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Logarithmic Sobolev inequalities for Dunkl operators with applications to functional inequalities for singular Boltzmann-Gibbs measures*

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Abstract

In this paper we study several inequalities of log-Sobolev type for Dunkl operators. After proving an equivalent of the classical inequality for the usual Dunkl measure μ_k, we also study a number of inequalities for probability measures of Boltzmann type of the form $e^{-|x|^p}\mathrm{d}\mu_k$. These are obtained using the method of U-bounds. Poincaré inequalities are obtained as consequences of the log-Sobolev inequality. The connection between Poincaré and log-Sobolev inequalities is further examined, obtaining in particular tight log-Sobolev inequalities. Finally, we study application to exponential integrability and to functional inequalities for a class of singular Boltzmann-Gibbs measures.

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1 Introduction

The non-tight logarithmic Sobolev inequality (or log-Sobolev, for short), on a general measure space $(\Omega, \mathcal{F}, \mu)$ with a quadratic form Q defined on a suitable space of functions on Ω, states that

$$\int_\Omega f^2 \log \frac{f^2}{\int_\Omega f^2\,\mathrm{d}\mu}\,\mathrm{d}\mu \le CQ(f) + D\int_\Omega f^2\,\mathrm{d}\mu, \tag{1.1}$$

for some constants C and D. If $D = 0$, we say that (1.1) is tight and we call it simply the log-Sobolev inequality. Although this inequality was used before, it was first explicitly

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recognised in Gross's seminal paper [10]. His main result was the equivalence of log-Sobolev inequalities to hypercontractivity. For more information about the properties and uses of log-Sobolev inequalities, as well as some historical background, see [4] and [11] and references therein.

Dunkl operators are differential-difference operators which generalise the usual partial derivatives by including difference terms defined in terms of a finite reflection group. Although originally introduced to study special functions with certain symmetries, they have found other applications, for example in mathematical physics where they have been used to study Calogero-Moser-Sutherland (CMS) models of interacting particles. A short introduction to the theory of Dunkl operators is given below in Section 2. More information about applications to CMS models can be found in [18], and an overview of their use in probability theory is contained in [9]. For more recent work on functional inequalities for Dunkl operators see [19], [20], [3], [1], [7], [21], [15], and references therein.

We begin our study with a log-Sobolev inequality for the Dunkl measure μ_k which we prove using the Sobolev inequality for Dunkl operators and Jensen's inequality:

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \,\mathrm{d}\mu_k} \,\mathrm{d}\mu_k \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\mu_k. \tag{1.2}$$

Here $\mathrm{d}\mu_k = w_k \,\mathrm{d}x$ is the Dunkl measure with weight w_k and ∇_k is the Dunkl gradient (see Section 2 for a definition of these terms and an introduction to Dunkl theory). Here and in most inequalities proved in the paper, the constants we obtain depend on the dimension N and are not necessarily optimal.

The log-Sobolev inequality (1.2) will be the basis of many of the subsequent inequalities. Our main aim is to study functional inequalities for the Boltzmann-type probability measures

$$\mathrm{d}\nu_U = \frac{1}{Z} e^{-U} \,\mathrm{d}\mu_k, \tag{1.3}$$

where Z is just a normalising constant and U is a function (in this paper, we mainly consider $U(x) = |x|^p$ for some $p > 1$). The strategy to prove (non-tight) log-Sobolev inequalities for such measures is to apply inequality (1.2) to a function with a suitable weight. This will indeed almost prove the inequality we desire, except for a few residual terms. In order to estimate these terms we use U-bounds, which were introduced in [12] as part of powerful machinery to study quite general functional inequalities.

We also exploit the connection between the log-Sobolev and Poincaré inequalities. In general, it is known that the tight log-Sobolev inequality implies the Poincaré inequality. On the other hand, a non-tight log-Sobolev inequality, in the presence of a Poincaré inequality, can be improved to obtain a tight log-Sobolev inequality. For a detailed discussion of this connection, see [4]. We use these ideas both to produce new Poincaré inequalities for Dunkl operators, and to deduce tight log-Sobolev inequalities from our previous results.

Let us summarise our main results. Firstly, for $\mathrm{d}\nu_U$ defined by (1.3) with $U(x) = |x|^p$ for some $p > 1$, we shall prove in Proposition 6.4 the Poincaré inequality

$$\int_{\mathbb{R}^N} \left| f - \int_{\mathbb{R}^N} f \,\mathrm{d}\nu_U \right|^2 \mathrm{d}\nu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U.$$

For the same measures but for $p \geq 2$, we shall prove in Theorem 7.1 the tight log-Sobolev inequality

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \,\mathrm{d}\nu_U} \,\mathrm{d}\nu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U.$$

Such an inequality cannot hold for $1 < p < 2$ (see the Remark at the end of Section 8.1), and in this range we shall prove in Theorem 7.3 a more general tight Φ-Sobolev inequality

$$\int_{\mathbb{R}^N} \Phi(f^2)\,\mathrm{d}\nu_U - \Phi\left(\int_{\mathbb{R}^N} f^2\,\mathrm{d}\nu_U\right) \le C\int_{\mathbb{R}^N} |\nabla_k f|^2\,\mathrm{d}\nu_U,$$

where $\Phi(x) = x(\log(x+1))^s$ and $s = 2\frac{p-1}{p}$. Finally, we also prove, in Theorem 7.3, a log-Sobolev type inequality in L^1:

$$\int_{\mathbb{R}^N} f\left|\log\frac{|f|}{\int|f|\,\mathrm{d}\nu_U}\right|^s \mathrm{d}\nu_U \le C_1\int_{\mathbb{R}^N} |\nabla_k f|\,\mathrm{d}\nu_U + C_2\int_{\mathbb{R}^N} |f|\,\mathrm{d}\nu_U,$$

where $p \ge 1$ and $s = \frac{p-1}{p}$. Note that here, and everywhere below, we use the shorthand $\int := \int_{\mathbb{R}^N}$ in order not to overcomplicate the notation.

In terms of applications, we first prove exponential integrability of Lipschitz functions for probability measures of the form (1.3), as well as a Gaussian measure concentration property for the same family of measures.

Finally, we also study applications of our inequalities to singular Boltzmann-Gibbs measures. A good expository paper on functional inequalities for such measures is [5]. In this paper, the probability measures in question are of the form $\frac{1}{Z}\mathbb{1}_D e^{-V}\,\mathrm{d}x$, where Z is a normalising constant, $\mathbb{1}_D$ is the indicator function of $D = \{x \in \mathbb{R}^N | x_1 > x_2 > \ldots > x_N\}$, and

$$V(x) = U(x) + \sum_{i<j} W(x_i - x_j).$$

In this notation, U is the confinement potential and it is assumed to be strongly convex (i.e., $U - m|x|^2$ is convex for some $m > 0$), and $W : (0,\infty) \to \mathbb{R}$ is the interaction potential, which is assumed to be convex. This setting can be naturally interpreted in terms of Dunkl theory, with particular interaction potentials arising from the weight function w_k that appears in the Dunkl measure μ_k. Indeed, the set D corresponds to a Weyl chamber associated to the root system A_{N-1} (see (2.2)), and the canonical choice $W(u) = -2k\log u$ produces exactly the Dunkl measure for the same root system:

$$e^{-\sum_{i<j} W(x_i-x_j)} = \prod_{i<j}(x_i - x_j)^{2k} = w_k(x).$$

Using this idea, from our results we obtain functional inequalities similar to those of [5], for confinement potentials of the form $U(x) = |x|^p$. Note that in our case U is not strongly-convex for $1 < p < 2$, so our results complement those of [5].

Moreover, our results hold for any root system and so they allow for different new types of interaction potentials. For example, using the root system B_N (see (2.2)), we obtain the probability measure

$$\mathrm{d}\nu_{U,H} = \frac{1}{Z_H}\mathbb{1}_{\{x_1>\ldots>x_N>0\}} e^{-|x|^p}\prod_{i=1}^N |x_i|^{2k_1}\prod_{i<j}(x_i - x_j)^{2k_2}\prod_{i<j}(x_i + x_j)^{2k_2},$$

for some $k_1, k_2 \ge 0$. A more detailed discussion of examples corresponding to different root systems is Section 8.2.

This paper is organised as follows. After a short introduction to Dunkl theory in Section 2, we prove the main log-Sobolev inequality for the Dunkl measure μ_k in Section 3. In order to introduce the method of U-bounds in a simple framework, in the same section we also prove the log-Sobolev inequality for the Gaussian measure $e^{-|x|^2}\,\mathrm{d}\mu_k$. More general U-bounds are proved in Section 4, which we then apply in Section 5 to obtain the desired non-tight log-Sobolev inequalities for Boltzmann measures. In

Section 6 we prove Poincaré inequalities which we then use in Section 7 to obtain tight log-Sobolev inequalities. Finally, in Section 8 we discuss applications to exponential integrability and singular Gibbs measures.

2 Introduction to Dunkl theory

In this section we will present a very quick introduction to Dunkl operators. For proofs of fundamental results mentioned below and for more details see the survey papers [16] and [2].

A root system is a finite set $R \subset \mathbb{R}^N \setminus \{0\}$ such that $R \cap \alpha\mathbb{R} = \{-\alpha, \alpha\}$ and $\sigma_\alpha(R) = R$ for all $\alpha \in R$. Here σ_α is the reflection in the hyperplane orthogonal to the root α, i.e.,

$$\sigma_\alpha x = x - 2\frac{\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}\alpha. \tag{2.1}$$

The group generated by all the reflections σ_α for $\alpha \in R$ is a finite subgroup of the orthogonal group $O(N)$, and we denote it by G.

We say that a root system is irreducible if it cannot be written as a disjoint union of two orthogonal root systems. Irreducible root systems are fully classified and they consist of four infinite families and a number of exceptional cases. The first three infinite families can be defined in R^N for $N \geq 2$:

$$\begin{aligned} A_{N-1} &= \{e_i - e_j | 1 \leq i < j \leq N\} \\ B_N &= \{\sqrt{2}e_i | 1 \leq i \leq N\} \cup \{e_i \pm e_j | 1 \leq i < j \leq N\} \\ D_N &= \{e_i \pm e_j | 1 \leq i < j \leq N\}, \end{aligned} \tag{2.2}$$

where the $e_1, e_2, \ldots, e_N$ are the standard orthonormal basis of $\mathbb{R}^N$. The fourth infinite family is best described in $\mathbb{C}$ and it is given by

$$I_2(m) = \{e^{i\frac{j\pi}{m}} | 0 \leq j < m\}.$$

Let $k : R \to [0, \infty)$ be a G-invariant function, i.e., $k(\alpha) = k(g\alpha)$ for all $g \in G$ and all $\alpha \in R$. We will normally write $k_\alpha = k(\alpha)$ as these will be the coefficients in our Dunkl operators. We can write the root system R as a disjoint union $R = R_+ \cup (-R_+)$, where R_+ and $-R_+$ are separated by a hyperplane through the origin and we call R_+ a positive subsystem; this decomposition is not unique, but the particular choice of positive subsystem does not make a difference in the definitions below because of the G-invariance of the coefficients k.

The Weyl chambers associated to the root system R are the connected components of $\{x \in \mathbb{R}^N : \langle \alpha, x\rangle \neq 0 \ \forall \alpha \in R\}$. It can be checked that the reflection group G acts simply transitively on the set of Weyl chambers so, in particular, the number of Weyl chambers equals the order of the group, $|G|$.

From now on we fix a root system in $\mathbb{R}^N$ with positive subsystem R_+. We also assume without loss of generality that $|\alpha|^2 = 2$ for all $\alpha \in R$. For $i = 1, \ldots, N$ we define the Dunkl operator on $C^1(\mathbb{R}^N)$ by

$$T_i f(x) = \partial_i f(x) + \sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}.$$

We will denote by $\nabla_k = (T_1, \ldots, T_N)$ the Dunkl gradient, and $\Delta_k = \sum_{i=1}^{N} T_i^2$ will denote the Dunkl Laplacian. Note that for $k \equiv 0$ Dunkl operators reduce to partial derivatives, and $\nabla_0 = \nabla$ and $\Delta_0 = \Delta$ are the usual gradient and Laplacian. Note also that if f is G-invariant, i.e., $f(x) = f(gx)$ for all $g \in G$, then $T_i f = \partial_i f$, so, in particular, $\nabla_k f = \nabla f$.

We can express the Dunkl Laplacian in terms of the usual gradient and Laplacian using the following formula:

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k_\alpha \left[\frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2} \right]. \tag{2.3}$$

The weight function naturally associated to Dunkl operators is

$$w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k_\alpha}.$$

This is a homogeneous function of degree 2γ, where

$$\gamma := \sum_{\alpha \in R_+} k_\alpha.$$

We will work in spaces $L^p(\mu_k)$, where $\mathrm{d}\mu_k = w_k(x)\,\mathrm{d}x$ is the weighted measure; the norm of these spaces will be written simply $\|\cdot\|_p$. With respect to this weighted measure we have the integration by parts formula

$$\int_{\mathbb{R}^N} T_i(f) g \,\mathrm{d}\mu_k = -\int_{\mathbb{R}^N} f T_i(g)\,\mathrm{d}\mu_k. \tag{2.4}$$

This formula holds for $f \in C_c^\infty(\mathbb{R}^N)$ and $g \in C^1(\mathbb{R}^N)$ and can be extended (see below) to $f, g \in H_k^1(\mathbb{R}^N)$.

For any $f \in L^1_{\mathrm{loc}}(\mu_k)$ we say that $T_i f$ exists in a weak sense if there exists $g \in L^1_{\mathrm{loc}}(\mu_k)$ such that

$$\int_{\mathbb{R}^N} f T_i \varphi \,\mathrm{d}\mu_k = -\int_{\mathbb{R}^N} g\varphi \,\mathrm{d}\mu_k \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^N)$$

and we write $T_i f = g$. Higher order derivatives are defined similarly and we use the notation $T^\eta f = T_1^{\eta_1} T_2^{\eta_2} \dots T_N^{\eta_N} f$ for $\eta \in \mathbb{N}_0^N$. We can then define a Dunkl Sobolev space $W_k^{n,p}(\mathbb{R}^N)$ for all $n \in \mathbb{N}$ and $1 \le p \le \infty$ as the space of all functions $f \in L^p(\mu_k)$ for which $T^\eta f$ exists in a weak sense and $T^\eta f \in L^p(\mu_k)$ for all $\eta \in \mathbb{N}_0^N$ with $|\eta| \le n$. It can be checked (following, for example, the ideas of [8, Section 5.2]) that this is a Banach space under the norm

$$\|f\|_{W_k^{n,p}(\mathbb{R}^N)} := \left(\sum_{\eta \in \mathbb{N}_0^N, |\eta| \le n} \|T^\eta f\|_p^p \right)^{1/p}.$$

In the particular case $p = 2$, we write $H_k^n(\mathbb{R}^N) := W_k^{n,2}(\mathbb{R}^N)$. More generally, for any measure μ we can define $W_k^{n,p}(\mu)$ as the space for which $T^\eta f \in L^p(\mu)$ for all $0 \le |\eta| \le n$.

Note that for $k \equiv 0$, when Dunkl operators reduce to usual partial derivatives, then Dunkl Sobolev spaces also reduce to the standard Sobolev spaces, i.e., $W_0^{n,p}(\mu) = W^{n,p}(\mu)$.

One of the main differences between Dunkl operators and classical partial derivatives is that the Leibniz rule does not hold in general. Instead, we have the following.

Lemma 2.1. *If one of the functions f, g is G-invariant, then we have the Leibniz rule*

$$T_i(fg) = fT_i g + gT_i f.$$

In general, we have

$$T_i(fg)(x) = T_i f(x) g(x) + f(x) T_i g(x) - \sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{(f(x) - f(\sigma_\alpha x))(g(x) - g(\sigma_\alpha x))}{\langle \alpha, x \rangle}.$$

A Sobolev inequality is available for the Dunkl gradient (see [19], which also contains a discussion about the optimal constant C_{DS}):

Proposition 2.2. *Let $1 \le p < N + 2\gamma$ and $q = \frac{p(N+2\gamma)}{N+2\gamma-p}$. Then there exists a constant $C_{DS} > 0$ such that we have the inequality*

$$\|f\|_q \le C_{DS} \|\nabla_k f\|_p \qquad \forall f \in W_k^{1,p}(\mathbb{R}^N).$$

3 The main Log-Sobolev inequalities

To begin with, we have the following Dunkl equivalent of the classical log-Sobolev inequality.

Theorem 3.1. *Assume that $N + 2\gamma > 2$. Then, there exists a constant $c \in \mathbb{R}$ such that for any $\delta > 0$ and for any $f \in H_k^1(\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \,\mathrm{d}\mu_k} \,\mathrm{d}\mu_k \le \delta \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\mu_k + C(\delta) \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\mu_k, \tag{3.1}$$

where $C(\delta) = \frac{N+2\gamma}{2}(\log \frac{1}{\delta} - c)$.

In particular, by choosing $\delta = e^{-c}$, we obtain the following tight log-Sobolev inequality: there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \,\mathrm{d}\mu_k} \,\mathrm{d}\mu_k \le C \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\mu_k, \tag{3.2}$$

holds for any $f \in H_k^1(\mathbb{R}^N)$.

Proof. Fix $f \in H_k^1(\mathbb{R}^N)$, $f \not\equiv 0$. Then $\frac{f^2}{\int_{\mathbb{R}^N} f^2 \,\mathrm{d}\mu_k} \,\mathrm{d}\mu_k$ is a probability measure, and so by Jensen's inequality we have, for any $r > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \,\mathrm{d}\mu_k} \,\mathrm{d}\mu_k &= \frac{1}{r} \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\mu_k \cdot \int_{\mathbb{R}^N} \frac{f^2}{\int f^2 \,\mathrm{d}\mu_k} \log \left(\frac{f^2}{\int f^2 \,\mathrm{d}\mu_k} \right)^r \mathrm{d}\mu_k \\
&\le \frac{1}{r} \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\mu_k \cdot \log \int_{\mathbb{R}^N} \left(\frac{f^2}{\int f^2 \,\mathrm{d}\mu_k} \right)^{1+r} \mathrm{d}\mu_k \\
&= \frac{r+1}{r} \|f\|_2^2 \log \frac{\|f\|_{2+2r}^2}{\|f\|_2^2}.
\end{aligned}$$

We then use the elementary inequality

$$\log x \le \delta x + \log \frac{1}{\delta} - 1,$$

which holds for all $x, \delta > 0$. Thus

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \,\mathrm{d}\mu_k} \,\mathrm{d}\mu_k \le \frac{r+1}{r} \left[\delta \|f\|_{2+2r}^2 + (\log \frac{1}{\delta} - 1) \|f\|_2^2 \right].$$

Finally, by choosing $r > 0$ such that $2 + 2r = \frac{2(N+2\gamma)}{N+2\gamma-2}$, we can apply Sobolev's inequality to deduce

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \,\mathrm{d}\mu_k} \,\mathrm{d}\mu_k \le \frac{r+1}{r} C_{DS}^2 \delta \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\mu_k + \frac{r+1}{r} (\log \frac{1}{\delta} - 1) \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\mu_k,$$

where C_{DS} is the Sobolev constant. A simple relabelling of the constants finishes the proof. $\square$

Remark 3.2. From the proof above we can compute the constant c appearing in $C(\delta)$ in terms of the Sobolev constant: $c = 1 - \log\left(\frac{N+2\gamma}{2} C_{DS}^2\right)$. Thus, we can also compute the constant in the tight log-Sobolev inequality (3.2): $C = \frac{N+2\gamma}{2e} C_{DS}^2$. Note that this constant is not necessarily optimal.

Using the results of [6], from the non-tight inequality (3.1) which holds for any $\delta > 0$, we can deduce a more general L^p result, as well as the ultracontractivity property. Here we use the fact that the Dunkl heat semigroup $(H_t)_{t\geq 0}$ has generator Δ_k and associated quadratic form $Q(f) = \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\mu_k$ (for a discussion of this semigroup, see [16, Section 4.2]).

Corollary 3.3. *Assume that $N + 2\gamma > 0$ and let $2 < p < \infty$. Then, for any $\delta > 0$ and for any $f \in C_c^\infty(\mathbb{R}^N)$ such that $f \geq 0$, we have*

$$\int_{\mathbb{R}^N} f^p \log \frac{f^p}{\int f^p \,\mathrm{d}\mu_k} \,\mathrm{d}\mu_k \leq \delta \int_{\mathbb{R}^N} \nabla_k f \cdot \nabla_k (f^{p-1}) \,\mathrm{d}\mu_k + C\left(\frac{2\delta}{p}\right) \int_{\mathbb{R}^N} f^p \,\mathrm{d}\mu_k,$$

where $C(\delta)$ is as in the previous theorem.

Proof. This follows from the previous theorem and [6, Lemma 2.2.6]. □

Finally, we recover the ultracontractivity property for the Dunkl heat semigroup. This was already established in [19] using properties of the heat kernel; using the new log-Sobolev approach, no a priori bounds on the heat kernel are necessary.

Corollary 3.4. *The Dunkl heat semigroup $(H_t)_{t\geq 0}$ on $L^2(\mu_k)$ with generator Δ_k is ultracontractive. More precisely, there exists a constant $C > 0$ such that for all $t > 0$ and for all $f \in L^2(\mu_k)$ we have*

$$\|H_t f\|_\infty \leq C t^{-\frac{N+2\gamma}{4}} \|f\|_2 .$$

Proof. This follows from the previous Corollary and [6, Theorem 2.2.7]. □

In what follows, we want to study inequalities for probability measures of the form

$$\mathrm{d}\nu_U := \frac{1}{Z} e^{-U} \,\mathrm{d}\mu_k,$$

where $U(x) = |x|^p$ and $Z = \displaystyle\int_{\mathbb{R}^N} e^{-U} \,\mathrm{d}\mu_k$. To illustrate the method and to motivate the study of U-bounds in the next section, we first consider Gaussian weight in the following theorem. This result will be further refined and generalised in the next sections, but the main lines of the proof will remain the same.

Theorem 3.5. *Assume that $N + 2\gamma > 2$ and consider the probability measure ν_U with $U(x) = |x|^2$. Then, there exist constants $C_1, C_2 > 0$ such that the following inequality holds for all $f \in H_k^1(\mathbb{R}^N)$:*

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \,\mathrm{d}\nu_U} \,\mathrm{d}\nu_U \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U + C_2 \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U.$$

Proof. Applying inequality (3.2) to the function $\frac{1}{\sqrt{Z}} f e^{-U/2}$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \,\mathrm{d}\nu_U} \,\mathrm{d}\nu_U \leq{}& \frac{1}{Z} C \int_{\mathbb{R}^N} |\nabla_k (f e^{-U/2})|^2 \,\mathrm{d}\mu_k \\ &+ \log Z \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U + \int_{\mathbb{R}^N} f^2 U \,\mathrm{d}\nu_U.\end{aligned} \tag{3.3}$$

Since U is G-invariant, we can use the Leibniz rule from Lemma 2.1 to obtain the identity

$$\nabla_k(fe^{-U/2}) = e^{-U/2}\nabla_k f - \frac{1}{2}fe^{-U/2}\nabla U. \tag{3.4}$$

Thus, we have

$$\begin{aligned}\frac{1}{Z}\int_{\mathbb{R}^N}|\nabla_k(fe^{-U/2})|^2\,\mathrm{d}\mu_k &= \int_{\mathbb{R}^N}|\nabla_k f - \frac{1}{2}f\nabla U|^2\,\mathrm{d}\nu_U\\ &\leq 2\int_{\mathbb{R}^N}|\nabla_k f|^2\,\mathrm{d}\nu_U + \frac{1}{2}\int_{\mathbb{R}^N}f^2|\nabla U|^2\,\mathrm{d}\nu_U\\ &= 2\int_{\mathbb{R}^N}|\nabla_k f|^2\,\mathrm{d}\nu_U + 2\int_{\mathbb{R}^N}f^2U\,\mathrm{d}\nu_U,\end{aligned}$$

where, in the last line, we used the fact that $\nabla U(x) = 2x$, so $|\nabla U|^2 = 4U$. Replacing this inequality in formula (3.3), we have

$$\begin{aligned}\int_{\mathbb{R}^N} f^2\log\frac{f^2}{\int f^2\,\mathrm{d}\nu_U}\,\mathrm{d}\nu_U \leq 2C\int_{\mathbb{R}^N}|\nabla_k f|^2\,\mathrm{d}\nu_U&\\ + \log Z\int_{\mathbb{R}^N}f^2\,\mathrm{d}\nu_U + (1+2C)\int_{\mathbb{R}^N}f^2U\,\mathrm{d}\nu_U&.\end{aligned} \tag{3.5}$$

Using the identity (3.4) again, we deduce

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla_k f|^2\,\mathrm{d}\nu_U = \frac{1}{Z}\int_{\mathbb{R}^N}|\nabla_k(fe^{-U/2})|^2\,\mathrm{d}\mu_k + \frac{1}{4}\int_{\mathbb{R}^N}f^2|\nabla U|^2\,\mathrm{d}\nu_U&\\ + \frac{1}{Z}\int_{\mathbb{R}^N}f\nabla U\cdot\nabla_k(fe^{-U/2})e^{-U/2}\,\mathrm{d}\mu_k&.\end{aligned} \tag{3.6}$$

Keeping in mind that $\nabla U(x) = 2x$, this equality implies that

$$\int_{\mathbb{R}^N}|\nabla_k f|^2\,\mathrm{d}\nu_U \geq \int_{\mathbb{R}^N}f^2U\,\mathrm{d}\nu_U + A, \tag{3.7}$$

where

$$A := \frac{1}{Z}\int_{\mathbb{R}^N}f\nabla U\cdot\nabla_k(fe^{-U/2})e^{-U/2}\,\mathrm{d}\mu_k$$

We now compute the quantity A. Firstly, by the integration by parts formula (2.4) applied to the pairs of functions $fe^{-U/2}, x_i fe^{-U/2} \in H_k^1(\mathbb{R}^N)$ for each $1 \leq i \leq N$, we have

$$A = -\sum_{i=1}^N\frac{1}{Z}\int_{\mathbb{R}^N}fe^{-U/2}T_i(2x_i fe^{-U/2})\,\mathrm{d}\mu_k(x).$$

Using Lemma 2.1, we have

$$\begin{aligned}&T_i(x_i fe^{-U/2})\\ &\quad= x_iT_i(fe^{-U/2}) + fe^{-U/2}T_i(x_i) - \sum_{\alpha\in R_+}k_\alpha\alpha_i e^{-U(x)/2}\frac{(f(x)-f(\sigma_\alpha x))(x_i-(\sigma_\alpha x)_i)}{\langle\alpha,x\rangle}\\ &\quad= x_iT_i(fe^{-U/2}) + fe^{-U/2}(1+\sum_{\alpha\in R_+}k_\alpha\alpha_i^2) - \sum_{\alpha\in R_+}k_\alpha\alpha_i^2e^{-U(x)/2}(f(x)-f(\sigma_\alpha x)).\end{aligned} \tag{3.8}$$

Thus, recalling that $|\alpha|^2 = 2$,

$$A = -A - 2(N+2\gamma)\int_{\mathbb{R}^N}f^2\,\mathrm{d}\nu_U + 4\sum_{\alpha\in R_+}k_\alpha\int_{\mathbb{R}^N}f(x)(f(x)-f(\sigma_\alpha x))\,\mathrm{d}\nu_U(x),$$

and so

$$A = -N \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U - 2 \sum_{\alpha \in R_+} k_\alpha \int_{\mathbb{R}^N} f(x) f(\sigma_\alpha x) \,\mathrm{d}\nu_U(x).$$

Using the elementary inequality $2XY \leq X^2 + Y^2$ and the fact that the measure ν_U is G-invariant, we obtain

$$A \geq -(N + 2\gamma) \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U.$$

Replacing this in equation (3.7), we obtain

$$\int_{\mathbb{R}^N} f^2 U \,\mathrm{d}\nu_U \leq \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U + (N + 2\gamma) \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U. \tag{3.9}$$

Finally, using this in (3.5), we have

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \,\mathrm{d}\nu_U} \,\mathrm{d}\nu_U \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U + C_2 \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U,$$

for some constants $C_1, C_2 > 0$, as required. □

4 U-bounds

Looking back at the proof of the weighted log-Sobolev inequality in Theorem 3.5, we can see that inequality (3.9) was the key element. Inequalities of this form are called U-bounds (cf. [12]). In this section we will prove more general U-bounds by adapting our proof slightly, and these will later be used to deduce log-Sobolev inequalities.

Proposition 4.1. *Let $p > 1$ and consider the probability measure ν_U with $U(x) = |x|^p$. Then, there exist some constants $C, D > 0$ such that for any $f \in H_k^1(\nu_U)$, we have the inequality*

$$\int_{\mathbb{R}^N} f^2 |x|^{2(p-1)} \,\mathrm{d}\nu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U + D \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U. \tag{4.1}$$

Proof. We follow more closely the proof of (3.9). We have

$$\nabla U(x) = p|x|^{p-2} x.$$

From (3.6), we obtain

$$\frac{p^2}{4} \int_{\mathbb{R}^N} f^2 |x|^{2(p-1)} \,\mathrm{d}\nu_U \leq \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U - A,$$

where

$$A = \frac{1}{Z} \int_{\mathbb{R}^N} f \nabla U \cdot \nabla_k (f e^{-U/2}) e^{-U/2} \,\mathrm{d}\mu_k.$$

As in the proof of Theorem 3.5, using integration by parts and Lemma 2.1, we have

$$\begin{aligned} 2A &= -\left[p(p-2) + p(N + 2\gamma) - 2\gamma p\right] \int_{\mathbb{R}^N} f^2 |x|^{p-2} \,\mathrm{d}\nu_U \\ &\quad - 2p \sum_{\alpha \in R_+} k_\alpha \int_{\mathbb{R}^N} |x|^{p-2} f(x) f(\sigma_\alpha x) \,\mathrm{d}\nu_U(x) \\ &\geq -\left[p^2 + p(N + 2\gamma - 2)\right] \int_{\mathbb{R}^N} f^2 |x|^{p-2} \,\mathrm{d}\nu_U. \end{aligned}$$

Thus

$$\int_{\mathbb{R}^N} f^2 |x|^{2(p-1)} \,\mathrm{d}\nu_U \leq \frac{4}{p^2} \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U + \frac{2\left[p^2 + p(N + 2\gamma - 2)\right]}{p^2} \int_{\mathbb{R}^N} f^2 |x|^{p-2} \,\mathrm{d}\nu_U. \tag{4.2}$$

Assume first that $p > 2$ and let $\epsilon > 0$. Then, using Hölder's inequality with coefficients $\tilde{p} := \frac{2(p-1)}{p}$ and $\tilde{q} := \frac{2(p-1)}{p-2}$, and then Young's inequality $XY \leq \frac{X^{\tilde{p}}}{\tilde{p}} + \frac{Y^{\tilde{q}}}{\tilde{q}}$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} f^2|x|^{p-2}\,\mathrm{d}\nu_U &\leq \left(\int_{\mathbb{R}^N} f^2\,\mathrm{d}\nu_U\right)^{\frac{p}{2(p-1)}}\left(\int_{\mathbb{R}^N} f^2|x|^{2(p-1)}\,\mathrm{d}\nu_U\right)^{\frac{p-2}{2(p-1)}}\\ &\leq \frac{p}{2(p-1)}\epsilon^{-\frac{p-2}{p}}\int_{\mathbb{R}^N} f^2\,\mathrm{d}\nu_U + \frac{p-2}{2(p-1)}\epsilon\int_{\mathbb{R}^N} f^2|x|^{2(p-1)}\,\mathrm{d}\nu_U.\end{aligned}$$

Thus, by choosing $\epsilon > 0$ small enough such that

$$1 > \frac{(p-2)[p^2+p(N+2\gamma-2)]}{p^2(p-1)}\epsilon,$$

we obtain inequality (4.1) for some constants $C, D > 0$.

The case $1 < p < 2$ requires more care. Let $\phi : \mathbb{R}^2 \to [0,1]$ be defined by

$$\phi(x) = \begin{cases} 0, & |x| < 1\\ |x|-1, & 1 \leq |x| \leq 2\\ 1, & |x| > 2.\end{cases} \tag{4.3}$$

Note that ϕ is radial, so G-invariant, and $\nabla\phi(x) = \frac{x}{|x|}$ on $1 \leq |x| \leq 2$, and it vanishes elsewhere; in particular, $|\nabla\phi| \leq 1$ on $\mathbb{R}^N$. Then, writing $f = f\phi + f(1-\phi)$, we have

$$\int_{\mathbb{R}^N} f^2|x|^{2(p-1)}\,\mathrm{d}\nu_U \leq 2\int_{\mathbb{R}^N} |f\phi|^2|x|^{2(p-1)}\,\mathrm{d}\nu_U + 2\int_{\mathbb{R}^N} |f(1-\phi)|^2|x|^{2(p-1)}\,\mathrm{d}\nu_U, \tag{4.4}$$

and we estimate each of the terms on the right hand side separately. Firstly, by (4.2), we have

$$\int_{\mathbb{R}^N} |f\phi|^2|x|^{2(p-1)}\,\mathrm{d}\nu_U \leq \frac{4}{p^2}\int_{\mathbb{R}^N} |\nabla_k(f\phi)|^2\,\mathrm{d}\nu_U + C_p\int_{\mathbb{R}^N} |f\phi|^2|x|^{p-2}\,\mathrm{d}\nu_U \tag{4.5}$$

where $C_p := \frac{2[p^2+p(N+2\gamma-2)]}{p^2}$. By the Leibniz rule (since ϕ is G-invariant) and using the properties of the function ϕ, we have

$$|\nabla_k(f\phi)|^2 \leq 2\phi^2|\nabla_k f|^2 + 2f^2|\nabla\phi|^2 \leq 2|\nabla_k f|^2 + 2f^2. \tag{4.6}$$

Moreover, note that $f\phi = 0$ on $|x| \leq 1$ and outside this region we have $|x|^{p-2} \leq 1$ (since $p < 2$), so

$$\int_{\mathbb{R}^N} |f\phi|^2|x|^{p-2}\,\mathrm{d}\nu_U \leq \int_{\mathbb{R}^N} f^2\,\mathrm{d}\nu_U. \tag{4.7}$$

Thus, combining inequalities (4.5), (4.6) and (4.7), we have obtained

$$\int_{\mathbb{R}^N} |f\phi|^2|x|^{2(p-1)}\,\mathrm{d}\nu_U \leq \frac{8}{p^2}\int_{\mathbb{R}^N} |\nabla_k f|^2\,\mathrm{d}\nu_U + (\frac{8}{p^2} + C_p)\int_{\mathbb{R}^N} f^2\,\mathrm{d}\nu_U. \tag{4.8}$$

We now turn to the second term in (4.4). Here we simply note that the function $f(1-\phi)$ is supported on $|x| \leq 2$, and thus

$$\int_{\mathbb{R}^N} |f(1-\phi)|^2|x|^{2(p-1)}\,\mathrm{d}\nu_U \leq 2^{2(p-1)}\int_{\mathbb{R}^N} |f(1-\phi)|^2\,\mathrm{d}\nu_U \leq 2^{2(p-1)}\int_{\mathbb{R}^N} f^2\,\mathrm{d}\nu_U. \tag{4.9}$$

Therefore, putting (4.4), (4.8) and (4.9) together, we obtain the U-bound (4.1) for some $C, D > 0$. □

From this result we can obtain another type of U-bound which will be essential in the later study of log-Sobolev inequalities. Note however that this bound holds in the more restricted range $p \geq 2$.

Corollary 4.2. *Let $p \geq 2$ and consider the probability measure ν_U with $U(x) = |x|^p$. Then, there exist some constants $C, D > 0$ such that for any $f \in H^1_k(\mathbb{R}^N)$ we have the inequality*

$$\int_{\mathbb{R}^N} f^2 |x|^p \,\mathrm{d}\nu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U + D \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U. \tag{4.10}$$

Proof. We employ the same idea as in the last part of the previous result. Namely, let ϕ be the function defined by (4.3) and consider the decomposition $f = f\phi + f(1-\phi)$. We have

$$\int_{\mathbb{R}^N} f^2 |x|^p \,\mathrm{d}\nu_U \leq 2 \int_{\mathbb{R}^N} |f\phi|^2 |x|^p \,\mathrm{d}\nu_U + 2 \int_{\mathbb{R}^N} |f(1-\phi)|^2 |x|^p \,\mathrm{d}\nu_U. \tag{4.11}$$

The function $f\phi$ vanishes on $|x| \leq 1$ and outside this region we have $|x|^p \leq |x|^{2(p-1)}$ (since $p > 2$). Thus

$$\int_{\mathbb{R}^N} |f\phi|^2 |x|^p \,\mathrm{d}\nu_U \leq \int_{\mathbb{R}^N} |f\phi|^2 |x|^{2(p-1)} \,\mathrm{d}\nu_U \leq \int_{\mathbb{R}^N} f^2 |x|^{2(p-1)} \,\mathrm{d}\nu_U.$$

On the other hand, the function $f(1-\phi)$ is supported on $|x| \leq 2$ and thus

$$\int_{\mathbb{R}^N} |f(1-\phi)|^2 |x|^p \,\mathrm{d}\nu_U \leq 2^p \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U.$$

Putting these inequalities together, we obtain

$$\int_{\mathbb{R}^N} f^2 |x|^p \,\mathrm{d}\nu_U \leq 2 \int_{\mathbb{R}^N} f^2 |x|^{2(p-1)} \,\mathrm{d}\nu_U + 2^{p+1} \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U.$$

Finally, using inequality (4.1) for the first term on the right hand side, we obtain (4.10), as required. $\square$

The two bounds we have proved so far are both in $L^2(\nu_U)$. In the last result of this section we prove an $L^1(\nu_U)$ bound whose proof will require a slightly different approach.

Proposition 4.3. *Let $p > 1$ and consider the probability measure ν_U with $U(x) = |x|^p$. Then, there exist some constants $C, D > 0$ such that for any $f \in W^{1,1}_k(\mathbb{R}^N)$ we have the inequality*

$$\int_{\mathbb{R}^N} |f| \cdot |x|^{p-1} \,\mathrm{d}\nu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f| \,\mathrm{d}\nu_U + D \int_{\mathbb{R}^N} |f| \,\mathrm{d}\nu_U.$$

Proof. In order to avoid a singularity that will arise at the origin, we first consider a function f that vanishes on the unit ball. As before, we start with identity (3.4). Noting that in this case we have

$$\nabla U(x) = p|x|^{p-1} \nabla(|x|),$$

the identity in the previous section now reads

$$\nabla_k (f e^{-U}) = e^{-U} \nabla_k f - p f |x|^{p-1} e^{-U} \nabla(|x|).$$

Taking inner product with $\nabla(|x|)$ and integrating on both sides, we have

$$\begin{aligned} &\frac{1}{Z} \int_{\mathbb{R}^N} \nabla(|x|) \cdot \nabla_k (f e^{-U}) \,\mathrm{d}\mu_k \\ &\qquad = \int_{\mathbb{R}^N} \nabla(|x|) \cdot \nabla_k f \,\mathrm{d}\nu_U - p \int_{\mathbb{R}^N} |\nabla(|x|)|^2 f |x|^{p-1} \,\mathrm{d}\nu_U. \end{aligned} \tag{4.12}$$

We can use integration by parts on the left hand side to obtain

$$\frac{1}{Z}\int_{\mathbb{R}^N}\nabla(|x|)\cdot\nabla_k(fe^{-U})\,\mathrm{d}\mu_k=-\int_{\mathbb{R}^N}\Delta_k(|x|)f\,\mathrm{d}\nu_U.$$

Replacing this in (4.12), and using also the fact that $|\nabla(|x|)|=1$ for $x\neq 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}f|x|^{p-1}\,\mathrm{d}\nu_U&=\frac{1}{p}\int_{\mathbb{R}^N}\nabla(|x|)\cdot\nabla_k f\,\mathrm{d}\nu_U+\frac{1}{p}\int_{\mathbb{R}^N}\Delta_k(|x|)f\,\mathrm{d}\nu_U\\&\leq\frac{1}{p}\int_{\mathbb{R}^N}|\nabla_k f|\,\mathrm{d}\nu_U+\frac{1}{p}\int_{\mathbb{R}^N}\Delta_k(|x|)f\,\mathrm{d}\nu_U.\end{aligned}$$

Finally, we have

$$T_i^2(|x|)=T_i\left(\frac{x_i}{|x|}\right)=\frac{1}{|x|}-\frac{x_i^2}{|x|^3}+\sum_{\alpha\in R_+}k_\alpha\frac{\alpha_i^2}{|x|},$$

so

$$\Delta_k(|x|)=(N+2\gamma-1)\frac{1}{|x|}.$$

Therefore, from the above we deduce that (recall that f vanishes on the unit ball)

$$\int_{\mathbb{R}^N}f|x|^{p-1}\,\mathrm{d}\nu_U\leq\frac{1}{p}\int_{\mathbb{R}^N}|\nabla_k f|\,\mathrm{d}\nu_U+\frac{N+2\gamma-1}{p}\int_{\mathbb{R}^N}|f|\,\mathrm{d}\nu_U.$$

Writing $f=f_+-f_-$, where $f_+(x)=\max(f(x),0)$ and $f_-(x)=-\min(f(x),0)$, we can apply this inequality to f_+ and f_- separately. Adding the two resulting inequalities, we have

$$\int_{\mathbb{R}^N}|f|\cdot|x|^{p-1}\,\mathrm{d}\nu_U\leq C_1\int_{\mathbb{R}^N}|\nabla_k f|\,\mathrm{d}\nu_U+D_1\int_{\mathbb{R}^N}|f|\,\mathrm{d}\nu_U,$$

where $C_1=\frac{1}{p}$ and $D_1=\frac{N+2\gamma-1}{p}$.

Having proved the result for functions that vanish on the unit ball, let us now consider a general function $f\in L^1(\mathrm{d}\nu_U)$. To prove this more general result, we once again employ the method from the end of the proof of Proposition 4.1. More precisely, let ϕ be the function defined in (4.3) and consider $f=\phi f+(1-\phi)f$; the first term vanishes on the unit ball so the above can be applied to it, while the second term has compact support and it is easy to bound. We have

$$\begin{aligned}\int_{\mathbb{R}^N}|f|\cdot|x|^{p-1}\,\mathrm{d}\nu_U&\leq\int_{\mathbb{R}^N}|\phi f|\cdot|x|^{p-1}\,\mathrm{d}\nu_U+\int_{\mathbb{R}^N}|(1-\phi)f|\cdot|x|^{p-1}\,\mathrm{d}\nu_U\\&\leq C_1\int_{\mathbb{R}^N}|\nabla_k(\phi f)|\,\mathrm{d}\nu_U+D_1\int_{\mathbb{R}^N}|\phi f|\,\mathrm{d}\nu_U+2^{p-1}\int_{\mathbb{R}^N}|f|\,\mathrm{d}\nu_U\\&\leq C_1\int_{\mathbb{R}^N}|\nabla_k f|\,\mathrm{d}\nu_U+(C_1+D_1+2^{p-1})\int_{\mathbb{R}^N}|f|\,\mathrm{d}\nu_U.\end{aligned}$$

Here, in the last step, we used the Leibniz rule (which holds because ϕ is G-invariant) and the fact that $|\nabla\phi|\leq 1$. This completes the proof. □

5 Non-tight log-Sobolev inequalities for Boltzmann-type measures

In this section we use the U-bounds obtained above to deduce non-tight log-Sobolev inequalities. To begin with, from Proposition 4.1 and Corollary 4.2 we obtain a log-Sobolev inequality for probability measures $\mathrm{d}\nu_U=\frac{1}{Z}e^{-U(x)}\,\mathrm{d}\mu_k$, for $U(x)=|x|^p$ and with $p\geq 2$, and in the range $1\leq p\leq 2$ we prove a Φ-Sobolev inequality. Similarly, from Proposition 4.3 we obtain a Φ-Sobolev inequality in $L^1(\nu_U)$ for general $1<p<\infty$. The approach in these results is similar to that of Theorem 3.1: first employing Jensen's inequality to take the logarithm outside the integral, and then using the classical Sobolev inequality. U-bounds will be used to control residual terms arising from the introduction of a weight.

Theorem 5.1. *Assume $N + 2\gamma > 2$. Let $p \geq 2$ and consider the probability measure $\mathrm{d}\nu_U$ with $U(x) = |x|^p$. Then there exist some constants $C_1, C_2 > 0$ such that for all $f \in H_k^1(\nu_U)$, we have the inequality*

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \,\mathrm{d}\nu_U} \,\mathrm{d}\nu_U \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U + C_2 \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U. \tag{5.1}$$

Proof. We apply inequality (3.2) of Theorem 3.1 to the function $\frac{1}{\sqrt{Z}} f e^{-U/2}$ and thus we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \,\mathrm{d}\nu_U} \,\mathrm{d}\nu_U \leq 2C \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U + \log Z \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U \\ + \int_{\mathbb{R}^N} f^2 U \,\mathrm{d}\nu_U + 2C \int_{\mathbb{R}^N} f^2 |\nabla U|^2 \,\mathrm{d}\nu_U. \end{aligned}$$

Note that in this case we have $|\nabla U|^2 = |x|^{2(p-1)}$ and thus by applying Proposition 4.1 and Corollary 4.2 (hence the restriction $p \geq 2$), we obtain inequality (5.1) for some constants $C_1, C_2 > 0$, as required. $\square$

Theorem 5.2. *Assume $N + 2\gamma > 2$. Let $1 < p \leq 2$ and consider the probability measure ν_U with $U(x) = |x|^p$. Let $s = 2\frac{p-1}{p}$. Then there exist some constants $C_1, C_2 > 0$ such that for all $f \in H_k^1(\nu_U)$, we have the inequality*

$$\int_{\mathbb{R}^N} f^2 \left|\log \frac{f^2}{\int f^2 \,\mathrm{d}\nu_U}\right|^s \mathrm{d}\nu_U \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U + C_2 \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U. \tag{5.2}$$

Proof. Consider the function $h = \frac{1}{\sqrt{Z}} f e^{-U/2}$. Then $\displaystyle\int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U = \int_{\mathbb{R}^N} h^2 \,\mathrm{d}\mu_k$, and

$$\begin{aligned} \int_{\mathbb{R}^N} f^2 \left|\log \frac{f^2}{\int f^2 \,\mathrm{d}\nu_U}\right|^s \mathrm{d}\nu_U &= \int_{\mathbb{R}^N} h^2 \left|\log \frac{h^2}{\int h^2 \,\mathrm{d}\mu_k} + U + \log Z\right|^s \mathrm{d}\mu_k \\ &\leq \int_{\mathbb{R}^N} h^2 \left|\log \frac{h^2}{\int h^2 \,\mathrm{d}\mu_k}\right|^s \mathrm{d}\mu_k + \int_{\mathbb{R}^N} h^2 U^s \,\mathrm{d}\mu_k + |\log Z|^s \int_{\mathbb{R}^N} h^2 \,\mathrm{d}\mu_k, \end{aligned} \tag{5.3}$$

where in the last inequality we used the fact that since $s \in (0, 1]$, then the function $X \mapsto X^s$ is subadditive, i.e., we have $(X + Y)^s \leq X^s + Y^s$ for $X, Y > 0$.

Before we start the usual procedure of applying Jensen's inequality, we note that the function $x \left|\log x\right|^s$ is bounded on $(0, 1)$, and let

$$D = \sup_{x \in (0,1)} x \left|\log x\right|^s < \infty.$$

Consider now the function $\log_+ x := \max\{0, \log x\}$. Then the above observation implies that

$$x \left|\log x\right|^s \leq x(\log_+ x)^s + D \qquad \text{for all } x > 0.$$

With this in mind, we have

$$\begin{aligned} \int_{\mathbb{R}^N} h^2 \left|\log \frac{h^2}{\int h^2 \,\mathrm{d}\mu_k}\right|^s \mathrm{d}\mu_k &= \int_{\mathbb{R}^N} h^2 \,\mathrm{d}\mu_k \cdot \int_{\mathbb{R}^N} \frac{h^2}{\int h^2 \,\mathrm{d}\mu_k} \left|\log \frac{h^2}{\int h^2 \,\mathrm{d}\mu_k}\right|^s \mathrm{d}\mu_k \\ &\leq \int_{\mathbb{R}^N} h^2 \,\mathrm{d}\mu_k \cdot \left[\int_{\mathbb{R}^N} \frac{h^2}{\int h^2 \,\mathrm{d}\mu_k} \left(\log_+ \frac{h^2}{\int h^2 \,\mathrm{d}\mu_k}\right)^s \mathrm{d}\mu_k + D\right]. \end{aligned} \tag{5.4}$$

For the fixed function h the measure $\frac{h^2}{\int h^2 \,\mathrm{d}\mu_k} \,\mathrm{d}\mu_k$ is a probability measure. Thus, we can apply Jensen's inequality to the concave function $(\log_+ t)^s$ in the below as follows

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{h^2}{\int h^2 \,\mathrm{d}\mu_k} \left(\log_+ \frac{h^2}{\int h^2 \,\mathrm{d}\mu_k}\right)^s \mathrm{d}\mu_k &= \frac{1}{\delta^s} \int_{\mathbb{R}^N} \frac{h^2}{\int h^2 \,\mathrm{d}\mu_k} \left(\log_+ \left(\frac{h^2}{\int h^2 \,\mathrm{d}\mu_k}\right)^\delta\right)^s \mathrm{d}\mu_k \\ &\le \frac{1}{\delta^s} \left(\log_+ \int_{\mathbb{R}^N} \left(\frac{h^2}{\int h^2 \,\mathrm{d}\mu_k}\right)^{1+\delta} \mathrm{d}\mu_k\right)^s \\ &= \frac{(1+\delta)^s}{\delta^s} \left(\log_+ \frac{\|h\|_{2+2\delta}^2}{\|h\|_2^2}\right)^s.\end{aligned}$$

By standard calculus methods we can show that

$$c_s := \sup_{x>0} \frac{|\log_+ x|^s}{x} < \infty.$$

Applying this in the previous inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{h^2}{\int h^2 \,\mathrm{d}\mu_k} \left(\log_+ \frac{h^2}{\int h^2 \,\mathrm{d}\mu_k}\right)^s \mathrm{d}\mu_k &\le c_s \frac{(1+\delta)^s}{\delta^s} \frac{\|h\|_{2+2\delta}^2}{\|h\|_2^2} \\ &\le c_s \frac{(1+\delta)^s}{\delta^s} \frac{1}{\int h^2 \,\mathrm{d}\mu_k} \epsilon C_{DS}^2 \|\nabla_k h\|_2^2 .\end{aligned} \tag{5.5}$$

Here in the last step we used the Sobolev inequality of Proposition 2.2, which holds if we choose $\delta > 0$ such that $2 + 2\delta = \frac{2(N+2\gamma)}{N+2\gamma-2}$.

Next, we have

$$\|\nabla_k h\|_2^2 = \frac{1}{Z} \int_{\mathbb{R}^N} |\nabla_k (f e^{-U/2})|^2 \,\mathrm{d}\mu_k \le 2 \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U + \frac{1}{2} \int_{\mathbb{R}^N} f^2 |\nabla U|^2 \,\mathrm{d}\nu_U. \tag{5.6}$$

Combining (5.3), (5.4), (5.5) and (5.6), we have

$$\begin{aligned}\int_{\mathbb{R}^N} f^2 \left|\log \frac{f^2}{\int f^2 \,\mathrm{d}\nu_U}\right|^s \mathrm{d}\nu_U \le c_s \left(\frac{N+2\gamma}{2}\right)^s 2C_{DS}^2 \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U + |\log Z|^s \int_{\mathbb{R}^N} f^2 \,\mathrm{d}\nu_U \\ + c_s \left(\frac{N+2\gamma}{2}\right)^s \frac{C_{DS}^2}{2} \int_{\mathbb{R}^N} f^2 |\nabla U|^2 \,\mathrm{d}\nu_U + \int_{\mathbb{R}^N} f^2 U^s \,\mathrm{d}\nu_U.\end{aligned}$$

But $|\nabla U| = p|x|^{p-1}$ and $U^s = |x|^{2(p-1)}$, so the last two terms can be computed as follows

$$\int_{\mathbb{R}^N} f^2 U^s \,\mathrm{d}\nu_U = \int_{\mathbb{R}^N} f^2 |x|^{2(p-1)} \,\mathrm{d}\nu_U = \frac{1}{p^2} \int_{\mathbb{R}^N} f^2 |\nabla U|^2 \,\mathrm{d}\nu_U.$$

Finally, from Proposition 4.1 we obtain inequality (5.2) for some constants $C_1, C_2 > 0$ which depend on γ and the dimension N. This concludes the proof. □

Theorem 5.3. *Assume $N + 2\gamma > 1$. Let $1 \le p < \infty$ and consider the probability measure ν_U with $U(x) = |x|^p$. Let $s = \frac{p-1}{p}$. Then there exist some constants $C_1, C_2 > 0$ such that for all $f \in W_k^{1,1}(\nu_U)$, we have the inequality*

$$\int_{\mathbb{R}^N} f \left|\log \frac{|f|}{\int |f| \,\mathrm{d}\nu_U}\right|^s \mathrm{d}\nu_U \le C_1 \int_{\mathbb{R}^N} |\nabla_k f| \,\mathrm{d}\nu_U + C_2 \int_{\mathbb{R}^N} |f| \,\mathrm{d}\nu_U.$$

Proof. The proof is similar to that of the previous result except for in this case we rely on the Sobolev inequality

$$\|h\|_q \le C \|\nabla_k h\|_1$$

where $q = \frac{N+2\gamma}{N+2\gamma-1}$ (see Proposition 2.2), and the U-bound of Proposition 4.3. □

6 Poincaré inequalities

In this section we discuss Poincaré inequalities for Dunkl operators. These will be used in the next section to improve some of our previous log-Sobolev inequalities, but are also of independent interest.

Using a standard argument (see for example [4]), the non-tight log-Sobolev inequality (3.1) of Theorem 3.1 implies the following Poincaré inequality.

Theorem 6.1. *Assume $N+2\gamma > 2$. Let $R > 0$ and consider the ball $B_R := \{x \in \mathbb{R}^N : |x| \leq R\}$. There exists a constant $C > 0$ independent of R such that for any $f \in H_k^1(B_R, \mu_k)$ we have the inequality*

$$\int_{B_R} \left| f - \frac{1}{\mu_k(B_R)} \int_{B_R} f \,\mathrm{d}\mu_k \right|^2 \mathrm{d}\mu_k \leq CR^2 \int_{B_R} |\nabla_k f|^2 \,\mathrm{d}\mu_k.$$

Proof. To simplify the notation, let $\tilde{\mu}_R := \frac{1}{\mu_k(B_R)}\mu_k$ be the Dunkl probability measure on the ball B_R. Note that it is enough to prove the theorem for f that satisfies the additional assumption $\int_{B_R} f \,\mathrm{d}\tilde{\mu}_R = 0$, in which case the inequality takes the form

$$\int_{B_R} f^2 \,\mathrm{d}\tilde{\mu}_R \leq CR^2 \int_{B_R} |\nabla_k f|^2 \,\mathrm{d}\tilde{\mu}_R. \tag{6.1}$$

To obtain the general case it is then enough to replace f by $f - \int_{B_R} f \,\mathrm{d}\tilde{\mu}_R$ in (6.1).

For any $\epsilon > 0$ consider the function $g = 1 + \epsilon f$. A Taylor expansion shows that

$$g^2 \log \frac{g^2}{\int_{B_R} g^2 \,\mathrm{d}\tilde{\mu}_R} = 2\epsilon f + 3\epsilon^2 f^2 - \epsilon^2 \int_{B_R} f^2 \,\mathrm{d}\tilde{\mu}_R + o(\epsilon^2),$$

and thus

$$\int_{B_R} g^2 \log \frac{g^2}{\int_{B_R} g^2 \,\mathrm{d}\tilde{\mu}_R} \,\mathrm{d}\tilde{\mu}_R = 2\epsilon^2 \int_{B_R} f^2 \,\mathrm{d}\tilde{\mu}_R + o(\epsilon^2), \tag{6.2}$$

as $\epsilon \to 0$.

From Theorem 3.1 we have that

$$\int_{B_R} g^2 \log \frac{g^2}{\int_{B_R} g^2 \,\mathrm{d}\tilde{\mu}_R} \,\mathrm{d}\tilde{\mu}_R \leq \delta \int_{B_R} |\nabla_k g|^2 \,\mathrm{d}\tilde{\mu}_R + (C(\delta) + \log(\mu_k(B_R))) \int_{B_R} g^2 \,\mathrm{d}\tilde{\mu}_R,$$

holds for all $\delta > 0$. However, using the fact that $\mu_k(B_R) = \mu_k(B_1)R^{N+2\gamma}$, we find that $\delta = c'R^2$, for a constant $c' > 0$, solves the equation $C(\delta) + \log(\mu_k(B_R)) = 0$ (the exact formula for c' is given in the remark below). Therefore, we have the tight log-Sobolev inequality

$$\int_{B_R} g^2 \log \frac{g^2}{\int_{B_R} g^2 \,\mathrm{d}\tilde{\mu}_R} \,\mathrm{d}\tilde{\mu}_R \leq c'R^2 \int_{B_R} |\nabla_k g|^2 \,\mathrm{d}\tilde{\mu}_R. \tag{6.3}$$

Combining (6.2) and (6.3), and letting $\epsilon \to 0$, we have obtained (6.1), as required. □

Remark 6.2. The constant c' obtained above can be computed and we obtain

$$c' = \frac{N + 2\gamma}{2e} \mu_k(B_1)^{2/(N+2\gamma)} C_{DS}^2,$$

where C_{DS} is the Sobolev constant. The volume of the unit ball $\mu_k(B_1)$ can also be computed explicitly using the Macdonald-Mehta integral, see formulas (2.4) and (2.5) in [19] for details.

Remark 6.3. This Poincaré inequality corresponds to the classical Neumann-Poincaré inequality. A Dirichlet-Poincaré inequality for Dunkl operators was also proved in [20]. Namely, we have the result:

$$\int_\Omega |f|^2 \,\mathrm{d}\mu_k \leq C(\Omega) \int_\Omega |\nabla_k f|^2 \,\mathrm{d}\mu_k,$$

which holds on any bounded domain $\Omega \subset \mathbb{R}^N$ for a constant $C(\Omega) > 0$ and for all $f \in C_0^\infty(\mathbb{R}^N)$.

We can now use the previous result together with the U-bounds proved above to obtain a Poincaré inequality for the weighted measure ν_U.

Proposition 6.4. *Assume $N + 2\gamma > 2$. Let $p > 1$ and consider the weighted probability measure ν_U with $U(x) = |x|^p$. Then, there exists a constant $C > 0$ such that for any $f \in H_k^1(\nu_U)$ we have*

$$\int_{\mathbb{R}^N} \left| f - \int_{\mathbb{R}^N} f \,\mathrm{d}\nu_U \right|^2 \mathrm{d}\nu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U. \tag{6.4}$$

Proof. It is known that

$$\mathrm{var}_{\nu_U}(f) := \int_{\mathbb{R}^N} \left| f - \int_{\mathbb{R}^N} f \,\mathrm{d}\nu_U \right|^2 \mathrm{d}\nu_U = \min_{\zeta \in \mathbb{R}} \int_{\mathbb{R}^N} |f - \zeta|^2 \,\mathrm{d}\nu_U. \tag{6.5}$$

Indeed, this can be proved by considering the minimum of the quadratic function $\zeta \mapsto \int_{\mathbb{R}^N} |f - \zeta|^2 \,\mathrm{d}\nu_U$ over $\mathbb{R}$. Thus, it is enough to prove the inequality

$$\int_{\mathbb{R}^N} |f - \zeta|^2 \,\mathrm{d}\nu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U \tag{6.6}$$

for some $\zeta \in \mathbb{R}$.

Let $R > 0$ and let $B_R = \{|x| \leq R\}$. We will prove (6.6) with $\zeta = \frac{1}{\mu_k(B_R)} \int_{B_R} f \,\mathrm{d}\mu_k$ for large enough R. Firstly, we have

$$\begin{aligned} \int_{B_R} |f - \zeta|^2 \,\mathrm{d}\nu_U &\leq \frac{1}{Z} \int_{B_R} \left| f - \frac{1}{\mu_k(B_R)} \int_{B_R} f \,\mathrm{d}\mu_k \right|^2 \mathrm{d}\mu_k \\ &\leq \frac{C}{Z} R^2 \int_{B_R} |\nabla_k f|^2 \,\mathrm{d}\mu_k \\ &\leq C R^2 e^{R^p} \int_{B_R} |\nabla_k f|^2 \,\mathrm{d}\nu_U. \end{aligned} \tag{6.7}$$

Here we used the Poincaré inequality of Theorem 6.1 and the bounds $e^{-R^p} \leq e^{-U} \leq 1$ on B_R.

On the other hand, we can use Proposition 4.1 applied to the function $(f - \zeta)\mathbb{1}_{\mathbb{R}^N \setminus B_R}$, where $\mathbb{1}_X$ is the indicator function of set X, to obtain

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R} |f - \zeta|^2 \,\mathrm{d}\nu_U &\leq R^{-2(p-1)} \int_{|x| \geq R} |f(x) - \zeta|^2 |x|^{2(p-1)} \,\mathrm{d}\nu_U(x) \\ &\leq C R^{-2(p-1)} \int_{|x| \geq R} |\nabla_k f|^2 \,\mathrm{d}\nu_U + D R^{-2(p-1)} \int_{|x| \geq R} |f - \zeta|^2 \,\mathrm{d}\nu_U. \end{aligned}$$

But R was an arbitrary positive number so we are free to choose it such that $DR^{-2(p-1)} < 1$. Then we have

$$\int_{\mathbb{R}^N \setminus B_R} |f - \zeta|^2 \,\mathrm{d}\nu_U \leq \frac{C R^{-2(p-1)}}{1 - D R^{-2(p-1)}} \int_{|x| \geq R} |\nabla_k f|^2 \,\mathrm{d}\nu_U. \tag{6.8}$$

Adding the inequalities (6.7) and (6.8), we obtain (6.6), and therefore, by the observation above, the Proposition is proved. □

7 Tight log-Sobolev inequalities

We now have all the ingredients to obtain tight log-Sobolev inequalities. The first is a tight version of the log-Sobolev inequality from Theorem 5.1.

Theorem 7.1. *Assume $N + 2\gamma > 2$. Let $p \geq 2$ and consider the probability measure ν_U with $U(x) = |x|^p$. Then there exists a constant $C > 0$ such that the inequality*

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \, \mathrm{d}\nu_U} \, \mathrm{d}\nu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 \, \mathrm{d}\nu_U \tag{7.1}$$

holds for all $f \in H^1_k(\nu_U)$.

In order to prove this result we will need the following inequality, known as Rothaus's lemma (see [17, Lemma 9]).

Lemma 7.2. *Recall that*

$$\mathrm{Ent}(g) := \int_{\mathbb{R}^N} g \log g \, \mathrm{d}\nu_U - \int_{\mathbb{R}^N} g \, \mathrm{d}\nu_U \log \int_{\mathbb{R}^N} g \, \mathrm{d}\nu_U,$$

for $g \geq 0$. Then, for all f measurable with $\int_{\mathbb{R}^N} f \, \mathrm{d}\nu_U = 0$, we have the inequality

$$\mathrm{Ent}((f + c)^2) \leq \mathrm{Ent}(f^2) + 2 \int_{\mathbb{R}^N} f^2 \, \mathrm{d}\nu_U,$$

for all $c \in \mathbb{R}$.

Proof of Theorem 7.1. By Rothaus's lemma we have

$$\mathrm{Ent}(f^2) \leq \mathrm{Ent}\left(\left(f - \int_{\mathbb{R}^N} f \, \mathrm{d}\nu_U\right)^2\right) + 2 \int_{\mathbb{R}^N} \left(f - \int_{\mathbb{R}^N} f \, \mathrm{d}\nu_U\right)^2 \mathrm{d}\nu_U.$$

Furthermore, from Theorem 5.1, we have

$$\mathrm{Ent}(f^2) \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 \, \mathrm{d}\nu_U + (2 + C_2) \int_{\mathbb{R}^N} \left(f - \int_{\mathbb{R}^N} f \, \mathrm{d}\nu_U\right)^2 \mathrm{d}\nu_U.$$

Finally, using the Poincaré inequality of Proposition 6.4, we obtain

$$\mathrm{Ent}(f^2) \leq (C_1 + C(2 + C_2)) \int_{\mathbb{R}^N} |\nabla_k f|^2 \, \mathrm{d}\nu_U,$$

which is exactly what we wanted to prove. □

As we shall see in the next section, the condition $p \geq 2$ in the previous theorem is necessary. However, in the range $1 < p < 2$ we can still obtain a Φ-Sobolev inequality. This is the object of the following theorem, which is a tight version of the generalised log-Sobolev inequality of Theorem 5.2, and it is obtained from this result in a manner very similar to the proof that we have just seen.

Theorem 7.3. *Assume $N + 2\gamma > 2$. Let $1 < p < 2$ and $s = 2\frac{p-1}{p}$ and consider the probability measure ν_U with $U(x) = |x|^p$. Let also*

$$\Phi(x) = x(\log(x + 1))^s.$$

Then there exists a constant $C > 0$ such that the inequality

$$\int_{\mathbb{R}^N} \Phi(f^2) \, \mathrm{d}\nu_U - \Phi\left(\int_{\mathbb{R}^N} f^2 \, \mathrm{d}\nu_U\right) \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 \, \mathrm{d}\nu_U,$$

holds for all $f \in H^1_k(\nu_U)$.

As before, we need the following generalisation of Rothaus's lemma (see [14, Lemma A.1]).

Lemma 7.4. *Let Φ be as in the statement of the theorem and define, for $g \geq 0$,*

$$\mathrm{Ent}_\Phi(g) := \int_{\mathbb{R}^N} \Phi(g)\,\mathrm{d}\nu_U - \Phi\left(\int_{\mathbb{R}^N} g\,\mathrm{d}\nu_U\right).$$

Then there exist constants $A_1, B_1 > 0$ such that for any f with $\int_{\mathbb{R}^N} f\,\mathrm{d}\nu_U = 0$ we have

$$\mathrm{Ent}_\Phi((f+c)^2) \leq A_1 \mathrm{Ent}_\Phi(f^2) + B_1 \int_{\mathbb{R}^N} f^2\,\mathrm{d}\nu_U$$

for all $c \in \mathbb{R}$.

Proof of Theorem 7.3. The proof of this goes along the same lines as the proof of Theorem 7.1. From the previous Lemma we have

$$\mathrm{Ent}_\Phi(f^2) \leq \mathrm{Ent}_\Phi\left(\left(f - \int_{\mathbb{R}^N} f\,\mathrm{d}\nu_U\right)^2\right) + 2\int_{\mathbb{R}^N}\left(f - \int_{\mathbb{R}^N} f\,\mathrm{d}\nu_U\right)^2 \mathrm{d}\nu_U. \tag{7.2}$$

However, here we cannot apply Theorem 5.2 directly to bound $\mathrm{Ent}_\Phi\left((f - \int f\,\mathrm{d}\nu_U)^2\right)$ since the quantity on the left-hand side of (5.2) is not the same as $\mathrm{Ent}_\Phi(f)$.

Instead, we note that

$$\begin{aligned}\mathrm{Ent}_\Phi(g) &= \int_{\mathbb{R}^N} g\left[(\log(1+g))^s - \left(\log\left(1 + \int_{\mathbb{R}^N} g\right)\right)^s\right]\mathrm{d}\nu_U \\ &\leq \int_{\mathbb{R}^N} g\left|\log\frac{g+1}{\int g\,\mathrm{d}\nu_U + 1}\right|^s \mathrm{d}\nu_U,\end{aligned}$$

where we used the inequality $(a+b)^s \leq a^s + b^s$ which holds for all $a, b \geq 0$ since $s \in [0,1]$. We compute the integral on the right hand side separately over $X = \left\{x : g(x) \geq \int_{\mathbb{R}^N} g\,\mathrm{d}\nu_U\right\}$ and $\overline{X} = \mathbb{R}^N \setminus X$. On X we have

$$1 \leq \frac{g+1}{\int g\,\mathrm{d}\nu_U + 1} \leq \frac{g}{\int g\,\mathrm{d}\nu_U},$$

so

$$\int_X g\left|\log\frac{g+1}{\int g\,\mathrm{d}\nu_U + 1}\right|^s \mathrm{d}\nu_U \leq \int_{\mathbb{R}^N} g\left|\log\frac{g}{\int g\,\mathrm{d}\nu_U}\right|^s \mathrm{d}\nu_U.$$

On the other hand, on $\overline{X}$ we have

$$1 \leq \frac{\int g\,\mathrm{d}\nu_U + 1}{g+1} \leq 1 + \frac{\int g\,\mathrm{d}\nu_U}{g},$$

so

$$\begin{aligned}\int_{\overline{X}} g\left|\log\frac{g+1}{\int g\,\mathrm{d}\nu_U + 1}\right|^s \mathrm{d}\nu_U &= \int_{\overline{X}} g\left(\log\frac{\int g\,\mathrm{d}\nu_U + 1}{g+1}\right)^s \mathrm{d}\nu_U \\ &\leq \int_{\overline{X}} g\left(\frac{\int g\,\mathrm{d}\nu_U}{g}\right)^s \mathrm{d}\nu_U \leq \int_{\mathbb{R}^N} g\,\mathrm{d}\nu_U,\end{aligned}$$

where we first used the inequality $\log(1+x) \leq x$ for all $x \geq 0$, and then the fact that $s \leq 1$, so $\left(\frac{\int g\,\mathrm{d}\nu_U}{g}\right)^s \leq \frac{\int g\,\mathrm{d}\nu_U}{g}$. Thus

$$\mathrm{Ent}_\Phi(g) \leq \int_{\mathbb{R}^N} g\left|\log\frac{g}{\int g\,\mathrm{d}\nu_U}\right|^s \mathrm{d}\nu_U + \int_{\mathbb{R}^N} g\,\mathrm{d}\nu_U.$$

We can now apply the same strategy as before. First, by Theorem 5.2, we have

$$\mathrm{Ent}_\Phi(g^2) \le C_1 \int_{\mathbb{R}^N} |\nabla_k g|^2 \,\mathrm{d}\nu_U + (C_2+1) \int_{\mathbb{R}^N} g^2 \,\mathrm{d}\nu_U.$$

Taking $g = f - \int_{\mathbb{R}^N} f \,\mathrm{d}\nu_U$ in this inequality and applying (7.2), we have

$$\mathrm{Ent}_\Phi(f^2) \le A_1 C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 \,\mathrm{d}\nu_U + (A_1(C_2+1)+B_1) \int_{\mathbb{R}^N} \left(f - \int_{\mathbb{R}^N} f \,\mathrm{d}\nu_U \right)^2 \mathrm{d}\nu_U.$$

Finally, using the Poincaré inequality of Proposition 6.4, the proof is complete. □

8 Applications

8.1 Exponential integrability and measure concentration

As a consequence of the tight log-Sobolev inequality of Theorem 7.1, we can prove exponential integrability for Lipschitz functions (note that we say a function f is a-Lipschitz if $|f(x) - f(y)| \le a|x-y|$). The proof of this fact uses the classical Herbst argument (see [4]); for completeness, we give a sketch of the argument here.

Theorem 8.1. *Assume $N + 2\gamma > 2$. Let $p \ge 2$ and consider the probability measure ν_U with $U(x) = |x|^p$. For any a-Lipschitz function f and for any $b < \sqrt{\frac{2}{a^2 C}}$ (where C is the constant in* (7.1)*) we have*

$$\int_{\mathbb{R}^N} e^{b^2 f^2/2} \,\mathrm{d}\nu_U < \infty.$$

Proof. **Step 1:** assume f is G-invariant (in addition to the assumptions above). Then, for any $s \in \mathbb{R}$ we have

$$\int_{\mathbb{R}^N} e^{sf} \,\mathrm{d}\nu_U \le \exp\left(s \int_{\mathbb{R}^N} f \,\mathrm{d}\nu_U + a^2 C \frac{s^2}{4} \right). \tag{8.1}$$

It is enough to prove this inequality for a bounded function f. Indeed, the general case can then be obtained by defining $f_n(x) = \max\{\min\{f(x), n\}, -n\}$ for all $n \in \mathbb{N}$, and taking the limit $n \to \infty$ in (8.1) using Fatou's lemma.

From inequality (7.1) applied to the function $e^{sf/2}$ (recall that f is G-invariant, so $\nabla_k(e^{sf/2}) = \frac{s}{2} e^{sf/2} \nabla f$) we obtain

$$\int_{\mathbb{R}^N} e^{sf} \log e^{sf} \,\mathrm{d}\nu_U - \int_{\mathbb{R}^N} e^{sf} \,\mathrm{d}\nu_U \log \int_{\mathbb{R}^N} e^{sf} \,\mathrm{d}\nu_U \le C \frac{s^2}{4} \int_{\mathbb{R}^N} e^{sf} |\nabla f|^2 \,\mathrm{d}\nu_U. \tag{8.2}$$

Define $X(s) = \int_{\mathbb{R}^N} e^{sf} \,\mathrm{d}\nu_U$ and hence $X'(s) = s \int_{\mathbb{R}^N} f e^{sf} \,\mathrm{d}\nu_U$. Using this new notation, inequality (8.2) becomes

$$sX'(s) - X(s) \log X(s) \le a^2 C \frac{s^2}{4} X(s).$$

Here we also used the fact that since f is a-Lipschitz, then $|\nabla f| \le a$ a.e. Letting $Y(s) = \frac{1}{s} \log X(s)$ (with $Y(0) = \int f \,\mathrm{d}\nu_U$), this further becomes

$$Y'(s) \le \frac{a^2 C}{4}.$$

Integrating this inequality we obtain (8.1).

Multiplying (8.1) with $e^{-s^2/(2b^2)}$ we obtain

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^N} e^{-\frac{s^2}{2b^2}+sf}\,\mathrm{d}\nu_U\,\mathrm{d}s \le \int_{-\infty}^{\infty} e^{-\frac{s^2}{2b^2}+a^2C\frac{s^2}{4}+s\int f\,\mathrm{d}\nu_U}\,\mathrm{d}s.$$

Using Fubini's theorem and computing the integrals with respect to s, it follows that

$$\int_{\mathbb{R}^N} e^{b^2f^2/2}\,\mathrm{d}\nu_U \le \frac{\sqrt{2}}{\sqrt{2-b^2a^2C}}\exp\left(\frac{c^2}{2-b^2a^2C}\left(\int_{\mathbb{R}^N} f\,\mathrm{d}\nu_U\right)^2\right).$$

To conclude the proof in this case, it is enough to check that f is integrable. We refer to the proof of [4, Proposition 4.4.2] for a discussion of this using the Poincaré inequality.

Step 2: general f (not necessarily G-invariant). Before we prove this case, let us note here that for any G-invariant function g and any Weyl chamber H we have

$$\int_{\mathbb{R}^N} g\,\mathrm{d}\mu_k = \sum_{H'}\int_{H'} g\,\mathrm{d}\mu_k = |G|\int_H g\,\mathrm{d}\mu_k, \tag{8.3}$$

where the sum goes over all Weyl chambers H' and recall that $|G|$ is the number of Weyl chambers. Indeed, this is because for any Weyl chamber H' there exists $\alpha \in R$ such that $H' = \sigma_\alpha H$, so by a change of variables $y = \sigma_\alpha x$ we have

$$\int_H g\,\mathrm{d}\mu_k = \int_{H'} g\,\mathrm{d}\mu_k.$$

For any Weyl chamber H, let $f\big|_H : H \to \mathbb{R}$ be the restriction of f to H, and let $\tilde{f}_H : \mathbb{R}^N \to \mathbb{R}$ be the G-invariant function equal to $f\big|_H$ on each Weyl chamber, i.e.,

$$\tilde{f}_H(\sigma_\alpha x) = f\big|_H(x) \qquad \forall x \in H,\ \forall \alpha \in R_+.$$

Then $\tilde{f}_H$ is also a-Lipschitz. Indeed, let $x, y \in \mathbb{R}^N$. As the Weyl group G acts simply transitively on the set of Weyl chambers, there exist $g_1, g_2 \in G$ such that g_1x, g_1y are both in H. Then, from the definition of $\tilde{f}_H$ and using the fact that f is a-Lipschitz, we obtain

$$|\tilde{f}_H(x) - \tilde{f}_H(y)| = |f\big|_H(g_1x) - f\big|_H(g_2y)| = |f(g_1x) - f(g_2y)| \le a|g_1x - g_2y|. \tag{8.4}$$

Since g_1x and g_2y belong to the same Weyl chamber, by Lemma 8.2 below we have

$$|g_1x - g_2y| \le |g_1x - (g_1g_2^{-1})g_2y| = |g_1x - g_1y| = |x - y|, \tag{8.5}$$

where in the last step we used the fact that G is a subgroup of the orthogonal group $O(N)$, so it is distance-preserving. From (8.4) and (8.5) it follows that

$$|\tilde{f}_H(x) - \tilde{f}_H(y)| \le a|x - y|,$$

which proves that $\tilde{f}_H$ is a-Lipschitz.

As $\tilde{f}_H$ is a-Lipschitz, from Step 1 above, we have

$$|G|\int_H e^{b^2f^2/2}\,\mathrm{d}\nu_U = \int_{\mathbb{R}^N} e^{b^2\tilde{f}_H^2}\,\mathrm{d}\nu_U < \infty.$$

Here in the first equality we used property (8.3) and the fact that $\tilde{f}_H = f$ on H. Finally, we have

$$\int_{\mathbb{R}^N} e^{b^2f^2/2}\,\mathrm{d}\nu_U = \sum_H \int_H e^{b^2f^2/2}\,\mathrm{d}\nu_U < \infty,$$

where the sum goes over all the Weyl chambers H. This completes the proof. □

For the reader's convenience, we include the proof of the following lemma. For a similar approach, see the proof of Theorem 2.12 in [13, Chapter VII].

Lemma 8.2. *Let $x, y \in \mathbb{R}^N$ belong to the same Weyl chamber associated to the Weyl group G. Then, we have*

$$\min_{g \in G} |x - gy| = |x - y|.$$

Proof. Suppose for contradiction that $\min_{g\in G} |x - gy| = |x - g'y|$ for some non-identity element $g' \in G$. Then x and $g'y$ must belong to different Weyl chambers, so, by definition, there exists a root $\alpha \in R$ such that $\langle \alpha, x\rangle$ and $\langle \alpha, g'y\rangle$ have different signs, i.e.,

$$\langle \alpha, x\rangle \cdot \langle \alpha, g'y\rangle < 0. \tag{8.6}$$

On the other hand, we can compute

$$|x - \sigma_\alpha g'y|^2 = \langle x - \sigma_\alpha g'y, x - \sigma_\alpha g'y\rangle = |x|^2 + |y|^2 - 2\langle x, \sigma_\alpha g'y\rangle, \tag{8.7}$$

and similarly

$$|x - g'y|^2 = |x|^2 + |y|^2 - 2\langle x, g'y\rangle. \tag{8.8}$$

Furthermore, using formula (2.1) (recall that $|\alpha|^2 = 2$), we obtain

$$\langle x, \sigma_\alpha g'y\rangle = \langle x, g'y\rangle - \langle \alpha, g'y\rangle\langle \alpha, x\rangle.$$

This implies, using (8.6), that $\langle x, \sigma_\alpha g'y\rangle > \langle x, g'y\rangle$, so, from equations (8.7) and (8.8), we obtain

$$|x - \sigma_\alpha g'y| < |x - g'y|,$$

which contradicts the choice of g'. This concludes the proof. □

As a by-product of the proof of Theorem 8.1, we next obtain a Gaussian measure concentration property.

Corollary 8.3. *Assume $N + 2\gamma > 2$. Let $p \geq 2$ and consider the probability measure ν_U with $U(x) = |x|^p$. For any G-invariant a-Lipschitz function f and for any $r \geq 0$ we have*

$$\nu_U\left(f \geq \int_{\mathbb{R}^N} f \,\mathrm{d}\nu_U + r\right) \leq e^{-r^2/(a^2C)}. \tag{8.9}$$

Proof. By Markov's inequality and (8.1) we have, for any $s \in \mathbb{R}$,

$$\begin{aligned}\nu_U\left(f \geq \int_{\mathbb{R}^N} f \,\mathrm{d}\nu_U + r\right) &= \nu_U\left(e^{sf} \geq \exp\left(s\int_{\mathbb{R}^N} f \,\mathrm{d}\nu_U + sr\right)\right)\\ &\leq e^{-s\int f \,\mathrm{d}\nu_U - sr}\int_{\mathbb{R}^N} e^{sf}\,\mathrm{d}\nu_U \leq e^{-sr + a^2C\frac{s^2}{4}}.\end{aligned}$$

The right hand side is minimised for $s = \frac{2r}{a^2C}$, and replacing this in the above inequality we obtain exactly (8.9), as required. □

Remark 8.4. Using exponential integrability we can see that the condition $p \geq 2$ in Theorem 7.1 is necessary. Indeed, assuming that (5.1) holds for some $1 < p < 2$, then Theorem 8.1 can be extended in exactly the same way to this case. In other words, this shows that $e^{b^2f^2/2}$ is integrable if f is a Lipschitz function. In particular, taking $f(x) = |x|$ which is 1-Lipschitz, we have that

$$\int_{\mathbb{R}^N} e^{-|x|^p + b^2|x|^2/2}\,\mathrm{d}\mu_k < \infty,$$

for some $b > 0$. Since the weight w_k of the measure μ_k is homogeneous of degree 2γ, using polar coordinates (see, for example, formula (2.4) in [19]), the above implies

$$\int_0^\infty r^{N+2\gamma-1} e^{-r^p + b^2 r^2/2} \, \mathrm{d}r < \infty,$$

which contradicts the assumption $p < 2$.

8.2 Functional inequalities for singular Boltzmann-Gibbs measures

As discussed in the introduction, the Dunkl setting allows us to rephrase some functional inequalities related to Boltzmann-Gibbs measures. We exploit this connection here to obtain such applications. The inequalities in this subsection are all stated for the classical gradient ∇f, and the probability measures we consider are supported on the closure of a Weyl chamber H, and take the form

$$\mathrm{d}\nu_{U,H} = \frac{1}{Z_H} \mathbb{1}_H e^{-|x|^p} \, \mathrm{d}\mu_k, \tag{8.10}$$

where $Z_H = \displaystyle\int_{\mathbb{R}^N} e^{-|x|^p} \mathbb{1}_H \, \mathrm{d}\mu_k$ is a normalising constant, 1_H is the indicator function of any Weyl chamber, and $p > 1$. The spaces $H^1(\nu_{U,H})$ used in this section are classical Sobolev spaces defined in terms of partial derivatives.

Firstly, as a corollary of Proposition 6.4, we obtain a Poincaré inequality for this setting.

Theorem 8.5. *Assume $N + 2\gamma > 2$. Let $p > 1$. Let H be any Weyl chamber associated with the root system R and consider the probability measure $\mathrm{d}\nu_{U,H}$ defined by* (8.10)*. Then there exists a constant $\tilde{C} > 0$ such that for any $f \in H^1(\nu_{U,H})$ we have*

$$\int_{\mathbb{R}^N} \left| f - \int_{\mathbb{R}^N} f \, \mathrm{d}\nu_{U,H} \right|^2 \mathrm{d}\nu_{U,H} \le \tilde{C} \int_{\mathbb{R}^N} |\nabla f|^2 \, \mathrm{d}\nu_{U,H}. \tag{8.11}$$

Proof. As in (6.5) we have

$$\int_{\mathbb{R}^N} \left| f - \int_{\mathbb{R}^N} f \, \mathrm{d}\nu_{U,H} \right|^2 \mathrm{d}\nu_{U,H} \le \int_{\mathbb{R}^N} |f - \zeta|^2 \, \mathrm{d}\nu_{U,H} \tag{8.12}$$

for any $\zeta \in \mathbb{R}$.

Let $f\big|_H : H \to \mathbb{R}$ be the restriction of f to H, and let $\tilde{f} : \mathbb{R}^N \to \mathbb{R}$ be the G-invariant function equal to $f\big|_H$ on each Weyl chamber, i.e.,

$$\tilde{f}(\sigma_\alpha x) = f\big|_H(x) \qquad \forall x \in H, \ \forall \alpha \in R_+. \tag{8.13}$$

Since $\tilde{f}$ is G-invariant, we have $\nabla_k \tilde{f} = \nabla \tilde{f}$. Moreover, since $f \in H^1(\mu_{U,H})$, it can be checked that also $\tilde{f} \in H^1_k(\mu_U)$.

Applying the Poincaré inequality (6.4) to the function $\tilde{f}$ we obtain

$$\int_{\mathbb{R}^N} \left| \tilde{f} - \int_{\mathbb{R}^N} \tilde{f} \, \mathrm{d}\nu_U \right|^2 \mathrm{d}\nu_U \le C \int_{\mathbb{R}^N} |\nabla_k \tilde{f}|^2 \, \mathrm{d}\nu_U. \tag{8.14}$$

Using $\nabla_k f = \nabla f$ and property (8.3), the inequality (8.14) becomes

$$\int_H \left| f - |G| \int_H f \, \mathrm{d}\nu_U \right|^2 \mathrm{d}\nu_U \le C \int_H |\nabla f|^2 \, \mathrm{d}\nu_U.$$

Using now the fact that $\mathbb{1}_H \, \mathrm{d}\nu_U = \frac{Z_H}{Z} \, \mathrm{d}\nu_{U,H}$, this inequality becomes

$$\int_{\mathbb{R}^N} \left| f - |G| \frac{Z_H}{Z} \int_{\mathbb{R}^N} f \, \mathrm{d}\nu_{U,H} \right|^2 \mathrm{d}\nu_{U,H} \leq C \int_{\mathbb{R}^N} |\nabla f|^2 \, \mathrm{d}\nu_{U,H}.$$

Taking $\zeta = |G| \frac{Z_H}{Z} \int_{\mathbb{R}^N} f \, \mathrm{d}\nu_{U,H}$ in (8.12) together with the previous inequality imply (8.11) with $\tilde{C} = 4C$. □

Similarly, from Theorem 7.1 we obtain a tight log-Sobolev inequality for this setting when $p \geq 2$.

Theorem 8.6. *Assume $N + 2\gamma > 2$. Let $p \geq 2$. Let H be any Weyl chamber associated with the root system R and consider the probability measure $\mathrm{d}\nu_{U,H}$ defined by (8.10). Then there exists a constant $C > 0$ such that the inequality*

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \, \mathrm{d}\nu_{U,H}} \, \mathrm{d}\nu_{U,H} \leq C \int_{\mathbb{R}^N} |\nabla f|^2 \, \mathrm{d}\nu_{U,H} \tag{8.15}$$

holds for all $f \in H^1(\nu_{U,H})$.

Proof. Consider the G-invariant function $\tilde{f} : \mathbb{R}^N \to \mathbb{R}$ defined by (8.13). Applying the log-Sobolev inequality (7.1) to the function $\tilde{f}$ and using property (8.3), we obtain

$$\int_H f^2 \log \frac{f^2}{\int_H f^2 \, \mathrm{d}\nu_U} \, \mathrm{d}\nu_U \leq C \int_H |\nabla f|^2 \, \mathrm{d}\nu_U + \log |G| \int_H f^2 \, \mathrm{d}\nu_U.$$

Using now the fact that $\mathbb{1}_H \, \mathrm{d}\nu_U = \frac{Z_H}{Z} \, \mathrm{d}\nu_{U,H}$, this inequality becomes

$$\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \, \mathrm{d}\nu_{U,H}} \, \mathrm{d}\nu_{U,H} \leq C \int_{\mathbb{R}^N} |\nabla f|^2 \, \mathrm{d}\nu_{U,H} + \log \left(|G| \frac{Z_H}{Z} \right) \int_{\mathbb{R}^N} f^2 \, \mathrm{d}\nu_{U,H}.$$

To obtain a tight log-Sobolev inequality we use the same method as in the proof of Theorem 7.1, making use of the Rothaus lemma and the Poincaré inequality (8.11). □

Example 8.7. Let us consider the case of root system A_{N-1} where we have $R_+ = \{e_i - e_j | 1 \leq i < j \leq N\}$ and one choice of Weyl chamber is $H = \{x \in \mathbb{R}^N | x_1 > x_2 > \ldots > x_N\}$. In this case, all roots belong to the same orbit of the reflection group $G = S_N$, so the multiplicity function reduces to a constant, i.e., $k_\alpha = k \geq 0$ for all $\alpha \in R_+$, and $w_k(x) = \prod_{i<j} (x_i - x_j)^{2k}$. Thus, the measure $\nu_{U,H}$ becomes

$$\mathrm{d}\nu_{U,H} = \frac{1}{Z_H} \mathbb{1}_{\{x_1 > x_2 > \ldots > x_N\}} e^{-|x|^p} \prod_{1 \leq i < j \leq N} (x_i - x_j)^{2k}.$$

Example 8.8. In the case of root system B_N we have $R_+ = \{\sqrt{2} e_i | 1 \leq i \leq N\} \cup \{e_i \pm e_j | 1 \leq i < j \leq N\}$ and a choice of Weyl chamber is $H = \{x \in \mathbb{R}^N | x_1 > x_2 > \ldots > x_N > 0\}$. Here, the multiplicity function reduces to two constants, say $k_1, k_2 \geq 0$ (depending on whether the root is of the form $\sqrt{2} e_i$, or $e_i \pm e_j$) and the Dunkl weight becomes $w_k(x) = 2^{k_1 N} \prod_{i=1}^N |x_i|^{2k_1} \prod_{i<j} (x_i - x_j)^{2k_2} \prod_{i<j} (x_i + x_j)^{2k_2}$. Thus, the measure (8.10) in this case equals

$$\mathrm{d}\nu_{U,H} = \frac{1}{Z_H} \mathbb{1}_{\{x_1 > \ldots > x_N > 0\}} e^{-|x|^p} \prod_{i=1}^N |x_i|^{2k_1} \prod_{i<j} (x_i - x_j)^{2k_2} \prod_{i<j} (x_i + x_j)^{2k_2}.$$

Finally, we note that we can obtain a Φ-Sobolev inequality in the range $1 < p < 2$ which complements Theorem 8.6. The proof of this fact uses Theorem 7.3 and goes along the same lines as above so we omit it here.

Theorem 8.9. *Assume $N+2\gamma>2$. Let $1<p<2$ and $s=2\frac{p-1}{p}$. Let H be any Weyl chamber associated with the root system R and consider the probability measure $\mathrm{d}\nu_{U,H}$ defined by* (8.10)*. Define*

$$\Phi(x)=x(\log(x+1))^s.$$

Then there exists a constant $C>0$ such that the inequality

$$\int_{\mathbb{R}^N}\Phi(f^2)\,\mathrm{d}\nu_{U,H}-\Phi\left(\int_{\mathbb{R}^N}f^2\,\mathrm{d}\nu_{U,H}\right)\leq C\int_{\mathbb{R}^N}|\nabla f|^2\,\mathrm{d}\nu_{U,H} \tag{8.16}$$

holds for all $f\in H^1(\nu_{U,H})$.

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