/n)$ and Markov's inequality ensures that with high probability, G does not contain any strongly connected subgraph other than cycles or trivial subgraphs. \square

The eigenvalues of A are thus 0 and eigenvalues of directed cycles, which are roots of unity. In particular, all the eigenvalues of A have modulus zero or 1 with probability going to 1 as $n \rightarrow \infty$, which closes the proof of the $d < 1$ case in Theorem 2.6. \square

8 Properties of the Poisson multiplicative function

8.1 Radii of convergence of f and \mathcal{P}

The radius of convergence r_g of a series $g(z) = \sum a_k z^k$ is given by Hadamard's formula, $r_g = (\limsup |a_k|^{1/k})^{-1}$. For the statements in Theorem 2.7 related to radii of convergence of f and \mathcal{P} , we thus need to study $\limsup |X_k/k|^{1/k}$ and $\limsup |(X_k - \tau_k)/k|^{1/k}$. Since $k^{1/k} \rightarrow 1$, it will be enough to study $\limsup X_k^{1/k}$ and $\limsup |X_k - \tau_k|^{1/k}$.

Case $d < 1$

The probability of a $\text{Poi}(\lambda)$ random variable being nonzero is $1 - e^{-\lambda}$. Since $1 - e^x \leq -x$ for any x , we thus have $\mathbf{P}(Y_\ell \neq 0) \leq d^\ell/\ell$, and in particular

$$\sum_{\ell=1}^{\infty} \mathbf{P}(Y_\ell \neq 0) < \infty.$$

The Borel-Cantelli lemma shows that almost surely, there is a random ℓ_0 such that for all $\ell > \ell_0$, $Y_\ell = 0$. As a consequence,

$$X_k = \sum_{\ell|k} \ell Y_\ell \leq \ell_0(Y_1 + \dots + Y_{\ell_0}) =: C_0,$$

a random variable which is almost surely finite. Then, Hadamard's formula shows that the radius of convergence of f is ≥ 1 . If at least one of the Y_ℓ is nonzero, then there is a subsequence k_j such that $X_{k_j} \geq 1$, and consequently $\limsup X_k^{1/k} = 1$. But it is not always the case. In fact, the probability that *all* the Y_ℓ are equal to zero is precisely $\prod_{\ell=1}^{\infty} e^{-d^\ell/\ell} = e^{\log(1-d)} = 1-d$.

We also observe that τ_k is smaller than $1 + d + \dots + d^k \leq (1-d)^{-1}$: this shows that $X_k - \tau_k$ is also bounded, and that the radius of convergence of \mathcal{P} is also ≥ 1 .

Case $d = 1$

Since $\mathbf{P}(Y_\ell = 1) = e^{-1/\ell}/\ell$, which is not summable, the converse of the Borel-Cantelli lemma tells us that there is an infinite number of Y_ℓ which are equal to 1. However,

$$\mathbf{P}(Y_\ell \geq 2) = 1 - e^{-1/\ell} - \frac{e^{-1/\ell}}{\ell} \leq \frac{1}{\ell^2}$$

which is summable, so the Borel-Cantelli lemma shows that there is an ℓ_0 such that for all $\ell > \ell_0$, $Y_\ell \leq 1$. We deduce the existence of a random constant C_0 such that $X_k \leq k + C_0$, showing that $\limsup X_k^{1/k}$ is smaller than 1, and since there is a subsequence k_j with $X_{k_j} \geq 1$, this limsup is equal to 1.

Case $d > 1$

We begin with two useful lemmas.

Lemma 8.1. *Almost surely, there is an integer ℓ_0 such that for every $\ell \geq \ell_0$,*

$$\left| Y_\ell - \frac{d^\ell}{\ell} \right| \leq 2 \ln(\ell) \sqrt{\frac{d^\ell}{\ell}}. \quad (8.1)$$

Proof. The Chernoff bound for Poisson variables (see [11] section 2.2) can be written as follows: if $Z \sim \text{Poi}(\lambda)$, then

$$\mathbf{P}(|Z - \lambda| \geq t) \leq \exp \left\{ -\lambda h \left(\frac{t}{\lambda} \right) \right\} + \exp \left\{ -\lambda h \left(-\frac{t}{\lambda} \right) \right\} \quad (8.2)$$

where $h(x) = (1+x) \ln(1+x) - x$. Let us apply this to our family of Poisson random variables $Y_\ell \sim \text{Poi}(\lambda_\ell)$, where $\lambda_\ell = d^\ell/\ell$. We set $t_\ell = 2 \ln(\ell) \sqrt{\lambda_\ell}$; the right-hand side of (8.2) is then $O(\ell^{-2})$, and in particular it is summable. The Borel-Cantelli lemma ensures the result. \square

Lemma 8.2. *If $d > 1$, $\tau_k = d^k(1 + o(1))$ and almost surely there is a random constant c such that for all k ,*

$$|X_k - \tau_k| \leq ck^2 d^{k/2}. \quad (8.3)$$

Proof. The statement on τ_k is trivial. For the other one, note that

$$\left| X_k - \sum_{\ell|k} d^\ell \right| \leq \sum_{\ell=1}^k \ell \left| Y_\ell - \frac{d^\ell}{\ell} \right|.$$

Statement (8.1) shows that almost surely, $|Y_\ell - d^\ell/\ell|$ is smaller than $2 \ln(\ell) \sqrt{d^\ell/\ell}$ for $\ell \geq \ell_0$, and this is again smaller than $2 \ln(k) \sqrt{d^k/k}$, so

$$|X_k - \tau_k| \leq \sum_{\ell=1}^{\ell_0} \ell \left| Y_\ell - \frac{d^\ell}{\ell} \right| + 2k^2 \ln(k) \sqrt{\frac{d^k}{k}}. \quad (8.4)$$

If we note c' the first term, the LHS is smaller than $(c' + 1) \times 2k^2 d^{k/2}$, a crude bound but which is largely enough for our needs. \square

From this, we can easily deal with the convergence properties of f and \mathcal{P} stated in Theorem 2.7.

- If $d < 1$, the upper bound in (8.1) goes to 0 with ℓ , and in particular if ℓ is large enough, it is strictly smaller than $1/2$. Since d^ℓ/ℓ is also smaller than $1/2$ when $\ell > 1$, we see that the integer Y_ℓ must be strictly smaller than 1, therefore it is zero: almost surely, only a finite number of Y_ℓ can be nonzero. As a consequence, the random variables X_k are almost surely bounded, and $\limsup X_k^{1/k} \leq 1$.
- If $d = 1$, similar arguments as before apply with the difference that the probability of all the Y_ℓ are equal to zero is zero.
- If $d > 1$, we can apply Lemma 8.2 and we easily obtain that $X_k = \tau_k(1 + o(1))$, so clearly $\limsup X_k^{1/k} = d$. Let us now check that the radius of convergence of \mathcal{P} is greater than $1/\sqrt{d}$. For this, we only need to prove that

$$\limsup |X_k - \tau_k|^{1/k} \leq \sqrt{d}. \quad (8.5)$$

But this is a straightforward consequence of the inequality $|X_k - \tau^k| \leq 2ck^2 \sqrt{d}^k$ from the lemma.

8.2 Infinite product representations

We now prove (2.11), (2.12) and (2.13). In all the sequel, \log will refer to the principal branch of the complex logarithm, the one defined on $\mathbb{C} \setminus \mathbb{R}_-$. We first see that for every z in the disk of convergence of f , which in any case is contained in $D(0, 1)$, so

$$\begin{aligned} f(z) &= \sum_{k, \ell=1}^{\infty} \ell Y_\ell \frac{z^k}{k} \mathbf{1}_{\ell|k} = \sum_{\ell=1}^{\infty} Y_\ell \sum_{j=1}^{\infty} \frac{(z^\ell)^j}{j} \\ &= - \sum_{\ell=1}^{\infty} Y_\ell \log(1 - z^\ell). \end{aligned}$$

The series inversions performed in these equalities are justified by the uniform convergence of f on compact subsets of the disk of convergence of f , but they do not necessarily hold outside. As a consequence, e^{-f} is itself well-defined on this disk, analytic, and

$$e^{-f(z)} = \lim_{N \rightarrow \infty} e^{\sum_{\ell=1}^N Y_\ell \log(1 - z^\ell)} = \lim_{N \rightarrow \infty} \prod_{\ell=1}^N (1 - z^\ell)^{Y_\ell} = (2.12).$$

As a first consequence, we see that when $d < 1$, since only a finite number of Y_ℓ are nonzero, $F = e^{-f}$ must actually be a polynomial. Now, we introduce $L(z) = \mathbf{E}[f(z)]$, which can be written as

$$\sum_{k=1}^{\infty} \sum_{\ell|k} d^\ell \frac{z^k}{k}.$$

This is almost a Lambert function, but the presence of a k in the denominator actually makes it closer to a ‘log-Lambert’ function. Since the series above is uniformly convergent on compact subsets of $D(0, 1/d)$, we can reorder like we just did for f , but in a

slightly different way:

$$\begin{aligned}
 L(z) &= \sum_{\ell=1}^{\infty} \sum_{j=1}^{\infty} d^{\ell} \frac{z^{j\ell}}{j^{\ell}} \\
 &= \sum_{\ell=1}^{\infty} \sum_{j=1}^{\infty} \frac{(dz^j)^{\ell}}{j^{\ell}} \\
 &= \sum_{j=1}^{\infty} \frac{1}{j} \sum_{\ell=1}^{\infty} \frac{(dz^j)^{\ell}}{\ell} \\
 &= - \sum_{j=1}^{\infty} \frac{\log(1 - dz^j)}{j}.
 \end{aligned} \tag{8.6}$$

By definition, $-f(z) = \mathcal{P}(z) - L(z)$, so the formula (2.13) in Theorem 2.7 is a consequence of the following proposition, whose main point is to extend the convergence of the preceding sum (minus the first term) to $D(0, 1/\sqrt{d})$.

Proposition 8.3. *If $d > 1$, then for every $|z| < 1/d$,*

$$e^{-L(z)} = \prod_{\ell=1}^{\infty} \sqrt[\ell]{1 - dz^{\ell}}$$

and the infinite product is uniformly convergent on compact subsets of $D(0, 1/\sqrt{d})$.

Proof. The fact that this identity holds for $|z| < 1/d$ follows from (8.6), since

$$\begin{aligned}
 \exp(-L(z)) &= \exp \left(\lim_{J \rightarrow \infty} \sum_{j=1}^J j^{-1} \log(1 - dz^j) \right) \\
 &= \lim_{J \rightarrow \infty} \exp \left(\sum_{j=1}^J j^{-1} \log(1 - dz^j) \right) \\
 &= \lim_{J \rightarrow \infty} (1 - dz) \sqrt[2]{1 - dz^2} \sqrt[3]{1 - dz^3} \cdots \sqrt[J]{1 - dz^J} \\
 &= \prod_{\ell=1}^{\infty} \sqrt[\ell]{1 - dz^{\ell}}.
 \end{aligned}$$

To upgrade this convergence in $D(0, 1/d)$ to uniform convergence on $D(0, 1/\sqrt{d})$, we note that

$$\prod_{\ell=1}^{\infty} \sqrt[\ell]{1 - dz^{\ell}} = (1 - dz) e^{\sum_{j=2}^{\infty} j^{-1} \log(1 - dz^j)}$$

so it is sufficient to prove that the series of logs started at $j = 2$ is uniformly convergent on compact subsets of $D(0, 1/\sqrt{d})$. This is done by noting that if K is a compact in $D(0, s)$ for some $s < 1/\sqrt{d}$, then for $z \in K$ one has $|dz^j| < ds^2 < 1$. Note $r = ds^2 < 1$. There is a constant c_r such that for every $|z| < r$ one has $|\log(1 + z)| < c_r|z|$, and consequently $\sum_{j=2}^{\infty} |j^{-1} \log(1 - dz^j)| \leq c_r d \sum_{j=2}^{\infty} j^{-1} r^j$ so uniform convergence on K follows. \square

Remark 8.4. By the same argument one can extend the uniform convergence of the infinite product to the set of $z \in D(0, 1)$ such that for every $j \geq 2$, $dz^j \notin \mathbb{R}_-$ since we use the principal logarithm. This set can be described as $D(0, 1)$ deprived of every segment semi-infinite line $\{td^{-1/j}\theta_j : t \geq 1\}$ where θ_j is a j -th root of unity and j spans $\{2, 3, \dots\}$.

8.3 Secular moments and generating function

In the representation

$$F(z) = e^{-f(z)} = \prod_{\ell=1}^{\infty} (1 - z^{\ell})^{Y_{\ell}},$$

the product is uniformly convergent on compact subsets of the disk of convergence of f . We now seek the secular coefficients, ie the coefficient of z^n in this series, noted $c_n = [z^n]F(z)$. Clearly, $c_{n_1} \cdots c_{n_r} = [z_1^{n_1} \cdots z_r^{n_r}]F(z_1) \cdots F(z_r)$. We note $F(\mathbf{z}) = F(z_1) \cdots F(z_r)$. Our goal is to express this as simply as possible to extract the coefficients. We have

$$F(\mathbf{z}) = \prod_{\ell=1}^{\infty} \left(\prod_{s=1}^r (1 - z_s^{\ell}) \right)^{Y_{\ell}}.$$

We recall that if $X \sim \text{Poi}(\lambda)$, then $\mathbf{E}[z^X] = e^{\lambda(z-1)}$. By independence,

$$\begin{aligned} \mathbf{E}[F(\mathbf{z})] &= \prod_{\ell=1}^{\infty} \mathbf{E} \left[\left(\prod_{s=1}^r (1 - z_s^{\ell}) \right)^{Y_{\ell}} \right] \\ &= \prod_{\ell=1}^{\infty} e^{\frac{d^{\ell}}{\ell} [\prod_{s=1}^r (1 - z_s^{\ell}) - 1]} \\ &= \exp \left\{ \sum_{\ell=1}^{\infty} \frac{d^{\ell}}{\ell} \left(\prod_{s=1}^r (1 - z_s^{\ell}) - 1 \right) \right\}. \end{aligned}$$

With the convention that a product over an empty set is equal to 1, we have

$$\prod_{s=1}^r (1 - z_s^{\ell}) = \sum_{S \subset [r]} (-1)^{|S|} \prod_{s \in S} z_s^{\ell} = \sum_{S \subset [r]} (-1)^{|S|} \left(\prod_{s \in S} z_s \right)^{\ell}$$

and consequently the sum inside the integral is equal to

$$\begin{aligned} \sum_{\ell=1}^{\infty} \frac{d^{\ell}}{\ell} \left(\prod_{s=1}^r (1 - z_s^{\ell}) - 1 \right) &= \sum_{S \subset [r], S \neq \emptyset} (-1)^{|S|} \sum_{\ell=1}^{\infty} \frac{(d \prod_{s \in S} z_s)^{\ell}}{\ell} \\ &= \sum_{S \subset [r], S \neq \emptyset} (-1)^{|S|-1} \log \left(1 - d \prod_{s \in S} z_s \right) \\ &= \sum_{S \in \text{Odd}_r} \log \left(1 - d \prod_{s \in S} z_s \right) - \sum_{S \in \text{Even}_r} \log \left(1 - d \prod_{s \in S} z_s \right) \end{aligned}$$

where $\text{Odd}_r, \text{Even}_r$ denote the sets of nonempty subsets of $[r]$ with an odd number of elements or with an even number of elements. We obtain

$$\mathbf{E}[F(\mathbf{z})] = \frac{\prod_{S \in \text{Odd}_r} (1 - d \prod_{s \in S} z_s)}{\prod_{S \in \text{Even}_r} (1 - d \prod_{s \in S} z_s)} \quad (8.7)$$

which is Theorem 2.16. We gather some remarks on the combinatorics of set-partitions of multisets in Subsection 8.6.

When $r = 1$, there is only one nonempty subset of $\{1\}$, so

$$\mathbf{E}[F(z)] = 1 - zd. \quad (8.8)$$

Equivalently, $\mathbf{E}[c_0] = 1$, $\mathbf{E}[c_1] = -d$ and $\mathbf{E}[c_k] = 0$ for $k > 1$. When $r = 2$, the formula says that

$$\mathbf{E}[F(z)F(w)] = \frac{(1 - zd)(1 - wd)}{(1 - zwd)}. \quad (8.9)$$

This is also equal to $\sum_{k=0}^{\infty} d^k z^k w^k (1 - zd - wd + d^2 zw)$, so

$$\mathbf{E}[c_n^2] = d^n + d^{n+1}. \quad (8.10)$$

8.4 Sobolev regularity

Remember that the Sobolev norm was defined in (2.15) as

$$\|F\|_s^2 = \sum_{n \in \mathbb{Z}} (1 + n^2)^s \frac{|c_n|^2}{d^n}.$$

Since F is analytic, the sum only spans $n \geq 0$, and by the preceding section, $\mathbf{E}[|c_n|^2] = d^n(d+1)$. As long as $s < -1/2$, we have $\mathbf{E}[\|F\|_s^2] < \infty$ and consequently $\|F\|_s^2 < \infty$ almost surely, as requested in Theorem 2.10.

One can ask if the PHC_d is *not* s -Sobolev for $s > -1/2$. The trick used in [26], Section 6.1 can indeed be applied to our setting: therein, it is proved by elementary means that for $s < 0$, there is a constant $c_s > 0$ such that for $r < 1$,

$$\|F\|_s^2 \geq \frac{c_s}{|\log r|^{2s}} \sum_{n=0}^{\infty} r^{2n} |c_n|^2 = \frac{c_s}{|\log r|^{2s}} \frac{1}{2\pi} \int_0^{2\pi} |F(re^{it})|^2 dt = \frac{c_s}{|\log r|^{2s}} \frac{1}{2\pi} \int_0^{2\pi} e^{2\text{Re}(f(z))} dt \quad (8.11)$$

where the middle equality is Parseval's identity and the last one is the definition of F . Now, when f is a Gaussian analytic function, the limit of this last integral with a suitable normalization term exists, and it is a real random variable representing the total mass of the Gaussian Multiplicative Chaos. The existence of a limit is not trivial, and the identification of the distribution of the limit (the Fyodorov-Bouchaud-Lie formula) was proved recently, see [28] and [26]. Given these, [26] saw that in the Gaussian case, the right-hand side of (8.11) converges to a random number times $|\log 0|$, hence $\|F\|_s^2 = \infty$ and F is not s -Sobolev for $s \in [-1/2, 0)$. In our case where f is the Poisson function given by (2.10), the existence of a limit when $r \rightarrow d^{-1/2}$ of

$$\frac{1}{2\pi} \int_0^{2\pi} e^{2 \sum_{k=1}^{\infty} X_k \frac{r^k \cos(kt)}{k}} dt$$

is not known.

8.5 The correlation of the Poisson field

We close this section by a small remark which strengthens the analogy between F and the Gaussian holomorphic chaos.

Proposition 8.5. *f is log-correlated in the following sense: for every z, w in the disk of convergence of f ,*

$$\text{Cov}(f(z), f(w)) = \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \log \left(\frac{1}{1 - dz^{\alpha} \bar{w}^{\beta}} \right) \frac{1}{\alpha\beta}. \quad (8.12)$$

Note that in this work, the covariance between two complex random variables X, Y is defined as $\mathbf{E}[(X - \mathbf{E}[X])(\overline{Y - \mathbf{E}[Y]})]$.

Proof. Let us note $Z_\ell = Y_\ell - d^\ell/\ell$. Since the Y_i are independent, so are the Z_i , and in particular $E[Z_i Z_j] = 0$ if $i \neq j$, and $E[Z_i^2] = \text{Var}(Y_i) = d^i/i$, and so

$$\begin{aligned} E[\mathcal{P}(z)\overline{\mathcal{P}(w)}] &= \sum_{k,h=1}^{\infty} \frac{z^k \bar{w}^h}{kh} E\left[\sum_{\substack{i|k \\ j|h}} ij Z_i Z_j\right] \\ &= \sum_{k,h=1}^{\infty} \frac{z^k \bar{w}^h}{kh} \sum_{i \in \text{div}(h,k)} i^2 E[Z_i^2] \\ &= \sum_{k,h=1}^{\infty} \frac{z^k \bar{w}^h}{kh} \sum_{\ell \in \text{div}(h,k)} \ell d^\ell. \end{aligned}$$

where $\text{div}(a, b)$ denotes the set of common divisors of a and b . We can reorder the sum, and we get

$$\begin{aligned} E[\mathcal{P}(z)\overline{\mathcal{P}(w)}] &= \sum_{\ell=1}^{\infty} \sum_{a,b=1}^{\infty} \frac{z^{a\ell} \bar{w}^{b\ell}}{a\ell \times b\ell} \ell d^\ell \\ &= \sum_{a,b=1}^{\infty} \frac{1}{ab} \sum_{\ell=1}^{\infty} \frac{z^{a\ell} \bar{w}^{b\ell} d^\ell}{\ell} \\ &= - \sum_{a,b=1}^{\infty} \frac{\log(1 - dz^a \bar{w}^b)}{ab}. \end{aligned} \quad \square$$

8.6 Set-partitions of multisets

In this informal section, we give some remarks on the combinatorics of the generating function (2.16). Let $r \geq 1$ be an integer. We follow the terminology by Bender [4]: given a multiset \mathcal{M} on r elements, that is, $\mathcal{M} = \{1^{n_1}, \dots, r^{n_r}\}$, a *set-partition* of \mathcal{M} is a multiset $B = \{B_1, \dots, B_h\}$ of nonempty indistinguishable sets $B_1, \dots, B_h \subset [r]$, possibly repeated, such that their multiset-union is \mathcal{M} . In other words, the B_j are subsets of $[r]$ such that for every $i \in [r]$,

$$\sum_{j=1}^h \mathbf{1}_{i \in B_j} = n_i.$$

The number of blocks in the multiset B will be noted $|B|$.

Example 8.6. If $\mathcal{M} = \{1, \dots, r\}$ is a set, then this corresponds to the classical set-partitions of $[r]$, counted by the Bell numbers B_r . If $\mathcal{M} = \{1^n\}$ (1 repeated n times) then the only set-partition of \mathcal{M} is $\{1\}, \{1\}, \dots, \{1\}$. If $\mathcal{M} = \{1^4, 2^1, 3^2, 4^2\}$, then a possible set-partition of \mathcal{M} is

$$\{1, 2, 3\}, \{1, 3, 4\}, \{1\}, \{1\}, \{3, 4\}$$

and another one is

$$\{1, 2, 3, 4\}, \{1, 3\}, \{1, 4\}, \{1\}.$$

Now, let z_1, \dots, z_r be complex variables and $q \geq 1$ a parameter. The multivariable function

$$G(z_1, \dots, z_r) = \prod_{S \subset [r]} \frac{1}{1 - q \prod_{s \in S} z_s}$$

is the generating function of set-partitions of multisets in the following sense; let us note

$$\text{par}(\mathcal{M}, q) = \sum_{B \vdash \mathcal{M}} q^{|B|}$$

where the $B \vdash \mathcal{M}$ means that the sum runs over all the set-partitions of the multiset \mathcal{M} . Then,

$$G(z) = \sum_{\{\{1^{n_1}, \dots, r^{n_r}\}\}} \text{par}(\{\{1^{n_1}, \dots, r^{n_r}\}\}, q) \times z_1^{n_1} \dots z_r^{n_r}. \quad (8.13)$$

For instance, the coefficient of $z_1 \dots z_r$ in G , when $q = 1$, is nothing but the classical number of partitions of $[r]$, which is counted by the Bell numbers. Then, the generating function (2.16) is a weighted sum of set-partitions of multisets, such that the blocks with odd cardinality are all distincts. The weight is $(-1)^k d^{|B|}$, where k is the number of blocks with odd cardinality.

Remark 8.7. A celebrated formula in combinatorics states that $\prod_{i \geq 0} (1 - z^{2i+1}) = \sum (-1)^n p_s(n) z^n$, with $p_s(n)$ the number of self-conjugate partitions of n , see for instance [1, eqn. (11) page 127]. There is a chance that products like the $\prod_{S \in \mathcal{V}_r^{\text{odd}}} (1 - dz_S)$ appearing in (2.16) have a similar simple interpretation, but for the moment it is not clear for the author.

9 Proof of Theorem 2.2

The goal of this section is to prove Theorem 2.2. We chose to follow standard methods in path-counting combinatorics, but an alternative proof could use the Stein method to get a convergence speed.

9.1 Usual preliminaries

Notations When a is a positive integer, $[a]$ is the set $\{1, \dots, a\}$. The falling factorials of a are $(a)_1 = a$, $(a)_2 = a(a-1)$, \dots , $(a)_k = a(a-1) \dots (a-k+1)$ and the binomial numbers $\binom{a}{k} = (a)_k / k!$. When \mathcal{E} is a set, $\#\mathcal{E}$ is the number of its elements. A k -tuple of elements in a set \mathcal{E} is a sequence (i_1, \dots, i_k) where the i_t are elements of \mathcal{E} , while a k -set of elements in \mathcal{E} is a set of k distincts elements in \mathcal{E} . We note \mathcal{E}_k the set of k -tuples of elements in $[n] = \{1, \dots, n\}$, that is:

$$\mathcal{E}_k = \{\mathbf{i} = (i_1, \dots, i_k) : i_s \in [n]\}.$$

We will never indicate the dependencies in n , the size of the matrix A . However, every object encountered in the sequel depends on n , and every limit is with respect to $n \rightarrow \infty$.

In general, we will adopt the following notational rules:

- (i) Calligraphic letters are for sets.
- (ii) Boldface letters are for tuples, for instance $\mathbf{i} = (i_1, \dots, i_k)$.

Directed graphs The matrix A is an $n \times n$ matrix with 0/1 entries. It represents the adjacency matrix of a digraph, on the vertex set $V = [n]$. The edge set is $E = \{(i, j) \in [n] \times [n] : A_{i,j} = 1\}$. We insist on the fact that $G = (V, E)$ is directed and possibly has loops. We say that a digraph $G' = (V', E')$ is weakly connected if, for any pair of distinct vertices i, j , there is a (weak) path from i to j , ie a sequence $i_0 = i, i_1, \dots, i_k = j$ such that for each s , $(i_{s-1}, i_s) \in E'$ or $(i_s, i_{s-1}) \in E'$ or both. Naturally, if this is the case one has $\#E' - \#V' \geq -1$, with equality if and only if G' has no cycle.

A-sub notation For any tuple $\mathbf{i} = (i_1, \dots, i_k) \in \mathcal{E}_k$, the shorthand $A_{\mathbf{i}}$ stands for:

$$A_{\mathbf{i}} := A_{i_1, i_2} \times \dots \times A_{i_{k-1}, i_k} A_{i_k, i_1}. \quad (9.1)$$

It is the indicator that the cycle induced by \mathbf{i} is present in G . With this notation,

$$\text{tr}(A^k) = \sum_{\mathbf{i} \in \mathcal{E}_k} A_{\mathbf{i}}.$$

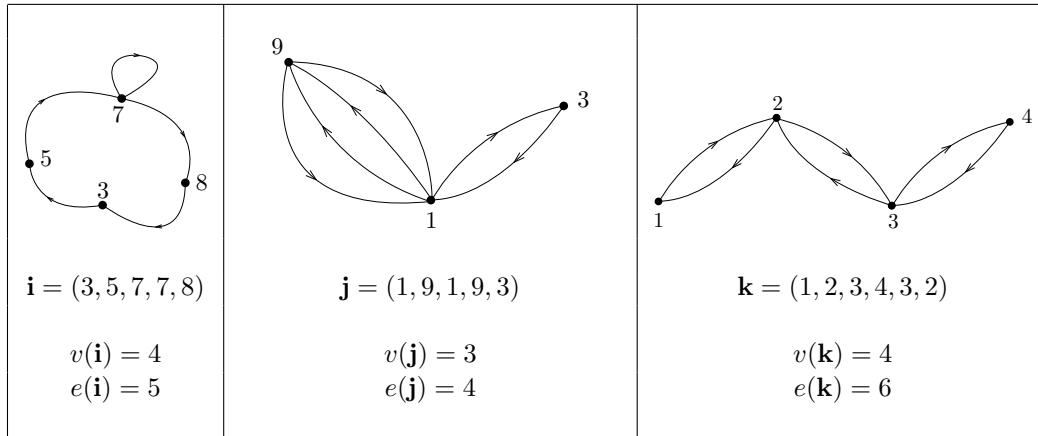
9.2 Reduction to cycle count

For every $\mathbf{i} = (i_1, \dots, i_k) \in \mathcal{E}_k$, we note

$$V(\mathbf{i}) = \{i_1, \dots, i_k\} \quad v(\mathbf{i}) = \#V(\mathbf{i}) \quad (9.2)$$

$$E(\mathbf{i}) = \{(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k), (i_k, i_1)\} \quad e(\mathbf{i}) = \#E(\mathbf{i}). \quad (9.3)$$

Here are simple examples with a graphical representations. If an edge (i, j) is crossed twice or more in \mathbf{i} , we represent it as many times, but it counts as one edge in $E(\mathbf{i})$, see the middle figure.



One can interpret $v(\mathbf{i})$ as the number of ‘vertices’ of $[n]$ that are visited by \mathbf{i} , and $e(\mathbf{i})$ as the number of distinct ‘edges’ of \mathbf{i} . The digraph $(V(\mathbf{i}), E(\mathbf{i}))$ contains a loop, so necessarily $v(\mathbf{i}) \leq e(\mathbf{i})$. On the other hand, there can be no more edges than the length of \mathbf{i} , that is, $e(\mathbf{i}) \leq k$ when $\mathbf{i} \in \mathcal{E}_k$. Since the entries of A are independent 0/1 random variables, for every fixed \mathbf{i} we have

$$\mathbf{E}[A_{\mathbf{i}}] = \left(\frac{d}{n}\right)^{e(\mathbf{i})}. \quad (9.4)$$

For every $v \leq e \leq k$, we set $\mathcal{E}_k(v, e) = \{\mathbf{i} \in \mathcal{E}_k, v(\mathbf{i}) = v, e(\mathbf{i}) = e\}$, and

$$T_k = \sum_{\substack{e \leq k \\ v=e}} \sum_{\mathbf{i} \in \mathcal{E}_k(v, e)} A_{\mathbf{i}}, \quad R_k = \sum_{\substack{e \leq k \\ v < e}} \sum_{\mathbf{i} \in \mathcal{E}_k(v, e)} A_{\mathbf{i}} \quad (9.5)$$

so that naturally $\text{tr}(A^k) = T_k + R_k$. Before stating a few technical lemmas, we recall the fact that k is a fixed integer.

Lemma 9.1. $\#\mathcal{E}_k(v, e) \leq k^k n^v$.

Proof. We first choose which v vertices will be used, which gives $\binom{n}{v} \leq n^v$. Then, we organize them in a k -tuple so that exactly e edges appear, but the number of ways to do this is certainly smaller than $v^k \leq k^k$. \square

Lemma 9.2. $\mathbf{E}[R_k] \leq d^k k^{k+2}/n$. Consequently, $R_k \rightarrow 0$ in probability and in distribution.

Proof. We use (9.4):

$$\mathbf{E}[R_k] = \sum_{v < e \leq k} \#\mathcal{E}_k(v, e) \times \left(\frac{d}{n}\right)^e.$$

By the preceding lemma, $\mathbf{E}[R_k] \leq \sum_{v < e} k^k d^e n^v / n^e$. It is crudely bounded by $d^k k^{k+2} n^{-1}$. The rest follows from the Markov inequality. \square

Lemma 9.3. $\mathcal{E}_k(v, v)$ is empty if v is not a divisor of k . Otherwise, if $k = vq$, then the elements of $\mathcal{E}_k(v, v)$ are exactly the sequences

$$(i_1, i_2, \dots, i_v, i_1, \dots, i_v, \dots, i_1, \dots, i_v) \quad (9.6)$$

where i_1, \dots, i_v are all distinct, and the subsequence $\mathbf{i}' = (i_1, \dots, i_v)$ is repeated q times.

Before the proof, let us precise that we adopt the cyclic notation for tuples: that is, if $\mathbf{i} = (i_1, \dots, i_k)$ is a k -tuple, we set $i_{k+1} = i_1, i_{k+2} = i_2$ and so on.

Proof. Let \mathbf{i} be in $\mathcal{E}_k(v, v)$. In the sequel, we will note $s \in [k + 1]$ the first time a vertex is revisited, that is:

$$s = \min\{t \in \{2, \dots, k + 1\} : i_t \in \{i_1, \dots, i_{t-1}\}\}.$$

If $s = k + 1$, then each i_t is present exactly once in \mathbf{i} , so \mathbf{i} is a cycle on k distinct vertices and $v(\mathbf{i}) = e(\mathbf{i}) = k$. Otherwise, $s \leq k$, and $\mathbf{i}' := (i_1, \dots, i_s)$ has $v(\mathbf{i}') = e(\mathbf{i}') = s$.

If $v(\mathbf{i}) > v(\mathbf{i}')$, then \mathbf{i} visits a 'new' vertex $j \notin \mathbf{i}'$ after s ; but then, there will be a second cycle in \mathbf{i} , and $(V(\mathbf{i}), E(\mathbf{i}))$ will have at least two cycles; if this was true, then we should have $e(\mathbf{i}) \geq v(\mathbf{i}) + 1$, a contradiction.

No other vertices than the s vertices of \mathbf{i}' will thus be present in \mathbf{i} ; but since $e(\mathbf{i}) = v(\mathbf{i}) = v(\mathbf{i}') = s$, it also means that no other edge than $E(\mathbf{i}')$ will be present in $E(\mathbf{i})$. From this, it is easily seen that \mathbf{i} consists in q repetitions of \mathbf{i}' for some q , and consequently that $qs = k$. \square

We now make a crucial observation. Let us fix some integer ℓ . Let $k = q\ell$ be a multiple of ℓ , and let $\mathbf{i} \in \mathcal{E}_k(\ell, \ell)$; by the lemma above, \mathbf{i} is just some $\mathbf{i}' = (i_1, \dots, i_\ell)$ repeated q times. But then, since A has only zero/one entries,

$$A_{\mathbf{i}} = (A_{i_1, i_2} \times \dots \times A_{i_\ell, i_1})^q = A_{\mathbf{i}'}$$

On the other hand, the lemma also shows that the elements in $\mathcal{E}_k(\ell, \ell)$ are fully determined by their first ℓ terms, and thus $\#\mathcal{E}_k(\ell, \ell) = \#\mathcal{E}_\ell(\ell, \ell)$. Consequently, for every multiple k of ℓ ,

$$\sum_{\mathbf{i} \in \mathcal{E}_k(\ell, \ell)} A_{\mathbf{i}} = \sum_{\mathbf{j} \in \mathcal{E}_\ell(\ell, \ell)} A_{\mathbf{j}}. \quad (9.7)$$

In other words, the preceding sum does not depend on k , but only on ℓ . But one can go further into simplifying this expression: the set $\mathcal{E}_\ell(\ell, \ell)$ is nothing but the set of ℓ -tuples of distinct elements, say (i_1, \dots, i_ℓ) ; but if \mathbf{j} is another such tuple, obtained from \mathbf{i} by a cyclic permutation, say $\mathbf{j} = (i_{1+a}, i_{2+a}, \dots, i_a)$, then $A_{\mathbf{i}} = A_{\mathbf{j}}$. For each \mathbf{i} , there are exactly ℓ cyclic permutations of \mathbf{i} ; noting \mathcal{C}_ℓ the set of ordered ℓ -tuples of distinct elements of $[n]$ up to cyclic permutation, it is now clear that for every k multiple of ℓ ,

$$\sum_{\mathbf{i} \in \mathcal{E}_k(\ell, \ell)} A_{\mathbf{i}} = \ell \sum_{\mathbf{i} \in \mathcal{C}_\ell} A_{\mathbf{i}}.$$

This is why we introduce the k -free notation S_ℓ :

$$S_\ell := \sum_{\mathbf{i} \in \mathcal{C}_\ell} A_{\mathbf{i}}. \quad (9.8)$$

With this notation and the discussion above,

$$T_k = \sum_{\ell|k} \ell S_\ell.$$

We now show that the random variables S_ℓ are asymptotically Poisson independent random variables with parameters d^ℓ/ℓ , using the method of falling factorial moments.

Proposition 9.4. *Let $m, \ell_1, p_1, \dots, \ell_m, p_m$ be positive integers, with the ℓ_i all distinct. Then,*

$$\lim_{n \rightarrow \infty} \mathbf{E}[(S_{\ell_1})_{p_1} \times \dots \times (S_{\ell_m})_{p_m}] = \left(\frac{d^{\ell_1}}{\ell_1}\right)^{p_1} \times \dots \times \left(\frac{d^{\ell_m}}{\ell_m}\right)^{p_m}. \quad (9.9)$$

The proof of Proposition 9.4 will be the content of the next subsection; before that, we show how it directly implies the main result in Theorem 2.2. We recall from Definition 2.1 that Y_ℓ is a family of independent Poisson random variables, with parameter d^ℓ/ℓ .

Proof of Theorem 2.2. The Poisson distribution $Z \sim \text{Poi}(\lambda)$ is the unique probability distribution on nonnegative integers such that $\mathbf{E}[(Z)_k] = \lambda^k$ for all $k \geq 0$. Consequently, the RHS of (9.9) is nothing but

$$\mathbf{E}[(Y_{\ell_1})_{p_1} \times \dots \times (Y_{\ell_m})_{p_m}].$$

The joint convergence of the factorial moments described in (9.9) implies the convergence

$$(S_{\ell_1}, \dots, S_{\ell_m}) \xrightarrow{\text{law}} (Y_{\ell_1}, \dots, Y_{\ell_m})$$

and in particular $(S_1, \dots, S_k) \xrightarrow{\text{law}} (Y_1, \dots, Y_k)$ for any k . Since $T_k = \sum_{\ell|k} \ell S_\ell$, there is a fixed linear function $g: \mathbb{R}^k \rightarrow \mathbb{R}^k$ such that $(T_1, \dots, T_k) = g(S_1, \dots, S_k)$; more precisely,

$$g(x_1, \dots, x_k) = \left(x_1, x_1 + 2x_2, x_1 + 3x_3, x_1 + 2x_2 + 4x_4, \dots, \sum_{i|k} ix_i \right).$$

Since $\text{tr}(A^k) = T_k + R_k$, we obtain

$$(\text{tr}(A^1), \dots, \text{tr}(A^k)) = g(S_1, \dots, S_k) + (R_1, \dots, R_k).$$

Lemma 9.2 says that $R_i \rightarrow 0$ in probability for every fixed i , so $(R_1, \dots, R_k) \rightarrow (0, \dots, 0)$ in probability for every fixed k . Since g is continuous, Slutsky's lemma implies that

$$(T_1, \dots, T_k) \xrightarrow{\text{law}} g(Y_1, \dots, Y_k) = (X_1, \dots, X_k)$$

which is the claim in Theorem 2.2. \square

9.3 Proof of Proposition 9.4

For all the proof, we definitely fix the integer $m \geq 1$, as well as the lengths ℓ_1, \dots, ℓ_m and the powers p_1, \dots, p_m ; all these numbers are assumed to be nonzero integers.

Definition 9.5. *Let ℓ, p be two integers. The (ℓ, p) -loopsoup, noted $\mathcal{S}_{\ell, p}$, is the collection of ordered p -tuples of distinct cycles of length ℓ , that is, elements in \mathcal{C}_ℓ . The (ℓ, p) loopsoup associated with $\ell = (\ell_1, \dots, \ell_m)$, $p = (p_1, \dots, p_m)$ is the set $\mathcal{S}_{\ell, p} = \mathcal{S}_{\ell_1, p_1} \times \dots \times \mathcal{S}_{\ell_m, p_m}$, whose elements are m -tuples (c_1, \dots, c_m) , with $c_i \in \mathcal{S}_{\ell_i, p_i}$. The total number of cycles in a loopsoup is $p_1 + \dots + p_m$. The maximum number of $i \in [n]$ that appear in an element of $\mathcal{S}_{\ell, p}$ is*

$$M = M(\ell, p) = \ell_1 p_1 + \dots + \ell_m p_m. \quad (9.10)$$

Example 9.6. A typical element of $\mathcal{S}_{\ell, p}$ is a tuple of tuples. Let us take for instance $\ell = (2, 4)$ and $p = (3, 6)$. An element of $\mathcal{S}_{\ell, p}$ is (c_1, c_2) where $c_1 = (i_1, i_2, i_3)$ and i_s are distinct 2-cycles, and $c_2 = (j_1, j_2, j_3, j_4, j_5, j_6)$ where the j_s are distinct 4-cycles.

If $c = (c_1, \dots, c_m) \in \mathcal{S}_{\ell, p}$, we refer to the j -th element of c_i as $c_{i,j}$ and we note

$$e(c) = \bigcup_{i \in [m]} E(c_{i,j} : i \in [m], j \in [p_i]) \quad e(c) = \#E(c).$$

It is the set of ‘edges’ appearing in one of the $p_1 + \dots + p_m$ cycles appearing in c . Since each $c_{i,j}$ is a cycle and thus has the same number of vertices and edges, we have $e(c) \leq M$.

Lemma 9.7. For any $m \geq 1$ and $\ell = (\ell_1, \dots, \ell_m)$, $p = (p_1, \dots, p_m)$, one has

$$\mathbf{E} \left[\prod_{i=1}^m (S_{\ell_i})_{p_i} \right] = \sum_{c \in \mathcal{S}_{\ell,p}} \left(\frac{d}{n} \right)^{e(c)}. \quad (9.11)$$

Proof. S_k is the number of k -cycles that appear in the graph induced by A , so $(S_k)_h$ is the number of ways to choose an h -tuple of k -cycles that appear in the graph induced by A , that is: $(S_k)_k = \sum_{(i_1, \dots, i_h) \in \mathcal{S}_{k,h}} A_{i_1} \dots A_{i_h}$. We develop the product and take the expectation:

$$\mathbf{E} \left[\prod_{i=1}^m (S_{\ell_i})_{p_i} \right] = \sum_{c \in \mathcal{S}_{\ell,p}} \mathbf{E} \prod_{i=1}^m \prod_{j=1}^{p_i} A_{c_{i,j}} = \sum_{c \in \mathcal{S}_{\ell,p}} \left(\frac{d}{n} \right)^{e(c)}. \quad \square$$

In (9.11), we are going to split the sum according to the value of $e(c) \in [M]$: for every $k \leq M$, we set $\mathcal{S}_{\ell,p}(k) = \{c \in \mathcal{S}_{\ell,p}, e(c) = k\}$. It is clear that when we sort the summands in (9.11) according to this value of e , we obtain that it is equal to $Z_1 + \dots + Z_M$, where

$$Z_k = \left(\frac{d}{n} \right)^k \#(\mathcal{S}_{\ell,p}(k)).$$

As expected, the dominant term in (9.11) is Z_M . This is the content of the following proposition, which, considering the above discussion, finishes the proof of Theorem 9.4.

Proposition 9.8. There is a constant $c = c(\ell, p, d) > 0$ such that for any n and $k < M$,

$$\#(\mathcal{S}_{\ell,p}(k)) \leq cn^{k-1}. \quad (9.12)$$

Moreover,

$$\#(\mathcal{S}_{\ell,p}(M)) = (1 + o(1)) \prod_{i=1}^m \left(\frac{n^{\ell_i}}{\ell_i} \right)^{p_i}. \quad (9.13)$$

Proof of (9.13). Let \mathcal{S}' be the subset of $\mathcal{S}_{\ell,p}(M)$ composed of those c which are completely vertex-disjoint, ie $V(c_{i,k}) \cap V(c_{j,h}) = \emptyset$ when $(i, h) \neq (j, k)$. By elementary counting,

$$\#\mathcal{S}' = \frac{(n)_M}{\ell_1^{p_1} \times \dots \times \ell_m^{p_m}}.$$

Since M is fixed, this expression is equivalent to $n^M / \prod \ell_i^{p_i}$ when $n \rightarrow \infty$. We shall now prove that the number of elements in $\mathcal{S}'' = \mathcal{S}_{\ell,p}(M) \setminus \mathcal{S}'$ is $o(n^M)$, which is well enough for our needs since $\#\mathcal{S}_{\ell,p}(M) = \#\mathcal{S}' + \#\mathcal{S}''$. If c is in \mathcal{S}'' , it means that two among the M distinct cycles of c visit one common vertex, and consequently that $v(c) \leq M - 1$. Since ℓ and c are fixed, for every set $V \subset [n]$, there is a constant c such that there are less than c elements $c \in \mathcal{S}_{\ell,p}$ such that $V(c) = V$ (one can take $c = M^M$ for instance), and so the number of elements in \mathcal{S}'' is smaller than

$$c + c \binom{n}{2} + \dots + c \binom{n}{M-1},$$

an extremely crude bound, but still smaller than cn^{M-1} upon adjusting the constant c . \square

Proof of (9.12). Consider an element $c \in \mathcal{S}_{\ell,p}(k)$ for $k < M$. If all of its cycles were vertex-disjoint, then c would have M edges, so at least two of its cycles share a common vertex, and consequently the number of vertices $v(c)$ present in c cannot be greater than $k - 1$. But for any subset $V \subset [n]$ with $v(c)$ elements, there is a constant c such that there are no more than c elements in $\mathcal{S}_{\ell,p}$ with $V(c) = V$. Consequently,

$$\#\mathcal{S}_{\ell,p}(k) \leq c + c \binom{n}{2} + \cdots + c \binom{n}{k-1}.$$

This is smaller than cn^{k-1} for some (other) constant c , depending only on ℓ and p . \square

10 Proofs of our results when $d_n \rightarrow \infty$

10.1 Tightness and weak convergence: proof of Theorem 3.2

The proof strategy is the same as for the sparse case: the main difference is in the identification of the distributional limits of the traces. We will adopt the following notation:

$$\hat{q}_n(z) = \frac{\det(\mathbf{I} - zA_n/\sqrt{d_n})}{\sqrt{d_n}}. \quad (10.1)$$

We first show that the sequence (\hat{q}_n) is tight on $D(0, 1)$ and then we identify that its limit is $-z\sqrt{1-z^2}G(z)$ by finding the distributional limit of the traces of A^k .

Lemma 10.1. *The sequence of random polynomials \hat{q}_n is tight in \mathbb{H}_1 .*

Proof. We simply use our criterion from Lemma 5.2. First of all, we note that $\hat{q}_n(z) = \sum (-1)^k z^k \hat{\Delta}_k$, where $\hat{\Delta}_k = \Delta_k d_n^{-(k+1)/2}$ and Δ_k is exactly (6.4). Consequently, Proposition 6.2 shows that

$$\mathbb{E}[|\hat{\Delta}_k|^2] \leq \frac{(n)_k}{d_n^{k+1}} \left(\frac{d_n}{n}\right)^k (1 - d_n/n + d_n) \leq \frac{1}{d_n} (1 + d_n) \leq 2$$

provided that n is large enough to ensure that $d_n \geq 1$. Lemma 5.2 leads to the conclusion. \square

We must now identify the finite-dimensional limits of \hat{q}_n . Using (6.3), we have $\hat{q}_n(z) = e^{-f_n(z)}/\sqrt{d_n}$, where

$$f_n(z) = \sum_{k=1}^{\infty} \operatorname{tr} \left(\left(\frac{A}{\sqrt{d_n}} \right)^k \right) \frac{z^k}{k}.$$

Consequently, for fixed n and every $|z| < 1/\sqrt{d_n}$, using the expansion of the complex logarithm on $D(0, 1)$ we get

$$\hat{q}_n(z) = \frac{1 - z\sqrt{d_n}}{\sqrt{d_n}} \frac{e^{-f_n(z)}}{1 - z\sqrt{d_n}} = \frac{1 - z\sqrt{d_n}}{\sqrt{d_n}} e^{-\sum_{k=1}^{\infty} \alpha_k \frac{z^k}{k}}$$

with

$$\alpha_k := \frac{\operatorname{tr}(A^k)}{\sqrt{d_n}^k} - \sqrt{d_n}^k \quad (10.2)$$

and the log-exp formula (6.3) gives

$$\hat{q}_n(z) = \frac{1 - z\sqrt{d_n}}{\sqrt{d_n}} \left(1 + \sum_{k=1}^n P_k(\alpha_1, \dots, \alpha_k) \frac{z^k}{k!} \right), \quad (10.3)$$

an identity which is valid for $|z| < 1/\sqrt{d_n}$, but which extends to every z by analyticity. Thanks to this identity, we see that proving the finite-dimensional convergence of \hat{q}_n only reduces to studying the distributional limits of the α_i and this is done in Theorem 3.1 and its proof thereafter. Given this theorem, we can finish the proof of Theorem 3.2 by a simple application of Theorem 3.1 which says that α_k converges in distribution towards $1 + \sqrt{k}N_k$ if k is even and $\sqrt{k}N_k$ if k is odd, where N_k is a real standard Gaussian; consequently,

$$e^{-\sum_{k=1}^{\infty} \alpha_k \frac{z^k}{k}} \xrightarrow{n \rightarrow \infty} e^{-\sum_{k=1}^{\infty} \frac{z^{2k}}{2k} - \sum_{k=1}^{\infty} \frac{N_k}{\sqrt{k}} z^k} = e^{\frac{1}{2} \log(1-z^2) - \sum_{k=1}^{\infty} \frac{N_k}{\sqrt{k}} z^k} = \sqrt{1-z^2} \times G(z).$$

The sequence of functions $z \mapsto (1 - z\sqrt{d_n})/\sqrt{d_n}$ converges uniformly on \mathbb{C} towards $-z$, so by continuity

$$\hat{q}_n(z) = \frac{1 - z\sqrt{d_n}}{\sqrt{d_n}} \times e^{-\sum_{k=1}^{\infty} \alpha_k \frac{z^k}{k}} \xrightarrow[n \rightarrow \infty]{\text{law}} -z \times \sqrt{1-z^2} G(z).$$

10.2 Proof of Theorem 3.3

It is the same proof as for Theorem 2.6 in Section 7. The limiting function is $z\sqrt{1-z^2}G(z)$ and G has no zeroes, hence \hat{q}_n has exactly one root which converges towards 0, and all the other roots are with high probability outside $D(0, 1-\varepsilon)$. The result follows from the fact that the roots of \hat{q}_n are the inverse eigenvalues of $A_n/\sqrt{d_n}$.

10.3 Proof of Theorem 3.1: trace asymptotics

We now prove the trace limits in Theorem 3.1. The proof globally follows the same lines as for the sparse regimes and could possibly be simplified. We will stick to the same notations as in Section 9, most of them being gathered at Page 27. The starting point is the identity

$$\text{tr}(A^k) = \sum_{\mathbf{i} \in \mathcal{E}_k} A_{\mathbf{i}}$$

where \mathcal{E}_k is the set of k -tuples of elements of $[n]$. If \mathbf{i} has $e(\mathbf{i})$ ‘edges’ as in (9.2), then by independence

$$\mathbf{E}[A_{\mathbf{i}}] = \left(\frac{d_n}{n}\right)^{e(\mathbf{i})}.$$

For every $v \leq e \leq k$, we set $\mathcal{E}_k(v, e) = \{\mathbf{i} \in \mathcal{E}_k, v(\mathbf{i}) = v, e(\mathbf{i}) = e\}$, and

$$T_k = \sum_{\substack{e \leq k \\ v=e}} \sum_{\mathbf{i} \in \mathcal{E}_k(v, e)} A_{\mathbf{i}}, \quad R_k = \sum_{\substack{e \leq k \\ v < e}} \sum_{\mathbf{i} \in \mathcal{E}_k(v, e)} A_{\mathbf{i}}$$

so that naturally $\text{tr}(A^k) = T_k + R_k$. We repeat Lemmas 9.2 and surrounding results adapted to our regime.

Lemma 10.2. $\mathbf{E}[R_k] \leq d_n^k k^{k+2}/n$. Consequently, $R_k \rightarrow 0$ in probability and in distribution as long as $d_n = n^{o(1)}$.

Lemma 10.3. $\mathcal{E}_k(v, v)$ is empty if v is not a divisor of k . Otherwise, if $k = vq$, then the elements of $\mathcal{E}_k(v, v)$ are exactly the sequences

$$(i_1, i_2, \dots, i_v, i_1, \dots, i_v, \dots, i_1, \dots, i_v)$$

where i_1, \dots, i_v are all distinct, and the subsequence $\mathbf{i}' = (i_1, \dots, i_v)$ is repeated q times.

We recall the notation S_{ℓ} :

$$S_{\ell} := \sum_{\mathbf{i} \in \mathcal{C}_{\ell}} A_{\mathbf{i}}$$

where \mathcal{C}_ℓ is the set of cycles of length ℓ on $[n]$. With this notation, we saw that

$$T_k = \sum_{\ell|k} \ell S_\ell.$$

We now show that the random variables S_ℓ are asymptotically Gaussian independent random variables with mean and variance d_n^ℓ/ℓ , using the method of moments — this is where the proof drifts away from the sparse/Poisson case. To do this, we define a normalized version of the S_{ℓ_i} : we set $B_i = A_i - d_n^\ell/n^\ell$, so that $\mathbf{E}[B_i] = 0$, and

$$W_\ell = \frac{\sum_{i \in \mathcal{C}_\ell} B_i}{\sqrt{d_n^\ell/\ell}} = \frac{S_\ell - \mathbf{E}[S_\ell]}{\sqrt{d_n^\ell/\ell}}.$$

The core of the proof is the following result.

Proposition 10.4. *We suppose that d_n satisfies (H). For any distinct nonzero integers ℓ_i and any integers p_i ,*

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\prod_{i=1}^m W_{\ell_i}^{p_i} \right] = \prod_{i=1}^m \frac{p_i!}{2^{p_i/2} (p_i/2)!} \mathbf{1}_{p_i \text{ is even.}} \quad (10.4)$$

Proof of Theorem 3.1. Let $(N_\ell : \ell \geq 1)$ be a family of independent standard Gaussian variables. The RHS of (10.4) is nothing but

$$\mathbf{E}[N_{\ell_1}^{p_1} \times \cdots \times N_{\ell_m}^{p_m}].$$

The joint convergence of the moments in (10.4) implies $(W_1, \dots, W_k) \xrightarrow{\text{law}} (N_1, \dots, \sqrt{k}N_k)$ for any k . Now, we have

$$\frac{\text{tr}(A^k)}{\sqrt{d_n^k}} = \frac{T_k}{\sqrt{d_n^k}} + \frac{R_k}{\sqrt{d_n^k}}$$

and we saw that $R_k \rightarrow 0$. Since $T_k = \sum_{\ell|k} \ell S_\ell$ and $S_\ell = \sqrt{d_n^\ell/\ell} W_\ell + d_n^\ell/\ell$,

$$\begin{aligned} \frac{T_k}{\sqrt{d_n^k}} - \sqrt{d_n^k} &= -\sqrt{d_n^k} + \frac{1}{\sqrt{d_n^k}} \sum_{\ell|k} (d_n^\ell + \sqrt{\ell d_n^{\ell-k}} W_\ell) \\ &= -\sqrt{d_n^k} + \sum_{\ell|k} d_n^{\ell-k/2} + \sum_{\ell|k} \sqrt{\ell d_n^{\ell-k}} W_\ell \\ &= \sum_{\ell|k, \ell < k} d_n^{\ell-k/2} + \sum_{\ell|k, \ell < k} \sqrt{\ell d_n^{\ell-k}} W_\ell + \sqrt{k} W_k. \end{aligned}$$

If ℓ is a divisor of k , it is either $k, k/2$ or smaller than $k/2$, so the first sum is equal to $1 + O(1/d_n) = 1 + o(1)$ if k is even and $o(1)$ if k is odd. It is easily seen that the second term goes to zero in probability as long as $d_n \rightarrow \infty$: in fact, by Proposition 10.4, we have $\mathbf{E}[W_\ell^2] \leq c$ for some c , hence if $\ell < k$,

$$\mathbf{E}[(\sqrt{\ell d_n^{\ell-k}} W_\ell)^2] \leq \frac{c'}{d_n} \rightarrow 0$$

and the whole sum in $\ell < k$ goes to zero in probability. We thus get

$$\frac{\text{tr}(A^k)}{\sqrt{d_n^k}} - \sqrt{d_n^k} = o(1) + \frac{T_k}{\sqrt{d_n^k}} - \sqrt{d_n^k} = o_{\mathbf{P}}(1) + \mathbf{1}_{k \text{ even}} + \sqrt{k} W_k,$$

and the claim in Theorem 3.1 follows from the joint convergence of the W_k . \square

10.4 Proof of Proposition 10.4

For all the proof, we fix the integer $m \geq 1$, as well as the ℓ_1, \dots, ℓ_m and the powers p_1, \dots, p_m ; these numbers are assumed to be nonzero integers and the ℓ_i are distinct. We will frequently use the shorthand $\ell = (\ell_1, \dots, \ell_m)$ and $\mathbf{p} = (p_1, \dots, p_m)$.

Definition 10.5. Let $\mathcal{U}_{\ell, \mathbf{p}}$ be the cartesian product of p_i copies of \mathcal{C}_{ℓ_i} for each $i = 1, \dots, m$, in other words:

$$\mathcal{U}_{\ell, \mathbf{p}} = \bigtimes_{i=1}^m \bigtimes_{j=1}^{p_i} \mathcal{C}_{\ell_i}.$$

A typical element of $\mathcal{U}_{\ell, \mathbf{p}}$ is a $(p_1 + \dots + p_m)$ -tuple of cycles with p_i of them of length ℓ_i . We will note $\mathbf{c} = (c_{i,j})$ with $i \in [m]$, $j \in [p_i]$ and $c_{i,j} \in \mathcal{C}_{\ell_i}$. We note $v(\mathbf{c})$ the total number of vertices present in the cycles of \mathbf{c} , and

$$E(\mathbf{c}) = \bigcup_{i \in [m]} E(c_{i,j} : i \in [m], j \in [p_i]) \quad e(\mathbf{c}) = \#E(\mathbf{c}).$$

It is the set of ‘edges’ appearing in one of the $p_1 + \dots + p_m$ cycles appearing in \mathbf{c} ; in particular, if one edge appears in multiple cycles $c_{i,j}$, we only count it once. Since each $c_{i,j}$ is a cycle and thus has the same number of vertices and edges, we have $e(\mathbf{c}) \leq M$ where

$$M = M(\ell, \mathbf{p}) = \ell_1 p_1 + \dots + \ell_m p_m. \quad (10.5)$$

We also recall that $B_{\mathbf{i}} = A_{\mathbf{i}} - (d/\ell)^\ell$, for any $\mathbf{i} \in \mathcal{C}_\ell$, and that

$$W_\ell = \frac{1}{\sigma_\ell} \sum_{\mathbf{i} \in \mathcal{C}_\ell} B_{\mathbf{i}} \quad (10.6)$$

where $\sigma_\ell = \sqrt{d^\ell/\ell}$.

Lemma 10.6. For any $m \geq 1$ and $\ell = (\ell_1, \dots, \ell_m)$, $\mathbf{p} = (p_1, \dots, p_m)$, one has

$$\mathbb{E} \left[\prod_{i=1}^m W_{\ell_i}^{p_i} \right] = \sum_{\mathbf{c} \in \mathcal{U}_{\ell, \mathbf{p}}} \mathbb{E} \left[\prod_{i=1}^m \prod_{j=1}^{p_i} \frac{B_{c_{i,j}}}{\sigma_{\ell_i}} \right]. \quad (10.7)$$

Proof. One only has to expand the sum defining W_ℓ . \square

Since \mathbf{c} is a collection of cycles, it is customary that $v(\mathbf{c}) \leq e(\mathbf{c})$, and clearly $e(\mathbf{c}) \leq M$. We decompose the sum in (10.7) as follows:

$$\mathbb{E} \left[\prod_{i=1}^m W_{\ell_i}^{p_i} \right] = \sum_{\substack{\mathbf{c} \in \mathcal{U}_{\ell, \mathbf{p}} \\ v(\mathbf{c}) < e(\mathbf{c})}} \mathbb{E} \left[\prod_{i=1}^m \prod_{j=1}^{p_i} \frac{B_{c_{i,j}}}{\sigma_{\ell_i}} \right] + \sum_{k=1}^M \sum_{\substack{\mathbf{c} \in \mathcal{U}_{\ell, \mathbf{p}} \\ v(\mathbf{c}) = e(\mathbf{c}) = k}} \mathbb{E} \left[\prod_{i=1}^m \prod_{j=1}^{p_i} \frac{B_{c_{i,j}}}{\sigma_{\ell_i}} \right]. \quad (10.8)$$

Our task will now consist in finding the dominant terms in this sum. We first start with Lemma 10.7, a rough estimate on the expectations in the above sums that will be used everywhere after. Then, in Lemma 10.8 we prove that the sum over $v < e$ is negligible. In Lemma 10.9 we prove that in the second sum, all the terms with $k < M/2$ are negligible and in Lemma 10.10 we show that the terms with $k > M/2$ are also negligible. Finally, in Lemma 10.11 we study the limit of the only remaining term, the one for $k = M/2$.

Lemma 10.7. For any $\mathbf{c} \in \mathcal{U}_{\ell, \mathbf{p}}$,

$$\mathbb{E} \left[\prod_{i=1}^m \prod_{j=1}^{p_i} \frac{B_{c_{i,j}}}{\sigma_{\ell_i}} \right] \leq c(\ell, \mathbf{p}) \left(\frac{d}{n} \right)^{e(\mathbf{c})} \frac{1}{d^{\frac{M}{2}}} \quad (10.9)$$

where $c(\ell, \mathbf{p}) \leq 2^M (\max \ell)^{M/2}$.

The proof is in Appendix 11.

Lemma 10.8. Under (H),

$$\lim_{n \rightarrow \infty} \sum_{\substack{\mathbf{c} \in \mathcal{U}_{\ell, \mathbf{p}} \\ v(\mathbf{c}) < e(\mathbf{c})}} \mathbf{E} \left[\prod_{i=1}^m \prod_{j=1}^{p_i} \frac{B_{\mathbf{c}_{i,j}}}{\sigma_{\ell_i}} \right] = 0. \quad (10.10)$$

Proof. By the preceding lemma, the sum is bounded by a constant times

$$\sum_{h < k \leq M} \#\{\mathbf{c} \in \mathcal{U}_{\ell, \mathbf{p}}, v(\mathbf{c}) = h, e(\mathbf{c}) = k\} \times d^{k-M/2} n^{-k}.$$

By counting arguments similar to those already used before, there are constants c (depending on h, k, ℓ, \mathbf{p}) such that

$$\#\{\mathbf{c} \in \mathcal{U}_{\ell, \mathbf{p}}, v(\mathbf{c}) = h, e(\mathbf{c}) = k\} \leq cn^h.$$

All things together yield a bound of order $\sum_{h < k \leq M} d^{k-M/2} n^{h-k} = O(d^{k-M/2}/n)$, which is $o(1)$ when $d \leq n^{o(1)}$. \square

Lemma 10.9. Under (H), if $k < M/2$, then

$$\lim_{n \rightarrow \infty} \sum_{\substack{\mathbf{c} \in \mathcal{U}_{\ell, \mathbf{p}} \\ v(\mathbf{c}) = e(\mathbf{c}) = k}} \mathbf{E} \left[\prod_{i=1}^m \prod_{j=1}^{p_i} \frac{B_{\mathbf{c}_{i,j}}}{\sigma_{\ell_i}} \right] = 0. \quad (10.11)$$

Proof. The same proof as the preceding lemma yields a bound of order $d^{k-M/2} n^{k-k} = d^{k-M/2}$, which is $o(1)$ when $k < M/2$. \square

Lemma 10.10. Under (H), if $k > M/2$ then

$$\lim_{n \rightarrow \infty} \sum_{\substack{\mathbf{c} \in \mathcal{U}_{\ell, \mathbf{p}} \\ v(\mathbf{c}) = e(\mathbf{c}) = k}} \mathbf{E} \left[\prod_{i=1}^m \prod_{j=1}^{p_i} \frac{B_{\mathbf{c}_{i,j}}}{\sigma_{\ell_i}} \right] = 0. \quad (10.12)$$

Proof. Remember that elements in $\mathcal{U}_{\ell, \mathbf{p}}$ are collections of cycles. Consequently, if $v(\mathbf{c}) = e(\mathbf{c})$, then \mathbf{c} can consist only in cycles which are vertex-disjoint, but possibly repeated multiple times. If $k > M/2$, then all those cycles cannot be repeated, because otherwise \mathbf{c} would contain $2k > M$ cycles. So, at least one of them is not repeated, and disjoint from all the others, say $\mathbf{c}_{i,j}$. But then, $B_{\mathbf{c}_{i,j}}$ is independent from all the others $B_{\mathbf{c}_{i',j'}}$: the expectation inside the sum above splits, and since $B_{\mathbf{c}_{i,j}}$ is centered, it is equal to zero. Consequently, the whole sum above is actually zero. \square

Lemma 10.11. Under (H),

$$\lim_{n \rightarrow \infty} \sum_{\substack{\mathbf{c} \in \mathcal{U}_{\ell, \mathbf{p}} \\ v(\mathbf{c}) = e(\mathbf{c}) = M/2}} \mathbf{E} \left[\prod_{i=1}^m \prod_{j=1}^{p_i} \frac{B_{\mathbf{c}_{i,j}}}{\sigma_{\ell_i}} \right] = \prod_{i=1}^m \frac{p_i!}{2^{p_i/2} (p_i/2)!} \mathbf{1}_{p_i \text{ is even}}. \quad (10.13)$$

Proof. The proof of the preceding lemma shows that the only elements contributing to the sum above are precisely the \mathbf{c} consisting in $M/2$ vertex-disjoint cycles, each one being repeated at least once; but there are M elements in \mathbf{c} , each one is actually repeated exactly twice. First of all, this is not possible if one of the p_i is not even, so in this case

the whole sum is zero. Let us assume that all the p_i are even. Let \mathcal{V} be the subset of $\mathcal{U}_{\ell,p}$ matching these constraints; we will write its elements \mathbf{c} as multisets,

$$\mathbf{c} = \{\{c_{1,1}, c_{1,1}, \dots, c_{1,p_1/2}, c_{1,p_1/2}, c_{2,1}, c_{2,1}, \dots\}\}$$

and the $M/2$ distinct $c_{i,j}$ are vertex-disjoint. The expectation inside the sum is then equal to

$$\mathbf{E} \left[\prod_{i=1}^m \prod_{j=1}^{p_i/2} \frac{B_{c_{i,j}}^2}{\sigma_{\ell_i}^2} \right] = \prod_{i=1}^m \prod_{j=1}^{p_i/2} \mathbf{E} \left[\frac{B_{c_{i,j}}^2}{\sigma_{\ell_i}^2} \right] = \prod_{i=1}^m \prod_{j=1}^{p_i/2} \frac{(d/n)^{\ell_i} (1 - (d/n)^{\ell_i})}{\sigma_{\ell_i}^2}$$

where we used that the $B_{c_{i,j}}$ are independent Bernoulli $((d/n)^{\ell_i})$ random variables. Now, we recall that $\sum p_i \ell_i = M$ and $\sigma_{\ell} = d^{\ell/2} \ell^{-1/2}$, so the last expression is

$$(1 + o(1)) \frac{(d/n)^{M/2}}{\prod_{i=1}^m \prod_{j=1}^{p_i/2} \sigma_{\ell_i}^2} = (1 + o(1)) \frac{\prod \ell_i^{p_i}}{n^{M/2}}.$$

We now need to count the elements in \mathcal{V} . The choice of the $M/2$ cycles is exactly

$$\frac{(n)_{M/2}}{\prod_{i=1}^m \ell_i^{p_i}}$$

and they must be repeated twice in \mathbf{c} , which gives

$$\prod_{i=1}^m \frac{p_i!}{2^{p_i/2} (p_i/2)!}$$

possibilities for any choice of the $M/2$ cycles. Putting it all together, we get that

$$\#\mathcal{V} = (1 + o(1)) n^{M/2} \times \prod_{i=1}^m \frac{p_i!}{2^{p_i/2} (p_i/2)! \ell_i^{p_i}}$$

and (10.13) follows. \square

Supplementary material

11 Proof of Lemma 10.7

The proof of Lemma 10.7 is a consequence of the following application of the FKG inequality.

Lemma 11.1. *Let G be an Erdős-Rényi digraph (loops allowed) with connectivity p . Let $\mathfrak{X}_1, \dots, \mathfrak{X}_r$ be any number of graphs on n vertices and let $X_i = \mathbf{1}_{\mathfrak{X}_i \in G}$. Then,*

$$\mathbf{E} \left[\prod_{i=1}^r (X_i - \mathbf{E}[X_i]) \right] \leq 2^r p^E \quad (11.1)$$

where E is the total number of distinct edges in $\cup \mathfrak{X}_i$.

Proof. Note $x_i = \mathbf{E}[X_i]$. We have

$$\mathbf{E} \left[\prod (X_i - x_i) \right] \leq \mathbf{E} \left[\prod (X_i + x_i) \right] = \sum_{I \subseteq [r]} \mathbf{E} \left[\prod_{i \in I} X_i \right] \prod_{i \notin I} x_i. \quad (11.2)$$

The random variables X_i are increasing with the addition of edges, as well as any of their products. The FKG inequality (see [22, Section 2.2]) then states that for any $I \subset [r]$ we have

$$\prod_{i \in I} x_i \leq \mathbf{E} \left[\prod_{i \in I} X_i \right].$$

The FKG inequality also ensures that

$$\mathbf{E} \left[\prod_{i \in I} X_i \right] \mathbf{E} \left[\prod_{i \notin I} X_i \right] \leq \mathbf{E} \left[\prod_{i \in [r]} X_i \right].$$

Using these two inequalities for every $I \subset [r]$ in (11.2) ends the proof, because $\prod_{i \in [r]} X_i$ is a product of E independent Bernoulli(p) random variables and there are 2^r subsets of $[r]$. \square

Proof of Lemma 10.7. We apply the preceding theorem to the family of graphs $\mathfrak{X}_{i,j} = \mathbf{c}_{i,j}$ and to the random Erdős-Rényi digraph G induced by A . Since $A_{\mathbf{c}_{i,j}} = \mathbf{1}_{\mathfrak{X}_{i,j} \subset G}$ and $B_{\mathbf{c}_{i,j}} = A_{\mathbf{c}_{i,j}} - \mathbf{E}[A_{\mathbf{c}_{i,j}}]$, so

$$\prod_{i=1}^m \prod_{j=1}^{p_i} B_{\mathbf{c}_{i,j}} = \prod_{i=1}^m \prod_{j=1}^{p_i} (X_{i,j} - \mathbf{E}[X_{i,j}]).$$

There are M cycles in \mathbf{c} and $e(\mathbf{c})$ is the total number of distinct edges present in any of them, so the bound in the preceding lemma is $2^M (d/n)^{e(\mathbf{c})}$. Finally,

$$\prod_{i=1}^m \prod_{j=1}^{p_i} \frac{1}{\sigma_{\ell_i}} = \prod_{i=1}^m \prod_{j=1}^{p_i} \sqrt{\frac{\ell_i}{d^{\ell_i}}} \leq (\max \ell)^{M/2} d^{-M/2}.$$

\square

References

- [1] Martin Aigner, *A course in enumeration*, vol. 238, Springer Science & Business Media, 2007. MR2339282
- [2] Anirban Basak, Mark Rudelson, et al., *The circular law for sparse non-hermitian matrices*, *Annals of Probability* **47** (2019), no. 4, 2359–2416. MR3980923
- [3] Anirban Basak and Ofer Zeitouni, *Outliers of random perturbations of toeplitz matrices with finite symbols*, *Probability Theory and Related Fields* **178** (2020), no. 3, 771–826. MR4168388
- [4] Edward A. Bender, *Partitions of multisets*, *Discrete Mathematics* **9** (1974), no. 4, 301–311. MR0406817
- [5] Charles Bordenave, *A new proof of friedman’s second eigenvalue theorem and its extension to random lifts*, arXiv preprint arXiv:1502.04482 (2015). MR4203039
- [6] Charles Bordenave, Pietro Caputo, Djalil Chafaï, Konstantin Tikhomirov, et al., *On the spectral radius of a random matrix: An upper bound without fourth moment*, *Annals of Probability* **46** (2018), no. 4, 2268–2286. MR3813992
- [7] Charles Bordenave, Djalil Chafaï, and David García-Zelada, *Convergence of the spectral radius of a random matrix through its characteristic polynomial*, 2020. MR4408512
- [8] Charles Bordenave and Benoît Collins, *Eigenvalues of random lifts and polynomials of random permutation matrices*, *Ann. of Math. (2)* **190** (2019), no. 3, 811–875. MR4024563
- [9] Charles Bordenave, Simon Coste, and Raj Rao Nadakuditi, *Detection thresholds in very sparse matrix completion*, arXiv preprint arXiv:2005.06062 (2020).

- [10] Charles Bordenave, Marc Lelarge, and Laurent Massoulié, *Non-backtracking spectrum of random graphs: community detection and non-regular ramanujan graphs*, 2015 IEEE 56th Annual Symposium on Foundations of Computer Science, IEEE, 2015, pp. 1347–1357. MR3473374
- [11] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart, *Concentration inequalities: A nonasymptotic theory of independence*, Oxford university press, 2013. MR3185193
- [12] Gerandy Brito, Ioana Dumitriu, and Kameron Decker Harris, *Spectral gap in random bipartite biregular graphs and applications*, 2020. MR4390294
- [13] Simon Coste, *The spectral gap of sparse random digraphs*, Annales de l’Institut Henri Poincaré, Probabilités et Statistiques, vol. 57, Institut Henri Poincaré, 2021, pp. 644–684. MR4260478
- [14] Simon Coste, Gaultier Lambert, and Yizhe Zhu, *The characteristic polynomial of sums of random permutations and regular digraphs*, 2022.
- [15] Simon Coste and Ludovic Stephan, *A simpler spectral approach for clustering in directed networks*, 2021. MR3922407
- [16] Persi Diaconis and Alex Gamburd, *Random matrices, magic squares and matching polynomials*, Electron. J. Combin. **11** (2004/06), no. 2, Research Paper 2, 26. MR2120097
- [17] Persi Diaconis and Mehrdad Shahshahani, *On the eigenvalues of random matrices*, vol. 31A, 1994, Studies in applied probability, pp. 49–62. MR1274717
- [18] Ioana Dumitriu, Tobias Johnson, Soumik Pal, and Elliot Paquette, *Functional limit theorems for random regular graphs*, Probability Theory and Related Fields **156** (2013), no. 3-4, 921–975. MR3078290
- [19] Joel Friedman, *A proof of alon’s second eigenvalue conjecture and related problems*, American Mathematical Soc., 2008. MR2437174
- [20] Z. Füredi and J. Komlós, *The eigenvalues of random symmetric matrices*, Combinatorica **1** (1981), no. 3, 233–241. MR0637828
- [21] Yan V Fyodorov and Jonathan P Keating, *Freezing transitions and extreme values: random matrix theory, and disordered landscapes*, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences **372** (2014), no. 2007, 20120503. MR3151088
- [22] Svante Janson, Tomasz Luczak, and Andrzej Rucinski, *Random graphs*, vol. 45, John Wiley & Sons, 2011. MR1782847
- [23] Tiefeng Jiang and Sho Matsumoto, *Moments of traces of circular beta-ensembles*, The Annals of Probability **43** (2015), no. 6, 3279–3336. MR3433582
- [24] Jean-Pierre Kahane, *Some random series of functions*, vol. 5, Cambridge University Press, 1993. MR0833073
- [25] Fernando Lucas Metz, Izaak Neri, and Tim Rogers, *Spectral theory of sparse non-hermitian random matrices*, Journal of Physics A: Mathematical and Theoretical **52** (2019), no. 43, 434003. MR4024905
- [26] Joseph Najnudel, Elliot Paquette, and Nick Simm, *Secular coefficients and the holomorphic multiplicative chaos*, 2020.
- [27] Alon Nilli, *On the second eigenvalue of a graph*, Discrete Mathematics **91** (1991), no. 2, 207–210. MR1124768
- [28] Guillaume Remy et al., *The fyodorov-bouchaud formula and liouville conformal field theory*, Duke Mathematical Journal **169** (2020), no. 1, 177–211. MR4047550
- [29] Rémi Rhodes and Vincent Vargas, *Gaussian multiplicative chaos and applications: a review*, 2013. MR3274356
- [30] Mark Rudelson and Konstantin Tikhomirov, *The sparse circular law under minimal assumptions*, Geometric and Functional Analysis **29** (2019), no. 2, 561–637. MR3945840
- [31] Tomoyuki Shirai, *Limit theorems for random analytic functions and their zeros: Dedicated to the late professor yasunori okabe (functions in number theory and their probabilistic aspects)*, RIMS Kokyuroku Bessatsu **34** (2012), 335–359. MR3014854

- [32] Barry Simon et al., *A comprehensive course in analysis*, American Mathematical Society Providence, Rhode Island, 2015. MR3364090
- [33] Yizhe Zhu, *On the second eigenvalue of random bipartite biregular graphs*, arXiv preprint arXiv:2005.08103 (2020).

Acknowledgments. The author thanks the three authors of [7] as well as Yizhe Zhu and Ludovic Stephan for various discussions on this method and comments on the paper.