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Using moment approximations to study the density of jump driven SDEs*

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Abstract

In order to study the regularity of the density of a solution of a infinite activity jump driven stochastic differential equation we consider the following two-step approximation method. First, we use the solution of the moment problem in order to approximate the small jumps by another whose Lévy measure has finite support. In a second step we replace the approximation of the first two moments by a small noise Brownian motion based on the Assmussen-Rosiński approach. This approximation needs to satisfy certain properties in order to apply the "balance" method which allows the study of densities for the solution process based on Malliavin Calculus for the Brownian motion. Our results apply to situations where the Lévy measure is absolutely continuous with respect to the Lebesgue measure or purely atomic measures or combinations of them.

Keywords: moment problem; Lévy driven sde's; interpolation method; smoothness of densities.
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1 Introduction

In this article we consider a jump driven stochastic differential equation (sde) of the type

$$X_t = x + \int_0^t \int_{|z|\leq 1} c(X_{s-}, z)\widetilde{N}(ds, dz) + \int_0^t \int_{|z|>1} c(X_{s-}, z)N(ds, dz). \tag{1.1}$$

Here N is a homogeneous Poisson random measure on $\mathbb{R}^m$ with compensator $\widehat{N}(ds, dz) = ds\nu(dz)$ and compensated measure $\widetilde{N}(ds, dz)$. We make the following assumptions:

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Assumption (H)

- $\int_{\mathbb{R}^m} 1 \wedge |z|^2 \nu(dz) < \infty$ and $\int_{|z|>1} |z|^p \nu(dz) < \infty$ for all $p > 1$.
- For all $\varepsilon \in (0,1)$, we assume that $\int_{|z|\leq\varepsilon} |z|^2 \nu(dz) > 0$ and $\nu(\{0\}) = 0$.
- The coefficient function $c : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ is smooth, bounded with bounded derivatives and it satisfies that $c(x,0) = 0$.

Under the above hypotheses one has the existence and uniqueness of solutions for the equation (1.1). A general reference for jump driven sde's is Chapter 6 in [1].

The goal of the present article is to discuss an alternative method in order to obtain sufficient conditions for the regularity of the law of X_t. That is,

$$P(X_t \in dy) = p_t(y)dy \text{ with } p_t \in C^\infty(\mathbb{R}^d).$$

This problem has been extensively discussed in the literature. Various approaches based on the amplitude of the jumps have been introduced in [8], [23], [24], [7] and [9]. In these references, a recurrent hypothesis is to assume that $\nu(dz) = h(z)dz$ with various degrees of regularity or assumptions on the function h together with a uniform ellipticity condition of the type $\nabla_z c(x,z)(\nabla_z c(x,z))^* \geq \lambda_0 I > 0$ for any $(x,z) \in \mathbb{R}^d \times \mathbb{R}^m$ and some $\lambda_0 > 0$. The main argument is based on Malliavin type calculus associated with the law of the amplitude of the jumps h which is then used to study the regularity of $P \circ X_t^{-1}(dy)$.

An extension of this line of research is to consider the equivalent of the hypoelliptic hypothesis as in the Hörmander theorem. This question does not have a unique answer as it can be seen in [18], [11], [28], [31] and [29].

A second approach started by [10] and extended in [6], [13], [20] and [19] uses a Malliavin type calculus based on the exponential density of the jump times in order to study the same problem.

A third approach is to use the parametrix method as explained in [14] which has been extended in [17], [21].

Finally, a fourth approach initiated in [26] (see also [15] for more references and extensions) constructs a Malliavin type calculus based on a difference operator instead of the Malliavin differential operator and obtains the regularity of the law of $P \circ X_t^{-1}$ under the hypotheses that there exists $\alpha \in (0,2)$ and $c_0 > 0$ with

$$\rho^{-\alpha} \int_{|z|\leq\rho} z^2 \nu(dz) \geq c_0 > 0 \tag{1.2}$$

and c is uniformly elliptic [1] in the sense that $\int_{\mathbb{R}^m} cc^*(x,z)\nu(dz) \geq \lambda_0 I > 0$ for any $x \in \mathbb{R}^d$ and some $\lambda_0 > 0$. Note that this class of measures ν could be purely atomic in contrast with the hypothesis that $\nu(dz) << dz$.

The method of proof in [26] is based on a duality principle associated to the difference operator which arises naturally in the case of Poisson random measures. This method has been largely extended. For recent developments and further references we refer the reader to [22], [15] and [16].

Our results also apply to these situations and therefore they can be compared to those in [26] but our method is different and is based on an approximation of small jumps by a Brownian motion.

More precisely, we propose to use the Assmussen-Rosiński approximation in order to obtain the regularity of densities for jump driven sde's. This approximation was introduced in [2] and it uses a d-dimensional Brownian motion B in order to approximate

[1] Some additional regularity conditions are also required. For details, we refer to Corollary 4.4. in [26].

the small jumps of a Lévy process with infinite activity around zero. In fact, define X^ε to be the solution of the following sde:

$$X_t^\varepsilon = x + \int_0^t \sigma_\varepsilon(X_s^\varepsilon)dB_s + \int_0^t \int_{\varepsilon<|z|\le 1} c(X_{s-}^\varepsilon, z)\widetilde{N}(ds,dz) + \int_0^t \int_{|z|>1} c(X_{s-}^\varepsilon, z)N(ds,dz). \tag{1.3}$$

Here[2] $\sigma_\varepsilon(x)$ is an appropriate approximation of the second moment of the jump terms in the set $\{z; |z| \le \varepsilon\}$.

It is not difficult to prove that $X_t^\varepsilon \xrightarrow{\mathcal{L}} X_t$. In order to explain how the definition of X^ε is obtained, we use the language of semigroups and their associated infinitesimal operators. First, the generator of X is defined for $f \in C_b^1(\mathbb{R}^d)$ as

$$\begin{aligned} Lf(x) &:= \int_{\mathbb{R}^m} \Delta_1 f(x,z)\nu(dz) \\ \Delta_1 f(z,x) &:= f(x+c(x,z)) - f(x) - \langle \nabla f(x), c(x,z)\rangle . \end{aligned}$$

If one considers the non-local operator L on the neighborhood of $z=0$, one obtains as approximation

$$\int_{|z|\le\varepsilon} \Delta_1 f(x,z)\nu(dz) = \frac{1}{2}\mathrm{Tr}(D^2 f(x) a_\varepsilon(x)) + O\left(\int_{|z|\le\varepsilon} |z|^3 \nu(dz)\right).$$

Here, $D^2 f$ denotes the Hessian matrix and a_ε is a suitable symmetric positive definite matrix defined through (4.8). This matrix being positive definite has a well defined square root matrix which gives σ_ε. As the regularization noise in the equation (1.1) is generated by the small jumps one hopes that B will provide the required regularity. But it is also clear that as $\varepsilon \downarrow 0$ then the upper bound density estimates of X_t^ε obtained with this argument will blow up.

In order to solve this blow up problem we use the balance method introduced in [5]. Roughly speaking the strategy is the following: suppose that we have a family of random variables $F_\varepsilon \to F$ in law so that the law of F_ε is smooth. That is, $P(F_\varepsilon \in dx) = p_\varepsilon(x)dx$. If we know that $p_\varepsilon \to p$ then it is clear that we obtain $P(F \in dx) = p(x)dx$.

But the striking point of the balance method is that even if the upper estimates for p_ε explode, we may still obtain the regularity of the law of F if the rate of convergence is fast enough.

In our problem, as $\sigma_\varepsilon \to 0$, the upper bound estimates for p_ε grow to infinity. The "balance" argument consists in proving that, if the speed of convergence $F_\varepsilon \to F$ is much faster than the blow up of p_ε (this is the "balance") then we are still able to prove the regularity of the law of F. A precise quantitative expression of this balance is given in Theorem 5.2 below. Once we decide to use this strategy, it is clear that we have to produce a family of approximations F_ε which converges as fast enough to F so as to achieve the desired regularity properties. With this goal in mind, we use the moment method in order to accelerate the rate of convergence.

In fact, the sole application of the Assmussen-Rosiński approximation does not give any regularity results because the rate of convergence is not good enough (i.e. the rate $O\left(\int_{|z|\le\varepsilon} |z|^3\nu(dz)\right)$ does not suffice). Therefore we need to improve the approximation method defined in (1.3). This is an essential point in order to apply the balance method.

In order to improve the approximation method, we will introduce another jump measure which matches higher moments of ν near zero in a sense to be described in

[2] For more details, see (4.8)

(3.2). With this in mind, we apply the solution of the Hamburger moment problem to define a new approximation.

The moment problem is a classical problem in mathematics which has applications in many fields. A standard reference with many historical remarks is [27]. However, a direct application of this technique does not give a solution to our problem as we explain below.

Roughly speaking, we need to find a measure $\bar{\nu}_{M,\varepsilon}(dz)$ with support on the set $\{|z| \leq \varepsilon\}$ which approximates the renormalized measure $\bar{\nu}_\varepsilon(dz) = \frac{|z|^2}{\varepsilon^2} 1_{\{|z|\leq\varepsilon\}}\nu(dz)$ in the following sense: for every ε small enough, the associated moments of order $k = 1, \ldots, M$ of $\bar{\nu}_\varepsilon(dz)$ and $\bar{\nu}_{M,\varepsilon}(dz)$ coincide. It is important that the total mass of $\bar{\nu}_{M,\varepsilon}(dz)$ has to be smaller than the total mass of $\bar{\nu}_\varepsilon$ in order to use the remaining mass to construct the Brownian component in the approximation. This is the heuristic meaning of (3.2).

This latter condition makes this moment problem non-trivial. This problem is solved in Section 3 which is the mathematical core of the present research. The authors are not aware of any results on this type of moment problems.

Let $C \in (0, 1)$ be the quantity associated with the reduction of total mass. Then one proves the existence of $\bar{\nu}_{M,\varepsilon}(dz)$ satisfying the above restrictions and which after suitable renormalization leads to the approximation

$$\int_{\{|z|\leq\varepsilon\}} \Delta_1 f(x,z)\nu(dz) = \frac{C}{2}\mathrm{Tr}(D^2 f(x) a_\varepsilon(x)) + \int_{\{|z|\leq\varepsilon\}} \Delta_1 f(x,z)\nu_{M,\varepsilon}(dz) + O\Big(\int_{|z|\leq\varepsilon} |z|^{M+1}\nu(dz)\Big).$$

Here, $\nu_{M,\varepsilon}(dz)$ is the renormalization of $\bar{\nu}_{M,\varepsilon}(dz)$. So, using this moment result, we obtain a high order approximation which preserves a large portion of the second order noise (see Lemma 4.3). We define $\sigma_\varepsilon = (Ca_\varepsilon)^{1/2}$ and then we consider the process that approximates X as

$$\begin{aligned} X_t^{M,\varepsilon} &= x + \int_0^t \sigma_\varepsilon(X_s^{M,\varepsilon}) dB_s + \int_0^t \int_{|z|\leq\varepsilon} c(X_{s-}^{M,\varepsilon}, z)\widetilde{N}^{\nu_{M,\varepsilon}}(ds, dz) \\ &\quad + \int_0^t \int_{\varepsilon<|z|\leq 1} c(X_{s-}^{M,\varepsilon}, z)\widetilde{N}(ds, dz) + \int_0^t \int_{|z|>1} c(X_{s-}^{M,\varepsilon}, z)N(ds, dz), \end{aligned}$$

in which B is a Brownian motion and $\widetilde{N}^{\nu_{M,\varepsilon}}(ds, dz)$ is a compensated Poisson random measure associated with the Lévy measure $\nu_{M,\varepsilon}$. We then study the regularity properties of the density of the resulting approximative solutions using classical Malliavin Calculus based on the Wiener process product of the second order term of the approximation. Combining this with the rate of convergence of the proposed approximation one obtains that X_t, $t > 0$ has a density and that this density is smooth.

In the above sense, our method of proof is different from Picard's method: as Asmussen-Rosiński have already pointed out, the behavior of small jumps of an infinite activity Lévy process is close to be Gaussian. Therefore it is natural to think that Malliavin Calculus for the Wiener process may be an alternative method for this problem.

2 Notations

The multidimensional closed ball of center $a \in \mathbb{R}^d$ and radius $r \geq 0$ is denoted by $B_r(a)$. As usual, we also use the notation $S_{d-1} = \partial B_1(0)$.

Let A be an open or closed subset of $\mathbb{R}^d$. Spaces of continuous differentiable functions of order k on the set A are denoted by $C^k(A)$ and $C_b^k(A)$ if furthermore all derivatives up to order k are bounded. In the case that A is a closed set all regularity results at the points of A which are also boundary points are considered as regularity from the interior of the set A. Positive constants and their value may change from one line to the next. On the other hand, $C \in (0,1)$ is a constant that it is fixed through the text and appears in Lemma 3.1.

3 A moment problem

In this section, we introduce the arguments that lead to the definition of the approximating Lévy measure in the multidimensional case. This approximation is based on the method of moments. We refer the reader to [27] for a detailed introduction to the subject.

In order to introduce the approximation, we use a general "polar" type representation for any $z \in \mathbb{R}^m$ as $z = (\rho, r)$ with $\rho \in E$ and $r \in \mathbb{R}_+$. The set E is a compact set and in most examples it is given by $E = S_{m-1} = \{\rho \in \mathbb{R}^{m-1} : |\rho| = 1\}$. Therefore without loss of generality, we will assume the latter in what follows.

The length of z is given by $r = |z|$. The structure of E is not strongly used in our results and in many statements is a parameter. In the case of norm assumptions these assumptions are always supposed to be satisfied uniformly in ρ. For example, $c(\cdot,(*,\cdot)) \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}_+)$ means that the derivatives of the function $c(x,(\rho,r))$ with respect to the variables x and r exist and they are uniformly bounded with respect to $\rho \in E$. We will suppose in what follows that ν is a Lévy measure of the form (see [30])

$$\nu(dz) = \lambda(d\rho) g_\rho(dr)$$

where λ is a non negative finite measure on a compact set E and $g_\rho(dr)$ is a non negative σ-finite measure on $\mathbb{R}_+$ which is measurable with respect to ρ, it does not charge zero and it satisfies $\int (1 \wedge r^2) g_\rho(dr) < \infty$ for λ-a.e. and $\int_{\{r>1\}} r^p g_\rho(dr) < \infty$ for every $p > 1$.

We will use the following notation for the moment sequence associated with any measure g on $\mathbb{R}_+$

$$m_k(g,\varepsilon) := \int_{\{r \le \varepsilon\}} r^k g(dr).$$

In the particular case that $\varepsilon = \infty$, we let $m_k(g) \equiv m_k(g,\infty)$. Recall that from $(\mathbf{H})$ we have that $m_2(g_\rho,\varepsilon) > 0$ for all $\varepsilon \in (0,1)$.

We construct the rescaled measures and restricted moments in order to set-up the approximative moment problems using probability measures on $[0,1]$:

$$g_{\rho,\varepsilon}(dr) \quad = \quad \frac{\varepsilon^2}{m_2(g_\rho,\varepsilon)} 1_{\{r\le 1\}} r^2 g_\rho \circ \eta_\varepsilon^{-1}(dr) \qquad \text{with } \eta_\varepsilon(r) = \frac{r}{\varepsilon}. \tag{3.1}$$

Note that, by construction, for $k = 0,1,2,\ldots$

$$m_k(g_{\rho,\varepsilon}) = \frac{1}{\varepsilon^k} \times \frac{m_{k+2}(g_\rho,\varepsilon)}{m_2(g_\rho,\varepsilon)}.$$

The next property characterizes a measure $h_{\rho,\varepsilon}$ which has the same moments as $g_{\rho,\varepsilon}$ except that its total mass is smaller than one. This measure will be used to build an appropriate approximation of the Lévy measure of the driving jump measure. The constant C represents the Assmussen-Rosiński approximation on which we will base our infinite dimensional analysis. In the rest of the article, we will frequently assume this important property for the Lévy measure ν.

Germ Property $\boldsymbol{GP(M,C,\varepsilon_*)}$: Here $C, \varepsilon_* \in (0,1)$ and $M \in \mathbb{N}$. Furthermore $\nu(dz) = \lambda(d\rho) g_\rho(dr)$ and for each $\rho \in E$, g_ρ is a non negative measure g_ρ on $\mathbb{R}_+$ which satisfies that for every $\varepsilon < \varepsilon_*$ there exists a non negative measure $h_{\rho,\varepsilon} \equiv h_{\rho,\varepsilon}(M, C, \varepsilon_*)$ which is measurable in ρ such that

$$\begin{aligned} m_0(h_{\rho,\varepsilon}) =& m_0(g_{\rho,\varepsilon}) - C = 1 - C \qquad \text{and} \\ m_k(h_{\rho,\varepsilon}) =& m_k(g_{\rho,\varepsilon}) \qquad k = 1, 2, ..., M. \end{aligned} \tag{3.2}$$

In the case that the above property is satisfied we say that ν satisfies the $\boldsymbol{GP(M, C, \varepsilon_*)}$ (germ) property.

The following lemma provides sufficient conditions in order to ensure that the $\boldsymbol{GP(M, C, \varepsilon_*)}$ property is satisfied. For $k \in \mathbb{N}$, we define the moment sequence

$$s_k(\rho, \varepsilon) = m_k(g_{\rho,\varepsilon}) = \frac{1}{\varepsilon^k} \times \frac{m_{k+2}(g_\rho, \varepsilon)}{m_2(g_\rho, \varepsilon)} = (\int_{\{|r| \le \varepsilon\}} (r/\varepsilon)^2 g_\rho(dr))^{-1} \int_{\{|r| \le \varepsilon\}} (r/\varepsilon)^{k+2} g_\rho(dr).$$

We consider the following hypothesis:

Hypothesis $\boldsymbol{A(M)}$: For $N = \lfloor \frac{M}{2} \rfloor$, $q = (q_0, ..., q_N) \in \mathbb{R}^{N+1}$, $\rho \in E$ and $\varepsilon \in (0, 1]$ define for $Q(x; q) = \sum_{i=0}^N q_i x^i$, the function

$$\begin{aligned} \Phi(\rho, \varepsilon, q) :=& \int Q^2(r; q) dg_{\rho,\varepsilon}(r) = (\int_{\{|r| \le \varepsilon\}} g_\rho(dr))^{-1} \int_{\{|r| \le \varepsilon\}} Q^2(r/\varepsilon; q) g_\rho(dr) \\ =& \sum_{i,j=0}^N q_i q_j s_{i+j}(\rho, \varepsilon). \end{aligned}$$

Assume that Φ satisfies for any $(\rho_0, \varepsilon_0, q_0) \in E \times [0, 1] \times \mathbb{R}^{N+1}$:

1. If $\varepsilon_0 \in (0, 1)$ then $\Phi(\rho_0, \varepsilon_0, q_0) = \lim_{n \to \infty} \Phi(\rho_n, \varepsilon_n, q_n)$ for any sequence $(\rho_n, \varepsilon_n, q_n) \to (\rho_0, \varepsilon_0, q_0)$ as $n \to \infty$ with $\varepsilon_n \ge \varepsilon_0$. We denote this convergence as $(\rho_n, \varepsilon_n, q_n) \to (\rho_0, \varepsilon_0+, q_0)$.
2. If $\varepsilon_0 \in (0, 1]$ then for any sequence $(\rho_n, \varepsilon_n, q_n) \to (\rho_0, \varepsilon_0, q_0)$ as $n \to \infty$ with $\varepsilon_n < \varepsilon_0$ (which we denote as $(\rho_n, \varepsilon_n, q_n) \to (\rho_0, \varepsilon_0-, q_0)$), we have

$$\begin{aligned} \Phi(\rho_0, \varepsilon_0-, q_0) :=& \lim_{n \to \infty} \Phi(\rho_n, \varepsilon_n, q_n) \\ =& (\int_{\{|r| < \varepsilon_0\}} r^2 g_\rho(dr))^{-1} \int_{\{|r| < \varepsilon_0\}} Q^2(r/\varepsilon_0; q_0) r^2 g_\rho(dr). \end{aligned}$$

3. In the particular case that $\varepsilon_0 = 0$, we assume that the limit $\lim_{(\rho,\varepsilon,q) \to (\rho_0, 0+, q_0)} \Phi(\rho, \varepsilon, q)$ exists and we define this value as $\Phi(\rho_0, 0, q_0)$.

This hypothesis is essential in the following result.

Lemma 3.1. *Suppose that the assumptions* $\boldsymbol{A(M+1)}$ *and* $(\mathbf{H})$ *hold true. Moreover, we assume*

$$\underline{\lim}_{(\rho',\varepsilon) \to (\rho, 0+)} \frac{m_2(g_{\rho'}, \delta\varepsilon)}{m_2(g_{\rho'}, \varepsilon)} > 0, \tag{3.3}$$

$$\overline{\lim}_{\delta \to 0} \overline{\lim}_{(\rho',\varepsilon) \to (\rho, 0+)} \frac{m_2(g_{\rho'}, \delta\varepsilon)}{m_2(g_{\rho'}, \varepsilon)} = 0. \tag{3.4}$$

Then there exists $C, \varepsilon_* \in (0, 1)$ *such that* ν *satisfies the* $\boldsymbol{GP(M, C, \varepsilon_*)}$ *property.*

In the proof that follows we will use some tools and results related to the moment problem. We refer the reader to Chapter 9 in [27] as a basic reference on the results to be used below. In particular, for a sequence of real numbers $s = (s_0, s_1, ..., s_{2N})$ we consider the so-called Hankel's matrices defined as $H_k(s) = (s_{i+j})_{i,j=0,...,k}$ and we denote

$$D_N(s) = \min_{k=1,...,N} \det H_k(s).$$

Proof: Step 1. We set $N = \lfloor \frac{M+1}{2} \rfloor$. Let us prove that $\Phi(\rho, \varepsilon, q) > 0$ for every $(\rho, \varepsilon, q) \in E \times [0,1] \times \mathbb{R}^{N+1}$. Suppose first that $\varepsilon > 0$ is fixed. Note that the polynomial $Q^2(r;q)$ has at most a finite number of roots in r. Denote by r_0 its smallest non-zero positive root. We consider the case $r_0 < 1$ as otherwise $\Phi(\rho, \varepsilon, q) > 0$. Then as we have assumed that $m_2(g_\rho, \delta) > 0$ for any $\delta > 0$ and $\rho \in E$ and $g_\rho(\{0\}) = 0$,we obtain that $g_\rho(B_\delta(0) - \{0\}) > 0$ for any $\delta > 0$. Taking $\delta < \varepsilon r_0$ one obtains that

$$\begin{aligned} \Phi(\rho, \varepsilon, q) =& (\int_{\{|r| \le \varepsilon\}} g_\rho(dr))^{-1} \int_{\{|r| \le \varepsilon\}} Q^2(r/\varepsilon; q) g_\rho(dr) \\ \ge& (\int_{\{|r| \le \varepsilon\}} g_\rho(dr))^{-1} \int_{\{|r| \le \delta\}} Q^2(r/\varepsilon; q) g_\rho(dr) > 0. \end{aligned} \tag{3.5}$$

Therefore the conclusion follows in this case. Replacing the set $\{|r| \le \varepsilon\}$ by $\{|r| < \varepsilon\}$ one also concludes that $\Phi(\rho, \varepsilon-, q) > 0$ for any $(\rho, \varepsilon, q) \in E \times (0,1] \times \mathbb{R}^{N+1}$.

Take now $\varepsilon = 0$. Notice that $\{g_{\rho,\varepsilon}, \varepsilon > 0, \rho \in E\}$ is a family of probability measures with support included in the compact set $[0,1]$, so it is a relatively compact sequence of probability measures. Then we can select a sub-sequence $(\rho_n, \varepsilon_n) \to (\rho, 0+)$ such that $\lim_{n\to\infty} g_{\rho_n,\varepsilon_n} = \mu_0$ for a probability measure μ_0 (this measure may be not unique but this is not required for the argument that follows). By our assumption (3.3),

$$\mu_0(r \le 2\delta) \ge \lim_{n\to\infty} g_{\rho_n,\varepsilon_n}(r \le \delta) = \underline{\lim}_{n\to\infty} \frac{m_2(g_{\rho_n}, \delta\varepsilon_n)}{m_2(g_{\rho_n}, \varepsilon_n)} > 0. \tag{3.6}$$

So μ_0 assigns positive probability to all small balls. Moreover, due to (3.4), we have

$$\overline{\lim}_{\delta\to 0} \mu_0(r < \delta) \le \overline{\lim}_{\delta\to 0} \overline{\lim}_{n\to\infty} g_{\rho_n,\varepsilon_n}(r \le 2\delta) = \overline{\lim}_{\delta\to 0} \overline{\lim}_{n\to\infty} \frac{m_2(g_{\rho_n}, 2\delta\varepsilon_n)}{m_2(g_{\rho_n}, \varepsilon_n)} = 0.$$

Therefore $\mu_0(\{0\}) = 0$. Combining this with (3.6) and the previous argument used to obtain (3.5) implies that $\int Q^2(r) d\mu_0(r) > 0$. Using the hypothesis $\mathbf{A}(\mathbf{M})3.$, we take limits in order to obtain

$$\Phi(\rho, 0, q) = \lim_n \Phi(\rho_n, \varepsilon_n, q) = \int Q^2(r;q) d\mu_0(r) > 0.$$

From the above arguments, we have proved that Φ and its left limits are strictly positive on the compact set $E \times [0,1] \times S_M$.

We will now prove that Φ has a global strictly positive lower bound Λ :

$$\Phi(\rho, \varepsilon, q) \ge \Lambda > 0 \qquad \forall (\rho, \varepsilon, q) \in E \times [0,1] \times S_M.$$

Using a proof by contradiction suppose that $\Lambda = 0$. From the fact that the set $E \times [0,1] \times S_M$ is compact, we obtain a sequence $(\rho_n, \varepsilon_n, q_n) \to (\rho^*, \varepsilon^*, q*) \in E \times [0,1] \times S_M$ so that $\lim_{n\to\infty} \Phi(\rho_n, \varepsilon_n, q_n) = \Lambda = 0$. Using subsequences if necessary one may assume that either $\varepsilon_n \ge \varepsilon^*$ or $\varepsilon_n < \varepsilon^*$. This contradicts the fact that Φ and its left limits are strictly positive on the compact set $E \times [0,1] \times S_M$.

We take $C = \frac{1}{2}\Lambda$ and we define the following perturbed sequence $s_0^C(\rho, \varepsilon) = 1 - C$ and $s_i^C(\rho, \varepsilon) = s_i(\rho, \varepsilon)$ for $i = 1, ..., 2N$. Then

$$\sum_{i,j=0}^N q_i q_j s_{i+j}^C(\rho, \varepsilon) = \sum_{i,j=0}^N q_i q_j s_{i+j}(\rho, \varepsilon) - C q_0^2 \ge \Lambda |q|^2 - C q_0^2 \ge \frac{1}{2}\Lambda |q|^2.$$

This means that the lowest eigenvalue of the Hankel matrix $(s_{i+j}^C)_{i,j=0,\dots,N}$ is larger than $\frac{\Lambda}{2}$ so the determinant is strictly positive. This is true for every fixed $(\rho, \varepsilon) \in E \times [0,1]$.

Finally, we will use the theory of the moment problem in order to prove in the next step that one can construct the measures $h_{\rho,\varepsilon}$ in the (M, C, ε_*)-germ property so that they are measurable with respect to (ρ, ε).

Step 2: In the truncated moment problem, one fixes $N \in \mathbb{N}$ and a finite sequence of moments $(s_1, ..., s_{2N})$ is given such that the Hankel matrices are positive definite. That is, $D_N(s) > 0$. Then one uses the Gaussian quadrature method as explained in Section 9.1, Theorem 9.4 in [27] where the explicit solution is given right before that theorem. That is, there exists $q_i \equiv q_i(\rho, \varepsilon) > 0$ and $x_i \equiv x_i(\rho, \varepsilon) \in \mathbb{R}_+$, $i = 1, ..., N+1$ which are measurable with respect to (ρ, ε) (due to the explicit construction[3]) and such that s_k is the k-th moment of the measure $h(dr) \equiv h_{\rho,\varepsilon}(dr) = \sum_{i=1}^{N+1} q_i \delta_{x_i}$. □

Remark 3.2. The proof provided in Section 9.1, Theorem 9.4 of [27] also gives that each $h_{\rho,\varepsilon}$ can be considered as a discrete measure supported by $N+1$ points, with $N = \lfloor \frac{M+1}{2} \rfloor$. And also, one may take $C = \frac{1}{2}\Lambda$ with

$$\Lambda = \inf \sum_{i,j=0}^{N} q_i q_j s_{i+j}(\rho, \varepsilon) > 0$$

where the infimum is taken over $(\rho, \varepsilon) \in E \times (0, 1]$ and $q = (q_0, ..., q_N) \in S_N$.

Example 3.3. Here, we will present some examples of Lévy measures which satisfy the conditions in Lemma 3.1. Consider

$$g_\rho(dr) = \frac{1}{r^{1+\theta}} |\log r|^p \, 1_{\{0<r\leq 1\}} dr,$$

with $\theta \in (-\infty, 2)$ and $p \geq 0$. Then (3.3) and (3.4) hold.

Note the following three sub-cases:

1. If $p = 0$ and $\theta \in (0, 2)$ then $g_\rho(dr) = \frac{dr}{r^{1+\theta}} 1_{\{0<r\leq 1\}}$ which corresponds to a stable like distribution. In this case, one has that $m_{k+2}(g_\rho, \varepsilon) = \frac{\varepsilon^{2+k-\theta}}{2+k-\theta}$ and the limit in (3.3) is $\delta^{2-\theta}$ which satisfies (3.4).
2. If $p = 0$ and $\theta = 0$ then $g_\rho(dr) = 1_{\{0<r\leq 1\}} r^{-1} |\log(r)| dr$. In this case, the limit in (3.3) is δ^2. Note that in this example, $m_{k+2}(g_\rho, \varepsilon) = \frac{\varepsilon^{k+2}}{k+2}(\frac{1}{k+2} - \log(\varepsilon))$.
3. If $\theta = -1$ and $p = 0$ then $g_\rho(dr) = 1_{\{0<r\leq 1\}} dr$ and $m_{k+2}(g_\rho, \varepsilon) = \frac{\varepsilon^{k+3}}{k+3}$ which also satisfies the hypotheses in Lemma 3.1.

Note that the condition $\boldsymbol{A(M)}$ can be checked using standard limit theorems for integrals as g_ρ does not depend on ρ. The condition 3. is due to the above moment expressions. Furthermore (3.3) and (3.4) in Lemma 3.1 only depend on the moment behavior of the measure g_ρ. Therefore the fact that in this example g_ρ has a density is somewhat irrelevant and one can also use this example as a reference in order to consider cases where the measure g_ρ may be purely atomic. This is done in the next example.

Example 3.4. For $\theta \in (-\infty, 2)$ and $p \geq 0$, define the following measure

$$\bar{g}_\rho(dr) = \sum_{i=1}^{\infty} i^{\theta-1} (\log(i))^p \delta_{i^{-1}}(dr).$$

Then using the measure g_ρ in the previous example, we have that for $\varepsilon \leq e^{-p/(3+k-\theta)}$ there exists constants c_1 and c_2 such that for $k \geq 0$:

$$0 < c_1 \leq \frac{m_{k+2}(\bar{g}_\rho, \varepsilon)}{m_{k+2}(g_\rho, \varepsilon)} \leq c_2.$$

[3] The explicit construction uses the so-called Haviland Theorem which in turn is based on Riesz representation theorem whose proof can be obtained in a constructive manner using the Radon-Nikodym theorem. For more details, see e.g. Theorem 1.8 in page 16. We do not use this explicit construction in the rest of the article.

In fact, it is enough to note that for $\varepsilon \le e^{-p/(3+k-\theta)}$ the function $x^{-3-k+\theta}|\log(x)|^p$ is decreasing for $x \ge \varepsilon^{-1}$ and therefore

$$\int_{\varepsilon^{-1}+1}^{\infty} x^{-3-k+\theta}|\log(x)|^p dx \le m_{k+2}(\bar{g}_\rho, \varepsilon) \le \int_{\varepsilon^{-1}}^{\infty} x^{-3-k+\theta}|\log(x)|^p dx.$$

Therefore the same cases as in Example 3.3 can be obtained here.

The condition $\boldsymbol{A(M)}$ is verified easily using limit theorems for integrals. For the third condition, it is easier to find $\lim_{(\rho,\varepsilon)\to(\rho_0,0+)} s_{i+j}(\rho,\varepsilon)$, $i,j=0,...,N$ with respect to $\bar{g}_\rho$ using the above inequalities.

Also note that in many situations this restriction can be weakened using Corollary 5.5.

4 An approximation result

The motivation of the hypothesis $\boldsymbol{A(M)}$ is that for a measure ν which verifies such a property we may use Lemma 3.1 in order to construct a suitable approximation to the Lévy measure ν. This is the goal of this section. Note that in order to do this, we have to undo the transformations defined at the beginning of Section 3, equation (3.1).

Recall that the $\boldsymbol{GP(M+1, C, \varepsilon_*)}$ property guarantees the existence of the measures $h_{\rho,\varepsilon}$ which satisfy (3.2) up to $M+1$ with respect to g_ρ. A renormalization of this measure from $[0,1]$ to $[0,\varepsilon]$ will give the approximation for g_ρ and therefore the approximation for ν will be defined in two steps. That is, note that $\eta_\varepsilon^{-1}(r) = r\varepsilon$ and define

$$\widehat{h}_{\rho,\varepsilon}(dr) = \frac{m_2(g_\rho,\varepsilon)}{\varepsilon^2}\frac{1}{r^2}h_{\rho,\varepsilon}(dr), \qquad \overline{h}_{\rho,\varepsilon}(dr) = \widehat{h}_{\rho,\varepsilon}\circ\eta_\varepsilon(dr).$$

Here $\overline{h}_{\rho,\varepsilon}$ is obtained by using the inverse transformation used in (3.1) in two steps and this is the approximation for g_ρ. Now we give the following moment properties associated to this approximation:

$$m_2(\widehat{h}_{\rho,\varepsilon}) = \frac{m_2(g_\rho,\varepsilon)}{\varepsilon^2}m_0(h_{\rho,\varepsilon}) = \frac{m_2(g_\rho,\varepsilon)}{\varepsilon^2}(1-C).$$

Therefore, as expected a non-negligible part of the second moment of $g_{\rho,\varepsilon}$ has been removed and it is still generated by the measure $\overline{h}_{\rho,\varepsilon}$ as follows:

$$m_2(\overline{h}_{\rho,\varepsilon}) = \varepsilon^2 m_2(\widehat{h}_{\rho,\varepsilon}) = m_2(g_\rho,\varepsilon)(1-C) = \int_{r\le\varepsilon} r^2 g_\rho(dr)(1-C). \tag{4.1}$$

As for higher moments, we have that for $k = 1,2,...,M-1$

$$\begin{aligned} m_{k+2}(\widehat{h}_{\rho,\varepsilon}) &= \frac{m_2(g_\rho,\varepsilon)}{\varepsilon^2}m_k(h_{\rho,\varepsilon}) = \frac{m_2(g_\rho,\varepsilon)}{\varepsilon^2}m_k(g_{\rho,\varepsilon}) = \int_{r\le 1} r^{k+2} g_\rho\circ\eta_\varepsilon^{-1}(dr) \\ &= \frac{1}{\varepsilon^{k+2}}\int_{r\le\varepsilon} r^{k+2}g_\rho(dr) \end{aligned}$$

so that

$$m_{k+2}(\overline{h}_{\rho,\varepsilon}) = \varepsilon^{k+2}m_{k+2}(\widehat{h}_{\rho,\varepsilon}) = \int_{r\le\varepsilon} r^{k+2}g_\rho(dr). \tag{4.2}$$

Finally, we define the approximation as the Lévy measure

$$\nu_{M,\varepsilon}(dz) = \lambda(d\rho)\overline{h}_{\rho,\varepsilon}(dr). \tag{4.3}$$

We give now estimates for the error when we use $\nu_{M,\varepsilon}$ instead of ν. For this, let us introduce some notation. For a smooth function $\phi : E \times \mathbb{R}_+ \to \mathbb{R}^m$ we use the norm notation $\|\phi\|_{k,\infty} = \sum_{i=1}^m \sum_{|\alpha|\le k} \left\|\partial^\alpha \phi^i\right\|_\infty$. Here, ∂^α is the derivative of order α and the derivative is taken with respect to the second argument of ϕ. That is, $\partial^\alpha \phi^i(\rho, x) \equiv \partial_x^\alpha \phi^i(\rho, x)$ and the supremum norm is taken with respect to both variables. We may sometimes simplify the notation by just writing $\partial^\alpha \phi^i(z)$. We will also use the general notation $R_M(\varepsilon)$ to denote residues. Their actual value may change from one line to the next unless explicitly stated.

Proposition 4.1. *Let ν satisfy the $\boldsymbol{GP(M+1, C, \varepsilon_*)}$ property and let $\nu_{M,\varepsilon}$ be the approximation of ν constructed in (4.3).*

There exists a universal constant K (depending on M) such that the following holds. For any sequence of smooth functions $\phi^i(\rho, \cdot) \in C_b^\infty(\mathbb{R}_+), i \in \mathbb{N}$ such that $\phi^i(\rho, 0) = 0$ then, for $0 < \varepsilon \le \varepsilon_*$, one has:

***A.** For every $i, j \in \{1, ..., m\}$*

$$\int_{\{|z|\le\varepsilon\}} \phi^i(z)\phi^j(z)\nu(dz) = Ca_\varepsilon^{i,j}(\phi) + \int \phi^i(z)\phi^j(z)\nu_{M,\varepsilon}(dz) + R_M(\varepsilon) \tag{4.4}$$

with a remainder $R_M(\varepsilon)$ which satisfies (4.7) for $k = 2$ and with

$$a_\varepsilon^{i,j}(\phi) = \int \lambda(d\rho) \int_{|r|\le\varepsilon} g_\rho(dr) r^2 \partial\phi^i(\rho, 0)\partial\phi^j(\rho, 0). \tag{4.5}$$

***B**. If $k \ge 3$ then*

$$\int_{\{|z|\le\varepsilon\}} \prod_{i=1}^k \phi^i(z)\nu(dz) = \int \prod_{i=1}^k \phi^i(z)\nu_{M,\varepsilon}(dz) + R_M^1(\varepsilon) \tag{4.6}$$

with

$$\left|R_M^1(\varepsilon)\right| \le K \prod_{i=1}^k \left\|\phi^i\right\|_{M+1,\infty} \times \int_{\{|z|\le\varepsilon\}} |z|^{M+1} \nu(dz). \tag{4.7}$$

Proof.

A. In the sequel, several remainders with the same property will appear and we will use $R_M(\varepsilon)$ as a generic notation for such remainders. Using the Taylor expansion of order M we write

$$\int \phi^i\phi^j(z)\nu_{M,\varepsilon}(dz) = \int \lambda(d\rho)(\sum_{p_1,p_2=1}^M \frac{1}{p_1!p_2!}\partial^{p_1}\phi^i(\rho,0)\partial^{p_2}\phi^j(\rho,0)r^{p_1+p_2})\overline{h}_{\rho,\varepsilon}(dr) + R_M(\varepsilon).$$

For $p_1 = p_2 = 1$ we have using (4.1) and (4.5),

$$\int \lambda(d\rho)\partial\phi^i(\rho,0)\partial\phi^j(\rho,0) \int r^2\overline{h}_{\rho,\varepsilon}(dr) = (1-C)a_\varepsilon^{i,j}(\phi).$$

Moreover, for $p_1 + p_2 \le M$,

$$\begin{aligned} &\int \lambda(d\rho) \sum_{M\ge p_1+p_2\ge3} \frac{1}{p_1!p_2!}\partial^{p_1}\phi^i(\rho,0)\partial^{p_2}\phi^j(\rho,0) \int r^{p_1+p_2}\overline{h}_{\rho,\varepsilon}(dr) \\ = &\int \lambda(d\rho) \sum_{M\ge p_1+p_2\ge3} \frac{1}{p_1!p_2!}\partial^{p_1}\phi^i(\rho,0)\partial^{p_2}\phi^j(\rho,0) \int_{r\le\varepsilon} r^{p_1+p_2} g_\rho(dr). \end{aligned}$$

Then,

$$\int \phi^i\phi^j(z)\nu_{M,\varepsilon}(dz) = (1-C)a_\varepsilon^{i,j}(\phi)$$

$$+\int\lambda(d\rho)\sum_{M\geq p_1+p_2\geq 3}\frac{1}{p_1!p_2!}\partial^{p_1}\phi^i(\rho,0)\partial^{p_2}\phi^j(\rho,0)\int_{r\leq\varepsilon}r^{p_1+p_2}g_\rho(dr)+R_M(\varepsilon).$$

A similar argument with Taylor expansion also gives

$$\begin{aligned}&\int_{|z|\leq\varepsilon}\phi^i\phi^j(z)\nu(dz)\\ =&\int\lambda(d\rho)\sum_{p_1,p_2=1}^{M}\frac{1}{p_1!p_2!}\partial^{p_1}\phi^i(\rho,0)\partial^{p_2}\phi^j(\rho,0)\int_{r\leq\varepsilon}r^{p_1+p_2}g_\rho(dr)+R_M(\varepsilon)\\ =&a_\varepsilon^{i,j}(\phi)+\int\lambda(d\rho)\sum_{M\geq p_1+p_2\geq 3}\frac{1}{p_1!p_2!}\partial^{p_1}\phi^i(\rho,0)\partial^{p_2}\phi^j(\rho,0)\int_{r\leq\varepsilon}r^{p_1+p_2}g_\rho(dr)+R_M(\varepsilon).\end{aligned}$$

So (4.4) follows. The estimates on $R_M(\varepsilon)$ are easy to obtain. In fact, it is enough to use the remainders of Taylor expansions of the functions ϕ^i and ϕ^j and the definition of the norm $\|\cdot\|_{M+1,\infty}$.

B. We use Taylor's expansion of order M around $r=0$ and we write

$$\prod_{i=1}^k\phi^i(z)=\prod_{i=1}^k\phi^i(\rho,r)=\prod_{i=1}^k(\sum_{p_i=0}^M\partial^{p_i}\phi^i(\rho,0)\frac{r^{p_i}}{p_i!})+R_M(\varepsilon,z)$$

with

$$\begin{aligned}&\int|R_M(\varepsilon,z)|\,(d\nu_{M,\varepsilon}(z)+1_{\{|z|\leq\varepsilon\}}d\nu(z))\\ \leq&K\prod_{i=1}^k\left\|\phi^i\right\|_{M+1,\infty}\times\int|z|^{M+1}\,(d\nu_{M,\varepsilon}(z)+1_{\{|z|\leq\varepsilon\}}d\nu(z))\\ =&K\prod_{i=1}^k\left\|\phi^i\right\|_{M+1,\infty}\times\int|z|^{M+1}\,1_{\{|z|\leq\varepsilon\}}d\nu(z).\end{aligned}$$

the second inequality being true because we have (4.1) and (4.2) for $M+1$-th moment. In the sequel, several remainders with the same property will appear and we will use $R_M(\varepsilon,z)$ or $R_M(\varepsilon)$ as a generic notation for such remainders.

Since $\phi^i(\rho,0)=0$, we may take $p_i=1,...,M$ and we get

$$\prod_{i=1}^k(\sum_{p_i=1}^M\partial^{p_i}\phi^i(\rho,0)\frac{r^{p_i}}{p_i!})=\sum_{p_1,\ldots,p_k=1}^M\left(\prod_{i=1}^k\frac{1}{p_i!}\partial^{p_i}\phi^i(\rho,0)\right)r^{p_1+\ldots+p_k}.$$

Since $p_i\geq 1$, we have $p_1+....+p_k\geq k\geq 3$ so that (with $R_M(\varepsilon)=\int R_M(\varepsilon,z)d\nu_{M,\varepsilon}(z)$)

$$\begin{aligned}&\int\prod_{i=1}^k\phi^i(z)d\nu_{M,\varepsilon}(z)\\ =&R_M(\varepsilon)+\sum_{\substack{p_1,\ldots,p_k=1\\ p_1+\ldots+p_k\leq M}}^M\int\lambda(d\rho)\left(\prod_{i=1}^k\frac{1}{p_i!}\partial^{p_i}\phi^i(\rho,0)\right)\int r^{p_1+\ldots+p_k}\overline{h}_{\rho,\varepsilon}(dr)\\ =&R_M(\varepsilon)+\sum_{\substack{p_1,\ldots,p_k=1\\ p_1+\ldots+p_k\leq M}}^M\int\lambda(d\rho)\left(\prod_{i=1}^k\frac{1}{p_i!}\partial^{p_i}\phi^i(\rho,0)\right)\int_{|r|\leq\varepsilon}r^{p_1+\ldots+p_k}g_\rho(dr)\\ =&R_M(\varepsilon)+R'_M(\varepsilon)+\int_{\{|z|\leq\varepsilon\}}\prod_{i=1}^k\phi^i(z)d\nu(z).\end{aligned}$$

Here we have used the identity (4.2). In the last line a new remainder appears because we apply the same steps in reverse in order to obtain the last integral with respect to ν. This finishes the proof of **B**. □

Now, we explain how to use the previous approximation result in order to introduce the approximating coefficients associated with the Wiener components. We consider the coefficient $c : \mathbb{R}^d \times E \times \mathbb{R}_+ \to \mathbb{R}^d$ which is smooth in the argument $r \in \mathbb{R}_+$ of $c(x, \rho, r)$. Recall that c is the coefficient function in (1.1) under the convention introduced at the beginning of Section 3. We define the norm

$$\left\|c_{(x)}\right\|_{k,\infty} = \sum_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^d} \sup_{(\rho,r)\in E\times\mathbb{R}_+} |\partial^\alpha c(x,z)|\,.$$

We also define $a_\varepsilon^{i,j}(x) := a_\varepsilon^{i,j}(c(x,\cdot))$. That is, from (4.5),

$$a_\varepsilon^{i,j}(x) = \int \lambda(d\rho) \int_{|r|\leq\varepsilon} g_\rho(dr) r^2 \partial c^i(x,(\rho,0))\partial c^j(x,(\rho,0)). \tag{4.8}$$

Then, for any function $f \in C_b^\infty(\mathbb{R}^d)$, we define

$$\Delta_1 f(x,z) := f(x+c(x,z)) - f(x) - \sum_{i=1}^d \partial_i f(x) c^i(x,z)$$

and

$$L_\varepsilon f(x) \;=\; \int_{\{|z|\leq\varepsilon\}} \Delta_1 f(x,z)\nu(dz) \tag{4.9}$$

$$L_{M,\varepsilon} f(x) \;=\; \int_{\{|z|\leq\varepsilon\}} \Delta_1 f(x,z)\nu_{M,\varepsilon}(dz) + \frac{C}{2}\sum_{i,j=1}^d a_\varepsilon^{i,j}(x)\partial_{i,j} f(x). \tag{4.10}$$

We remark that in the above we have used the notation ∂_i and in general ∂_α to denote derivatives with respect to the usual space derivatives in order to distinguish them from high order derivatives with respect to $r \in \mathbb{R}_+$ which are denoted by ∂^k, $k \in \mathbb{N}$.

We recall the reader that the existence of the above universal constant $\frac{C}{2}$ is assured by Lemma 3.1 as the (M, C, ε_*)-germ property is satisfied. In the next corollary we give the approximation error between (4.9) and (4.10).

Corollary 4.2. *Suppose that ν satisfies the $\boldsymbol{GP(M+1, C, \varepsilon_*)}$ property, $c(\cdot,(*,\cdot)) \in C^\infty(\mathbb{R}^d \times \mathbb{R}_+)$ and $c(x,(\rho,0)) = 0$ for every $(x,\rho) \in \mathbb{R}^d \times E$. Then, for $0 < \varepsilon < \varepsilon_*$,*

$$\|(L_\varepsilon - L_{M,\varepsilon})f\|_\infty \leq KQ(f,c) \times \int_{\{|z|\leq\varepsilon\}} |z|^{M+1}\,\nu(dz) \tag{4.11}$$

with K a universal constant (depending on M) and

$$Q(f,c) = \|f\|_{M+1,\infty} \times \left\|c_{(x)}\right\|_{M+1,\infty}^{M+1}.$$

As we remarked in the Introduction, the balance method consists of two elements. The above result gives the first element: an improved speed of convergence by choosing M large enough. The uniform ellipticity condition on a_ε will be required later in order to obtain the second property: that is, the regularity property of the approximation laws.

Proof. Using Taylor expansion we obtain

$$\Delta_1 f(x,z) = \frac{1}{2}\sum_{i,j=1}^d \partial_{i,j} f(x) c^i c^j(x,z) + \sum_{3\leq|\alpha|\leq M} \frac{1}{\alpha!}\partial_\alpha f(x) c^\alpha(x,z) + R_M(x,z)$$

with

$$|R_M(x,z)| \leq K \|f\|_{M+1,\infty} \times \sum_{|\alpha|=M+1} |c^\alpha(x,z)|.$$

Here, c^α denotes the product of the component of the function c which are indicated through the multi-index α. Since $c(x,0)=0$ we have

$$\left|c^i(x,z)\right| = \left|c^i(x,z) - c^i(x,0)\right| \leq \left\|c_{(x)}\right\|_{1,\infty} |z|$$

so that, using the $\boldsymbol{GP(M+1, C, \varepsilon_*)}$ property, we have

$$\int |R_M(x,z)| \left(1_{\{|z|\leq\varepsilon\}}\nu(dz) + \nu_{M,\varepsilon}(dz)\right) \leq K \|f\|_{M+1,\infty} \left\|c_{(x)}\right\|_{1,\infty}^{M+1} \times \int_{\{|z|\leq\varepsilon\}} |z|^{M+1} \nu(dz).$$

We fix now a multi-index α with $3 \leq |\alpha| \leq M$ and $x \in \mathbb{R}^d$ and we use the inequality (4.6) for $\phi^i(z) = c^{\alpha_i}(x,z)$ in order to obtain

$$\left|\int c^\alpha(x,z)(1_{\{|z|\leq\varepsilon\}}\nu(dz) - \nu_{M,\varepsilon}(dz))\right| \leq K \left\|c_{(x)}\right\|_{M+1,\infty}^{|\alpha|} \int_{\{|z|\leq\varepsilon\}} |z|^{M+1} \nu(dz).$$

Finally we notice that, with the notation from (4.5), $a_\varepsilon(x) = a_\varepsilon(c(x,\cdot))$ so we use (4.4) in order to get

$$\left|\int c^i c^j(x,z)(1_{\{|z|\leq\varepsilon\}}\nu(dz) - \nu_{M,\varepsilon}(dz)) - Ca_\varepsilon^{i,j}(x)\right| \leq K \left\|c_{(x)}\right\|_{M+1,\infty}^{2} \int_{\{|z|\leq\varepsilon\}} |z|^{M+1} \nu(dz).$$

The proof is completed using (4.9) and (4.10). □

The previous result gives the definition of the coefficient function $a_\varepsilon = (a_\varepsilon^{i,j})$ in (4.8) whose properties are stated next.

We will use the non degeneracy of the matrix $a_\varepsilon(x)$. In order to simplify the notation we define

$$b_\varepsilon^{i,j}(x) = \int_{|z|\leq\varepsilon} c^i c^j(x,z)\nu(dz).$$

Moreover, we denote by λ_ε the smallest eigenvalue of the matrix $b_\varepsilon(x)$:

$$\lambda_\varepsilon = \inf_{x\in\mathbb{R}^d, |\xi|=1} \langle b_\varepsilon(x)\xi, \xi\rangle = \inf_{x\in\mathbb{R}^d, |\xi|=1} \int_{|z|\leq\varepsilon} \langle c(x,z), \xi\rangle^2 \nu(dz).$$

We assume that c is infinitely differentiable with respect to both x and r and we denote

$$\|c\|_{k,\infty} = \sum_{|\alpha|+|\beta|\leq k} \left\|\partial_\alpha \partial^\beta c\right\|_\infty.$$

Note that $\|c_{(x)}\|_{k,\infty} \leq \|c\|_{k,\infty}$. In what follows, we also let $m_2(\varepsilon) := \int_{|z|\leq\varepsilon} |z|^2 \nu(dz)$.

Lemma 4.3. *Suppose that c satisfies the following uniform ellipticity condition: There exists $\lambda_0 > 0$ and $\varepsilon_* > 0$ such that for $\varepsilon \in (0, \varepsilon_*)$ we have*

$$\frac{\lambda_\varepsilon}{m_2(\varepsilon)} \geq \lambda_0 > 0.$$

Then we may find $\varepsilon_{**} > 0$ such that for $\varepsilon < \varepsilon_{**}$ and $\xi \in \mathbb{R}^d$

$$\xi^* a_\varepsilon(x)\xi \geq \frac{1}{2}\lambda_0 m_2(\varepsilon)|\xi|^2 \geq 0.$$

As a consequence we may define $\sigma_\varepsilon(x) = \sqrt{Ca_\varepsilon(x)}$ and we have

$$\xi^* \sigma_\varepsilon \sigma_\varepsilon^*(x)\xi \geq \frac{C}{2}\lambda_0 m_2(\varepsilon)|\xi|^2.$$

Note that the main hypothesis on this Lemma is a weakened form of the uniform ellipticity condition (see e.g. (5.2)).

Proof. By Taylor expansion,

$$c^i(x,\rho,r) = \partial c^i(x,\rho,0)r + R_2(x,\rho,r)$$

with

$$\sup_{x,\rho} |R_2(x,\rho,r)| \leq K\|c\|_{2,\infty} r^2.$$

So we have from (4.5)

$$\frac{1}{m_2(\varepsilon)}\xi^* a_\varepsilon(x)\xi \geq \frac{1}{m_2(\varepsilon)}\xi^* b_\varepsilon(x)\xi - |R_2(x)|\,|\xi|^2 \geq \lambda_0|\xi|^2 - |R_2(x)|\,|\xi|^2$$

with

$$\sup_x |R_2(x)| \leq \frac{1}{m_2(\varepsilon)} K\|c\|_{2,\infty} \int \lambda(d\rho) \int_{|r|\leq\varepsilon} g_\rho(dr) r^3 \leq K\|c\|_{2,\infty}\varepsilon.$$

So we take $\varepsilon \leq \lambda_0/(2K\|c\|_{2,\infty})$ and we obtain the result. □

5 Application 1: densities of jump driven SDEs

In this section, we apply the previous approximations results in order to obtain the regularity of the law for X_t solution of (1.1).

5.1 The approximation driving process and its associated equation

We consider the d dimensional jump equation

$$X_t = x + \int_0^t \int_{|z|\leq 1} c(X_{s-},z)\widetilde{N}(ds,dz) + \int_0^t \int_{|z|>1} c(X_{s-},z)N(ds,dz) \tag{5.1}$$

where N is a Poisson point measure of intensity[4] ν and $c \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^m)$ with $|c(x,z)| \leq K|z|$. These hypotheses are usual hypotheses that guarantee existence and uniqueness of strong solutions. Furthermore, we assume the following ellipticity hypothesis: there exists $\lambda_0 > 0$ such that for every ε small,

$$\inf_{x\in\mathbb{R}^d, |\xi|=1} \int_{\{|z|\leq\varepsilon\}} \langle c(x,z),\xi\rangle^2 \nu(dz) \geq \lambda_0 \int_{\{|z|\leq\varepsilon\}} |z|^2 \nu(dz) > 0. \tag{5.2}$$

In order to carry out the approximation procedure, we assume that the intensity measure ν satisfies the $\boldsymbol{GP(M+1,C,\varepsilon_*)}$ property for some given M, C, ε_* and construct the matrix $a_\varepsilon \in C_b^\infty(\mathbb{R}^d)$ as in (4.8). Recall that in Lemma 4.3 we have proved that there exists ε_{**} such that for $\varepsilon < \varepsilon_{**}$ we have

$$a_\varepsilon(x) \geq \frac{\lambda_0}{2} m_2(\varepsilon) I > 0.$$

Then we define $\sigma_\varepsilon = \sqrt{Ca_\varepsilon}$ and consider the equation

$$\begin{aligned} X_t^{M,\varepsilon} = \quad & x + \int_0^t \sigma_\varepsilon(X_s^{M,\varepsilon})dB_s + \int_0^t \int_{|z|\leq\varepsilon} c(X_{s-}^{M,\varepsilon},z)\widetilde{N}^{\nu_{M,\varepsilon}}(ds,dz) \\ & + \int_0^t \int_{\varepsilon<|z|\leq 1} c(X_{s-}^{M,\varepsilon},z)\widetilde{N}(ds,dz) + \int_0^t \int_{|z|>1} c(X_{s-}^{M,\varepsilon},z)N(ds,dz). \end{aligned} \tag{5.3}$$

Here B denotes a Brownian motion and $N^{\nu_{M,\varepsilon}}$ denotes a Poisson point measure of intensity $\nu_{M,\varepsilon}$ with $\nu_{M,\varepsilon}$ defined in (4.3) and $\tilde{N}^{\nu_{M,\varepsilon}}$ denotes its compensated counterpart.

[4] If we want to insist in the fact that N depends on ν, we may write N^ν and $\widetilde{N}^\nu$.

The processes $B, N^{\nu_{M,\varepsilon}}, N$ are all independent. When necessary we may use the notation $\int_0^t \sigma_\varepsilon(X_s^{M,\varepsilon})dB_s = \int_0^t \sigma_\varepsilon^k(X_s^{M,\varepsilon})dB_s^k$ where we are using the convention of summation over doubly appearing indexes.

We denote by $P_t f(x) = E(f(X_t(x)))$ and $P_t^{M,\varepsilon} f(x) = E(f(X_t^{M,\varepsilon}(x)))$ the semigroups of the Markov processes $X_t(x)$ and $X_t^{M,\varepsilon}(x)$ respectively. We prove the following approximation result:

Lemma 5.1. *Let ν satisfy the $\boldsymbol{GP(M+1, C, \varepsilon_*)}$ property and $c(\cdot,(*,\cdot)) \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}_+)$ then the following error estimate is satisfied*

$$\left\| P_t f - P_t^{M,\varepsilon} f \right\|_\infty \leq Q_M(c,f) \int_{|z|\leq\varepsilon} |z|^{N+1} \nu(dz) \tag{5.4}$$

with

$$Q_M(c,f) = K \|f\|_{M+1,\infty} \|c\|_{M+1,\infty}^{M+1}.$$

Proof. We notice that

$$\|P_t f\|_\infty + \left\| P_t^{M,\varepsilon} f \right\|_\infty \leq C \|f\|_\infty .$$

and, as a consequence of (4.11), if L is the infinitesimal generator associated to P_t and $\overline{L}_{M,\varepsilon}$ is the infinitesimal operator associated to $P_t^{M,\varepsilon}$ then we have the error estimate for $f \in C_b^{M+1}(\mathbb{R}^d)$

$$\left\| Lf - \overline{L}_{M,\varepsilon} f \right\|_\infty \leq Q_M(c,f) \int_{|z|\leq\varepsilon} |z|^{M+1} \nu(dz).$$

This gives using the second fundamental theorem of calculus for $P_{t-s}P_s^{M,\varepsilon} f(x)$, $s \in [0,t]$ and the properties of infinitesimal generators the following estimate:

$$\begin{aligned} \left\| P_t f - P_t^{M,\varepsilon} f \right\|_\infty &= \left\| \int_0^t \partial_s \left(P_{t-s} P_s^{M,\varepsilon} f \right) ds \right\|_\infty \\ &\leq \int_0^t \left\| P_{t-s}(L - \overline{L}_{M,\varepsilon}) P_s^{M,\varepsilon} f \right\|_\infty ds \\ &\leq Q_M(c,f) \int_{|z|\leq\varepsilon} |z|^{M+1} \nu(dz). \end{aligned}$$

Here we have used that $Q_M(c, P_s^{M,\varepsilon} f) \leq K Q_M(c,f)$ for some constant $K > 0$ independent of M, ε and $s \in [0,t]$. This proof uses the fact that the flows defined by $X^{M,\varepsilon}$ are well defined, their moments are finite and independent of M and ε. □

5.2 The balance method

In this section, we formalize the proof of our main applications which uses the so called balance method. For a complete presentation of this subject, including proofs, we refer to [5] and [4]. Here, we use a criterion introduced in [3] (for details, see the Appendix therein).

We start defining the following norms for $k \in \mathbb{N}$, $h, p > 0$,

$$\|f\|_{k,h,p} = \sum_{|\alpha|\leq k} (\int |\partial_\alpha f(x)|^p (1+|x|)^h dx)^{1/p}, \quad \|f\|_{k,\infty} = \sum_{|\alpha|\leq k} \|\partial_\alpha f\|_\infty .$$

$$d_k(\mu,\nu) = \sup \left\{ \left| \int f(x)(d\mu - d\nu)(dx) \right| : \|f\|_{k,\infty} \leq 1 \right\}.$$

The following theorem is Lemma 4.3 in [3] and may be called a "balance method". In what follows p_* denotes the conjugate of p and $W^{q,p}$ denotes the Sobolev space defined as the subset of functions $f \in L^p(\mathbb{R})$ such that f and its weak derivatives up to order q have a finite $L^p(\mathbb{R})$ norm.

Theorem 5.2. *Let $p > 1, k, q, d \in \mathbb{N}$ and $h \in \mathbb{N}_*$ be fixed and let us denote*

$$\rho_h(q) = \frac{k + q + d/p_*}{2h}.$$

We consider a sequence $\theta(n) \uparrow \infty$ such that, for some $\Theta \geq 1$,

$$\theta(n+1) \leq \Theta \times \theta(n). \tag{5.5}$$

We also consider a sequence of functions $f_n : \mathbb{R}^d \to \mathbb{R}$ such that

$$\|f_n\|_{2h+q,2h,p} \leq \theta(n), \quad n \in \mathbb{N} \tag{5.6}$$

and we denote $\mu_n(dx) = f_n(x)dx$. Suppose that there exists a measure μ and $\delta > 0$ such that

$$\overline{\lim_n} d_k(\mu, \mu_n) \times \theta^{\rho_h(q)+\delta}(n) < \infty. \tag{5.7}$$

Then $\mu(dx) = f(x)dx$ with $f \in W^{q,p}$.

Remark 5.3.

1. The reason why the above result is called a "balance method" requires to explain every element in the above theorem: d is the space dimension of the laws $\{\mu_n; n \in \mathbb{N}\}$, q is the order of derivation that we hope to obtain for the law μ and $p > 1$ is the $L^p(\mathbb{R}^d)$ in which we consider the derivatives of the law μ.
2. The fact that we use $\theta(n)$ instead $\|f_n\|_{2h+q,2h,p}$ in (5.6) has an important meaning in the result as the value of Θ plays a central role in the estimates. In general, it seems difficult to prove that $\|f_{n+1}\|_{2h+q,2h,p} \leq \Theta \|f_n\|_{2h+q,2h,p}$.
3. Finally, the "balance" appears because on the one hand one expects that $d_k(\mu, \mu_n) \to 0$ as $n \to \infty$. We assume that the approximations do not need to approximate the limit densities and in fact the derivatives may even explode at a rate controlled by $\theta(n)$. If (5.7) is satisfied, then one obtains some regularity of the density.

We will make the following "sector type hypothesis" (called the order condition in [26]): there exists $\eta \in (0, 2)$ and $\varepsilon_* > 0$ such that, for $\varepsilon < \varepsilon_*$

$$m_2(\varepsilon) > \varepsilon^{2-\eta}. \tag{5.8}$$

The following hypothesis (sometimes called the flow condition) is needed to ensure that the first derivatives of the flows do not degenerate[5]:

$$\inf_{(x,z)\in(B_1(0)\times\mathbb{R}^m)\cup\left(\bar{B}_1(0)^c\times\mathbb{R}^m\right)} |\det(J_x c(x, z) + I)| > 0. \tag{5.9}$$

Here, $J_x c(x, z)$ denotes the Jacobian with respect to the x.

Theorem 5.4. *Suppose that for each M there are some $C, \varepsilon_* \in (0, 1)$ such that the intensity measure ν satisfies the $\boldsymbol{GP(M+1, C, \varepsilon_*)}$ property. Suppose also that the ellipticity condition (5.2), the sector condition (5.8) and the non-degeneracy hypothesis* (5.9) *hold true. Then $P_t(x, dy) = p_t(x, y)dy$ with $p_t(x, \cdot) \in C^\infty(\mathbb{R}^d)$.*

More precisely, consider $M \in \mathbb{N}_*$, $q \in \mathbb{N}$ and $p > 1$ such that

$$M > \frac{4}{\eta^2}(1 + (1 - \frac{\eta^2}{4})(1 + q + d/p_*)).$$

Assume that ν satisfies the $\boldsymbol{GP(M+1, C, \varepsilon_*)}$ property then $p_t(x, \cdot) \in W^{q,p}$.

[5] Clearly, this condition is relatively independent of the fact that the radius of localization for the stochastic equation representation is 1.

Proof. We will use Theorem 5.2 for the process given by (5.3) with semigroup $P_t^{M,\varepsilon}$ and density $\mu_n(dy) = p_t^{\varepsilon_n}(x,y)dy$. Fixed $q \in \mathbb{N}$ and $k = M+1$. We take $\varepsilon_n = \frac{1}{n}$ and $\theta(n) = \varepsilon_n^{-(2-\eta)(2h+q+d)/2}$.

In fact, the existence of $p_t^{\varepsilon_n}$ and the required estimates for its derivatives are obtained using the integration by parts on the Brownian motion through Malliavin Calculus (for details on Malliavin Calculus for the Wiener process, see [7]) and hypothesis (5.8). That is, we obtain the estimate

$$\|p_t^{\varepsilon_n}(x,\cdot)\|_{2h+q,h,p} \leq \frac{K(q,h,p)}{(Cm_2(\varepsilon_n)t)^{(2h+q+d)/2}} \leq C\theta(n). \tag{5.10}$$

In fact, the details of the above proof are long but standard in the literature. We only point out the main steps:

1. In order to obtain the above results one has to use what is known as partial Malliavin Calculus. That is, Malliavin calculus with respect to the Brownian motion B conditioned on the jump process. In fact, denote by D^k the stochastic derivative with respect to B^k. Then conditioned on the jump noises, one has that the stochastic derivative of (5.3) exists and is a solution of the following linear equation (here $s \leq t$):

$$\begin{aligned} D_s^k X_t^{M,\varepsilon} =& \sigma_\varepsilon^k(X_s^{M,\varepsilon}) + \int_s^t \nabla\sigma_\varepsilon^\ell(X_u^{M,\varepsilon}) D_s^k X_u^{M,\varepsilon} dB_u^\ell \\ &+ \int_s^t \int_{|z|\leq\varepsilon} \nabla c(X_{u-}^{M,\varepsilon}, z) D_s^k X_u^{M,\varepsilon} \widetilde{N}^{\nu_{M,\varepsilon}}(du,dz) \\ &+ \int_s^t \int_{\varepsilon<|z|\leq 1} \nabla c(X_{u-}^{M,\varepsilon}, z) D_s^k X_u^{M,\varepsilon} \widetilde{N}(du,dz) \\ &+ \int_s^t \int_{|z|>1} \nabla c(X_{u-}^{M,\varepsilon}, z) D_s^k X_u^{M,\varepsilon} N(du,dz). \end{aligned}$$

   The solution of the above equation can be rewritten using the variation of constants formula as $D_s^k X_t^{M,\varepsilon} = \sigma_\varepsilon^k(X_s^{M,\varepsilon})\mathcal{E}_{s,t}$ where $\mathcal{E}_{s,t}$ is the solution of the same linear matrix equation satisfied by $D_s^k X_t^{M,\varepsilon}$, $t \geq s$ with initial condition I. Furthermore, the process $\mathcal{E}_{u,t} = \mathcal{E}_{0,t}\mathcal{E}_{0,u}^{-1}$ where $\mathcal{E}_{0,t}$ is the so-called Doléans-Dade exponential. The existence of the inverse of $\mathcal{E}$ is assured because of condition (5.9) and their moment estimates are obtained using the stochastic equations satisfied by the inverse of the Doléans-Dade exponential. For details, see [7].
2. The moment estimates for the stochastic derivatives are obtained applying Gronwall inequalities or the arguments in [7].
3. The basic tool in order to obtain the existence of p^{ε_n} is the integration by parts formula of Malliavin Calculus which requires estimates for the so-called Malliavin covariance matrix given by $\Sigma_t = \int_0^t \sum_{k=1}^d (D_s^k X_u^{M,\varepsilon})(D_s^k X_u^{M,\varepsilon})^T du$.
4. The invertibility of the Malliavin covariance matrix is ensured by the ellipticity condition (5.2) and the sector condition (5.8). In fact, one has $\det(\Sigma_t) \geq Cm_2(\varepsilon_n)t$.
5. Finally, one uses the integration by parts formulas which are the same as the ones appearing in [25] and [12]. For this, one uses the previous estimate on $\det(\Sigma_t)$ and the moment estimates obtained in previous steps. This gives (5.10).

Now, we verify (5.7). In fact, note first that the estimate (5.4) reads

$$d_{M+1}(P_t(x,dy), P_t^{M,\varepsilon}(x,dy)) \leq K \int_{|z|\leq\varepsilon} |z|^{M+1} \nu(dz).$$

The restriction (5.5) is satisfied with $\Theta = 2$ and n large enough. Moreover

$$\begin{aligned} d_{M+1}(\mu,\mu_n) \times \theta^{\rho_h(q)+\delta}(n) &\leq K \int_{|z|\leq\varepsilon_n} |z|^{M+1}\, \nu(dz)\varepsilon_n^{-(2-\eta)(2h+q+d)(\rho_h(q)+\delta)/2} \\ &\leq K\varepsilon_n^{M-1-(2-\eta)(2h+q+d)(\rho_h(q)+\delta)/2} m_2(\varepsilon_n). \end{aligned}$$

Finally, the proof finishes by analyzing the above exponent. After some algebraic simplifications one needs to check that

$$M-1 \geq (1-\frac{\eta}{2})\left(1+\frac{q+d}{2h}\right)\left(M+1+q+\frac{d}{p_*}+2h\delta\right).$$

In order for the above inequality to be satisfied it is sufficient to choose first $h \geq \frac{q+d}{\eta}$ and then

$$M \geq \frac{4}{\eta^2}(1+(1-\frac{\eta^2}{4})(1+q+d/p_*+2h\delta)).$$

Now take $\delta > 0$ small enough and therefore is sufficient to assume that

$$M > \frac{4}{\eta^2}(1+(1-\frac{\eta^2}{4})(1+q+d/p_*)).$$

In fact, these conditions imply that

$$(1-\frac{\eta}{2})\left(1+\frac{q+d}{2h}\right)\left(M+1+q+\frac{d}{p_*}+2h\delta\right) \leq (1-\frac{\eta^2}{4})\left(M+1+q+\frac{d}{p_*}+2h\delta\right) \leq M-1.$$

So the exponent is strictly positive and therefore (5.7) follows. □

The following straightforward extension of the previous result indicates that although the hypothesis $\boldsymbol{A(M+1)}$ contains a continuity requirement on s_k, this can be circumvented with the following approximation procedure.

In (5.1), we have replaced $1_{\{|z|\leq\varepsilon\}}\widetilde{N}(ds,dz)$ by $\sigma_\varepsilon(X_s^\varepsilon)dB_s + 1_{\{|z|\leq\varepsilon\}}\widetilde{N}^{\nu_{M,\varepsilon}}(ds,dz)$. We may repeat the same argument with only the continuous part of ν which therefore implies that no regularity condition on ν is required.

Then, instead of the equation (5.3) we would consider the approximation

$$\begin{aligned} X_t^{M,\varepsilon} =& x + \int_0^t \sigma_\varepsilon^\mu(X_s^{M,\varepsilon})dB_s + \int_0^t\int_{|z|\leq\varepsilon} c(X_{s-}^{M,\varepsilon},z)\widetilde{N}^{\mu_{M,\varepsilon}}(ds,dz) \\ &+ \int_0^t\int_{\varepsilon<|z|\leq1} c(X_{s-}^{M,\varepsilon},z)\widetilde{N}^{\mu}(ds,dz) + \int_0^t\int_{|z|\leq1} c(X_{s-}^{M,\varepsilon},z)\widetilde{N}^{\nu-\mu}(ds,dz) \\ &+ \int_0^t\int_{|z|>1} c(X_{s-}^{M,\varepsilon},z)N(ds,dz) \end{aligned}$$

with σ_ε^μ constructed as above, but with ν replaced by μ. This gives much more freedom in choosing a measure μ which should verify the $\boldsymbol{GP(M+1,C,\varepsilon_*)}$ property. For example, this is the case if the following inequality is valid: There exists $\eta > 0$ such that for any set $A \subset B_\eta(0)$

$$\nu(E\times A) \geq \int_A \frac{dr}{r^{1+\alpha}}.$$

Then we may take μ to be the stable law of index α and, using Lemma 3.1, this measure verifies the $\boldsymbol{GP(M,C,\varepsilon_*)}$ property for every M. The price to be paid is that in the non-degeneracy condition (5.2) we have to replace ν by μ but on the other hand ν may be an "irregular" measure. The proof is the same as Theorem 5.4 with the corresponding changes of ν by μ in the appropriate places.

Corollary 5.5. *Let μ be a measure such that $\mu \leq \nu$ and it satisfies the conditions of Theorem 5.4. Then $P_t(x,dy) = p_t(x,y)dy$ with $p_t(x,\cdot) \in C^\infty(\mathbb{R}^d)$.*

6 Application 2: a state dependent Lévy measure

In this section we treat an stochastic equation driven by a Poisson random measure which is state dependent. We show that a general jump transformation that may allow to treat these cases which at first are not treated in Section 3. Specifically, we will deal with jump equations with Lévy measure given by $\nu(x,dz) = \frac{dz}{z^{1+\alpha(x)}}$ in the case where $d = m = 1$.

We give the main steps of the analysis leading to the regularity of the density of the associated stochastic equation. First, we consider a change of variables problem:

Step 1: Given $\alpha_0, a \in (0,2)$ and $\alpha \in C_b^\infty(\mathbb{R}, (a,2))$ then there exists a strictly increasing function $\phi_x : \mathbb{R}_+ \to \mathbb{R}_+$, $x \in \mathbb{R}$, such that it verifies that $\phi_x(0) = 0$, $\phi_x(\infty) = \infty$ and such that for every bounded measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $|f(z)| \leq K|z|^2$ in a neighborhood of $z = 0$, we have

$$\int_0^\infty f(\phi_x(z))\frac{dz}{z^{1+\alpha_0}} = \int_0^\infty f(z)\frac{dz}{z^{1+\alpha(x)}}. \tag{6.1}$$

Proof. Consider the change of variable $y = \phi_x(z)$ which also reads $z = \phi_x^{-1}(y)$. Then

$$dz = \frac{dy}{\phi_x'(\phi_x^{-1}(y))}$$

and the above equality (6.1) becomes

$$\int_0^\infty f(y)\frac{dy}{\phi_x^{-1}(y)^{1+\alpha_0}\phi_x'(\phi_x^{-1}(y))} = \int_0^\infty f(z)\frac{dz}{z^{1+\alpha(x)}}.$$

Therefore in order for (6.1) to be satisfied, we just need to solve the ordinary differential equation

$$\frac{1}{\phi_x^{-1}(y)^{1+\alpha_0}\phi_x'(\phi_x^{-1}(y))} = \frac{1}{y^{1+\alpha(x)}}.$$

If we denote $u = \phi_x^{-1}(y)$ then the above equality reads

$$\frac{1}{u^{1+\alpha_0}\phi_x'(u)} = \frac{1}{\phi_x(u)^{1+\alpha(x)}}.$$

Therefore the simplified ode becomes:

$$\frac{1}{u^{1+\alpha_0}} = \frac{\phi_x'(u)}{\phi_x(u)^{1+\alpha(x)}}, \quad \phi_x(0) = 0.$$

The solution to the above equation is

$$\phi_x(u) = \left(\frac{\alpha_0}{\alpha(x)}\right)^{\frac{1}{\alpha(x)}} \times u^{\frac{\alpha_0}{\alpha(x)}}.$$

Step 2: Infinitesimal operator

We consider the effect of the change of variables in Step 1 in order to redefine the infinitesimal generators associated with the stochastic differential equation under consideration.

For simplicity, we consider the infinitesimal operator

$$Lf(x) = \int_0^\infty (f(x+c(x,z)) - f(x) - f'(x)c(x,z))\frac{dz}{z^{1+\alpha(x)}}.$$

We assume that $c \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^m)$ with $|c(x,z)| \leq K|z|$. The above integral in view of (6.1) may be rewritten as

$$Lf(x) = \int_0^\infty (f(x+c(x,\phi_x(z))) - f(x) - f'(x)c(x,\phi_x(z)))\frac{dz}{z^{1+\alpha_0}}.$$

So, instead of the couple $(c(x,z), \frac{dz}{z^{1+\alpha(x)}})$ we work with the representation $(\overline{c}(x,z), \frac{dz}{z^{1+\alpha_0}})$ with

$$\overline{c}(x,z) = c(x,\phi_x(z)).$$

Recall that that $0 < a < \alpha(x) \leq \alpha_0$ and $\alpha \in C_b^\infty$. Then $x \mapsto \phi_x(u)$ is infinitely differentiable and we have

$$\partial_x^{(k)}\phi_x(u) = \phi_x(u) \times \Theta_k(x)$$

with $\Theta_k \in C_b^\infty$. This means that all the properties of regularity which we impose on $x \mapsto c(x,z)$ will hold for $x \mapsto \overline{c}(x,z)$ as well. So we just need to apply the same argument as in the previous example for the pair $(\overline{c}(x,z), \frac{dz}{z^{1+\alpha_0}})$.

This gives the following result.

Theorem 6.1. *Let $N(x,ds,dz)$ be the Poisson random measure associated with the state dependent Lévy measure $\nu(x,dz) = \frac{dz}{z^{1+\alpha(x)}}$. Consider the equation:*

$$X_t = x + \int_0^t \int_{|z|\leq 1} \overline{c}(X_{s-},z)d\widetilde{N}(x,ds,dz) + \int_0^t \int_{|z|>1} c(X_{s-},z)dN(x,ds,dz).$$

Besides the above conditions, suppose also that the ellipticity condition (5.2) and the flow condition (5.9) *hold true for the above coefficients. Then $P_t(x,dy) = p_t(x,y)dy$ with $p_t(x,\cdot) \in C^\infty(\mathbb{R}^d)$.*

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