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Quasi-stationary distribution for the Langevin process in cylindrical domains, part II: overdamped limit*

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Abstract

Consider the Langevin process, described by a vector (positions and momenta) in $\mathbb{R}^d \times \mathbb{R}^d$. Let $\mathcal{O}$ be a $\mathcal{C}^2$ open bounded and connected set of $\mathbb{R}^d$. Recent works showed the existence of a unique quasi-stationary distribution (QSD) of the Langevin process on the domain $D := \mathcal{O} \times \mathbb{R}^d$. In this article, we study the overdamped limit of this QSD, i.e. when the friction coefficient goes to infinity. In particular, we show that the marginal law in position of the overdamped limit is the QSD of the overdamped Langevin process on the domain $\mathcal{O}$.

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1 Introduction

In statistical physics, the evolution of a molecular system at a given temperature is typically modeled by the Langevin dynamics

$$\begin{cases} \mathrm{d}q_t = M^{-1}p_t\mathrm{d}t, \\ \mathrm{d}p_t = F(q_t)\mathrm{d}t - \gamma M^{-1}p_t\mathrm{d}t + \sqrt{2\gamma\beta^{-1}}\mathrm{d}B_t, \end{cases} \tag{1.1}$$

where $d = 3N$ for a number N of particles, $(q_t, p_t) \in \mathbb{R}^d \times \mathbb{R}^d$ denotes the set of positions and momenta of the particles, $M \in \mathbb{R}^{d\times d}$ is the mass matrix, $F : \mathbb{R}^d \to \mathbb{R}^d$ is the force acting on the particles, $\gamma > 0$ is the friction parameter, and $\beta^{-1} = k_B T$ with k_B the Boltzmann constant and T the temperature of the system. Alternatively, the overdamped Langevin dynamics

$$\mathrm{d}\overline{q}_t = F(\overline{q}_t)\mathrm{d}t + \sqrt{2\beta^{-1}}\mathrm{d}B_t, \tag{1.2}$$

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may also be employed. Notice that both processes are related by the fact that when the force field F is conservative, that is to say when there exists $V:\mathbb{R}^d\to\mathbb{R}$ such that $F=-\nabla V$, then the stationary distribution of $(\overline{q}_t)_{t\geq 0}$ writes

$$\overline{\nu}(\mathrm{d}q)=\frac{1}{Z}\mathrm{e}^{-\beta V(q)},\qquad Z=\int_{\mathbb{R}^d}\mathrm{e}^{-\beta V(q)}\mathrm{d}q, \tag{1.3}$$

while the stationary distribution of $(q_t,p_t)_{t\geq 0}$ has the product structure

$$\nu(\mathrm{d}q\mathrm{d}p)=\overline{\nu}(\mathrm{d}q)\frac{\mathrm{e}^{-\frac{\beta|p|^2}{2}}}{(2\pi\beta^{-1})^{\frac{d}{2}}}\mathrm{d}p. \tag{1.4}$$

The dynamics presented above are used in particular to compute thermodynamic and dynamic quantities, with numerous applications in biology, chemistry and materials science. In many practical situations of interest, the system remains trapped for very long times in subsets of the phase space, called metastable states, see for example [16, Sections 6.3 and 6.4]. Typically, these states are defined in terms of positions only, and are thus open sets $\mathcal{O}$ of $\mathbb{R}^d$ for (1.2) or cylinders of the form $D=\mathcal{O}\times\mathbb{R}^d$ for (1.1). In such a case, it is expected that the process reaches a local equilibrium distribution within the metastable state before leaving it. This distribution is called the quasi-stationary distribution (QSD). The existence of this limiting behavior has been shown recently in [18], using compactness arguments. Similar results can also be found in the recent works: [20, Chapter 4] based on criterias developed in [2] by N. Champagnat and D. Villemonais and in [9] using a Lyapunov and an Harnack inequality argument.

The motivation for this work comes from the well-known fact that, when $\gamma\to\infty$, for all $T>0$, the process $(q_{\gamma t})_{t\in[0,T]}$ converges in distribution to $(\overline{q}_t)_{t\in[0,T]}$, hence the name overdamped Langevin dynamics for (1.2) (see [14, Proposition 2.15] and [12, 4] for instance) on the space of continuous functions on $[0,T]$, endowed with the supremum norm on $[0,T]$. Therefore, it is expected that the marginal law in position of the QSD on D of $(q_t,p_t)_{t\geq 0}$ converges weakly to the QSD on $\mathcal{O}$ of the overdamped Langevin process. We actually prove a more general result by perturbing the Langevin dynamics to obtain an independent couple, which will allow us to consider the marginals separately, making the proof much easier. To the best of our knowledge, this result is the first to provide an overdamped limit of the couple (position,velocity) for the Langevin process. We are then able to identify the weak limit of the QSD on D, from which we can easily deduce the weak convergence of the marginal distributions.

More precisely, we study the limit of the QSD on D, of $(q_t,p_t)_{t\geq 0}$, when the friction parameter γ goes to infinity and show that it converges to the product measure

$$\mu^{(\infty)}(\mathrm{d}q\mathrm{d}p)=\overline{\mu}(\mathrm{d}q)\frac{\mathrm{e}^{-\frac{\beta|p|^2}{2}}}{(2\pi\beta^{-1})^{\frac{d}{2}}}\mathrm{d}p, \tag{1.5}$$

where $\overline{\mu}$ is the QSD of the overdamped Langevin process $(\overline{q}_t)_{t\geq 0}$ in $\mathcal{O}$. This result is stated in Section 2 and it relies on recent results on the Langevin process which are recalled in Section 3.

2 Main results

We first introduce in Section 2.1 the notion of quasi-stationary distribution (QSD) and recall well-known results for the QSD of the overdamped Langevin process on a smooth bounded domain $\mathcal{O}$. We also recall in Section 2.2 recent results from the companion paper [18] related to the existence of a unique QSD of the Langevin process on the domain $D:=\mathcal{O}\times\mathbb{R}^d$. Finally, we state our main result regarding the overdamped limit of the Langevin QSD on D in Section 2.3.

2.1 Quasi-stationary distribution for the overdamped Langevin process

The notion of quasi-stationary distribution (QSD) is central in this text. We recall here its definition in a general setting, and refer to [3, 19] for a complete introduction.

Let E be a Polish space endowed with its Borel σ-algebra $\mathcal{B}(E)$, and let $(X_t)_{t\geq 0}$ be a time-homogeneous, strong Markov process in E with continuous sample-paths. For any $x \in E$, we denote by $\mathbb{P}_x$ the probability measure under which $X_0 = x$ almost surely, and for any probability measure θ on E, we define

$$\mathbb{P}_\theta(\cdot) := \int_E \mathbb{P}_x(\cdot)\theta(\mathrm{d}x).$$

Let D be an open subset of E and τ_∂ be the stopping time defined by

$$\tau_\partial := \inf\{t > 0 : X_t \notin D\}.$$

Definition 2.1 (QSD). *A probability measure μ on D is said to be a QSD on D of the process $(X_t)_{t\geq 0}$, if for all $A \in \mathcal{B}(D) := \{A \cap D, A \in \mathcal{B}(E)\}$, for all $t \geq 0$,*

$$\mathbb{P}_\mu(X_t \in A, \tau_\partial > t) = \mu(A)\mathbb{P}_\mu(\tau_\partial > t). \tag{2.1}$$

When $\mathbb{P}_\mu(\tau_\partial > t) > 0$, the identity (2.1) equivalently writes $\mathbb{P}_\mu(X_t \in A|\tau_\partial > t) = \mu(A)$. Now let $\beta > 0$ and $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ satisfying the following assumption.

Hypothesis 2.2. $F \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and $(B_t)_{t\geq 0}$ a d-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion. Under Hypothesis 2.2, the vector field F is locally Lipschitz continuous and therefore the stochastic differential equation (1.2) possesses a unique strong solution $(\overline{q}_t)_{0\leq t<\overline{\tau}_\infty}$ defined up to some explosion time $\overline{\tau}_\infty \in (0, +\infty]$. Let $\mathcal{O}$ be an open set of $\mathbb{R}^d$ satisfying the following assumption.

Hypothesis 2.3. *$\mathcal{O}$ is an open $\mathcal{C}^2$ bounded connected set of $\mathbb{R}^d$.*

Let $\overline{\tau}_\partial := \inf\{t > 0 : \overline{q}_t \notin \mathcal{O}\}$ be the first exit time from $\mathcal{O}$ of the process $(\overline{q}_t)_{0\leq t<\overline{\tau}_\infty}$. Under Hypotheses 2.2 and 2.3, the vector field F is Lipschitz continuous on $\mathcal{O}$ and therefore $\overline{\tau}_\partial \leq \overline{\tau}_\infty$.

It has been shown in [1, 7, 13, 11] that the overdamped Langevin process admits a unique QSD on $\mathcal{O}$, which moreover satisfies the following properties.

Theorem 2.4 (QSD of the overdamped Langevin process). *Under Hypotheses 2.2 and 2.3, there exists a unique QSD $\overline{\mu}$ on $\mathcal{O}$ of the process $(\overline{q}_t)_{t\geq 0}$. Furthermore,*

1. *there exists $\overline{\psi} \in \mathcal{C}^2(\mathcal{O}) \cap \mathcal{C}^b(\overline{\mathcal{O}})$ such that $\overline{\mu}(\mathrm{d}q) = \overline{\psi}(q)\mathrm{d}q$, where $\mathrm{d}q$ is the Lebesgue measure on $\mathbb{R}^d$,*
2. *there exists $\overline{\lambda}_0 > 0$ such that, if $\overline{q}_0$ is distributed according to $\overline{\mu}$, then $\overline{\tau}_\partial$ follows the exponential law with parameter $\overline{\lambda}_0$.*

2.2 Quasi-stationary distribution for the Langevin process

In this section we recall some results from [18] that will be used henceforth. Let $\gamma > 0$ and $\beta > 0$ independent of γ. Consider now the following Langevin process

$$\begin{cases} \mathrm{d}q_t^{(\gamma)} = p_t^{(\gamma)}\mathrm{d}t, \\ \mathrm{d}p_t^{(\gamma)} = F(q_t^{(\gamma)})\mathrm{d}t - \gamma p_t^{(\gamma)}\mathrm{d}t + \sqrt{2\gamma\beta^{-1}}\mathrm{d}B_t, \end{cases} \tag{2.2}$$

Under Hypothesis 2.2, the stochastic differential equation (2.2) possesses a unique strong solution $(X_t^{(\gamma)} = (q_t^{(\gamma)}, p_t^{(\gamma)}))_{0\leq t<\tau_\infty^{(\gamma)}}$, defined up to some explosion time $\tau_\infty^{(\gamma)} \in (0, +\infty]$.

Notice that, compared to (1.1), we consider here and henceforth the mass to be identity, so that momentum is identified with velocity.

Let $\tau_\partial^{(\gamma)}$ be the first exit time from D of the Langevin process $(X_t^{(\gamma)})_{t\geq 0}$ in (2.2), i.e.

$$\tau_\partial^{(\gamma)} = \inf\{t > 0 : X_t^{(\gamma)} \notin D\}.$$

Under Hypotheses 2.2 and 2.3, F is Lipschitz continuous on $\mathcal{O}$ and therefore $\tau_\partial^{(\gamma)} \leq \tau_\infty^{(\gamma)}$. Concerning the existence of a QSD on the domain $D := \mathcal{O} \times \mathbb{R}^d$, similar proofs, as in the overdamped Langevin case, do not apply here. In fact, the infinitesimal generator of the process $(X_t^{(\gamma)})_{t\geq 0}$ is not elliptic but only hypoelliptic, and the natural domain $D = \mathcal{O} \times \mathbb{R}^d$ is not bounded, even if $\mathcal{O}$ is bounded. However, using a compactness argument, analogous results to Theorem 2.4 for the Langevin process (2.2) are obtained in [18]:

Theorem 2.5 (QSD of the Langevin process). *Under Hypotheses 2.2 and 2.3, there exists a unique QSD $\mu^{(\gamma)}$ on D of the process $(X_t^{(\gamma)})_{t\geq 0}$. Furthermore,*

1. *there exists $\psi^{(\gamma)} \in \mathcal{C}^2(D) \cap \mathcal{C}^b(\overline{D})$ such that $\mu^{(\gamma)}(\mathrm{d}q\mathrm{d}p) = \psi^{(\gamma)}(q,p)\mathrm{d}q\mathrm{d}p$, where $\mathrm{d}q\mathrm{d}p$ is the Lebesgue measure on $\mathbb{R}^{2d}$,*
2. *there exists $\lambda_0^{(\gamma)} > 0$ such that, if $X_0^{(\gamma)}$ is distributed according to $\mu^{(\gamma)}$, then $\tau_\partial^{(\gamma)}$ follows the exponential law with parameter $\lambda_0^{(\gamma)}$.*

2.3 Main result: overdamped limit of the Quasi-stationary distribution of the Langevin process

To state the main results of this work, it is more convenient to keep track of the initial value q (resp. $x = (q,p)$) of the solution to (1.2) (resp. to (2.2)) by denoting the latter by $(\overline{q}_t^q)_{t\geq 0}$ (resp. $(X_t^{(\gamma),x} = (q_t^{(\gamma),x}, p_t^{(\gamma),x}))_{t\geq 0}$). Moreover, we need the following strengthening of Hypothesis 2.2.

Hypothesis 2.6. *$F \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and F is bounded and globally Lipschitz continuous on $\mathbb{R}^d$.*

The following theorem will be the key to obtain the overdamped limit of the QSD. It is an extension of the well-known convergence of the position marginal $(q_{\gamma t})_{t\in[0,T]}$ for the Langevin to the couple $((q_{\gamma t})_{t\in[0,T]}, p_{\gamma T})$ using a novel perturbative argument.

Theorem 2.7 (Generalization of the overdamped limit of the Langevin process). *Assume that Assumption 2.6 holds. Let $T > 0$ and $x = (q,p) \in \mathbb{R}^{2d}$. Let $Z \sim \mathcal{N}_d(0, \beta^{-1} I_d)$ be a Gaussian vector independent of the process $(\overline{q}_t^q)_{t\in[0,T]}$. The law of the couple $((q_{\gamma t}^{(\gamma),x})_{t\in[0,T]}, p_{\gamma T}^{(\gamma),x})$ converges weakly to the law of $((\overline{q}_t^q)_{t\in[0,T]}, Z)$ when $\gamma \to \infty$.*

Using this convergence, we are then able to obtain the overdamped limit of the QSD.

Theorem 2.8 (QSD overdamped limit). *Let Hypotheses 2.2 and 2.3 hold. The QSD $\mu^{(\gamma)}$ converges weakly, when $\gamma \to \infty$, to the probability measure $\mu^{(\infty)}$ on D defined by:*

$$\mu^{(\infty)}(\mathrm{d}q\mathrm{d}p) := \overline{\mu}(\mathrm{d}q)\frac{\mathrm{e}^{-\frac{\beta|p|^2}{2}}}{(2\pi\beta^{-1})^{\frac{d}{2}}}\mathrm{d}p. \tag{2.3}$$

Furthermore, the eigenvalue $\lambda_0^{(\gamma)}$ associated with the QSD satisfies

$$\lambda_0^{(\gamma)} \underset{\gamma\to\infty}{\sim} \frac{\overline{\lambda}_0}{\gamma},$$

where $\overline{\lambda}_0$ is defined in Theorem 2.4.

Theorem 2.7 is proven in Section 3.1 and Theorem 2.8 is proven in Section 3.2.

3 Proofs

We are interested in the behavior of the QSD of the Langevin process defined in (2.2) when γ goes to infinity. We shall use the following notation: under Assumption 2.6, for any $x = (q,p) \in \mathbb{R}^d$, we denote by $(X_t^{(\gamma),x} = (q_t^{(\gamma),x}, p_t^{(\gamma),x}))_{t\geq 0}$ the solution to (2.2) with initial condition x, and by $(\overline{q}_t^{(\gamma),q})_{t\geq 0}$ the solution to the stochastic differential equation (1.2) with initial condition q and driven by the Brownian motion $(B_t^{(\gamma)})_{t\geq 0} = (\frac{B_{\gamma t}}{\sqrt{\gamma}})_{t\geq 0}$. All these processes are defined on the same probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and it is more convenient to keep track of the initial condition of each process with the superscript notation rather than in the probability measure. We also emphasize the fact that under Assumption 2.6, uniqueness in distribution holds for the stochastic differential equation (1.2) and therefore the law of the process $(\overline{q}_t^{(\gamma),q})_{t\geq 0}$ does not depend on γ.

3.1 Proof of Theorem 2.7

Let $x = (q,p) \in \mathbb{R}^{2d}$, $T > 0$. First, let us briefly show the scheme of proof for the convergence of the marginal laws $(q_{\gamma t}^{(\gamma),x})_{t\in[0,T]}$ and $p_{\gamma T}^{(\gamma),x}$, which is standard in the litterature. Second, we introduce a perturbed Langevin dynamics having the same overdamped limit as the dynamics (2.2). The perturbed dynamics being an independent couple, we shall retrieve its overdamped limit through the overdamped limit of the marginals, from which we will conclude the proof of Theorem 2.7.

Let us start by considering the convergence of the marginal laws of $((q_{\gamma t}^{(\gamma),x})_{t\in[0,T]}, p_{\gamma T}^{(\gamma),x})$. Considering (2.2), we have almost surely, for $t \in [0,T]$,

$$q_{\gamma t}^{(\gamma),x} = q - \frac{p_{\gamma t}^{(\gamma),x} - p}{\gamma} + \int_0^t F(q_{\gamma s}^{(\gamma),x})\mathrm{d}s + \sqrt{2\beta^{-1}} B_t^{(\gamma)}. \tag{3.1}$$

Using Gronwall's lemma, we are able to deduce from this equality the inequalities (1) and (2) in Lemma 3.2, which ensure that the difference $(q_{\gamma t}^{(\gamma),x})_{t\in[0,T]} - (\overline{q}_t^{(\gamma),q})_{t\in[0,T]}$ converges in probability to 0, in the space of the bounded continuous functions on $[0,T]$. Furthermore, the process $(\overline{q}_t^{(\gamma),q})_{t\in[0,T]}$ shares the same law as the process $(\overline{q}_t^{q})_{t\in[0,T]}$, which does not depend on γ. Therefore, the law of the process $(q_{\gamma t}^{(\gamma),x})_{t\in[0,T]}$ converges weakly to the law of $(\overline{q}_t^{q})_{t\in[0,T]}$ when γ goes to infinity.

Moreover, it follows from (2.2) that for all $t \geq 0$,

$$p_t^{(\gamma),x} = p\mathrm{e}^{-\gamma t} + \mathrm{e}^{-\gamma t}\int_0^t \mathrm{e}^{\gamma s} F(q_s^{(\gamma),x})\mathrm{d}s + \sqrt{2\gamma\beta^{-1}}\mathrm{e}^{-\gamma t}\int_0^t \mathrm{e}^{\gamma s}\mathrm{d}B_s. \tag{3.2}$$

For $t \geq 0$, let

$$Y_t^{(\gamma)} := \sqrt{2\gamma\beta^{-1}}\mathrm{e}^{-\gamma^2 t}\int_0^{\gamma t} \mathrm{e}^{\gamma s}\mathrm{d}B_s, \tag{3.3}$$

then evaluating (3.2) at $t = \gamma T$ for $T \geq 0$, we get

$$p_{\gamma T}^{(\gamma),x} = p\mathrm{e}^{-\gamma^2 T} + \gamma\mathrm{e}^{-\gamma^2 T}\int_0^T \mathrm{e}^{\gamma^2 s} F(q_{\gamma s}^{(\gamma),x})\mathrm{d}s + Y_T^{(\gamma)}. \tag{3.4}$$

Under Assumption 2.6, F is bounded. Besides, $Y_T^{(\gamma)} \sim \mathcal{N}_d(0, \beta^{-1}(1 - \mathrm{e}^{-2\gamma^2 T})I_d)$. Therefore, $Y_T^{(\gamma)} \xrightarrow[\gamma\to\infty]{\mathcal{L}} Z$ where $Z \sim \mathcal{N}_d(0, \beta^{-1}I_d)$ and $p_{\gamma T}^{(\gamma),x} \xrightarrow[\gamma\to\infty]{\mathcal{L}} Z$ by Slutsky's theorem.

The arguments above give the limit in law of the marginals of the couple $((q_{\gamma t}^{(\gamma),x})_{t\in[0,T]}, p_{\gamma T}^{(\gamma),x})$. To prove Theorem 2.7, it remains to show that, in the limit $\gamma \to \infty$, the two random variables $(q_{\gamma t}^{(\gamma),x})_{t\in[0,T]}$ and $p_{\gamma T}^{(\gamma),x}$ are independent. This is done by introducing a perturbed Langevin process defined later. Let $h_T^{(\gamma)} : [0,T] \mapsto \mathbb{R}$ and the process $(Z_{t,T}^{(\gamma)})_{t\in[0,T]}$ be defined as follows:

$$\forall t \in [0,T], \qquad h_T^{(\gamma)}(t) := \frac{2}{\gamma}\frac{\mathrm{e}^{-\gamma^2(T-t)} - \mathrm{e}^{-\gamma^2 T}}{1-\mathrm{e}^{-2\gamma^2 T}}, \tag{3.5}$$

$$Z_{t,T}^{(\gamma)} := \sqrt{2\beta^{-1}}B_t^{(\gamma)} - h_T^{(\gamma)}(t)Y_T^{(\gamma)}.$$

Let $(\mathcal{F}_t^{(\gamma),Z})_{t\in[0,T]}$ be the natural filtration of $(Z_{t,T}^{(\gamma)})_{t\in[0,T]}$. Under Assumption 2.6, Itô's fixed point argument [10, Thm 2.9 p. 289] shows that the stochastic differential equation

$$\begin{cases} \mathrm{d}w_t^{(\gamma),q} = F(w_t^{(\gamma),q})\mathrm{d}t + \mathrm{d}Z_{t,T}^{(\gamma)}, \\ w_0^{(\gamma),q} = q, \end{cases} \tag{3.6}$$

possesses a unique strong solution $(w_t^{(\gamma),q})_{t\in[0,T]}$, which is thus adapted to $(\mathcal{F}_t^{(\gamma),Z})_{t\in[0,T]}$.

The process $((w_t^{(\gamma),q})_{t\in[0,T]}, Y_T^{(\gamma)})$ satisfies the following lemmata.

Lemma 3.1 (Independence)**.** *Under Assumption 2.6, for all $T>0$, the process $(w_t^{(\gamma),q})_{t\in[0,T]}$ is independent of the random variable $Y_T^{(\gamma)}$.*

Proof. Let $T>0$. Since $(w_t^{(\gamma),q})_{t\in[0,T]}$ is $\mathcal{F}_T^{(\gamma),Z}$-measurable, it is sufficient to prove that the process $(Z_{t,T}^{(\gamma)})_{t\in[0,T]}$ is independent of $Y_T^{(\gamma)}$. It is clear that for any $t_1,\dots,t_k \in [0,T]$, the vector $(Z_{t_1,T}^{(\gamma)},\dots,Z_{t_k,T}^{(\gamma)}, Y_T^{(\gamma)})$ is Gaussian, therefore the independence is satisfied if and only if for all $t\in[0,T]$, the covariance matrix of $(Z_{t,T}^{(\gamma)}, Y_T^{(\gamma)})$ is null, which is indeed the case by an easy computation. □

Lemma 3.2 (Perturbed Langevin)**.** *Let Assumption 2.6 hold. There exists $C>0$ such that for all $T>0$, $x=(q,p)\in\mathbb{R}^{2d}$, $\gamma>1$,*

1. $\mathbb{E}\left[\sup_{t\in[0,T]}\left|q_{\gamma t}^{(\gamma),(q,p)} - w_t^{(\gamma),q}\right|\right] \le \frac{C}{\gamma}\left(1+|p|+\sqrt{\log(1+\gamma^2 T)}\right)\mathrm{e}^{CT}$,
2. $\mathbb{E}\left[\sup_{t\in[0,T]}\left|w_t^{(\gamma),q} - \overline{q}_t^{(\gamma),q}\right|\right] \le \frac{C}{\gamma}\mathrm{e}^{CT}$.

The proof of Lemma 3.2 is postponed to Section 3.3. These two lemmata now yield the following proof of Theorem 2.7.

Proof of Theorem 2.7. Let $T>0$, $x=(q,p)\in\mathbb{R}^{2d}$. Let Φ be a bounded k_Φ-Lipschitz continuous function on $\mathcal{C}([0,T],\mathbb{R}^d)$ equipped with the supremum norm on $[0,T]$ and let g be a bounded k_g-Lipschitz continuous function on $\mathbb{R}^d$. Our goal is to prove the following convergence:

$$\mathbb{E}\left[\Phi((q_{\gamma t}^{(\gamma),x})_{t\in[0,T]})g(p_{\gamma T}^{(\gamma),x})\right] \underset{\gamma\to\infty}{\longrightarrow} \mathbb{E}\left[\Phi((\overline{q}_t^{q})_{t\in[0,T]})\right]\mathbb{E}\left[g(Z)\right], \tag{3.7}$$

where, in the right-hand side, $(\overline{q}_t^q)_{t\in[0,T]}$ refers to the solution of (1.2) (which we recall has the same law as all processes $(\overline{q}_t^{(\gamma),q})_{t\in[0,T]}$ for $\gamma>0$).

By (1) in Lemma 3.2 and (3.4), there exists $C'>0$, depending on T, such that for all $\gamma>1$,

$$\begin{aligned}&\left|\mathbb{E}\left[\Phi((q_{\gamma t}^{(\gamma),x})_{t\in[0,T]})g(p_{\gamma T}^{(\gamma),x})\right] - \mathbb{E}\left[\Phi((w_t^{(\gamma),q})_{t\in[0,t]})g(Y_T^{(\gamma)})\right]\right| \\ &\le k_\Phi\|g\|_\infty\frac{C'}{\gamma}\left(1+|p|+\sqrt{\log(1+\gamma^2 T)}\right) + k_g\|\Phi\|_\infty\left(|p|\mathrm{e}^{-\gamma^2 T} + \frac{\|F\|_\infty}{\gamma}\right),\end{aligned}$$

which converges to 0 when $\gamma \to \infty$. Furthermore, by Lemma 3.1,

$$\mathbb{E}\left[\Phi((w_t^{(\gamma),q})_{t\in[0,T]})g(Y_T^{(\gamma)})\right] = \mathbb{E}\left[\Phi((w_t^{(\gamma),q})_{t\in[0,T]})\right]\mathbb{E}\left[g(Y_T^{(\gamma)})\right].$$

Since, $Y_T^{(\gamma)} \sim \mathcal{N}_d(0, \beta^{-1}(1-\mathrm{e}^{-2\gamma^2 T})I_d)$ then $Y_T^{(\gamma)} \underset{\gamma\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} Z$ with $Z \sim \mathcal{N}_d(0, \beta^{-1}I_d)$. As a result, $\mathbb{E}[g(Y_T^{(\gamma)})] \underset{\gamma\to\infty}{\longrightarrow} \mathbb{E}[g(Z)]$. Besides, using (2) in Lemma 3.2, one obtains that

$$\mathbb{E}\left[\Phi((w_t^{(\gamma),q})_{t\in[0,T]})\right] - \mathbb{E}\left[\Phi((\overline{q}_t^{(\gamma),q})_{t\in[0,T]})\right] \underset{\gamma\to\infty}{\longrightarrow} 0.$$

Moreover, $\mathbb{E}[\Phi((\overline{q}_t^{(\gamma),q})_{t\in[0,T]})] = \mathbb{E}[\Phi((\overline{q}_t^{q})_{t\in[0,T]})]$, since $(\overline{q}_t^{(\gamma),q})_{t\in[0,T]}$ and $(\overline{q}_t^{q})_{t\in[0,T]}$ share the same law, which concludes the proof of (3.7). □

3.2 Proof of Theorem 2.8

We consider in this section the weak limit, when $\gamma \to \infty$, of the QSD $\mu^{(\gamma)}$ of the Langevin process on D. Furthermore, we only assume here that F satisfies Hypothesis 2.2. In fact, we consider here the QSD on D of the process (2.2) which only depends on the values of the process inside D, hence on the values of F inside $\mathcal{O}$ by Friedman's uniqueness result [5, Theorem 5.2.1.]. As a result, one can extend F arbitrarily outside of $\mathcal{O}$ so that it satisfies Assumption 2.6. The notation for the overdamped Langevin process and its QSD remains the same as in Theorem 2.4.

The idea of the proof of Theorem 2.8 is the following. We pick an arbitrary sequence $(\gamma_n)_{n\geq 1}$ of positive numbers going to infinity. We first prove that the sequence of probability measures $(\mu^{(\gamma_n)})_{n\geq 1}$ is tight. Then using Prokhorov's theorem we obtain a convergent subsequence $(\mu^{(\gamma'_n)})_{n\geq 1}$ to a probability measure μ' on D. It is then left to prove that such a μ' is necessarily $\mu^{(\infty)}$ (see (2.3)), whatever the sequence $(\gamma'_n)_{n\geq 1}$. As a result, $\mu^{(\gamma)}$ necessarily converges weakly, when γ goes to infinity, to $\mu^{(\infty)}$ defined by (2.3).

This approach allows us to obtain the existence of a weak limit for the QSD and to identify it. However, it does not provide a speed of convergence, which can be interesting in applied contexts for instance. Nonetheless, in the simpler case of a stationary distribution, we are able to obtain a speed of convergence in Wasserstein distance for the overdamped limit of the stationary distribution, using estimates from Lemma 3.2 and Theorem 2.7, see [15]. Obtaining a speed of convergence for the QSD instead is still an open problem which is being looked at.

Now, let $(\gamma_n)_{n\geq 1}$ be an arbitrary sequence of positive numbers going to infinity. Let us first prove that the sequence $(\mu^{(\gamma_n)})_{n\geq 1}$ is tight. This is the consequence of the following lemma which is proven in Section 3.3.

Proposition 3.3 (Estimates on $\psi^{(\gamma)}$)**.** *Under Hypotheses 2.2 and 2.3, the density $\psi^{(\gamma)}$ of the QSD $\mu^{(\gamma)}$ of* (2.2) *satisfies the following properties:*

1. $\limsup_{\gamma\to\infty} \left\|\psi^{(\gamma)}\right\|_\infty < \infty$,
2. $\limsup_{\gamma\to\infty} \sup_{q\in\mathcal{O}} \int_{\mathbb{R}^d} \psi^{(\gamma)}(q,p)\mathrm{d}p < \infty$,
3. $\limsup_{\gamma\to\infty} \iint_D |p|\psi^{(\gamma)}(q,p)\mathrm{d}p\mathrm{d}q < \infty$.

Corollary 3.4 (Tightness)**.** *Under Hypotheses 2.2 and 2.3, the sequence of probability measures $(\mu^{(\gamma_n)})_{n\geq 1}$ is tight.*

Proof. Recall that for any $n \geq 1$, $\mu^{(\gamma_n)}$ is supported in D. For $k \geq 1$, let K_k be the compact subset of D defined by

$$K_k := \left\{(q,p) \in D : |p| \leq k, \mathrm{d}_\partial(q) \geq \frac{1}{k}\right\},$$

where d_∂ is the Euclidean distance to the boundary $\partial\mathcal{O}$. Let $K_k^c := D \setminus K_k = \{(q,p) \in D : |p| > k\} \cup \{(q,p) \in D : \mathrm{d}_\partial(q) < \frac{1}{k}\}$. Let us prove the following limit

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu^{(\gamma_n)}(K_k^c) = 0, \tag{3.8}$$

which immediately yields the required tightness.

Let $\mathcal{O}_k := \{q \in \mathcal{O} : \mathrm{d}_\partial(q) < \frac{1}{k}\}$. For all $n \geq 1$,

$$\begin{aligned}
\mu^{(\gamma_n)}(K_k^c) &\leq \iint_{D\cap\{|p|>k\}} \psi^{(\gamma_n)}(q,p)\mathrm{d}p\mathrm{d}q + \iint_{D\cap\{\mathrm{d}_\partial(q)<\frac{1}{k}\}} \psi^{(\gamma_n)}(q,p)\mathrm{d}p\mathrm{d}q \\
&\leq \iint_{D\cap\{|p|>k\}} \psi^{(\gamma_n)}(q,p)\frac{|p|}{k}\mathrm{d}p\mathrm{d}q + \int_{\mathrm{d}_\partial(q)<\frac{1}{k}} \left(\int_{\mathbb{R}^d} \psi^{(\gamma_n)}(q,p)\mathrm{d}p\right)\mathrm{d}q \\
&\leq \frac{\iint_D \psi^{(\gamma_n)}(q,p)|p|\mathrm{d}p\mathrm{d}q}{k} + |\mathcal{O}_k| \sup_{q\in\mathcal{O}} \int_{\mathbb{R}^d} \psi^{(\gamma_n)}(q,p)\mathrm{d}p.
\end{aligned}$$

The convergence (3.8) then follows from Proposition 3.3, which concludes the proof. □

Last, we state and prove here the following lemma which is used later in the proof of Theorem 2.8.

Lemma 3.5 (Convergence in distribution)**.** *Let Assumptions 2.6 and 2.3 hold. Let* $f \in \mathcal{C}^b(\mathcal{O})$, $g \in \mathcal{C}^b(\mathbb{R}^d)$. *For all* $(q,p) \in D$ *and* $t > 0$,

$$\mathbb{E}\left[f(q_{\gamma t}^{(\gamma),(q,p)})g(p_{\gamma t}^{(\gamma),(q,p)})\mathbb{1}_{\tau_\partial^{(\gamma),(q,p)}>\gamma t}\right] \underset{\gamma\to\infty}{\longrightarrow} \mathbb{E}\left[f(\overline{q}_t^q)\mathbb{1}_{\overline{\tau}_\partial^q>t}\right]\mathbb{E}\left[g(Z)\right]. \tag{3.9}$$

Proof. Let $(q,p) \in D$ and $T > 0$. Since $\mathcal{O}$ is an open set, we have for any $\gamma > 0$,

$$f(q_{\gamma T}^{(\gamma),(q,p)})g(p_{\gamma T}^{(\gamma),(q,p)})\mathbb{1}_{\tau_\partial^{(\gamma),(q,p)}>\gamma T} = \Phi\left((q_{\gamma t}^{(\gamma),(q,p)})_{t\in[0,T]}, p_{\gamma T}^{(\gamma),(q,p)}\right),$$

where $\Phi : \mathcal{C}([0,T],\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ is defined by

$$\Phi\left((q_t)_{t\in[0,T]}, z\right) = f(q_T)g(z)\mathbb{1}_{\inf_{t\in[0,T]}\mathrm{d}_\partial(q_t)>0},$$

and we take the convention that $\mathrm{d}_\partial(q') = 0$ if $q' \notin \mathcal{O}$. The functional Φ is not continuous on the space $\mathcal{C}([0,T],\mathbb{R}^d) \times \mathbb{R}^d$, which prevents us from applying Theorem 2.7 directly. Indeed, take for example a continuous trajectory $(q_t)_{t\in[0,T]}$ on $[0,T]$ which hits the boundary $\partial\mathcal{O}$ and is reflected back into the domain $\mathcal{O}$. One can construct a sequence of functions $((q_t^{(n)})_{t\in[0,T]})_{n\geq 1}$ converging in the supremum norm to $(q_t)_{t\in[0,T]}$ such that for all $n \geq 1$, $\inf_{t\in[0,T]}\mathrm{d}_\partial(q_t^{(n)}) > 0$. As a result, $(q_t)_{t\in[0,T]}$ is an example of a discontinuity point of the function Φ.

The discontinuity points of Φ are contained in the set of discontinuity points of $\mathbb{1}_{\inf_{t\in[0,T]}\mathrm{d}_\partial(q_t)>0}$, which can be characterized as follows. They correspond to the trajectories $(q_t)_{t\in[0,T]}$ which hit the boundary and remain on the boundary $\partial\mathcal{O}$ or come back inside $\mathcal{O}$. In fact if $(q_t)_{t\in[0,T]}$ is such that $\inf_{t\in[0,T]}\mathrm{d}_\partial(q_t) > 0$ or $\sup_{t\in[0,T]}\mathrm{dist}(q_t, \mathbb{R}^d \setminus \overline{\mathcal{O}}) > 0$, then taking a sequence of functions $(q_t^{(n)})_{t\in[0,T]}$ in $\mathcal{C}([0,T],\mathbb{R}^d)$ such that $\|q^{(n)} - q\|_\infty \leq \frac{\inf_{t\in[0,T]}\mathrm{d}_\partial(q_t)}{2}$ or $\|q^{(n)} - q\|_\infty \leq \frac{\sup_{t\in[0,T]}\mathrm{dist}(q_t,\mathbb{R}^d\setminus\overline{\mathcal{O}})}{2}$ then it follows from the $1-$Lipschitz continuity of the Euclidean distances $\mathrm{d}_\partial(\cdot)$ and $\mathrm{dist}(\cdot, \mathbb{R}^d \setminus \overline{\mathcal{O}})$ that

$$\mathbb{1}_{\inf_{t\in[0,T]}\mathrm{d}_\partial(q_t^{(n)})>0} \underset{n\to\infty}{\longrightarrow} \mathbb{1}_{\inf_{t\in[0,T]}\mathrm{d}_\partial(q_t)>0}.$$

As a consequence, the set of discontinuities of Φ is included in the set S of continuous trajectories $(q_t)_{t\in[0,T]}$ such that there exists $t_\partial \in [0,T]$ for which $q_{t_\partial} \in \partial\mathcal{O}$ but for all

$t \in [0,T]$, $q_t \in \overline{\mathcal{O}}$. Let us now justify that for all $q \in \mathcal{O}$, $\mathbb{P}((\overline{q}^q_t)_{t\in[0,T]} \in S) = 0$. Using the strong Markov property at $\overline{\tau}^q_\partial$, this is the case if for all $q \in \partial\mathcal{O}$, $\mathbb{P}(\overline{\tau}^q_\partial > 0) = 0$. This is clearly the case since for all $t > 0$, $q \in \partial\mathcal{O}$, $\mathbb{P}(\overline{\tau}^q_\partial \leq t) = 1$, see [6, p. 347]. Thus, the continuous mapping theorem ensures that $\Phi((q^{(\gamma),(q,p)}_{\gamma t})_{t\in[0,T]}, p^{(\gamma),(q,p)}_{\gamma t})$ converges in distribution to

$$\Phi\left((\overline{q}^q_t)_{t\in[0,T]}, Z\right) = \mathbb{E}\left[f(\overline{q}^q_t)\mathbb{1}_{\overline{\tau}^q_\partial > t}\right]\mathbb{E}\left[g(Z)\right],$$

which completes the proof. □

Proof of Theorem 2.8. Notice that since the QSD $\mu^{(\gamma)}$ does not depend on the values of F outside of $\mathcal{O}$, we can consider here, up to a modification of F outside of $\mathcal{O}$, that F satisfies Assumption 2.6. Therefore, the result of Theorem 2.7 still applies in the current setting.

By Corollary 3.4, the sequence $(\mu^{(\gamma_n)})_{n\geq 1}$ is tight, and therefore it is sequentially compact by Prokhorov's theorem. Let us consider a subsequence $(\gamma'_n)_{n\geq 1}$ such that the sequence $(\mu^{(\gamma'_n)})_{n\geq 1}$ converges weakly to a probability measure μ' on D when n goes to infinity. Let us now prove that $\mu' = \mu^{(\infty)}$ defined in (2.3) whatever the sequence $(\gamma'_n)_{n\geq 1}$, which will conclude the proof.

By Definition 2.1 of a QSD, one easily deduce that for all $f \in \mathcal{C}^b(\mathcal{O})$, $g \in \mathcal{C}^b(\mathbb{R}^d)$ and all $t > 0$,

$$\begin{aligned}
&\iint_D \mu^{(\gamma'_n)}(\mathrm{d}q\mathrm{d}p)\mathbb{E}\left[f(q^{(\gamma'_n),(q,p)}_{\gamma'_n t})g(p^{(\gamma'_n),(q,p)}_{\gamma'_n t})\mathbb{1}_{\tau^{(\gamma'_n),(q,p)}_\partial > \gamma'_n t}\right] \\
&= \mathrm{e}^{-\lambda_0^{(\gamma'_n)}\gamma'_n t}\underbrace{\iint_D f(q)g(p)\mu^{(\gamma'_n)}(\mathrm{d}q\mathrm{d}p)}_{\underset{n\to\infty}{\longrightarrow} \iint_D f(q)g(p)\mu'(\mathrm{d}q\mathrm{d}p)},
\end{aligned} \tag{3.10}$$

where $\tau^{(\gamma'_n),(q,p)}_\partial$ denotes the exit time from D for the process $(X^{(\gamma'_n),(q,p)}_t)_{t\geq 0}$.

Let $Z \sim \mathcal{N}_d(0, \beta^{-1}I_d)$ be a Gaussian vector independent of the process $(\overline{q}^q_t)_{t\in[0,T]}$ defined in (1.2). Let us prove that the term in the left-hand side of the equality (3.10) converges to $\iint_D \mathbb{E}[f(\overline{q}^q_t)\mathbb{1}_{\overline{\tau}^q_\partial > t}]\mathbb{E}[g(Z)]\mu'(\mathrm{d}q\mathrm{d}p)$. Considering the difference between the term in the left-hand side of the equality (3.10) and $\iint_D \mathbb{E}[f(\overline{q}^q_t)\mathbb{1}_{\overline{\tau}^q_\partial > t}]\mathbb{E}[g(Z)]\mu^{(\gamma'_n)}(\mathrm{d}q\mathrm{d}p)$ and partitioning the set $\{p \in \mathbb{R}^d\}$ into $\{|p| \leq K\}$ and $\{|p| > K\}$ for $K > 0$, one obtains

$$\begin{aligned}
&\left|\iint_D \mu^{(\gamma'_n)}(\mathrm{d}q\mathrm{d}p)\left(\mathbb{E}\left[f(q^{(\gamma'_n),(q,p)}_{\gamma'_n t})g(p^{(\gamma'_n),(q,p)}_{\gamma'_n t})\mathbb{1}_{\tau^{(\gamma'_n),(q,p)}_\partial > \gamma'_n t}\right] - \mathbb{E}\left[f(\overline{q}^q_t)\mathbb{1}_{\overline{\tau}^q_\partial > t}\right]\mathbb{E}\left[g(Z)\right]\right)\right| \\
&= \left|\iint_D \psi^{(\gamma'_n)}(q,p)\Bigg(\mathbb{E}\left[f(q^{(\gamma'_n),(q,p)}_{\gamma'_n t})g(p^{(\gamma'_n),(q,p)}_{\gamma'_n t})\mathbb{1}_{\tau^{(\gamma'_n),(q,p)}_\partial > \gamma'_n t}\right]\right. \\
&\quad \left. - \mathbb{E}\left[f(\overline{q}^q_t)\mathbb{1}_{\overline{\tau}^q_\partial > t}\right]\mathbb{E}\left[g(Z)\right]\Bigg)\mathrm{d}p\mathrm{d}q\right| \\
&\leq \|\psi^{(\gamma'_n)}\|_\infty \iint_{\mathcal{O}\times\{|p|\leq K\}} \\
&\quad \times \left|\underbrace{\mathbb{E}\left[f(q^{(\gamma'_n),(q,p)}_{\gamma'_n t})g(p^{(\gamma'_n),(q,p)}_{\gamma'_n t})\mathbb{1}_{\tau^{(\gamma'_n),(q,p)}_\partial > \gamma'_n t}\right] - \mathbb{E}\left[f(\overline{q}^q_t)\mathbb{1}_{\overline{\tau}^q_\partial > t}\right]\mathbb{E}\left[g(Z)\right]}_{\underset{n\to\infty}{\longrightarrow} 0 \text{ by Lemma } 3.5}\right|\mathrm{d}p\mathrm{d}q \\
&\quad + 2\|f\|_\infty\|g\|_\infty \iint_{\mathcal{O}\times\{|p|>K\}} \psi^{(\gamma'_n)}(q,p)\mathrm{d}p\mathrm{d}q.
\end{aligned}$$

Therefore, using Proposition 3.3 (i) and the dominated convergence theorem to get that the limsup of the first term in the right-hand side is zero,

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\iint_D \mu^{(\gamma_n')}(\mathrm{d}q\mathrm{d}p)\left(\mathbb{E}\left[f(q_{\gamma_n' t}^{(\gamma_n'),(q,p)})g(p_{\gamma_n' t}^{(\gamma_n'),(q,p)})\mathbb{1}_{\tau_\partial^{(\gamma_n'),(q,p)}>\gamma_n' t}\right]\right.\right.\\
&\quad\left.\left. -\mathbb{E}\left[f(\overline{q}_t^q)\mathbb{1}_{\overline{\tau}_\partial^q>t}\right]\mathbb{E}\left[g(Z)\right]\right)\right|\\
&\leq 2\|f\|_\infty\|g\|_\infty \limsup_{n\to\infty}\iint_{\mathcal{O}\times\{|p|>K\}}\psi^{(\gamma_n')}(q,p)\mathrm{d}p\mathrm{d}q\\
&\leq 2\|f\|_\infty\|g\|_\infty \limsup_{n\to\infty}\iint_{\mathcal{O}\times\{|p|>K\}}\psi^{(\gamma_n')}(q,p)\frac{|p|}{K}\mathrm{d}p\mathrm{d}q\\
&\leq \frac{2\|f\|_\infty\|g\|_\infty}{K}\limsup_{n\to\infty}\iint_D \psi^{(\gamma_n')}(q,p)|p|\mathrm{d}p\mathrm{d}q \underset{K\to\infty}{\longrightarrow} 0,
\end{aligned}$$

using Proposition 3.3 (iii).

Consequently,

$$\begin{aligned}
&\iint_D \mu^{(\gamma_n')}(\mathrm{d}q\mathrm{d}p)\left(\mathbb{E}\left[f(q_{\gamma_n' t}^{(\gamma_n'),(q,p)})g(p_{\gamma_n' t}^{(\gamma_n'),(q,p)})\mathbb{1}_{\tau_\partial^{(\gamma_n'),(q,p)}>\gamma_n' t}\right]\right.\\
&\quad\left. -\mathbb{E}\left[f(\overline{q}_t^q)\mathbb{1}_{\overline{\tau}_\partial^q>t}\right]\mathbb{E}\left[g(Z)\right]\right) \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

In addition,

$$\begin{aligned}
\iint_D \mu^{(\gamma_n')}(\mathrm{d}q\mathrm{d}p)\mathbb{E}\left[f(\overline{q}_t^q)\mathbb{1}_{\overline{\tau}_\partial^q>t}\right]\mathbb{E}\left[g(Z)\right] &= \mathbb{E}\left[g(Z)\right]\int_{\mathcal{O}}\mathbb{E}\left[f(\overline{q}_t^q)\mathbb{1}_{\overline{\tau}_\partial^q>t}\right]\left(\int_{p\in\mathbb{R}^d}\mu^{(\gamma_n')}(\mathrm{d}q\mathrm{d}p)\right)\\
&\underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[g(Z)\right]\int_{\mathcal{O}}\mathbb{E}\left[f(\overline{q}_t^q)\mathbb{1}_{\overline{\tau}_\partial^q>t}\right]\left(\int_{p\in\mathbb{R}^d}\mu'(\mathrm{d}q\mathrm{d}p)\right),
\end{aligned}$$

since $q\in\mathcal{O}\mapsto\mathbb{E}\left[f(\overline{q}_t^q)\mathbb{1}_{\overline{\tau}_\partial^q>t}\right]$ is a bounded and continuous function on $\mathcal{O}$, see [5, Theorem 6.5.2]. Consequently, taking $n\to\infty$ in the left-hand side of the equation (3.10) and choosing $t=1$, it follows that $\lambda_0^{(\gamma_n')}\gamma_n'$ converges to a value $\lambda'\in[0,\infty)$. Hence, taking $n\to\infty$ again in Equation (3.10), it follows that for all $t>0$,

$$\mathbb{E}\left[g(Z)\right]\int_{\mathcal{O}}\mathbb{E}\left[f(\overline{q}_t^q)\mathbb{1}_{\overline{\tau}_\partial^q>t}\right]\left(\int_{p\in\mathbb{R}^d}\mu'(\mathrm{d}q\mathrm{d}p)\right)=\mathrm{e}^{-\lambda' t}\iint_D f(q)g(p)\mu'(\mathrm{d}q\mathrm{d}p). \tag{3.11}$$

Let $\mu'_{\mathcal{O}}$ be the probability measure on $\mathcal{O}$ defined by:

$$\mu'_{\mathcal{O}}(\mathrm{d}q)=\int_{p\in\mathbb{R}^d}\mu'(\mathrm{d}q\mathrm{d}p).$$

Taking $g=1$ and $f=1$ in (3.11), we obtain that $\mathbb{P}_{\mu'_{\mathcal{O}}}(\overline{\tau}_\partial>t)=\exp(-\lambda' t)$. Since the equality can also be extended to all functions $f\in\mathrm{L}^\infty(\mathcal{O})$, using the density of $\mathcal{C}^b(\mathcal{O})$ in $\mathrm{L}^\infty(\mathcal{O})$, one gets for $g=1$ and $f=\mathbb{1}_A$ in (3.11) with $A\in\mathcal{B}(\mathcal{O})$,

$$\frac{\mathbb{P}_{\mu'_{\mathcal{O}}}(\overline{q}_t\in A,\overline{\tau}_\partial>t)}{\mathbb{P}_{\mu'_{\mathcal{O}}}(\overline{\tau}_\partial>t)}=\mu'_{\mathcal{O}}(A).$$

Therefore, $\mu'_{\mathcal{O}}$ is the unique QSD on $\mathcal{O}$ of $(\overline{q}_t)_{t\geq 0}$ by Theorem 2.4, which admits the density $\overline{\psi}$ with respect to the Lebesgue measure on $\mathcal{O}$. In particular, one has that $\lambda'=\overline{\lambda}_0$. Finally, reinjecting this equality into (3.11), we obtain that μ' satisfies the equality (2.3) since $Z\sim\mathcal{N}_d(0,\beta^{-1}I_d)$, which concludes the proof. □

3.3 Proofs of the technical results

This section gathers the proofs of the technical results stated previously: Lemma 3.2 and Proposition 3.3.

3.3.1 Proof of Lemma 3.2

Proof of Lemma 3.2. Let $T > 0$, $x = (q,p) \in \mathbb{R}^{2d}$. Let us prove (1). We recall from (3.1) that almost surely, for all $t \in [0,T]$, for all $\gamma > 1$,

$$q_{\gamma t}^{(\gamma),x} = q - \frac{p_{\gamma t}^{(\gamma),x} - p}{\gamma} + \int_0^t F(q_{\gamma s}^{(\gamma),x})\mathrm{d}s + \sqrt{2\beta^{-1}}B_t^{(\gamma)}.$$

Furthermore, by (3.6), almost surely, for all $t \in [0,T]$,

$$w_t^{(\gamma),q} = q + \int_0^t F(w_s^{(\gamma),q})\mathrm{d}s + Z_{t,T}^{(\gamma)},$$

where we recall $Z_{t,T}^{(\gamma)} = \sqrt{2\beta^{-1}}B_t^{(\gamma)} - h_T^{(\gamma)}(t)Y_T^{(\gamma)}$, with $Y_T^{(\gamma)}$ defined by (3.3). It follows from (3.5) that for all $T > 0$, $\gamma > 0$ and $t \in [0,T]$,

$$\begin{aligned} h_T^{(\gamma)}(t) &\leq \frac{2}{\gamma}\frac{1 - \mathrm{e}^{-\gamma^2 T}}{1 - \mathrm{e}^{-2\gamma^2 T}} \\ &\leq \frac{2}{\gamma}. \end{aligned}$$

Therefore, by Grönwall's Lemma, since F is globally Lipschitz continuous with a Lipschitz coefficient $C_1 > 0$,

$$\sup_{t\in[0,T]} \left| q_{\gamma t}^{(\gamma),x} - w_t^{(\gamma),q} \right| \leq \left(\frac{\sup_{t\in[0,T]} \left| p_{\gamma t}^{(\gamma),x} - p \right|}{\gamma} + \frac{2}{\gamma}\left| Y_T^{(\gamma)} \right| \right) \mathrm{e}^{C_1 T}.$$

Moreover, by (3.2) and (3.3), almost surely, for $t \in [0,T]$,

$$\frac{p_{\gamma t}^{(\gamma),x} - p}{\gamma} = -\frac{1 - \mathrm{e}^{-\gamma^2 t}}{\gamma}p + \mathrm{e}^{-\gamma^2 t}\int_0^t \mathrm{e}^{\gamma^2 s}F(q_{\gamma s}^{(\gamma),x})\mathrm{d}s + \frac{Y_t^{(\gamma)}}{\gamma}.$$

Therefore, since $\gamma > 1$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \frac{\left| p_{\gamma t}^{(\gamma),x} - p \right|}{\gamma} \right] \leq \frac{|p| + \|F\|_\infty + \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^{(\gamma)}| \right]}{\gamma}. \tag{3.12}$$

Let $(H_t^{(\gamma)} = ((H_t^{(\gamma)})_1, \ldots, (H_t^{(\gamma)})_d))_{t\in[0,T]}$ be the strong solution on $\mathbb{R}^d$ of the following Ornstein-Uhlenbeck SDE:

$$\mathrm{d}H_t^{(\gamma)} = -\gamma H_t^{(\gamma)}\mathrm{d}t + \mathrm{d}B_t, \qquad H_0^{(\gamma)} = 0,$$

then it is easy to see that, almost surely, for $t \in [0,T]$, $Y_t^{(\gamma)} = \sqrt{2\gamma\beta^{-1}}H_{\gamma t}^{(\gamma)}$. Therefore, the Minkowski inequality applied to the Euclidean norm on $\mathbb{R}^d$ of $|Y_t^{(\gamma)}|$ ensures that

$$\sup_{t\in[0,T]} |Y_t^{(\gamma)}| \leq \sqrt{2\gamma\beta^{-1}} \sum_{i=1}^d \sup_{t\in[0,\gamma T]} |(H_t^{(\gamma)})_i|. \tag{3.13}$$

A sharp inequality on the expectation in the summand above is provided in [8] and ensures the existence of a universal constant $C_2 > 0$ such that for all $t \in [0,T]$, $\gamma > 0$ and $i \in [\![1,d]\!]$,

$$\mathbb{E}\left[\sup_{t\in[0,\gamma T]} |(H_t^{(\gamma)})_i|\right] \leq \frac{C_2}{\sqrt{\gamma}}\sqrt{\log(1+\gamma^2 T)}.$$

Reinjecting into (3.13), one gets $\mathbb{E}[\sup_{t\in[0,T]}|Y_t^{(\gamma)}|] \leq dC_2\sqrt{2\beta^{-1}}\sqrt{\log(1+\gamma^2T)}$. Therefore, the inequality (3.12) ensures the existence of a constant $C_3 > 0$ such that for all $\gamma > 1$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \frac{\left|p_{\gamma t}^{(\gamma),x} - p\right|}{\gamma}\right] \leq \frac{C_3}{\gamma}\left(1 + |p| + \sqrt{\log(1+\gamma^2 T)}\right).$$

Using the Cauchy-Schwarz inequality and the Itô isometry, one easily gets that $\mathbb{E}[|Y_T^{(\gamma)}|] \leq \sqrt{d\beta^{-1}}$. Therefore, for all $\gamma > 1$, $T > 0$, $t \in [0,T]$ and $x = (q,p) \in \mathbb{R}^{2d}$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|q_{\gamma t}^{(\gamma),x} - w_t^{(\gamma),q}\right|\right] \leq \frac{C_4}{\gamma}\left(1 + |p| + \sqrt{\log(1+\gamma^2 T)}\right)\mathrm{e}^{C_1 T}.$$

This concludes the proof of (1) and the proof of (2) also follows from the use of Gronwall's Lemma along with the previous estimates. □

3.3.2 Proof of Proposition 3.3

Let us now prove Proposition 3.3. In order to do so, we resort to the two following results.

Proposition 3.6 (Principal eigenvalue)**.** *Under Hypotheses 2.2 and 2.3,*

$$\limsup_{\gamma\to\infty} \lambda_0^{(\gamma)}\gamma < \infty. \tag{3.14}$$

The proof of Proposition 3.6 is postponed to the next section. In order to state the next lemma, let us first recall some results obtained in [17] related to the transition density of the Langevin process (2.2) absorbed at the boundary ∂D.

The transition kernel P_t^D of the process $(X_t)_{t\geq 0}$ absorbed at the boundary ∂D is defined by:

$$\forall t > 0, \quad \forall x \in \overline{D}, \quad \forall A \in \mathcal{B}(D), \qquad \mathrm{P}_t^D(x,A) := \mathbb{P}_x(X_t \in A, \tau_\partial > t).$$

It has been shown in [17, Theorem 2.20] that P_t^D admits a smooth transition density

$$(t,x,y) \in \mathbb{R}_+^* \times \overline{D} \times \overline{D} \mapsto \mathrm{p}_t^D(x,y) \in \mathcal{C}^\infty(\mathbb{R}_+^* \times D \times D) \cap \mathcal{C}(\mathbb{R}_+^* \times \overline{D} \times \overline{D}),$$

which admits the following Gaussian upper-bound, see [17, Theorem 2.19].

Theorem 3.7 (Gaussian upper-bound)**.** *Under Hypotheses 2.2 and 2.3, the transition density $\mathrm{p}_t^D(x,y)$ is such that for all $\alpha \in (0,1)$, there exists $c_\alpha > 0$, depending only on α, such that for all $t > 0$, for all $x, y \in D$,*

$$\mathrm{p}_t^D(x,y) \leq \frac{1}{\alpha^d}\sum_{j=0}^{\infty}\frac{\left(\|F\|_\infty c_\alpha\sqrt{\pi t}\right)^j}{(2\gamma\beta^{-1})^{j/2}\Gamma\left(\frac{j+1}{2}\right)}\widehat{\mathrm{p}}_t^{(\alpha)}(x,y), \tag{3.15}$$

where Γ is the Gamma function and $\widehat{\mathrm{p}}_t^{(\alpha)}(x,y)$ is the transition density of the Gaussian process $(\widehat{q}_t^{(\alpha)}, \widehat{p}_t^{(\alpha)})_{t\geq 0}$ defined by

$$\begin{cases} \mathrm{d}\widehat{q}_t^{(\alpha)} = \widehat{p}_t^{(\alpha)}\mathrm{d}t, \\ \mathrm{d}\widehat{p}_t^{(\alpha)} = -\gamma \widehat{p}_t^{(\alpha)}\mathrm{d}t + \dfrac{\sqrt{2\gamma\beta^{-1}}}{\sqrt{\alpha}}\mathrm{d}B_t. \end{cases} \tag{3.16}$$

Remark 3.8. Notice that, in particular, for all $\alpha \in (0,1)$, there exists $c_\alpha > 0$, depending only on α, such that for all $t > 0$, for all $x, y \in D$,

$$\mathrm{p}_{\gamma t}^D(x,y) \leq \frac{1}{\alpha^d}\sum_{j=0}^{\infty} \frac{\left(\|F\|_\infty c_\alpha \sqrt{\pi t}\right)^j}{(2\beta^{-1})^{j/2}\Gamma\left(\frac{j+1}{2}\right)} \widehat{\mathrm{p}}_{\gamma t}^{(\alpha)}(x,y),$$

where the prefactor is now independent of γ.

The purpose of the next lemma is to give some estimates satisfied by the transition density $\widehat{\mathrm{p}}_t^{(\alpha)}$ introduced in Theorem 3.7, which will prove to be useful for the proof of Proposition 3.3.

Let Φ_1, Φ_2 be the following positive continuous functions on $\mathbb{R}$:

$$\Phi_1 : \rho \in \mathbb{R} \mapsto \begin{cases} \frac{1-\mathrm{e}^{-\rho}}{\rho} & \text{if } \rho \neq 0, \\ 1 & \text{if } \rho = 0, \end{cases} \tag{3.17}$$

$$\Phi_2 : \rho \in \mathbb{R} \mapsto \begin{cases} \frac{3}{2\rho^3}\left[2\rho - 3 + 4\mathrm{e}^{-\rho} - \mathrm{e}^{-2\rho}\right] & \text{if } \rho \neq 0, \\ 1 & \text{if } \rho = 0. \end{cases} \tag{3.18}$$

One can show, see [17, Section 5.1], that for all $t \geq 0$ and $\alpha \in (0,1]$, the vector $(\widehat{q}_t^{(\alpha)}, \widehat{p}_t^{(\alpha)})$ admits the following law under $\mathbb{P}_{(q,p)}$

$$\begin{pmatrix} \widehat{q}_t^{(\alpha)} \\ \widehat{p}_t^{(\alpha)} \end{pmatrix} \sim \mathcal{N}_{2d}\left(\begin{pmatrix} m_q(t) \\ m_p(t) \end{pmatrix}, \frac{C(t)}{\alpha} \right), \tag{3.19}$$

where the mean vector is

$$m_q(t) := q + tp\Phi_1(\gamma t), \qquad m_p(t) := p\mathrm{e}^{-\gamma t},$$

and the matrix $C(t)$ is defined by:

$$C(t) := \begin{pmatrix} c_{qq}(t)I_d & c_{qp}(t)I_d \\ c_{qp}(t)I_d & c_{pp}(t)I_d \end{pmatrix},$$

where I_d is the identity matrix in $\mathbb{R}^{d\times d}$ and

$$c_{qq}(t) := \frac{\sigma^2 t^3}{3}\Phi_2(\gamma t), \qquad c_{qp}(t) := \frac{\sigma^2 t^2}{2}\Phi_1(\gamma t)^2, \qquad c_{pp}(t) := \sigma^2 t \Phi_1(2\gamma t). \tag{3.20}$$

The determinant of the covariance matrix $\frac{C(t)}{\alpha}$ is $\det(\frac{C(t)}{\alpha}) = (\frac{\sigma^4 t^4}{12\alpha}\phi(\gamma t))^d$ where ϕ is the positive continuous function defined by

$$\phi : \rho \in \mathbb{R} \mapsto 4\Phi_2(\rho)\Phi_1(2\rho) - 3\Phi_1(\rho)^4 = \begin{cases} \frac{6(1-\mathrm{e}^{-\rho})}{\rho^4}\left[-2 + \rho + (2+\rho)\mathrm{e}^{-\rho}\right] & \text{if } \rho \neq 0, \\ 1 & \text{if } \rho = 0. \end{cases} \tag{3.21}$$

Let us now prove the following lemma.

Lemma 3.9 (Properties of the transition densities)**.** *For any $t > 0$, $\alpha \in (0,1)$, there exist $C_t > 0$ and $\gamma_t > 1$ such that for all $\gamma \geq \gamma_t$ and $(q,p),(q',p') \in \mathbb{R}^{2d}$,*

$$\widehat{\mathrm{p}}^{(\alpha)}_{\gamma t}((q,p),(q',p')) \leq C_t, \tag{3.22}$$

and

$$\sup_{q' \in \mathcal{O}} \int_{\mathbb{R}^d} \widehat{\mathrm{p}}^{(\alpha)}_{\gamma t}((q,p),(q',p'))\mathrm{d}p' \leq C_t. \tag{3.23}$$

Proof of Lemma 3.9. Let $t > 0$ and $\alpha \in (0,1)$. The law of the Gaussian vector $(\widehat{q}^{(\alpha)}_{\gamma t}, \widehat{p}^{(\alpha)}_{\gamma t})$ detailed above ensures that for all $(q,p),(q',p') \in \mathbb{R}^{2d}$,

$$\mathrm{p}^{(\alpha)}_{\gamma t}((q,p),(q',p')) \leq \frac{1}{\sqrt{(2\pi)^{2d}\det\left(\frac{C(\gamma t)}{\alpha}\right)}} = \frac{1}{\sqrt{(2\pi)^{2d}\left(\frac{(2\gamma\beta^{-1})^2(\gamma t)^4}{12\alpha}\phi(\gamma^2 t)\right)^d}}.$$

Besides, since $\gamma^6 t^4 \phi(\gamma^2 t) \underset{\gamma\to\infty}{\longrightarrow} 6t$, the estimate (3.22) easily follows.

Let us now prove (3.23). Since $\widehat{\mathrm{p}}^{(\alpha)}_{\gamma t}((q,p),(q',p'))$ is the density of the Gaussian vector $(\widehat{q}^{(\alpha)}_{\gamma t}, \widehat{p}^{(\alpha)}_{\gamma t})$, the expression of $\int_{\mathbb{R}^d} \widehat{\mathrm{p}}^{(\alpha)}_t((q,p),(q',p'))\mathrm{d}p'$ corresponds to the marginal density of $\widehat{q}^{(\alpha)}_{\gamma t}$ under $\mathbb{P}_{(q,p)}$. Besides, under $\mathbb{P}_{(q,p)}$,

$$\widehat{q}^{(\alpha)}_{\gamma t} \sim \mathcal{N}_d\left(q + \gamma t p \Phi_1(\gamma^2 t), \frac{c_{qq}(\gamma t)}{\alpha} I_d\right), \quad \frac{c_{qq}(\gamma t)}{\alpha} = \frac{2\beta^{-1}t^3}{3\alpha}\gamma^4\Phi_2(\gamma^2 t),$$

so that

$$\begin{aligned}\int_{\mathbb{R}^d} \widehat{\mathrm{p}}^{(\alpha)}_{\gamma t}((q,p),(q',p'))\mathrm{d}p' &= \frac{(3\alpha)^{d/2}}{\left(4\pi\beta^{-1}t^3\gamma^4\Phi_2(\gamma^2 t)\right)^{d/2}} \mathrm{e}^{-\frac{3\alpha}{4\beta^{-1}t^3\gamma^4\Phi_2(\gamma^2 t)}\left|q'-q-\gamma t p\Phi_1(\gamma^2 t)\right|^2} \\ &\leq \frac{(3\alpha)^{d/2}}{\left(4\pi\beta^{-1}t^3\gamma^4\Phi_2(\gamma^2 t)\right)^{d/2}}.\end{aligned}$$

Since $t^3\gamma^4\Phi_2(\gamma^2 t) \underset{\gamma\to\infty}{\longrightarrow} 3t$, the upper bound (3.23) immediately follows. $\square$

Using the Gaussian upper-bound recalled in Remark 3.8, we are now able to prove Proposition 3.3.

Proof of Proposition 3.3. For any $\alpha \in (0,1)$ and any $T > 0$, there exists $C_{\alpha,T} > 0$ such that for all $\gamma > 0$, for all $t \in (0,T]$, for all $x, y \in D$,

$$\mathrm{p}^D_{\gamma t}(x,y) \leq C_{\alpha,T}\widehat{\mathrm{p}}^{(\alpha)}_{\gamma t}(x,y), \tag{3.24}$$

where $\widehat{\mathrm{p}}^{(\alpha)}_s(x,y)$ is the transition density of the Gaussian process $(\widehat{q}^{(\alpha)}_s, \widehat{p}^{(\alpha)}_s)_{s\geq 0}$ defined in (3.16).

Let $\gamma > 0$, by Definition 2.1 of a QSD, $\mu^{(\gamma)}$ is such that for all $A \in \mathcal{B}(D)$,

$$\mathbb{P}_{\mu^{(\gamma)}}(X^{(\gamma)}_\gamma \in A, \tau^{(\gamma)}_\partial > \gamma) = \mu^{(\gamma)}(A)\mathrm{e}^{-\lambda^{(\gamma)}_0\gamma},$$

since $\mathbb{P}_{\mu^{(\gamma)}}(\tau^{(\gamma)}_\partial > \gamma) = \mathrm{e}^{-\lambda^{(\gamma)}_0\gamma}$ because $\tau^{(\gamma)}_\partial$ follows the exponential law of parameter $\lambda^{(\gamma)}_0$, see [18, Theorem 2.13].

The equality above being satisfied for any $A \in \mathcal{B}(D)$, and since $\mu^{(\gamma)}$ has the continuous density $\psi^{(\gamma)}$ with respect to the Lebesgue measure on D, one deduces that for all $(q',p') \in D$,

$$\psi^{(\gamma)}(q',p') = \mathrm{e}^{\lambda^{(\gamma)}_0\gamma}\iint_D \psi^{(\gamma)}(q,p)\mathrm{p}^D_\gamma((q,p),(q',p'))\mathrm{d}p\mathrm{d}q.$$

Let $\alpha \in (0,1)$. Using Remark 3.8, there exists $C > 0$ such that for all $\gamma > 0$ and $(q',p') \in D$,

$$\psi^{(\gamma)}(q',p') \leq C\mathrm{e}^{\lambda_0^{(\gamma)}\gamma} \iint_D \psi^{(\gamma)}(q,p)\widehat{\mathrm{p}}_\gamma^{(\alpha)}((q,p),(q',p'))\mathrm{d}p\mathrm{d}q, \tag{3.25}$$

where $\widehat{\mathrm{p}}_t^{(\alpha)}$ is the transition density of the process $(\widehat{q}_t^{(\alpha)}, \widehat{p}_t^{(\alpha)})_{t\geq 0}$ defined in (3.16). By Proposition 3.6 and the upper-bounds (3.22) and (3.23) in Lemma 3.9, the first two estimates in Proposition 3.3 follow from (3.25) and the fact that $\psi^{(\gamma)}$ is the density of a probability measure on D. It remains now to prove the last estimate in Proposition 3.3.

It follows from Fubini-Tonelli's theorem and the inequality (3.25) that

$$\iint_D \psi^{(\gamma)}(q',p')|p'|\mathrm{d}p'\mathrm{d}q' \leq C\mathrm{e}^{\lambda_0^{(\gamma)}\gamma} \iint_D \psi^{(\gamma)}(q,p) \left(\iint_D \widehat{\mathrm{p}}_\gamma^{(\alpha)}((q,p),(q',p'))|p'|\mathrm{d}p'\mathrm{d}q'\right)\mathrm{d}p\mathrm{d}q. \tag{3.26}$$

Let us now prove that

$$\limsup_{\gamma\to\infty} \sup_{(q,p)\in D} \iint_D \widehat{\mathrm{p}}_\gamma^{(\alpha)}((q,p),(q',p'))|p'|\mathrm{d}p'\mathrm{d}q' < \infty,$$

this will conclude the proof using (3.14) and (3.26).

Let us start by rewriting, for any $(q,p) \in D$ and $\gamma > 0$,

$$\begin{aligned}\iint_D \widehat{\mathrm{p}}_\gamma^{(\alpha)}((q,p),(q',p'))|p'|\mathrm{d}p'\mathrm{d}q' &= \mathbb{E}_{(q,p)}\left[\mathbb{1}_{\widehat{q}_\gamma^{(\alpha)}\in\mathcal{O}}|\widehat{p}_\gamma^{(\alpha)}|\right] \\ &\leq \mathbb{E}_{(q,p)}\left[|\widehat{p}_\gamma^{(\alpha)} - p\mathrm{e}^{-\gamma^2}|\right] + |p|\mathrm{e}^{-\gamma^2}\mathbb{P}_{(q,p)}\left(\widehat{q}_\gamma^{(\alpha)} \in \mathcal{O}\right),\end{aligned}$$

and recall that under $\mathbb{P}_{(q,p)}$, $\widehat{q}_\gamma^{(\alpha)}$ and $\widehat{p}_\gamma^{(\alpha)}$ have marginal distributions

$$\widehat{q}_\gamma^{(\alpha)} \sim \mathcal{N}_d\left(q + \gamma p\Phi_1(\gamma^2), \frac{2\beta^{-1}\gamma^4}{3\alpha}\Phi_2(\gamma^2)I_d\right), \qquad \widehat{p}_\gamma^{(\alpha)} \sim \mathcal{N}_d\left(p\mathrm{e}^{-\gamma^2}, \frac{2\beta^{-1}\gamma^2\Phi_1(2\gamma^2)}{\alpha}I_d\right).$$

As a consequence, we deduce from the Cauchy-Schwarz inequality that

$$\mathbb{E}_{(q,p)}\left[|\widehat{p}_\gamma^{(\alpha)} - p\mathrm{e}^{-\gamma^2}|\right] \leq \sqrt{\frac{2d\beta^{-1}\gamma^2\Phi_1(2\gamma^2)}{\alpha}},$$

the right-hand side of which is uniform in (q,p) and is bounded when $\gamma \to \infty$. On the other hand, let us define $\delta := \sup_{q,q'\in\mathcal{O}} |q - q'|$ (which is finite since $\mathcal{O}$ is bounded) and note that

$$\begin{aligned}\mathbb{P}_{(q,p)}\left(\widehat{q}_\gamma^{(\alpha)} \in \mathcal{O}\right) &\leq \mathbb{P}_{(q,p)}\left(|\widehat{q}_\gamma^{(\alpha)} - q| \leq \delta\right) \\ &= \mathbb{P}\left(\left|\gamma p\Phi_1(\gamma^2) + \sqrt{\frac{2\beta^{-1}\gamma^4}{3\alpha}\Phi_2(\gamma^2)}Z\right| \leq \delta\right),\end{aligned}$$

where $Z \sim \mathcal{N}_d(0, I_d)$. By the triangle, Markov and Cauchy-Schwarz inequalities, if $|p| \neq 0$ then

$$\begin{aligned}\mathbb{P}\left(\left|\gamma p\Phi_1(\gamma^2) + \sqrt{\frac{2\beta^{-1}\gamma^4}{3\alpha}\Phi_2(\gamma^2)}Z\right| \leq \delta\right) &\leq \mathbb{P}\left(\sqrt{\frac{2\beta^{-1}\gamma^4}{3\alpha}\Phi_2(\gamma^2)}|Z| + \delta \geq \gamma|p|\Phi_1(\gamma^2)\right) \\ &\leq \frac{\sqrt{\frac{2d\beta^{-1}\gamma^4}{3\alpha}\Phi_2(\gamma^2)} + \delta}{\gamma|p|\Phi_1(\gamma^2)},\end{aligned}$$

so that

$$|p|\mathrm{e}^{-\gamma^2}\mathbb{P}_{(q,p)}\left(\widehat{q}_\gamma^{(\alpha)} \in \mathcal{O}\right) \leq \mathrm{e}^{-\gamma^2}\frac{\sqrt{\frac{2d\beta^{-1}\gamma^4}{3\alpha}\Phi_2(\gamma^2)} + \delta}{\gamma\Phi_1(\gamma^2)},$$

the right-hand side of which is uniform in (q,p) and vanishes when $\gamma \to \infty$. □

3.3.3 Proof of Proposition 3.6

Let us finally prove Proposition 3.6. We will need the following intermediate lemma.

Lemma 3.10 (Uniform velocity tightness)**.** *Let Assumption 2.6 hold. For every* $\epsilon > 0$*, there exists* $M > 0$ *such that for all* $\gamma \geq 4$*,*

$$\sup_{(q,p)\in\mathcal{O}\times\mathrm{B}(0,M)} \mathbb{P}\left(p_\gamma^{(\gamma),(q,p)} \notin \mathrm{B}(0,M)\right) \leq \epsilon, \tag{3.27}$$

where $\mathrm{B}(0,M) := \{p \in \mathbb{R}^d : |p| < M\}$*.*

Proof. Let $\epsilon > 0$. Let us take $M \geq \frac{2\sqrt{d\beta^{-1}}}{\epsilon} + \|F\|_\infty$. By (2.2), for all $x = (q,p) \in \mathcal{O}\times\mathrm{B}(0,M)$ and $\gamma \geq 4$ (so that $\frac{M}{\gamma^2} + \frac{M}{\gamma} \leq \frac{M}{2}$),

$$\begin{aligned} \left|p_\gamma^{(\gamma),x}\right| &= \left|p\mathrm{e}^{-\gamma^2} + \gamma\mathrm{e}^{-\gamma^2}\int_0^1 \mathrm{e}^{\gamma^2 s}F(q_{\gamma s}^{(\gamma),x})\mathrm{d}s + Y_1^{(\gamma)}\right| \\ &\leq \frac{M}{\gamma^2}\underbrace{\gamma^2\mathrm{e}^{-\gamma^2}}_{<1} + \frac{\|F\|_\infty}{\gamma} + \left|Y_1^{(\gamma)}\right| \\ &< \frac{M}{2} + \left|Y_1^{(\gamma)}\right| \end{aligned}$$

since $M \geq \|F\|_\infty$ and $\gamma \geq 4$. Besides,

$$\mathbb{P}\left(\left|Y_1^{(\gamma)}\right| > M/2\right) \leq \frac{\mathbb{E}\left[\left|Y_1^{(\gamma)}\right|\right]}{M/2} \leq \frac{2\sqrt{d\beta^{-1}}}{M} \leq \epsilon$$

by definition of M. Therefore, for all $(q,p) \in \mathcal{O}\times\mathrm{B}(0,M)$,

$$\mathbb{P}\left(p_\gamma^{(\gamma),(q,p)} \notin \mathrm{B}(0,M)\right) \leq \epsilon. \qquad \square$$

Let us now prove Proposition 3.6.

Proof of Proposition 3.6. Let $q_0 \in \mathcal{O}$. Let $r \in (0,1)$ such that $\mathrm{B}(q_0, 2r) \subset \mathcal{O}$. For $q \in \mathbb{R}^d$, we define the following stopping time:

$$\overline{\tau}_0^{(\gamma),q} = \inf\{t > 0 : \overline{q}_t^{(\gamma),q} \notin \mathrm{B}(q_0, 3r/2)\}.$$

Let also $a := \inf_{q\in\mathrm{B}(q_0,r)} \mathbb{P}(\overline{q}_1^{(\gamma),q} \in \mathrm{B}(q_0, r/2), \overline{\tau}_0^{(\gamma),q} > 1)$. Notice that $a > 0$ since it is well known that the function $q \in \overline{\mathrm{B}(q_0,r)} \mapsto \mathbb{P}(\overline{q}_1^{(\gamma),q} \in \mathrm{B}(q_0,r/2), \overline{\tau}_0^{(\gamma),q} > 1)$ is continuous and positive on the compact set $\overline{\mathrm{B}(q_0,r)}$. Besides, a does not depend on γ since the law of the process $(\overline{q}_t^{(\gamma),q})_{t\geq 0}$ does not depend on γ. Let us take $\epsilon \in (0, \frac{a}{4})$ and $M > 0$ such that (3.27) in Lemma 3.10 is satisfied.

Step 1: Let us prove that there exists $\gamma_1 > 1$ such that

$$c := \inf_{\gamma\geq\gamma_1} \inf_{(q,p)\in\mathrm{B}(q_0,r)\times\mathrm{B}(0,M)} \mathbb{P}\left(X_\gamma^{(\gamma),(q,p)} \in \mathrm{B}(q_0,r)\times\mathrm{B}(0,M), \tau_\partial^{(\gamma),(q,p)} > \gamma\right) > 0. \tag{3.28}$$

For $(q,p) \in \mathrm{B}(q_0,r)\times\mathrm{B}(0,M)$,

$$\begin{aligned} &\mathbb{P}\left(X_\gamma^{(\gamma),(q,p)} \in \mathrm{B}(q_0,r)\times\mathrm{B}(0,M), \tau_\partial^{(\gamma),(q,p)} > \gamma\right) \\ &\geq \mathbb{P}\left(X_\gamma^{(\gamma),(q,p)} \in \mathrm{B}(q_0,r)\times\mathrm{B}(0,M), \tau_\partial^{(\gamma),(q,p)} > \gamma, \sup_{t\in[0,1]}\left|q_{\gamma t}^{(\gamma),(q,p)} - \overline{q}_t^{(\gamma),q}\right| \leq r/2\right). \end{aligned} \tag{3.29}$$

By (1) and (2) in Lemma 3.2, there exists $C_1 > 0$, depending on M, such that for all $\gamma > 4$,

$$\sup_{(q,p)\in \mathrm{B}(q_0,r)\times \mathrm{B}(0,M)} \mathbb{E}\left[\sup_{t\in[0,1]}\left|q^{(\gamma),(q,p)}_{\gamma t} - \overline{q}^{(\gamma),q}_t\right|\right] \leq C_1 \frac{1+\sqrt{\log(1+\gamma^2)}}{\gamma}. \tag{3.30}$$

Moreover, by (3.27) in Lemma 3.10,

$$\begin{aligned}
&\mathbb{P}\left(X^{(\gamma),(q,p)}_\gamma \in \mathrm{B}(q_0,r)\times \mathrm{B}(0,M), \tau^{(\gamma),(q,p)}_\partial > \gamma, \sup_{t\in[0,1]}\left|q^{(\gamma),(q,p)}_{\gamma t} - \overline{q}^{(\gamma),q}_t\right| \leq r/2\right)\\
&\geq \mathbb{P}\left(\overline{q}^{(\gamma),q}_1 \in \mathrm{B}(q_0,r/2), \overline{\tau}^{(\gamma),q}_0 > 1, \sup_{t\in[0,1]}\left|q^{(\gamma),(q,p)}_{\gamma t} - \overline{q}^{(\gamma),q}_t\right| \leq r/2\right) - \epsilon,
\end{aligned}$$

by definition of $\overline{\tau}^{(\gamma),q}_0$ and since $\mathrm{B}(q_0,2r)\subset \mathcal{O}$. Using (3.30) and the Markov inequality, it follows that for all $(q,p)\in \mathrm{B}(q_0,r)\times \mathrm{B}(0,M)$,

$$\begin{aligned}
&\mathbb{P}\left(\overline{q}^{(\gamma),q}_1 \in \mathrm{B}(q_0,r/2), \overline{\tau}^{(\gamma),q}_0 > 1, \sup_{t\in[0,1]}\left|q^{(\gamma),(q,p)}_{\gamma t} - \overline{q}^{(\gamma),q}_t\right| \leq r/2\right)\\
&\geq \mathbb{P}\left(\overline{q}^{(\gamma),q}_1 \in \mathrm{B}(q_0,r/2), \overline{\tau}^{(\gamma),q}_0 > 1\right) - \frac{2C_1}{\gamma r}(1+\sqrt{\log(1+\gamma^2)})\\
&\geq a - \frac{2C_1}{\gamma r}(1+\sqrt{\log(1+\gamma^2)}).
\end{aligned}$$

As a result, by (3.29) and by definition of a and ϵ, for all $\gamma > 4$,

$$\begin{aligned}
&\inf_{(q,p)\in \mathrm{B}(q_0,r)\times \mathrm{B}(0,M)} \mathbb{P}\left(X^{(\gamma),(q,p)}_\gamma \in \mathrm{B}(q_0,r)\times \mathrm{B}(0,M), \tau^{(\gamma),(q,p)}_\partial > \gamma\right)\\
&\geq a - \frac{2C_1}{\gamma r}(1+\sqrt{\log(1+\gamma^2)}) - \frac{a}{4}.
\end{aligned}$$

Hence, there exists $\gamma_1 > 4$ such that for all $\gamma \geq \gamma_1$,

$$\inf_{(q,p)\in \mathrm{B}(q_0,r)\times \mathrm{B}(0,M)} \mathbb{P}\left(X^{(\gamma),(q,p)}_\gamma \in \mathrm{B}(q_0,r)\times \mathrm{B}(0,M), \tau^{(\gamma),(q,p)}_\partial > \gamma\right) \geq \frac{a}{2}.$$

Step 2: Now let us prove (3.14). By (3.28), for all $\gamma \geq \gamma_1$,

$$\begin{aligned}
&\mathrm{e}^{\lambda^{(\gamma)}_0\gamma} \iint_{\mathrm{B}(q_0,r)\times \mathrm{B}(0,M)} \psi^{(\gamma)}(q,p)\mathbb{P}(X^{(\gamma),(q,p)}_\gamma \in \mathrm{B}(q_0,r)\times \mathrm{B}(0,M), \tau^{(\gamma),(q,p)}_\partial > \gamma)\mathrm{d}q\mathrm{d}p\\
&\geq c\mathrm{e}^{\lambda^{(\gamma)}_0\gamma} \iint_{\mathrm{B}(q_0,r)\times \mathrm{B}(0,M)} \psi^{(\gamma)}(q,p)\mathrm{d}q\mathrm{d}p.
\end{aligned}$$

Since $\psi^{(\gamma)}$ is the density of the QSD of the Langevin process $(X^{(\gamma)}_t)_{t\geq 0}$ then

$$\begin{aligned}
&\mathrm{e}^{\lambda^{(\gamma)}_0\gamma} \iint_{\mathrm{B}(q_0,r)\times \mathrm{B}(0,M)} \psi^{(\gamma)}(q,p)\mathbb{P}(X^{(\gamma),(q,p)}_\gamma \in \mathrm{B}(q_0,r)\times \mathrm{B}(0,M), \tau^{(\gamma),(q,p)}_\partial > \gamma)\mathrm{d}q\mathrm{d}p\\
&\leq \mathrm{e}^{\lambda^{(\gamma)}_0\gamma} \iint_{D} \psi^{(\gamma)}(q,p)\mathbb{P}(X^{(\gamma),(q,p)}_\gamma \in \mathrm{B}(q_0,r)\times \mathrm{B}(0,M), \tau^{(\gamma),(q,p)}_\partial > \gamma)\mathrm{d}q\mathrm{d}p\\
&= \iint_{\mathrm{B}(q_0,r)\times \mathrm{B}(0,M)} \psi^{(\gamma)}(q,p)\mathrm{d}q\mathrm{d}p.
\end{aligned}$$

Consequently, for $\gamma \geq \gamma_1$,

$$c\mathrm{e}^{\lambda^{(\gamma)}_0\gamma} \iint_{\mathrm{B}(q_0,r)\times \mathrm{B}(0,M)} \psi^{(\gamma)}(q,p)\mathrm{d}q\mathrm{d}p \leq \iint_{\mathrm{B}(q_0,r)\times \mathrm{B}(0,M)} \psi^{(\gamma)}(q,p)\mathrm{d}q\mathrm{d}p$$

which concludes the proof since $\iint_{\mathrm{B}(q_0,r)\times \mathrm{B}(0,M)} \psi^{(\gamma)}(q,p)\mathrm{d}q\mathrm{d}p > 0$. □

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