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Zooming in at the root of the stable tree

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Abstract

We study the shape of the normalized stable Lévy tree $\mathcal{T}$ near its root. We show that, when zooming in at the root at the proper speed with a scaling depending on the index of stability, we get the unnormalized Kesten tree. In particular the limit is described by a tree-valued Poisson point process which does not depend on the initial normalization. We apply this to study the asymptotic behavior of additive functionals of the form

$$\mathbf{Z}_{\alpha,\beta} = \int_{\mathcal{T}} \mu(\mathrm{d}x) \int_0^{H(x)} \sigma_{r,x}^{\alpha} \mathfrak{h}_{r,x}^{\beta} \, \mathrm{d}r$$

as $\max(\alpha, \beta) \to \infty$, where μ is the mass measure on $\mathcal{T}$, $H(x)$ is the height of x and $\sigma_{r,x}$ (resp. $\mathfrak{h}_{r,x}$) is the mass (resp. height) of the subtree of $\mathcal{T}$ above level r containing x. Such functionals arise as scaling limits of additive functionals of the size and height on conditioned Bienaymé-Galton-Watson trees.

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1 Introduction

Stable trees are special instances of Lévy trees which were introduced by Le Gall and Le Jan [23] in order to generalize Aldous' Brownian tree [4]. More precisely, stable trees are compact weighted rooted real trees depending on a parameter $\gamma \in (1, 2]$, with $\gamma = 2$ corresponding to the Brownian tree, which encode the genealogical structure of continuous-state branching processes with branching mechanism $\psi(\lambda) = \lambda^{\gamma}$. As such, they are the possible scaling limits of Bienaymé-Galton-Watson trees with critical offspring distribution belonging to the domain of attraction of a stable distribution with index $\gamma \in (1, 2]$, see Duquesne [10] and Kortchemski [22]. They also appear as scaling limits of various models of trees and graphs, see e.g. Haas and Miermont [20], and are intimately related to fragmentation and coalescence processes, see Miermont [25, 26]

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and Berestycki, Berestycki and Schweinsberg [5]. Stable trees can be defined via the normalized excursion of the so-called height process which is a local time functional of a spectrally positive Lévy process. We refer to Duquesne and Le Gall [11] for a detailed account. See also Duquesne and Winkel [14], Goldschmidt and Haas [18], Marchal [24] for alternative constructions.

In the present paper, we study the shape of the normalized stable tree $\mathcal{T}$ (*i.e.* the stable tree conditioned to have total mass 1) near its root. More precisely we show that, after zooming in at the root of $\mathcal{T}$ and rescaling, one gets the continuous analogue of the Kesten tree, that is a random real tree consisting of an infinite branch on which subtrees are grafted according to a Poisson point process. In particular, the (rescaled) subtrees near the root of $\mathcal{T}$ are independent and the conditioning for the total mass to be equal to 1 disappears when zooming in. This idea to zoom in at the root of the stable tree is closely related to the small time asymptotics – present in the works of Miermont [25] and Haas [19] – of the self-similar fragmentation process $F^{-}(t)$ obtained from the stable tree by removing vertices located under height t. See Remark 4.5 in this direction. As a consequence, we obtain the asymptotic behavior of additive functionals on $\mathcal{T}$ of the form

$$\mathbf{Z}_{\alpha,\beta} = \int_{\mathcal{T}} Z_{\alpha,\beta}(x)\,\mu(\mathrm{d}x) \qquad \text{with} \quad \forall x \in \mathcal{T},\ Z_{\alpha,\beta}(x) = \int_0^{H(x)} \sigma_{r,x}^{\alpha}\mathfrak{h}_{r,x}^{\beta}\,\mathrm{d}r, \tag{1.1}$$

where μ is the mass measure on $\mathcal{T}$ which is a uniform measure supported by the set of leaves, $H(x)$ is the height of $x \in \mathcal{T}$, that is its distance to the root, and $\sigma_{r,x}$ (resp. $\mathfrak{h}_{r,x}$) is the mass (resp. height) of the subtree of $\mathcal{T}$ above level r containing x.

Before stating our results, we first introduce some notations. Let $\mathbb{T}$ be the space of weighted rooted compact real trees, that is the set of compact real trees (T,d) endowed with a distinguished vertex $\emptyset$ called the root and with a nonnegative finite measure μ. We equip the set $\mathbb{T}$ with the Gromov-Hausdorff-Prokhorov topology, see Section 2 for a precise definition.

Define a rescaling map $R_\gamma\colon \mathbb{T}\times(0,\infty)\to\mathbb{T}$ by

$$R_\gamma\left((T,\emptyset,d,\mu),a\right) = \left(T,\emptyset,ad,a^{\gamma/(\gamma-1)}\mu\right). \tag{1.2}$$

In words, $R_\gamma((T,\emptyset,d,\mu),a)$ is the tree obtained from $(T,\emptyset,d,\mu)$ by multiplying all distances by a and all masses by $a^{\gamma/(\gamma-1)}$. Moreover, define for every $(T,\emptyset,d,\mu)\in\mathbb{T}$

$$\mathsf{norm}_\gamma(T) = R_\gamma(T,\mu(T)^{-1+1/\gamma}), \tag{1.3}$$

which is the tree T normalized to have total mass 1 and where distances are rescaled accordingly. Denote by $\mathbb{N}^{(1)}$ the distribution of the normalized stable tree with total mass 1, see Section 3 for a precise definition. Under $\mathbb{N}^{(1)}$, let U be a uniformly chosen leaf, that is U is a $\mathcal{T}$-valued random variable with distribution μ. Denote by $\mathcal{T}_i$, $i\in I_U$ the trees grafted on the branch $[\![\emptyset,U]\!]$ joining the root $\emptyset$ to the leaf U, each one at height h_i and with total mass $\sigma_i=\mu(\mathcal{T}_i)$, see Figure 1. Fix $\mathfrak{f}\colon(0,\infty)\to(0,\infty)$ (this represents the speed at which we zoom in) and define for every $\varepsilon>0$ a point measure on $[0,\infty)^2\times\mathbb{T}$ by

$$\mathcal{N}_\varepsilon^{\mathfrak{f}}(U) = \sum_{h_i\leqslant\mathfrak{f}(\varepsilon)H(U)} \delta_{\left(\varepsilon^{-1}h_i,\varepsilon^{-\gamma/(\gamma-1)}\sigma_i,\mathsf{norm}_\gamma(\mathcal{T}_i)\right)}. \tag{1.4}$$

Finally, for any metric space X, we denote by $\mathcal{M}_p(X)$ the space of point measures on X equipped with the topology of vague convergence.

Our first main result states that the measure $\mathcal{N}_\varepsilon^{\mathfrak{f}}(U)$ converges to a Poisson point process which is independent of the underlying tree $\mathcal{T}$ and of $H(U)$.

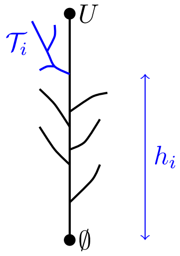

Figure 1: The subtrees $\mathcal{T}_i$ grafted on the branch $[\![\emptyset, U]\!]$ at height h_i.

Theorem 1.1. *Let $\mathcal{T}$ be the normalized stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1,2]$. Conditionally on $\mathcal{T}$, let U be a $\mathcal{T}$-valued random variable with distribution μ under $\mathbb{N}^{(1)}$. Let $(\mathsf{T}'_s,\ s \geqslant 0)$ be a Poisson point process with intensity $\mathbb{N}^{\mathrm{B}}$ given by (4.1), independent of $(\mathcal{T}, H(U))$. Let $\Phi\colon [0,\infty)^2 \times \mathbb{T} \to [0,\infty)$ be a measurable function such that there exists $C > 0$ such that for every $h \geqslant 0$ and $T \in \mathbb{T}$, we have*

$$|\Phi(h,b,T) - \Phi(h,a,T)| \leqslant C|b-a|.$$

(i) If $\lim_{\varepsilon\to 0} \varepsilon^{-1/2}\mathfrak{f}(\varepsilon) = 0$ and $\lim_{\varepsilon \to 0} \varepsilon^{-1}\mathfrak{f}(\varepsilon) = \infty$, then we have the following convergence in distribution

$$\left(\mathcal{T}, H(U), \langle \mathcal{N}_\varepsilon^{\mathfrak{f}}(U), \Phi\rangle\right) \xrightarrow[\varepsilon\to 0]{(d)} \left(\mathcal{T}, H(U), \sum_{s\geqslant 0} \Phi\left(s, \mu(\mathsf{T}'_s), \mathsf{norm}_\gamma\left(\mathsf{T}'_s\right)\right)\right) \tag{1.5}$$

in the space $\mathbb{T} \times [0,\infty) \times [0,\infty]$. In particular, we have the following convergence in distribution in $\mathbb{T} \times [0,\infty) \times \mathcal{M}_p([0,\infty) \times \mathbb{T})$

$$\left(\mathcal{T}, H(U), \sum_{h_i \leqslant \mathfrak{f}(\varepsilon) H(U)} \delta_{(\varepsilon^{-1}h_i, R_\gamma(\mathcal{T}_i, \varepsilon^{-1}))}\right) \xrightarrow[\varepsilon\to 0]{(d)} \left(\mathcal{T}, H(U), \sum_{s\geqslant 0} \delta_{(s,\mathsf{T}'_s)}\right). \tag{1.6}$$

(ii) If $\mathfrak{f}(\varepsilon) = \varepsilon$, then we have the following convergence in distribution

$$\left(\mathcal{T}, H(U), \langle \mathcal{N}_\varepsilon^{\mathfrak{f}}(U), \Phi\rangle\right) \xrightarrow[\varepsilon\to 0]{(d)} \left(\mathcal{T}, H(U), \sum_{s\leqslant H(U)} \Phi\left(s, \mu(\mathsf{T}'_s), \mathsf{norm}_\gamma\left(\mathsf{T}'_s\right)\right)\right) \tag{1.7}$$

in the space $\mathbb{T} \times [0,\infty) \times [0,\infty]$.

In other words, zooming in at the speed $\mathfrak{f}(\varepsilon) = \varepsilon$ gives a *finite* branch on which subtrees are grafted in a Poissonian manner, whereas zooming in at a slower speed gives an *infinite* branch at the limit. Notice that the convergence (1.5) is stronger than convergence in distribution for the vague topology (1.6) as it holds for functions Φ with very few regularity assumptions: $\Phi(h,a,T)$ is only Lipschitz-continuous with respect to a instead of (Lipschitz-)continuous with respect to (h,a,T) with bounded support. In particular, this could allow to consider local time functionals of the tree.

As an application of this result, we study the asymptotic behavior as $\max(\alpha,\beta) \to \infty$ of additive functionals $\mathbf{Z}_{\alpha,\beta}$ on the stable tree $\mathcal{T}$. Such functionals arise as scaling limits of additive functionals of the size and height on conditioned Bienaymé-Galton-Watson trees, see Delmas, Dhersin and Sciauveau [9] or Abraham, Delmas and Nassif [1] where it is shown that $\mathbf{Z}_{\alpha,\beta} < \infty$ a.s. if (and only if) $\gamma\alpha + (\gamma-1)(\beta+1) > 0$, see Corollary 6.10 therein. In the present paper, we only consider $\alpha, \beta \geqslant 0$ which guarantees in particular the finiteness of $\mathbf{Z}_{\alpha,\beta}$. For example, let us mention the total path length and the Wiener index which when properly scaled converge respectively to $\mathbf{Z}_{0,0}$ and $\mathbf{Z}_{1,0}$. Fill and Janson [16] considered the case $\gamma = 2$ and $\beta = 0$ (*i.e.* functionals of the mass on the Brownian tree) and proved that there is convergence in distribution as $\alpha \to \infty$ of $\mathbf{Z}_{\alpha,0}$ properly normalized to

$$\int_0^\infty \mathrm{e}^{-S_t}\,\mathrm{d}t,$$

where $(S_t,\, t \geqslant 0)$ is a $1/2$-stable subordinator. Their proof relies on the connection between the normalized Brownian excursion which codes the Brownian tree and the three-dimensional Bessel bridge. Our aim is twofold: we extend their result to the non-Brownian stable case $\gamma \in (1,2)$ while also considering polynomial functionals depending on both the mass and the height. We use a different approach relying on the Bismut decomposition of the stable tree.

Going back to the connection with the self-similar fragmentation process $F^-(t) = (F_1^-(t), F_2^-(t), \ldots)$, it is not hard to see that the additive functional $\mathbf{Z}_{\alpha,0}$ can be expressed in terms of F^- as

$$\mathbf{Z}_{\alpha,0} = \sum_{i\geqslant 1} \int_0^\infty F_i^-(t)^{\alpha+1}\,\mathrm{d}t.$$

Once this is established, one can argue that only the largest fragment F_1^- contributes to the limit, the others being negligible, then use [19, Corollary 17] which implies that $1 - F_1^-$ properly normalized converges in distribution to a $(1-1/\gamma)$-stable subordinator S, to get the convergence of $\mathbf{Z}_{\alpha,0}$ to $\int_0^\infty \mathrm{e}^{-S_t}\,\mathrm{d}t$. In the present paper, we do not adopt this approach as it does not allow to consider functionals of the height (that is $\beta \neq 0$).

We distinguish two regimes according to the behavior of $\beta/\alpha^{1-1/\gamma}$. The regime $\beta/\alpha^{1-1/\gamma} \to c \in [0,\infty)$ is related to Theorem 1.1 and the result in that case can be stated as follows, see Theorem 5.4 for a more general statement.

Theorem 1.2. *Assume that $\alpha \to \infty$, $\beta \geqslant 0$ and $\beta/\alpha^{1-1/\gamma} \to c \in [0,\infty)$. Let $\mathcal{T}$ be the normalized stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1,2]$ and denote by $\mathfrak{h}$ its height. Then we have the following convergence in distribution under $\mathbb{N}^{(1)}$*

$$\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\mathbf{Z}_{\alpha,\beta} \xrightarrow[\alpha\to\infty]{(d)} \int_0^\infty \mathrm{e}^{-S_t - ct/\mathfrak{h}}\,\mathrm{d}t, \tag{1.8}$$

where $(S_t, t \geqslant 0)$ is a stable subordinator with Laplace exponent $\varphi(\lambda) = \gamma\lambda^{1-1/\gamma}$, independent of $\mathcal{T}$.

Let us briefly explain why we get a subordinator S at the limit. It is well known that μ is supported on the set of leaves of $\mathcal{T}$. Let $x \in \mathcal{T}$ be a leaf and recall that $\sigma_{r,x}$ is the mass of the subtree above level r containing x. Since the total mass of the stable tree is 1, the main contribution to $Z_{\alpha,\beta}(x)$ as $\alpha \to \infty$ comes from large subtrees $\mathcal{T}_{r,x}$ with r close to 0. The height $\mathfrak{h}_{r,x}$ of such subtrees is approximately $\mathfrak{h} - r$. On the other hand, their mass is equal to 1 minus the mass we discarded from the subtrees grafted on the branch $[\![\emptyset, x]\!]$ at height less than r. By Theorem 1.1, subtrees are grafted on $[\![\emptyset, x]\!]$ according to a point process which is approximately Poissonian, at least close to the root $\emptyset$. Thus the mass $\sigma_{r,x}$ is approximately $1 - S_r$.

Theorem 5.4 is slightly more general: we prove joint convergence in distribution of $\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\mathbf{Z}_{\alpha,\beta}$ and $\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}Z_{\alpha,\beta}(U)$, where $U \in \mathcal{T}$ is a leaf chosen uniformly at random (*i.e.* according to the measure μ), to the same random variable. In other words, taking the average of $Z_{\alpha,\beta}(x)$ over all leaves yields the same asymptotic behavior as taking a leaf uniformly at random. This is due to the following observations: a) a uniform leaf U is not too close to the root with high probability in the sense that its most recent common ancestor with x^* has height greater than ε, where x^* is the heighest leaf of $\mathcal{T}$,

b) when taking the average over all leaves, the contribution of those leaves whose most recent common ancestor with x^* has height less than ε is negligible, and c) for those $x \in \mathcal{T}$ whose most recent common ancestor with x^* has height greater than ε, the main contribution to $Z_{\alpha,\beta}(x)$ comes from large subrees $\mathcal{T}_{r,x}$ with $r \leqslant \varepsilon$, these subtrees are common to all such leaves as $\mathcal{T}_{r,x} = \mathcal{T}_{r,x^*}$. This is made rigorous in Lemma 5.3.

Let us make a connection with Theorem 1.18 of Fill and Janson [16]. Recall that the normalized Brownian tree with branching mechanism $\psi(\lambda) = \lambda^2$ is coded by $\sqrt{2}B^{\mathrm{ex}}$ where B^{ex} is the normalized Brownian excursion, see [11]. Thanks to the representation formula of [9, Lemma 8.6], we see that Fill and Janson's $Y(\alpha) = \sqrt{2}\mathbf{Z}_{\alpha-1,0}$. Thus, we recover their result in the Brownian case $\gamma = 2$ when $\beta = 0$ (in which case $c = 0$).

Notice that as long as the exponent β of the height does not grow too quickly, viz. $\beta/\alpha^{1-1/\gamma} \to 0$, the additional dependence on the height makes no contribution at the limit. On the other hand, in the regime $\beta/\alpha^{1-1/\gamma} \to \infty$, the height $\mathfrak{h}_{r,x}^{\beta}$ dominates the mass $\sigma_{r,x}^{\alpha}$ so we get the convergence in probability of $\mathbf{Z}_{\alpha,\beta}$ with a different scaling and there is no longer a subordinator at the limit. See Theorem 6.1 for a more general statement.

Theorem 1.3. *Assume that $\beta \to \infty$, $\alpha \geqslant 0$ and $\alpha^{1-1/\gamma}/\beta \to 0$. Let $\mathcal{T}$ be the normalized stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1, 2]$. Then we have the following convergence in $\mathbb{N}^{(1)}$-probability*

$$\lim_{\beta\to\infty} \beta\mathfrak{h}^{-\beta}\mathbf{Z}_{\alpha,\beta} = \mathfrak{h}. \tag{1.9}$$

Remark 1.4. Assume that $\alpha, \beta \to \infty$ and $\beta/\alpha^{1-1/\gamma} \to c \in (0, \infty)$ so that Theorem 1.2 applies. Then we have the convergence in distribution under $\mathbb{N}^{(1)}$

$$\beta\mathfrak{h}^{-\beta}\mathbf{Z}_{\alpha,\beta} = \frac{\beta}{\alpha^{1-1/\gamma}}\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\mathbf{Z}_{\alpha,\beta} \xrightarrow[\beta\to\infty]{(d)} c\int_0^\infty \mathrm{e}^{-S_t - ct/\mathfrak{h}}\,\mathrm{d}t = \mathfrak{h}\int_0^\infty \mathrm{e}^{-S_{\mathfrak{h}t/c} - t}\,\mathrm{d}t.$$

Now letting $c \to \infty$, the right-hand side converges to $\mathfrak{h}\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t = \mathfrak{h}$. Thus, one may view Theorem 1.3 as a special case of Theorem 1.2 by saying that, if $\beta \to \infty$ and $\beta/\alpha^{1-1/\gamma} \to c \in (0, \infty]$, then we have the convergence in distribution under $\mathbb{N}^{(1)}$

$$\beta\mathfrak{h}^{-\beta}\mathbf{Z}_{\alpha,\beta} \xrightarrow[\beta\to\infty]{(d)} c\int_0^\infty \mathrm{e}^{-S_t - ct/\mathfrak{h}}\,\mathrm{d}t,$$

where the measure $c\mathrm{e}^{-ct/\mathfrak{h}}\,\mathrm{d}t$ on $[0, \infty)$ should be understood as $\mathfrak{h}\delta_0$ if $c = \infty$.

We conclude the introduction by giving a decomposition of a general (compact) Lévy tree used in the proof of Theorem 1.2 which is of independent interest. Consider a Lévy tree $\mathcal{T}$ under its excursion measure $\mathbb{N}$ associated with a branching mechanism $\psi(\lambda) = a\lambda + b\lambda^2 + \int_0^\infty (\mathrm{e}^{-\lambda r} - 1 + \lambda r)\,\pi(\mathrm{d}r)$ where $a, b \geqslant 0$ and π is a σ-finite measure on $(0, \infty)$ satisfying $\int_0^\infty (r \wedge r^2)\,\pi(\mathrm{d}r) < \infty$. We further assume that the Grey condition holds $\int^\infty \mathrm{d}\lambda/\psi(\lambda) < \infty$ which is equivalent to the compactness of the Lévy tree. We refer to [11, Section 1] for a complete presentation of the subject. For every $x \in \mathcal{T}$ and every $0 \leqslant r < r' \leqslant H(x)$, we let $\mathcal{T}_{[r,r'),x} = (\mathcal{T}_{r,x} \setminus \mathcal{T}_{r',x}) \cup \{x_{r'}\}$ where $x_{r'}$ is the unique ancestor of x at height $H(x_{r'}) = r'$ and $\mathcal{T}_{r,x}$ is the subtree of $\mathcal{T}$ above level r containing x. The following result states that, when $x \in \mathcal{T}$ and $0 =: r_0 < r_1 < \ldots < r_n < r_{n+1} := H(x)$ are chosen "uniformly" at random under $\mathbb{N}$, then the random trees $\mathcal{T}_{[r_{i-1},r_i),x}$, $1 \leqslant i \leqslant n+1$ are independent and distributed as $\mathcal{T}$ under $\mathbb{N}[\sigma\bullet]$, see Figure 2. In particular, this generalizes [1, Lemma 6.1] which corresponds to $n = 1$.

Theorem 1.5. *Let $\mathcal{T}$ be the Lévy tree with a general branching mechanism ψ satisfying the Grey condition $\int^\infty \mathrm{d}\lambda/\psi(\lambda) < \infty$ under its excursion measure $\mathbb{N}$. Then for every*

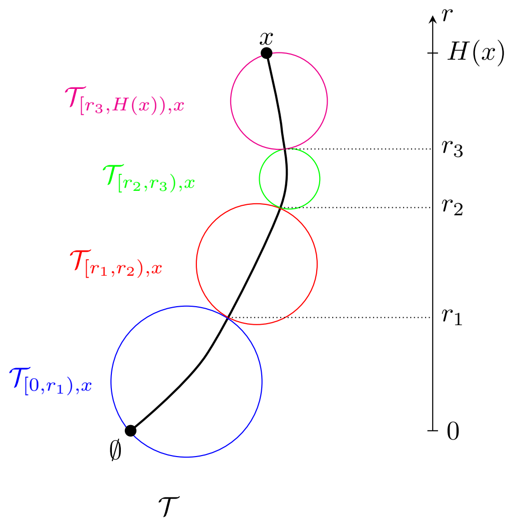

Figure 2: The decomposition of $\mathcal{T}$ under $\mathbb{N}$ into $n+1$ (with $n=3$) subtrees along the ancestral line of a uniformly chosen leaf x.

$n \geqslant 1$ and all nonnegative measurable functions $f_i,\ 1 \leqslant i \leqslant n+1$ defined on $[0,\infty) \times \mathbb{T}$, we have with $r_0 = 0$ and $r_{n+1} = H(x)$

$$\mathbb{N}\left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} f_i\left(r_i - r_{i-1}, \mathcal{T}_{[r_{i-1},r_i),x}\right) \prod_{i=1}^{n} \mathrm{d}r_i\right] = \prod_{i=1}^{n+1} \mathbb{N}\left[\int_{\mathcal{T}} \mu(\mathrm{d}x) f_i(H(x), \mathcal{T})\right].$$

In particular, for every nonnegative measurable functions g_i, $1 \leqslant i \leqslant n+1$ defined on $\mathbb{T}$, we have

$$\mathbb{N}\left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} g_i\left(\mathcal{T}_{[r_{i-1},r_i),x}\right) \prod_{i=1}^{n} \mathrm{d}r_i\right] = \prod_{i=1}^{n+1} \mathbb{N}\left[\sigma g_i(\mathcal{T})\right].$$

A consequence of this decomposition is the following result giving the joint distribution of $\mathcal{T}_y$, the subtree of $\mathcal{T}$ above vertex $y \in \mathcal{T}$, and $H(y)$ when y is chosen according to the length measure $\ell(\mathrm{d}y)$ on the stable tree $\mathcal{T}$ (which roughly speaking is the Lebesgue measure on the branches of $\mathcal{T}$). In particular, this generalizes [1, Proposition 1.6].

Corollary 1.6. *Let $\mathcal{T}$ be the normalized stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1,2]$. Let f and g be nonnegative measurable functions defined on $\mathbb{T}$ and $[0,\infty)$ respectively. We have*

$$\mathbb{N}^{(1)}\left[\int_{\mathcal{T}} f(\mathcal{T}_y) g(H(y))\, \ell(\mathrm{d}y)\right] = \mathbb{N}\left[\mathbf{1}_{\{\sigma<1\}}(1-\sigma)^{-1/\gamma} G(1-\sigma) f(\mathcal{T})\right] \tag{1.10}$$

where

$$G(a) = \mathbb{N}^{(1)}\left[\int_{\mathcal{T}} \mu(\mathrm{d}x) g\left(a^{1-1/\gamma} H(x)\right)\right], \quad \forall a > 0.$$

The paper is organized as follows. In Section 2 we define the space of real trees and the Gromov-Hausdorff-Prokhorov topology. In Section 3, we introduce the stable tree, recall some of its properties and prove Theorem 1.5 as well as some other useful results.

In Section 4, we prove Theorem 1.1. Sections 5 and 6 deal with the asymptotic behavior of $\mathbf{Z}_{\alpha,\beta}$ when $\beta/\alpha^{1-1/\gamma} \to c \in [0,\infty)$ and $\beta/\alpha^{1-1/\gamma} \to \infty$ respectively. Finally, we gather some technical proofs in Section 7.

2 Real trees and the Gromov-Hausdorff-Prokhorov topology

2.1 Real trees

We recall the formalism of real trees, see [15]. A metric space (T,d) is a real tree if the following two properties hold for every $x,y \in T$.

(i) (Unique geodesics). There exists a unique isometric map $f_{x,y}\colon [0,d(x,y)] \to T$ such that $f_{x,y}(0) = x$ and $f_{x,y}(d(x,y)) = y$.

(ii) (Loop-free). If φ is a continuous injective map from $[0,1]$ into T such that $\varphi(0) = x$ and $\varphi(1) = y$, then we have

$$\varphi([0,1]) = f_{x,y}\left([0,d(x,y)]\right).$$

A weighted rooted real tree $(T,\emptyset,d,\mu)$ is a real tree (T,d) with a distinguished vertex $\emptyset \in T$ called the root and equipped with a nonnegative finite measure μ. Let us consider a weighted rooted real tree $(T,\emptyset,d,\mu)$. The range of the mapping $f_{x,y}$ described above is denoted by $[\![x,y]\!]$ (this is the line segment between x and y in the tree). In particular, $[\![\emptyset,x]\!]$ is the path going from the root to x which we will interpret as the ancestral line of vertex x. We define a partial order on the tree by setting $x \preccurlyeq y$ (x is an ancestor of y) if and only if $x \in [\![\emptyset,y]\!]$. If $x,y \in T$, there is a unique $z \in T$ such that $[\![\emptyset,x]\!] \cap [\![\emptyset,y]\!] = [\![\emptyset,z]\!]$. We write $z = x \wedge y$ and call it the most recent common ancestor to x and y. For every vertex $x \in T$, we define its height by $H(x) = d(\emptyset,x)$. The height of the tree is defined by $\mathfrak{h}(T) = \sup_{x\in T} H(x)$. Note that if (T,d) is compact, then $\mathfrak{h}(T) < \infty$.

Let $x \in T$ be a vertex. For every $r \in [0,H(x)]$, we denote by $x_r \in T$ the unique ancestor of x at height r. Furthermore, we define the subtree $T_{r,x}$ of T above level r containing x as

$$T_{r,x} = \{y \in T\colon\ H(x \wedge y) \geqslant r\}. \tag{2.1}$$

Equivalently, $T_{r,x} = \{y \in T\colon\ x_r \preccurlyeq y\}$ is the subtree of T above x_r. Then $T_{r,x}$ can be naturally viewed as a weighted rooted real tree, rooted at x_r and endowed with the distance d and the measure $\mu_{|T_{r,x}}$ (the restriction of μ to $T_{r,x}$). Note that $T_{0,x} = T$. We also define the subtree of T above x by $T_x \coloneqq T_{H(x),x}$. Denote by

$$\sigma_{r,x}(T) = \mu(T_{r,x}) \quad \text{and} \quad \mathfrak{h}_{r,x}(T) = \mathfrak{h}(T_{r,x}) \tag{2.2}$$

the total mass and the height of $T_{r,x}$. For every $\alpha,\beta \geqslant 0$, we define

$$Z^T_{\alpha,\beta}(x) = \int_0^{H(x)} \sigma_{r,x}(T)^\alpha \mathfrak{h}_{r,x}(T)^\beta \,\mathrm{d}r, \quad \forall x \in T. \tag{2.3}$$

We shall omit the dependence on T when there is no ambiguity, simply writing $\sigma_{r,x}$, $\mathfrak{h}_{r,x}$ and $Z_{\alpha,\beta}(x)$. For every $0 \leqslant r < r' \leqslant H(x)$, we also introduce the notation

$$T_{[r,r'),x} = (T_{r,x} \setminus T_{r',x}) \cup \{x_{r'}\} = \{y \in T\colon\ r \leqslant H(x \wedge y) < r'\} \cup \{x_{r'}\}, \tag{2.4}$$

which defines a weighted rooted real tree, equipped with the distance and the measure it inherits from T and naturally rooted at x_r.

The next lemma, whose proof is elementary, relates $\mathfrak{h}_{r,x}(T)$, the height of the subtree of T above level r containing x, to the total height $\mathfrak{h}(T)$.

Lemma 2.1. *Let T be a compact real tree. For every $x \in T$ and $r \in [0, H(x)]$, we have*

$$\mathfrak{h}(T) \geqslant \mathfrak{h}_{r,x}(T) + r. \tag{2.5}$$

Furthermore, if $x^* \in T$ is such that $H(x^*) = \mathfrak{h}(T)$, then for every $r \in [0, H(x \wedge x^*)]$, we have

$$\mathfrak{h}(T) = \mathfrak{h}_{r,x}(T) + r. \tag{2.6}$$

2.2 The Gromov-Hausdorff-Prokhorov topology

We denote by $\mathbb{T}$ the set of (measure-preserving, root-preserving isometry classes of) compact real trees. We will often identify a class with an element of this class. So we shall write $(T, \emptyset, d, \mu) \in \mathbb{T}$ for a weighted rooted compact real tree.

Let us define the Gromov-Hausdorff-Prokhorov (GHP) topology on $\mathbb{T}$. Let $(T, \emptyset, d, \mu)$, $(T', \emptyset', d', \mu') \in \mathbb{T}$ be two compact real trees. Recall that a correspondence between T and T' is a subset $\mathcal{R} \subset T \times T'$ such that for every $x \in T$, there exists $x' \in T'$ such that $(x, x') \in \mathcal{R}$, and conversely, for every $x' \in T'$, there exists $x \in T$ such that $(x, x') \in \mathcal{R}$. In other words, if we denote by $p\colon T \times T' \to T$ (resp. $p'\colon T \times T' \to T'$) the canonical projection on T (resp. on T'), a correspondence is a subset $\mathcal{R} \subset T \times T'$ such that $p(\mathcal{R}) = T$ and $p'(\mathcal{R}) = T'$. If $\mathcal{R}$ is a correspondence between T and T', its distortion is defined by

$$\mathrm{dis}(\mathcal{R}) = \sup \left\{ |d(x,y) - d'(x',y')| : \ (x,x'), (y,y') \in \mathcal{R} \right\}.$$

Next, for any nonnegative finite measure m on $T \times T'$, we define its discrepancy with respect to μ and μ' by

$$\mathrm{D}(m; \mu, \mu') = d_{\mathrm{TV}}(m \circ p^{-1}, \mu) + d_{\mathrm{TV}}(m \circ {p'}^{-1}, \mu'),$$

where d_{TV} denotes the total variation distance. Then the GHP distance between T and T' is defined as

$$d_{\mathrm{GHP}}(T, T') = \inf \left\{ \frac{1}{2} \mathrm{dis}(\mathcal{R}) \vee \mathrm{D}(m; \mu, \mu') \vee m(\mathcal{R}^c) \right\}, \tag{2.7}$$

where the infimum is taken over all correspondences $\mathcal{R}$ between T and T' such that $(\emptyset, \emptyset') \in \mathcal{R}$ and all nonnegative finite measures m on $T \times T'$. It can be verified that d_{GHP} is indeed a distance on $\mathbb{T}$ and that the space $(\mathbb{T}, d_{\mathrm{GHP}})$ is complete and separable, see e.g. [3].

The next lemma gives an upper bound for the GHP distance between a tree $(T, \emptyset, d, \mu) \in \mathbb{T}$ and the tree $(T, \emptyset, ad, b\mu)$ obtained from T by multiplying all distances by $a > 0$ and the measure μ by $b > 0$. The proof is elementary and is left to the reader.

Lemma 2.2. *For every $T \in \mathbb{T}$ and $a, b > 0$, we have*

$$d_{\mathrm{GHP}}\left((T, \emptyset, d, \mu), (T, \emptyset, ad, b\mu)\right) \leqslant 2|a-1|\mathfrak{h}(T) + |b-1|\mu(T). \tag{2.8}$$

3 Preleminary results on general compact Lévy trees and stable trees

3.1 Two decompositions of the general Lévy tree

Although in this paper we are only interested in the stable case $\psi(\lambda) = \lambda^\gamma$, we state the results of this section in the general Lévy case. Let $\mathcal{T}$ denote a Lévy tree under its excursion measure $\mathbb{N}$ associated with a branching mechanism

$$\psi(\lambda) = a\lambda + b\lambda^2 + \int_0^\infty (\mathrm{e}^{-\lambda r} - 1 + \lambda r)\, \pi(\mathrm{d}r) \tag{3.1}$$

where $a, b \geqslant 0$ and π is a σ-finite measure on $(0,\infty)$ satisfying $\int_0^\infty (r \wedge r^2)\,\pi(\mathrm{d}r) < \infty$. We further assume that $\int^\infty \mathrm{d}\lambda/\psi(\lambda) < \infty$ so that the Lévy tree is compact.

Remark 3.1. The Brownian case $\psi(\lambda) = \lambda^2$ corresponds to $a = 0$, $b = 1$ and $\pi = 0$ while the non-Brownian stable case $\psi(\lambda) = \lambda^\gamma$ with $\gamma \in (1,2)$ corresponds to $a = b = 0$ and

$$\pi(\mathrm{d}r) = \frac{\gamma(\gamma-1)}{\Gamma(2-\gamma)}\,\frac{\mathrm{d}r}{r^{1+\gamma}}. \tag{3.2}$$

We shall need Bismut's decomposition of the stable tree on several occasions. This is a decomposition of the tree along the ancestral line of a uniformly chosen leaf. We refer the reader to [12, Theorem 4.5] and [2, Theorem 2.1] for more details. We will also need the probability measure $\mathbb{P}_r$ on $\mathbb{T}$ which is the distribution of the Lévy tree starting from an initial mass $r > 0$. More precisely, take $\sum_{i\in I} \delta_{\mathcal{T}_i}$ a Poisson point measure on $\mathbb{T}$ with intensity $r\,\mathbb{N}$ and define $\mathbb{P}_r$ as the distribution of the random tree $\mathcal{T}$ obtained by gluing together the trees $\mathcal{T}_i$ at their root. See [2, Section 2.6] for further details.

Before stating the result, we first introduce some notations. Let $(T, \emptyset, d, \mu)$ be a (class representative of a)compact real tree and let $x \in T$. Denote by $(x_i,\, i \in I_x)$ the branching points of T which lie on the branch $[\![\emptyset, x]\!]$, that is those points $y \in [\![\emptyset, x]\!]$ such that $T \setminus \{y\}$ has at least three connected components. For every $i \in I_x$, define the tree grafted on the branch $[\![\emptyset, x]\!]$ at x_i by $T_i = \{y \in T\colon x \wedge y = x_i\}$. We consider T_i as an element of $\mathbb{T}$ in the obvious way. Let $h_i = H(x_i)$ and define a point measure on $[0,\infty) \times \mathbb{T}$ by

$$\mathcal{M}_x^T = \sum_{i\in I_x} \delta_{(h_i, T_i)}.$$

We can now state Bismut's decomposition, see [12, Theorem 4.5] or [2, Theorem 2.1].

Theorem 3.2. *Let $\mathcal{T}$ be the Lévy tree with a general branching mechanism* (3.1) *satisfying the Grey condition $\int^\infty \mathrm{d}\lambda/\psi(\lambda) < \infty$ under its excursion measure $\mathbb{N}$. For every $\lambda \geqslant 0$ and every nonnegative measurable function Φ on $[0,\infty) \times \mathbb{T}$, we have*

$$\mathbb{N}\left[\int_{\mathcal{T}} \mu(\mathrm{d}x)\mathrm{e}^{-\lambda H(x) - \langle \mathcal{M}_x^{\mathcal{T}}, \Phi\rangle}\right] = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-(\lambda+a)t}\, \mathbb{E}\left[\mathrm{e}^{-\sum_{0\leqslant s\leqslant t} \Phi(s, \mathsf{T}_s)}\right], \tag{3.3}$$

where $(\mathsf{T}_s,\, 0 \leqslant s \leqslant t)$ is a Poisson point process with intensity $\mathbb{N}^{\mathrm{B}}[\mathrm{d}\mathcal{T}] = 2b\,\mathbb{N}[\mathrm{d}\mathcal{T}] + \int_0^\infty r\pi(\mathrm{d}r)\,\mathbb{P}_r(\mathrm{d}\mathcal{T})$.

Remark 3.3. Bismut's decomposition states the following: let $\mathcal{T}$ be the Lévy tree under its excursion measure $\mathbb{N}$ and, conditionally on $\mathcal{T}$, let U be a leaf chosen uniformly at random, *i.e.* according to the distribution $\sigma^{-1}\mu$. Then, under $\mathbb{N}[\sigma\bullet]$, the random variable $H(U)$ has "distribution" $\mathrm{e}^{-at}\,\mathrm{d}t$ on $(0,\infty)$ and, conditionally on $H(U) = t$, the point measure $\mathcal{M}_U^{\mathcal{T}}$ is distributed as $\sum_{s\leqslant t} \delta_{(s,\mathsf{T}_s)}$. One can make this claim rigorous by introducing the space of compact weighted rooted real trees with an additional marked vertex and considering the semidirect product measure $\mathbb{N} \times \sigma^{-1}\mu$ on it which corresponds to the distribution of the pair $(\mathcal{T}, U)$. Under this measure, the distribution of the random pair $(H(U), \mathcal{M}_U^{\mathcal{T}})$ does not depend on the particular choice of representative in the class of $\mathcal{T}$.

As a first application of Bismut's decomposition, we give a decomposition of the Lévy tree into $n+1$ subtrees which generalizes [1, Lemma 6.1].

Theorem 3.4. *Let $\mathcal{T}$ be the Lévy tree with a general branching mechanism* (3.1) *under its excursion measure $\mathbb{N}$. Then for every $n \geqslant 1$ and all nonnegative measurable functions $f_i,\ 1 \leqslant i \leqslant n+1$ defined on $[0,\infty) \times \mathbb{T}$, we have with $r_0 = 0$ and $r_{n+1} = H(x)$*

$$\mathbb{N}\left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} f_i\left(r_i - r_{i-1}, \mathcal{T}_{[r_{i-1}, r_i), x}\right) \prod_{i=1}^{n} \mathrm{d}r_i\right]$$

$$= \prod_{i=1}^{n+1} \mathbb{N}\left[\int_{\mathcal{T}} \mu(\mathrm{d}x) f_i(H(x),\mathcal{T})\right]. \quad (3.4)$$

Proof. Recall from (3.7) the definition of $\mathsf{T}^{\downarrow}$. By Theorem 3.2, we have

$$\mathbb{N}\left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} f_i\left(r_i - r_{i-1}, \mathcal{T}_{[r_{i-1},r_i),x}\right) \prod_{i=1}^{n} \mathrm{d}r_i\right]$$
$$= \int_0^\infty \mathrm{d}r_{n+1}\, \mathrm{e}^{-ar_{n+1}}\, \mathbb{E}\left[\int_{0<r_1<\ldots<r_n<r_{n+1}} \prod_{i=1}^{n+1} f_i\left(r_i - r_{i-1}, \mathsf{T}_{[r_{i-1},r_i)}\right) \prod_{i=1}^{n} \mathrm{d}r_i\right],$$

where we set $\mathsf{T}_{[r,r')} = (\mathsf{T}^{\downarrow}_{t-r} \setminus \mathsf{T}^{\downarrow}_{t-r'}) \cup \{t-r'\}$ for every $0<r<r'<t$. Since $(\mathsf{T}_s,\, 0 \leqslant s \leqslant t)$ is a Poisson point process, we get that the $\mathsf{T}_{[r_{i-1},r_i)}$ are independent and distributed as $\mathsf{T}_{[0,r_i-r_{i-1})}$. We deduce that

$$\mathbb{N}\left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} f_i\left(r_i - r_{i-1}, \mathcal{T}_{[r_{i-1},r_i),x}\right) \prod_{i=1}^{n} \mathrm{d}r_i\right]$$
$$= \int_{0<r_1<\ldots<r_n<r_{n+1}} \prod_{i=1}^{n+1} \mathrm{e}^{-a(r_i-r_{i-1})}\, \mathbb{E}\left[f_i\left(r_i - r_{i-1}, \mathsf{T}_{[0,r_i-r_{i-1})}\right)\right]\, \mathrm{d}r_i$$
$$= \int_{[0,\infty)^{n+1}} \prod_{i=1}^{n+1} \mathrm{e}^{-as_i}\, \mathbb{E}\left[f_i\left(s_i, \mathsf{T}_{[0,s_i)}\right)\right]\, \mathrm{d}s_i$$
$$= \prod_{i=1}^{n+1} \mathbb{N}\left[\int_{\mathcal{T}} \mu(\mathrm{d}x) f_i(H(x),\mathcal{T})\right],$$

where we made the change of variables $(s_1, s_2, \ldots, s_{n+1}) = (r_1, r_2 - r_1, \ldots, r_{n+1} - r_n)$ for the second equality and used Bismut's decomposition (3.12) together with the fact that $\mathsf{T}_{[0,t)} = \mathsf{T}^{\downarrow}_t$ $\mathbb{P}$-a.s. for the last. □

3.2 The stable tree and its scaling property

Here, we define the stable tree and recall some of its properties. We refer to [12] for background. We shall work with the stable tree $\mathcal{T}$ with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1,2]$ under its excursion measure $\mathbb{N}$: more explicitly, using the coding of compact real trees by height functions, one can define a σ-finite measure $\mathbb{N}$ on $\mathbb{T}$ with the following properties.

(i) **Mass measure.** $\mathbb{N}$-a.e. the mass measure μ is supported by the set of leaves $\mathrm{Lf}(\mathcal{T}) \coloneqq \{x \in \mathcal{T}\colon\ \mathcal{T} \setminus \{x\} \text{ is connected}\}$ and the distribution on $(0,\infty)$ of the total mass $\sigma \coloneqq \mu(\mathcal{T})$ is given by

$$\mathbb{N}[\sigma \in \mathrm{d}a] = \frac{1}{\gamma\Gamma(1-1/\gamma)} \frac{\mathrm{d}a}{a^{1+1/\gamma}}.$$

(ii) **Height.** $\mathbb{N}$-a.e. there exists a unique leaf $x^* \in \mathcal{T}$ realizing the height, that is $H(x^*) = \mathfrak{h}(\mathcal{T})$, and the distribution on $(0,\infty)$ of the height $\mathfrak{h} \coloneqq \mathfrak{h}(\mathcal{T})$ is given by

$$\mathbb{N}[\mathfrak{h} \in \mathrm{d}a] = (\gamma-1)^{-\gamma/(\gamma-1)} \frac{\mathrm{d}a}{a^{\gamma/(\gamma-1)}}.$$

We will make extensive use of the scaling property of the stable tree under $\mathbb{N}$. Recall from (1.2) the definition of R_γ and note that if T has total mass σ and height $\mathfrak{h}$ then

$R_\gamma(T,a)$ has total mass $a^{\gamma/(\gamma-1)}\sigma$ and height $a\mathfrak{h}$. Furthermore, it is straightforward to show that for all $x \in T$, $r \in [0, H(x)]$ and $a > 0$:

$$\begin{aligned}\sigma_{ar,x}(R_\gamma(T,a)) &= a^{\gamma/(\gamma-1)}\sigma_{r,x}(T),\\ \mathfrak{h}_{ar,x}(R_\gamma(T,a)) &= a\mathfrak{h}_{r,x}(T),\\ Z^{R_\gamma(T,a)}_{\alpha,\beta}(x) &= a^{\alpha\gamma/(\gamma-1)+\beta+1} Z^T_{\alpha,\beta}(x).\end{aligned} \tag{3.5}$$

The scaling property of the stable tree can be written as follows:

$$R_\gamma(\mathcal{T},a) \quad \text{under } \mathbb{N} \quad \overset{(d)}{=} \quad \mathcal{T} \quad \text{under } a^{1/(\gamma-1)}\,\mathbb{N}, \tag{3.6}$$

see e.g. [13, Eq. (40)]. Using this, one can define a regular conditional probability measure $\mathbb{N}^{(a)} = \mathbb{N}[\bullet|\sigma = a]$ such that $\mathbb{N}^{(a)}$-a.s. $\sigma = a$ and

$$\mathbb{N}[\bullet] = \frac{1}{\gamma\Gamma(1-1/\gamma)} \int_0^\infty \mathbb{N}^{(a)}[\bullet]\, \frac{\mathrm{d}a}{a^{1+1/\gamma}}.$$

Informally, $\mathbb{N}^{(a)}$ can be seen as the distribution of the stable tree $\mathcal{T}$ with total mass a.

The next result is a restatement of [17, Proposition 5.7] in terms of trees which gives a version of the scaling property for the stable tree conditioned on its total mass. Recall from (1.3) the definition of norm_γ.

Lemma 3.5. *Let $\mathcal{T}$ be the stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1,2]$.*

(i) For every measurable function $F\colon \mathbb{T} \to [0,\infty]$, we have

$$\mathbb{N}^{(1)}\left[F(\mathcal{T})\right] = \Gamma(1-1/\gamma)\,\mathbb{N}\left[\mathbf{1}_{\{\sigma>1\}} F(\mathsf{norm}_\gamma(\mathcal{T}))\right].$$

(ii) Under $\mathbb{N}^{(a)}$, the random tree $\mathcal{T}$ is distributed as $R_\gamma(\mathcal{T}, a^{1-1/\gamma})$ under $\mathbb{N}^{(1)}$ for every $a > 0$.

3.3 Preliminary results on the stable tree

Let $(\mathsf{T}_s, 0 \leqslant s \leqslant t)$ be a Poisson point process on $\mathbb{T}$ with intensity $\mathbb{N}^{\mathrm{B}}$ given by

$$\mathbb{N}^{\mathrm{B}}[\mathrm{d}\mathcal{T}] = \begin{cases} 2\,\mathbb{N}[\mathrm{d}\mathcal{T}] & \text{if } \gamma = 2,\\ \int_0^\infty r\pi(\mathrm{d}r)\,\mathbb{P}_r(\mathrm{d}\mathcal{T}) & \text{if } \gamma \in (1,2),\end{cases}$$

and denote by

$$\mathsf{T}^{\downarrow}_r \coloneqq [t-r,t] \circledast_{t-r\leqslant s\leqslant t} (\mathsf{T}_s, s)\,, \quad \forall 0 \leqslant r \leqslant t \tag{3.7}$$

the random real tree obtained by grafting T_s on a branch $[t-r,t]$ at height s for every $t-r \leqslant s \leqslant t$ and rooted at $t-r$, see Figure 3. We refer the reader to [2, Section 2.4] for a precise definition of the grafting procedure. Let

$$\tau_r \coloneqq \mu(\mathsf{T}^{\downarrow}_r) = \sum_{t-r\leqslant s\leqslant t} \mu(\mathsf{T}_s) \quad \text{and} \quad \eta_r \coloneqq \mathfrak{h}(\mathsf{T}^{\downarrow}_r) = \max_{t-r\leqslant s\leqslant t} \left(\mathfrak{h}(\mathsf{T}_s) + s - (t-r)\right) \tag{3.8}$$

denote its mass and height. Finally, let

$$S_r \coloneqq \sum_{s\leqslant r} \mu(\mathsf{T}_s). \tag{3.9}$$

It is shown in the proof of [9, Lemma 4.6], see Section 8.6 and more precisely (8.20) therein, that in the stable case $\psi(\lambda) = \lambda^\gamma$, both τ and S are subordinators defined on $[0,t]$ with Laplace exponent

$$\varphi(\lambda) = \gamma\lambda^{1-1/\gamma}. \tag{3.10}$$

In particular, thanks to [30, Section 4] or [31, Eq. (2.1.8)], we have for every $p \in (-\infty, 1-1/\gamma)$,

$$\mathbb{E}\left[\tau_1^p\right] < \infty. \tag{3.11}$$

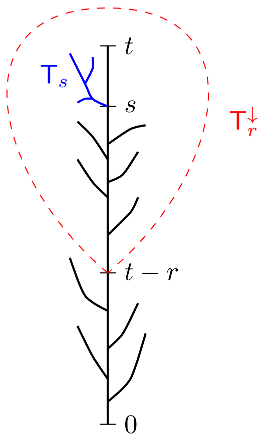

Figure 3: The real tree $\mathsf{T}_r^{\downarrow}$ obtained by grafting the atoms T_s of a Poisson point process on a branch $[t-r, t]$ at height s.

We now give the following form of Bismut's decomposition which we will use throughout the paper. Denote by $D[0,\infty)$ the space of cadlag functions on $[0,\infty)$ endowed with the Skorokhod $J1$ topology. By Theorem 3.2 we have, for every measurable function $F\colon [0,\infty)^3 \times \mathbb{T} \times D[0,\infty)^2 \to [0,\infty]$,

$$\begin{aligned}\mathbb{N}\left[\int_{\mathcal{T}} \mu(\mathrm{d}x) F\left(H(x), \sigma, \mathfrak{h}, \mathcal{T}, \left(\sigma_{H(x)-r,x},\, 0 \leqslant r \leqslant H(x)\right), \left(\mathfrak{h}_{H(x)-r,x},\, 0 \leqslant r \leqslant H(x)\right)\right)\right] \\ = \int_0^\infty \mathrm{d}t\, \mathbb{E}\left[F\left(t, \tau_t, \eta_t, \mathsf{T}_t^{\downarrow}, (\tau_r,\, 0 \leqslant r \leqslant t), (\eta_r,\, 0 \leqslant r \leqslant t)\right)\right]. \end{aligned}\tag{3.12}$$

Notice that by definition $\tau_t = S_t$ and $S_{r-} = \tau_t - \tau_{t-r}$ for every $r \in [0,t]$. This will be used implicitly in the sequel. In particular, the following computation will be useful

$$\int_0^\infty \mathbb{E}\left[\frac{1}{S_t}\mathbf{1}_{\{S_t>1\}}\right] \mathrm{d}t = \int_0^\infty \mathbb{E}\left[\frac{1}{\tau_t}\mathbf{1}_{\{\tau_t>1\}}\right] \mathrm{d}t = \mathbb{N}\left[\sigma > 1\right] = \frac{1}{\Gamma(1-1/\gamma)}, \tag{3.13}$$

where in the last equality we used Lemma 3.5-(i) with $F \equiv 1$.

Next, as an application of Theorem 3.4, we give the decomposition of the *normalized* stable tree into $n+1$ subtrees. For functions f, g defined on $(0,\infty)$, we denote by $f * g$ their convolution defined by

$$f * g(t) = \int_0^t f(s) g(t-s)\, \mathrm{d}s, \quad \forall t > 0.$$

Proposition 3.6. *Let $\mathcal{T}$ be the stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1,2]$. For every $n \geqslant 1$ and all nonnegative measurable functions f_i, $1 \leqslant i \leqslant n+1$ defined on $[0,\infty) \times \mathbb{T}$, we have with $r_0 = 0$ and $r_{n+1} = H(x)$*

$$\mathbb{N}^{(1)}\left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} f_i\left(r_i - r_{i-1}, \mathcal{T}_{[r_{i-1}, r_i), x}\right) \prod_{i=1}^{n} \mathrm{d}r_i\right]$$

$$= \frac{1}{\gamma^n \Gamma(1-1/\gamma)^n} F_1 * \cdots * F_{n+1}(1), \quad (3.14)$$

where R_γ *is defined in* (1.2) *and*

$$F_i(a) = a^{-1/\gamma}\, \mathbb{N}^{(1)} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) f_i \left(a^{1-1/\gamma} H(x), R_\gamma \left(\mathcal{T}, a^{1-1/\gamma} \right) \right) \right], \quad \forall a > 0.$$

In particular, for every $n \geqslant 1$ *and all nonnegative measurable functions* $g_i,\ 1 \leqslant i \leqslant n+1$ *defined on* $[0,\infty) \times [0,1]$*, we have*

$$\mathbb{N}^{(1)} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} g_i \left(r_i - r_{i-1}, \sigma_{r_{i-1},x} - \sigma_{r_i,x} \right) \prod_{i=1}^{n} \mathrm{d}r_i \right]$$
$$= \frac{1}{\gamma^n \Gamma(1-1/\gamma)^n} G_1 * \cdots * G_{n+1}(1), \quad (3.15)$$

where

$$G_i(a) = a^{-1/\gamma}\, \mathbb{N}^{(1)} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) g_i \left(a^{1-1/\gamma} H(x), a \right) \right], \quad \forall a > 0.$$

Proof. Let $f_i \colon [0,\infty) \times \mathbb{T} \to \mathbb{R}$ be continuous and bounded for $1 \leqslant i \leqslant n+1$. By Theorem 3.4, we have for $\lambda > 0$

$$\begin{aligned}
&\prod_{i=1}^{n+1} \mathbb{N} \left[\mathrm{e}^{-\lambda\sigma} \int_{\mathcal{T}} \mu(\mathrm{d}x) f_i(H(x), \mathcal{T}) \right] \\
&= \mathbb{N} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} \mathrm{e}^{-\lambda \mu\left(\mathcal{T}_{[r_{i-1},r_i),x}\right)} f_i \left(r_i - r_{i-1}, \mathcal{T}_{[r_{i-1},r_i),x} \right) \prod_{i=1}^{n} \mathrm{d}r_i \right] \\
&= \mathbb{N} \left[\mathrm{e}^{-\lambda\sigma} \int_{\mathcal{T}} \mu(\mathrm{d}x) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} f_i \left(r_i - r_{i-1}, \mathcal{T}_{[r_{i-1},r_i),x} \right) \prod_{i=1}^{n} \mathrm{d}r_i \right].
\end{aligned} \quad (3.16)$$

Disintegrating with respect to σ and using the scaling property from Lemma 3.5-(ii), we have

$$\begin{aligned}
&\mathbb{N} \left[\mathrm{e}^{-\lambda\sigma} \int_{\mathcal{T}} \mu(\mathrm{d}x) f_i(H(x), \mathcal{T}) \right] \\
&\qquad = \frac{1}{\gamma\Gamma(1-1/\gamma)} \int_0^\infty \mathrm{e}^{-\lambda a}\, \mathbb{N}^{(a)} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) f_i(H(x), \mathcal{T}) \right] \frac{\mathrm{d}a}{a^{1+1/\gamma}} \\
&\qquad = \frac{1}{\gamma\Gamma(1-1/\gamma)} \mathcal{L}F_i(\lambda),
\end{aligned} \quad (3.17)$$

where $\mathcal{L}$ denotes the Laplace transform on $[0,\infty)$.

On the other hand, again disintegrating with respect to σ, we have

$$\begin{aligned}
&\gamma\Gamma(1-1/\gamma)\, \mathbb{N} \left[\mathrm{e}^{-\lambda\sigma} \int_{\mathcal{T}} \mu(\mathrm{d}x) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} f_i \left(r_i - r_{i-1}, \mathcal{T}_{[r_{i-1},r_i),x} \right) \prod_{i=1}^{n} \mathrm{d}r_i \right] \\
&= \int_0^\infty \frac{\mathrm{d}a}{a^{1+1/\gamma}} \mathrm{e}^{-\lambda a}\, \mathbb{N}^{(a)} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} f_i \left(r_i - r_{i-1}, \mathcal{T}_{[r_{i-1},r_i),x} \right) \prod_{i=1}^{n} \mathrm{d}r_i \right] \\
&= \int_0^\infty \mathrm{d}a\, a^{(n+1)(1-1/\gamma)-1} \mathrm{e}^{-\lambda a} F(a),
\end{aligned} \quad (3.18)$$

where we set

$$F(a) = \mathbb{N}^{(1)} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \right.$$
$$\left. \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} f_i \left(a^{1-1/\gamma}(r_i - r_{i-1}), R_\gamma \left(\mathcal{T}_{[r_{i-1},r_i),x}, a^{1-1/\gamma} \right) \right) \prod_{i=1}^{n} \mathrm{d}r_i \right].$$

Putting together (3.16)–(3.18) yields

$$\begin{aligned} \frac{1}{\gamma^n \Gamma(1-1/\gamma)^n} \mathcal{L}(F_1 * \ldots * F_{n+1})(\lambda) &= \frac{1}{\gamma^n \Gamma(1-1/\gamma)^n} \prod_{i=1}^{n+1} \mathcal{L}F_i(\lambda) \\ &= \int_0^\infty \mathrm{d}a\, a^{(n+1)(1-1/\gamma)-1} \mathrm{e}^{-\lambda a} F(a). \end{aligned}$$

Since this holds for every $\lambda > 0$, we deduce that $\mathrm{d}a$-a.e. on $(0,\infty)$,

$$\frac{1}{\gamma^n \Gamma(1-1/\gamma)^n} F_1 * \ldots * F_{n+1}(a) = a^{(n+1)(1-1/\gamma)-1} F(a). \tag{3.19}$$

Thanks to Lemma 2.2, the mapping $a \mapsto R_\gamma(T, a^{1-1/\gamma})$ is continuous on $(0,\infty)$ for every $T \in \mathbb{T}$. We deduce from the dominated convergence theorem that the F_i are continuous on $(0,\infty)$ and thus $F_1 * \ldots * F_{n+1}$ too. Similarly, the right-hand side of (3.19) is continuous with respect to a. Therefore the equality holds for every $a \in (0,\infty)$. In particular, taking $a = 1$ proves (3.14) for continuous bounded functions $f_i \colon [0,\infty)\times\mathbb{T} \to \mathbb{R}$. This extends to measurable functions $f_i \colon [0,\infty) \times \mathbb{T} \to \mathbb{R}$ thanks to the monotone class theorem. Finally, (3.15) is a direct consequence of (3.14). □

In particular, the following corollary will be useful.

Corollary 3.7. *We have*

$$\sup_{\alpha \geqslant 0} \alpha^{2-2/\gamma}\, \mathbb{N}^{(1)} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \left(\int_0^{H(x)} \sigma^\alpha_{r,x}\, \mathrm{d}r \right)^2 \right] < \infty. \tag{3.20}$$

Proof. Applying (3.15) with $n = 2$, $g_1(r,a) = g(1-a)$, $g_2(r,a) = 1$ and $g_3(r,a) = g(a)$ yields, for every measurable function $g \colon [0,1] \to [0,\infty]$,

$$\begin{aligned} &\mathbb{N}^{(1)} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \left(\int_0^{H(x)} g(\sigma_{r,x})\, \mathrm{d}r \right)^2 \right] \\ &\qquad = \frac{2}{\gamma^2 \Gamma(1-1/\gamma)^2} \int_0^1 g(y)(1-y)^{-1/\gamma}\, \mathrm{d}y \int_0^y g(z) z^{-1/\gamma}(y-z)^{-1/\gamma}\, \mathrm{d}z. \end{aligned} \tag{3.21}$$

Taking $g(a) = a^\alpha$, we get

$$\begin{aligned} &\alpha^{2-2/\gamma}\, \mathbb{N}^{(1)} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \left(\int_0^{H(x)} \sigma^\alpha_{r,x}\, \mathrm{d}r \right)^2 \right] \\ &\qquad = \frac{2\alpha^{2-2/\gamma}}{\gamma^2 \Gamma(1-1/\gamma)^2} \int_0^1 y^\alpha (1-y)^{-1/\gamma}\, \mathrm{d}y \int_0^y z^{\alpha-1/\gamma}(y-z)^{-1/\gamma}\, \mathrm{d}z \\ &\qquad = \frac{2\alpha^{2-2/\gamma}}{\gamma^2 \Gamma(1-1/\gamma)^2} \mathrm{B}\left(2\alpha + 2 - 2/\gamma, 1 - 1/\gamma\right) \mathrm{B}\left(\alpha + 1 - 1/\gamma, 1 - 1/\gamma\right), \end{aligned}$$

where B is the Beta function. Using that $\mathrm{B}(x, 1-1/\gamma) \sim \Gamma(1-1/\gamma) x^{-1+1/\gamma}$ as $x \to \infty$, (3.20) readily follows. □

As a consequence of Proposition 3.6, we are able to compute the intensity measure of the random measure $\Psi_{\mathcal{T}}$ appearing in [1], see Proposition 6.3 therein.

Corollary 3.8. *Let $\mathcal{T}$ be the normalized stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1, 2]$. Let f and g be nonnegative measurable functions defined on $\mathbb{T}$ and $[0, \infty)$ respectively. We have*

$$\gamma\Gamma(1-1/\gamma)\,\mathbb{N}^{(1)}\left[\int_{\mathcal{T}}\mu(\mathrm{d}x)\int_0^{H(x)} f(\mathcal{T}_{r,x})g(r)\,\mathrm{d}r\right]$$
$$= \int_0^1 \frac{\mathrm{d}a}{a^{1/\gamma}(1-a)^{1/\gamma}}\,\mathbb{N}^{(1)}\left[f\circ R_\gamma\left(\mathcal{T}, a^{1-1/\gamma}\right)\right]\mathbb{N}^{(1)}\left[\int_{\mathcal{T}}\mu(\mathrm{d}x)g\left((1-a)^{1-1/\gamma}H(x)\right)\right]. \tag{3.22}$$

Another application of Theorem 3.2 is the following result giving the moments of the height $H(U)$ of a uniformly distributed leaf $U \in \mathcal{T}$ (*i.e.* according to μ) under $\mathbb{N}^{(1)}$. In particular, this allows to give a nontrivial upper bound for the size of the ball with radius $\varepsilon > 0$ centered around the root of the normalized stable tree. Let us mention that this result is not new since the distribution of $H(U)$ under $\mathbb{N}^{(1)}$ is known: in the Brownian case $\gamma = 2$, H is distributed as $\sqrt{2}e$ where e is the Brownian excursion so $\sqrt{2}H(U)$ has Rayleigh distribution; in the case $\gamma \in (1, 2)$, $H(U)$ is distributed as a multiple of the local time at 0 of the Bessel bridge of dimension $2/\gamma$, see [21, Corollary 10].

Lemma 3.9. *Let $\mathcal{T}$ be the normalized stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1, 2]$. For every $p \in (-\infty, 2)$, we have*

$$\mathbb{N}^{(1)}\left[\int_{\mathcal{T}} H(x)^{-p}\,\mu(\mathrm{d}x)\right] = \frac{(\gamma-1)\gamma^{p-1}\Gamma(1-1/\gamma)\Gamma(2-p)}{\Gamma(1-(p-1)(1-1/\gamma))} < \infty. \tag{3.23}$$

Proof. Using Bismut's decomposition (3.12), we have for every $\lambda > 0$

$$\mathbb{N}\left[\sigma \mathrm{e}^{-\lambda\sigma}\int_{\mathcal{T}} H(x)^{-p}\,\mu(\mathrm{d}x)\right] = \int_0^\infty t^{-p}\,\mathbb{E}\left[\tau_t \mathrm{e}^{-\lambda\tau_t}\right]\,\mathrm{d}t = \varphi'(\lambda)\int_0^\infty t^{1-p}\mathrm{e}^{-t\varphi(\lambda)}\,\mathrm{d}t.$$

On the other hand, disintegrating with respect to σ and using Lemma 3.5-(ii), we have

$$\begin{aligned}\mathbb{N}\left[\sigma \mathrm{e}^{-\lambda\sigma}\int_{\mathcal{T}} H(x)^{-p}\,\mu(\mathrm{d}x)\right] &= \frac{1}{\gamma\Gamma(1-1/\gamma)}\int_0^\infty a\mathrm{e}^{-\lambda a}\,\mathbb{N}^{(a)}\left[\int_{\mathcal{T}} H(x)^{-p}\,\mu(\mathrm{d}x)\right]\frac{\mathrm{d}a}{a^{1+1/\gamma}}\\ &= \frac{1}{\gamma\Gamma(1-1/\gamma)}\int_0^\infty \mathrm{e}^{-\lambda a}\,\frac{\mathrm{d}a}{a^{(p-1)(1-1/\gamma)}}\,\mathbb{N}^{(1)}\left[\int_{\mathcal{T}} H(x)^{-p}\,\mu(\mathrm{d}x)\right]\\ &= \frac{\Gamma(1-(p-1)(1-1/\gamma))}{\gamma\Gamma(1-1/\gamma)\lambda^{1-(p-1)(1-1/\gamma)}}\,\mathbb{N}^{(1)}\left[\int_{\mathcal{T}} H(x)^{-p}\,\mu(\mathrm{d}x)\right].\end{aligned}$$

Using (3.10), it follows that

$$\begin{aligned}\mathbb{N}^{(1)}\left[\int_{\mathcal{T}} H(x)^{-p}\,\mu(\mathrm{d}x)\right] &= \frac{\gamma\Gamma(1-1/\gamma)\lambda^{1-(p-1)(1-1/\gamma)}\varphi'(\lambda)}{\Gamma(1-(p-1)(1-1/\gamma))}\int_0^\infty t^{1-p}\mathrm{e}^{-t\varphi(\lambda)}\,\mathrm{d}t\\ &= \frac{(\gamma-1)\gamma^{p-1}\Gamma(1-1/\gamma)\Gamma(2-p)}{\Gamma(1-(p-1)(1-1/\gamma))}.\end{aligned}$$

□

Remark 3.10. Conditionally on $\mathcal{T}$, let $U \in \mathcal{T}$ be a uniformly distributed leaf. Then we can rewrite (3.23) as follows:

$$\frac{1}{c_\gamma}\,\mathbb{N}^{(1)}\left[\frac{1}{H(U)}\,(\gamma H(U))^p\right] = \frac{\Gamma(p+1)}{\Gamma(p(1-1/\gamma)+1)}, \quad \forall p > -1, \tag{3.24}$$

where $c_\gamma = (\gamma-1)\Gamma(1-1/\gamma)$. This implies that, under the probability measure $c_\gamma^{-1}\,\mathbb{N}^{(1)}[H(U)^{-1}\bullet]$, the random variable $\gamma H(U)$ has Mittag-Leffler distribution with index $1-1/\gamma$, see [27, Eq. (0.42)].

4 Zooming in at the root of the stable tree

In this section, we study the shape of the stable tree in a small neighborhood of its root. The main result, Theorem 4.2, states that after zooming in and rescaling, one sees a branch on which trees are grafted according to a Poisson point process on $\mathbb{T}$ with intensity $\mathbb{N}^{\mathrm{B}}$ given by

$$\mathbb{N}^{\mathrm{B}}[\mathrm{d}\mathcal{T}] = \begin{cases} 2\,\mathbb{N}[\mathrm{d}\mathcal{T}] & \text{if } \gamma = 2, \\ \int_0^\infty r\pi(\mathrm{d}r)\,\mathbb{P}_r(\mathrm{d}\mathcal{T}) & \text{if } \gamma\in(1,2), \end{cases} \tag{4.1}$$

where we recall from Section 3 that π is given by (3.2) and $\mathbb{P}_r$ is the distribution of the random tree $\mathcal{T}$ obtained by gluing together at their roots a family of trees distributed according to a Poisson point measure with intensity $r\,\mathbb{N}$.

We start with the following result giving the scaling property of the stable tree under $\mathbb{N}^{\mathrm{B}}$.

Lemma 4.1. *The following identity holds for every $a>0$*

$$R_\gamma(\mathcal{T},a) \quad \textit{under } \mathbb{N}^{\mathrm{B}} \quad \overset{(d)}{=} \quad \mathcal{T} \quad \textit{under } a\,\mathbb{N}^{\mathrm{B}}\,. \tag{4.2}$$

Proof. The case $\gamma=2$ reduces to the scaling property (3.6) so we only need to prove the case $\gamma\in(1,2)$. Thanks to (3.6), we deduce that $R_\gamma(\mathcal{T},a)$ under $\mathbb{P}_r$ has distribution $\mathbb{P}_{a^{1/(\gamma-1)}r}$. It follows from (3.2) that under $\mathbb{N}^{\mathrm{B}}$, $R_\gamma(\mathcal{T},a)$ has distribution

$$\int_0^\infty r\pi(\mathrm{d}r)\,\mathbb{P}_{a^{1/(\gamma-1)}r}(\mathrm{d}\mathcal{T}) = a\int_0^\infty s\pi(\mathrm{d}s)\,\mathbb{P}_s(\mathrm{d}\mathcal{T}) = a\,\mathbb{N}^{\mathrm{B}}[\mathrm{d}\mathcal{T}]. \qquad \square$$

Let $(T,\emptyset,d,\mu)$ be a compact real tree and let $x\in T$. Recall from Section 3 that T_i, $i\in I_x$ are the trees grafted on the branch $[\![\emptyset,x]\!]$, each one at height h_i. Fix $\mathfrak{f}\colon(0,\infty)\to(0,\infty)$ and define for every $\varepsilon>0$ a point measure on $[0,\infty)^2\times\mathbb{T}$ by

$$\mathcal{N}_\varepsilon^{\mathfrak{f}}(x) = \sum_{h_i\leqslant \mathfrak{f}(\varepsilon)H(x)} \delta_{\left(\varepsilon^{-1}h_i,\,\varepsilon^{-\gamma/(\gamma-1)}\sigma_i,\,\mathsf{norm}_\gamma(T_i)\right)}. \tag{4.3}$$

We are now in a position to give the main result of this section.

Theorem 4.2. *Let $\mathcal{T}$ be the normalized stable tree with branching mechanism $\psi(\lambda)=\lambda^\gamma$ where $\gamma\in(1,2]$. Conditionally on $\mathcal{T}$, let U be a $\mathcal{T}$-valued random variable with distribution μ under $\mathbb{N}^{(1)}$. Let $(\mathsf{T}'_s,\, s\geqslant 0)$ be a Poisson point process with intensity $\mathbb{N}^{\mathrm{B}}$, independent of $(\mathcal{T},H(U))$. Let $\Phi\colon[0,\infty)^2\times\mathbb{T}\to[0,\infty)$ be a measurable function such that there exists $C>0$ such that for every $h\geqslant 0$ and $T\in\mathbb{T}$, we have*

$$|\Phi(h,b,T)-\Phi(h,a,T)| \leqslant C|b-a|. \tag{4.4}$$

(i) If $\lim_{\varepsilon\to0}\varepsilon^{-1/2}\mathfrak{f}(\varepsilon)=0$ and $\lim_{\varepsilon\to0}\varepsilon^{-1}\mathfrak{f}(\varepsilon)=\infty$, then we have the following convergence in distribution

$$\left(\mathcal{T},H(U),\langle\mathcal{N}_\varepsilon^{\mathfrak{f}}(U),\Phi\rangle\right) \xrightarrow[\varepsilon\to0]{(d)} \left(\mathcal{T},H(U),\sum_{s\geqslant0}\Phi\left(s,\mu(\mathsf{T}'_s),\mathsf{norm}_\gamma\left(\mathsf{T}'_s\right)\right)\right) \tag{4.5}$$

in the space $\mathbb{T}\times[0,\infty)\times[0,\infty]$.

(ii) If $\mathfrak{f}(\varepsilon) = \varepsilon$, then we have the following convergence in distribution

$$\left(\mathcal{T}, H(U), \langle \mathcal{N}_\varepsilon^{\mathfrak{f}}(U), \Phi\rangle\right) \xrightarrow[\varepsilon\to 0]{(d)} \left(\mathcal{T}, H(U), \sum_{s\leqslant H(U)} \Phi\left(s, \mu(\mathsf{T}'_s), \mathsf{norm}_\gamma\left(\mathsf{T}'_s\right)\right)\right) \tag{4.6}$$

in the space $\mathbb{T}\times[0,\infty)\times[0,\infty]$.

Proof. We only prove (i), the proof of (ii) being similar. Let $f\colon \mathbb{T}\to\mathbb{R}$ and $g\colon [0,\infty)\to\mathbb{R}$ be Lipschitz-continuous and bounded and assume that $\Phi\colon [0,\infty)^2\times\mathbb{T}\to[0,\infty)$ is measurable and satisfies (4.4). We shall consider the following modification of the measure $\mathcal{N}_\varepsilon^{\mathfrak{f}}(U)$:

$$\widehat{\mathcal{N}}_\varepsilon^{\mathfrak{f}}(U) \coloneqq \sum_{h_i\leqslant \mathfrak{f}(\varepsilon)H(U)} \delta_{\left(\varepsilon^{-1}h_i/H(U), \varepsilon^{-\gamma/(\gamma-1)}\sigma_i, \mathsf{norm}_\gamma(T_i)\right)}.$$

Step 1. Set

$$\begin{aligned} F(\varepsilon) &\coloneqq \mathbb{N}^{(1)}\left[f(\mathcal{T})g(H(U))\exp\left\{-\left\langle\widehat{\mathcal{N}}_\varepsilon^{\mathfrak{f}}(U),\Phi\right\rangle\right\}\right] \\ &= \mathbb{N}^{(1)}\left[\int_{\mathcal{T}}\mu(\mathrm{d}x)f(\mathcal{T})g(H(x)) \right.\\ &\qquad\left.\times\exp\left\{-\sum_{h_i\leqslant\mathfrak{f}(\varepsilon)H(x)}\Phi\left(\varepsilon^{-1}h_i/H(x),\varepsilon^{-\gamma/(\gamma-1)}\sigma_i,\mathsf{norm}_\gamma(\mathcal{T}_i)\right)\right\}\right]. \end{aligned}$$

Using Lemma 3.5-(i) and Theorem 3.2, we have

$$\begin{aligned} \frac{F(\varepsilon)}{\Gamma(1-1/\gamma)} &= \mathbb{N}\left[\frac{1}{\sigma}\mathbf{1}_{\{\sigma>1\}}\int_{\mathcal{T}}\mu(\mathrm{d}x)f\circ\mathsf{norm}_\gamma\left(\mathcal{T}\right)g\left(\sigma^{-1+1/\gamma}H(x)\right)\right.\\ &\qquad\left.\times\exp\left\{-\sum_{h_i\leqslant\mathfrak{f}(\varepsilon)H(x)}\Phi\left(\varepsilon^{-1}h_i/H(x),\varepsilon^{-\gamma/(\gamma-1)}\sigma^{-1}\sigma_i,\mathsf{norm}_\gamma\left(\mathcal{T}_i\right)\right)\right\}\right] \\ &= \int_0^\infty \mathrm{d}t\,\mathbb{E}\left[\frac{1}{\tau_t}\mathbf{1}_{\{\tau_t>1\}}f\circ\mathsf{norm}_\gamma\left(\mathsf{T}_t^{\downarrow}\right)g\left(\tau_t^{-1+1/\gamma}t\right)\right. \qquad (4.7)\\ &\qquad\left.\times\exp\left\{-\sum_{s\leqslant\mathfrak{f}(\varepsilon)t}\Phi\left(\varepsilon^{-1}s/t,\varepsilon^{-\gamma/(\gamma-1)}\tau_t^{-1}\mu(\mathsf{T}_s),\mathsf{norm}_\gamma\left(\mathsf{T}_s\right)\right)\right\}\right]. \end{aligned}$$

Step 2. The proof of the following lemma is postponed to Section 7.1. To simplify notation, we introduce $\mathfrak{g}(\varepsilon) = 1-\mathfrak{f}(\varepsilon)$.

Lemma 4.3. *Assume that* $\lim_{\varepsilon\to 0}\varepsilon^{-1/2}\mathfrak{f}(\varepsilon) = 0$. *Let* $f\colon\mathbb{T}\to\mathbb{R}$ *and* $g\colon[0,\infty)\to\mathbb{R}$ *be Lipschitz-continuous and bounded and assume that* $\Phi\colon[0,\infty)^2\times\mathbb{T}\to[0,\infty)$ *is measurable and satisfies* (4.4). *We have*

$$\begin{aligned} \lim_{\varepsilon\to 0}\Gamma(1-1/\gamma)^{-1}F(\varepsilon) - \int_0^\infty\mathrm{d}t\,\mathbb{E}&\left[\frac{1}{\tau_{\mathfrak{g}(\varepsilon)t}}\mathbf{1}_{\{\tau_{\mathfrak{g}(\varepsilon)t}>1\}}f\circ\mathsf{norm}_\gamma\left(\mathsf{T}^{\downarrow}_{\mathfrak{g}(\varepsilon)t}\right)g\left(\tau_{\mathfrak{g}(\varepsilon)t}^{-1+1/\gamma}t\right)\right.\\ &\left.\times\exp\left\{-\sum_{s\leqslant\mathfrak{f}(\varepsilon)t}\Phi\left(\varepsilon^{-1}s/t,\varepsilon^{-\gamma/(\gamma-1)}\tau^{-1}_{\mathfrak{g}(\varepsilon)t}\mu(\mathsf{T}_s),\mathsf{norm}_\gamma\left(\mathsf{T}_s\right)\right)\right\}\right] = 0. \end{aligned}$$

Since $(\mathsf{T}_s,\, 0\leqslant s\leqslant t)$ is a Poisson point process, it follows from the definition of $\mathsf{T}^{\downarrow}_{\mathfrak{g}(\varepsilon)t}$ that $(\mathsf{T}_s,\, 0\leqslant s\leqslant\mathfrak{f}(\varepsilon)t)$ is independent of $\mathsf{T}^{\downarrow}_{\mathfrak{g}(\varepsilon)t}$. Thus, denoting by $(\mathsf{T}'_s,\, s\geqslant 0)$ a Poisson point process with intensity $\mathbb{N}^{\mathrm{B}}$ which is independent of $\mathsf{T}^{\downarrow}_{\mathfrak{g}(\varepsilon)t}$, recalling that $\tau_{\mathfrak{g}(\varepsilon)t}$ is a measurable function of $\mathsf{T}^{\downarrow}_{\mathfrak{g}(\varepsilon)t}$ and making the change of variable $u=\mathfrak{g}(\varepsilon)t$, we have

$$\lim_{\varepsilon\to 0}\left|\Gamma(1-1/\gamma)^{-1}F(\varepsilon)-\mathfrak{g}(\varepsilon)^{-1}\int_0^\infty \mathrm{d}u\,\mathbb{E}\left[Y_\varepsilon(u)\right]\right|=0, \tag{4.8}$$

where

$$\begin{aligned}Y_\varepsilon(u)=&\frac{1}{\tau_u}\mathbf{1}_{\{\tau_u>1\}}f\circ\mathsf{norm}_\gamma\left(\mathsf{T}^{\downarrow}_u\right)g\left(\mathfrak{g}(\varepsilon)^{-1}\tau_u^{-1+1/\gamma}u\right)\\&\times\mathbb{E}\left[\exp\left\{-\sum_{s\leqslant\mathfrak{f}(\varepsilon)\mathfrak{g}(\varepsilon)^{-1}u}\Phi\left(\varepsilon^{-1}\mathfrak{g}(\varepsilon)s/u,\varepsilon^{-\gamma/(\gamma-1)}\tau_u^{-1}\mu(\mathsf{T}'_s),\mathsf{norm}_\gamma\left(\mathsf{T}'_s\right)\right)\right\}\middle|\mathsf{T}^{\downarrow}_u\right].\end{aligned} \tag{4.9}$$

Step 3. For fixed $\lambda>0$, we have

$$\begin{aligned}\mathbb{E}&\left[\exp\left\{-\sum_{s\leqslant\mathfrak{f}(\varepsilon)\mathfrak{g}(\varepsilon)^{-1}u}\Phi\left(\varepsilon^{-1}\mathfrak{g}(\varepsilon)s/u,\varepsilon^{-\gamma/(\gamma-1)}\lambda^{-1}\mu(\mathsf{T}'_s),\mathsf{norm}_\gamma\left(\mathsf{T}'_s\right)\right)\right\}\right]\\&=\exp\left\{-\int_0^{\mathfrak{f}(\varepsilon)\mathfrak{g}(\varepsilon)^{-1}u}\mathrm{d}s\,\mathbb{N}^{\mathrm{B}}\left[1-\mathrm{e}^{-\Phi\left(\varepsilon^{-1}\mathfrak{g}(\varepsilon)s/u,\varepsilon^{-\gamma/(\gamma-1)}\lambda^{-1}\sigma,\mathsf{norm}_\gamma(\mathcal{T})\right)}\right]\right\}\\&=\exp\left\{-\mathfrak{g}(\varepsilon)^{-1}\int_0^{\varepsilon^{-1}\mathfrak{f}(\varepsilon)\lambda^{-1+1/\gamma}u}\mathrm{d}r\,\mathbb{N}^{\mathrm{B}}\left[1-\mathrm{e}^{-\Phi\left(\lambda^{1-1/\gamma}r/u,\sigma,\mathsf{norm}_\gamma(\mathcal{T})\right)}\right]\right\},\end{aligned}$$

where we made the change of variable $r=\varepsilon^{-1}\mathfrak{g}(\varepsilon)\lambda^{-1+1/\gamma}s$ and used Lemma 4.1 with $a=\varepsilon\lambda^{1-1/\gamma}$. (Notice that $\mathsf{norm}_\gamma(\mathcal{T})$ has the same distribution under $a\,\mathbb{N}^{\mathrm{B}}$ for every $a>0$). Thus, we deduce that a.s. for every $u>0$

$$\begin{aligned}\lim_{\varepsilon\to 0}&\,\mathbb{E}\left[\exp\left\{-\sum_{s\leqslant\mathfrak{f}(\varepsilon)\mathfrak{g}(\varepsilon)^{-1}u}\Phi\left(\varepsilon^{-1}\mathfrak{g}(\varepsilon)s/u,\varepsilon^{-\gamma/(\gamma-1)}\tau_u^{-1}\mu(\mathsf{T}'_s),\mathsf{norm}_\gamma\left(\mathsf{T}'_s\right)\right)\right\}\middle|\mathsf{T}^{\downarrow}_u\right]\\&=\lim_{\varepsilon\to 0}\exp\left\{-\mathfrak{g}(\varepsilon)^{-1}\int_0^{\varepsilon^{-1}\mathfrak{f}(\varepsilon)\lambda^{-1+1/\gamma}u}\mathrm{d}r\,\mathbb{N}^{\mathrm{B}}\left[1-\mathrm{e}^{-\Phi\left(\lambda^{1-1/\gamma}r/u,\sigma,\mathsf{norm}_\gamma(\mathcal{T})\right)}\right]_{|\lambda=\tau_u}\right\}\\&=\exp\left\{-\int_0^{\infty}\mathrm{d}r\,\mathbb{N}^{\mathrm{B}}\left[1-\mathrm{e}^{-\Phi\left(\lambda^{1-1/\gamma}r/u,\sigma,\mathsf{norm}_\gamma(\mathcal{T})\right)}\right]_{|\lambda=\tau_u}\right\}\\&=\mathbb{E}\left[\exp\left\{-\sum_{s\geqslant 0}\Phi\left(\tau_u^{1-1/\gamma}s/u,\mu(\mathsf{T}'_s),\mathsf{norm}_\gamma\left(\mathsf{T}'_s\right)\right)\right\}\middle|\mathsf{T}^{\downarrow}_u\right].\end{aligned}$$

Step 4. We deduce that a.s. for every $u>0$

$$\begin{aligned}\lim_{\varepsilon\to 0}Y_\varepsilon(u)=&\frac{1}{\tau_u}\mathbf{1}_{\{\tau_u>1\}}f\circ\mathsf{norm}_\gamma\left(\mathsf{T}^{\downarrow}_u\right)g\left(\tau_u^{-1+1/\gamma}u\right)\\&\times\mathbb{E}\left[\exp\left\{-\sum_{s\geqslant 0}\Phi\left(\tau_u^{1-1/\gamma}s/u,\mu(\mathsf{T}'_s),\mathsf{norm}_\gamma\left(\mathsf{T}'_s\right)\right)\right\}\middle|\mathsf{T}^{\downarrow}_u\right].\end{aligned} \tag{4.10}$$

Since $|Y_\varepsilon(u)|\leqslant\|f\|_\infty\|g\|_\infty\tau_u^{-1}\mathbf{1}_{\{\tau_u>1\}}$ where the right-hand side is integrable with respect to $\mathbf{1}_{(0,\infty)}(u)\,\mathrm{d}u\otimes\mathbb{P}$ thanks to (3.13), it follows by dominated convergence that

$$\lim_{\varepsilon\to 0}\int_0^\infty\mathrm{d}u\,\mathbb{E}\left[Y_\varepsilon(u)\right]=\int_0^\infty\mathrm{d}u\,\mathbb{E}\left[\frac{1}{\tau_u}\mathbf{1}_{\{\tau_u>1\}}f\circ\mathsf{norm}_\gamma\left(\mathsf{T}^{\downarrow}_u\right)g\left(\tau_u^{-1+1/\gamma}u\right)\right.$$

$$\times \exp\left\{-\sum_{s\geqslant 0}\Phi\left(\tau_u^{1-1/\gamma}s/u,\mu(\mathsf{T}'_s),\mathsf{norm}_\gamma\left(\mathsf{T}'_s\right)\right)\right\}\Bigg]. \quad (4.11)$$

Step 5. Using Theorem 3.2 and Lemma 3.5-(i) again, we get that

$$\begin{aligned}\lim_{\varepsilon\to 0}F(\varepsilon) =\ & \Gamma(1-1/\gamma)\int_0^\infty \mathrm{d}u\,\mathbb{E}\left[\frac{1}{\tau_u}\mathbf{1}_{\{\tau_u>1\}}f\circ \mathsf{norm}_\gamma\left(\mathsf{T}_u^{\downarrow}\right)g\left(\tau_u^{-1+1/\gamma}u\right)\right.\\ &\left.\times\exp\left\{-\sum_{s\geqslant 0}\Phi\left(\tau_u^{1-1/\gamma}s/u,\mu(\mathsf{T}'_s),\mathsf{norm}_\gamma\left(\mathsf{T}'_s\right)\right)\right\}\right]\\ =\ & \Gamma(1-1/\gamma)\,\mathbb{N}\left[\frac{1}{\sigma}\mathbf{1}_{\{\sigma>1\}}\int_{\mathcal{T}}\mu(\mathrm{d}x)f\circ\mathsf{norm}_\gamma\left(\mathcal{T}\right)g\left(\sigma^{-1+1/\gamma}H(x)\right)\right.\\ &\left.\times\exp\left\{-\sum_{s\geqslant 0}\Phi\left(\sigma^{1-1/\gamma}s/H(x),\mu(\mathsf{T}'_s),\mathsf{norm}_\gamma\left(\mathsf{T}'_s\right)\right)\right\}\right]\\ =\ & \mathbb{N}^{(1)}\left[\int_{\mathcal{T}}\mu(\mathrm{d}x)f(\mathcal{T})g(H(x))\exp\left\{-\sum_{s\geqslant 0}\Phi\left(s/H(x),\mu(\mathsf{T}'_s),\mathsf{norm}_\gamma\left(\mathsf{T}'_s\right)\right)\right\}\right]\\ =\ & \mathbb{N}^{(1)}\left[f(\mathcal{T})g(H(U))\exp\left\{-\sum_{s\geqslant 0}\Phi\left(s/H(U),\mu(\mathsf{T}'_s),\mathsf{norm}_\gamma\left(\mathsf{T}'_s\right)\right)\right\}\right],\end{aligned}$$

where, with a slight abuse of notation, we denote by $(\mathsf{T}'_s,\, s\geqslant 0)$ a Poisson point process with intensity $\mathbb{N}^{\mathrm{B}}$ under $\mathbb{N}^{(1)}$, independent of $(\mathcal{T},H(U))$. Since $H(U)$ and $(\mathsf{T}'_s,\, s\geqslant 0)$ are independent, this concludes the proof. □

As a consequence of Theorem 4.2, the next result gives the asymptotic behavior of the total mass of the subtrees grafted near the root of the stable tree.

Corollary 4.4. *Let $\mathcal{T}$ be the normalized stable tree with branching mechanism $\psi(\lambda)=\lambda^\gamma$ where $\gamma\in(1,2]$. Conditionally on $\mathcal{T}$, let U be $\mathcal{T}$-valued random variable with distribution μ under $\mathbb{N}^{(1)}$. Assume that $\lim_{\varepsilon\to 0}\varepsilon^{-1/2}\mathfrak{f}(\varepsilon)=0$ and $\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathfrak{f}(\varepsilon)=\infty$. Define a process S^ε by*

$$S_t^\varepsilon := \sum_{h_i\leqslant \varepsilon t\wedge \mathfrak{f}(\varepsilon)H(U)}\varepsilon^{-\gamma/(\gamma-1)}\sigma_i,\quad t\geqslant 0.$$

Then we have the following convergence in distribution

$$(\mathcal{T},H(U),(S_t^\varepsilon,\,t\geqslant 0))\xrightarrow[\varepsilon\to 0]{(d)}(\mathcal{T},H(U),(S_t,\,t\geqslant 0)) \quad (4.12)$$

in the space $\mathbb{T}\times\mathbb{R}\times D[0,\infty)$, where S is a stable subordinator with Laplace exponent φ given by (3.10)*, independent of $(\mathcal{T},H(U))$.*

Proof. We adapt the arguments of [28, Chapter VII, Section 7.2], see also Theorem 3.1 and Corollary 3.4 in [29]. Since the process S has no fixed points of discontinuity, it is enough to show that the convergence (4.12) holds in $\mathbb{T}\times\mathbb{R}\times D[0,r]$ for every $r>0$.

Fix $r>0$ and let $\delta>0$. Define

$$S_t^{\varepsilon,\delta} := \sum_{h_i\leqslant \varepsilon t\wedge \mathfrak{f}(\varepsilon)H(U)}\varepsilon^{-\gamma/(\gamma-1)}\sigma_i\mathbf{1}_{\left\{\varepsilon^{-\gamma/(\gamma-1)}\sigma_i>\delta\right\}},\quad t\geqslant 0.$$

Recall that for a metric space X, we denote by $\mathcal{M}_p(X)$ the space of point measures on X equipped with the topology of vague convergence. It is known (see [28, p. 215]) that the restriction mapping

$$m\mapsto m_{|[0,\infty)\times(\delta,\infty)}$$

is a.s. continuous from $\mathcal{M}_p([0,\infty)^2)$ to $\mathcal{M}_p([0,\infty)\times(\delta,\infty))$ with respect to the distribution of the Poisson random measure $\sum_{s\geqslant 0}\delta_{(s,\mu(\mathsf{T}'_s))}$. Furthermore, the summation mapping

$$m \mapsto \left(\int_{[0,t]\times(\delta,\infty)} x\, m(\mathrm{d}s,\mathrm{d}x),\ 0\leqslant t\leqslant r\right)$$

is a.s. continuous from $\mathcal{M}_p([0,\infty)\times(\delta,\infty))$ to $D[0,r]$ with respect to the same distribution. We deduce from Theorem 4.2-(i) and the continuous mapping theorem the following convergence in distribution

$$\left(\mathcal{T}, H(U), \left(S_t^{\varepsilon,\delta},\ 0\leqslant t\leqslant r\right)\right) \xrightarrow[\varepsilon\to 0]{(d)} \left(\mathcal{T}, H(U), \left(\sum_{s\leqslant t}\mu(\mathsf{T}'_s)\mathbf{1}_{\{\mu(\mathsf{T}'_s)>\delta\}},\ 0\leqslant t\leqslant r\right)\right) \tag{4.13}$$

in $\mathbb{T}\times\mathbb{R}\times D[0,r]$, where $(\mathsf{T}'_s, s\geqslant 0)$ is a Poisson point process with intensity $\mathbb{N}^{\mathrm{B}}$, independent of $(\mathcal{T}, H(U))$.

Furthermore, since $\sum_{s\leqslant r}\mu(\mathsf{T}'_s)$ is $\mathbb{N}^{(1)}$-a.s. finite, it is clear by the dominated convergence theorem that $\mathbb{N}^{(1)}$-a.s.

$$\lim_{\delta\to 0}\sup_{t\leqslant r}\left|\sum_{s\leqslant t}\mu(\mathsf{T}'_s) - \sum_{s\leqslant t}\mu(\mathsf{T}'_s)\mathbf{1}_{\{\mu(\mathsf{T}'_s)>\delta\}}\right| = \lim_{\delta\to 0}\sum_{s\leqslant r}\mu(\mathsf{T}'_s)\mathbf{1}_{\{\mu(\mathsf{T}'_s)\leqslant\delta\}} = 0.$$

Since uniform convergence on $[0,T]$ implies convergence for the Skorokhod $J1$ topology, we deduce that

$$\left(\mathcal{T}, H(U), \left(\sum_{s\leqslant t}\mu(\mathsf{T}'_s)\mathbf{1}_{\{\mu(\mathsf{T}'_s)>\delta\}},\ 0\leqslant t\leqslant r\right)\right) \xrightarrow[\delta\to 0]{(d)} \left(\mathcal{T}, H(U), (S_t,\ 0\leqslant t\leqslant r)\right), \tag{4.14}$$

where $S_t=\sum_{s\leqslant t}\mu(\mathsf{T}'_s)$ is a stable subordinator with Laplace exponent φ, independent of $(\mathcal{T}, H(U))$.

Finally, we shall prove that for every $\eta>0$

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbb{N}^{(1)}\left[\sup_{0\leqslant t\leqslant T}\left|S_t^{\varepsilon}-S_t^{\varepsilon,\delta}\right|\geqslant\eta\right]=0. \tag{4.15}$$

Let $f\colon[0,\infty)\to[0,\infty)$ be Lipschitz-continuous such that $x\mathbf{1}_{[0,\delta]}(x)\leqslant f(x)\leqslant x\mathbf{1}_{[0,2\delta]}(x)$. We have

$$\begin{aligned}\sup_{0\leqslant t\leqslant r}\left|S_t^{\varepsilon}-S_t^{\varepsilon,\delta}\right| &= \sum_{h_i\leqslant\varepsilon r\wedge\mathfrak{f}(\varepsilon)H(U)}\varepsilon^{-\gamma/(\gamma-1)}\sigma_i\mathbf{1}_{\left\{\varepsilon^{-\gamma/(\gamma-1)}\sigma_i\leqslant\delta\right\}}\\ &\leqslant \sum_{h_i\leqslant\varepsilon r\wedge\mathfrak{f}(\varepsilon)H(U)} f\left(\varepsilon^{-\gamma/(\gamma-1)}\sigma_i\right).\end{aligned}$$

It follows that

$$\begin{aligned}\limsup_{\varepsilon\to 0}\mathbb{N}^{(1)}\left[\sup_{0\leqslant t\leqslant r}\left|S_t^{\varepsilon}-S_t^{\varepsilon,\delta}\right|\geqslant\eta\right] &\leqslant \limsup_{\varepsilon\to 0}\mathbb{N}^{(1)}\left[\sum_{h_i\leqslant\varepsilon r\wedge\mathfrak{f}(\varepsilon)H(U)} f\left(\varepsilon^{-\gamma/(\gamma-1)}\sigma_i\right)\geqslant\eta\right]\\ &\leqslant \mathbb{N}^{(1)}\left[\sum_{s\leqslant r} f\left(\mu(\mathsf{T}'_s)\right)\geqslant\eta\right]\\ &\leqslant \mathbb{N}^{(1)}\left[\sum_{s\leqslant r}\mu(\mathsf{T}'_s)\mathbf{1}_{\{\mu(\mathsf{T}'_s)\leqslant 2\delta\}}\geqslant\eta\right],\end{aligned} \tag{4.16}$$

where in the second inequality we used the Portmanteau theorem together with the following convergence in distribution

$$\sum_{h_i \leqslant \varepsilon r \wedge \mathfrak{f}(\varepsilon) H(U)} f\left(\varepsilon^{-\gamma/(\gamma-1)}\sigma_i\right) \xrightarrow[\varepsilon \to 0]{(d)} \sum_{s \leqslant r} f\left(\mu(\mathsf{T}'_s)\right),$$

which holds thanks to Theorem 4.2-(i) applied with $\Phi(h,a,T) = \mathbf{1}_{\{h \leqslant r\}} f(a)$. But, by the dominated convergence theorem, we have that $\mathbb{N}^{(1)}$-a.s.

$$\lim_{\delta \to 0} \sum_{s \leqslant r} \mu(\mathsf{T}'_s) \mathbf{1}_{\{\mu(\mathsf{T}'_s) \leqslant 2\delta\}} = 0.$$

Together with (4.16), this implies (4.15).

Putting together (4.13)–(4.15), it follows from the second converging together theorem, see e.g. [8, Theorem 3.2], that

$$(\mathcal{T}, H(U), (S^\varepsilon_t,\, 0 \leqslant t \leqslant r)) \xrightarrow[\varepsilon \to 0]{(d)} (\mathcal{T}, H(U), (S_t,\, 0 \leqslant t \leqslant r))$$

in $\mathbb{T} \times \mathbb{R} \times D[0,r]$. This finishes the proof. □

Remark 4.5. Let us comment on the connection between Theorem 4.2 and the small time asymptotics of the fragmentation at height of the stable tree F^-, see [7, Section 4] for the Brownian case $\gamma = 2$ and [25] for the case $\gamma \in (1,2)$. We briefly recall its definition. Consider the normalized stable tree $\mathcal{T}$ and denote by $(\mathcal{T}_j,\, j \in J_t)$ the connected components of the set $\{x \in \mathcal{T} \colon\ H(x) > t\}$ obtained from $\mathcal{T}$ by removing vertices located at height $\leqslant t$. Then $F^-(t) = (F^-_1(t), F^-_2(t), \ldots)$ is defined as the decreasing sequence of masses $(\mu(\mathcal{T}_j),\, j \in J_t)$. In [19, Section 5.1], Haas obtains the following functional convergence in distribution as a consequence of a more general result

$$\varepsilon^{-\gamma/(\gamma-1)}(1 - F^-_1(\varepsilon \cdot), (F^-_2(\varepsilon \cdot), F^-_3(\varepsilon \cdot), \ldots)) \xrightarrow[\varepsilon \to 0]{(d)} (S, FI), \tag{4.17}$$

where the convergence holds with respect to the Skorokhod $J1$ topology. Here FI is a fragmentation process with immigration and S is a stable subordinator with index $1 - 1/\gamma$ representing the total mass of immigrants.

At least heuristically, this can be recovered from Theorem 4.2. Let $U \in \mathcal{T}$ be a leaf chosen uniformly at random. It is not difficult to see that for $0 \leqslant t \leqslant H(U)$, with high probability as $\varepsilon \to 0$, the biggest fragment at time εt is the one containing U. Thus we get $1 - F^-_1(\varepsilon t) = \sum_{h_i \leqslant \varepsilon t} \sigma_i$ and

$$(F^-_2(\varepsilon t), F^-_3(\varepsilon t), \ldots) = (\mu(\mathcal{T}_i^{\geqslant \varepsilon t - h_i}),\, h_i \leqslant \varepsilon t)^{\downarrow}$$

is the decreasing rearrangement of the masses of $\mathcal{T}_i^{\geqslant \varepsilon t - h_i}$ for the subtrees grafted at height $h_i \leqslant \varepsilon t$. Here we denote by $T^{\geqslant r} = T \setminus T^{<r} = \{x \in T \colon\ H(x) \geqslant r\}$ the set of vertices of T above height r. To recover (4.17), we may prove the joint convergence of

$$\left(\sum_{h_i \leqslant \varepsilon \cdot \wedge \varepsilon H(U)} \varepsilon^{-\gamma/(\gamma-1)} \sigma_i, \sum_{h_i \leqslant \varepsilon H(U)} \delta_{\left(\mathbf{1}_{\{h_i \leqslant \varepsilon t\}} \varepsilon^{-\gamma/(\gamma-1)} \mu(\mathcal{T}_i^{\geqslant \varepsilon t - h_i}),\, t \geqslant 0\right)} \right), \tag{4.18}$$

then argue that the convergence of the point measure in (4.18) implies that of the rearranged atoms. Notice that we may obtain the convergence of the first coordinate in (4.18) using Theorem 4.2-(ii), similarly to how we proved Corollary 4.4 using Theorem 4.2-(i). For the convergence of the second coordinate, the idea is to consider $\Phi(h,a,T) = F\left((\mathbf{1}_{\{h \leqslant t\}} a \mu(T^{\geqslant a^{-1+1/\gamma}(t-h)}),\, t \geqslant 0)\right)$, where $F \colon D[0,\infty) \to [0,\infty)$ is

Lipschitz-continuous with compact support. However, Φ is not Lipschitz-continuous with respect to a so our result does not apply directly. Similarly, to get the convergence of the dust, notice that

$$\mu(\mathcal{T}^{<\varepsilon t}) = \sum_{h_i \leqslant \varepsilon t} \mu(\mathcal{T}_i^{<\varepsilon t - h_i}).$$

Thus the idea is to apply Theorem 4.2-(ii) with $\Phi(h, a, T) = \mathbf{1}_{\{h \leqslant t\}} a \mu(T^{<a^{-1+1/\gamma}(t-h)})$ which again does not satisfy the assumptions.

5 Asymptotic behavior of $\mathrm{Z}_{\alpha,\beta}$ in the case $\beta/\alpha^{1-1/\gamma} \to c \in [0, \infty)$

We start by showing that if $U \in \mathcal{T}$ is a leaf chosen uniformly at random, $Z_{\alpha,\beta}(U)$ defined in (1.1) converges in disrtibution after proper rescaling.

Proposition 5.1. *Assume that $\alpha \to \infty$, $\beta \geqslant 0$ and $\beta/\alpha^{1-1/\gamma} \to c \in [0, \infty)$. Let $\mathcal{T}$ be the normalized stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1, 2]$. Conditionally on $\mathcal{T}$, let U be a $\mathcal{T}$-valued random variable with disribution μ under $\mathbb{N}^{(1)}$. Then we have the following convergence in distribution*

$$\left(\mathcal{T}, H(U), \alpha^{1-1/\gamma} \mathfrak{h}^{-\beta} Z_{\alpha,\beta}(U)\right) \xrightarrow[\alpha\to\infty]{(d)} \left(\mathcal{T}, H(U), \int_0^\infty \mathrm{e}^{-S_t - ct/\mathfrak{h}} \, \mathrm{d}t\right), \tag{5.1}$$

where $(S_t,\, t \geqslant 0)$ is a stable subordinator with Laplace exponent φ given by (3.10), independent of $(\mathcal{T}, H(U))$.

Proof. Set

$$\varepsilon = \varepsilon(\alpha) := \alpha^{(\delta-1)(1-1/\gamma)} \tag{5.2}$$

with $\delta \in (0, 1/3)$ so that $\varepsilon \to 0$ as $\alpha \to \infty$. Define

$$I_\alpha := \alpha^{1-1/\gamma} \int_0^{\varepsilon H(U)} \mathrm{e}^{-\alpha(1-\sigma_{r,U}) - \beta r/\mathfrak{h}} \, \mathrm{d}r. \tag{5.3}$$

Lemma 5.2. *We have the following convergence in $\mathbb{N}^{(1)}$-probability*

$$\lim_{\alpha\to\infty} \left(\alpha^{1-1/\gamma} \mathfrak{h}^{-\beta} Z_{\alpha,\beta}(U) - I_\alpha\right) = 0.$$

The proof is postponed to Section 7.2. Using this together with Slutsky's theorem, it is clear that the proof of (5.1) reduces to showing the following convergence in distribution

$$(\mathcal{T}, H(U), I_\alpha) \xrightarrow[\alpha\to\infty]{(d)} \left(\mathcal{T}, H(U), \int_0^\infty \mathrm{e}^{-S_t - ct/\mathfrak{h}} \, \mathrm{d}t\right). \tag{5.4}$$

Making the change of variable $t = \alpha^{1-1/\gamma} r$, notice that

$$I_\alpha = \int_0^{\alpha^{1-1/\gamma}\varepsilon H(U)} \exp\left\{-\alpha\left(1 - \sigma_{\alpha^{-1+1/\gamma}t, U}\right) - \beta\alpha^{-1+1/\gamma} t/\mathfrak{h}\right\} \mathrm{d}t, \tag{5.5}$$

Let $A > 0$. Notice that, applying Corollary 4.4, we get the following convergence in distribution

$$\left(\mathcal{T}, H(U), \left(\sum_{h_i \leqslant \alpha^{-1+1/\gamma} t \wedge \varepsilon H(U)} \alpha\sigma_i,\, 0 \leqslant t \leqslant A\right)\right) \xrightarrow[\alpha\to\infty]{(d)} \left(\mathcal{T}, H(U), (S_t,\, 0 \leqslant t \leqslant A)\right), \tag{5.6}$$

where S is a subordinator with Laplace exponent φ, independent of $(\mathcal{T}, H(U))$. Moreover, on the event $\Omega_\alpha \coloneqq \{\alpha^{-1+1/\gamma} A \leqslant \varepsilon H(U)\}$, we have for every $t \in [0, A]$

$$\sum_{h_i \leqslant \alpha^{-1+1/\gamma} t \wedge \varepsilon H(U)} \sigma_i = \sum_{h_i \leqslant \alpha^{-1+1/\gamma} t} \sigma_i = 1 - \sigma_{\alpha^{-1+1/\gamma} t, U}. \tag{5.7}$$

Since $\alpha^{1-1/\gamma}\varepsilon \to \infty$, it is clear that $\lim_{\alpha\to\infty} \mathbb{N}^{(1)}[\Omega_\alpha] = 1$. Thus, it follows from (5.6) and (5.7) that

$$\left(\mathcal{T}, H(U), \left(\alpha\left(1 - \sigma_{\alpha^{-1+1/\gamma} t, U}\right), \, 0 \leqslant t \leqslant A\right)\right) \xrightarrow[\alpha\to\infty]{(d)} \left(\mathcal{T}, H(U), (S_t, \, 0 \leqslant t \leqslant A)\right).$$

Now a simple application of the continuous mapping theorem gives

$$\left(\mathcal{T}, H(U), \int_0^A \exp\left\{-\alpha\left(1 - \sigma_{\alpha^{-1+1/\gamma} t, U}\right) - \beta \alpha^{-1+1/\gamma} t/\mathfrak{h}\right\} \mathrm{d}t\right) \xrightarrow[\alpha\to\infty]{(d)} \left(\mathcal{T}, H(U), \int_0^A \mathrm{e}^{-S_t - ct/\mathfrak{h}} \, \mathrm{d}t\right). \tag{5.8}$$

On the other hand, applying (3.22) with $f(T) = \mathrm{e}^{-\alpha(1-\mu(T))}$ and $g(r) = \mathbf{1}_{\{r \geqslant \alpha^{-1+1/\gamma} A\}}$, we get

$$\begin{aligned}
&\mathbb{N}^{(1)}\left[\int_A^{\alpha^{1-1/\gamma}\varepsilon H(U)} \exp\left\{-\alpha\left(1 - \sigma_{\alpha^{-1+1/\gamma} t, U}\right) - \beta \alpha^{-1+1/\gamma} t/\mathfrak{h}\right\} \mathrm{d}t\right] \\
&\qquad \leqslant \alpha^{1-1/\gamma} \, \mathbb{N}^{(1)}\left[\int_{\alpha^{-1+1/\gamma} A}^{H(U)} \exp\left\{-\alpha\left(1 - \sigma_{r,U}\right)\right\} \mathrm{d}r\right] \\
&\qquad = \frac{\alpha^{1-1/\gamma}}{\gamma\Gamma(1-1/\gamma)} \int_0^1 x^{-1/\gamma}(1-x)^{-1/\gamma} \mathrm{e}^{-\alpha x} \, \mathbb{N}^{(1)}\left[(\alpha x)^{1-1/\gamma} H(U) \geqslant A\right] \mathrm{d}x \\
&\qquad = \frac{1}{\gamma\Gamma(1-1/\gamma)} \int_0^\alpha y^{-1/\gamma}\left(1 - \frac{y}{\alpha}\right)^{-1/\gamma} \mathrm{e}^{-y} \, \mathbb{N}^{(1)}\left[y^{1-1/\gamma} H(U) \geqslant A\right] \mathrm{d}y.
\end{aligned}$$

By the dominated convergence theorem, we have

$$\begin{aligned}
\lim_{\alpha\to\infty} \int_0^{\alpha/2} y^{-1/\gamma}\left(1 - \frac{y}{\alpha}\right)^{-1/\gamma} \mathrm{e}^{-y} \, \mathbb{N}^{(1)}\left[y^{1-1/\gamma} H(U) \geqslant A\right] \mathrm{d}y& \\
= \int_0^\infty y^{-1/\gamma} \mathrm{e}^{-y} \, \mathbb{N}^{(1)}\left[y^{1-1/\gamma} H(U) \geqslant A\right] \mathrm{d}y&.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\int_{\alpha/2}^\alpha y^{-1/\gamma}\left(1 - \frac{y}{\alpha}\right)^{-1/\gamma} \mathrm{e}^{-y} \, \mathbb{N}^{(1)}\left[y^{1-1/\gamma} H(U) \geqslant A\right] \mathrm{d}y& \\
\leqslant \mathrm{e}^{-\alpha/2} \int_{\alpha/2}^\alpha y^{-1/\gamma}\left(1 - \frac{y}{\alpha}\right)^{-1/\gamma} \mathrm{d}y& \\
= \alpha^{1-1/\gamma} \mathrm{e}^{-\alpha/2} \int_{1/2}^1 z^{-1/\gamma}(1-z)^{-1/\gamma} \, \mathrm{d}z&,
\end{aligned}$$

where the last term converges to 0 as $\alpha \to \infty$. We deduce that

$$\limsup_{\alpha\to\infty} \mathbb{N}^{(1)}\left[\int_A^{\alpha^{1-1/\gamma}\varepsilon H(U)} \exp\left\{-\alpha\left(1 - \sigma_{\alpha^{-1+1/\gamma} t, U}\right) - \beta \alpha^{-1+1/\gamma} t/\mathfrak{h}\right\} \mathrm{d}t\right]$$

$$\leqslant \frac{1}{\gamma\Gamma(1-1/\gamma)} \int_0^\infty y^{-1/\gamma} \mathrm{e}^{-y}\, \mathbb{N}^{(1)} \left[y^{1-1/\gamma} H(U) \geqslant A \right] \mathrm{d}y,$$

and, thanks to the dominated convergence theorem,

$$\lim_{A\to\infty} \limsup_{\alpha\to\infty} \mathbb{N}^{(1)} \left[\int_A^{\alpha^{1-1/\gamma}\varepsilon H(U)} \exp\left\{ -\alpha\left(1-\sigma_{\alpha^{-1+1/\gamma}t,U}\right) - \beta\alpha^{-1+1/\gamma} t/\mathfrak{h} \right\} \mathrm{d}t \right] = 0. \tag{5.9}$$

Combining (5.8) and (5.9) and applying [8, Theorem 3.2], (5.4) readily follows. This finishes the proof. □

The next lemma, whose proof is postponed to Section 7.3, states that taking a leaf uniformly at random or taking the average over all leaves yields the same limiting behavior for $Z_{\alpha,\beta}(x)$. Recall from (1.1) the definition of $\mathbf{Z}_{\alpha,\beta}$.

Lemma 5.3. *Under the assumptions of Theorem 5.1, we have the convergence in $\mathbb{N}^{(1)}$-probability*

$$\lim_{\alpha\to\infty} \alpha^{1-1/\gamma}\mathfrak{h}^{-\beta} \left(Z_{\alpha,\beta}(U) - \mathbf{Z}_{\alpha,\beta} \right) = 0. \tag{5.10}$$

Combining Proposition 5.1 and Lemma 5.3, we get the following result using Slutsky's theorem.

Theorem 5.4. *Assume that $\alpha \to \infty$, $\beta \geqslant 0$ and $\beta/\alpha^{1-1/\gamma} \to c \in [0,\infty)$. Let $\mathcal{T}$ be the stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1,2]$. Conditionally on $\mathcal{T}$, let U be a $\mathcal{T}$-valued random variable with distribution μ under $\mathbb{N}^{(1)}$. Then we have the following convergence in distribution*

$$\left(\mathcal{T}, H(U), \alpha^{1-1/\gamma}\mathfrak{h}^{-\beta} Z_{\alpha,\beta}(U), \alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\mathbf{Z}_{\alpha,\beta}\right) \xrightarrow[\alpha\to\infty]{(d)} \left(\mathcal{T}, H(U), \int_0^\infty \mathrm{e}^{-S_t - ct/\mathfrak{h}}\, \mathrm{d}t, \int_0^\infty \mathrm{e}^{-S_t - ct/\mathfrak{h}}\, \mathrm{d}t\right), \tag{5.11}$$

where S is a stable subordinator with Laplace exponent φ given by (3.10)*, independent of $(\mathcal{T}, H(U))$.*

6 Asymptotic behavior of $\mathbf{Z}_{\alpha,\beta}$ in the case $\beta/\alpha^{1-1/\gamma} \to \infty$

We treat the case $\beta/\alpha^{1-1/\gamma} \to \infty$. Intuitively, this assumption guarantees that $\mathfrak{h}^\beta_{r,x}$ dominates $\sigma^\alpha_{r,x}$, thus we get a different asymptotic behavior and there is no longer a subordinator in the limit.

Theorem 6.1. *Assume that $\beta \to \infty$, $\alpha \geqslant 0$ and $\alpha^{1-1/\gamma}/\beta \to 0$. Let $\mathcal{T}$ be the stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1,2]$. Then we have the following convergence in $\mathbb{N}^{(1)}$-probability*

$$\lim_{\beta\to\infty} \beta\mathfrak{h}^{-\beta}\mathbf{Z}_{\alpha,\beta} = \mathfrak{h}. \tag{6.1}$$

Furthermore, if $\alpha^{1-1/\gamma}/\beta^\rho \to 0$ for some $\rho \in (0,1)$, then the convergence holds $\mathbb{N}^{(1)}$-almost surely.

Proof. We start by assuming that $\alpha \to \infty$ and $\alpha^{1-1/\gamma}/\beta \to 0$ (the case α bounded from above is covered by the second part of the theorem). Setting $\varepsilon = (\alpha^{1-1/\gamma}\beta)^{-1/2}$, it is straightforward to check that $\varepsilon \to 0$, $\beta\varepsilon \to \infty$ and $\alpha^{1-1/\gamma}\varepsilon \to 0$. Write

$$\beta\mathfrak{h}^{-\beta}\mathbf{Z}_{\alpha,\beta} = E_\beta + \sum_{i=1}^4 F^i_\beta \tag{6.2}$$

where

$$\begin{aligned}
F_\beta^1 &= \beta \int_{\mathcal{T}} \mathbf{1}_{\{H(x)<2\varepsilon\}}\, \mu(\mathrm{d}x) \int_0^{H(x)} \sigma_{r,x}^\alpha \left(\frac{\mathfrak{h}_{r,x}}{\mathfrak{h}}\right)^\beta \mathrm{d}r,\\
F_\beta^2 &= \beta \int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}}\, \mu(\mathrm{d}x) \int_\varepsilon^{H(x)} \sigma_{r,x}^\alpha \left(\frac{\mathfrak{h}_{r,x}}{\mathfrak{h}}\right)^\beta \mathrm{d}r,\\
F_\beta^3 &= \beta \int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}}\, \mu(\mathrm{d}x) \int_0^{\varepsilon} \sigma_{r,x}^\alpha \left[\left(\frac{\mathfrak{h}_{r,x}}{\mathfrak{h}}\right)^\beta - \left(1-\frac{r}{\mathfrak{h}}\right)^\beta\right] \mathrm{d}r,\\
F_\beta^4 &= \beta \int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}}\, \mu(\mathrm{d}x) \int_0^{\varepsilon} \sigma_{r,x}^\alpha \left[\left(1-\frac{r}{\mathfrak{h}}\right)^\beta - \mathrm{e}^{-\beta r/\mathfrak{h}}\right] \mathrm{d}r,\\
E_\beta &= \beta \int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}}\, \mu(\mathrm{d}x) \int_0^{\varepsilon} \sigma_{r,x}^\alpha \mathrm{e}^{-\beta r/\mathfrak{h}}\, \mathrm{d}r.
\end{aligned}$$

We shall prove that $\lim_{\beta\to\infty} F_\beta^i = 0$ in $\mathbb{N}^{(1)}$-probability for every $i \in \{1,2,3,4\}$.

Let $p \in (1,2)$. Using that $\sigma_{r,x} \leqslant 1$ and $\mathfrak{h}_{r,x} \leqslant \mathfrak{h}$ and applying the Markov inequality, it is clear that

$$F_\beta^1 \leqslant 2\beta\varepsilon \int_{\mathcal{T}} \mathbf{1}_{\{H(x)<2\varepsilon\}}\, \mu(\mathrm{d}x) \leqslant 2^{1+p}\beta\varepsilon^{1+p} \int_{\mathcal{T}} H(x)^{-p}\, \mu(\mathrm{d}x).$$

Since the last integral has a finite first moment by Lemma 3.9 and $\beta\varepsilon^{1+p} \to 0$, we deduce that $\mathbb{N}^{(1)}$-a.s. $\lim_{\beta\to\infty} F_\beta^1 = 0$.

Next, using (2.5), we get

$$\begin{aligned}
F_\beta^2 &= \beta \int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}}\, \mu(\mathrm{d}x) \int_\varepsilon^{H(x)} \sigma_{r,x}^\alpha \left(\frac{\mathfrak{h}_{r,x}}{\mathfrak{h}}\right)^\beta \mathrm{d}r\\
&\leqslant \beta \left(1-\frac{\varepsilon}{\mathfrak{h}}\right)^\beta \int_{\mathcal{T}} \mu(\mathrm{d}x) \int_0^{H(x)} \sigma_{r,x}^\alpha\, \mathrm{d}r. \qquad (6.3)
\end{aligned}$$

By [1, Corollary 6.6], we have

$$\mathbb{N}^{(1)}\left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \int_0^{H(x)} \sigma_{r,x}^\alpha\, \mathrm{d}r\right] = \frac{1}{|\Gamma(-1/\gamma)|} \mathrm{B}\left(\alpha+1-1/\gamma, 1-1/\gamma\right),$$

where B is the beta function. Using that $\mathrm{B}(x, 1-1/\gamma) \sim \Gamma(1-1/\gamma)x^{-1+1/\gamma}$ as $x\to\infty$, we deduce that

$$\sup_{\alpha\geqslant 0} \mathbb{N}^{(1)}\left[\alpha^{1-1/\gamma} \int_{\mathcal{T}} \mu(\mathrm{d}x) \int_0^{H(x)} \sigma_{r,x}^\alpha\, \mathrm{d}r\right] < \infty. \qquad (6.4)$$

On the other hand, let $\theta > 1$. Since the function $x \mapsto x^{1+\theta}\mathrm{e}^{-x}$ is bounded on $[0,\infty)$, it follows that

$$\frac{\beta}{\alpha^{1-1/\gamma}} \left(1-\frac{\varepsilon}{\mathfrak{h}}\right)^\beta \leqslant \frac{\beta}{\alpha^{1-1/\gamma}} \mathrm{e}^{-\beta\varepsilon/\mathfrak{h}} \leqslant C \frac{\mathfrak{h}^{1+\theta}}{\beta^\theta \varepsilon^{1+\theta}\alpha^{1-1/\gamma}} \qquad (6.5)$$

for some constant $C > 0$. Notice that $\beta^\theta\varepsilon^{1+\theta}\alpha^{1-1/\gamma} \to \infty$ since $\theta > 1$. Thus the right-hand side of (6.5) goes to 0 almost surely. Now putting together (6.3), (6.4) and (6.5), we deduce that $\lim_{\beta\to\infty} F_\beta^2 = 0$ in $\mathbb{N}^{(1)}$-probability.

Let $x \in \mathcal{T}$. Recall from (2.5) and (2.6) that $\mathfrak{h}_{r,x} \leqslant \mathfrak{h} - r$ for every $r \in [0, H(x)]$ and that the equality holds for $r \in [0, H(x\wedge x^*)]$. Therefore, we get

$$|F_\beta^3| = \beta \int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon,\, H(x\wedge x^*)<\varepsilon\}}\, \mu(\mathrm{d}x) \int_{H(x\wedge x^*)}^{\varepsilon} \sigma_{r,x}^\alpha \left[\left(1-\frac{r}{\mathfrak{h}}\right)^\beta - \left(\frac{\mathfrak{h}_{r,x}}{\mathfrak{h}}\right)^\beta\right] \mathrm{d}r$$

$$\begin{aligned}
&\leqslant \beta \int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon,\, H(x\wedge x^*)<\varepsilon\}}\, \mu(\mathrm{d}x) \int_{H(x\wedge x^*)}^{\varepsilon} \left(1-\frac{r}{\mathfrak{h}}\right)^{\beta} \mathrm{d}r \\
&\leqslant \beta \int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon,\, H(x\wedge x^*)<\varepsilon\}}\, \mu(\mathrm{d}x) \int_{H(x\wedge x^*)}^{\varepsilon} \mathrm{e}^{-\beta r/\mathfrak{h}}\, \mathrm{d}r \\
&\leqslant \mathfrak{h} \int_{\mathcal{T}} \mathrm{e}^{-\beta H(x\wedge x^*)/\mathfrak{h}}\, \mu(\mathrm{d}x).
\end{aligned}$$

Since $H(x \wedge x^*) > 0$ for μ-a.e. $x \in \mathcal{T}$, a simple application of the dominated convergence theorem gives that $\mathbb{N}^{(1)}$-a.s. $\lim_{\beta\to\infty} F_\beta^3 = 0$.

Furthermore, using the inequality $|\mathrm{e}^b - \mathrm{e}^a| \leqslant |b-a|\mathrm{e}^b$ for $a \leqslant b$ together with the fact that $j\colon y \mapsto -(y+\log(1-y))/y^2$ is increasing on $[0,1)$, we get for $r \in [0,\varepsilon]$

$$\left| \mathrm{e}^{-\beta r/\mathfrak{h}} - \left(1-\frac{r}{\mathfrak{h}}\right)^{\beta} \right| \leqslant \beta \left| \frac{r}{\mathfrak{h}} + \log\left(1-\frac{r}{\mathfrak{h}}\right) \right| \mathrm{e}^{-\beta r/\mathfrak{h}} \leqslant \beta \left(\frac{r}{\mathfrak{h}}\right)^2 \mathrm{e}^{-\beta r/\mathfrak{h}} j\left(\frac{\varepsilon}{\mathfrak{h}}\right).$$

Therefore, we deduce that

$$|F_\beta^4| \leqslant j\left(\frac{\varepsilon}{\mathfrak{h}}\right) \int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}}\, \mu(\mathrm{d}x) \int_0^{\varepsilon} \left(\frac{\beta r}{\mathfrak{h}}\right)^2 \mathrm{e}^{-\beta r/\mathfrak{h}}\, \mathrm{d}r \leqslant Cj\left(\frac{\varepsilon}{\mathfrak{h}}\right)\varepsilon,$$

where we used that $y \mapsto y^2\mathrm{e}^{-y}$ is bounded on $[0,\infty)$ by some constant $C < \infty$ for the second inequality. Since $\lim_{y\to 0} j(y) = 1/2$, we get $\mathbb{N}^{(1)}$-a.s. $\lim_{\beta\to\infty} F_\beta^4 = 0$. We deduce the following convergence in $\mathbb{N}^{(1)}$-probability

$$\lim_{\beta\to\infty} \sum_{i=1}^{4} F_\beta^i = 0. \tag{6.6}$$

Notice that

$$E_\beta \leqslant \beta \int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}}\, \mu(\mathrm{d}x) \int_0^{\varepsilon} \mathrm{e}^{-\beta r/\mathfrak{h}}\, \mathrm{d}r = \mathfrak{h}\left(1-\mathrm{e}^{-\beta\varepsilon/\mathfrak{h}}\right) \int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}}\, \mu(\mathrm{d}x) \leqslant \mathfrak{h}. \tag{6.7}$$

On the other hand, using that $\sigma_{r,x} \geqslant \sigma_{\varepsilon,x}$ for every $x \in \mathcal{T}$ such that $H(x) \geqslant 2\varepsilon$ and every $r \in [0,\varepsilon]$, we get

$$E_\beta \geqslant \mathfrak{h}\left(1-\mathrm{e}^{-\beta\varepsilon/\mathfrak{h}}\right) \int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}} \sigma_{\varepsilon,x}^{\alpha}\, \mu(\mathrm{d}x). \tag{6.8}$$

We now shall prove the following convergence in $\mathbb{N}^{(1)}$-probability

$$\lim_{\beta\to\infty} \int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}} \sigma_{\varepsilon,x}^{\alpha}\, \mu(\mathrm{d}x) = 1. \tag{6.9}$$

Using Lemma 3.5-(i) and Bismut’s decomposition (3.12), we have

$$\begin{aligned}
&\mathbb{N}^{(1)}\left[\int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}} \sigma_{\varepsilon,x}^{\alpha}\, \mu(\mathrm{d}x)\right] \\
&\quad = \Gamma(1-1/\gamma)\, \mathbb{N}\left[\frac{1}{\sigma}\mathbf{1}_{\{\sigma>1\}} \int_{\mathcal{T}} \mathbf{1}_{\left\{\sigma^{-1+1/\gamma} H(x)\geqslant 2\varepsilon\right\}} \left(\frac{\sigma_{\sigma^{1-1/\gamma}\varepsilon,x}}{\sigma}\right)^{\alpha} \mu(\mathrm{d}x)\right] \\
&\quad = \Gamma(1-1/\gamma) \int_0^{\infty} \mathrm{d}t\, \mathbb{E}\left[\frac{1}{S_t}\mathbf{1}_{\left\{S_t>1,\, t\geqslant 2\varepsilon S_t^{1-1/\gamma}\right\}} \left(1-\frac{S_{\varepsilon S_t^{1-1/\gamma}}}{S_t}\right)^{\alpha}\right].
\end{aligned} \tag{6.10}$$

Recall that S is a stable subordinator with index $1-1/\gamma$. Thus the process T defined by

$$T_r \coloneqq \frac{1}{\alpha} S_{\alpha^{1-1/\gamma} r}, \quad \forall r \geqslant 0$$

is distributed as S. Applying this, we get that

$$\alpha S\left(\varepsilon S_t^{1-1/\gamma}\right) \overset{(d)}{=} \alpha T\left(\varepsilon {T_t}^{1-1/\gamma}\right) = S\left(\varepsilon S^{1-1/\gamma}_{\alpha^{1-1/\gamma}t}\right). \tag{6.11}$$

Now notice that

$$\varepsilon S^{1-1/\gamma}_{\alpha^{1-1/\gamma}t} = \varepsilon\alpha^{1-1/\gamma}T_t^{1-1/\gamma} \overset{(d)}{=} \varepsilon\alpha^{1-1/\gamma}S_t^{1-1/\gamma}.$$

Since $\varepsilon\alpha^{1-1/\gamma} \to 0$, this clearly implies that $\varepsilon S^{1-1/\gamma}_{\alpha^{1-1/\gamma}t} \to 0$ in probability. As S is a.s. continuous at 0, we deduce that $S\left(\varepsilon S^{1-1/\gamma}_{\alpha^{1-1/\gamma}t}\right) \to 0$ in probability. Thus, it follows from (6.11) that $\alpha S\left(\varepsilon S_t^{1-1/\gamma}\right) \to 0$ in probability for every $t > 0$ and

$$\alpha \log\left(1 - \frac{S_{\varepsilon S_t^{1-1/\gamma}}}{S_t}\right) \sim -\alpha\frac{S\left(\varepsilon S_t^{1-1/\gamma}\right)}{S_t} \xrightarrow{\mathbb{P}} 0.$$

In particular, this implies the following convergence in probability for every $t > 0$

$$\frac{1}{S_t}\mathbf{1}_{\left\{S_t>1,\, t\geqslant 2\varepsilon S_t^{1-1/\gamma}\right\}}\left(1 - \frac{S_{\varepsilon S_t^{1-1/\gamma}}}{S_t}\right)^{\alpha} \to \frac{1}{S_t}\mathbf{1}_{\{S_t>1\}}.$$

Since we have the inequality

$$\frac{1}{S_t}\mathbf{1}_{\left\{S_t>1,\, t\geqslant 2\varepsilon S_t^{1-1/\gamma}\right\}}\left(1 - \frac{S_{\varepsilon S_t^{1-1/\gamma}}}{S_t}\right)^{\alpha} \leqslant \frac{1}{S_t}\mathbf{1}_{\{S_t>1\}}$$

where the right-hand side is integrable with respect to $\mathbf{1}_{(0,\infty)}(t)\,\mathrm{d}t\otimes\mathbb{P}$ thanks to (3.13), the dominated convergence theorem yields

$$\int_0^\infty \mathrm{d}t\,\mathbb{E}\left[\frac{1}{S_t}\mathbf{1}_{\left\{S_t>1,\, t\geqslant 2\varepsilon S_t^{1-1/\gamma}\right\}}\left(1 - \frac{S_{\varepsilon S_t^{1-1/\gamma}}}{S_t}\right)^{\alpha}\right] \to \int_0^\infty \mathrm{d}t\,\mathbb{E}\left[\frac{1}{S_t}\mathbf{1}_{\{S_t>1\}}\right] = \frac{1}{\Gamma(1-1/\gamma)}.$$

Together with (6.10) and the fact that

$$\int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}}\sigma^{\alpha}_{\varepsilon,x}\,\mu(\mathrm{d}x) \leqslant 1,$$

this proves (6.9).

Finally, since $\beta\varepsilon \to \infty$, it is clear that $\mathfrak{h}(1-\mathrm{e}^{-\beta\varepsilon/\mathfrak{h}}) \to \mathfrak{h}$ almost surely. In conjunction with (6.9), this gives the following convergence in $\mathbb{N}^{(1)}$-probability

$$\mathfrak{h}\left(1-\mathrm{e}^{-\beta\varepsilon/\mathfrak{h}}\right)\int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}}\sigma^{\alpha}_{\varepsilon,x}\,\mu(\mathrm{d}x) \to \mathfrak{h}.$$

Thus, using this together (6.7) and (6.8) yields $\lim_{\beta\to\infty} E_\beta = \mathfrak{h}$ in $\mathbb{N}^{(1)}$-probability. It follows from (6.2) and (6.6) that $\lim_{\beta\to\infty}\beta\mathfrak{h}^{-\beta}\mathbf{Z}_{\alpha,\beta} = \mathfrak{h}$ in $\mathbb{N}^{(1)}$-probability. This proves the first part of the theorem.

Next, we treat the case $\alpha^{1-1/\gamma}/\beta^{\rho} \to 0$ for some $\rho \in (0,1)$. The proof is similar and we only highlight the differences. Notice that there exists $p, q \in (0,1)$ and $\theta \in (0, \gamma/(\gamma-1))$ such that $(1+p)q > 1$ and $q\theta > \rho\gamma/(\gamma-1)$. Taking $\varepsilon = \beta^{-q}$, it is straighforward to check that $\varepsilon \to 0$, $\beta\varepsilon \to \infty$, $\beta\varepsilon^{1+p} \to 0$ and $\alpha\varepsilon^{\theta} \to 0$. As in the first part, we have that $\mathbb{N}^{(1)}$-a.s. $\lim_{\beta\to\infty} F^1_\beta + F^3_\beta + F^4_\beta = 0$.

Furthermore, using that $\sigma_{r,x} \leqslant 1$, it follows from (6.3) that

$$F^2_\beta \leqslant \beta\left(1-\frac{\varepsilon}{\mathfrak{h}}\right)^{\beta}\mathfrak{h} \leqslant \beta\mathrm{e}^{-\beta\varepsilon/\mathfrak{h}}\mathfrak{h} = \beta\mathrm{e}^{-\beta^{1-q}/\mathfrak{h}}.$$

This proves that $\mathbb{N}^{(1)}$-a.s. $\lim_{\beta\to\infty} F_\beta^2 = 0$.

Now we shall prove that $\mathbb{N}^{(1)}$-a.s. $\mu(\mathrm{d}x)$-a.s.

$$\lim_{\beta\to\infty} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}}\sigma_{\varepsilon,x}^\alpha = 1. \tag{6.12}$$

Using the same computation as in (6.10), we have the following identity in distribution

$$\begin{aligned}&\left(\mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}}\sigma_{\varepsilon,x}^\alpha,\, \varepsilon>0\right) \quad \text{under } \mathbb{N}^{(1)}\\ &\overset{(d)}{=} \quad \left(\mathbf{1}_{\left\{t\geqslant 2\varepsilon S_t^{1-1/\gamma}\right\}}\left(1-\frac{S\left(\varepsilon S_t^{1-1/\gamma}\right)}{S_t}\right)^\alpha,\, \varepsilon>0\right) \quad \text{under } \int_0^\infty \mathrm{d}t\, \mathbb{E}\left[\frac{1}{S_t}\mathbf{1}_{\{S_t>1\}}\bullet\right].\end{aligned} \tag{6.13}$$

Since $\theta<\gamma/(\gamma-1)$, [6, Chapter III, Theorem 9] guarantees that $\mathbb{P}$-a.s. $\limsup_{r\to 0} r^{-\theta}S_r = 0$. By composition, it follows that $\mathbb{P}$-a.s. for every $t>0$, $\lim_{\varepsilon\to 0}\varepsilon^{-\theta}S\left(\varepsilon S_t^{1-1/\gamma}\right)=0$. Thus we deduce that

$$\alpha\log\left(1-\frac{S\left(\varepsilon S_t^{1-1/\gamma}\right)}{S_t}\right)\sim -\alpha\frac{S\left(\varepsilon S_t^{1-1/\gamma}\right)}{S_t} = -\alpha\varepsilon^\theta\frac{\varepsilon^{-\theta}S\left(\varepsilon S_t^{1-1/\gamma}\right)}{S_t}\to 0$$

since $\alpha\varepsilon^\theta\to 0$. This proves that the process in the right-hand side of (6.13) goes to 1 $\mathbb{P}$-a.s. as $\varepsilon\to 0$, thus (6.12) follows.

Thanks to (6.12), since $\sigma_{\varepsilon,x}\leqslant 1$, a simple application of the dominated convergence theorem gives that $\mathbb{N}^{(1)}$-a.s.

$$\lim_{\beta\to\infty}\int_{\mathcal{T}} \mathbf{1}_{\{H(x)\geqslant 2\varepsilon\}}\sigma_{\varepsilon,x}^\alpha\,\mu(\mathrm{d}x) = 1.$$

This, together with the estimates (6.7) and (6.8) yields the $\mathbb{N}^{(1)}$-a.s. convergence $\lim_{\beta\to\infty} E_\beta = \mathfrak{h}$ which concludes the proof of the second part of the theorem. ☐

7 Technical lemmas

7.1 Proof of Lemma 4.3

Recall that $\mathfrak{g}(\varepsilon)=1-\mathfrak{f}(\varepsilon)$. Using the expression of $F(\varepsilon)$ from (4.7), we write

$$\begin{aligned}&\Gamma(1-1/\gamma)^{-1}F(\varepsilon) - \int_0^\infty \mathrm{d}t\,\mathbb{E}\Bigg[\frac{1}{\tau_{\mathfrak{g}(\varepsilon)t}}\mathbf{1}_{\left\{\tau_{\mathfrak{g}(\varepsilon)t}>1\right\}} f\circ\mathsf{norm}_\gamma\left(\mathsf{T}^{\downarrow}_{\mathfrak{g}(\varepsilon)t}\right) g\left(\tau_{\mathfrak{g}(\varepsilon)t}^{-1+1/\gamma}t\right)\\ &\times\exp\left\{-\sum_{s\leqslant \mathfrak{f}(\varepsilon)t}\Phi\left(\varepsilon^{-1}s/t, \varepsilon^{-\gamma/(\gamma-1)}\tau_{\mathfrak{g}(\varepsilon)t}^{-1}\mu(\mathsf{T}_s), \mathsf{norm}_\gamma\left(\mathsf{T}_s\right)\right)\right\}\Bigg] = \sum_{i=1}^4\int_0^\infty \mathrm{d}t\,\mathbb{E}\left[N_\varepsilon^i(t)\right],\end{aligned} \tag{7.1}$$

where

$$\begin{aligned}N_\varepsilon^1(t) = &\ \frac{1}{\tau_t}\mathbf{1}_{\{\tau_t>1\}}\left\{f\circ\mathsf{norm}_\gamma\left(\mathsf{T}_t^{\downarrow}\right) - f\circ\mathsf{norm}_\gamma\left(\mathsf{T}^{\downarrow}_{\mathfrak{g}(\varepsilon)t}\right)\right\} g\left(\tau_t^{-1+1/\gamma}t\right)\\ &\times\exp\left\{-\sum_{s\leqslant\mathfrak{f}(\varepsilon)t}\Phi\left(\varepsilon^{-1}s/t,\varepsilon^{-\gamma/(\gamma-1)}\tau_t^{-1}\mu(\mathsf{T}_s),\mathsf{norm}_\gamma\left(\mathsf{T}_s\right)\right)\right\},\\ N_\varepsilon^2(t) = &\ \frac{1}{\tau_t}\mathbf{1}_{\{\tau_t>1\}} f\circ\mathsf{norm}_\gamma\left(\mathsf{T}^{\downarrow}_{\mathfrak{g}(\varepsilon)t}\right)\left\{g\left(\tau_t^{-1+1/\gamma}t\right) - g\left(\tau_{\mathfrak{g}(\varepsilon)t}^{-1+1/\gamma}t\right)\right\}\\ &\times\exp\left\{-\sum_{s\leqslant\mathfrak{f}(\varepsilon)t}\Phi\left(\varepsilon^{-1}s/t,\varepsilon^{-\gamma/(\gamma-1)}\tau_t^{-1}\mu(\mathsf{T}_s),\mathsf{norm}_\gamma\left(\mathsf{T}_s\right)\right)\right\},\end{aligned}$$

$$\begin{aligned}N_\varepsilon^3(t) = &\frac{1}{\tau_t}\mathbf{1}_{\{\tau_t>1\}} f\circ \mathsf{norm}_\gamma\left(\mathsf{T}^\downarrow_{\mathfrak{g}(\varepsilon)t}\right) g\left(\tau_{\mathfrak{g}(\varepsilon)t}^{-1+1/\gamma}t\right)\\ &\times\left[\exp\left\{-\sum_{s\leqslant \mathfrak{f}(\varepsilon)t}\Phi\left(\varepsilon^{-1}s/t, \varepsilon^{-\gamma/(\gamma-1)}\tau_t^{-1}\mu(\mathsf{T}_s), \mathsf{norm}_\gamma\left(\mathsf{T}_s\right)\right)\right\}\right.\\ &\left.-\exp\left\{-\sum_{s\leqslant \mathfrak{f}(\varepsilon)t}\Phi\left(\varepsilon^{-1}s/t, \varepsilon^{-\gamma/(\gamma-1)}\tau_{\mathfrak{g}(\varepsilon)t}^{-1}\mu(\mathsf{T}_s), \mathsf{norm}_\gamma\left(\mathsf{T}_s\right)\right)\right\}\right],\\ N_\varepsilon^4(t) = &\left\{\frac{1}{\tau_t}\mathbf{1}_{\{\tau_t>1\}} - \frac{1}{\tau_{\mathfrak{g}(\varepsilon)t}}\mathbf{1}_{\left\{\tau_{\mathfrak{g}(\varepsilon)t}>1\right\}}\right\} f\circ \mathsf{norm}_\gamma\left(\mathsf{T}^\downarrow_{\mathfrak{g}(\varepsilon)t}\right) g\left(\tau_{\mathfrak{g}(\varepsilon)t}^{-1+1/\gamma}t\right)\\ &\times\exp\left\{-\sum_{s\leqslant \mathfrak{f}(\varepsilon)t}\Phi\left(\varepsilon^{-1}s/t, \varepsilon^{-\gamma/(\gamma-1)}\tau_{\mathfrak{g}(\varepsilon)t}^{-1}\mu(\mathsf{T}_s), \mathsf{norm}_\gamma\left(\mathsf{T}_s\right)\right)\right\}.\end{aligned}$$

Recall from (1.3) the definition of norm_γ and notice that since the total mass of $\mathsf{T}_t^\downarrow$ is τ_t, we have $\mathsf{norm}_\gamma(\mathsf{T}_t^\downarrow) = R_\gamma(\mathsf{T}_t^\downarrow, \tau_t^{-1+1/\gamma})$. It follows that

$$\begin{aligned}\left|N_\varepsilon^1(t)\right| \leqslant &\|f\|_{\mathrm{L}}\|g\|_\infty \frac{1}{\tau_t}\mathbf{1}_{\{\tau_t>1\}} d_{\mathrm{GHP}}\left(\mathsf{norm}_\gamma\left(\mathsf{T}_t^\downarrow\right), \mathsf{norm}_\gamma\left(\mathsf{T}^\downarrow_{\mathfrak{g}(\varepsilon)t}\right)\right)\\ \leqslant &\|f\|_{\mathrm{L}}\|g\|_\infty \mathbf{1}_{\{\tau_t>1\}}\left[d_{\mathrm{GHP}}\left(R_\gamma\left(\mathsf{T}_t^\downarrow, \tau_t^{-1+1/\gamma}\right), R_\gamma\left(\mathsf{T}^\downarrow_{\mathfrak{g}(\varepsilon)t}, \tau_t^{-1+1/\gamma}\right)\right)\right.\\ &\left.+ d_{\mathrm{GHP}}\left(R_\gamma\left(\mathsf{T}^\downarrow_{\mathfrak{g}(\varepsilon)t}, \tau_t^{-1+1/\gamma}\right), R_\gamma\left(\mathsf{T}^\downarrow_{\mathfrak{g}(\varepsilon)t}, \tau_{\mathfrak{g}(\varepsilon)t}^{-1+1/\gamma}\right)\right)\right],\end{aligned}$$

where $\|f\|_{\mathrm{L}}$ denotes the Lipschitz constant of f. Notice that, by construction, the tree $\mathsf{T}_t^\downarrow$ is obtained from $\mathsf{T}^\downarrow_{\mathfrak{g}(\varepsilon)t}$ by adding to the root a branch $[0, \mathfrak{f}(\varepsilon)t)$ onto which we graft T_s at height $0\leqslant s<\mathfrak{f}(\varepsilon)t$. It is clear that the added part has mass $\sum_{s<\mathfrak{f}(\varepsilon)t}\mu(\mathsf{T}_s) = S_{f(\varepsilon)t-}$ and height at most $\max_{s<\mathfrak{f}(\varepsilon)t}\mathfrak{h}(\mathsf{T}_s)+\mathfrak{f}(\varepsilon)t$. Thus, by definition (1.2) of the mapping R_γ, we deduce that

$$\begin{aligned}&d_{\mathrm{GHP}}\left(R_\gamma\left(\mathsf{T}_t^\downarrow, \tau_t^{-1+1/\gamma}\right), R_\gamma\left(\mathsf{T}^\downarrow_{\mathfrak{g}(\varepsilon)t}, \tau_t^{-1+1/\gamma}\right)\right)\\ &\qquad\leqslant \tau_t^{-1}S_{\mathfrak{f}(\varepsilon)t-} + \tau_t^{-1+1/\gamma}\left(\max_{s<\mathfrak{f}(\varepsilon)t}\mathfrak{h}(\mathsf{T}_s)+\mathfrak{f}(\varepsilon)t\right).\end{aligned}\tag{7.2}$$

Moreover, using Lemma 2.2 and again the definition of R_γ, we get

$$\begin{aligned}&d_{\mathrm{GHP}}\left(R_\gamma\left(\mathsf{T}^\downarrow_{\mathfrak{g}(\varepsilon)t}, \tau_t^{-1+1/\gamma}\right), R_\gamma\left(\mathsf{T}^\downarrow_{\mathfrak{g}(\varepsilon)t}, \tau_{\mathfrak{g}(\varepsilon)t}^{-1+1/\gamma}\right)\right)\\ &\qquad\leqslant 2\left(\tau_{\mathfrak{g}(\varepsilon)t}^{-1+1/\gamma} - \tau_t^{-1+1/\gamma}\right)\mathfrak{h}\left(\mathsf{T}^\downarrow_{\mathfrak{g}(\varepsilon)t}\right) + \left(\tau_{\mathfrak{g}(\varepsilon)t}^{-1} - \tau_t^{-1}\right)\mu\left(\mathsf{T}^\downarrow_{\mathfrak{g}(\varepsilon)t}\right).\end{aligned}\tag{7.3}$$

From (7.2) and (7.3), we deduce that

$$\begin{aligned}\left|N_\varepsilon^1(t)\right| \leqslant \|f\|_{\mathrm{L}}\|g\|_\infty&\left[S_{\mathfrak{f}(\varepsilon)t-} + \max_{s<\mathfrak{f}(\varepsilon)t}\mathfrak{h}(\mathsf{T}_s)+\mathfrak{f}(\varepsilon)t\right.\\ &\left.+2\left(\tau_{\mathfrak{g}(\varepsilon)t}^{-1+1/\gamma} - \tau_t^{-1+1/\gamma}\right)\mathfrak{h}\left(\mathsf{T}^\downarrow_{\mathfrak{g}(\varepsilon)t}\right) + \left(\tau_{\mathfrak{g}(\varepsilon)t}^{-1} - \tau_t^{-1}\right)\mu\left(\mathsf{T}^\downarrow_{\mathfrak{g}(\varepsilon)t}\right)\right].\end{aligned}$$

Therefore it follows that for every $t>0$ $\mathbb{P}$-a.s.

$$\lim_{\varepsilon\to 0} N_\varepsilon^1(t) = 0.\tag{7.4}$$

Furthermore, it is clear that

$$\left|N_\varepsilon^2(t)\right| \leqslant \|f\|_\infty\|g\|_{\mathrm{L}}\, t\left|\tau_t^{-1+1/\gamma} - \tau_{\mathfrak{g}(\varepsilon)t}^{-1+1/\gamma}\right|.$$

Thus, we have for every $t > 0$ $\mathbb{P}$-a.s.

$$\lim_{\varepsilon \to 0} N_\varepsilon^2(t) = 0. \tag{7.5}$$

Since

$$\left| N_\varepsilon^1(t) + N_\varepsilon^2(t) \right| \leqslant 4 \, \|f\|_\infty \, \|g\|_\infty \, \frac{1}{\tau_t} \mathbf{1}_{\{\tau_t > 1\}}$$

where the right-hand side is integrable with respect to $\mathbf{1}_{(0,\infty)}(t) \, \mathrm{d}t \otimes \mathbb{P}$ thanks to (3.13), it follows from (7.4) and (7.5) that

$$\lim_{\varepsilon \to 0} \int_0^\infty \mathrm{d}t \, \mathbb{E}\left[N_\varepsilon^1(t) + N_\varepsilon^2(t) \right] = 0. \tag{7.6}$$

Using the inequality $|\mathrm{e}^b - \mathrm{e}^a| \leqslant 1 \wedge |b - a|$ for $a \leqslant b \leqslant 0$, we have

$$\begin{aligned} \left| N_\varepsilon^3(t) \right| \leqslant \; & \|f\|_\infty \, \|g\|_\infty \, \frac{1}{\tau_t} \mathbf{1}_{\{\tau_t > 1\}} \left(1 \wedge \sum_{s \leqslant \mathfrak{f}(\varepsilon) t} \Big| \Phi\left(\varepsilon^{-1} s/t, \varepsilon^{-\gamma/(\gamma-1)} \tau_t^{-1} \mu(\mathsf{T}_s), \mathsf{norm}_\gamma\left(\mathsf{T}_s\right) \right) \right. \\ & \left. \qquad - \Phi\left(\varepsilon^{-1} s/t, \varepsilon^{-\gamma/(\gamma-1)} \tau_{\mathfrak{g}(\varepsilon)t}^{-1} \mu(\mathsf{T}_s), \mathsf{norm}_\gamma\left(\mathsf{T}_s\right) \right) \Big| \right) \\ \leqslant \; & \|f\|_\infty \, \|g\|_\infty \, \frac{1}{\tau_t} \mathbf{1}_{\{\tau_t > 1\}} \left(1 \wedge C \varepsilon^{-\gamma/(\gamma-1)} \left| \tau_t^{-1} - \tau_{\mathfrak{g}(\varepsilon)t}^{-1} \right| \sum_{s \leqslant \mathfrak{f}(\varepsilon) t} \mu(\mathsf{T}_s) \right) \\ = \; & \|f\|_\infty \, \|g\|_\infty \, \frac{1}{\tau_t} \mathbf{1}_{\{\tau_t > 1\}} \left(1 \wedge C \varepsilon^{-\gamma/(\gamma-1)} \frac{\left(\tau_t - \tau_{\mathfrak{g}(\varepsilon)t}\right)^2}{\tau_t \tau_{\mathfrak{g}(\varepsilon)t}} \right). \end{aligned} \tag{7.7}$$

Since τ is a stable subordinator with index $1 - 1/\gamma$, we get that

$$\varepsilon^{-\gamma/(\gamma-1)} \left(\tau_t - \tau_{\mathfrak{g}(\varepsilon)t} \right)^2 \overset{(d)}{=} \varepsilon^{-\gamma/(\gamma-1)} \tau_{\mathfrak{f}(\varepsilon)t}^2 \overset{(d)}{=} \left(\varepsilon^{-1} \mathfrak{f}(\varepsilon)^2 \right)^{\gamma/(\gamma-1)} \tau_t^2 \xrightarrow[\varepsilon \to 0]{(d)} 0$$

as $\varepsilon^{-1} \mathfrak{f}(\varepsilon)^2 \to 0$. We deduce the following convergence in $\mathbb{P}$-probability

$$\lim_{\varepsilon \to 0} \frac{1}{\tau_t} \mathbf{1}_{\{\tau_t > 1\}} \left(1 \wedge C \varepsilon^{-\gamma/(\gamma-1)} \frac{\left(\tau_t - \tau_{\mathfrak{g}(\varepsilon)t}\right)^2}{\tau_t \tau_{\mathfrak{g}(\varepsilon)t}} \right) = 0.$$

Thanks to (3.13), it follows from the dominated convergence theorem that

$$\lim_{\varepsilon \to 0} \int_0^\infty \mathrm{d}t \, \mathbb{E}\left[\frac{1}{\tau_t} \mathbf{1}_{\{\tau_t > 1\}} \left(1 \wedge C \varepsilon^{-\gamma/(\gamma-1)} \frac{\left(\tau_t - \tau_{\mathfrak{g}(\varepsilon)t}\right)^2}{\tau_t \tau_{\mathfrak{g}(\varepsilon)t}} \right) \right] = 0.$$

Together with (7.7), this gives

$$\lim_{\varepsilon \to 0} \int_0^\infty \mathrm{d}t \, \mathbb{E}\left[N_\varepsilon^3(t) \right] = 0. \tag{7.8}$$

Finally, notice that

$$\left| \int_0^\infty \mathrm{d}t \, \mathbb{E}\left[N_\varepsilon^4(t) \right] \right| \leqslant \|f\|_\infty \, \|g\|_\infty \int_0^\infty \mathrm{d}t \, \mathbb{E}\left[\frac{1}{\tau_t} \mathbf{1}_{\left\{\tau_{\mathfrak{g}(\varepsilon)t} \leqslant 1 < \tau_t\right\}} + \frac{\tau_t - \tau_{\mathfrak{g}(\varepsilon)t}}{\tau_t \tau_{\mathfrak{g}(\varepsilon)t}} \mathbf{1}_{\left\{\tau_{\mathfrak{g}(\varepsilon)t} > 1\right\}} \right]. \tag{7.9}$$

Thanks to (3.13) and the dominated convergence theorem, it is clear that

$$\lim_{\varepsilon \to 0} \int_0^\infty \mathrm{d}t \, \mathbb{E}\left[\frac{1}{\tau_t} \mathbf{1}_{\left\{\tau_{\mathfrak{g}(\varepsilon)t} \leqslant 1 < \tau_t\right\}} \right] = 0 \tag{7.10}$$

as the process τ is a.s. continuous at t. On the other hand, using the inequality

$$\frac{\tau_t - \tau_{\mathfrak{g}(\varepsilon)t}}{\tau_t \tau_{\mathfrak{g}(\varepsilon)t}} \mathbf{1}_{\{\tau_{\mathfrak{g}(\varepsilon)t}>1\}} \leqslant \left(\frac{\tau_t - \tau_{\mathfrak{g}(\varepsilon)t}}{\tau_t}\right)^{1-q} \frac{\left(\tau_t - \tau_{\mathfrak{g}(\varepsilon)t}\right)^q}{\tau_{\mathfrak{g}(\varepsilon)t}^{1+q}} \mathbf{1}_{\{\tau_{\mathfrak{g}(\varepsilon)t}>1\}}$$
$$\leqslant \frac{\left(\tau_t - \tau_{\mathfrak{g}(\varepsilon)t}\right)^q}{\tau_{\mathfrak{g}(\varepsilon)t}^{1+q}} \mathbf{1}_{\{\tau_{\mathfrak{g}(\varepsilon)t}>1\}}$$

where $q \in (0, 1-1/\gamma)$, we get that

$$\begin{aligned}
\int_0^\infty \mathrm{d}t\, \mathbb{E}\left[\frac{\tau_t - \tau_{\mathfrak{g}(\varepsilon)t}}{\tau_t \tau_{\mathfrak{g}(\varepsilon)t}} \mathbf{1}_{\{\tau_{\mathfrak{g}(\varepsilon)t}>1\}}\right] &\leqslant \int_0^\infty \mathrm{d}t\, \mathbb{E}\left[\frac{\left(\tau_t - \tau_{\mathfrak{g}(\varepsilon)t}\right)^q}{\tau_{\mathfrak{g}(\varepsilon)t}^{1+q}} \mathbf{1}_{\{\tau_{\mathfrak{g}(\varepsilon)t}>1\}}\right] \\
&= \int_0^\infty \mathrm{d}t\, \mathbb{E}\left[\tau_{\mathfrak{f}(\varepsilon)t}^q\right] \mathbb{E}\left[\frac{1}{\tau_{\mathfrak{g}(\varepsilon)t}^{1+q}} \mathbf{1}_{\{\tau_{\mathfrak{g}(\varepsilon)t}>1\}}\right] \\
&= \frac{\mathfrak{f}(\varepsilon)^{q\gamma/(\gamma-1)}}{\mathfrak{g}(\varepsilon)^{1+q\gamma/(\gamma-1)}} \mathbb{E}\left[\tau_1^q\right] \int_0^\infty \mathrm{d}r\, r^{q\gamma/(\gamma-1)} \mathbb{E}\left[\frac{1}{\tau_r^{1+q}} \mathbf{1}_{\{\tau_r>1\}}\right] \\
&= \frac{\mathfrak{f}(\varepsilon)^{q\gamma/(\gamma-1)}}{\mathfrak{g}(\varepsilon)^{1+q\gamma/(\gamma-1)}} \mathbb{E}\left[\tau_1^q\right] \mathbb{E}\left[\frac{1}{\tau_1^{1+q}} \int_{\tau_1^{-1+1/\gamma}}^\infty \mathrm{d}r\, r^{-\gamma/(\gamma-1)}\right] \\
&= \frac{\mathfrak{f}(\varepsilon)^{q\gamma/(\gamma-1)}}{\mathfrak{g}(\varepsilon)^{1+q\gamma/(\gamma-1)}} \mathbb{E}\left[\tau_1^q\right] \mathbb{E}\left[\tau_1^{-1-q+1/\gamma}\right], \qquad (7.11)
\end{aligned}$$

where we used that $\tau_t - \tau_{\mathfrak{g}(\varepsilon)t}$ is independent of $\tau_{\mathfrak{g}(\varepsilon)t}$ and is distributed as $\tau_{\mathfrak{f}(\varepsilon)t}$ for the first equality and that $\tau_t \overset{(d)}{=} t^{\gamma/(\gamma-1)}\tau_1$ for the second. Thanks to (3.11), we have $\mathbb{E}\left[\tau_1^q\right] < \infty$ and $\mathbb{E}\left[\tau_1^{-1+1/\gamma-q}\right] < \infty$. Thus, it follows from (7.11) that

$$\lim_{\varepsilon\to 0} \int_0^\infty \mathrm{d}t\, \mathbb{E}\left[\frac{\tau_t - \tau_{\mathfrak{g}(\varepsilon)t}}{\tau_t \tau_{\mathfrak{g}(\varepsilon)t}} \mathbf{1}_{\{\tau_{\mathfrak{g}(\varepsilon)t}>1\}}\right] = 0. \qquad (7.12)$$

Combining (7.9), (7.10) and (7.12), we deduce that

$$\lim_{\varepsilon\to 0} \int_0^\infty \mathrm{d}t\, \mathbb{E}\left[\left|N_\varepsilon^4(t)\right|\right] = 0. \qquad (7.13)$$

It follows from (7.1), (7.6), (7.8) and (7.13) that

$$\lim_{\varepsilon\to 0} \Gamma(1-1/\gamma)^{-1} F(\varepsilon) - \int_0^\infty \mathrm{d}t\, \mathbb{E}\Bigg[\frac{1}{\tau_{\mathfrak{g}(\varepsilon)t}} \mathbf{1}_{\{\tau_{\mathfrak{g}(\varepsilon)t}>1\}} f\circ R\left(\mathsf{T}^{\downarrow}_{\mathfrak{g}(\varepsilon)t}, \tau_{\mathfrak{g}(\varepsilon)t}^{-1}\right) g\left(\tau_{\mathfrak{g}(\varepsilon)t}^{-1+1/\gamma} t\right)$$
$$\times \exp\left\{-\sum_{s\leqslant \mathfrak{f}(\varepsilon)t} \Phi\left(\varepsilon^{-1}s/t, \varepsilon^{-\gamma/(\gamma-1)}\tau_{\mathfrak{g}(\varepsilon)t}^{-1}\mu(\mathsf{T}_s), R\left(\mathsf{T}_s, \mu(\mathsf{T}_s)^{-1}\right)\right)\right\}\Bigg] = 0.$$

7.2 Proof of Lemma 5.2

Recall from (5.3) the definition of I_α. Write $\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta} Z_{\alpha,\beta}(U) - I_\alpha = \sum_{i=1}^4 J_\alpha^i$ where

$$J_\alpha^1 = \alpha^{1-1/\gamma}\mathfrak{h}^{-\beta} \int_{\varepsilon H(U)}^{H(U)} \sigma_{r,U}^\alpha \mathfrak{h}_{r,U}^\beta\, \mathrm{d}r,$$
$$J_\alpha^2 = \alpha^{1-1/\gamma} \int_0^{\varepsilon H(U)} \sigma_{r,U}^\alpha \left\{\left(\frac{\mathfrak{h}_{r,U}}{\mathfrak{h}}\right)^\beta - \left(1-\frac{r}{\mathfrak{h}}\right)^\beta\right\} \mathrm{d}r,$$
$$J_\alpha^3 = \alpha^{1-1/\gamma} \int_0^{\varepsilon H(U)} \sigma_{r,U}^\alpha \left\{\left(1-\frac{r}{\mathfrak{h}}\right)^\beta - \mathrm{e}^{-\beta r/\mathfrak{h}}\right\} \mathrm{d}r,$$

$$J^4_\alpha = \alpha^{1-1/\gamma} \int_0^{\varepsilon H(U)} \left\{ \sigma^\alpha_{r,U} - \mathrm{e}^{-\alpha(1-\sigma_{r,U})} \right\} \mathrm{e}^{-\beta r/\mathfrak{h}} \, \mathrm{d}r.$$

We shall prove that for every $1 \leqslant i \leqslant 4$, $\lim_{\alpha\to\infty} J^i_\alpha = 0$ in $\mathbb{N}^{(1)}$-probability.

We start by showing that $\mathbb{N}^{(1)}$-a.s. $\mu(\mathrm{d}x)$-a.s.

$$\lim_{\alpha\to\infty} \alpha^{1-1/\gamma} \int_{\varepsilon H(x)}^{H(x)} \sigma^\alpha_{r,x} \, \mathrm{d}r = 0. \tag{7.14}$$

Recall from (3.9) the definition of S. Using Lemma 3.5-(i) and Bismut's decomposition (3.12), we have

$$\begin{aligned}
&\Gamma(1-1/\gamma)^{-1}\, \mathbb{N}^{(1)} \left[\mu\left(x \in \mathcal{T} \colon \limsup_{\alpha\to\infty} \alpha^{1-1/\gamma} \int_{\varepsilon H(x)}^{H(x)} \sigma^\alpha_{r,x} \, \mathrm{d}r > 0 \right) \right] \\
&\quad = \mathbb{N} \left[\frac{1}{\sigma} \mathbf{1}_{\{\sigma>1\}} \mu \left(x \in \mathcal{T} \colon \limsup_{\alpha\to\infty} \left(\frac{\alpha}{\sigma}\right)^{1-1/\gamma} \int_{\varepsilon H(x)}^{H(x)} \left(\frac{\sigma_{r,x}}{\sigma} \right)^\alpha \mathrm{d}r > 0 \right) \right] \\
&\quad = \int_0^\infty \mathrm{d}t \, \mathbb{E} \left[\frac{1}{\tau_t} \mathbf{1}_{\{\tau_t>1\}} ; \limsup_{\alpha\to\infty} \left(\frac{\alpha}{\tau_t} \right)^{1-1/\gamma} \int_{\varepsilon t}^t \left(1 - \frac{S_r}{\tau_t} \right)^\alpha \mathrm{d}r > 0 \right].
\end{aligned} \tag{7.15}$$

Let $t > 0$. It is clear that

$$\int_{\varepsilon t}^t \left(1 - \frac{S_r}{\tau_t} \right)^\alpha \mathrm{d}r \leqslant \int_{\varepsilon t}^t \mathrm{e}^{-\alpha S_r/\tau_t} \, \mathrm{d}r \leqslant t \mathrm{e}^{-\alpha S_{\varepsilon t}/\tau_t}. \tag{7.16}$$

According to [6, Chapter III, Theorem 11], we have that $\mathbb{P}$-a.s.

$$\liminf_{\varepsilon\to 0} \frac{S_{\varepsilon t}}{h(\varepsilon t)} = \gamma - 1 > 0,$$

where $h(r) = r^{\gamma/(\gamma-1)} \log\left(|\log r|\right)^{-1/(\gamma-1)}$. As a consequence, there exist a positive random variable $\rho = \rho(\omega)$ and a constant $c > 0$ such that $\mathbb{P}$-a.s. $S_{\varepsilon t} \geqslant c h(\varepsilon t)$ for every $\varepsilon \in (0, \rho)$. We deduce that for every $t > 0$, $\mathbb{P}$-a.s.

$$\begin{aligned}
\limsup_{\alpha\to\infty} \alpha^{1-1/\gamma} \mathrm{e}^{-\alpha S_{\varepsilon t}/\tau_t} &\leqslant \limsup_{\alpha\to\infty} \alpha^{1-1/\gamma} \mathrm{e}^{-c\alpha h(\varepsilon t)/\tau_t} \\
&= \limsup_{\alpha\to\infty} \alpha^{1-1/\gamma} \mathrm{e}^{-c t^{\gamma/(\gamma-1)} \alpha^\delta \log(|\log(\varepsilon t)|)^{-1}/\tau_t} = 0,
\end{aligned}$$

where in the second to last equality we used (5.2). In conjunction with (7.15) and (7.16), this yields (7.14).

Let $\eta > 0$. Using that $\mathfrak{h}_{r,U} \leqslant \mathfrak{h}$, we have

$$\begin{aligned}
\limsup_{\alpha\to\infty} \mathbb{N}^{(1)} \left[J^1_\alpha > \eta \right] &\leqslant \limsup_{\alpha\to\infty} \mathbb{N}^{(1)} \left[\alpha^{1-1/\gamma} \int_{\varepsilon H(U)}^{H(U)} \sigma^\alpha_{r,U} \, \mathrm{d}r > \eta \right] \\
&= \limsup_{\alpha\to\infty} \mathbb{N}^{(1)} \left[\mu\left(x \in \mathcal{T} \colon \alpha^{1-1/\gamma} \int_{\varepsilon H(x)}^{H(x)} \sigma^\alpha_{r,x} \, \mathrm{d}r > \eta \right) \right],
\end{aligned}$$

where the last term vanishes thanks to (7.14) and the dominated convergence theorem. This gives that $\lim_{\alpha\to\infty} J^1_\alpha = 0$ in $\mathbb{N}^{(1)}$-probability.

Under $\mathbb{N}^{(1)}$, let x^* be the unique leaf realizing the total height, that is the unique $x \in \mathcal{T}$ such that $H(x) = \mathfrak{h}$. Then $\mathbb{N}^{(1)}$-a.s. we have $H(U \wedge x^*) > 0$ and, thanks to (2.6), $\mathfrak{h}_{r,U} = \mathfrak{h} - r$ for every $r \in [0, \varepsilon H(U)]$ if $\varepsilon > 0$ is small enough (more precisely for $\varepsilon \leqslant H(U \wedge x^*)/H(U)$). In particular, this implies that $\mathbb{N}^{(1)}$-a.s. $\lim_{\alpha\to\infty} J^2_\alpha = 0$.

Next, we have

$$\begin{aligned}
|J_\alpha^3| &\leqslant \alpha^{1-1/\gamma}\int_0^{\varepsilon H(U)}\sigma_{r,U}^\alpha\left|\left(1-\frac{r}{\mathfrak{h}}\right)^\beta-\mathrm{e}^{-\beta r/\mathfrak{h}}\right|\mathrm{d}r\\
&\leqslant \alpha^{1-1/\gamma}\beta\int_0^{\varepsilon H(U)}\sigma_{r,U}^\alpha\left|\log\left(1-\frac{r}{\mathfrak{h}}\right)+\frac{r}{\mathfrak{h}}\right|\mathrm{e}^{-\beta r/\mathfrak{h}}\,\mathrm{d}r\\
&\leqslant \alpha^{1-1/\gamma}\beta j\left(\frac{\varepsilon H(U)}{\mathfrak{h}}\right)\int_0^{\varepsilon H(U)}\sigma_{r,U}^\alpha\frac{r^2}{\mathfrak{h}^2}\mathrm{e}^{-\beta r/\mathfrak{h}}\,\mathrm{d}r\\
&\leqslant CH(U)j\left(\varepsilon\right)\varepsilon^3\alpha^{2(1-1/\gamma)},
\end{aligned}$$

where we used that $|\mathrm{e}^b-\mathrm{e}^a|\leqslant|b-a|\mathrm{e}^b$ for $a\leqslant b$ for the second inequality, that the function $j\colon y\mapsto-(y+\log(1-y))/y^2$ is increasing on $[0,1)$ for the third and the fact that $H(U)\leqslant\mathfrak{h}$ and $\beta/\alpha^{1-1/\gamma}$ is bounded by some constant $C>0$ for the last. Using (5.2), notice that $\varepsilon^3\alpha^{2(1-1/\gamma)}=\alpha^{(3\delta-1)(1-1/\gamma)}\to0$ as $\delta<1/3$. Since $\lim_{y\to0}j(y)=1/2$, we deduce that $\mathbb{N}^{(1)}$-a.s. $\lim_{\alpha\to\infty}J_\alpha^3=0$.

Finally, we have

$$\begin{aligned}
|J_\alpha^4| &\leqslant \alpha^{2-1/\gamma}\int_0^{\varepsilon H(U)}\left|\log\left(\sigma_{r,U}\right)+1-\sigma_{r,U}\right|\mathrm{e}^{-\alpha(1-\sigma_{r,U})}\,\mathrm{d}r\\
&\leqslant j\left(1-\sigma_{\varepsilon H(U),U}\right)\alpha^{2-1/\gamma}\int_0^{\varepsilon H(U)}\left(1-\sigma_{r,U}\right)^2\mathrm{e}^{-\alpha(1-\sigma_{r,U})}\,\mathrm{d}r\\
&\leqslant CH(U)j\left(1-\sigma_{\varepsilon H(U),U}\right)\alpha^{-1/\gamma}\varepsilon,
\end{aligned}$$

where we used that $|\mathrm{e}^b-\mathrm{e}^a|\leqslant|b-a|\mathrm{e}^b$ for $a\leqslant b$ for the first inequality, that the function $j\colon x\mapsto-(x+\log(1-x))/x^2$ is increasing on $[0,1)$ for the second and that the function $x\mapsto x^2\mathrm{e}^{-x}$ is bounded on $[0,\infty)$ for the last. Since $\lim_{x\to0}j(x)=1/2$, $\lim_{\varepsilon\to0}\sigma_{\varepsilon H(U),U}=1$ and $\alpha^{-1/\gamma}\varepsilon\to0$, we deduce that $\mathbb{N}^{(1)}$-a.s. $\lim_{\alpha\to\infty}J_\alpha^4=0$.

7.3 Proof of Lemma 5.3

It is enough to show that for every Lipschitz-continuous and bounded function $f\colon[0,\infty)\to\mathbb{R}$

$$\lim_{\alpha\to\infty}\mathbb{N}^{(1)}\left[\int_{\mathcal{T}}\mu(\mathrm{d}x)f\left(\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\left(Z_{\alpha,\beta}(x)-\mathbf{Z}_{\alpha,\beta}\right)\right)\right]=f(0).$$

Let $\varepsilon=\alpha^{(\delta-1)(1-1/\gamma)}$ with $\delta\in(0,1/2)$. For every $x\in\mathcal{T}$ such that $H(x)\geqslant\varepsilon$, set

$$Z_{\alpha,\beta}^\varepsilon(x)=\int_0^\varepsilon\sigma_{r,x}^\alpha\mathfrak{h}_{r,x}^\beta\,\mathrm{d}r\quad\text{and}\quad\mathbf{Z}_{\alpha,\beta}^\varepsilon=\int_{\mathcal{T}}\mathbf{1}_{\{H(x)\geqslant\varepsilon\}}Z_{\alpha,\beta}^\varepsilon(x)\,\mu(\mathrm{d}x).$$

Let $x^*\in\mathcal{T}$ be the unique leaf realizing the height, that is $H(x^*)=\mathfrak{h}$. Using that $\mathfrak{h}\geqslant H(x\wedge x^*)$ and that $Z_{\alpha,\beta}^\varepsilon(x)=Z_{\alpha,\beta}^\varepsilon(x^*)$ if $\varepsilon\leqslant H(x\wedge x^*)$, write

$$\int_{\mathcal{T}}\mu(\mathrm{d}x)f\left(\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\left(Z_{\alpha,\beta}(x)-\mathbf{Z}_{\alpha,\beta}\right)\right)=\sum_{i=1}^4A_\alpha^i+B_\alpha,$$

where

$$\begin{aligned}
A_\alpha^1 &= \int_{\mathcal{T}}\mu(\mathrm{d}x)\mathbf{1}_{\{H(x\wedge x^*)<\varepsilon\}}f\left(\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\left(Z_{\alpha,\beta}(x)-\mathbf{Z}_{\alpha,\beta}\right)\right),\\
A_\alpha^2 &= \int_{\mathcal{T}}\mu(\mathrm{d}x)\mathbf{1}_{\{H(x\wedge x^*)\geqslant\varepsilon\}}\Big\{f\left(\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\left(Z_{\alpha,\beta}(x)-\mathbf{Z}_{\alpha,\beta}\right)\right)\\
&\qquad\qquad -f\left(\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\left(Z_{\alpha,\beta}^\varepsilon(x)-\mathbf{Z}_{\alpha,\beta}\right)\right)\Big\},
\end{aligned}$$

$$\begin{aligned}A_\alpha^3 = & \int_{\mathcal{T}} \mu(\mathrm{d}x) \mathbf{1}_{\{H(x\wedge x^*)\geqslant\varepsilon\}} \Big\{ f\left(\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\left(Z^\varepsilon_{\alpha,\beta}(x) - \mathbf{Z}_{\alpha,\beta}\right)\right) \\ & \qquad\qquad - f\left(\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\left(Z^\varepsilon_{\alpha,\beta}(x) - \mathbf{Z}^\varepsilon_{\alpha,\beta}\right)\right)\Big\}, \\ A_\alpha^4 = & -\mu\left(\{x\in\mathcal{T}:\ H(x\wedge x^*) < \varepsilon\}\right) f\left(\mathbf{1}_{\{\mathfrak{h}\geqslant\varepsilon\}}\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\left(Z^\varepsilon_{\alpha,\beta}(x^*) - \mathbf{Z}^\varepsilon_{\alpha,\beta}\right)\right), \\ B_\alpha = & f\left(\mathbf{1}_{\{\mathfrak{h}\geqslant\varepsilon\}}\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\left(Z^\varepsilon_{\alpha,\beta}(x^*) - \mathbf{Z}^\varepsilon_{\alpha,\beta}\right)\right).\end{aligned}$$

Thanks to the dominated convergence theorem, we have

$$\lim_{\alpha\to\infty} \mathbb{N}^{(1)}[|A_\alpha^1 + A_\alpha^4|] \leqslant 2\|f\|_\infty \lim_{\alpha\to\infty} \mathbb{N}^{(1)}\left[\int_{\mathcal{T}} \mu(\mathrm{d}x)\mathbf{1}_{\{H(x\wedge x^*)<\varepsilon\}}\right] = 0. \tag{7.17}$$

Next, notice that

$$\begin{aligned}\mathbb{N}^{(1)}[|A_\alpha^2|] &\leqslant \|f\|_{\mathrm{L}}\, \mathbb{N}^{(1)}\left[\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\int_{\mathcal{T}}\mu(\mathrm{d}x)\mathbf{1}_{\{H(x\wedge x^*)\geqslant\varepsilon)\}}\left(Z_{\alpha,\beta}(x) - Z^\varepsilon_{\alpha,\beta}(x)\right)\right] \\ &\leqslant \|f\|_{\mathrm{L}}\, \mathbb{N}^{(1)}\left[\alpha^{1-1/\gamma}\int_{\mathcal{T}}\mu(\mathrm{d}x)\mathbf{1}_{\{H(x)\geqslant\varepsilon\}}\int_\varepsilon^{H(x)}\sigma^\alpha_{r,x}\,\mathrm{d}r\right],\end{aligned} \tag{7.18}$$

where we used that $H(x\wedge x^*) \leqslant H(x)$ and $\mathfrak{h}_{r,x} \leqslant \mathfrak{h}$ for the second inequality. Now similarly to (7.14), we have $\mathbb{N}^{(1)}$-a.s. $\mu(\mathrm{d}x)$-a.s.

$$\lim_{\alpha\to\infty}\alpha^{1-1/\gamma}\mathbf{1}_{\{H(x)\geqslant\varepsilon\}}\int_\varepsilon^{H(x)}\sigma^\alpha_{r,x}\,\mathrm{d}r = 0. \tag{7.19}$$

Furthermore, applying Corollary 3.7, we have

$$\begin{aligned}\sup_{\alpha\geqslant 0}\alpha^{2-2/\gamma}\,\mathbb{N}^{(1)}&\left[\int_{\mathcal{T}}\mu(\mathrm{d}x)\left(\mathbf{1}_{\{H(x)\geqslant\varepsilon\}}\int_\varepsilon^{H(x)}\sigma^\alpha_{r,x}\,\mathrm{d}r\right)^2\right] \\ &\leqslant \sup_{\alpha\geqslant 0}\alpha^{2-2/\gamma}\,\mathbb{N}^{(1)}\left[\int_{\mathcal{T}}\mu(\mathrm{d}x)\left(\int_0^{H(x)}\sigma^\alpha_{r,x}\,\mathrm{d}r\right)^2\right] < \infty.\end{aligned}$$

We deduce that the family

$$\left(\alpha^{1-1/\gamma}\mathbf{1}_{\{H(x)\geqslant\varepsilon\}}\int_\varepsilon^{H(x)}\sigma^\alpha_{r,x}\,\mathrm{d}r:\ \alpha\geqslant 0\right)$$

is uniformly integrable under the measure $\mathbb{N}^{(1)}[\mathrm{d}\mathcal{T}]\mu(\mathrm{d}x)$. In conjunction with (7.19), this gives

$$\lim_{\alpha\to\infty}\mathbb{N}^{(1)}\left[\alpha^{1-1/\gamma}\int_{\mathcal{T}}\mathbf{1}_{\{H(x)\geqslant\varepsilon\}}\,\mu(\mathrm{d}x)\int_\varepsilon^{H(x)}\sigma^\alpha_{r,x}\,\mathrm{d}r\right] = 0, \tag{7.20}$$

which, thanks to (7.18), implies that

$$\lim_{\alpha\to\infty}\mathbb{N}^{(1)}[|A_\alpha^2|] = 0. \tag{7.21}$$

We have

$$\begin{aligned}\mathbb{N}^{(1)}[|A_\alpha^3|] &\leqslant \|f\|_{\mathrm{L}}\,\mathbb{N}^{(1)}\left[\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\int_{\mathcal{T}}\mu(\mathrm{d}x)\mathbf{1}_{\{H(x\wedge x^*)\geqslant\varepsilon\}}\left(\mathbf{Z}_{\alpha,\beta} - \mathbf{Z}^\varepsilon_{\alpha,\beta}\right)\right] \\ &\leqslant \|f\|_{\mathrm{L}}\,\mathbb{N}^{(1)}\left[\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\left(\mathbf{Z}_{\alpha,\beta} - \mathbf{Z}^\varepsilon_{\alpha,\beta}\right)\right]\end{aligned}$$

$$\leqslant \|f\|_{\mathrm{L}}\,\mathbb{N}^{(1)}\left[\alpha^{1-1/\gamma}\int_{\mathcal{T}}\mathbf{1}_{\{H(x)\geqslant\varepsilon\}}\,\mu(\mathrm{d}x)\int_{\varepsilon}^{H(x)}\sigma^{\alpha}_{r,x}\,\mathrm{d}r\right]$$
$$+\|f\|_{\mathrm{L}}\,\mathbb{N}^{(1)}\left[\alpha^{1-1/\gamma}\int_{\mathcal{T}}\mathbf{1}_{\{H(x)<\varepsilon\}}\,\mu(\mathrm{d}x)\int_{0}^{H(x)}\sigma^{\alpha}_{r,x}\,\mathrm{d}r\right], \tag{7.22}$$

where we used that $\mathfrak{h}_{r,x}\leqslant\mathfrak{h}$ for the last inequality. Let $p\in(1,2)$ and notice that $\varepsilon^{1+p}\alpha^{1-1/\gamma}\to 0$. Using that $\sigma_{r,x}\leqslant 1$ together with the Markov inequality, we get

$$\mathbb{N}^{(1)}\left[\alpha^{1-1/\gamma}\int_{\mathcal{T}}\mathbf{1}_{\{H(x)<\varepsilon\}}\,\mu(\mathrm{d}x)\int_{0}^{H(x)}\sigma^{\alpha}_{r,x}\,\mathrm{d}r\right]\leqslant\mathbb{N}^{(1)}\left[\varepsilon\alpha^{1-1/\gamma}\int_{\mathcal{T}}\mathbf{1}_{\{H(x)<\varepsilon\}}\,\mu(\mathrm{d}x)\right]$$
$$\leqslant\varepsilon^{1+p}\alpha^{1-1/\gamma}\,\mathbb{N}^{(1)}\left[\int_{\mathcal{T}}H(x)^{-p}\,\mu(\mathrm{d}x)\right].$$

By Lemma 3.9, the last term is finite. This, in conjunction with (7.20) and (7.22), implies that

$$\lim_{\alpha\to\infty}\mathbb{N}^{(1)}[|A^3_\alpha|]=0. \tag{7.23}$$

It remains to show that $\lim_{\alpha\to\infty}\mathbb{N}^{(1)}[B_\alpha]=f(0)$, which is equivalent to the following convergence in $\mathbb{N}^{(1)}$-probability

$$\lim_{\alpha\to\infty}\mathbf{1}_{\{\mathfrak{h}\geqslant\varepsilon\}}\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\left(Z^{\varepsilon}_{\alpha,\beta}(x^*)-\mathbf{Z}^{\varepsilon}_{\alpha,\beta}\right)=0. \tag{7.24}$$

Again using that $Z^{\varepsilon}_{\alpha,\beta}(x)=Z^{\varepsilon}_{\alpha,\beta}(x^*)$ if $\varepsilon\leqslant H(x\wedge x^*)$, we write

$$\mathbf{1}_{\{\mathfrak{h}\geqslant\varepsilon\}}\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\left(Z^{\varepsilon}_{\alpha,\beta}(x^*)-\mathbf{Z}^{\varepsilon}_{\alpha,\beta}\right)=B^1_\alpha+B^2_\alpha,$$

where

$$B^1_\alpha=\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\left(\mathbf{1}_{\{\mathfrak{h}\geqslant\varepsilon\}}Z^{\varepsilon}_{\alpha,\beta}(x^*)-\int_{\mathcal{T}}\mu(\mathrm{d}x)\mathbf{1}_{\{H(x\wedge x^*)\geqslant\varepsilon\}}Z^{\varepsilon}_{\alpha,\beta}(x^*)\right),$$
$$B^2_\alpha=\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}\left(\int_{\mathcal{T}}\mu(\mathrm{d}x)\mathbf{1}_{\{H(x\wedge x^*)\geqslant\varepsilon\}}Z^{\varepsilon}_{\alpha,\beta}(x)-\mathbf{1}_{\{\mathfrak{h}\geqslant\varepsilon\}}\mathbf{Z}^{\varepsilon}_{\alpha,\beta}\right).$$

Recall that $\varepsilon=\alpha^{(\delta-1)(1-1/\gamma)}\to 0$ as $\alpha\to\infty$. Fix $\eta>0$ and let $\alpha_0>0$ be large enough so that for every $\alpha\geqslant\alpha_0$

$$\mathbb{N}^{(1)}\left[\int_{\mathcal{T}}\mu(\mathrm{d}x)\mathbf{1}_{\{H(x\wedge x^*)<\varepsilon\}}\right]\leqslant\eta.$$

Then we have for every $\alpha\geqslant\alpha_0$ and $C>0$

$$\begin{aligned}
&\mathbb{N}^{(1)}\left[\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}Z^{\varepsilon}_{\alpha,\beta}(x^*)\mathbf{1}_{\{\mathfrak{h}\geqslant\varepsilon\}}\geqslant C\right]\\
&\quad\leqslant\mathbb{N}^{(1)}\left[\int_{\mathcal{T}}\mu(\mathrm{d}x)\mathbf{1}_{\left\{\alpha^{1-1/\gamma}\mathfrak{h}^{-\beta}Z^{\varepsilon}_{\alpha,\beta}(x)\geqslant C,\,H(x\wedge x^*)\geqslant\varepsilon\right\}}\right]+\mathbb{N}^{(1)}\left[\int_{\mathcal{T}}\mu(\mathrm{d}x)\mathbf{1}_{\{H(x\wedge x^*)<\varepsilon\}}\right]\\
&\quad\leqslant\frac{\alpha^{2-2/\gamma}}{C^2}\,\mathbb{N}^{(1)}\left[\int_{\mathcal{T}}\mu(\mathrm{d}x)\mathbf{1}_{\{H(x\wedge x^*)\geqslant\varepsilon\}}\left(\mathfrak{h}^{-\beta}Z^{\varepsilon}_{\alpha,\beta}(x)\right)^2\right]+\eta\\
&\quad\leqslant\frac{\alpha^{2-2/\gamma}}{C^2}\,\mathbb{N}^{(1)}\left[\int_{\mathcal{T}}\mu(\mathrm{d}x)\left(\int_0^{H(x)}\sigma^{\alpha}_{r,x}\,\mathrm{d}r\right)^2\right]+\eta\\
&\quad\leqslant\frac{M}{C^2}+\eta
\end{aligned} \tag{7.25}$$

for some constant $M > 0$, where we used that $Z^{\varepsilon}_{\alpha,\beta}(x^*) = Z^{\varepsilon}_{\alpha,\beta}(x)$ for every $x \in \mathcal{T}$ such that $H(x \wedge x^*) \geqslant \varepsilon$ for the first inequality, the Markov inequality for the second and Corollary 3.7 for the last. Thus, we get that the family $(\mathbf{1}_{\{\mathfrak{h} \geqslant \varepsilon\}} \alpha^{1-1/\gamma} \mathfrak{h}^{-\beta} Z^{\varepsilon}_{\alpha,\beta}(x^*) \colon \alpha \geqslant \alpha_0,\, \beta \geqslant 0)$ is tight. Since $\mathbb{N}^{(1)}$-a.s.

$$\lim_{\alpha\to\infty} \int_{\mathcal{T}} \mu(\mathrm{d}x) \mathbf{1}_{\{H(x\wedge x^*)<\varepsilon\}} = 0,$$

we deduce the following convergence in $\mathbb{N}^{(1)}$-probability

$$\lim_{\alpha\to\infty} B^1_\alpha = \lim_{\alpha\to\infty} \mathbf{1}_{\{\mathfrak{h}\geqslant\varepsilon\}} \alpha^{1-1/\gamma} \mathfrak{h}^{-\beta} Z^{\varepsilon}_{\alpha,\beta}(x^*) \int_{\mathcal{T}} \mu(\mathrm{d}x) \mathbf{1}_{\{H(x\wedge x^*)<\varepsilon\}} = 0.$$

Furthermore, we have

$$\begin{aligned}
\mathbb{N}^{(1)}[|B^2_\alpha|] &= \alpha^{1-1/\gamma}\, \mathbb{N}^{(1)} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \mathbf{1}_{\{H(x)\geqslant\varepsilon,\, H(x\wedge x^*)<\varepsilon\}} \mathfrak{h}^{-\beta} Z^{\varepsilon}_{\alpha,\beta}(x) \right] \\
&\leqslant \alpha^{1-1/\gamma}\, \mathbb{N}^{(1)} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \left(\mathbf{1}_{\{H(x\wedge x^*)<\varepsilon\}} \int_0^{H(x)} \sigma^{\alpha}_{r,x}\, \mathrm{d}r \right) \right] \\
&\leqslant \alpha^{1-1/\gamma}\, \mathbb{N}^{(1)} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \left(\int_0^{H(x)} \sigma^{\alpha}_{r,x}\, \mathrm{d}r \right)^2 \right]^{1/2} \mathbb{N}^{(1)} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \mathbf{1}_{\{H(x\wedge x^*)<\varepsilon\}} \right]^{1/2} \\
&\leqslant C\, \mathbb{N}^{(1)} \left[\int_{\mathcal{T}} \mu(\mathrm{d}x) \mathbf{1}_{\{H(x\wedge x^*)<\varepsilon\}} \right]^{1/2}
\end{aligned}$$

for some constant $C > 0$, where we used the Cauchy-Schwarz inequality for the second inequality and Corollary 3.7 for the last. It follows from the dominated convergence theorem that $\lim_{\alpha\to\infty} \mathbb{N}^{(1)}[|B^2_\alpha|] = 0$. This finishes the proof of (7.24).

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