The effect of avoiding known infected neighbors on the persistence of a recurring infection process[*]

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Abstract

We study a generalization of the classical contact process (SIS epidemic model) on a directed graph G. Our model is a continuous-time interacting particle system in which at every time, each vertex is either healthy or infected, and each oriented edge is either active or inactive. Infected vertices become healthy at rate 1 and pass the infection along each active outgoing edge at rate λ. At rate α, healthy individuals deactivate each incoming edge from their infected neighbors, and an inactive edge becomes active again as soon as its tail vertex becomes healthy. When $\alpha = 0$, this model is the same as the classical contact process on a static graph. We study the persistence time of this epidemic model on the lattice $\mathbb{Z}$, the n-cycle $\mathbb{Z}_n$, and the n-star graph. We show that on $\mathbb{Z}$, for every $\alpha > 0$, there is a phase transition in λ between almost sure extinction and positive probability of indefinite survival; on $\mathbb{Z}_n$ we show that there is a phase transition between poly-logarithmic and exponential survival time as the size of the graph increases. On the star graph, we show that the survival time is $n^{\Delta+o(1)}$ for an explicit function $\Delta(\alpha, \lambda)$ whenever $\alpha > 0$ and $\lambda > 0$. In the cases of $\mathbb{Z}$ and $\mathbb{Z}_n$, our results qualitatively match what has been shown for the classical contact process, while in the case of the star graph, the classical contact process exhibits exponential survival for all $\lambda > 0$, which is qualitatively different from our result. This model presents a challenge because, unlike the classical contact process, it has not been shown to be monotonic in the infection parameter λ or the initial infected set.

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1 Introduction

The contact process is a stochastic model for an epidemic process on a graph, G, which has received a lot of recent attention [1, 3, 4, 5, 13, 19, 20, 23]. The *classical contact process* has a single parameter, λ, which controls the infection rate across each edge of G. At any time, the vertices of G can be either infected or healthy. Each healthy vertex becomes infected at rate λ times the number of infected neighbors that it has, while infected vertices become healthy at rate 1. Much is known about this model, especially when $G = \mathbb{Z}^d$, and when G is a finite random graph (see below for more background). In this paper, we study the *contact process with avoidance*, in which, in addition to the classical dynamics, each healthy individual attempts to temporarily deactivate each of the edges that it shares with its infected neighbors at rate α. A deactivated edge becomes active again when the infected neighbor becomes healthy. This avoidance behavior is intended to model the tendency of healthy individuals to try and avoid visibly infected individuals in a population.

The main ingredients in many proofs about the classical contact process, and many of its variants that have been studied, are duality and additivity. For rigorous definitions and proofs in the case of the classical contact process, see [18]. Informally, duality refers to the existence of a time-reversal process that is Markov; the classical contact process is self-dual. Additivity says that if x_t^A is the infected set of vertices at time t with initially infected set A, then for every $A, B \subseteq V$, there exists a coupling between x_t^A, x_t^B and $x_t^{A \cup B}$ such that $x_t^{A \cup B} = x_t^A \cup x_t^B$ for all $t \geq 0$. The contact process with avoidance is not known to possess these properties (we suspect it does not), and this is a notable technical challenge in deriving rigorous results about this process.

Our main results indicate that this model exhibits a phase transition similar to the classical contact process on $\mathbb{Z}$ and on the cycle $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$, but with a critical infection parameter that grows linearly in α. However, it exhibits drastically different behavior on the star graph with n leaves, where the classical contact process survives exponentially long for any $\lambda > 0$, while the contact process with avoidance survives only polynomially long (for every $\lambda, \alpha > 0$; the case $\alpha = 0$ corresponds to the classical contact process). We note that rigorous results for interacting particle systems that coevolve with the underlying topology, such as the CPA, are still scarce in the literature. We discuss notable examples in Section 1.3.

1.1 Main results

Let $G = (\mathbb{V}, \mathbb{E})$ be a graph with vertex set $\mathbb{V}$ and directed edge set $\mathbb{E}$. Now we formally define the *contact process with avoidance (CPA)* $(\mathcal{X}_t)_{t \geq 0}$ on the graph G, where $\mathcal{X}_t = (x_t, e_t)$ takes values in $\{0,1\}^{\mathbb{V}} \times \{0,1\}^{\mathbb{E}}$. The state of vertex $i \in \mathbb{V}$ at time t is given by $x_t(i) \in \{0,1\}$, where 0 indicates that i is susceptible (healthy) and 1 indicates that i is infected at time t. The state of the directed edge $(i,j) \in \mathbb{E}$ at time t is given by $e_t(i,j) \in \{0,1\}$, where 0 indicates that (i,j) is inactive (blocked) and 1 indicates that (i,j) is active (open) at time t. Given the parameters $\lambda, \alpha \geq 0$ governing the per edge infection and deactivation rates, the process $(\mathcal{X}_t)_{t \geq 0}$ evolves according to the following update rules.

1. For each $i \in \mathbb{V}$, $x_t(i)$ goes from $0 \to 1$ at rate $\lambda \sum_{j \in \mathbb{V}} x_t(j) e_t(j,i) \mathbb{1}\{(j,i) \in \mathbb{E}\}$.

2. For each $i \in \mathbb{V}$, $x_t(i)$ goes from $1 \to 0$ at rate 1.

3. For each $(i,j) \in \mathbb{E}$, $e_t(i,j)$ goes from $1 \to 0$ at rate α if $x_t(j) = 0$ and $x_t(i) = 1$, and at rate 0 otherwise.

4. For each $(i,j) \in \mathbb{E}$, $e_t(i,j)$ goes from $0 \to 1$ when $x_t(i) = 0$.

We denote the law of the process $(\mathcal{X}_t)$ starting with initial condition $\mathcal{X}_0$ by $\mathbb{P}^{\mathcal{X}_0}$. We abuse notation and identify $x_t \in \{0, 1\}^{\mathbb{V}}$ with the set of infected vertices $\{i \in \mathbb{V} : x_t(i) = 1\}$, and write $|x_t|$ to denote the cardinality of this set.

Consider the one dimensional lattice $G = (\mathbb{V}, \mathbb{E})$ where $\mathbb{V} = \mathbb{Z}$ and $\mathbb{E} = \{(i, j) : |i - j| = 1\}$. Define $\mathscr{X}$ to be the collection of initial configurations $\mathcal{X}_0 = (x_0, e_0)$ that satisfy the following conditions:

- $|x_0| < \infty$,
- there exists $i \in \mathbb{Z}$ such that $x_0(i) = 1$ and $e_0(i, i-1) + e_0(i, i+1) > 0$,
- for each $i \in \mathbb{Z}$ and $j = i \pm 1$, if $e_0(i, j) = 0$, then $x_0(i) = 1$, and
- there does not exist $i \in \mathbb{Z}$ such that $e_0(i, i+1) = e_0(i+1, i) = 0$.

The first condition is imposed so the CPA cannot trivially persist for all time by starting with infinitely many infected vertices. The second condition guarantees that there is some infected vertex initially that can spread the infection to one of its neighbors with positive probability – for initial configurations that do not satisfy this condition, the infection cannot persist. The third condition guarantees that each inactive edge corresponds to an infected vertex that is being avoided by its neighbor – inactive edges oriented toward healthy vertices would instantly become active anyhow. The last condition rules out a local configuration in which two neighboring vertices are avoiding one another. This configuration cannot be produced by the dynamics if it is not present initially, and even then the configuration is transient. To see why, note that for the edge $(i, i+1)$ to become inactive at time t, the vertex i must be healthy and $i + 1$ must be infected at time t, but edge $(i + 1, i)$ can only remain inactive at time t if vertex i is infected. We therefore restrict the initial configuration, which also simplifies the proof that the infection persists for large λ.

For each fixed $\alpha > 0$, we are unable to prove that the process is monotone in λ or in the initial configuration, so we define several critical values for λ as follows. For a collection of events $\{A_t\}_{t \in [0, \infty)}$, we say that A_t occurs *unboundedly often (u.o.)* if A_t occurs for an unbounded collection of times $t \in [0, \infty)$. That is, $\{A_t \text{ u.o.}\} = \{\omega : \{t \geq 0 : \omega \in A_t\}$ is unbounded$\}$. Now we define the critical values

$$
\begin{aligned}
\lambda_\alpha^- &:= \inf\{\lambda : \mathbb{P}^{\mathcal{X}_0}\left(|x_t| \geq 1 \ \forall t > 0\right) > 0 \text{ for some } \mathcal{X}_0 \in \mathscr{X}\}, \\
\lambda_{\alpha,w}^+ &:= \sup\{\lambda : \mathbb{P}^{\mathcal{X}_0}\left(|x_t| \geq 1 \ \forall t > 0\right) = 0 \ \forall \mathcal{X}_0 \in \mathscr{X}\}, \\
\lambda_\alpha^+ &:= \sup\{\lambda : \mathbb{P}^{\mathcal{X}_0}\left(x_t(0) = 1 \text{ u.o.}\right) = 0 \ \forall \mathcal{X}_0 \in \mathscr{X}\}.
\end{aligned}
\tag{1.1}
$$

When $\{|x_t| \geq 1 \ \forall t > 0\}$ occurs we say the process *survives weakly*, when $\{x_t(0) = 1 \text{ u.o.}\}$ occurs we say the process *survives strongly*, and when $\{|x_t| = 0 \text{ for some } t > 0\}$ occurs we say the process *dies out*. The w in the subscript of $\lambda_{\alpha,w}^+$ stands for "weak survival". Clearly $\lambda_\alpha^- \leq \lambda_{\alpha,w}^+$, and since $\{x_t(0) = 1 \text{ u.o.}\} \subseteq \{|x_t| \geq 1 \ \forall t > 0\}$, it follows that $\lambda_{\alpha,w}^+ \leq \lambda_\alpha^+$. The definitions are such that if $\lambda < \lambda_\alpha^-$, then the process dies out almost surely; if $\lambda > \lambda_{\alpha,w}^+$, then the process survives weakly with positive probability; and if $\lambda > \lambda_\alpha^+$, then the process survives strongly with positive probability.

When $\alpha = 0$, the classical contact process on $\mathbb{Z}^d$ either dies out or survives strongly [18], and by monotonicity all three critical values are equal: $\lambda_0^- = \lambda_{0,w}^+ = \lambda_0^+$. However, [21] showed that on trees, the contact process may die out, survive weakly but not strongly or survive strongly, depending on λ and the tree structure. A natural open question for the CPA is whether $\lambda_\alpha^- = \lambda_{\alpha,w}^+ = \lambda_\alpha^+$, so that there is a single critical value λ_α separating extinction and strong survival for the contact process with avoidance on $\mathbb{Z}^d$.

We now state our main results.

Theorem 1.1. *Let $G = (\mathbb{V}, \mathbb{E})$, where $\mathbb{V} = \mathbb{Z}$ and $\mathbb{E} = \{(i,j) : |i-j| = 1\}$. There are finite universal constants $a_1, a_2 > 0$ large enough for which $1 + \alpha \leq \lambda_\alpha^- \leq \lambda_\alpha^+ \leq a_1 + a_2\alpha$ for all $\alpha > 0$.*

Theorem 1.1 says there is a phase transition (in λ) between almost sure extinction and positive probability of strong survival of the infection on $\mathbb{Z}$. Moreover, both the upper and lower critical values are linear in α. Next we focus on the CPA on the n-cycle $\mathbb{Z}_n$.

Remark 1.2. Conservative choices for a_1 and a_2 are $a_1 = 51642$ and $a_2 = 51617$. These values are obtained by taking $p = 0.025$ and computing a_1, a_2 in equation (3.7). The choice of p is made to ensure that the inequalities in equation (3.6) are satisfied. See the proof of Theorem 1.1(upper bound) in Section 3. We did not attempt to optimize these values of a_1 and a_2, as the goal of our argument is to provide an upper bound for λ_α^+ which is linear in α.

Theorem 1.3. *Let $G = (\mathbb{V}, \mathbb{E})$, where $\mathbb{V} = \mathbb{Z}_n$ and $\mathbb{E} = \{(i,j) : |i-j| = 1 \bmod n\}$. Let $\tau = \inf\{t : |x_t| = 0\}$ be the time to extinction. Then there are finite universal constants $a_1, a_2 > 0$ large enough and constants $C = C(\lambda, \alpha), \gamma = \gamma(\lambda, \alpha) > 0$ that depend on λ and α for which the following holds. Starting from the initial configuration $\mathcal{X}_0$, which is given by $x_0(i) = 1 \ \forall i \in \mathbb{V}$ and $e_0(i,j) = 1 \ \forall (i,j) \in \mathbb{E}$,*
  (a) $\mathbb{P}^{\mathcal{X}_0} \left(\tau > C(\log n)^2 \right) \to 0$ as $n \to \infty$ for any $\alpha > 0$ and $\lambda < 1 + \alpha$,
  (b) $\mathbb{P}^{\mathcal{X}_0} \left(\tau \leq e^{\gamma n} \right) \to 0$ as $n \to \infty$ for any $\alpha > 0$ and $\lambda > a_1 + a_2\alpha$.

Remark 1.4. The universal constants a_1 and a_2 can be chosen large enough so that both assertions of Theorem 1.1 and 1.3 hold. In particular, the choices of a_1 and a_2 mentioned in Remark 1.2 satisfy both theorems.

Theorem 1.3 says there is a phase transition in the order of the limiting survival time on $\mathbb{Z}_n$, and the upper and lower critical values are linear in α. Finally, we consider the CPA on star graphs and find dramatically different behavior.

Theorem 1.5. *Let $G = (\mathbb{V}, \mathbb{E})$ where $\mathbb{V} = \{0, 1, \ldots, n-1\}$ and $\mathbb{E} = \{(0,j) : j \neq 0\} \cup \{(j,0) : j \neq 0\}$, and initial condition $\mathcal{X}_0$ such that $x_0(i) = 1 \ \forall i \in \mathbb{V}$ and $e_0(i,j) = 1 \ \forall (i,j) \in \mathbb{E}$. Let $\tau_{star} = \inf\{t : |x_t| = 0\}$ be the extinction time of the infection, and define*

$$\Delta = 2 \left[(\lambda + \alpha + 1) - \sqrt{(\lambda + \alpha + 1)^2 - 4\alpha} \right]^{-1}.$$

Then there exists N such that

$$\lim_{K \to \infty} \inf_{n \geq N} \mathbb{P} \left(\frac{1}{K} \left(\frac{n}{\log(n)^4} \right)^\Delta \leq \tau_{star} \leq K n^\Delta \right) = 1.$$

Theorem 1.5 says the survival time on the star graph is of polynomial order in n and the exponent depends on λ and α.

1.2 Graphical construction

One popular technique for analyzing contact process models is the Harris construction, which we define here and use throughout the paper. Consider each edge and each vertex on its time axis, and define events using Poisson processes as follows. Figure 1 gives a graphical example of the Harris construction.

1. Define a Poisson process with intensity λ on each directional edge. Then the waiting time starting from time s until the next arrival along the edge (j,k) is $I(s; j, k) \sim \mathrm{Exp}(\lambda)$. These arrivals can be thought of as vertex j attempting to infect vertex k, and the infection only occurs if $x_{t-}(j) = 1, x_{t-}(k) = 0$, and $e_{t-}(j,k) = 1$ just before time $t = s + I(s; j, k)$.
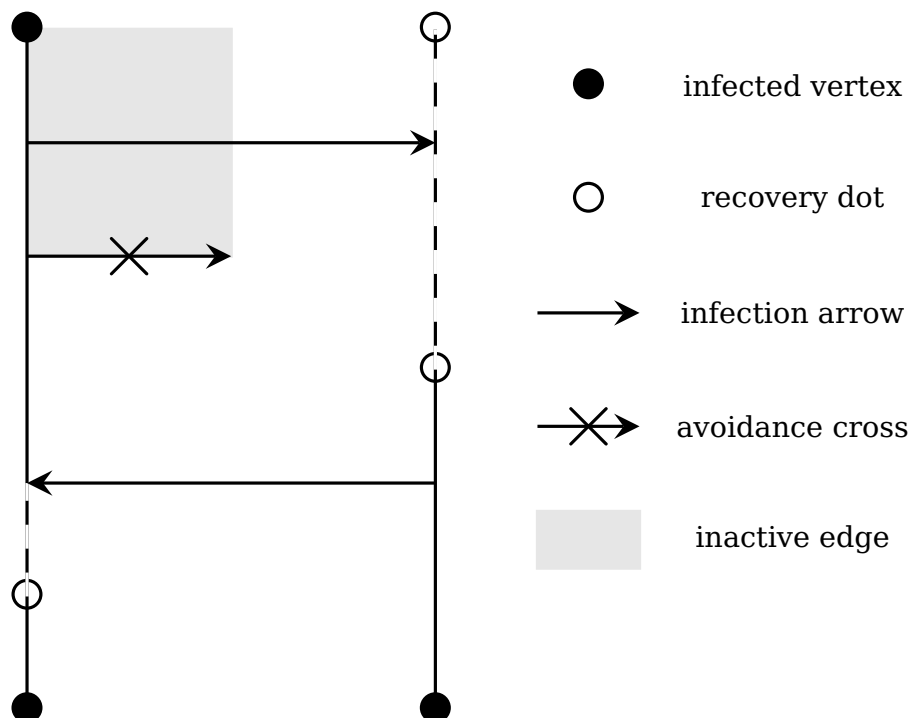
Figure 1: The Harris construction

2. Define a Poisson process with intensity 1 on each vertex. Then the waiting time starting from time s until the next arrival is $r(s; j) \sim \text{Exp}(1)$. These arrivals can be thought of as vertex j "attempting" to recover, with a recover only occuring if $x(j) = 1$ at time $s + r(s; j)-$.

3. Define a Poisson process with intensity α on each directed edge (j, k). Then the waiting time starting from time s until the next arrival is $b(s; j, k) \sim \text{Exp}(\alpha)$. These arrivals can be thought of as vertex k "attempting" to avoid vertex j, and the avoidance only occurs if $x(j) = 1, x(k) = 0$, and $e(j, k) = 1$ at time $s + b(s; j, k)-$.

For the classical contact process, the Harris construction provides a coupling of all initial states, which preserves the partial ordering of containment. This monotonicity (or attractiveness) is used to derive many of the known results. The CPA does not appear to possess this kind of monotonicity. Although we do not have a proof of this claim, certainly the Harris construction fails to preserve the partial order on vertex states. An example of the non-monotonicty of the CPA with respect to the set of infected vertices in the Harris construction is shown in Figure 2. Although the initial infected set is larger in the bottom figure, the final infected set is smaller. Nonetheless, this graphical construction of the CPA will be useful in our proofs.

1.3 Background and related results

It is well-known that the classical contact process on $\mathbb{Z}$ has a critical value $\lambda_c > 0$, such that when $\lambda > \lambda_c$ the infection survives forever with positive probability on $\mathbb{Z}$ (and has survival time $e^{\Theta(n)}$ on $\mathbb{Z}_n$), and when $\lambda < \lambda_c$ the infection dies almost surely on $\mathbb{Z}$ (and survives for $O(\log n)$ time on $\mathbb{Z}_n$). For more on the classical contact process on $\mathbb{Z}$
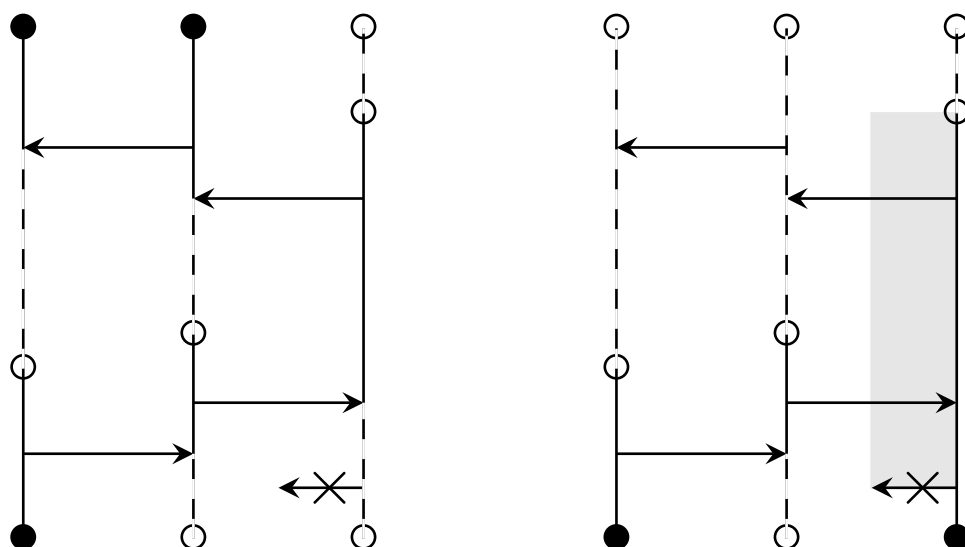
Figure 2: The contact process with avoidance is not monotonic in the Harris construction. The event times are the same in each realization, but the additional initially infected vertex in the right realization leads to fewer infected vertices at time t.

and $\mathbb{Z}_n$, see Liggett [18]. In contrast, on the star graph and on random graphs having power law degree distributions the limiting survival time is exponential for all $\lambda > 0$, and the metastable densities have been derived for a number of models [1, 3, 4, 5, 20].

Our model bears resemblance to the adaptive SIS model proposed by Gross, D'Lima, and Blasius [11], wherein edges between susceptible and infected individuals are 'rewired', rather than deactivated. This model has been of considerable interest in the physics literature [10, 25]. Study of this model and its variants has to date been restricted to mean field approximations, moment closures, and simulation results.

Guo, Trajanovsky, van de Bovenkamp, Wang, and Mieghem [12] study a variant of the adaptive SIS model in [11] more closely related to our contact process with avoidance in which healthy-infected neighbor pairs deactivate the two-way edge between them at rate α, and when both vertices are healthy, reactivate the edge at rate ξ. They study this model on the complete graph and derive an epidemic threshold using differential equation approximations. Szabó-Solticzky, Berthouze, Kiss and Simon [24] study another variant of the adaptive SIS model where SI edges are deleted at rate α and SS edges are created at rate ξ by independent processes, and study the existence of stable oscillations for this model.

The SIR epidemic, in which infected vertices are removed from the graph upon recovery, has also been studied on evolving graphs. Jacobsen, Burch, Tien, and Rempala [15] study a model in which infected vertices are able to activate and deactivate their edges using ODE and pair approximation. Jiang, Kassem, York, Junge, and Durrett [16] study the evoSIR model, in which SI edges rewire at some rate α, and find a critical infection rate λ_c above which there is positive probability a large epidemic occurs. The long term behavior of the SIR epidemic on evolving graphs tends to be easier to understand than that of the SIS epidemic because in the former case each vertex can become infected at most once.

Remenik [22] proposed an ecologically inspired contact process model, in which sites

of $\mathbb{Z}$ may become uninhabitable, thereby blocking passage and eliminating infection. His model differs from ours in that the appearance of an uninhabitable site does not depend on the state of neighboring vertices, and the lifetime of uninhabitable sites can be controlled independently of the other process dynamics. This model is monotonic in its parameters when each is viewed individually (although changing multiple together can create incomparable processes), and he proves phase transitions in both the infection rate and decay rate of uninhabitable sites. Jacob and Mörters [14] consider a contact process on evolving scale free networks, and prove that $\lambda_c > 0$ on the evolving graph for certain power-law degree distributions (and sufficiently fast rewiring dynamics) where $\lambda_c = 0$ on the static graph. However, in their model, vertices rewire independently of the state of the graph, and so given the current edges, the future edges are independent of the vertices. This is not true for the contact process with avoidance. Foxall [9] considers an SEIS model on $\mathbb{Z}$, in which infected vertices have an incubation period prior to becoming infectious. He claims this model is also not likely to be attractive, and he proceeds to prove existence of a phase transition. However, edges in this model do not evolve. Durrett and Neuhauser [6] study the SIRS epidemic, in which infected vertices enter a removed state for some time after recovery, during which they do not interact at all with other vertices. This model is not monotonic in the usual sense, but their results are limited to the case of $\mathbb{Z}^2$ and rely on isoperimetric properties specific to this lattice.

One can also consider similar models with other modes of avoidance. In particular, two other models seem most natural to us in this regard. First, one could consider a model in which infected vertices rather than healthy vertices do the avoiding. This reflects situations in which infected individuals are quarantined to prevent the spread of infection. In this case, an infected node deactivates all edges from itself when it avoids and remains avoiding until it recovers. Another possibility is a model with undirected edges where a healthy vertex avoids an infected vertex by deactivating the bidirectional edge between the two and remains avoiding until the next time both vertices are healthy. In the case of the star graph, we believe that similar results hold for both these alternative models. Applying the heuristic argument we give in section 5 suggests the survival time should still be polynomial in n, but with a different power depending on the choice of model. On $\mathbb{Z}$ and $\mathbb{Z}_n$ however, our upper bound proof techniques do not appear to work for these alternative models.

The proof of Theorem 1.3 only requires the existence in G of a self-avoiding path of length $\Omega(n)$ to conclude exponential survival in n of the CPA. This implies the existence of a supercritical regime on any graph satisfying this condition. Recent work [13, 2] has shown that the classical contact process has a subcritical regime only on finite graphs whose degree distributions have exponential tails. These results provide insight into many useful classes of graphs, including power law random graphs and Galton-Watson trees. However, the proofs of exponential survival for all $\lambda > 0$ in the subexponential tails case uses the behavior of the contact process on star graphs as a key ingredient. Because the contact process with avoidance exhibits qualitatively different behavior on stars, whether there exist graphs whose degree distributions have subexponential tails for which the contact process with avoidance has a subcritical regime is an open question. In particular, the cases of power law random graphs and Galton-Watson trees are of interest.

To further explore the phase transition on $\mathbb{Z}$ and $\mathbb{Z}_n$ we simulated the CPA model for a range of values of λ and α on $\mathbb{Z}_n$ with $n = 500$ vertices. Simulation results appear to indicate that the model is stochastically ordered in λ for fixed α, in which case a single λ_α would exist. It also appears that λ_α is linear in α with a slope between 1.9 and 2.1. Figure 3 shows a survival heatmap for various combinations of λ and α. We performed 30 iterations of each combination of λ and α, and the greyscale intensity indicates the
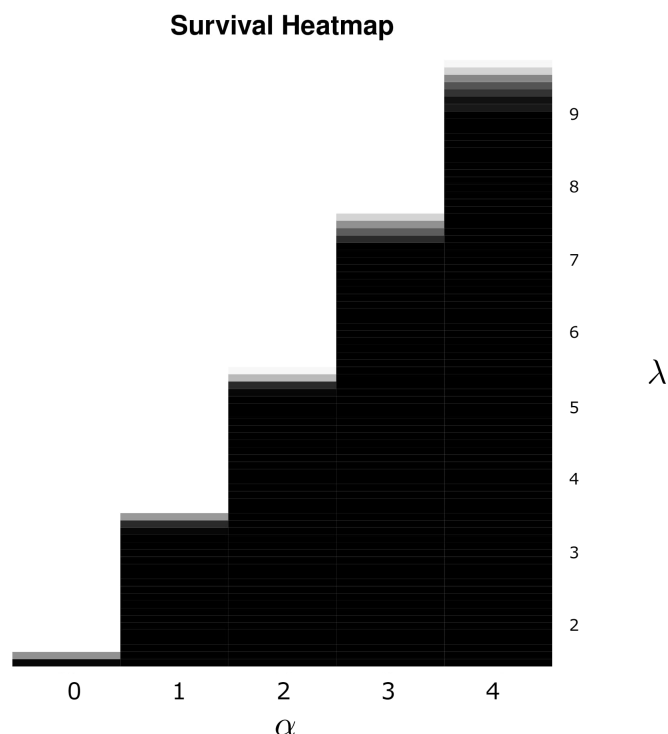
Survival Heatmap

Figure 3: Shading indicates the proportion of 30 simulations of the CPA on $\mathbb{Z}_{10000}$ that survived for at least 10000 units of time for parameter values $\alpha \in \{1, 2, 3, 4\} \times \lambda \in \{1.5, 1.6, \ldots, 9.6, 9.7\}$. White indicates 0 and black indicates 1.

proportion of iterations that survived. When $\alpha = 0$ the simulation identifies that the critical value, which is known to be approximately 1.65 [18], is between 1.5 and 1.7. Simulations with large λ and α are expensive, and so we did not simulate as extensively in that case. However, when $\lambda = 191.5$ and $\alpha = 100$ the process appears to die out, while for $\lambda = 211.7$ and $\alpha = 100$ the process appears to survive, which is consistent with a slope between 1.9 and 2.1.

The remainder of the paper is devoted to proving our three main theorems.

2 Lower bound for λ_α^- on $\mathbb{Z}$

Recall the definition of λ_α^- in equation (1.1), and the definition in the preceding paragraph of $\mathscr{X}$, the collection of allowable initial configurations.

Lemma 2.1. *Fix $\alpha > 0$. Then $\lambda_\alpha^- \geq 1 + \alpha$.*

Proof. Consider $(\mathcal{X}_t)_{t \geq 0}$ with initial configuration $\mathcal{X}_0 \in \mathscr{X}$, so that $|x_0| < \infty$. Since $|x_0| < \infty$, x_0 must have leftmost and rightmost infected vertices whose locations we will denote by l_0 and r_0. Let $(l_t)_{t \geq 0}$ and $(r_t)_{t \geq 0}$ track the locations of the leftmost and rightmost infected vertices in $\mathcal{X}_t$ (with the convention that $l_t = \infty$ and $r_t = -\infty$ if $x_t \equiv 0$). We now define an embedded discrete time process $(L_s)_{s \in \mathbb{Z}_+}$ of $(l_t)_{t \geq 0}$ as follows. A step in the chain L_s occurs when either

1. Vertex L_s infects vertex $L_s - 1$, in which case $L_{s+1} = L_s - 1$, or

2. Vertex L_s recovers, in which case $L_{s+1} = l_{t+}$ where l_{t+} is the new leftmost infected vertex at time t immediately after vertex L_s recovers.

Now observe that $L_{s+1} < L_s$ will hold only if L_s attempts to infect L_{s-1} before either L_s recovers or L_{s-1} avoids L_s. So then for $\lambda < 1 + \alpha$

$$\mathbb{P}\left(L_{s+1} < L_s\right) < 1/2$$

By symmetry we can construct an analogous discrete time process R_s starting from r_0 such that

$$\mathbb{P}\left(R_{s+1} > R_s\right) < 1/2$$

As long as $\mathcal{X}_t$ persists we are assured $L_s \leq R_s$. We thus observe that by the first time $R_s < L_s$ the process $\mathcal{X}_t$ must have reached its absorbing state. By our choice of $\lambda < 1 + \alpha$, L_s and R_s are dominated by random walks with positive and negative drifts respectively and $L_0 \leq R_0$ and so with probability 1 they will eventually cross and $\mathcal{X}_t$ will have died out. $\qquad\square$

3 Upper bound for λ_α^+ on $\mathbb{Z}$

For the classical contact process the supercritical regime can be proved by comparison with an oriented percolation process. The idea is to divide up spacetime into nonoverlapping boxes and declare a box "good" if the infection can successfully pass through on the time axis. The boxes can then be thought of as sites in an oriented site percolation model, which is known to survive strongly when the occupation probability is sufficiently large [8]. If the oriented percolation model is supercritical, then the infection survives strongly by propagating through the good regions with positive probability.

In the case of the classical contact process we know from monotonicity that "goodness" of regions is positively correlated. Thus, if we can show that a region is good with probability at least $1 - p$ for some $p > 0$ using only events in the part of the Harris construction contained in that region, we can then dominate an oriented site percolation with occupancy probability $1 - p$. However, the contact process with avoidance is not monotonic in the Harris construction, and so we must deal with the dependence among regions in a different way. We do this by finding a uniform bound on the probability that a given region is good regardless of what happens on its spacetime boundary and show this probability can be made arbitrarily close to 1. In this section, we formalize and prove this assertion.

We begin by defining our regions. Let $\tau = \tau(\alpha) > 0$ be a fixed timescale, which will be chosen later (see (3.7) for the specific definition). For each $k \in \mathbb{Z}$ and integer $\ell \geq 0$ such that $k + \ell$ is even, define the spacetime region $R_{k,\ell} = \{i : 2k \leq i \leq 2k+3\} \times \{(i,j) : 2k \leq i, j \leq 2k+3\} \times [\ell\tau, (\ell+1)\tau)$, which is a subset of $\mathbb{Z} \times \mathbb{E} \times \mathbb{R}_+$. Note that each block, $R_{k,\ell}$, contains 4 vertices and the edges between them over a time interval of length τ. We will consider waiting times to events using the Harris construction defined in Section 1.

We now define some notation to use for diagrams of states of vertices and edges among $\{0, 1, 2, 3\}$. Let $\bullet$ denote an infected vertex, let $\circ$ denote a healthy (susceptible) vertex, and let $?$ denote a vertex that is either healthy or infected. Let $\nrightarrow$ denote a blocked right-pointing edge, that is, $e_t(i, i+1) = 0$, so the vertex $i + 1$ is avoiding the infected vertex i. Similarly, let $\nleftarrow$ denote a blocked left-pointing edge, and let $\Leftrightarrow$ indicate that both the left- and right-pointing edges are active (open). Let $-$ indicate any of the three possible states for the pair of edges between i and $i + 1$. Note that under our dynamics, we can never have $e_t(i, i+1) = e_t(i+1, i) = 0$ or $e_t(i, j) = x_t(i) = 0$ for $j = i \pm 1$. For $k \in \mathbb{Z}$, let $\mathcal{Y}_k := \{2k, 2k+1, 2k+2, 2k+3\} \times \{(i,j) : i, j \in \{2k, 2k+1, 2k+2, 2k+3\}, |i - j| = 1\}$. Define the following collections of configurations in terms of their restrictions to $\mathcal{Y}_k$.

1. Let $A_{2,L}^k$ denote the set of all configurations whose restriction to $\mathcal{Y}_k$ has the form

   $\bullet \Leftrightarrow \bullet \nleftarrow ?{-}?.$

2. Let $A_{2,R}^k$ denote the set of all configurations whose restriction to $\mathcal{Y}_k$ has the form
   $$\bullet \Leftrightarrow \bullet \looparrowright \bullet - ?.$$

3. Let $A_{2,O}^k$ denote the set of all configurations whose restriction to $\mathcal{Y}_k$ has the form
   $$\bullet \Leftrightarrow \bullet \Leftrightarrow ?-?$$

4. Let $A_{2,*}^k$ denote the union of the sets defined in 1-3 and their reflections across the middle edge.

5. Let $A_{3,L}^k$ denote the set of all configurations whose restriction to $\mathcal{Y}_k$ has the form
   $$\bullet \Leftrightarrow \bullet \Leftrightarrow \bullet \looparrowleft ?.$$

6. Let $A_{3,R}^k$ denote the set of all configurations whose restriction to $\mathcal{Y}_k$ has the form
   $$\bullet \Leftrightarrow \bullet \Leftrightarrow \bullet \looparrowright \bullet.$$

7. Let $A_{3,O}^k$ denote the set of all configurations whose restriction to $\mathcal{Y}_k$ has the form
   $$\bullet \Leftrightarrow \bullet \Leftrightarrow \bullet \Leftrightarrow ?.$$

8. Let $A_{3,*}^k$ denote the union of the sets defined in 5-7 and their reflections across the middle edge.

9. Let A_4^k denote the set of all configurations whose restriction to $\mathcal{Y}_k$ has the form
   $$\bullet \Leftrightarrow \bullet \Leftrightarrow \bullet \Leftrightarrow \bullet.$$

We call the region $R_{k,\ell}$ **good** if starting from any of the configurations in $A_{2,*}^k$ at time $\ell\tau$ we reach one of the configurations in A_4^k at time $(\ell+1)\tau$. For brevity, we drop the superscript 0 from the notation $A_{2,L}^0$ (resp. $A_{2,R}^0, \ldots, A_4^0$), and write $A_{2,L}$ (resp. $A_{2,R}, \ldots, A_4$) instead. The following lemmas identify a sequence of events in the region $R_{0,0}$ such that starting from any initial configuration in $A_{2,*}$ at time 0 we reach A_4 at time τ regardless of what happens on the external spacetime boundary of $R_{0,0}$, and that for fixed $\alpha > 0$ this probability of the sequence of events mentioned above can be made arbitrarily close to 1 with appropriate choices of τ and λ. See (3.7) for the specific definition of τ and a lower bound for λ for which the above holds. Thus, we will obtain a uniform lower bound, which is arbitrarily close to 1, for the probability that $R_{0,0}$ is good regardless of what happens on its space-time boundary. The same assertion holds for all regions $R_{k,\ell}, k \in \mathbb{Z}, \ell \geq 0$ using analogous argument.

Definition 3.1. *We say that an event E is internal to $R_{k,\ell}$ if it depends only on the Harris construction marks within $R_{k,\ell}$.*

Lemma 3.2. *Fix $\alpha > 0$ and $p \in (0,1)$, and suppose $\lambda \geq (72/p)\log(72/p) \cdot (1+\alpha)$. There is an event E internal to $R_{0,0}$ such that, starting from any initial state $\mathcal{X}_0 \in A_{2,*}$, on the event E the process hits a state in $A_{3,*}$ by time $\log(72/p) + 1$, and $\mathbb{P}(E) \geq 1 - p$.*

Proof. Let $t = t_1 + t_2 + t_3$, where

$$t_1 = \log(72/p), \quad \text{and} \quad t_2 = t_3 = \frac{p}{72(1+\alpha)}.$$

Note that $t < \log(72/p) + 1$. By symmetry, it suffices to consider the scenario where the left two vertices are infected initially. In that scenario, the initial configuration can belong to $A_{2,O}$, $A_{2,L}$, or $A_{2,R}$. We consider the three corresponding cases separately. In all cases, we identify an event which (a) is internal to $R_{0,0}$, (b) ensures that $\mathcal{X}_t$ reaches a state in A_{3*} by time t, and (c) has probability $\geq 1 - p/6$. This would complete the proof of the lemma using the union bound, as there are six cases in total.

Recall that $I(t;i,j), r(t;i)$ and $b(t;i,j)$ denote the waiting times after time t until the next infection arrow at (i,j), recovery dot at i, and avoidance mark at (i,j), respectively; see Section 1.2.

$$R_{k,l}$$
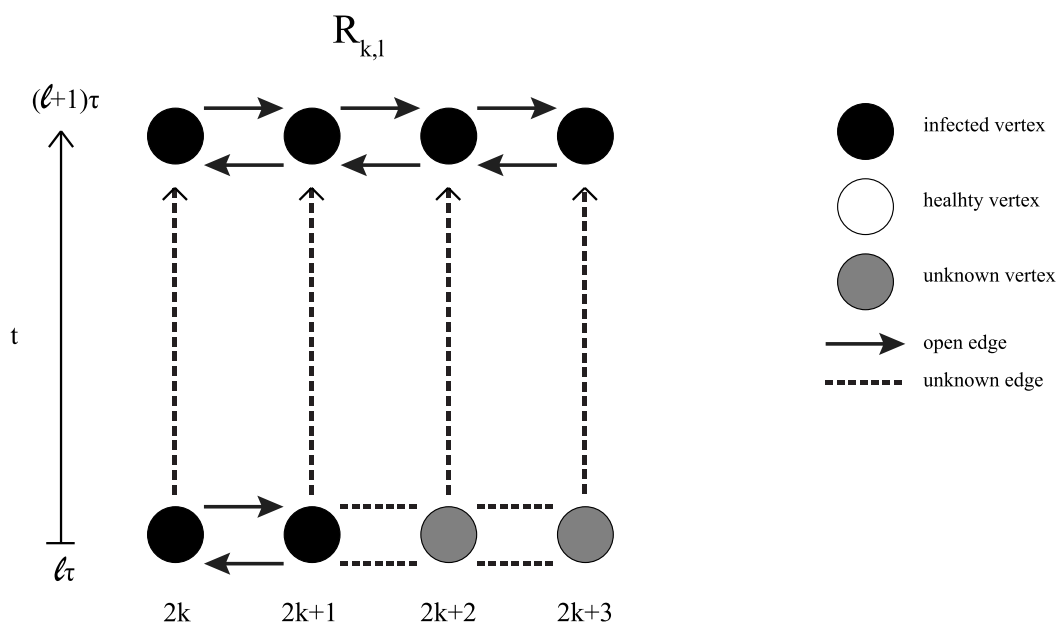
Figure 4: A scenario which will make the region $R_{k,\ell}$ good.

1. $A_{2,O}$: Suppose the initial configuration is $\mathcal{X}_0 \in A_{2,O}$. We reach $A_{3,*}$ by time t if vertex 1 attempts to infect vertex 2 before time t, and this infection event occurs before vertex 0 recovers, vertex 1 recovers, or the edge $(1,2)$ becomes inactive. Letting E_1 denote this event,

$$\mathbb{P}^{\mathcal{X}_0}\left(E_1^c\right) \leq \mathbb{P}\left(I(0;0,1) > t\right) + \mathbb{P}\left(I(0;0,1) > \min(r(0;0), r(0;1), b(0;1,2))\right)$$
$$= e^{-\lambda t} + \frac{2+\alpha}{\lambda + 2 + \alpha}$$
$$< \frac{p}{6}, \tag{3.1}$$

since $\lambda t > \log(12/p)$ and $\lambda > 12(2 + 2\alpha)/p$.

2. $A_{2,L}$: Suppose the initial configuration is $\mathcal{X}_0 \in A_{2,L}$. We reach $A_{3,*}$ by time $t = t_1 + t_2 + t_3$ if vertex 1 recovers before time t_1, vertex 0 is infected when this recovery occurs, vertex 0 attempts to infect vertex 1 within time t_2 of vertex 1's recovery, this infection occurs before vertex 0 recovers or the edge $(0,1)$ becomes inactive, vertex 1 attempts to infect vertex 2 within time t_3 of vertex 1's reinfection, and this infection occurs before either vertex 0 or 1 recovers (again) or the edge $(1,2)$ attempts to become inactive again. Let E_2 denote this event. Let $s_1 = r(0;1)$ be the first time vertex 1 recovers and $s_2 = s_1 + I(s_1;0,1)$ be the first time after s_1 that vertex 0 attempts to infect vertex 1. We have

$$\mathbb{P}^{\mathcal{X}_0}\left(E_2^c\right)$$
$$\leq \mathbb{P}\left(r(0;1) > t_1\right)$$
$$\quad + \mathbb{P}\left(\text{vertex 0 is healthy when vertex 1 first recovers}, r(0;1) \leq t_1\right)$$
$$\quad + \mathbb{P}\left(I(s_1;0,1) > t_2\right) + \mathbb{P}\left(I(s_1;0,1) > \min(r(s_1;0), b(s_1;0,1))\right)$$
$$\quad + \mathbb{P}\left(I(s_2;1,2) > t_3\right) + \mathbb{P}\left(I(s_2;1,2) > \min(r(s_2;0), r(s_2;1), b(s_2;1,2))\right)$$
$$= e^{-t_1} + \mathbb{P}\left(\text{vertex 0 is healthy when vertex 1 first recovers}, r(0;1) \leq t_1\right)$$
$$\quad + e^{-\lambda t_2} + \frac{1+\alpha}{\lambda + 1 + \alpha} + e^{-\lambda t_3} + \frac{2+\alpha}{\lambda + 2 + \alpha}$$
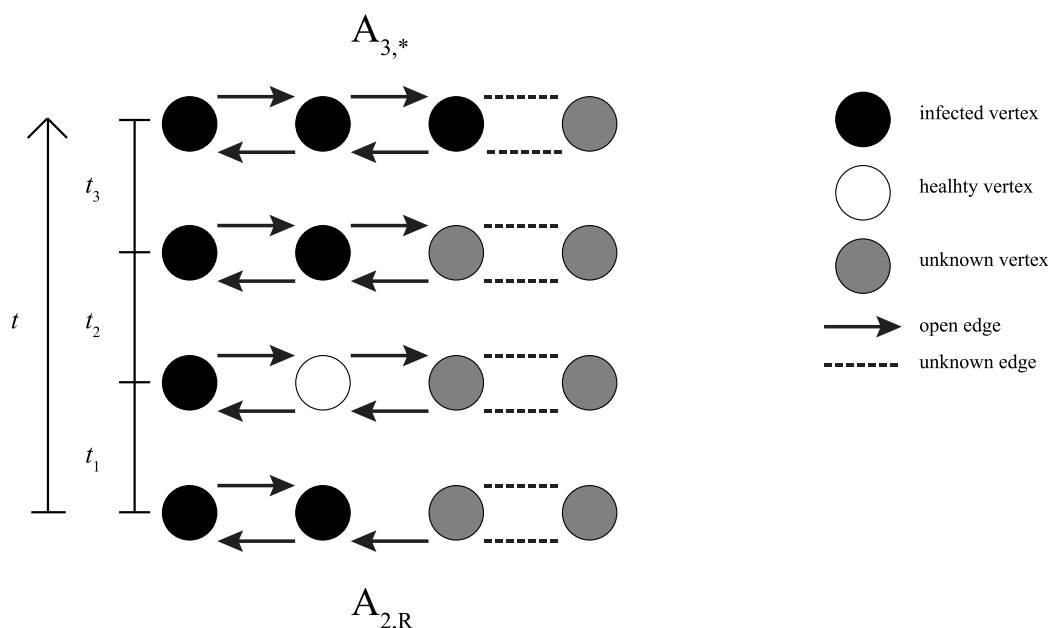
Figure 5: A sequence of events leading from $A_{2,R}$ to $A_{3,*}$

$$\leq p/72 + \mathbb{P}\left(\text{vertex 0 is healthy when vertex 1 first recovers}, r(0;1) \leq t_1\right)$$

$$+ p/72 + \frac{1+\alpha}{\lambda} + p/72 + \frac{2+\alpha}{\lambda}$$

$$= p/12 + \mathbb{P}\left(\text{vertex 0 is healthy when vertex 1 first recovers}, r(0;1) \leq t_1\right), \quad (3.2)$$

since $t_1 = \log(72/p)$, $\lambda t_2 = \lambda t_3 \geq \log(72/p)$, and $\lambda \geq 24(3+2\alpha)/p$. Vertex 0 is infected when vertex 1 recovers at time $r(0;1) \leq t_1$ if for a fixed k vertex 0 recovers at most k times by time t_1, and for the first k recoveries of vertex 0 (up to time t_1), vertex 1 successfully reinfects vertex 0 before vertex 1 recovers or the edge $(1,0)$ becomes inactive, and $r(0;1) \leq t_1$. The number of recovery marks at vertex 0 by time t_1 has Poisson(t_1) distribution. Let $r_0 = 0$ and $r_\ell = r_{\ell-1} + r(r_{\ell-1}, 0)$ for $\ell \geq 1$ denote the time of the ℓ-th recovery mark at vertex 0 in the Harris construction. Choose $k = \lfloor 3t_1 \rfloor$ such that if $X \sim$ Poisson(t_1), then $\mathbb{P}\left(X > k\right) \leq e^{-t_1} = p/72$ by a standard Chernoff bound. So

$$\mathbb{P}\left(\text{vertex 0 is healthy when vertex 1 recovers}, r(0;1) \leq t_1\right)$$

$$\leq \mathbb{P}\left(X > k\right) + \sum_{\ell=0}^{k-1} \mathbb{P}\left(I(r_\ell; 1, 0) > \min(r(r_\ell; 0), b(r_\ell; 1, 0))\right)$$

$$\leq p/72 + k \cdot \frac{1+\alpha}{1+\alpha+\lambda}$$

$$\leq p/72 + k \cdot \frac{1+\alpha}{\lambda}$$

$$\leq p/72 + p/18 < p/12, \quad (3.3)$$

since $\lambda \geq 18k(1+\alpha)/p$. Hence, we have

$$\mathbb{P}^{\mathcal{X}_0}\left(E_2^c\right) \leq p/6.$$

3. $A_{2,R}$: Suppose the initial configuration is $\mathcal{X}_0 \in A_{2,R}$. In this case we know vertex 2 must be infected since the edge (1,2) is inactive. We reach $A_{3,*}$ by time $t_1 + t_2 < t$ if vertex 2 recovers before time t_1, vertices 0 and 1 are infected when this recovery occurs, vertex 1 attempts to infect vertex 2 within t_2 time units of vertex 2's recovery, and this infection occurs before either vertex 0 or vertex 1 recovers or the edge $(1,2)$ becomes inactive. Let $s_1 = r(0;2)$ be the time that vertex 2 first recovers. Similar to (3.2), we have

$$\mathbb{P}^{\mathcal{X}_0} (T_{3,*} > t)$$
$$\leq \mathbb{P}\left(r(0;3) > t_1\right)$$
$$\qquad + \mathbb{P}\left(\text{vertex 0 or 1 is healthy when 2 recovers}, r(0;2) \leq t_1\right)$$
$$\qquad + \mathbb{P}\left(I(s_1;1,2) > t_2\right) + \mathbb{P}\left(I(s_1;1,1) > \min(r(s_1;0), r(s_1;1), b(s_1;1,2))\right)$$
$$= e^{-t_1} + \mathbb{P}\left(\text{vertex 0 or 1 is healthy when 2 recovers}, r(0;2) \leq t_1\right) + e^{-\lambda t_2}$$
$$\qquad + \frac{2+\alpha}{2+\alpha+\lambda}$$
$$\leq p/72 + \mathbb{P}\left(\text{vertex 0 or 1 is healthy when 2 recovers}, r(0;2) \leq t_1\right) + p/72$$
$$\qquad + \frac{2+\alpha}{\lambda}$$
$$\leq p/18 + \mathbb{P}\left(\text{vertex 0 or 1 is healthy when 2 recovers}, r(0;2) \leq t_1\right), \qquad (3.4)$$

since $t_1 = \log(72/p)$, $\lambda t_2 \geq \log(72/p)$, and $\lambda \geq 36(2+\alpha)/p$. Vertices 0 and 1 are infected when vertex 2 recovers at time $r(0;2) \leq t_1$ if for fixed k vertex 0 recovers at most k times, and vertex 1 recovers at most k times, and for the first k recoveries of vertex 0, vertex 1 successfully reinfects vertex 0 before vertex 1 recovers or the edge $(1,0)$ becomes inactive, and for the first k recoveries of vertex 1, vertex 0 successfully reinfects vertex 1 before vertex 0 recovers or the edge $(0,1)$ becomes inactive, and $r(0;2) \leq t_1$. The numbers of recovery marks at vertices 0 and 1 by time t_1 are independent Poisson(t_1) random variables. Define the times $(r_\ell^0, \ell \geq 0)$ (resp. $(r_\ell^1, \ell \geq 0)$) of the recoveries at vertex 0 (resp. 1) as follows. $r_0^0 = 0$, $r_\ell^0 = r_{\ell-1}^0 + r(r_{\ell-1}^0;0)$ for $\ell \geq 1$, $r_0^1 = 0$, and $r_\ell^1 = r_{\ell-1}^1 + r(r_{\ell-1}^1;1)$ for $\ell \geq 1$. Choose $k = \lfloor 3t_1 \rfloor$ such that if $X \sim$ Poisson(t_1), then $\mathbb{P}\left(X > k\right) \leq e^{-t_1} = p/72$ as mentioned before. So

$$\mathbb{P}\left(0 \text{ or } 1 \text{ is healthy when 2 recovers}, r(0;2) \leq t_1\right)$$
$$\leq 2\mathbb{P}\left(X > k\right) + \sum_{\ell=0}^{k-1} \mathbb{P}\left(I(r_\ell^0;1,0) > \min(r(r_\ell^0;1), b(r_\ell^0;1,0))\right)$$
$$\qquad + \sum_{\ell=0}^{k-1} \mathbb{P}\left(I(r_\ell^1;0,1) > \min(r(r_\ell^1;0), b(r_\ell^1;0,1))\right)$$
$$\leq p/36 + 2k\frac{1+\alpha}{1+\alpha+\lambda} \leq p/36 + 2k(1+\alpha)/\lambda \leq p/9, \qquad (3.5)$$

since $\lambda \geq 24k(1+\alpha)/p$. So we have

$$\mathbb{P}^{\mathcal{X}_0} (E_3^c) \leq p/6.$$

Finally, if all three of the events described above, $E_1 \cap E_2 \cap E_3$, occur, then for each initial configuration $\mathcal{X}_0 \in A_{2*}$ with the left two vertices infected, a configuration in A_{3*} is reached by time t. Moreover, $\mathbb{P}^{\mathcal{X}_0} (E_1 \cap E_2 \cap E_3) \geq 1 - p/2$. By symmetry, the same assertion holds for each initial configuration $\mathcal{X}_0 \in A_{2*}$ with the right two vertices infected. This completes the proof of the lemma using union bound. $\qquad \square$

Lemma 3.3. *Fix $\alpha > 0$ and $p \in (0,1)$, and suppose $\lambda \geq (162/p) \log(72/p) \cdot (1+\alpha)$. There is an event E internal to $R_{0,0}$ such that, starting from any initial state $\mathcal{X}_0 \in A_{3,*}$, on the event E the process hits a state in A_4 by time $\log(72/p) + 1$, and $\mathbb{P}(E) \geq 1 - p$.*

Proof. The proof of this lemma follows the same arguments as the proof of the previous lemma by examining the cases in of starting in $A_{3,O}$, $A_{3,L}$, and $A_{3,R}$. Because there are more vertices and edges, the rates of potential recoveries and edge deactivations are greater, so the value of λ must be taken a bit larger. $\qquad\square$

Lemmas 3.2 and 3.3 show that for fixed $\alpha > 0$ and $p \in (0,1)$, if $\lambda \geq (162/p) \log(72/p) \cdot (1+\alpha)$, then we can reach A_4 starting from any configuration in $A_{2,*}$ by time $\hat{\tau} := 2\log(72/p) + 2$ using only internal marks in the Harris construction with probability at least $1 - 2p$. However, for the region $R_{k,\ell}$ to be good, we must have a time τ so that we are in state A_4 at time τ. The following lemmas show that we can make this occur for a suitable choice of $\tau > \hat{\tau}$ with large probability.

Lemma 3.4. *Let B_3 denote the set of states for which exactly three of the vertices in $\{0,1,2,3\}$ are infected and all of the edges in $\{(i,j) : i,j \in \{0,1,2,3\}\}$ are active. For $\alpha > 0$ and $p \in (0,1)$, let $\hat{\tau} := 2\log(72/p) + 2$ and $\lambda_1^* = 24\hat{\tau}(3 + 2\alpha)/p$. There is an event E, which is internal to $R_{0,0}$ and satisfies $\mathbb{P}(E) \geq 1 - p$ for all $\lambda \geq \lambda_1^*$, such that the following holds on the event E. Starting from any initial state $\mathcal{X}_0 \in A_4$, the process $\mathcal{X}_s$ stays within $A_4 \cup B_3$ during time $[0, \hat{\tau}]$.*

Proof. Starting from A_4, the process $\mathcal{X}_s$ stays within $A_4 \cup B_3$ during time $[0, \hat{\tau}]$ if whenever a vertex in $\{0,1,2,3\}$ recovers either it becomes reinfected (via one of the edges in $\{(i,j) : i,j \in \{0,1,2,3\}\}$) before any of the other vertices recover or any of its edges become inactive or there are no more avoidances or recoveries before time $\hat{\tau}$. The number of recoveries of each vertex during $[0, \hat{\tau}]$ has Poisson($\hat{\tau}$) distribution independent of other events. Now, if $X \sim \text{Poisson}(\hat{\tau})$, then

$$\mathbb{P}(X > 3\hat{\tau}) \leq e^{-\hat{\tau}} \leq p/8.$$

For $i \in \{0,1,2,3\}$, define $r_0^i = 0$ and $r_\ell^i = r_{\ell-1}^i + r(r_{\ell-1}^i; i)$ for $\ell \geq 1$ to be the times of the recovery marks at i. Based on these random times and letting $k = \lfloor 3\hat{\tau} \rfloor$, define

$$E' := \bigcap_{i \in \{0,\cdots,3\}} \left\{ r_{k+1}^i > \hat{\tau} \right\} \bigcap_{j \in \{0,\cdots,3\}, m \in \{0,\cdots,k\}}$$
$$\left\{ \min_{h:|j-h|=1} \{I(r_m^j; h, j)\} \leq \min \left\{ \min_{h \in \{0,\ldots,3\}\setminus\{j\}} \{r(r_m^j; h)\}, \min_{h:|j-h|=1} \{b(r_m^j; h, j)\} \right\} \right\}.$$

Then, using union bound,

$$\mathbb{P}((E')^c) \leq 4\mathbb{P}(X > k) + \sum_{j=0}^{3} \sum_{m=0}^{k}$$
$$\mathbb{P}\left(\min_{h:|j-h|=1} \{I(r_m^j; h, j)\} > \min \left\{ \min_{h \in \{0,\ldots,3\}\setminus\{j\}} \{r(r_m^j; h)\}, \min_{h:|j-h|=1} \{b(r_m^j; h, j)\} \right\} \right)$$
$$\leq 4(p/8) + \sum_{j=0}^{3} \sum_{m=0}^{k} \frac{3 + 2\alpha}{3 + 2\alpha + \lambda} \leq p/2 + 4k \frac{3 + 2\alpha}{3 + 2\alpha + \lambda_1^*} \leq p,$$

E' is not internal to $R_{0,0}$ because $\min_{h:|j-1|=1}\{I(r_k^j, h, j)\}$ and

$\min\left\{\min_{h\in\{0,\dots,3\}\setminus\{j\}}\{r(r_m^j;h)\}, \min_{h:|j-h|=1}\{b(r_m^j;h,j)\}\right\}$ may be greater than $\hat{\tau}$. But note that

$$E' = \bigcap_{i\in\{0,\cdots,3\}}\{r_{k+1}^i > \hat{\tau}\}\bigcap_{j\in\{0,\cdots,3\},m\in\{0\cdots,k\}}$$

$$\left\{\min_{h:|j-h|=1}\{I(r_m^j;h,j)\} \leq \min\left\{\min_{h\in\{0,\dots,3\}\setminus\{j\}}\{r(r_m^j;h)\}, \min_{h:|j-h|=1}\{b(r_m^j;h,j)\}\right\}\right\}$$

$$= \bigcap_{i\in\{0,\cdots,3\}}\{r_{k+1}^i > \hat{\tau}\}\bigcap_{j\in\{0,\cdots,3\},m\in\{0\cdots,k\}}$$

$$\left\{\left\{\min_{h:|j-h|=1}\{I(r_m^j;h,j)\} \leq \min\left\{\min_{h\in\{0,\dots,3\}\setminus\{j\}}\{r(r_m^j;h)\}, \min_{h:|j-h|=1}\{b(r_m^j;h,j)\}\right\}\right\}\right.$$

$$\cap \left\{\min\left\{\min_{h\in\{0,\dots,3\}\setminus\{j\}}\{r(r_m^j;h)\}, \min_{h:|j-h|=1}\{b(r_m^j;h,j)\}\right\} \leq \hat{\tau}\right\}$$

$$\cup \left\{\min_{h:|j-h|=1}\{I(r_m^j;h,j)\} \leq \min\left\{\min_{h\in\{0,\dots,3\}\setminus\{j\}}\{r(r_m^j;h)\}, \min_{h:|j-h|=1}\{b(r_m^j;h,j)\}\right\}\right\}$$

$$\left.\cap \left\{\min\left\{\min_{h\in\{0,\dots,3\}\setminus\{j\}}\{r(r_m^j;h)\}, \min_{h:|j-h|=1}\{b(r_m^j;h,j)\}\right\} > \hat{\tau}\right\}\right\}$$

$$\subseteq \bigcap_{i\in\{0,\cdots,3\}}\{r_{k+1}^i > \hat{\tau}\}\bigcap_{j\in\{0,\cdots,3\},m\in\{0\cdots,k\}}$$

$$\left\{\left\{\min_{h:|j-h|=1}\{I(r_m^j;h,j)\} \leq \min\left\{\min_{h\in\{0,\dots,3\}\setminus\{j\}}\{r(r_m^j;h)\}, \min_{h:|j-h|=1}\{b(r_m^j;h,j)\}\right\}\right\}\right.$$

$$\cap \left\{\min\left\{\min_{h\in\{0,\dots,3\}\setminus\{j\}}\{r(r_m^j;h)\}, \min_{h:|j-h|=1}\{b(r_m^j;h,j)\}\right\} \leq \hat{\tau}\right\}$$

$$\left.\cup \left\{\min\left\{\min_{h\in\{0,\dots,3\}\setminus\{j\}}\{r(r_m^j;h)\}, \min_{h:|j-h|=1}\{b(r_m^j;h,j)\}\right\} > \hat{\tau}\right\}\right\}$$

$$=: E$$

where the event E is internal to $R_{0,0}$. E is the event that during time $[0,\hat{\tau}]$ whenever a vertex in $\{0,1,2,3\}$ recovers either it becomes reinfected (via one of the edges in $\{(i,j): i,j \in \{0,1,2,3\}\}$) before any of the other vertices recover or any of its edges become inactive or there are no more avoidances or recoveries before time $\hat{\tau}$, and on the event E starting from any initial state $\mathcal{X}_0 \in A_4$, the process $\mathcal{X}_s$ stays within $A_4 \cup B_3$ during time $[0,\hat{\tau}]$. Since $E' \subseteq E$, we have $P(E) \geq P(E') \geq 1-p$, completing the proof. $\square$

Lemma 3.5. *Let B_3 be the set of configurations defined in Lemma 3.4. Fix $\alpha > 0, p \in (0,1)$, and let $\check{\tau} := p/12$ and $\lambda_2^* := \max\{(12/p)\log(3/p), 6\alpha/p\}$. There is an event E, which is internal to $R_{0,0}$ and satisfies $\mathbb{P}(E) \geq 1-p$ for all $\lambda \geq \lambda_2^*$, such that on the event E, if initially $\mathcal{X}_0 \in A_4 \cup B_3$, then $\mathcal{X}_{\check{\tau}} \in A_4$.*

Proof. It is easy to see that $\mathcal{X}_{\check{\tau}} \in A_4$ if the following conditions (a), (b1), and (b2) hold.

(a) None of the vertices in $\{0, \dots, 3\}$ attempts to recover before time $\check{\tau}$,

(b) If $\mathcal{X}_0 \in B_3$, then

  (b1) a neighbor of the initially uninfected vertex attempts to infect it before time $\check{\tau}$, and

  (b2) this infection occurs before any edge attached to the uninfected vertex become inactive.

The total number of attempted recoveries of all 4 vertices before time $\check{\tau}$ follows a Poisson($4\check{\tau}$) distribution. Let $X \sim$ Poisson($4\check{\tau}$). Then

$$\mathbb{P}\left(X > 0\right) \leq E(X) = 4\check{\tau} = p/3.$$

If we call the (possibly) uninfected vertex k, then

$$\mathbb{P}^{\mathcal{X}_0}\left(\mathcal{X}_{\check{\tau}} \notin A_4\right) \leq \mathbb{P}\left(X > 0\right) + \mathbb{P}\left(\min_{j:|j-k|=1} I(0;j,k) > \check{\tau}\right)$$

$$+ \mathbb{P}\left(\min_{j:|j-k|=1} I(0;j,k) > \min_{j:|j-k|=1} b(0;j,k)\right)$$

$$\leq p/3 + e^{-\lambda\check{\tau}} + \frac{2\alpha}{2\alpha + \lambda} \leq p/3 + e^{-\lambda_2^*\check{\tau}} + \frac{2\alpha}{2\alpha + \lambda_2^*} \leq p/3 + p/3 + p/3 = p,$$

as $\lambda \geq \lambda_2^*$. This completes the proof. $\qquad\square$

Proof of Theorem 1.1 (Upper bound). Fix $\alpha > 0$, and choose $p \in (0,1)$ small enough such that

$$1 - 4p > p_c^s, \tag{3.6}$$

where p_c^s is the critical probability for oriented site percolation on $\mathbb{Z}^2$ on $\mathbb{Z}^2$. It follows from the rigorous upper bound $p_c^s < \frac{8}{9}$ from [8] that $p = 0.025$ satisfies (3.6). Having chosen p, let $\lambda_1^* = \lambda_1^*(\alpha, p)$ (resp. $\lambda_2^* = \lambda_2^*(\alpha, p)$) and $\hat{\tau} = \hat{\tau}(p)$ (resp. $\check{\tau} = \check{\tau}(p)$) be the numbers defined in Lemma 3.4 (resp. 3.5). Clearly, $\lambda_1^* = b_1(p) + b_2(p) \cdot \alpha$ and $\lambda_2^* = \max\{c_1(p), c_2(p) \cdot \alpha\}$, where $b_1 = (2\log(72/p) + 2)(72/p), b_2 = (2\log(72/p) + 2)(48/p), c_1 = (12/p)\log(3/p), c_2 = 6/p$. Define

$$\tau := \hat{\tau} + \check{\tau} \quad \text{and} \quad \lambda^*(\alpha) := a_1 + a_2\alpha, \text{ where}$$
$$a_1 := \max\{b_1, c_1, (162/p)\log(72/p)\} \quad \text{and} \quad a_2 := \max\{b_2, c_2, (162/p)\log(72/p)\}. \tag{3.7}$$

Also, let E be the intersection of the internal events defined in Lemmas 3.2-3.5. Clearly, $\mathbb{P}(E) \geq 1 - 4p$ for all $\lambda \geq \lambda^*(\alpha)$. Combining Lemmas 3.2–3.5 we see that the following sequence of events (a)-(c) occur on the event E. (a) Starting from any configuration in $A_{2,*}$ $\mathcal{X}_t$ reaches A_4 before time $\hat{\tau}$, (b) $\mathcal{X}_t$ reaches $B_3 \cup A_4$ at time $\hat{\tau}$, and (c) $\mathcal{X}_\tau \in A_4$.

Lemmas 3.2-3.5 identify a sequence of events that ensures a region $R_{k,\ell}$ is good regardless of what happens on the external spacetime boundary of $R_{k,\ell}$ and shows that the probability of this sequence is at least $1 - 4p$ for any choice of $\lambda \geq \lambda^*$ and the specific choice of τ. Thus, if we choose such λ and τ, the good regions stochastically dominate an oriented site percolation with occupancy probability $1 - 4p > p_c^s$ for each site independently. Then applying the result for the oriented site percolation in [8] we get the desired upper bound. $\qquad\square$

Remark 3.6. If we only wanted to prove weak survival, then it would be sufficient to define overlapping regions of 3 vertices and follow the infection in a single direction, applying the results of [17] to the resulting dependent percolation. Because the regions are now smaller, this would yield some improvement to the constants in the proof. However, it is not obvious how to obtain strong survival from this construction, and the constants will still be relatively large, so we do not feel the improvement is very meaningful.

4 Extension to $\mathbb{Z}_n$

We now consider the contact process with avoidance on $\mathbb{Z}_n$. We can adapt the arguments developed in previous two sections and combine them with some known results about oriented percolation to show a phase transition on $\mathbb{Z}_n$ and prove Theorem 1.3.

4.1 Subcritical proof: $\lambda < 1 + \alpha$

The proof of the subcritical regime on $\mathbb{Z}_n$ is again notably complicated by the fact that the CPA is not an attractive particle system. The basic strategy is to break $\mathbb{Z}_n$ into regions and establish that with high probability many regions will be cleared of the infection quickly and stay clear of the infection for a long time. However, as was the case on $\mathbb{Z}$, the dependence among regions is complex and substantial work is needed to establish bounding processes that allow us to treat the regions as if they were independent. The following lemmas allow us to do this.

Lemma 4.1. *Let R be a region of consecutive vertices $\{1, \ldots, k\}$ and all edges pointing those vertices. Let ω and ω' be two realizations of the graphical construction such that ω and ω' agree on R. If there are no symbols on the edges $(0, 1)$ and $(k+1, k)$ in the time interval $[0, t]$, then the states of the vertices 1 and k (the boundary vertices of R) are the same in both ω and ω' up to time t.*

Proof. We first show that if we know the state of a vertex v and all edges pointing toward v at time 0, the locations of all recovery dots in the graphical construction on v from time 0 to t, the locations of all infection arrows and avoidance crosses on the edges pointing to v from time 0 to t, and the states of vertices $v-1$ and $v+1$ from time 0 to t and additionally that the number of all such recovery dots, infection arrows, and avoidance crosses is finite and no pair of symbols occur at the same time, we can determine the state of v at any time $t' \in [0, t]$.

Start from time 0 in the graphical construction and continue forward in time until we encounter a symbol in the graphical construction on either v or one of the edges pointing toward v or we observe a change in state of either $v-1$ or $v+1$. There are four possibilities.

1. We encounter a recovery dot on v. In this case, set the state of v to healthy.
2. We encounter an infection arrow on an edge (w, v) pointing to v. In this case, set the state of v to infected if the vertex w is infected and the edge (w, v) is active.
3. We encounter an avoidance cross on an edge (w, v) pointing to v. In this case, set the edge (w, v) to inactive if w in infected and v is healthy.
4. We observe a change in state in a vertex w where w is either $v-1$ or $v+1$. In this case, set the edge (w, v) to active if w changed from infected to healthy.

We can update states at first occurrence in time of any of these four possibilities since we know the initial states of v and the edges pointing to it at time 0 and the states of $v-1$ and $v+1$ for all $t' \in [0, t]$. After the we can then proceed by induction. After the kth occurrence, the states of $v-1$, v, $v+1$, and the edges pointing to v remain unchanged until the $k+1$st occurrence, and we can update these states at the $k+1$st occurrence. We continue in this way until we have exhausted all occurrences up to time t', at which point the current state of v will be its state at time t'.

We now apply this same strategy to a region of containing a collection of vertices. To determine the states of all vertices in R up to time t, we need to know

1. the initial states of all vertices v_i in R and all edges pointing toward all v_i in R,
2. the locations of all recovery dots in the graphical construction on the v_i in R up to time t,
3. the locations of all infection arrows and avoidance crosses on the edges pointing toward all v_i in R up to time t, and
4. the states of vertices outside R that neighbor vertices in R through time t, so in this case the states of vertices 0 and $k+1$ through time t.

Items 1, 2 and 3 are assumed to be the same in both ω and ω'. Item 4 may differ, but we only use the information in 4 when we observe an infection arrow or avoidance cross on either of the edges $(0,1)$ or $(k+1,k)$. Since we have assumed no such marks exist in either ω or ω' up to time t, we in fact do not need 4. Therefore, the states of the vertices in 1 and k are the same in both ω and ω' up to time t. $\qquad\square$

Consider the CPA on $\mathbb{Z}_n$ starting from all vertices infected and suppose $\lambda < 1 + \alpha$. We divide $\mathbb{Z}_n$ into regions as follows. Let $C^*, K > 0$ be constants to be chosen later. The ith region consists of the vertices $\{(3C^* + 2K)(i-1)\log n, \ldots, (3C^* + 2K)i\log n - 1\}$, all edges among these vertices, and in addition the edges $((3C^* + 2K)(i-1)\log n - 1, (3C^* + 2K)(i-1)\log n)$ and $((3C^* + 2K)i\log n, 3(3C^* + 2K)i\log n - 1)$. Note that each region contains $N := (3C^* + 2K)\log n$ consecutive vertices. Of course N will generally not be an integer, and our convention will be to interpret this as the floor $\lfloor N \rfloor$, but as rounding will not affect the estimates below, we omit the floor from our notation. Any vertices left over after dividing n by N will not be part of any region, and these extra vertices will number at most N. We will further divide each region into two buffers and an interior. The left buffer consists of the vertices $\{(3C^* + 2K)(i-1)\log n, \ldots, (3C^* + 2K)(i-1)\log n + 2K\log n - 1\}$ and all edges oriented towards these vertices, and the right buffer consists of the vertices $\{(3C^* + 2K)i\log n - 2K\log n + 1, \ldots, (3C^* + 2K)i\log n\}$ and edges oriented towards these vertices.

We will further divide the interior of the region i into left, center, and right subregions consisting of $C^*\log n$ vertices each, so the left subregion of region i contains the vertices $\{(3C^* + 2K)(i-1)\log n + 2K\log n, \ldots, (3C^* + 2K)(i-1)\log n + (C^* + 2K)\log n - 1\}$, the center subregion contains vertices $\{(3C^* + 2K)(i-1)\log n + (C^* + 2K)\log n, \ldots, (3C^* + 2K)(i-1)\log n + (2C^* + 2K)\log n - 1\}$, and the right subregion contains vertices $\{(3C^* + 2K)(i-1)\log n + (2C^* + 2K)\log n, \ldots, (3C^* + 2K)(i-1)\log n + (3C^* + 2K)\log n - 1\}$.

Definition 4.2. *Let the spacetime region R_i consist of the vertices and edges of the ith region through time $C\log n$, where C will be chosen later depending on α, λ, and C^*. We call region R_i **broken** if it contains no infected vertices in the center subregion at time $C\log n$.*

Our goal is to identify an event internal to R_i such that when that event occurs, R_i is broken.

Our first step is to demonstrate an event in the graphical construction of the buffers such that when the event occurs, the interior of R_i is independent of everything that occurs outside R_i from time 0 to $C\log n$. Informally, the idea is as follows. Information passes along edges when either a vertex infects its neighbor or a vertex avoids its neighbor. We can identify events that potentially pass information by looking at the symbols in the graphical construction. If a symbol exists on an edge between two vertices at some time t, there is the possibility that information is passed at this time. However, in the immediate sense this can only happen with neighboring vertices. If in a particular spacetime region there is no path of symbols between a pair of vertices, they cannot influence each other's states.

Lemma 4.3. *Number the vertices in R_i by $\{1, \ldots, N\}$. Let $\mathcal{B}_i$ be the event that there is no increasing sequence of times $t_1 < \ldots < t_{K\log n} \le C\log n$ such that there is a symbol on the collections of edges $\{(0,1), (N+1, N)\}, \ldots \{(K\log n, K\log n + 1), (N - K\log n - 2, K\log n - 1)\}$ at times $t_1, \ldots, t_{K\log n}$ respectively. Fix an initial condition $\mathcal{X}_0$ and let $\omega, \omega' \in \mathcal{B}_i$ be two different realizations of the graphical construction such that ω and ω' are the same on R_i during the time interval $[0, C\log n]$. Then the states of all vertices in the interior of R_i are the same for both ω and ω' from time 0 to $C\log n$.*

Proof. Let t_1 be the first time there is a difference in the state of either vertex 1 or vertex

N in the dynamics following the instructions in ω versus ω'. Lemma 4.1 states that t_1 is at least the first time there is an infection arrow or avoidance cross on either of the edges $(0,1)$ or $(N+1, N)$. Now proceed by induction. Suppose t_m is the first time there is a difference in the state of either vertex m or vertex $N - m + 1$ in ω and ω'. We identify the first time t_{m+1} at which there could potentially be a difference in the state of either vertex $m+1$ or vertex $N - m$. Because ω and ω' agree on the subregion of R_i consisting of the vertices $\{m+1, \ldots, N-m\}$ up to time $C \log n$ and by the definition of t_m the states of vertices m and $N - m - 1$ agree up to time t_m, Lemma 4.1 states we must have $t_{m+1} \geq \min(C \log n, t_m)$. If $t_m < C \log n$, then Lemma 4.1 states that $t_{m+1} \geq \min(C \log n, t_m + t_m^*)$ where t_m^* is the first time after t_m that there is an infection arrow or avoidance cross on either of the edges $(m, m+1)$ or $(N - m + 1, N - m)$. Suppose now that $t_{K \log n} \leq C \log n$. Then our induction argument implies that there are symbols on the collections of edges $\{(0,1), (N+1, N)\}, \ldots \{(K \log n, K \log n + 1), (N - K \log n - 2, K \log n - 1)\}$ at time $t_1 < \ldots < t_{K \log n} \leq C \log n$, which contradicts our assumption that $\omega, \omega' \in \mathcal{B}_i$. $\square$

We next show that the probability of the event $\mathcal{B}_i$ can be made as large as needed.

Lemma 4.4. *Fix $\epsilon \in (0,1)$ and choose $K = \frac{1}{\epsilon}(C+1)(2\lambda + 2\alpha)$. Then $\mathbb{P}(\mathcal{B}_i) \geq 1 - \epsilon$.*

Proof. The possible symbols that can appear along an edge are an infection arrow and an avoidance cross. In the graphical construction, these symbols appear in each of our collections of two edges combined rate $2\lambda + 2\alpha$, and they appear independently on each edge and so independently on each of our collections of two edges. Since our process is memoryless, starting from any point in time, the time until a symbol appears on each collection of edges is $X_i \sim \text{Exp}(2\lambda + 2\alpha)$. The fastest sequences of times $t_1 < \ldots < t_{K \log n} \leq C \log n$ requires us to encounter a symbol on the first collection, then encounter a symbol on the second collection after the time we encountered a symbol on the first collection, and so on. Thus $t_{K \log n} = \sum_{i=1}^{K \log n} X_i$. Since the X_i are independent, we have $t_{K \log n} = \sum_{i=1}^{K \log n} X_i \sim \text{Gamma}(K \log n, 2\lambda + 2\alpha)$. Then if we choose

$$K = \frac{1}{\epsilon}(C+1)(2\lambda + 2\alpha)$$

and apply Chebyshev's inequality,

$$\mathbb{P}\left(t_{K \log n} > C \log n\right) \geq \mathbb{P}\left(t_{K \log n} - \frac{1}{\epsilon}(C+1)\log n > -\sqrt{\frac{1}{\epsilon}}\sqrt{\frac{1}{\epsilon}(C+1)\log n}\right)$$

$$\geq \mathbb{P}\left(|t_{K \log n} - \frac{1}{\epsilon}(C+1)\log n| < \sqrt{\frac{1}{\epsilon}}\sqrt{\frac{1}{\epsilon}(C+1)\log n}\right)$$

$$\geq 1 - \epsilon.$$

This finishes the proof. $\square$

Our next step is to show that starting from all vertices infected, the indicators of the events that the ith region is broken for $1 \leq i < n/((3C^* + 2K) \log n)$ stochastically dominate a collection of independent Bernoulli(p) random variables for some $p > 0$ (not depending on n). To this end, we will use the following lemma, which identifies an event internal to R_i that implies R_i is broken.

Lemma 4.5. *The event $\mathcal{B}_i \cap \{R_i \text{ is broken}\}$ is internal to R_i.*

Proof. Let ω and ω' be two realizations of the graphical construction such that $\omega \in \mathcal{B}_i \cap \{R_i \text{ is broken}\}$ and ω and ω' are the same on R_i from time 0 to $C \log n$. Because the event $\mathcal{B}_i$ depends only on symbols appearing on edges in R_i from time 0 to $C \log n$ and

ω and ω' have the same such symbols, $\omega' \in \mathcal{B}_i$. We then apply Lemma 4.3 to conclude that the states of all vertices in the interior of R_i are the same for both ω and ω' from time 0 to $C \log n$. Since the event $\{R_i \text{ is broken}\}$ is determined by the states of the vertices in the interior of R_i from time 0 to $C \log n$ and these are the same for ω and ω', $\omega' \in \{R_i \text{ is broken}\}$. Thus $\omega' \in \mathcal{B}_i \cap \{R_i \text{ is broken}\}$ and so $\mathcal{B}_i \cap \{R_i \text{ is broken}\}$ is internal to R_i. $\qquad\square$

We now bound from below the probability that R_i is broken. We define the **gap-edge process** (analogous to the process described in Section 2) in the interior of R_i as follows. Suppose the middle vertex of the interior of R_i (a designated vertex closest to the midpoint of the region) is healthy at time 1. Define the left gap-edge process, $l_g(t) \leq 0$, to track the displacement from the middle vertex to its closest infected neighbor to the left in the interior of R_i at time $t \geq 1$, with the conventions that $l_g(t) = -\infty$ if there are no such infected neighbors in R_i and $l_g(t)$ "hits" 0 if the middle vertex of the interior of R_i is reinfected from the left. The states $-\infty$ and 0 act as absorbing states for $l_g(t)$. Likewise, define the right gap-edge process, $r_g(t)$, to track the displacement from the middle vertex to its closest infected neighbor to the right in interior of R_i at time $t \geq 1$, with the convention that $r_g(t) = \infty$ if no such vertex exists. We make use of the standard couplings of these processes with biased random walks to prove the following lemma.

Lemma 4.6. *For each region R_i, $i = 1, \ldots, \frac{n}{N}$, we have*

$$\mathbb{P}(R_i \text{ is broken}) \geq \frac{1}{4} e^{-2\lambda} (1 - e^{-1}) \left(1 - \frac{\lambda}{2(1+\alpha)} \right)^2 =: 2p. \tag{4.1}$$

Proof. We define the following events.

1. The middle vertex of the center subregion of the interior R_i is healthy at time 1. Call this event $\mathcal{A}_1^i$.

2. If $\mathcal{A}_1^i$ occurs, at time 1, consider the left and right gap-edge processes around the middle vertex during time $[1, C \log n]$. These edge processes both leave the interior of R_i before infecting the middle vertex (hitting 0) and before time $C \log n$. Call this event $\mathcal{A}_2^i$.

3. Let $0 \leq t_1 < t_2 < \cdots < t_k \leq C \log n$ be all the times at which there are infection arrows along the leftmost edge in the interior of R_i, $((3C^* + 2K)(i-1)\log n + 2K \log n - 1, (3C^* + 2K)(i-1)\log n + 2K \log n)$, up to time $C \log n$. We say that an uninterrupted path of infection exists in the left subregion of the interior of R_i during time $[0, C \log n]$ if there exists $j \leq k$ such that, starting at time t_j with all edges active and all vertices healthy except for a single infection at $(3C^* + 2K)(i-1)\log n + 2K \log n - 1$, in the graphical construction restricted to R_i the vertex $(3C^* + 2K)(i-1)\log n + (C^* + 2K)\log n$ gets infected by time $C \log n$. Call the complementary event (that no uninterrupted path of infection exists) $\mathcal{A}_3^i$.

4. Analogously define an uninterrupted path of infection in the right subregion of the interior of R_i, and let $\mathcal{A}_4^i$ be the event that no such path exists.

If $\mathcal{A}_1^i \cap \mathcal{A}_2^i$ occurs, the left and right gap-edge processes will leave the interior of R_i at random times $\tau_l, \tau_r \leq C \log n$ and before reinfecting the middle vertex. If $\tau_l \leq \tau_r$, then at time τ_l all vertices to the left of the middle vertex in the interior of R_i are healthy and all edges (with possible exception of the leftmost edge) to the left of the middle vertex in the interior of R_i are active. Therefore, if $\mathcal{A}_3^i$ also occurs, the vertex $(3C^* + 2K)(i-1)\log n + (C^* + 2K)\log n$ cannot be reinfected during time interval $[\tau_l, \tau_r]$, and at time τ_r all vertices in the middle and right subregions of the interior of R_i are healthy and all edges (except possibly the rightmost edge) in the middle and right

subregions of the interior of R_i are active. Therefore, if $\mathcal{A}_4^i$ also occurs, then no vertices in the middle subregion of the interior of R_i will be infected at time $C \log n$. The case $\tau_l > \tau_r$ is similar, and we have

$$\bigcap_{j=1}^4 \mathcal{A}_j^i \subset \{R_i \text{ is broken}\}. \tag{4.2}$$

We next bound the probability of the event $\cap_{j=1}^4 \mathcal{A}_j^i$. For $\mathcal{A}_1^i$ to occur, it is sufficient for the middle vertex to recover at some time in $[0,1]$ and have no incoming infection arrows during $[0,1]$. Thus,

$$\mathbb{P}(\mathcal{A}_1^i) \geq e^{-2\lambda}(1 - e^{-1}). \tag{4.3}$$

Next we bound from below the probability of $\mathcal{A}_2^i$ given $\mathcal{A}_1^i$. On the event $\mathcal{A}_1^i$, consider the left gap-edge process, $l_g(t)$. Observe that $l_g(t)$ increases (by 1) only if the rightmost infected vertex to the left of the middle infects its neighbor to the right before either recovering or being avoided by its neighbor to the right (deactivation of the edge); otherwise $l_g(t)$ will decrease by at least 1. Note that when $l_g(t)$ jumps to the left, the size of the jump may not be 1, but until $l_g(t) \in \{-\infty, 0\}$, we have that $l_g(t)$ takes steps to the right with probability at most $\frac{\lambda}{\lambda + 1 + \alpha} < \frac{1}{2}$ and otherwise steps left. Therefore, the sequence of locations of $l_g(t)$ after successive jumps (the embedded discrete-time "chain", which is not Markov) is stochastically dominated by a simple random walk that steps left with probability

$$\xi = \frac{1 + \alpha}{1 + \alpha + \lambda} > 1/2,$$

right with probability $1 - \xi$, and starts at -1. (One can explicitly couple these stochastic processes until $l_g(t)$ hits either $-\infty$ or 0, after which the random walk process can be extended independently for all time.)

By a standard Gambler's Ruin analysis, the dominating random walk process never returns to 0 with probability $1 - \frac{\lambda}{1+\alpha} > 0$. Moreover, the location of the random walk after $\frac{3}{2\xi-1}C^* \log n$ steps is dominated by $\frac{3}{2\xi-1}C^* \log n - 2X$ where $X \sim \text{Bin}(\frac{3}{2\xi-1}C^* \log n, \xi)$. Since $E(\frac{3}{2\xi-1}C^* \log n - 2X) = \frac{3}{2\xi-1}C^* \log n(1 - 2\xi) = -3C^* \log n$ and $\text{Var}(\frac{3}{2\xi-1}C^* \log n - 2X) = 4\frac{3}{2\xi-1}C^* \log n \xi(1 - \xi) \leq \frac{3}{2\xi-1}C^* \log n$, Chebychev's inequality implies that the probability that the random walk has not crossed $-2C^* \log n$ after $\frac{3}{2\xi-1}C^* \log n$ steps is at most $\frac{3}{(2\xi-1)C^* \log n}$. Next, observe that $l_g(t)$ makes jumps at least as frequently as the arrivals of recovery dots, and by independently generating recovery dots at rate 1 after $l_g(t)$ is absorbed (to emulate additional jumps), the number of jumps made by $l_g(t)$ by time $C \log n$ stochastically dominates $\text{Poisson}(C \log n)$. Therefore, letting

$$C = \frac{6}{2\xi - 1}C^*,$$

the number of jumps made by time $C \log n$ exceeds $\frac{3}{2\xi-1}C^* \log n$ with probability at least $1 - \frac{2(2\xi-1)}{3C^* \log n}$ by Chebychev's inequality. Thus, $l_g(t)$ hits $-\infty$ by time $C \log n$ with probability at least $1 - \frac{\lambda}{1+\alpha} - \frac{3}{(2\xi-1)C^* \log n} - \frac{2(2\xi-1)}{3C^* \log n}$. An analogous argument shows $r_g(t)$ hits ∞ by time $C \log n$ with at least the same probability, and does so independently. We conclude that for large n,

$$\mathbb{P}(A_2^i | A_1^i) \geq \left(1 - \frac{\lambda}{2(1+\alpha)}\right)^2. \tag{4.4}$$

To see that $\mathcal{A}_3^i$ occurs with high probability first observe that k, the number of infection arrows along $((3C^* + 2K)(i-1) \log n + 2K \log n - 1, (3C^* + 2K)(i-1) \log n + 2K \log n)$, which is the edge into the left subregion of the interior R_i, during time interval $[0, C \log n]$

is Poisson($\lambda C \log n$) distributed, and therefore $k \leq 2\lambda C \log n$ with probability at least $1 - \frac{1}{\lambda C \log n}$. For each $1 \leq j \leq 2\lambda C \log n$, at the time t_j of the jth infection arrow this edge, we begin tracking an infection process in the graphical construction and with only $(3C^* + 2K)(i-1) \log n + 2K \log n - 1$, which is the left neighbor of the interior of R_i, infected at time t_j. (Note that for different j, these processes may "overlap" in their use of the symbols in the graphical construction, but this dependence will not matter.)

Consider the jth such process, started at time t_j. As in the previous argument, the sequence of locations of the rightmost infected vertex in R_i after each jump in its location is stochastically dominated by a simple random walk that moves left by 1 with probability $\xi > 1/2$ and right by 1 with probability $1 - \xi$. By the Gambler's Ruin, it follows that this random walk, when started from $3(3C^* + 2K)(i-1) \log n + 2K \log n$, hits $(3C^* + 2K)(i-1) \log n + (2K + C^*) \log n$ before hitting $(3C^* + 2K)(i-1) \log n + 2K \log n - 1$ with probability at most

$$\left(\frac{\lambda}{1+\alpha} \right)^{C^* \log n} = n^{-C^* \log((1+\alpha)/\lambda)}.$$

If the random walk hits $(3C^* + 2K)(i-1) \log n + 2K \log n - 1$ before hitting $(3C^* + 2K)(i-1) \log n + (2K + C*) \log n$, then the infection processes started at time t_j never reaches the middle subregion of R_i. Therefore, the probability that any of the first $2\lambda C \log n$ such infection attempts ever reaches the middle subregion of R_i is at most

$$2\lambda C (\log n) n^{-C^* \log((1+\alpha)/\lambda)},$$

so for large n,

$$\mathbb{P}(\mathcal{A}_3^i) \geq 1 - \frac{2}{\lambda C \log n}. \tag{4.5}$$

The same (symmetric) argument implies

$$\mathbb{P}(\mathcal{A}_4^i) \geq 1 - \frac{2}{\lambda C \log n}. \tag{4.6}$$

Combining equations (4.2)–(4.6), we have for all large n,

$$\mathbb{P}\left(\cap_{j=1}^4 \mathcal{A}_j^i \right) \geq \frac{1}{2} e^{-2\lambda} (1 - e^{-1}) \left(1 - \frac{\lambda}{2(1+\alpha)} \right)^2. \tag{4.7}$$

Since $\mathbb{P}\left(R_i \text{ is broken} \right) \geq \mathbb{P}\left(\cap_{j=1}^4 \mathcal{A}_j^i \right)$, this finished the proof of Lemma 4.6. $\qquad\square$

We are now ready to prove the first part of Theorem 1.3.

Proof of Theorem 1.3 for $\lambda < 1 + \alpha$. Choosing $\epsilon = p$ in Lemma 4.4 and combining with Lemma 4.6, we have

$$\mathbb{P}(\{R_i \text{ is broken}\} \cap \mathcal{B}_i) \geq p. \tag{4.8}$$

Now, let $i_1 < \cdots < i_M$ be all of the random indices such that $\{R_{i_\ell} \text{ is broken}\} \cap \mathcal{B}_{i_\ell}$ occurs for $1 \leq \ell \leq M$, and let $v_1, \ldots, v_M$ be the middle vertices (in the middle subregions) of each R_{i_ℓ}. By Lemma 4.5 and (4.8), the collection of indicators of broken regions dominates a collection of independent Bernoulli(p) random variables, of which there are fewer than n. The longest run of 0's in fewer than n independent Bernoulli(p) trials arranged in a cycle is smaller than $\frac{4}{\log((1-p)^{-1})} \log n$ with probability at least $1 - n^{-1}$. So with probability at least $1 - n^{-1}$ we have

$$\text{dist}(v_1, v_M) \vee \max_\ell \text{dist}(v_\ell, v_{\ell+1}) \leq \frac{5}{\log((1-p)^{-1})} \log n \cdot (3C^* \log n), \tag{4.9}$$

where dist is the shortest path distance on $\mathbb{Z}_n$, and the 5 in the numerator accounts for vertices that are not in any region (between v_M and v_1).

We now sketch an argument, which is very similar to the arguments above, to show that for a large enough $C' > 0$, during the time interval $[C \log n, C \log n + C'(\log n)^2]$, each interval of vertices between v_ℓ and $v_{\ell+1}$ will fully recover without ever interacting with neighboring intervals. This will complete the proof of the subcritical result.

For $\ell = 1, \ldots M - 1$, in the interval between v_ℓ and $v_{\ell+1}$ (and between v_M and v_1) there are no infected vertices within $\frac{1}{2}C^* \log n$ of v_ℓ or $v_{\ell+1}$ at time $C \log n$. Starting at time $C \log n$, the sequence of distances from v_ℓ to the leftmost infected vertex in $[v_\ell, v_{\ell+1}]$ after each jump in its location dominates a random walk that moves to the right with probability $\xi > 1/2$ (with the convention that the distance is ∞ if there are no infected vertices in the interval, and we ignore infections coming from outside the interval, but as we will see, there are none). Choosing

$$C^* = \frac{4}{\log((1 + \alpha)/\lambda)},$$

it follows that the probability that the leftmost infected vertex in this interval ever reaches v_ℓ is at most

$$\left(\frac{\lambda}{1 + \alpha}\right)^{(C^*/2)\log n} = n^{-2}.$$

Likewise, the probability that the rightmost infected vertex in this interval ever reaches $v_{\ell+1}$ is at most n^{-2}, so the probability that there exists an ℓ such that v_ℓ is ever reinfected is at most $2n^{-1}$ (and on the complementary event, we are justified in ignoring potential infections between neighboring intervals).

The leftmost infected vertex between v_ℓ and $v_{\ell+1}$ attempts to jump at rate at least 1 (extending the 'jump' process, as before, beyond the first time that either there are no infected vertices in the interval or v_ℓ gets infected), so the number of jumps during the time interval $[C \log n, C \log n + C'(\log n)^2]$ dominates a Poisson$(C'(\log n)^2)$ distribution. Therefore, by a standard lower tail estimate for the Poisson distribution, the probability that the number of attempted (potential) jumps by the leftmost infected vertex is less than $\frac{1}{2}C'(\log n)^2$ is at most $e^{-(\log n^2)} \le n^{-2}$ for C' and n sufficiently large. After $\frac{1}{2}C'(\log n)^2$ jumps, the displacement of a simple random walk that moves right with probability ξ is $2Y - \frac{1}{2}C'(\log n)^2$ where $Y \sim \text{Bin}(\frac{1}{2}C'(\log n)^2, \xi)$, and by a Chernoff bound, $2Y - \frac{1}{2}C'(\log n)^2$ exceeds $\frac{1}{4}(2\xi - 1)C'(\log n)^2$ with probability at least $1 - n^{-2}$. The displacement of the leftmost infected vertex after $\frac{1}{2}C'(\log n)^2$ jumps exceeds this, so on the event in (4.9), if $C' \ge \frac{60C^*}{(2\xi-1)\log((1-p)^{-1})}$, then the leftmost infected vertex will exceed $v_{\ell+1}$, which implies the interval between v_ℓ and $v_{\ell+1}$ is fully recovered. Finally, by a union bound, the probability that one of the $M \le n$ intervals has not cleared the infection by time $C \log n + C'(\log n)^2$ is at most $Mn^{-2} \to 0$ as $n \to \infty$. □

4.2 Supercritical proof (large λ)

To prove the upper bound, we will use our result from section 3 that the contact process with avoidance can stochastically dominate an oriented site percolation with probability of occupancy p for any chosen $p < 1$ so long as λ is chosen to be sufficiently large, along with some facts about oriented percolation. We begin by briefly describing the models, introducing some notation, and stating some results about oriented percolation that we need.

Oriented percolation is defined on the sites $\{(x, t) \in \mathbb{Z} \times \mathbb{N} : x = (t \mod 2) \mod 2\}$ where (x, t) and (y, s) are neighbors when $|x - y| = 1$ and $|t - s| = 1$. x can be thought of as space and t as time. In site percolation each site is either occupied with probability

p_s or unoccupied with probability $1 - p_s$ independently, and two sites are connected if they are neighbors and both are occupied. In bond percolation, bonds between neighboring sites are active with probability p_b and inactive with probability $1 - p_b$ independently, and two sites are connected if there is an active bond between them. Define $S_n^A = \{(x,t); t = n \text{ and there is a connected path from some } (y,s) \in A \text{ to } (x,t)\}$ for site percolation and B_n^A analogously for bond percolation. We write $\{A \to \infty\}$ to mean that there is an infinite oriented path starting from the set A. We denote the critical value for oriented site percolation by p_c^s where p_c^s is the unique value such that $\mathbb{P}(\{A \to \infty\}) > 0$ if and only if $p_s > p_c^s$ for site percolation and define p_c^b in the same way for bond percolation. We now state some results.

Proposition 4.7. *For any A, p_b, for all $p_s \geq p_b(2 - p_b)$, $B_n^A \underset{stoch.}{\subset} S_n^A$ for every n.*

This follows from a straightforward coupling argument. See [18] for details.

Proposition 4.8 ([8])**.** *For any set D, $\mathbb{P}(\{D \to \infty\}^c) \leq Ce^{-\gamma|D|}$ for some constants $C, \gamma > 0$.*

In words, the probability that an oriented site percolation dies is exponentially small in the size of the starting set. As [26] notes, the following is a corollary of Theorem 1 of [7].

Proposition 4.9. *Suppose $p_b > p_c^b$. Then for any $p^* < p_b$ and any finite set D of consecutive sites at time n, $\mathbb{P}\left(|B_n^{2\mathbb{Z}} \cap D| < p^*|D|\right) \leq Ce^{-\gamma|D|}$ for some constants $C, \gamma > 0$.*

The result is stated for bond percolation but using Proposition 4.7 we can also apply it to site percolation. Choose $p_b > p_c^b$ and then $p_s > p_b(2 - p_b)$. Then for any $p^* < p_b$, Propositions 4.7 and 4.9 together give that $\mathbb{P}\left(|S_n^{2\mathbb{Z}} \cap D| < p^*|D|\right) \leq Ce^{-\gamma|D|}$ for the constants $C, \gamma > 0$ in Proposition 4.9.

The previous result concerns oriented percolation on the infinite lattice $2\mathbb{Z}$ where each site or bond initially has some probability p of being occupied/active. However, in our comparison percolation the active sites are determined at the start of each cycle. We can remedy this technical difficulty by showing that for appropriately chosen D and any $k \in \mathbb{Z}_n$ we have with high probability $S_n^{2\mathbb{Z}} \cap D = S_n^{\{k\}}$ conditional on the event $F_k = \{\{k\} \to \infty\}$. To that end we require the following result.

Proposition 4.10 ([8])**.** *Define the right edge $r_n = \sup_x\{(x,t) : x \text{ is occupied and } t = n\}$ of a supercritical oriented site percolation starting from $\{k\}$ such that F_k occurs. Then there exists $a = a(p_s) \in (0,1)$ such that $\mathbb{P}(r_n \leq k + an) \leq Ce^{-\gamma n}$ for some constants $C, \gamma > 0$*

By symmetry an analogous result holds for the left edge l_n. Also note that oriented percolation is translation invariant so without loss of generality we can take $k = 0$.

Suppose $p > p_c^b(2 - p_c^b)$. Then if we take the set D in Proposition 4.9 to be $\{-an, \dots, an\}$ then

$$\mathbb{P}\left(|S_n^{2\mathbb{Z}} \cap D| \leq p^*2an|F_0\right) \leq Ce^{-\gamma 2an}.$$

Now note that $\mathbb{P}(r_n \leq an|F_0) \leq Ce^{-\gamma n}$ and $\mathbb{P}(l_n \geq -an|F_0) \leq Ce^{-\gamma n}$, so conditional on F_0 with probability $1 - Ce^{-\gamma n}$ any $0 \neq x \in S_0^{2\mathbb{Z}}$ for which there is a path from x to some $y \in D$ must intersect one of the edges of the percolation starting from $\{0\}$ in which case we have

$$S_n^{2\mathbb{Z}} \cap D|F_0 = S_n^{\{0\}} \cap D|F_0$$

and so

$$\mathbb{P}\left(|S_n^{\{0\}} \cap D| \leq p^*2an|F_0\right) \leq Ce^{-\gamma 2an}$$

We are now ready to commence the proof of the exponential survival regime. Note that while the values of the constants C and γ change from line to line, the values

themselves are uninteresting, and we only define finitely many different constants and so can take minima and maxima as needed.

Proof of Theorem 1.3 for $\lambda > a_1 + a_2\alpha$. Recall that we chose a_1 and a_2 by setting $p = 0.025$ and using equation (3.7). Thus for $\lambda > a_1 + a_2\alpha$, the block construction associated with the CPA stochastically dominates an oriented site percolation with occupancy probability $1 - 4p > p_c^s$, so we consider that process starting from all sites occupied. Divide $\mathbb{Z}_n$ into two halves $\{0, \dots n/2 - 1\}$ and $\{n/2 \dots n\}$. If we ignore for the moment the second half and consider only the first half, then by Proposition 4.8 a percolation on $\mathbb{Z}$ starting from the vertices $\{0, \dots n/2 - 1\}$ occupied survives forever on (and thus also until time $n/4$) with probability at least $1 - Ce^{-\gamma n/2}$ since the starting set has size $n/2$. Using translation invariance we can conclude from this that $\mathbb{P}(F_0) \geq 1 - Ce^{-\gamma n/2}$. Up until time $n/4$ the oriented percolation on $\mathbb{Z}_n$ started from the vertices $\{0, \dots n/2 - 1\}$ occupied can be coupled with the oriented percolation on $\mathbb{Z}$ starting from the vertices $\{0, \dots n/2 - 1\}$ occupied because the former has not yet had a chance to wrap back on itself.

Now suppose the percolation from the first half survives until time $n/4$ (which it does if F_0 occurs) and thus has had an opportunity to spread across the second half, but not to wrap back on itself. Choose $\epsilon < p/3$. It follows from the rigorous upper bounds $p_c^b \leq \frac{2}{3}$ from [18] that our choice of $p = 0.0025$ satisfies $1 - 4p > p_c^b(2 - p_c^b)$ so we may apply Propositions 4.7, 4.9, and 4.10 to see that

$$\mathbb{P}\left(|A_{n/2}^{\{0,\dots n/2-1\}}| < 2\epsilon n\right) \leq \mathbb{P}\left(|A_{n/2}^{\{0,\dots n/2-1\}}| < 2\epsilon n | F_0\right) + \mathbb{P}(F_0^c) \leq Ce^{-\gamma n}.$$

If we then have at least $2\epsilon n$ occupied sites on $\mathbb{Z}_n$, we must be able to take one half of $\mathbb{Z}_n$ that has at least ϵn occupied sites, and we can repeat the process, again starting from a set with size $O(n)$. Call each time through these steps a cycle. By a union bound on the probability of failure at each step, the probability of a successful cycle is at least $1 - Ce^{-\gamma n}$. Thus, τ stochastically dominates a geometric random variable with success probability $1 - Ce^{-\gamma n}$, and so there exist $C, \gamma > 0$ such that $\mathbb{P}(\tau \leq Ce^{\gamma n}) \to 0$ as $n \to \infty$. $\qquad\square$

5 Results for the star graph

The goal of this section is to prove Theorem 1.5. On the star graph it is possible to reformulate the CPA model by only assigning states to the center and the leaves and not individual edges, since the oriented edges (i, j) and (j, i) cannot both be in the avoiding state at the same time. We do this as follows. The center takes on values in $\{0, 1\}$ meaning healthy and infected as before. Leaves take on states in $\{0, 1\} \times \{A, D\}$ where 0 and 1 denote healthy and infected (vertex) states, and A and D denote active and inactive (edge) states. Active leaves can both receive and transmit the infection, while inactive leaves can do neither. Depending on the state of the center, the system follows different dynamics.

Definition 5.1. *When the center is infected (one-phase)*

1. $0A \to 1A$ *at rate* λ *(Center infects leaf)*

2. $1A \to 0A$ *at rate* 1. *(Leaf recovers)*

3. $0A \to 0D$ *at rate* α *(Leaf avoids center)*

4. $1D \to 0A$ *at rate* 1. *(Leaf that had been avoided by the center in a previous zero-phase recovers)*

5. *The center goes from* $1 \to 0$ *at rate* 1.

When the center is healthy (zero-phase)

1. *At the time of the center's recovery set all $0D$ to $0A$ (Leaves stop avoiding the center)*
2. *$1A \to 0A$ at rate 1. (Leaf recovers)*
3. *$1A \to 1D$ at rate α (Center avoids leaf)*
4. *The center goes from $0 \to 1$ at rate $m\lambda$ where m is the current number of $1A$ leaves.*

The system is perhaps most easily understood by referring to Figure 6. To see that these dynamics are equivalent to the dynamics of the CPA model on the star graph, consider the following transitions. A $0A \to 0D$ transition occurs when the directed edge from the center to a leaf becomes deactivated while the center is infected. All $0D$ return to $0A$ when the center recovers since all inactive edges from the center to leaves become active again at this time. A $1A \to 1D$ transition occurs when the edge from an infected leaf to the healthy center becomes deactivated. The subsequent $1D \to 0D$ occurs when an infected leaf whose edge to the center is inactive recovers, which both moves the leaf to state 0 and reactivates the edge.

The center plays a special role in the dynamics of the star graph, and so to understand the process dynamics on the star graph we consider in turn the dynamics when the center is infected (which we will call the one-phase,) the dynamics when the center is healthy (which we will call the zero-phase), and the change in the number of infected leaves between consecutive one-phases.

When the center is infected, the set of possible states of each of the other nodes is $S = \{1A, 0A, 0D, 1D\}$. Let X_t denote the state of a typical leaf node at time t assuming the center is held in the infected state through time t, and let

$$V_{i,j}(t) = \mathbb{P}\left(X_t = j | X_0 = i\right) \text{ for } i, j \in S \text{ and } \mathbf{V} = ((V_{i,j}))_{i,j \in S}.$$

Then using standard arguments for continuous time Markov chains it is easy to see that

$$\mathbf{V}'(t) = \mathbf{V}(t)\mathbf{A}, \text{ where } \mathbf{A} = \begin{array}{c} 1A \\ 0A \\ 0D \\ 1D \end{array} \left[\begin{array}{cccc} -1 & 1 & 0 & 0 \\ \lambda & -(\lambda + \alpha) & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{array} \right].$$

Note that $0D$ is an absorbing state for this 4-state Markov chain. In order to find the eigenvalues of $\mathbf{A}$ note that

$$\det(\mathbf{A} - \gamma I) = \gamma(-1 - \gamma)(\alpha\gamma + \alpha + \lambda\gamma + \gamma^2 + \gamma)$$

So the eigenvalues are $0, -\gamma_1, -\gamma_2$ and -1, where

$$\gamma_1 = \frac{1}{2}[(\lambda + \alpha + 1) - \sqrt{(\lambda + \alpha + 1)^2 - 4\alpha}], \gamma_2 = \frac{1}{2}[(\lambda + \alpha + 1) + \sqrt{(\lambda + \alpha + 1)^2 - 4\alpha}].$$

Simple algebra shows that

$$\gamma_1 + \gamma_2 = \lambda + \alpha + 1 \quad \text{and} \quad \gamma_1\gamma_2 = \alpha \quad \text{and}$$
$$\gamma_2 - \gamma_1 = [\lambda^2 + (\alpha - 1)^2 + 2\lambda(\alpha - 1)]^{1/2} \quad \text{and} \quad 0 \le \gamma_1 \le 1 \le \gamma_2 \le 1 + \lambda + \alpha. \quad (5.1)$$

We note that this differs from the case of the classical contact process, in which the corresponding matrix $\mathbf{A}$ with $\alpha = 0$ has rank 2 and eigenvalues $0, 0, -1$, and $-(1 + \lambda)$.

From the description of the process and the generator matrix, we can make a heuristic argument for why the survival time on the star is like n^Δ where $\Delta = 1/\gamma_1$. Essentially, the process dies when we observe a long one-phase during which all the leaves start
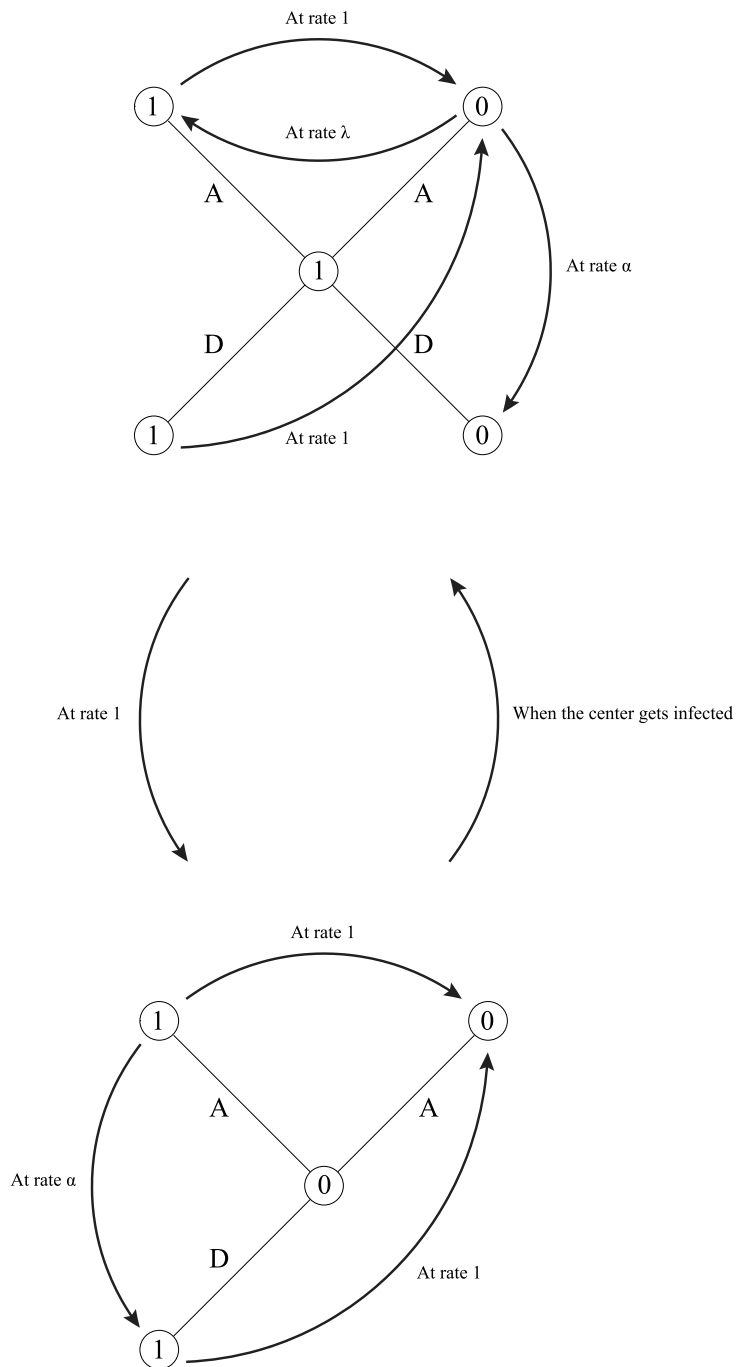
Figure 6: Star graph dynamics

avoiding the center, absorbing in the $0D$ state. Suppose we start at the beginning of a one-phase. Then this one-phase lasts for time T where $T \sim \mathrm{Exp}(1)$. At time T, we expect the number of leaves that have not yet been absorbed in the $0D$ state to be about $ne^{-\gamma_1 T}$, which is less than 1 when $T > (1/\gamma_1)\log n$. Since T is an $\mathrm{Exp}(1)$ random variable, $\mathbb{P}\left(T > (1/\gamma_1)\log n\right) = n^{-1/\gamma_1}$ so we should need about n^{1/γ_1} one-phases to observe a one-phase long enough for the process to die, and each one-phase-zero-phase cycle lasts on average for $O(1)$ time, suggesting a survival time of n^Δ. Of course, there are many details that need to be verified, and the remainder of this section is devoted to making this heuristic argument rigorous.

The right eigenvectors for the eigenvalues of $\mathbf{A}$ are the columns of the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & \zeta & \zeta & 0 \\ 1 & 1-\gamma_1 & 1-\gamma_2 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

where $\zeta = \frac{(1-\gamma_1)(\gamma_2-1)}{\lambda}$.

So $\mathbf{AB} = \mathbf{BD}$, where $\mathbf{D} = Diag(0, -\gamma_1, -\gamma_2, -1)$. Let $\mathbf{W}(t) = \mathbf{V}(t)\mathbf{B}$, so

$$\mathbf{W}'(t) = \mathbf{V}'(t)\mathbf{B} = \mathbf{V}(t)\mathbf{AB} = \mathbf{V}(t)\mathbf{BD} = \mathbf{W}(t)\mathbf{D}.$$

Since $\mathbf{D}$ is diagonal, we have $\mathbf{W}(t) = \mathbf{W}(0)\exp(\mathbf{D}t)$. Hence, using the fact that $\mathbf{V}(0) = I$

$$\mathbf{V}(t) = \mathbf{B}\exp(\mathbf{D}t)\mathbf{B}^{-1}.$$

Lemma 5.2. *If $u(t) := \mathbf{V}_{1A,1A}(t)$ and $v(t) := \mathbf{V}_{0A,1A}(t)$, then*

1. *$u(\cdot)$ is decreasing, $u(0) = 1$ and $u(t) \downarrow 0$ as $t \to \infty$ exponentially fast in t.,*
2. *$v(0) = 0$, $v(\cdot)$ is increasing (resp. decreasing) for $t \leq$ (resp. $\geq$) $(\log\gamma_2 - \log\gamma_1)/(\gamma_2 - \gamma_1)$ and $v(t) \to 0$ as $t \to \infty$ exponentially fast in t,*
3. *$v(t) \leq u(t)$ for all $t \geq 0$,*
4. *the map $\eta \mapsto f(\eta) := \int_0^\infty (\eta u(t) + (1-\eta)v(t))e^{-t}\,dt - \eta$ is monotonically decreasing, and $f(\eta) \geq 0$ (resp. ≤ 0) for $\eta \leq$ (resp. $\geq$) $\lambda/(\lambda + \alpha + 2)$*

Proof. By computation we see that

$$u(t) := \frac{(1-\gamma_1)e^{-\gamma_2 t} - (1-\gamma_2)e^{-\gamma_1 t}}{\gamma_2 - \gamma_1} = \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1}e^{-\gamma_1 t} + \frac{1-\gamma_1}{\gamma_2 - \gamma_1}e^{-\gamma_2 t}$$

$$v(t) := \left(\frac{(1-\gamma_1)(\gamma_2-1)}{\zeta(\gamma_2-\gamma_1)}\right)(e^{-\gamma_1 t} - e^{-\gamma_2 t}) = \left(\frac{\lambda}{\gamma_2 - \gamma_1}\right)(e^{-\gamma_1 t} - e^{-\gamma_2 t}).$$

1. From the properties of γ_1 and γ_2 in (5.1) it is clear that $\gamma_1, \gamma_2 > 0$ and the coefficients of $e^{-\gamma_1 t}$ and $e^{-\gamma_2 t}$ in $u(t)$ are both positive.

2. We observe (a) $0 < \gamma_1 < \gamma_2$, (b) $v(t)$ is a multiple of $e^{-\gamma_1 t} - e^{-\gamma_2 t}$, and (c) $v'(t)$ vanishes at $t = (\log\gamma_2 - \log\gamma_1)/(\gamma_2 - \gamma_1)$.

3. From the properties of γ_1 and γ_2 in (5.1)

$$v(t) \leq \frac{\lambda}{\gamma_2 - \gamma_1}e^{-\gamma_1 t} \leq \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1}e^{-\gamma_1 t} \leq u(t) \forall t \geq 0.$$

4. Since $\int_0^\infty e^{-(1+a)t}\,dt = (1+a)^{-1}$ for any $a > 0$,

$$\begin{aligned} f(\eta) &= \frac{1}{\gamma_2 - \gamma_1}\left[\eta\left(\frac{\gamma_2-1}{1+\gamma_1} + \frac{1-\gamma_1}{1+\gamma_2}\right) + (1-\eta)\left(\frac{\lambda}{1+\gamma_1} - \frac{\lambda}{1+\gamma_2}\right)\right] - \eta \\ &= \frac{1}{(1+\gamma_1)(1+\gamma_2)}[\eta(\gamma_2 + \gamma_1) + (1-\eta)\lambda] - \eta \\ f'(\eta) &= \frac{\gamma_1 + \gamma_2 - \lambda}{(1+\gamma_1)(1+\gamma_2)} - 1. \end{aligned}$$

Since $\gamma_1 + \gamma_2 = \lambda + \alpha + 1$ and $\gamma_1\gamma_2 = \alpha$, it is easy to see that $f'(\eta) < 0$ and

$$f(\eta) = 0 \Leftrightarrow \eta = \frac{\lambda}{(1 + \gamma_1)(1 + \gamma_2) - (\alpha + 1)} = \frac{\lambda}{\lambda + \alpha + 1}. \qquad \square$$

Note that in the case of the classical contact process, 1. and 2. above do not hold, and $u(t)$ and $v(t)$ do not converge to 0. It is the addition of avoidance allows the leaves to eventually avoid the center, after which they can no longer become infected during the current one-phase. This is the key difference driving the differing survival behaviors of the classical contact process and the CPA process on the star graph.

Next we will focus on the evolution of the number of nodes in different states of S. Note that at the beginning of each one-phase there is no node with state 0D and all nodes with state 0D at the end of each one-phase change their state to 0A at the beginning of the next zero-phase. Since the total number of nodes is n (the size of the star graph), it suffices to keep track of the number of nodes in states 1A and 1D at the beginning of the one-phases.

When there are m nodes in state 1A any time during a zero-phase, the rate at which the center gets infected is λm. Also the rates at which nodes change their states to 0A and 1D are m and αm respectively. Therefore, the time to the next event is exponentially distributed with mean $1/(\lambda + \alpha + 1)m$, and the probability that the next event is the center becoming infected (before any nodes change their states) is

$$\hat{\lambda} = \frac{\lambda}{1 + \alpha + \lambda}, \tag{5.2}$$

which does not depend on m. So if N is the number of 1A nodes lost during a zero-phase, then N has shifted Geometric distribution with success probability $\hat{\lambda}$.

$$\mathbb{P}(N = k) = (1 - \hat{\lambda})^k \hat{\lambda}, \quad k \geq 0. \tag{5.3}$$

Each of those nodes that changes its state from 1A becomes 1D or 0A with probability $\alpha/(1 + \alpha)$ and $1/(1 + \alpha)$ respectively. So the number of 1D nodes added during a 0-phase conditioned on N is Binomial with parameters N and $\alpha/(1 + \alpha)$. Unconditionally its distribution is shifted Geometric with success probability $\lambda/(\lambda + \alpha)$. Also if $\{T_i\}_{i \geq 0}$ denote the sequence of the above event times, then $T_0 = 0$ and $T_{i+1} - T_i$ has exponential distribution with mean $1/(\lambda + \alpha + 1)(m - i)$. If a node changes its state from 1A to 1D at time T_i, then it stays 1D till the end of the current 0-phase with probability $\exp(-(T_{N+1} - T_i))$, where N is the number of nodes lost from 1A.

On the other hand, if T is the duration of a 1-phase, then during this phase a node with state 1D does not change its state with probability e^{-T}. Some of the 1D nodes that change to 0A nodes could then also change to 1A nodes before the end of the 1 state. If we let $\{\sigma_i\}_{i \geq 0}$ denote the times when 1D nodes change to 0A where $\sigma_0 = 0$, then the number of 1D nodes that change to 1A is $\sum_{i=1}^{L - \tilde{L}} \mathrm{Ber}\left(v(T - \sigma_i)\right)$. Similarly using the $\{T_i\}_{i \geq 0}$ defined previously the number of 1A nodes that change to 1D and stay 1D until the end of a 0-phase is $\sum_{i=1}^{N} \mathrm{Ber}\left(\frac{\alpha}{1+\alpha}(T_{N+1} - T_i)\right)$

Using the above argument and the notation $u(t)$ and $v(t)$ as in Lemma 5.2, we see that the transition in the number of 1A and 1D states at the beginning of two consecutive

phase 1 states of the system can be described as follows.

$$
\begin{pmatrix} K \\ \\ L \end{pmatrix} \xrightarrow{\text{during 1-phase}} \begin{pmatrix} \tilde{K} = \text{Bin}(K, u(T)) + \text{Bin}(n - K - L, v(T)) \\ + \sum_{i=1}^{L-\tilde{L}} \text{Ber}\left(v(T - \sigma_i)\right) \\ \tilde{L} = \text{Bin}(L, e^{-T}) \end{pmatrix},
$$

$$
\begin{pmatrix} \tilde{K} \\ \tilde{L} \end{pmatrix} \xrightarrow{\text{during 0-phase}} \begin{pmatrix} \left(\tilde{K} - N\right)^+ \\ \text{Bin}(\tilde{L}, \exp(-T_{N+1})) + \sum_{i=1}^{N} \text{Ber}\left(\frac{\alpha}{1+\alpha}(T_{N+1} - T_i)\right) \end{pmatrix},
$$

where $T \sim \text{Exp}(1)$, $N \sim \text{Geom}(\hat{\lambda})$, $T_0 = 0$ and $(T_{i+1} - T_i) \sim \text{Exp}((\lambda + \alpha + 1)(\tilde{K} - i))$. Conditionally on $T, N, \{\sigma_i\}_{i \geq 0}$ and $\{T_i\}_{i \geq 0}$ all Binomial and Bernoulli random variables are independent.

In order to analyze the above Markov chain, let (K_i, L_i) be the number of 1A and 1D nodes at the beginning of the i-th one-phase. We would like to be able to ignore the 1D nodes and analyze K_i assuming that $L_i = 0$ for all i. To do this we first need to introduce some new processes.

If we assume transitions to 1D do not occur (so $L_i = 0$ for all i), then we obtain a new Markov chain $\{Z_i\}_{i=0}^\infty$, where Z_i is the number of 1A nodes at the start of the ith one-phase. The sequence $\{Z_i\}_{i=0}^\infty$ is defined by $Z_0 = n$ and for $i \geq 0$,

$$
Z_{i+1} = (X_i + Y_i - N_i)^+, \tag{5.4}
$$

where

$$
X_i \sim \text{Bin}(Z_i, u(T_i)), \quad Y_i \sim \text{Bin}(n - Z_i, v(T_i)),
$$
$$
T_i \sim \text{Exp}(1) \text{ and } \quad N_i \sim \text{shifted Geometric}(\hat{\lambda}) \text{ (as in (5.3))},
$$

where the coin flips involved in the Binomial expressions above are assumed to be conditionally independent of everything else given Z_i and T_i, and $\{T_i : i \geq 0\}$ and $\{N_i : i \geq 0\}$ are assumed to be i.i.d. sequences, independent of each other.

In order to justify studying the dynamics of Z_i rather than (K_i, L_i), we will show that there exists a good event G with probability going to 1 as $n \to \infty$ on which there exists a coupling such that

$$
Z_i^* \leq K_i \leq Z_i \tag{5.5}
$$

where $Z_i^* = Z_i - C^* \cdot (\log n)^3$ for all i and $C^* > 0$ is a constant, which will be chosen later to depend on α and λ.

In comparing Z_i and (K_i, L_i) we encounter two possible problems. First, we must establish that 1D vertices cannot accumulate in (K_i, L_i). Second, even if the number of 1D vertices is bounded above, there may still be a significant flow of vertices from state 1A to 1D to 0A, so we must establish that this drain of 1A vertices does not cause Z_i and K_i to drift too far apart. The good event G will ensure that neither of these happen.

To define G we first define another process $\{W_i\}_{i=0}^\infty$. Let $C > 0$ be a constant, and let $W_0 = n - C(\log n)^2$ and

$$
W_{i+1} = (X_i^W + Y_i^W - N_i^W)^+, \tag{5.6}
$$

where

$$
X_i^W \sim \text{Bin}(W_i - C(\log n)^2, u(T_i)),
$$
$$
Y_i^W \sim \text{Bin}(n - C(\log n)^2 - (W_i - C(\log n)^2), v(T_i)),
$$
$$
T_i \sim \text{Exp}(1),
$$
$$
\text{and } N_i^W \sim \text{shifted Geometric}(\hat{\lambda}) \text{ (as in (5.3))}.
$$

In essence $\{W_i\}_{i=0}^{\infty}$ has $C(\log n)^2$ vertices removed that can be thought of as being fixed as $1D$ and at the beginning of each one-phase $C(\log n)^2$ vertices are converted directly from state $1A$ to $0A$.

Lemma 5.3. *Let* $\lambda, \alpha, \gamma > 0$. *Then for any* $\epsilon > 0$, *there exist* $N > 0$ *such that*

1. $\mathbb{P}\left(\max_{i=1,\dots n^{\gamma}}(L_i) > (12\gamma + 1)\frac{(\lambda+\alpha)(\gamma+1)}{\alpha}(\log n)^2\right) < \epsilon$.

2. *Let* C *be the constant in the definition of the process* $\{W_i\}$, *and let* $R_{i,k}$ *be the number of the* $C(\log n)^2$ *vertices that were converted from* $1A$ *to* $0A$ *at the start of the* i*th one-phase that have not been reinfected by the start of the* k*th one-phase. Then*

$$\mathbb{P}\left(\max_{k=1\dots n^{\gamma}}\sum_{i=1}^{k} R_{i,k} \geq \left(\frac{2\gamma}{\log(\frac{1+\alpha+\lambda}{1+\lambda})} + 1\right)C(\log n)^3\right) < \epsilon \text{ for all } n \geq N.$$

Proof. In the process (K_i, L_i), we first observe that if J_i is the number of newly added $1D$ nodes during the i-th zero-phase, then J_i is stochastically dominated by a geometric distribution on $\{0, 1, \dots\}$ with success probability $\frac{\alpha}{\alpha+\lambda}$. Let A be the event $\{\max_{i=1,\dots,n^{\gamma}} J_i \leq \frac{(\lambda+\alpha)(\gamma+1)}{\alpha}\log n\}$. Then

$$\mathbb{P}(\mathsf{A}^c) \leq n^{\gamma}\frac{\lambda}{\lambda+\alpha}\left(\frac{1}{n}\right)^{\frac{(\lambda+\alpha)(\gamma+1)}{\alpha}\cdot\frac{\alpha}{\lambda+\alpha}} = o(1). \tag{5.7}$$

For the process W_i, the number of $1A$ nodes newly converted to $0A$ at the start of the ith one-phase, $R_{i,i}$, is deterministically $C(\log n)^2$.

For the process (K_i, L_i), if we ignore the conversion of $1D$ nodes to $0A$ nodes during zero-phases, then after the kth one-phase, the number of remaining $1D$ nodes from J_i is $J_{i,k} \sim \text{Bin}(J_i, \exp[-\sum_{\ell=i+1}^{k} T_{\ell}])$, where T_{ℓ}'s are iid $\text{Exp}(1)$ random variables. We now have

$$\sum_{i=1}^{k} J_{i,k} \succeq L_k. \tag{5.8}$$

In the case of $R_{i,k}$, a vertex that converted to $0A$ at the start of the ith one-phase becomes reinfected in a given future one-phase if the center attempts to infect it before either the center recovers or the vertex avoids the center. We can observe the probability that a vertex converted to $0A$ at the start of the ith one-phase becomes reinfected during the lth one-phase is $\frac{\lambda}{1+\alpha+\lambda}$.

Observe $\sum_{\ell=i+1}^{k} T_{\ell} \sim \text{Gamma}(k-i, 1)$, and so $\mathbb{P}\left(\sum_{\ell=i+1}^{k} T_{\ell} < (k-i)/2\right) \leq e^{-(k-i)/6}$. Let B$(k, k')$ be the event $\left\{\sum_{\ell=i+1}^{k} T_{\ell} \geq \frac{1}{2}(k-i) \text{ for all } 1 \leq i \leq k-k'\right\}$. Then for $0 \leq k' < k$, we have

$$\mathbb{P}(\mathsf{B}(k, k')^c) \leq (k-k')e^{-k'/6}, \tag{5.9}$$

and $\mathbb{P}(\mathsf{B}(k, k')^c) = 0$ if $k' \geq k$.

If we let $X_i \sim \text{Bin}(\frac{(\lambda+\alpha)(\gamma+1)}{\alpha}\log(n), e^{-(k-i)/2})$, then

$$J_{i,k}\mathbb{1}_{\mathsf{B}(k,k')\cap\mathsf{A}} \preceq X_i \text{ for all } i \leq k-k' \text{ and } k \leq n^{\gamma}, \tag{5.10}$$

Let D(k, k') be the event $\left\{J_{i,k}\mathbb{1}_{\mathsf{B}(k,k')\cap\mathsf{A}} = 0 \text{ for all } 1 \leq i \leq k-k'\right\}$. Then

$$\mathbb{P}(\mathsf{D}(k, k')^c) \leq \mathbb{P}(X_i > 0 \text{ for some } 1 \leq i \leq k-k')$$
$$\leq (k-k')(1-(1-e^{-k'/2})^{\frac{(\lambda+\alpha)(\gamma+1)}{\alpha}\log n}). \tag{5.11}$$

Observe that for each $1 \leq k \leq n^\gamma$,

$$\{\mathsf{A} \cap \mathsf{B}(k,k') \cap \mathsf{D}(k,k')\} \subseteq \left\{ \sum_{i=1}^{k} J_{i,k} \leq \frac{(\lambda+\alpha)(\gamma+1)}{\alpha}(\log n)k' \right\}. \tag{5.12}$$

To obtain part 1 of the Lemma, we choose $k' = (12\gamma+1)\log n$, so that the probabilities in (5.7), (5.9) and (5.11) are sufficiently small:

$$n^\gamma \left(\frac{1}{n}\right)^{\frac{(\lambda+\alpha)(\gamma+1)}{\alpha}\frac{\alpha}{\lambda+\alpha}} = o(1),$$

$$n^\gamma e^{-(12\gamma+1)\log n/6} = o(n^{-\gamma}),$$

$$\text{and} \quad n^\gamma(1 - (1 - e^{-(12\gamma+1)\log n})^{\frac{(\lambda+\alpha)(\gamma+1)}{\alpha}\log n}) = o(n^{-\gamma}). \tag{5.13}$$

Then by (5.8) and (5.13),

$$\mathbb{P}\left(\max_{i=1,\ldots n^\gamma} L_i > (12\gamma+1)\frac{(\lambda+\alpha)(\gamma+1)}{\alpha}(\log n)^2\right)$$

$$\leq \mathbb{P}\left(\bigcup_{k=1}^{n^\gamma}\{\mathsf{A} \cap \mathsf{B}(k,k') \cap \mathsf{D}(k,k')\}^c\right)$$

$$\leq \mathbb{P}(\mathsf{A}^c) + \sum_{k=1}^{n^\gamma}[\mathbb{P}(\mathsf{B}(k,k')^c) + \mathbb{P}(\mathsf{D}(k,k')^c)]$$

$$= o(1). \tag{5.14}$$

We now obtain part 2 of the Lemma by an analogous argument. Let $\mathsf{U}(k,k')$ be the event $\{R_{i,k} = 0 \text{ for all } 1 \leq i \leq k-k'\}$. First, note that for any $1 \leq i \leq k-k'$ we have

$$\mathbb{P}(R_{i,k} > 0) \leq C(\log n)^2\left(\frac{1+\alpha}{1+\alpha+\lambda}\right)^{k'}. \tag{5.15}$$

From this we see

$$\mathbb{P}(\mathsf{U}(k,k')^c) \leq (k-k')C(\log n)^2\left(\frac{1+\alpha}{1+\alpha+\lambda}\right)^{k'}, \tag{5.16}$$

so we can choose $k' = \left(\frac{2\gamma}{\log(\frac{1+\alpha+\lambda}{1+\lambda})}+1\right)\log n$ so that the probability in (5.16) is sufficiently small:

$$\sum_{k=1}^{n^\gamma}(k-k')C(\log n)^2\left(\frac{1+\alpha}{1+\alpha+\lambda}\right)^{k'} = o(1). \tag{5.17}$$

Finally, we observe

$$\mathbb{P}\left(\max_{k=1\ldots n^\gamma}\sum_{i=1}^{k} R_{i,k} \geq \left(\frac{2\gamma}{\log(\frac{1+\alpha+\lambda}{1+\lambda})}+1\right)C(\log n)^3\right) \leq \sum_{k=1}^{n^\gamma}\mathbb{P}(\mathsf{U}(k,k')^c) = o(1).$$

This completes the proof of the lemma. $\qquad\square$

Lemma 5.4. *Fix $\epsilon, \gamma > 0$, let G be the event*

$$\left\{\sum_{i=1}^{k} R_{ik} \leq C^*(\log n)^3 \ \forall k \leq n^\gamma\right\} \cap \left\{\max_{i=1,\ldots n^\gamma}(L_i) \leq C(\log n)^2\right\}$$

where $C = (12\gamma+1)\frac{(\lambda+\alpha)(\gamma+1)}{\alpha}$ in the definition of $\{W_i\}$ and $C^* = \left(\frac{2\gamma}{\log(\frac{1+\alpha+\lambda}{1+\lambda})}+1\right)C$ in the definition of $\{Z_i^*\}$, and let $\tau^* = \inf\{i \geq 0 : Z_i^* = 0\}$. Then for all sufficiently large n, we have $\mathbb{P}(\mathsf{G}) > 1 - \epsilon$. In addition

$$Z_i^* \mathbb{1}_\mathsf{G} \preceq K_i \mathbb{1}_\mathsf{G} \qquad \text{for } 0 \leq i \leq \tau^*, \text{ and}$$
$$K_i \mathbb{1}_\mathsf{G} \preceq Z_i \mathbb{1}_\mathsf{G} \qquad \text{for } 0 \leq i \leq \tau. \tag{5.18}$$

Proof. Lemma 5.3 implies $\mathbb{P}(\mathsf{G}) > 1 - \epsilon$.

We first describe a coupling between $W_i \mathbb{1}_\mathsf{G}$ and $Z_i \mathbb{1}_\mathsf{G}$ that holds for $0 \leq i \leq \tau_W$. Begin by expanding the probability space in the usual way so that we can track the states of individual vertices. Next, note that $W_i \mathbb{1}_\mathsf{G}$ and $Z_i \mathbb{1}_\mathsf{G}$ are embedded discrete time processes of $Z_t \mathbb{1}_\mathsf{G}$ and $W_t \mathbb{1}_\mathsf{G}$ where Z_t follows the process dynamics in Definition 5.1 except that it ignores transitions to the $1D$ state and W_t follows the same process dynamics, also ignores transitions to the $1D$ state, and converts $C(\log n)^2$ randomly chosen $1A$ vertices to $0A$ vertices instantaneously at the start of each one-phase. We describe a coupling for $Z_t \mathbb{1}_\mathsf{G}$ and $W_t \mathbb{1}_\mathsf{G}$ during the one-phase as follows:

1. At the start of each one-phase, pair every $1A$ vertex in $W_t \mathbb{1}_\mathsf{G}$ with a $1A$ vertex in $Z_t \mathbb{1}_\mathsf{G}$ and as many $0A$ vertices in $Z_t \mathbb{1}_\mathsf{G}$ with $0A$ vertices in $W_t \mathbb{1}_\mathsf{G}$ as possible. Paired vertices share all random variables that determine their possible state changes, and unpaired vertices evolve independently according to their marginals.

2. During a one-phase, whenever an unpaired $0A$ vertex in $W_t \mathbb{1}_\mathsf{G}$ becomes infected, pair it with an unpaired $1A$ vertex in $Z_t \mathbb{1}_\mathsf{G}$. Whenever an unpaired 1-state (infected) vertex in $Z_t \mathbb{1}_\mathsf{G}$ recovers, pair it with an unpaired $0A$ vertex in $W_t \mathbb{1}_\mathsf{G}$ if one exists.

3. During the zero-phase, instead couple the embedded discrete time processes $W_i \mathbb{1}_\mathsf{G}$ and $Z_i \mathbb{1}_\mathsf{G}$ by drawing a single N_i to determine the number of $1A$ vertices that recover and distributing those recoveries uniformly at random among the available $1A$ vertices in each process.

First observe that we have $W_t \mathbb{1}_\mathsf{G} \leq Z_t \mathbb{1}_\mathsf{G}$ which implies $W_i \mathbb{1}_\mathsf{G} \leq Z_i \mathbb{1}_\mathsf{G}$. Now note that in this coupling any vertices that are healthy in $W_i \mathbb{1}_\mathsf{G}$ but infected in $Z_i \mathbb{1}_\mathsf{G}$ must be vertices in the W_i process that were converted from $1A$ to $0A$ at the start of a one-phase and have never since been reinfected. Thus when G occurs, $Z_i - W_i \leq C^*(\log n)^3$, and so $Z_i \mathbb{1}_\mathsf{G} - W_i \mathbb{1}_\mathsf{G}$ is bounded above by $C^*(\log n)^3$, and so $Z_i^* \mathbb{1}_\mathsf{G} \leq W_i \mathbb{1}_\mathsf{G}$ for all $0 \leq i \leq \tau^*$. We also note that $Z_{\tau^*} \mathbb{1}_\mathsf{G} \leq C^*(\log n)^3$

Now define $\tau := \inf\{i \geq 0 : Z_i = 0\}$, and observe that $K_i \preceq Z_i$ for $0 \leq i \leq \tau$. Furthermore, for $0 \leq i \leq \tau^*$ we can couple $K_i \mathbb{1}_\mathsf{G}$ and $W_i \mathbb{1}_\mathsf{G}$ using the same coupling as for $W_i \mathbb{1}_\mathsf{G}$ and $Z_i \mathbb{1}_\mathsf{G}$ with the added stipulation that $1D$ vertices in $K_i \mathbb{1}_\mathsf{G}$ behave independently according to their marginals. When G occurs, the number of vertices that are not $1A$ in K_i because they are $1D$ is less than the number of removed vertices in W_i and the number of $1A$ vertices that change to $0A$ by first passing through the $1D$ state in K_i is less the number $1A$ vertices that W_i converts to $0A$ at the start of each one-phase. Thus in this coupling we have

$$Z_i^* \mathbb{1}_\mathsf{G} \leq W_i \mathbb{1}_\mathsf{G} \leq K_i \mathbb{1}_\mathsf{G} \text{ for } 0 \leq i \leq \tau^* \tag{5.19}$$

and so we conclude

$$Z_i^* \mathbb{1}_\mathsf{G} \preceq K_i \mathbb{1}_\mathsf{G} \text{ for } 0 \leq i \leq \tau^*, \tag{5.20}$$

and

$$K_i \mathbb{1}_\mathsf{G} \preceq Z_i \mathbb{1}_\mathsf{G} \text{ for } 0 \leq i \leq \tau. \tag{5.21}$$

$\square$

Since Z_i and Z_i^* differ by at most $C^*(\log n)^3$, we can now derive upper and lower bounds on τ, which, when combined with this coupling, will yield upper and lower bounds on τ_{star}.

We first consider the upper bound on τ. For this we need the following lemma about the transition probabilities of Z_i. The intuition is as follows: if a one-phase lasts for a long time, then the properties of $u(t)$ and $v(t)$ in Lemma 5.2 allow us to bound from below the probability that the entire process dies before the next one-phase.

Lemma 5.5. *For $k, l \geq 1$ if $p(k, l) := \mathbb{P}\left(Z_{i+1} = l | Z_i = k\right)$, then for any $\eta \in (0, 1]$ if $C_1 = e\hat{\lambda}(\alpha - \gamma_1)/(\gamma_2 - \gamma_1)$, then $p(\eta n, 0) \geq (1 + o(1))(C_1 n)^{-1/\gamma_1}$.*

Proof. From the definition of the Markov chain $\{Z_i\}_{t \leq \tau}$ it is easy to see that

$$
\begin{aligned}
p(k, l) &:= \mathbb{E}_T \mathbb{E}_{X,Y|T} \mathbb{P}\left(N = X + Y - l | T, X, Y\right) \\
&= \mathbb{E}_T \mathbb{E}_{X,Y|T} \hat{\lambda}(1 - \hat{\lambda})^{X+Y-l} \mathbf{1}_{X+Y \geq l}, \\
p(k, 0) &:= \sum_{l \leq 0} \mathbb{E}_T \mathbb{E}_{X,Y|T} \mathbb{P}\left(N = X + Y - l | T, X, Y\right) \\
&= \sum_{l \leq 0} \mathbb{E}_T \mathbb{E}_{X,Y|T} \hat{\lambda}(1 - \hat{\lambda})^{X+Y-l}.
\end{aligned}
$$

So, using the fact that

$$
E\left[s^{\mathrm{Bin}(k,p)}\right] = (1 - p(1 - s))^k \text{ for } s \in [0, 1], \tag{5.22}
$$

and writing $k = \eta n$,

$$
\begin{aligned}
p(\eta n, 0) &= \mathbb{E}_T \left[1 - \hat{\lambda}u(T)\right]^{\eta n} \left[1 - \hat{\lambda}v(T)\right]^{(1-\eta)n} \\
&= \int_0^\infty \left[(1 - \hat{\lambda}u(t))^\eta (1 - \hat{\lambda}v(t))^{1-\eta}\right]^n e^{-t} \, dt.
\end{aligned}
$$

To bound the above integral from below, let

$$
t_\epsilon = \frac{1}{\gamma_1} \log \frac{1}{\epsilon} \text{ be so that } \exp(-\gamma_1 t_\epsilon) = \epsilon.
$$

From property 1. and 3. of Lemma 5.2,

$$
p(\eta n, 0) \geq \int_{t_\epsilon}^\infty (1 - \hat{\lambda}u(t))^n e^{-t} \, dt \geq (1 - \hat{\lambda}u(t_\epsilon))^n \exp(-t_\epsilon) = (1 - c_1\epsilon - c_2\epsilon^{\gamma_2/\gamma_1})^n \epsilon^{1/\gamma_1},
$$

where $c_1 = \hat{\lambda}(\gamma_2 - 1)/(\gamma_2 - \gamma_1)$ and $c_2 = \hat{\lambda}(1 - \gamma_1)/(\gamma_2 - \gamma_1)$. Since $\gamma_2 > \gamma_1$, we ignore $\epsilon^{\gamma_2/\gamma_1}$ term and choose ϵ to maximize $(1 - c_1\epsilon)^n \epsilon^{1/\gamma_1}$. In order to do that, we set the derivative of the $\log[(1 - c_1\epsilon)^n \epsilon^{1/\gamma_1}]$ with respect to ϵ to 0 to have

$$
n\frac{c_1}{1 - c_1\epsilon} = \frac{1}{\gamma_1 \epsilon}, \text{ which gives } \epsilon = (c_1 + c_1\gamma_1 n)^{-1}.
$$

Plugging this value of ϵ,

$$
\begin{aligned}
p(\eta n, 0) &\geq \left[1 - (1 + \gamma_1 n)^{-1} - c_2(c_1 + c_1\gamma_1 n)^{-\gamma_2/\gamma_1}\right]^n (c_1 + c_1\gamma_1 n)^{-1/\gamma_1} \\
&= (c_1 e \gamma_1 n)^{-1/\gamma_1}(1 + o(1)). \tag{5.23}
\end{aligned}
$$

$\square$

We can now prove the upper bound for τ.

Proposition 5.6. *For the Markov chain $\{Z_i\}$ suppose $\tau = \inf\{t \geq 0 : Z_i = 0\}$. Fix $\epsilon > 0$. Then there exist constants N and C depending on λ and α such that for all $n \geq N$,*

$$
\mathbb{P}\left(\tau \leq Cn^{1/\gamma_1}\right) > 1 - \epsilon.
$$

Proof. Fix $\eta_0 \in (0, \hat{\lambda})$ and let C_1 be the constant in Lemma 5.5. From part 1 of Lemma 5.5, τ is stochastically dominated by a Geometric random variable with success probability $(1 + o(1))(C_1 n)^{-1/\gamma_1}$. Hence, for $k \geq 1$,

$$\mathbb{P}\left(\tau > k(C_1 n)^{1/\gamma_1}\right) \leq \left[1 - (1 + o(1))(C_1 n)^{-1/\gamma_1}\right]^{k(C_1 n)^{1/\gamma_1}} \leq e^{-(1+o(1))k}.$$

Now choose N so that the $o(1)$ term in the exponent is smaller than 1 for all $n \geq N$. Choosing k sufficiently large, and setting $C = kC_1^{1/\gamma_1}$ completes the proof. $\square$

Next we consider the lower bound on τ. Lemma 5.7 complements Lemma 5.5 by providing a matching-order upper bound on the probability of the infection dying during a one-phase. Lemma 5.8 will imply that the infection is exponentially unlikely (in the number of infected leaves) to die out in the zero-phase.

Lemma 5.7. *For $k, l \geq 1$ if $p(k, l) := \mathbb{P}(Z_{i+1} = l | Z_i = k)$ and $p(k, \leq l) := \sum_{l' \leq l} p(k, l')$, then for any $\epsilon, \eta_0 > 0$ satisfying*

$$\frac{2\epsilon}{\hat{\lambda}} \log \frac{1}{1 - \hat{\lambda}} \leq \eta_0 < \hat{\lambda},$$

there is a constant $C_2 = (1/\hat{\lambda}\eta_0) \log(1/(1 - \hat{\lambda})) > 0$ such that

$$p(\eta n, \leq \epsilon n) \leq 3(C_2 \epsilon)^{1/\gamma_1} \text{ for any } \eta \geq \eta_0.$$

Proof. Suppose s_ϵ is such that

$$\eta_0 \exp(-\gamma_1 s_\epsilon) = \frac{2\epsilon}{\hat{\lambda}} \log \frac{1}{1 - \hat{\lambda}}. \tag{5.24}$$

Then $s_\epsilon \in (0, \infty)$ by our hypothesis about η_0.

Now, it can be checked that the coefficient of $e^{-\gamma_2 t}$ in $\eta u(t) + (1 - \eta)v(t)$ is negative for $\eta < \lambda/(\lambda + 1 - \gamma_1)$. Hence, the coefficient of $e^{-\gamma_2 t}$ in $\eta_0 u(t) + (1 - \eta_0)v(t)$ is negative, as $\eta_0 < \lambda/(\lambda + \alpha + 1)$. So using the inequality $\gamma_2 > \gamma_1$, we get $\eta_0 u(t) + (1 - \eta_0)v(t) > \eta_0 e^{-\gamma_1 t}$. Combining this with (5.24) and the fact that $\eta \mapsto \eta u(t) + (1 - \eta)v(t)$ is increasing in η (by property 1. of Lemma 5.2),

$$\eta u(t) + (1 - \eta)v(t) \geq \frac{2\epsilon}{\hat{\lambda}} \log \frac{1}{1 - \hat{\lambda}} \text{ for any } \eta \geq \eta_0 \text{ and } t \leq s_\epsilon. \tag{5.25}$$

Now note that

$$p(\eta n, \leq \epsilon n) = \mathbb{E}_T \mathbb{E}_{X,Y|T}(1 - \hat{\lambda})^{(X+Y-\epsilon n)^+} = \int_0^\infty e^{-t} \mathbb{E}_{X,Y|T=t}(1 - \hat{\lambda})^{(X+Y-\epsilon n)^+} dt.$$

Let A be the event $\{X + Y \geq \epsilon n\}$. Then the quantity inside the expectation equals $(1 - \hat{\lambda})^{X+Y-\epsilon n} + \mathbf{1}_{A^c}$. Then, splitting the integral in the last display into two parts based on whether $t < s_\epsilon$ or not and using the fact that the integrand is almost 1, we get

$$p(\eta n, \epsilon n) \leq \int_0^{s_\epsilon} e^{-t} \mathbb{E}_{X,Y|T=t}[(1 - \hat{\lambda})^{X+Y-\epsilon n} + \mathbf{1}_{A^c}] dt + \exp(-s_\epsilon).$$

Using Markov inequality

$$\mathbb{E}_{X,Y|T=t}\mathbf{1}_{A^c} \leq \mathbb{E}_{X,Y|T=t}(1 - \hat{\lambda})^{X+Y-\epsilon n}.$$

Also using (5.22) and the inequality $1 - x \leq e^{-x}$,

$$\mathbb{E}_{X,Y|T=t}(1 - \hat{\lambda})^{X+Y-\epsilon n} \leq (1 - \hat{\lambda})^{-\epsilon n} \exp\left[-\hat{\lambda}n(\eta u(t) + (1 - \eta)v(t)\right].$$

Combining the last three displays and using (5.25),

$$
\begin{aligned}
p(\eta n, \epsilon n) &\le 2 \int_0^{s_\epsilon} e^{-t} (1 - \hat\lambda)^{-\epsilon n} \exp\left[-\hat\lambda n(\eta u(t) + (1 - \eta)v(t)) \right] \, dt + \exp(-s_\epsilon) \\
&\le 2 \int_0^{s_\epsilon} e^{-t} (1 - \hat\lambda)^{\epsilon n} \, dt + \exp(-s_\epsilon) \\
&\le 2(1 - \hat\lambda)^{\epsilon n} + \exp(-s_\epsilon).
\end{aligned}
$$

From (5.24), $\exp(-s_\epsilon) = (c\epsilon)^{1/\gamma_1}$ for an appropriate constant c. This proves the assertion. $\qquad\square$

Lemma 5.8. *Suppose $\eta_0 \in (0, \hat\lambda)$ and $\tilde\tau := \inf\{t \ge 0 : n^{-1} Z_i \notin (0, \eta_0)\}$. Then there is a $\vartheta > 0$ such that $U_t := \exp(-\vartheta Z_{t \wedge \tilde\tau})$ is a supermartingale.*

Proof. Suppose $Z_0 = \eta n$ for some $\eta \in (0, \eta_0)$. Define

$$
\varphi_\eta(\vartheta) := \left[\mathbb{E} \left(\exp(-\vartheta Z_1) \mid Z_0 = \eta n \right) \right]^{1/n} - e^{-\vartheta\eta}.
$$

Clearly $\varphi_\eta(0) = 0$ and $\varphi_\eta \in C^1[0, \infty)$ for any $\eta > 0$. We will show

$$
(a) \ \ \varphi'_{\eta_0}(0) < 0 \quad \text{and} \quad (b) \ \ \varphi'_\eta(0) \text{ is an increasing function of } \eta. \tag{5.26}
$$

Using continuity of φ'_{η_0} (a) will imply that there exists $\vartheta > 0$ such that $\varphi'_{\eta_0}(\beta) < 0$ for all $\beta \in [0, \vartheta]$. Also using the mean value theorem, $\varphi_\eta(\vartheta) = \varphi'_\eta(\beta_0)\vartheta$ for some $\beta_0 \in [0, \vartheta]$. Then (b) will imply $\varphi'_\eta(\beta_0) \le \varphi'_{\eta_0}(\beta_0) < 0$ for $\eta \le \eta_0$, which in turn implies $\varphi_\eta(\vartheta) < 0$ for $\eta \le \eta_0$.

In order to show (5.26) we will find an expression for $\varphi_\eta(\vartheta)$. Clearly,

$$
\varphi_\eta(\vartheta) = \left[\mathbb{E}_T \mathbb{E}_{X,Y|T} \mathbb{E}_{N|X,Y,T} \exp(-\vartheta(X + Y - N)^+) \right]^{1/n} - e^{-\vartheta\eta},
$$

where $T \sim \mathrm{Exp}(1)$, given $T = t$ $X \sim \mathrm{Bin}(\eta n, u(t))$, $Y \sim \mathrm{Bin}((1-\eta)n, v(t))$ and N is as in (5.3). Now

$$
\begin{aligned}
\mathbb{E}_{N|X,Y,T} &\exp(-\vartheta(X + Y - N)^+) \\
&= \mathbb{E}_{N|X,Y,T}[\exp(-\vartheta(X+Y-N))\mathbf{1}_{\{N < X+Y\}}] + \mathbb{E}_{N|X,Y,T}\mathbf{1}_{\{N \ge X+Y\}} \\
&= e^{-\vartheta(X+Y)} \sum_{j=0}^{X+Y-1} \hat\lambda[e^\vartheta(1-\hat\lambda)]^j + (1-\hat\lambda)^{X+Y} \\
&= \frac{\hat\lambda}{1 - e^\vartheta(1-\hat\lambda)} \left[e^{-\vartheta(X+Y)} - (1-\hat\lambda)^{X+Y} \right] + (1-\hat\lambda)^{X+Y}. \tag{5.27}
\end{aligned}
$$

Therefore, using (5.22)

$$
\begin{aligned}
\varphi_\eta(\vartheta) = &\left(\int_0^\infty e^{-t} \left[\frac{\hat\lambda}{1 - e^\vartheta(1-\hat\lambda)} \left(\chi(\eta, e^{-\vartheta}, t) - \chi(\eta, 1 - \hat\lambda, t) \right) + \chi(\eta, 1 - \hat\lambda, t) \right] dt \right)^{1/n} \\
&- e^{-\vartheta\eta},
\end{aligned}
$$

where

$$
\chi(\eta, s, t) := \left[(1 - (1-s)u(t))^\eta (1 - (1-s)v(t))^{1-\eta} \right]^n.
$$

Since $\chi(\eta, 1, t) = 1$,

$$
\varphi'_\eta(0) = \frac{1}{n} \int_0^\infty e^{-t} \left(\frac{1 - \hat\lambda}{\hat\lambda}[1 - \chi(\eta, 1 - \hat\lambda, t)] + \frac{d}{d\vartheta} \chi(\eta, e^{-\vartheta}, t) \Big|_{\vartheta=0} \right) dt + \eta.
$$

The first integrand is an increasing function of η, as $u(t) > v(t)$ by property 3. of Lemma 5.2. On the other hand, the second integrand is $n(-\eta u(t) - (1-\eta)v(t))$, and hence (b) of (5.26) holds by property 4. of Lemma 5.2. Also the first integrand is at most 1, so

$$\varphi'_{\eta_0}(0) \leq \frac{1}{n} - \int_0^\infty e^{-t}(\eta_0 u(t) + (1-\eta_0)v(t))\, dt + \eta < 0$$

using property 4. of Lemma 5.2 and the fact that $\eta_0 < \hat{\lambda}$. This proves (a) of (5.26) and proof of the lemma is complete. □

We can now prove the lower bound on τ.

Proposition 5.9. *For the Markov chain $\{Z_i\}$ suppose $\tau = \inf\{t \geq 0 : Z_i = 0\}$. Fix $\epsilon > 0$. Then there exist N, K_o such that for all $n \geq n$,*

$$\mathbb{P}\left(\frac{1}{K_0}\left(\frac{n}{\log(n)^4}\right)^{1/\gamma_1} \leq \tau \right) > 1 - \epsilon$$

Proof. Let C_2 and ϑ be the constants in Lemmas 5.7 and 5.8. Divide the interval $[0, n]$ into three parts

$$I_1 := [0, (\gamma_1\vartheta)^{-1}C(\log n)^4), \quad I_2 := [(\gamma_1\vartheta)^{-1}C(\log n)^4, \eta_0 n], \quad I_3 := (\eta_0 n, n]$$

and note that so long as Z_i is in I_2 or I_3 then the process Z_i^* defined in lemma 5.4 is greater than 0. Using $\epsilon = C\log(n)^4/(\gamma_1\vartheta n)$ in Lemma 5.7, it is easy to see that the number of times Z_i avoids jumping from I_3 to I_1 stochastically dominates a Geometric random variable with success probability $C(\log(n)^4/n)^{1/\gamma_1}$ for some constant $C > 0$.

Also, if $Z_0 \in I_2$, then applying the optional stopping theorem for the stopping time

$$\tilde{\tau} := \inf\{t \geq 0 : Z_i \notin (0, \eta_0 n)\}, \text{ and supermartingale } U_t := \exp(-\vartheta Z_t), 0 \leq t \leq \tilde{\tau},$$

we see that if $q := \mathbb{P}\left(Z_{\tilde{\tau}} = 0\right)$, then

$$q \leq \mathbb{E}\, U_{\tilde{\tau}} \leq U_0 \leq n^{-1/\gamma_1}.$$

So, the number of times Z_i jumps from I_2 to I_3 stochastically dominates a Geometric random variable with success probability n^{-1/γ_1}. Combining these two observations, τ stochastically dominates sum of two Geometric random variables with success probability $C(\log(n)^4/n)^{1/\gamma_1}$. Hence

$$\mathbb{P}\left(\tau < K^{-1}(n/\log(n)^4)^{1/\gamma_1}\right) \leq 2\left(1 - \left[1 - C\left(\log(n)^4/n\right)^{1/\gamma_1}\right]\right)^{(n/\log(n)^4)^{1/\gamma_1}/(2K)}$$
$$\leq C/K \to 0 \tag{5.28}$$

as $K \to \infty$. □

We now are ready to finish the proof of Theorem 1.5.

Proof of Theorem 1.5. Propositions 5.6 and 5.9 give bounds on τ and so it remains to compare τ and τ_{star}.

Let $\tau_K = \inf\{t \geq 0 : K_t = 0\}$ for the Markov chain (K_i, L_i) (without assuming $L_i = 0$ for all i). From Lemmas 5.3 and 5.4, the good event G has probability at least $1 - \epsilon$, $Z_i^* \mathbb{1}_G \preceq K_i \mathbb{1}_G \preceq Z_i \mathbb{1}_G$ for $0 \leq i \leq \tau^*$, and $K_i \preceq Z_i$ for $0 \leq i \leq \tau$. Furthermore, from Proposition (5.9), we see that $Z_i^* > 0$ so long as Z_i is not in the interval I_1. From this, we observe that the same holds for K_i and that we do not reach τ^* until Z_i jumps to

the interval I_1, and so it is sufficient for the coupling to hold until time τ^* for the lower bound. Thus we can conclude

$$
\begin{aligned}
\mathbb{P} &\left(\frac{1}{K_0} \left(\frac{n}{\log(n)^4} \right)^{1/\gamma_1} \leq \tau_K \leq K_0 (C_1 n)^{1/\gamma_1} \right) \\
&\geq \mathbb{P} \left(\left\{ \frac{1}{K_0} \left(\frac{n}{\log(n)^4} \right)^{1/\gamma_1} \leq \tau \leq K_0 (C_1 n)^{1/\gamma_1} \right\} \bigcap \{ \mathsf{G}^c \} \right) \\
&\geq 1 - \mathbb{P} \left(\left\{ \frac{1}{K_0} \left(\frac{n}{\log(n)^4} \right)^{1/\gamma_1} \leq \tau \leq K_0 (C_1 n)^{1/\gamma_1} \right\}^c \right) - \mathbb{P} \left(\mathsf{G}^c \right) \\
&> 1 - 2\epsilon.
\end{aligned}
\tag{5.29}
$$

However, τ_K counts the number of one-phases until the infection dies, and so we must bound the total amount of time this takes. The length of a one-phase is an $\text{Exp}(1)$ random variable and does not depend on the states of the vertices other than the center, so clearly $\sum_{i=1}^{\tau_K} X_i \preceq \tau_{star}$ where the X_i are iid $\text{Exp}(1)$ random variables gives a lower bound on the time to extinction. The distribution of the length of a zero-phase depends on the number of 1A nodes present at the start of the zero-phase. However, the length of a zero-phase is dominated by an $\text{Exp}(\min(\lambda, 1))$ random variable for any configuration. τ_K gives the number of zero-phases before the process dies, and let Z be the length of the last zero-phase. Then $\tau_{star} \preceq \sum_{i=1}^{\tau_K} (X_i + Y_i)$ where the X_i are as before and the Y_i are iid $\text{Exp}(\min(\lambda, 1))$ random variables gives an upper bound.

Using large deviation bounds for all m we have

$$
\mathbb{P} \left(\sum_{i=1}^m X_i \leq m/2 \right) \leq e^{-m/6},
$$

$$
\mathbb{P} \left(\sum_{i=1}^m X_i + Y_i \geq 2m(1 + \frac{1}{\min(\lambda, 1)}) \right) \leq 2e^{-m/\frac{6}{\min(\lambda, 1)}}.
$$

So then to get a lower bound we observe

$$
\begin{aligned}
\mathbb{P} \left(\tau_{star} \geq \tau_K/2 \right) &\geq \mathbb{P} \left(\sum_{i=1}^{\tau_K} X_i \geq \tau_K/2 \right) \\
&\geq \mathbb{P} \left(\{ \tau_K \geq M \} \bigcap \{ \sum_{i=1}^{\tau_K} X_i \geq \tau_K/2 \} \right) \\
&\geq \mathbb{P} \left(\{ \tau_K \geq M \} \bigcap \{ \sum_{i=1}^m X_i \geq m/2 \ \forall m \geq M \} \right) \\
&\geq 1 - \mathbb{P} \left(\tau_K < M \right) - \sum_{m=M}^\infty \mathbb{P} \left(\sum_{i=1}^m X_i \geq m/2 \right) \\
&= 1 - \mathbb{P} \left(\tau_K < M \right) - \frac{e^{(1-M)/6}}{e^{1/6} - 1}
\end{aligned}
\tag{5.30}
$$

Choosing $M = \frac{1}{K_0} \left(\frac{n}{\log(n)^4} \right)^{1/\gamma_1}$ and using (5.29), we conclude that there exists N such that for all $n \geq N$

$$
\mathbb{P} \left(\tau_{star} \geq \tau_K/2 \right) \geq 1 - 2\epsilon,
\tag{5.31}
$$

giving a lower bound on the survival time in terms of the number of cycles

To get an upper bound we observe

$$\mathbb{P}\left(\tau_{star} \le 2(1 + \frac{1}{\min(\lambda, 1)})\tau_K\right) \ge \mathbb{P}\left(\sum_{i=1}^{\tau_K} X_i + Y_i \le 2(1 + \frac{1}{\min(\lambda, 1)})\tau_K\right)$$

$$\ge \mathbb{P}\left(\{\tau_K \ge M\}\bigcap\{\sum_{i=1}^{\tau_K} X_i + Y_i \le 2(1 + \frac{1}{\min(\lambda, 1)})\tau_K\}\right)$$

$$\ge \mathbb{P}\left(\{\tau_K \ge M\}\bigcap\{\sum_{i=1}^{m} X_i + Y_i \le 2(1 + \frac{1}{\min(\lambda, 1)})m \ \ \forall m \ge M\}\right)$$

$$\ge 1 - \mathbb{P}\left(\tau_K < M\right) - \sum_{j=M}^{\infty} \mathbb{P}\left(\sum_{i=1}^{m} X_i + Y_i \le 2(1 + \frac{1}{\min(\lambda, 1)})m\right)$$

$$= 1 - \mathbb{P}\left(\tau_K < M\right) + \frac{e^{(1-M)/\frac{6}{\min(\lambda,1)}}}{e^{1/(\min(\lambda,1))} - 1} \tag{5.32}$$

Again choosing $M = \frac{1}{K_0}(\frac{n}{\log(n)^4})^{1/\gamma_1}$ and using (5.29), we conclude that there exists N such that for all $n \ge N$

$$\mathbb{P}\left(\tau_{star} \le 2(1 + \frac{1}{\min(\lambda, 1)})\tau_K\right) \ge 1 - 2\epsilon, \tag{5.33}$$

giving an upper bound on the survival time in terms of the number of cycles.

Combining our comparison of τ and τ_K with our comparison of τ_K and τ_{star}, we conclude for any $\lambda, \alpha > 0$ and any $\epsilon > 0$, there exist C, K, and N depending on λ and α such that for all $n \ge N$,

$$\mathbb{P}\left(\frac{1}{K}(\frac{n}{\log(n)^4})^{1/\gamma} \le \tau_{star} \le K(Cn)^{1/\gamma}\right) > 1 - \epsilon. \qquad \square$$

References

[1] Noam Berger, Christian Borgs, Jennifer Chayes, and Amin Saberi, *On the spread of viruses on the internet*, Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms (New York), ACM, 2005, pp. 301–310. MR2298278

[2] Shankar Bhamidi, Danny Nam, Oanh Nguyen, and Allan Sly, *Survival and extinction of epidemics on random graphs with general degrees*, Ann. Probab. **49** (2021), no. 1, 244–286. MR4203338

[3] Van Hao Can, *Metastability for the contact process on the preferential attachment graph*, Internet Math. (2017), 45pp. MR3683430

[4] Van Hao Can and Bruno Schapira, *Metastability for the contact process on the configuration model with infinite mean degree*, Electron. J. Probab. **20** (2015), no. 26, 22pp. MR3325096

[5] Shirshendu Chatterjee and Rick Durrett, *Contact processes on random graphs with power law degree distributions have critical value 0*, Ann. Probab. **37** (2009), no. 6, 2332–2356. MR2573560

[6] Rick Durrett and Claudia Neuhauser, *Epidemics with recovery in $d = 2$*, Ann. Appl. Probab. **1** (1991), no. 2, 189–206. MR1102316

[7] Rick Durrett and Roberto H. Schonmann, *Large deviations for the contact process and two dimensional percolation*, Probab. Theory Related Fields **77** (1988), no. 4, 583–603. MR0933991

[8] Richard Durrett, *Oriented percolation in two dimensions*, Ann. Probab. **12** (1984), no. 4, 999–1040. MR0757768

[9] Eric Foxall, *The SEIS model, or, the contact process with a latent stage*, J. Appl. Probab. **53** (2016), no. 3, 783–801. MR3570094

[10] Thilo Gross and Bernd Blasius, *Adaptive coevolutionary networks: a review*, J. R. Soc. Interface **5** (2008), no. 20, 259–271.

[11] Thilo Gross, Carlos J. Dommar D'Lima, and Bernd Blasius, *Epidemic dynamics on an adaptive network*, Phys. Rev. Lett. **96** (2006), 208701.

[12] Dongchao Guo, Stojan Trajanovski, Ruud Bovenkamp, Huijuan Wang, and Piet Van Mieghem, *Epidemic threshold and topological structure of susceptible-infectious-susceptible epidemics in adaptive networks*, Phys. Rev. E **88** (2013), 042802.

[13] Xiangying Huang and Rick Durrett, *The contact process on random graphs and galton–watson trees*, ALEA Lat. Am. J. Probab. Math. Stat., **17** (2020), 159–182. MR4057187

[14] Emmanuel Jacob and Peter Mörters, *The contact process on scale-free networks evolving by vertex updating*, R. Soc. Open Sci. **4** (2017), no. 5, 14pp. MR3666161

[15] Karly A. Jacobsen, Mark G. Burch, Joseph H. Tien, and Grzegorz A. Rempala, *The large graph limit of a stochastic epidemic model on a dynamic multilayer network*, J. Biol. Dyn. **12**, (2016), no 1, 746–788. MR3851224

[16] Yufeng Jiang, Remy Kassem, Grayson York, Matthew Junge, and Rick Durrett, *SIR epidemics on evolving graphs*, arXiv:1901.06568, (2019).

[17] T. M. Liggett, R. H. Schonmann, and A. M. Stacey, *Domination by product measures*, Ann. Probab. **25** (1997), no. 1, 71–95. MR1428500

[18] T.M. Liggett, *Stochastic interacting systems: Contact, voter and exclusion processes*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Springer, 1999. MR1717346

[19] Thomas Mountford, Jean-Christophe Mourrat, Daniel Valesin, and Qiang Yao, *Exponential extinction time of the contact process on finite graphs*, Stochastic Process. Appl. **126** (2016), no. 7, 1974–2013. MR3483744

[20] Thomas Mountford, Daniel Valesin, and Qiang Yao, *Metastable densities for the contact process on power law random graphs*, Electron. J. Probab. **18** (2013), 36 pp. MR3145050

[21] Robin Pemantle, *The contact process on trees*, Ann. Probab. **20** (1992), no. 4, 2089–2116. MR1188054

[22] Daniel Remenik, *The contact process in a dynamic random environment*, Ann. Appl. Probab. **18** (2008), no. 6, 28pp. MR2474541

[23] David Sivakoff, *Contact process on a graph with communities*, ALEA Lat. Am. J. Probab. Math. Stat. **14** (2017), 9–31. MR3612320

[24] András Szabó-Solticzky, Luc Berthouze, Istvan Z Kiss, and Péter L Simon, *Oscillating epidemics in a dynamic network model: stochastic and mean-field analysis*, J. Math. Biol. **72** (2016), no. 5, 1153–1176. MR3464198

[25] Ilker Tunc, Maxim S. Shkarayev, and Leah B. Shaw, *Epidemics in adaptive social networks with temporary link deactivation*, J. Stat. Phys. **151** (2013), no. 1–2, 355–366. MR3045823

[26] Achillefs Tzioufas, *Oriented percolation with density close to one*, arXiv:1311.2952v4 (2014).