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Gambler's Ruin and the ICM

Persi Diaconis and Stewart N. Ethier

"In the case where there are three players with limited fortunes, the various problems appear to be of quite a different order of difficulty than in the case of two players."

Louis Bachelier (1912)

Abstract. Consider gambler's ruin with three players, 1, 2, and 3, having initial capitals A, B, and C units. At each round a pair of players is chosen (uniformly at random) and a fair coin flip is made resulting in the transfer of one unit between these two players. Eventually, one of the players is eliminated and play continues with the remaining two. Let $\sigma \in S_3$ be the elimination order (e.g., $\sigma = 132$ means player 1 is eliminated first and player 3 is eliminated second, leaving player 2 with $A + B + C$ units).

We seek approximations (and exact formulas) for the elimination order probabilities $P_{A,B,C}(\sigma)$. Exact, as well as arbitrarily precise, computation of these probabilities is possible when $N := A + B + C$ is not too large. Linear interpolation can then give reasonable approximations for large N. One frequently used approximation, the independent chip model (ICM), is shown to be inadequate. A regression adjustment is proposed, which seems to give good approximations to the elimination order probabilities.

Key words and phrases: Gambler's ruin problem, tower problem, linear interpolation, independent chip model (ICM), Plackett–Luce model, linear regression.

1. INTRODUCTION

As motivation, first consider gambler's ruin with two players, 1 and 2, who initially have 1 and $N - 1$ units. At each round a fair coin flip is made resulting in the transfer of one unit from one player to the other. Eventually, one of the players goes broke. It is a classical result that

$$P_{1,N-1}(\text{player 2 goes broke}) = \frac{1}{N}.$$

Consider next the game with three players having initial fortunes 1, 1, $N - 2$. At each round a pair of players is chosen (uniformly at random) and a fair coin flip is made resulting in the transfer of one unit between these two players. What is

$$P_{1,1,N-2}(\text{player 3 goes broke first})?$$

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This basic problem has had little study. A first thought is, "Consider player 3 versus $\{1, 2\}$." This is like gambler's ruin with two players. Perhaps

$$P_{1,1,N-2}(\text{player 3 goes broke first}) \approx \frac{\text{constant}}{N}.$$

A well-studied scheme, the independent chip model (ICM), explained in Section 2.7 below, suggests

$$P_{1,1,N-2}(\text{player 3 goes broke first}) = \frac{2}{N(N-1)}.$$

We prove below that both of these are off. Indeed,

$$P_{1,1,N-2}(\text{player 3 goes broke first}) \approx \frac{\text{constant}}{N^3}.$$

It does not seem easy to give a simple heuristic for the N^3, and for $k \geq 4$ players, the correct order of decay is open.

Let the initial capitals be A, B, and C units, and put $N := A + B + C$. Let $\sigma \in S_3$ be the elimination order (e.g., $\sigma = 132$ means player 1 is eliminated first and player 3 is eliminated second, leaving player 2 with N units). Useful approximations to $P_{A,B,C}(\sigma)$ are important in widely played versions of tournament poker; if, at the final table, three players remain, the first-, second-, and third-place

TABLE 1
The final three in the 2019 *World Series of Poker Main Event*

Player	Chip count	Big blinds	Actual payoff
Dario Sammartino	67,600,000	33.8	$6,000,000
Alex Livingston	120,400,000	60.2	$4,000,000
Hossein Ensan	326,800,000	163.4	$10,000,000
Total	514,800,000	257.4	

TABLE 2
The approximate probabilities of the six possible elimination orders in the scenario of Table 1, *assuming chip counts* (*in units of* 400,000 *chips*, *or* 1/5 *of the big blind*) *equal to* $A = 169$, $B = 301$, *and* $C = 817$

σ	123	132	213	231	312	321
$P_{A,B,C}(\sigma)$	0.4196	0.2079	0.2152	0.1062	0.0260	0.0251

finishers get fixed amounts α, β, and γ, say, not depending on A, B, and C. Clearly, $P_{A,B,C}(\sigma)$ is crucial in evaluating an equitable split of the prize pool $\alpha+\beta+\gamma$, should the players decide to "settle." Such calculations are also required to evaluate the results of various actions throughout the game.

EXAMPLE 1.1. In the 2019 World Series of Poker Main Event, at the time the fourth-place finisher was eliminated, the three remaining players had chip counts as shown in Table 1 (WSOP, 2019a).

At this stage of the tournament, the standard unit bet—the big blind—was 2,000,000 chips. Initial capital, in big blinds, is shown for the three players in Table 1, but to avoid fractions we multiply these numbers by 5 to get $A = 169$, $B = 301$, and $C = 817$. In the ensuing competition, the elimination order turned out to be 213, leaving Hossein Ensan with all 514,800,000 chips and the $10 million first-place prize. The methods developed below (see Examples 2.3 and 3.2) give the chances shown in Table 2 for the six possible elimination orders, *assuming our random walk is a reasonable model for a no-limit Texas hold'em tournament*. Thus, the second most likely elimination order is what actually occurred.

Section 2 contains background on gambler's ruin and the independent chip model. We review the connections with absorbing Markov chain theory. This allows exact computation for N up to at least 200. Another approach, Jacobi iteration, allows virtually exact computation for N up to at least 300.

We also observe that the $N = 300$ data can be linearly interpolated to give reasonable approximations for arbitrary N. One other method of approximation, based on a Monte Carlo technique, is described.

Recent results for "nice" absorbing Markov chains (see Diaconis, Houston-Edwards and Saloff-Coste, 2021) allow crude but useful approximations of $P_{A,B,C}(\sigma)$ uniformly. The constant$/N^3$ result is proved as a consequence of that work.

A new approximation approach is introduced in Section 3. The ratio

$$P^{\mathrm{GR}}_{A,B,C}(\sigma)/P^{\mathrm{ICM}}_{A,B,C}(\sigma)$$

appears to be a smooth function of A, B, and C. A sixth-degree polynomial regression is fit to this ratio and seen to give good approximations to $P^{\mathrm{GR}}_{A,B,C}(\sigma)$. In the sequel, superscripts GR ("gambler's ruin") and ICM ("independent chip model") will be used only when there is a chance of confusion. (No superscript implicitly means GR.)

Section 4 gives some results for the gambler's ruin problem with $k \geq 4$ players as well as a conjecture, namely the *scaling conjecture*

$$P_{A',B',C'}(\sigma) \doteq P_{A,B,C}(\sigma) \quad \text{whenever } \frac{A'}{A} = \frac{B'}{B} = \frac{C'}{C},$$

where $\doteq$ denotes approximate equality. (The symbol $\approx$ has a different meaning; see Theorem 2.4 below.) An equivalent formulation,

$$P_{nA,nB,nC}(\sigma) \doteq P_{A,B,C}(\sigma), \quad n \geq 2, \tag{1.1}$$

may be preferable because it is closely related to the provable result that $\lim_{n\to\infty} P_{nA,nB,nC}(\sigma)$ exists; indeed, the limit can be expressed in terms of standard two-dimensional Brownian motion. A conjecture that is mathematically sharper than (1.1) appears in Section 4.1.

Finally, Section 5 summarizes the various methods of evaluating and approximating the elimination order probabilities. In all, six methods of approximation are studied, including a Brownian motion approximation along with the methods mentioned above.

2. BACKGROUND

This section contains background on gambler's ruin—in two and higher dimensions (three or more players). Exact computation of the Poisson kernel (harmonic measure) using absorbing Markov chains is taken up in Section 2.2, and arbitrarily precise computation by Jacobi iteration is the subject of Section 2.3. The use of barycentric coordinates to linearly interpolate these exact values is taken up in Section 2.4. Another approaches to approximate computation, Monte Carlo methods, is described in Section 2.5.

The asymptotics of the Poisson kernel are treated in Section 2.6, which includes a proof of $P_{1,1,N-2}$(player 3 goes broke first) $\approx$ constant$/N^3$. Finally, the ICM is introduced and its relation to the Plackett–Luce model is developed in Section 2.7.

2.1 Gambler's Ruin

With two players, gambler's ruin is a classical topic, well developed in Feller (1968), Chapter XIV, and Ethier (2010), Chapter 7. Important extensions to unfair coin flips and more-general step sizes are also well developed. See Song and Song (2013) for a historical survey.

For $k = 3$ players, the subject was first studied by Bachelier (1912). The first post-Bachelier reference we have found is a formulation in terms of Brownian motion in a triangle due to Cover (1987). This was solved by conformally mapping the triangle to a disk and using classical results for the Poisson kernel of the disk, by Hajek (1987) and later, independently, by Ferguson (1995). Further results for $k = 3$, including the poker connection, are in Kim (2005).

Martingale theory can be used to get information about the time to absorption. For three players, let T_1 be the first time one of the three players is eliminated. Bachelier (1912), Section 204, Engel (1993), and Stirzaker (1994) proved

$$E(T_1) = \frac{3ABC}{A+B+C}. \tag{2.1}$$

Thus, if $A = B = C = 100$, then $E(T_1) = 10{,}000$. If $A = B = 1$ and $C = 298$, then $E(T_1) = 2.98$. Bruss, Louchard and Turner (2003) and Stirzaker (2006) evaluated $\mathrm{Var}(T_1)$. Let T_2 be the first time two players are eliminated. Bachelier (1912), Section 209, Engel (1993), and Stirzaker (1994) showed that

$$E(T_2) = AB + AC + BC. \tag{2.2}$$

Thus, if $A = B = C = 100$, then $E(T_2) = 30{,}000$. If $A = B = 1$ and $C = 298$, then $E(T_2) = 597$. Actually, Bachelier and Engel used first-order linear partial difference equations, while Stirzaker used martingales. Bachelier's (1912), Section 209, proof of (2.2) is very much worth reading.

A standard theorem (Bachelier, 1912, Section 14) is

$$P(\text{player 3 wins all}) = \frac{C}{A+B+C}. \tag{2.3}$$

The results (2.2) and (2.3) generalize to k players. There is a related development in the language of the "Towers of Hanoi" problem (Bruss, Louchard and Turner, 2003, Ross, 2009). None of this literature addresses the position at the first absorption time.

2.2 Exact Computation by Markov Chain Methods

The gambler's ruin model is an example of an absorbing Markov chain in the state space

$$\mathscr{X} := \{(x_1, x_2, x_3) \in \mathbf{Z}^3 : x_1, x_2, x_3 \geq 0, \\ x_1 + x_2 + x_3 = N\}.$$

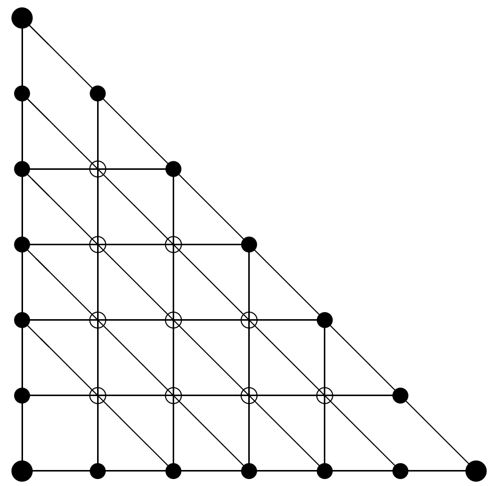

FIG. 1. *When $N = 6$, the state space $\mathscr{X}$ is represented by 28 dots, of which 10 are interior states (open dots), 15 are nonabsorbing boundary states (solid dots), and 3 are absorbing states (larger solid dots). Line segments show possible transitions. There are six from each interior state and two from each nonabsorbing boundary state.*

The first two coordinates determine things and the state space can be pictured (when $N = 6$) as in Figure 1. The classical stars and bars argument shows that

$$|\mathscr{X}| = \binom{N+2}{2},$$

and $\mathscr{X}$ has $\binom{N-1}{2}$ interior states, $3(N-1)$ nonabsorbing boundary states, and 3 absorbing states. The Markov chain stopped at time T_1 is itself a Markov chain whose transition matrix can be written in block form as

$$\begin{array}{r c} & \begin{array}{cc} \text{boundary} & \text{interior} \end{array} \\ \begin{array}{r} \text{boundary} \\ \text{interior} \end{array} & \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{S} & \boldsymbol{Q} \end{pmatrix} \end{array},$$

and elementary arguments yield the following theorem (Kemeny and Snell, 1976, Theorem 3.3.7).

THEOREM 2.1. *For $\boldsymbol{x} \in \mathrm{Int}(\mathscr{X})$ and $\boldsymbol{y}$ in the set of nonabsorbing boundary states of $\mathscr{X}$, define*

$$P(\boldsymbol{x}, \boldsymbol{y}) := P_{\boldsymbol{x}}(\textit{chain first reaches boundary at } \boldsymbol{y}),$$

so that $\boldsymbol{P}$ is an $\binom{N-1}{2} \times 3(N-1)$ matrix. Then

$$\boldsymbol{P} = (\boldsymbol{I} - \boldsymbol{Q})^{-1}\boldsymbol{S}.$$

The function $P(\boldsymbol{x}, \boldsymbol{y})$ is called the *Poisson kernel* or *harmonic measure*.

EXAMPLE 2.2. When $N = 6$, $|\mathscr{X}| = \binom{6+2}{2} = 28$, with the $\binom{6-1}{2} = 10$ interior states ordered 114, 123, 132, 141, 213, 222, 231, 312, 321, 411, and the $3(6-1) = 15$ nonabsorbing boundary states ordered 015, 024, 033, 042, 051, 105, 204, 303, 402, 501, 150, 240, 330, 420, 510. The Poisson kernel is given by Figure 2.

$$
\begin{pmatrix}
\frac{1711}{9456} & \frac{529}{2364} & \frac{87}{1576} & \frac{37}{2364} & \frac{31}{9456} & \frac{1711}{9456} & \frac{529}{2364} & \frac{87}{1576} & \frac{37}{2364} & \frac{31}{9456} & \frac{31}{9456} & \frac{25}{2364} & \frac{23}{1576} & \frac{25}{2364} & \frac{31}{9456} \\
\frac{135}{3152} & \frac{771}{3152} & \frac{397}{1576} & \frac{1}{16} & \frac{39}{3152} & \frac{135}{3152} & \frac{309}{3152} & \frac{125}{1576} & \frac{99}{3152} & \frac{23}{3152} & \frac{39}{3152} & \frac{115}{3152} & \frac{69}{1576} & \frac{85}{3152} & \frac{23}{3152} \\
\frac{39}{3152} & \frac{1}{16} & \frac{397}{1576} & \frac{771}{3152} & \frac{135}{3152} & \frac{39}{3152} & \frac{115}{3152} & \frac{69}{1576} & \frac{85}{3152} & \frac{23}{3152} & \frac{135}{3152} & \frac{309}{3152} & \frac{125}{1576} & \frac{99}{3152} & \frac{23}{3152} \\
\frac{31}{9456} & \frac{37}{2364} & \frac{87}{1576} & \frac{529}{2364} & \frac{1711}{9456} & \frac{31}{9456} & \frac{25}{2364} & \frac{23}{1576} & \frac{25}{2364} & \frac{31}{9456} & \frac{1711}{9456} & \frac{529}{2364} & \frac{87}{1576} & \frac{37}{2364} & \frac{31}{9456} \\
\frac{135}{3152} & \frac{309}{3152} & \frac{125}{1576} & \frac{99}{3152} & \frac{23}{3152} & \frac{135}{3152} & \frac{771}{3152} & \frac{397}{1576} & \frac{1}{16} & \frac{39}{3152} & \frac{23}{3152} & \frac{85}{3152} & \frac{69}{1576} & \frac{115}{3152} & \frac{39}{3152} \\
\frac{1}{48} & \frac{1}{12} & \frac{1}{8} & \frac{1}{12} & \frac{1}{48} & \frac{1}{48} & \frac{1}{12} & \frac{1}{8} & \frac{1}{12} & \frac{1}{48} & \frac{1}{48} & \frac{1}{12} & \frac{1}{8} & \frac{1}{12} & \frac{1}{48} \\
\frac{23}{3152} & \frac{99}{3152} & \frac{125}{1576} & \frac{309}{3152} & \frac{135}{3152} & \frac{23}{3152} & \frac{85}{3152} & \frac{69}{1576} & \frac{115}{3152} & \frac{39}{3152} & \frac{135}{3152} & \frac{771}{3152} & \frac{397}{1576} & \frac{1}{16} & \frac{39}{3152} \\
\frac{39}{3152} & \frac{115}{3152} & \frac{69}{1576} & \frac{85}{3152} & \frac{23}{3152} & \frac{39}{3152} & \frac{1}{16} & \frac{397}{1576} & \frac{771}{3152} & \frac{135}{3152} & \frac{23}{3152} & \frac{99}{3152} & \frac{125}{1576} & \frac{309}{3152} & \frac{135}{3152} \\
\frac{23}{3152} & \frac{85}{3152} & \frac{69}{1576} & \frac{115}{3152} & \frac{39}{3152} & \frac{23}{3152} & \frac{99}{3152} & \frac{125}{1576} & \frac{309}{3152} & \frac{135}{3152} & \frac{39}{3152} & \frac{1}{16} & \frac{397}{1576} & \frac{771}{3152} & \frac{135}{3152} \\
\frac{31}{9456} & \frac{25}{2364} & \frac{23}{1576} & \frac{25}{2364} & \frac{31}{9456} & \frac{31}{9456} & \frac{37}{2364} & \frac{87}{1576} & \frac{529}{2364} & \frac{1711}{9456} & \frac{31}{9456} & \frac{37}{2364} & \frac{87}{1576} & \frac{529}{2364} & \frac{1711}{9456}
\end{pmatrix}
$$

FIG. 2. *The Poisson kernel for $N = 6$. Rows are labeled by initial interior states* (114, 123, 132, 141, 213, 222, 231, 312, 321, 411), *and columns by nonabsorbing boundary states* (015, 024, 033, 042, 051, 105, 204, 303, 402, 501, 150, 240, 330, 420, 510).

From this, we have the chance that the first absorption occurs at a given boundary point. For the two remaining players, classical gambler's ruin gives the probability of the final outcome. Summing over the appropriate part of the boundary gives the chances of the various elimination orders. For $N = 6$, these are given in Figure 3. Here the row ordering is as before, whereas the column ordering is 123, 132, 213, 231, 312, 321.

Mathematica code, for arbitrary N, is provided in the Supplementary Materials (see Section 6). The only computationally difficult part of the program is inverting an $\binom{N-1}{2} \times \binom{N-1}{2}$ matrix. When $N = 200$ (the largest N for which we have results), this matrix is $19{,}701 \times 19{,}701$ and the program runtime (in double precision) was about 97 hours. A faster alternative is described in Section 2.3.

A very interesting paper by Swan and Bruss (2006) suggests that much larger problems might be tackled. Their ideas apply to more general absorbing chains, but let us specialize to the three-player gambler's ruin. They partition the transient states into disjoint "levels" and observe that the transition matrix can be written as a block tridiagonal matrix (up to "corner effects") with considerably smaller blocks. Their second idea is to derive a "folded" chain on the even blocks. This has the same block tridiagonal form and so recursion can be used. Finally the absorption probabilities for the chain started in the odd blocks can be filled in. They do an order of magnitude calculation of the number of operations involved (along the lines of "it takes order n^3 steps to invert an $n \times n$ matrix") and conclude that the new algorithm would run a factor of N^2 steps faster than the straightforward matrix inversion we have used above. The indexing is fairly sophisticated and we have not attempted to implement their fine ideas.

$$
\begin{pmatrix}
\frac{1}{3} & \frac{86}{591} & \frac{1}{3} & \frac{86}{591} & \frac{25}{1182} & \frac{25}{1182} \\
\frac{3287}{9456} & \frac{2521}{9456} & \frac{1441}{9456} & \frac{1007}{9456} & \frac{631}{9456} & \frac{569}{9456} \\
\frac{2521}{9456} & \frac{3287}{9456} & \frac{631}{9456} & \frac{569}{9456} & \frac{1441}{9456} & \frac{1007}{9456} \\
\frac{86}{591} & \frac{1}{3} & \frac{25}{1182} & \frac{25}{1182} & \frac{1}{3} & \frac{86}{591} \\
\frac{1441}{9456} & \frac{1007}{9456} & \frac{3287}{9456} & \frac{2521}{9456} & \frac{569}{9456} & \frac{631}{9456} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1007}{9456} & \frac{1441}{9456} & \frac{569}{9456} & \frac{631}{9456} & \frac{3287}{9456} & \frac{2521}{9456} \\
\frac{631}{9456} & \frac{569}{9456} & \frac{2521}{9456} & \frac{3287}{9456} & \frac{1007}{9456} & \frac{1441}{9456} \\
\frac{569}{9456} & \frac{631}{9456} & \frac{1007}{9456} & \frac{1441}{9456} & \frac{2521}{9456} & \frac{3287}{9456} \\
\frac{25}{1182} & \frac{25}{1182} & \frac{86}{591} & \frac{1}{3} & \frac{86}{591} & \frac{1}{3}
\end{pmatrix}
$$

FIG. 3. *The probabilities of the six elimination orders for $N = 6$. Rows are labeled by initial interior states* (114, 123, 132, 141, 213, 222, 231, 312, 321, 411) *and columns by elimination orders* (123, 132, 213, 231, 312, 321).

Using weighted directed multigraphs, David (2015) was able to reduce the number of transient states by about a factor of two. His results, with N as large as 192, are consistent with ours. For application of this approach to four-player gambler's ruin, see Marfil and David (2020).

Gilliland, Levental and Xiao (2007) found a way to avoid the inversion of large matrices in a one-dimensional gambler's ruin problem, but we have not been able to adapt their approach to the present setting.

These same techniques work for general absorbing Markov chains. We have used them (Supplementary Materials, Section 6) to compute the elimination order probabilities for $k = 4$ players, requiring the inverse of an $\binom{N-1}{3} \times \binom{N-1}{3}$ matrix. When $N = 50$ (the largest N for which we have results), this matrix is $18{,}424 \times 18{,}424$ and the program runtime (in single precision) was about 84.5 hours. Here the walk takes place in a discrete 4-simplex. Initial absorption is on one of the four triangular faces, and from there to final absorption one can apply the three-player results.

Our colleague Lexing Ying points out that the matrix $\boldsymbol{I} - \boldsymbol{Q}$ in Theorem 2.1 is sparse (it has at most seven nonzero entries per row). Sparse matrix inversion is a standard "off the shelf" tool in languages such as *MATLAB*. A useful textbook account is in Davis (2006). Using these techniques, Ying was able to write code that generates results for N as large as 3200 in about 2 minutes on a laptop computer. He graciously agreed to determine the probabilities of the six elimination orders for the WSOP data of Example 1.1, in which $N = 1287$, and we compare these numbers with those from our various algorithms in Table 8.

2.3 Arbitrarily Precise Computation by Jacobi Iteration

Fix an elimination order $\sigma \in S_3$ and total capital N. Let $P_{A,B}$ be short for $P_{A,B,N-A-B}(\sigma)$. Then, for $A, B \geq 1$ with $A + B \leq N - 1$,

$$P_{A,B} = \frac{1}{6}(P_{A-1,B+1} + P_{A+1,B-1} + P_{A-1,B} + P_{A+1,B} + P_{A,B-1} + P_{A,B+1})$$

with boundary conditions determined by σ. This may be used in two ways. Start with any values for the $P_{A,B}$ agreeing with the boundary conditions, say all $P_{A,B} = \frac{1}{6}$ except when $A = 0$, $B = 0$, or $A + B = N$. Then repeatedly iterate this recurrence. Again this may be done in two ways, either using (at stage n) P^{n+1} in terms of P^n or using updated values as they become available. This method was used by Kim (2005) and seen to converge well for small values of N (e.g., $N = 16$).

A second approach harnesses a monotonicity property of the recurrence. Let $P^*_{A,B}$ be the true gambler's ruin probabilities. If $P^n_{A,B} \leq P^*_{A,B}$ for all A, B, then $P^{n+1}_{A,B} \leq P^*_{A,B}$ for all A, B. Similarly for $P^n_{A,B} \geq P^*_{A,B}$. Thus, starting the recurrence off with the correct boundary values and all other $P^{0,-}_{A,B} \equiv 0$ and $P^{0,+}_{A,B} \equiv 1$ gives

$$P^{n,-}_{A,B} \leq P^*_{A,B} \leq P^{n,+}_{A,B} \quad \text{for all } A, B \text{ and } n.$$

When the lower and upper bounds are suitably close, this gives sharp control of $P^*_{A,B}$. For a proof of convergence and further development, history, and references, see Ethier (2010), Theorem 7.2.4.

We adopt the latter approach, and we find that we can ensure the desired accuracy (18 significant digits) with $2N^2$ iterations. *Mathematica* code (for arbitrary N) is provided in the Supplementary Materials. No matrix inversion is needed, so the program runs faster and uses much less memory than the one based on Markov chain methods. When $N = 200$, the program runtime (in double precision) was about 19 hours. When $N = 300$ (the largest N for which we have results) it was about 98.5 hours.

The output of this program is a list of $P_{A,B,C}(123)$ for all $A, B, C \geq 1$ with $A + B + C = N$. If, for example, we want $P_{1,1,N-2}(321)$, we simply look up $P_{N-2,1,1}(123)$ instead. Thus, there is no real loss of information in this condensed form of the output.

While this method allows for a larger N in evaluating the three-player elimination order probabilities than the Markov chain method does ($N = 300$ vs. $N = 200$), the improvement is more significant in the four-player setting. Here we generate the probabilities $P_{A,B,C,D}(1234)$ for all $A, B, C, D \geq 1$ with $A + B + C + D = N$ and again find that $2N^2$ iterations suffice to ensure the desired accuracy (9 significant digits). When $N = 100$ (the largest N for which we have results), the runtime was about 36 hours. *Mathematica* code is provided in the Supplementary Materials, but C++ code would run substantially faster.

2.4 Linear Interpolation from Exact Probabilities

The virtually exact results for $N = 300$ can be used to get useful approximations for other N. Given positive integers A, B, and C, let $N := A + B + C$ and

$$A_0 := A\frac{300}{N}, \qquad B_0 := B\frac{300}{N}, \qquad C_0 := C\frac{300}{N}.$$

Typically, these are not integers. Therefore, consider the four points

$$\begin{aligned}
\boldsymbol{v}_{00} &:= (\lfloor A_0 \rfloor, \lfloor B_0 \rfloor, 300 - \lfloor A_0 \rfloor - \lfloor B_0 \rfloor),\\
\boldsymbol{v}_{01} &:= (\lfloor A_0 \rfloor, \lceil B_0 \rceil, 300 - \lfloor A_0 \rfloor - \lceil B_0 \rceil),\\
\boldsymbol{v}_{10} &:= (\lceil A_0 \rceil, \lfloor B_0 \rfloor, 300 - \lceil A_0 \rceil - \lfloor B_0 \rfloor),\\
\boldsymbol{v}_{11} &:= (\lceil A_0 \rceil, \lceil B_0 \rceil, 300 - \lceil A_0 \rceil - \lceil B_0 \rceil),
\end{aligned}$$

belonging to $\mathscr{X}$, and discard the one ($\boldsymbol{v}_{00}$ or $\boldsymbol{v}_{11}$) whose third coordinate is neither $\lfloor C_0 \rfloor$ nor $\lceil C_0 \rceil$. The remaining three points, call them (A_1, B_1, C_1), (A_2, B_2, C_2), and (A_3, B_3, C_3), form a triangle with (A_0, B_0, C_0) belonging to its interior, and we can estimate $P_{A,B,C}(\sigma)$ by linear interpolation from the three values of $P_{A_i,B_i,C_i}(\sigma)$ $(i = 1, 2, 3)$.

The key idea is to represent (A_0, B_0, C_0) in barycentric coordinates. The relevant weights are

$$\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} := \begin{pmatrix}A_1 - A_3 & A_2 - A_3\\ B_1 - B_3 & B_2 - B_3\end{pmatrix}^{-1}\begin{pmatrix}A_0 - A_3\\ B_0 - B_3\end{pmatrix} \quad \text{and}$$

$$\lambda_3 := 1 - \lambda_1 - \lambda_2,$$

so that

$$(A_0, B_0, C_0) = \lambda_1(A_1, B_1, C_1) + \lambda_2(A_2, B_2, C_2) + \lambda_3(A_3, B_3, C_3),$$

and our interpolation estimate is then

$$\bar{P}_{A,B,C}(\sigma) := \lambda_1 P_{A_1,B_1,C_1}(\sigma) + \lambda_2 P_{A_2,B_2,C_2}(\sigma) + \lambda_3 P_{A_3,B_3,C_3}(\sigma).$$

EXAMPLE 2.3. As described in Example 1.1, the final three players in the 2019 WSOP Main Event had chip counts (in units of 400,000 chips, or 1/5 of the big blind) equal to $A = 169$, $B = 301$, and $C = 817$. Thus, $N = 1287$ and A, B, and C, multiplied by $300/N$, are $A_0 \doteq 39.39$, $B_0 \doteq 70.16$, and $C_0 \doteq 190.44$. It follows that $(A_1, B_1, C_1) = (39, 70, 191)$, $(A_2, B_2, C_2) = (39, 71, 190)$, and $(A_3, B_3, C_3) = (40, 70, 190)$. The weights can then be evaluated as

$$\lambda_1 = \frac{190}{429}, \qquad \lambda_2 = \frac{70}{429}, \qquad \lambda_3 = \frac{13}{33},$$

and we can look up the probabilities $P_{A_i,B_i,C_i}(\sigma)$ for $i = 1, 2, 3$ and each σ, with results shown in Table 3.

TABLE 3
Linearly interpolating elimination order probabilities from $N = 300$ data. Here $A = 169$, $B = 301$, and $C = 817$ from Example 1.1

σ	123	132	213	231	312	321
$P_{39,70,191}(\sigma)$	0.422050	0.207786	0.214617	0.105295	0.025547	0.024705
$P_{39,71,190}(\sigma)$	0.422204	0.210495	0.211129	0.104734	0.026172	0.025266
$P_{40,70,190}(\sigma)$	0.415774	0.206898	0.217559	0.107757	0.026436	0.025576
$\bar{P}_{A,B,C}(\sigma)$	0.419603	0.207878	0.215207	0.106174	0.025999	0.025139

The scaling conjecture and observed smoothness of $P_{A,B,C}(\sigma)$ in A, B, and C suggest that this will be a good approximation. One way to assess the accuracy of the method is to use it to estimate probabilities that are already known; we have done so in several cases, and it appears that the interpolated probabilities are accurate to four or five decimal places. See Example 3.2 below for an alternative approach.

Note that rounded proportions often do not sum precisely to 1. See Diaconis and Freedman (1979).

2.5 Monte Carlo Methods

While the interpolation method of Section 2.4 is our method of choice, this subsection records a further approximation method, Monte Carlo. Guanyang Wang suggested a straightforward Monte Carlo procedure that approximates $P_{A,B,C}(\sigma)$ for a given $A, B, C \geq 1$ and all $\sigma \in S_3$. Simply run the Markov chain, starting at (A, B, C), until it first reaches the boundary. If $N := A + B + C$ and the Markov chain first reaches the boundary at $(0, x, N - x)$, for example, then $\sigma = 123$ and $\sigma = 132$ are counted $(N - x)/N$ and x/N times, by virtue of the two-player gambler's ruin formula. Do this repeatedly, recording the proportion of times each $\sigma \in S_3$ occurs, and use these proportions as estimates. A difficulty is that this procedure is rather slow. For example, the expected number of steps for the Markov chain to first reach the boundary is given by (2.1), which is 96,876.4 when using the WSOP data of Examples 1.1 and 2.3 ($A = 169$, $B = 301$, and $C = 817$).

Wang suggested an optimization method to speed up the process. Starting from state (x_1, x_2, x_3), let $m = \min(x_1, x_2, x_3)$ and consolidate the next m steps of the Markov chain into a single step by simulating $(n_1, n_2, n_3) \sim \text{multinomial}(m, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, with n_i representing the number of matchups in the next m trials not involving player i, and $\zeta_i \sim \text{binomial}(n_i, \frac{1}{2})$ ($i = 1, 2, 3$), with ζ_i representing the number of the n_i matchups won by player $\text{mod}(i, 3) + 1$.

Wang has written *R* code and shown that it works well for quite large N (and also for $k = 4$). Starting with the just mentioned WSOP data ($A = 169$, $B = 301$, and $C = 817$), the standard Monte Carlo procedure requires 2.7 seconds per sample path or 7.5 hours for sample size 10^4, with the optimized procedure requiring only 0.0182 seconds per sample path or 3 minutes for sample size 10^4 (148 times faster). Example 3.2 below compares simulation results (optimized, with sample size 10^6) with other approximations.

2.6 Analytic Approximation

Some rather sophisticated analysis (John and inner uniform domains, Whitney covers, parabolic Harnack inequalities, Carlesson estimates) has been applied to get analytic approximations to the harmonic measure (Diaconis, Houston-Edwards and Saloff-Coste, 2021). The results apply to the k-player gambler's ruin problem, but we will content ourselves with the case $k = 3$. Code things up as in Figure 1 with two integer coordinates x_1, x_2 in the triangle $x_1, x_2 \geq 0$, $x_1 + x_2 \leq N$. This corresponds to $A = x_1$, $B = x_2$, and $C = N - x_1 - x_2$. By symmetry, it is enough to have approximations to

$$P(\boldsymbol{x}, (y, 0)) := P_{\boldsymbol{x}}(\text{walk first reaches boundary at } (y, 0))$$

with $\boldsymbol{x} = (x_1, x_2)$ in the interior of $\mathscr{X}$, satisfying $2x_1 + x_2 \leq N$. The boundary point $(y, 0)$ has $0 < y < N$.

THEOREM 2.4 (Diaconis, Houston-Edwards, Saloff-Coste, 2021). *For x_1, x_2, y as above,*

$$P(\boldsymbol{x}, (y, 0)) \approx \frac{x_1 x_2 (x_1 + x_2)(N - x_1 - x_2)(N - x_2) y^2 (N - y)^2}{N^4 (x_1 + d)^2 (x_2 + d)^2 (x_1 + x_2 + 2d)^2} \tag{2.4}$$

with d being the graph distance from $\boldsymbol{x}$ to $(y, 0)$. Here $a_N \approx b_N$ means there exist positive c and c' (universal) such that

$$c a_N \leq b_N \leq c' a_N$$

for all N. The constants implicit in (2.4) are uniform for all $\boldsymbol{x}$, y.

Let us illustrate this result by proving the $1/N^3$ result claimed in Section 1.

THEOREM 2.5.

$$P_{1,1,N-2}(\textit{player 3 goes broke first}) \approx \frac{1}{N^3}.$$

PROOF. To get things into the notation of Theorem 2.4, take $x_1 = 1$, $x_2 = N - 2$. Then for $0 < y < N$,

$$P(\boldsymbol{x}, (y, 0)) = P(\text{player 2 goes broke first at which time player 1 has } y).$$

For any y, $d \approx N$ so the denominator in (2.4) is $\approx N^{10}$. The numerator is $\approx N^2 y^2 (N - y)^2$. Thus,

$$P(\boldsymbol{x}, (y, 0)) \approx \frac{y^2(N-y)^2}{N^8} = \frac{1}{N^4}\left(\frac{y}{N}\right)^2\left(1 - \frac{y}{N}\right)^2.$$

Summing in y and reversing the roles of players 2 and 3,

$$\begin{aligned} &P_{1,1,N-2}(\text{player 3 goes broke first}) \\ &\quad \approx \frac{1}{N^3}\frac{1}{N}\sum_{y=1}^{N-1}\left(\frac{y}{N}\right)^2\left(1-\frac{y}{N}\right)^2 \sim \frac{B(3,3)}{N^3}, \end{aligned} \tag{2.5}$$

where B denotes the beta function. □

COROLLARY 2.6.

$$P_{1,1,N-2}(\textit{player 3 goes broke second}) \sim \frac{2}{N}.$$

PROOF. The desired probability is

$$\begin{aligned} &1 - P_{1,1,N-2}(\text{player 3 wins all}) \\ &\quad - P_{1,1,N-2}(\text{player 3 goes broke first}) \\ &= 1 - \frac{N-2}{N} - O(1/N^3) = \frac{2}{N} - O(1/N^3) \end{aligned}$$

by (2.3) and Theorem 2.5, and the result follows. □

REMARKS.

(1) The constant $B(3, 3) = 1/30$ in (2.5) is meaningless because of all the cruder approximations being used. Now

$$\begin{aligned} &P_{1,1,N-2}(\text{player 3 goes broke first}) \\ &\quad = P_{1,1,N-2}(312) + P_{1,1,N-2}(321), \end{aligned}$$

and because of symmetry,

$$P_{1,1,N-2}(312) = P_{1,1,N-2}(321),$$

so Theorem 2.5 implies

$$P_{1,1,N-2}(312) = P_{1,1,N-2}(321) \approx \frac{1}{N^3}.$$

(2) A similar calculation shows, for $1 \le i < N/2$,

$$P_{i,i,N-2i}(\text{player 3 goes broke first}) \approx \frac{i^3}{N^3},$$

uniformly in i. This is consistent with the scaling conjecture of Section 4.

(3) The asymptotics above may be supplemented by the exact computing of Sections 2.2 and 2.3. Table 4 gives $P_{1,1,N-2}(321)$ for $N = 50, 100, 150, 200, 250, 300$ as well as these values multiplied by N^3.

TABLE 4
The exact values of $P_{1,1,N-2}(321)$, rounded to 15 significant digits, suggesting that this quantity is asymptotic to c/N^3 for $c \doteq 4.5597945$

N	$P_{1,1,N-2}(321)$	$N^3 P_{1,1,N-2}(321)$
50	0.0000364783779008280	4.55979723760
100	0.00000455979467170448	4.55979467170
150	0.00000135105023226911	4.55979453391
200	0.000000569974313837992	4.55979451070
250	0.000000291826848279112	4.55979450436
300	0.000000168881277854908	4.55979450208

(4) In unpublished work, Sangchul Lee has used Ferguson's (1995) Brownian motion approximation to the discrete gambler's ruin problem to derive an analytical closed form expression for the constant $c \doteq 4.5597945$ in Table 4. He shows

$$c = \frac{\sqrt{\pi}}{3\sqrt{3}}\left(\frac{\Gamma(1/3)}{\Gamma(5/6)}\right)^3 \doteq 4.55979449996,$$

in remarkable agreement to the numbers in Table 4. The validity of the Brownian motion approximation has not been rigorously established to this degree. See Denisov and Wachtel (2015).

(5) Theorem 2.4 allows proof of similar asymptotics for other values of A, B, and C. For example, we have proved the following:

- For fixed $A, B \ge 1$ and $C_N := N - A - B$,

$$P_{A,B,C_N}(321) \approx P_{A,B,C_N}(312) \approx \frac{1}{N^3}.$$

- For $A = 1$, $B_N = \lfloor\sqrt{N}\rfloor$, and $C_N := N - 1 - B_N$,

$$P_{1,B_N,C_N}(\text{player 3 goes broke first}) \approx \frac{1}{N^2}.$$

Exact computations suggest that in the first case $N^3 \times P_{A,B,C_N}(321)$ and in the second case $N^2 P_{A,B_N,C_N}$(player 3 goes broke first) rapidly approach limits.

(6) Similarly, $P_{1,B_N,C_N}(321) \approx P_{1,B_N,C_N}(312) \approx 1/N^2$. This is a bit surprising. Of course, the event that the player with the big stack is eliminated first is a rare event but then the advantage that player 2 had over player 1 disappears. Indeed, numerical computations show that, for this case, given player 3 is eliminated first, the conditional gambler's ruin probability that 1 is eliminated second is 1/2 to remarkable approximation. For example, $P_{1,14,185}(321) \doteq 0.000059822$ and $P_{1,14,185}(312) \doteq 0.000059872$.

(7) The results of Diaconis, Houston-Edwards and Saloff-Coste (2021) were not intended to give good numerics. We hope that comparing them to data will allow better choices of omitted constants as in item (3) above. The following results are examples.

(8) For any N,

$$
\begin{aligned}
P_{1,1,N-2}(123) &= P_{1,1,N-2}(213) \\
&= \frac{1}{2} P_{1,1,N-2}(\text{player 3 wins all}) \\
&= \frac{1}{2}\frac{N-2}{N} = \frac{1}{2}\left(1-\frac{2}{N}\right).
\end{aligned}
$$

When $N = 200$, the right side is 0.495. Using Theorem 2.4,

$$
\begin{aligned}
P_{1,1,N-2}(123) &\approx \frac{1}{2}\sum_{y=1}^{N-1}\frac{(1-y/N)^3}{y^4} \\
&= \frac{\zeta(4)}{2}\big(1+o(1)\big) \doteq 0.5412.
\end{aligned}
$$

Similarly, if $1 \le i < N/2$,

$$
\begin{aligned}
P_{i,i,N-2i}(123) = P_{i,i,N-2i}(213) &= \frac{1}{2}\frac{N-2i}{N} \\
&= \frac{1}{2}\left(1-\frac{2i}{N}\right),
\end{aligned}
$$

confirming the scaling conjecture in this case. So perhaps not all hope is lost for using Theorem 2.4.

(9) Similarly, taking $x_1 = x_2 = 1$,

$$
\begin{aligned}
P_{1,1,N-2}(132) &= P_{1,1,N-2}(231) \\
&\approx \frac{1}{2}\sum_{y=1}^{N-1}\frac{(1-y/N)^2}{y^4}\frac{y}{N} \\
&= \frac{1}{2N}\sum_{y=1}^{N-1}\frac{(1-y/N)^2}{y^3} \sim \frac{\zeta(3)}{2N}.
\end{aligned}
$$

By Corollary 2.6, these probabilities are asymptotic to $1/N$, so this estimate is off by a factor of $\zeta(3)/2 \doteq 0.6010$.

2.7 The Independent Chip Model (ICM)

There are a variety of reasons for wanting to compute the chances of the various elimination orders. The most classical one, "The Problem of Points," has to do with splitting the capital in a k-player game when the game must be called off early. This is one of the problems that got Fermat and Pascal in correspondence—the start of modern probability theory. In tournament poker, we have seen three players decide to "settle," dividing the final prize money in proportion to their current chip totals. (As we will see, this is not the right way to do it.) Of course, calculating expectations for various decisions (mentioned earlier) is a key application.

The independent chip model (ICM), a popular scheme, originated in a 1986 article by Mason Malmuth in *Poker Player Newspaper*, which was reprinted in Malmuth (1987, 2004). Although the name came later, the concept was used to argue that rebuying in a percentage-payback poker tournament is mathematically correct, contrary to conventional wisdom at the time. Other implications of the ICM for poker tournaments were discussed by Gilbert (2009). See Aguilar (2016) for its use in "chopping" the prize pool in poker tournaments, using a poker ICM calculator (ICMizer, 2020).

ICM builds on a solid foundation: In the two-player gambler's ruin problem for fair coin-tossing, if player 1 starts with A and player 2 starts with B, the chance that player 1 (respectively, player 2) wins all is $A/(A+B)$ (resp., $B/(A+B)$). Now a heuristic step: Consider three players with initial capitals A, B, and C. The chance that a given player wins all is (rigorously) proportional to his initial capital (so the chance that player 1 wins all is A/N, where $N := A+B+C$). The ICM calculation conditions on this, uses the relative initial capitals of the two nonwinners to calculate the chance of being second eliminated, *and then multiplies*. This results in the chances shown in Table 5 assigned to the six elimination orders.

The probabilities for $k \ge 4$ players are determined similarly.

We can now be more explicit about how the prize pool is chopped when the last three players decide to settle. First, apportioning it in proportion to current chip totals does not take prize money into account and is unsupportable. For example, in Example 1.1, the chip leader would get about \$12.696 M, more than he would get by finishing first, and the player in third place would get about \$2.626 M, less than he would get by finishing third. So let $P(\sigma)$ be the probability of elimination order σ. Let α. β, and γ be the payouts for first, second, and third place. Then the expression for the amounts apportioned to players 1, 2, and 3 should be

$$
\begin{aligned}
&(\gamma,\beta,\alpha)P(123) + (\gamma,\alpha,\beta)P(132) \\
&\quad + (\beta,\gamma,\alpha)P(213) + (\alpha,\gamma,\beta)P(231) \\
&\quad + (\beta,\alpha,\gamma)P(312) + (\alpha,\beta,\gamma)P(321).
\end{aligned}
$$

TABLE 5
The ICM with three players

σ	123	132	213	231	312	321
$P^{\mathrm{ICM}}_{A,B,C}(\sigma)$	$\frac{C}{N}\frac{B}{A+B}$	$\frac{B}{N}\frac{C}{A+C}$	$\frac{C}{N}\frac{A}{A+B}$	$\frac{A}{N}\frac{C}{B+C}$	$\frac{B}{N}\frac{A}{A+C}$	$\frac{A}{N}\frac{B}{B+C}$

It is standard practice to use $P^{\mathrm{ICM}}_{A,B,C}(\sigma)$ in place of $P(\sigma)$. Alternatively, one could use $\bar{P}^{\mathrm{GR}}_{A,B,C}(\sigma)$. Of course, both are approximations.

EXAMPLE 2.7. Let us return to Example 1.1 with $A = 169$, $B = 301$, and $C = 817$; also $\alpha = \$10$ M, $\beta = \$6$ M, and $\gamma = \$4$ M. Using the ICM probabilities from Table 5, we get (\$5.325 M, \$6.287 M, \$8.388 M), which is consistent with ICMizer (2020), whereas using the interpolated gambler's ruin probabilities from Example 2.3, we get (\$5.270 M, \$6.293 M, \$8.437 M). The player with third-largest chip total gets about \$54 K more from an ICM chop than from a GR chop.

REMARKS.

(1) *ICM is different from gambler's ruin*. Consider $N = 6$ and initial capital $A = 1$, $B = 2$, and $C = 3$. What is the chance the elimination order is 321? Using the exact calculation in Figure 3 and the ICM formula yields

$$P^{\mathrm{GR}}_{1,2,3}(321) = \frac{569}{9456} \doteq 0.06017,$$

$$P^{\mathrm{ICM}}_{1,2,3}(321) = \frac{1}{6}\frac{2}{5} \doteq 0.06667.$$

(2) *The results can be of different orders of magnitude*. With starting capitals 1, 1, $N - 2$,

$$P^{\mathrm{GR}}_{1,1,N-2}(321) \approx \frac{1}{N^3},$$

$$P^{\mathrm{ICM}}_{1,1,N-2}(321) = \frac{1}{N}\frac{1}{N-1} \sim \frac{1}{N^2}.$$

(3) *Sometimes they agree*. With starting capitals i, i, $N - 2i$, where $1 \le i < N/2$,

$$P^{\mathrm{GR}}_{i,i,N-2i}(123) = P^{\mathrm{ICM}}_{i,i,N-2i}(123) = \frac{N-2i}{N}\frac{1}{2} = \frac{1}{2}\left(1 - \frac{2i}{N}\right).$$

(4) *They are often quite different*. In the next section, we calculate of the ratios

$$P^{\mathrm{GR}}_{A,B,C}(\sigma)/P^{\mathrm{ICM}}_{A,B,C}(\sigma)$$

for all $A, B, C \ge 1$ with $A + B + C = 300$ and all $\sigma \in S_3$. The ratios vary considerably, ranging from about 0.015 to about 1.15.

(5) *But* poker is a complicated game, particularly no limit where the bets can be arbitrary. The gambler's ruin model is based on single-unit bets. Why is this relevant? Some variants of the ± 1 transfer have been studied.

- *All in*: After two players out of the remaining k are chosen, if they have A and B respectively, the bet size is $\min(A, B)$. The player with the smaller chip count is eliminated or doubles up.
- *Occasionally all in*: This is a compromise between unit bets and all-in bets. After two players out of the remaining k are chosen, if they have A and B respectively, the bet size is chosen uniformly at random from $\{1, 2, \dots, \min(A, B)\}$.
- *Compulsive gambler* (Aldous, Lanoue and Salez, 2015): After two players out of the remaining k are chosen, one gets the other's money with probabilities given by the two-player gambler's ruin formula. That is, if the respective amounts are A and B, the player with A wins (and then has $A + B$) with probability $A/(A + B)$, or loses (and is eliminated) with probability $B/(A + B)$.

A fascinating effort at finding an optimal strategy for k-player gambler's ruin with all-in betting is in Ganzfried and Sandholm (2008). Interestingly, they use ICM as a starting evaluation of the value function and then sharpen this using fictitious play and value iteration.

These variants will (almost surely) result in different elimination order probabilities. The ICM assignment is different yet again. Thus, there are many distinct models. It would be worthwhile to look at some of the available data for tournament poker and compare. We wouldn't be surprised if all these models are inadequate.

The next section salvages something from these differences, using the ratios and regression to give a useful approximation to the gambler's ruin probabilities.

To finish this section, let us note that ICM is well studied as the Plackett–Luce model. This is a model allowing nonuniform distributions on S_k, the set of permutations of k distinct items, labeled $1, 2, \dots, k$. Each item i is assigned a weight $w_i > 0$ with $w_1 + w_2 + \cdots + w_k = w$. Now imagine these weights placed in an urn and the weights removed sequentially, each time with probability proportional to its size among the remaining weights. Thus,

$$P(\sigma) := \frac{w_{\sigma(1)}}{w}\frac{w_{\sigma(2)}}{w - w_{\sigma(1)}}\frac{w_{\sigma(3)}}{w - w_{\sigma(1)} - w_{\sigma(2)}} \cdots .$$

The model was introduced in perception psychology by R. Duncan Luce (1959, 1977). It has a variety of derivations: via the elimination by aspect axiom; as the distribution of the order statistics of independent exponential variables (the ith having mean w_i); and as the stationary distribution of the Tsetlin library. See Diaconis (1988), pages 174–175, for further references.

Later reinventions of the model were published by Harville (1973) and Plackett (1975), both of whom applied it to horse racing, and it seems to have a life of its own for this application (Stern, 2008). There is good available code for fitting this model to data (Turner et al., 2017) and many applications. Although the model is referred to in the literature as the Plackett–Luce model, perhaps Luce–Harville–Plackett–Malmuth would be chronologically more correct.

Finally, we note that enumerative combinatorics for the Plackett–Luce model can be interesting and challenging; What is the approximate distribution of the number of fixed points or cycles, and how does it depend on the weights?

3. ICM AND REGRESSION FOR GAMBLER'S RUIN

Here we show how to use the easy-to-compute ICM probabilities $P^{\mathrm{ICM}}_{A,B,C}(\sigma)$ to get surprisingly good approximations to the gambler's ruin probabilities $P^{\mathrm{GR}}_{A,B,C}(\sigma)$. Throughout, we work with $k = 3$ players, fair coin flips, and ± 1 transfers at each stage.

We can base the analysis on the $N = 300$ data, which gives $P_{A,B,C}(123)$ (in double precision) for all $A, B, C \geq 1$ with $A+B+C = 300$. There are $\binom{300-1}{2} = 44{,}551$ such points. Notice that, for $\sigma = \sigma(1)\sigma(2)\sigma(3)$,

$$P_{A_1,A_2,A_3}(\sigma) = P_{A_{\sigma(1)},A_{\sigma(2)},A_{\sigma(3)}}(123),$$

so there is no loss of information by restricting to $\sigma = 123$.

As efficient as this data set is, there is still some redundancy in the data, as has already been alluded to in (2.3) and elsewhere, namely

$$\begin{aligned} P_{A,B,C}(123) + P_{A,B,C}(213) &= \frac{C}{A+B+C}, \\ P_{A,B,C}(132) + P_{A,B,C}(312) &= \frac{B}{A+B+C}, \qquad (3.1) \\ P_{A,B,C}(231) + P_{A,B,C}(321) &= \frac{A}{A+B+C}, \end{aligned}$$

a consequence of the optional stopping theorem. The result is that it suffices to consider only one of the two probabilities in each row of (3.1). Incidentally, the equations in (3.1) hold trivially with superscript ICM.

We begin by evaluating, for $N = 300$ and $\sigma = 123$, the ratios

$$R_\sigma(A, B, C) := P^{\mathrm{GR}}_{A,B,C}(\sigma)/P^{\mathrm{ICM}}_{A,B,C}(\sigma) \qquad (3.2)$$

for all $A, B, C \geq 1$ with $A + B + C = N$. As already noted, there are 44,551 such ratios and all of them belong to $(0.015, 1.15)$. The function R_{123} is plotted in Figure 4.

Notice that R_{123} appears smooth as a function of (A, B) $(C = N - A - B)$, except for a singularity near $(1, 1)$. We can mitigate the effect of the singularity by considering R_σ over $1 \leq A \leq B \leq C$ with $A + B + C = N$ for $\sigma =$ 213, 312, and 321. (The number of such triples (A, B, C) is $N^2/12$ if N is divisible by 6, hence 7500 if $N = 300$.) In each case we fit a sextic polynomial in

$$x := \frac{A}{N} \quad \text{and} \quad y := \frac{B}{N}$$

to the function R_σ. A quadratic approximation does not give very good results, while a quartic approximation is quite good, and a sextic is even better. At the same time, the higher the degree, the closer the design matrix is to being less than full rank. An octic approximation results in some disturbingly large estimated regression coefficients, so we have settled on a sextic polynomial approximation. Thus, we want to approximate R_σ by the polynomial with 28 terms

$$p_\sigma(x, y) := \sum_{i,j \geq 0, i+j \leq 6} \beta_{ij} x^i y^j.$$

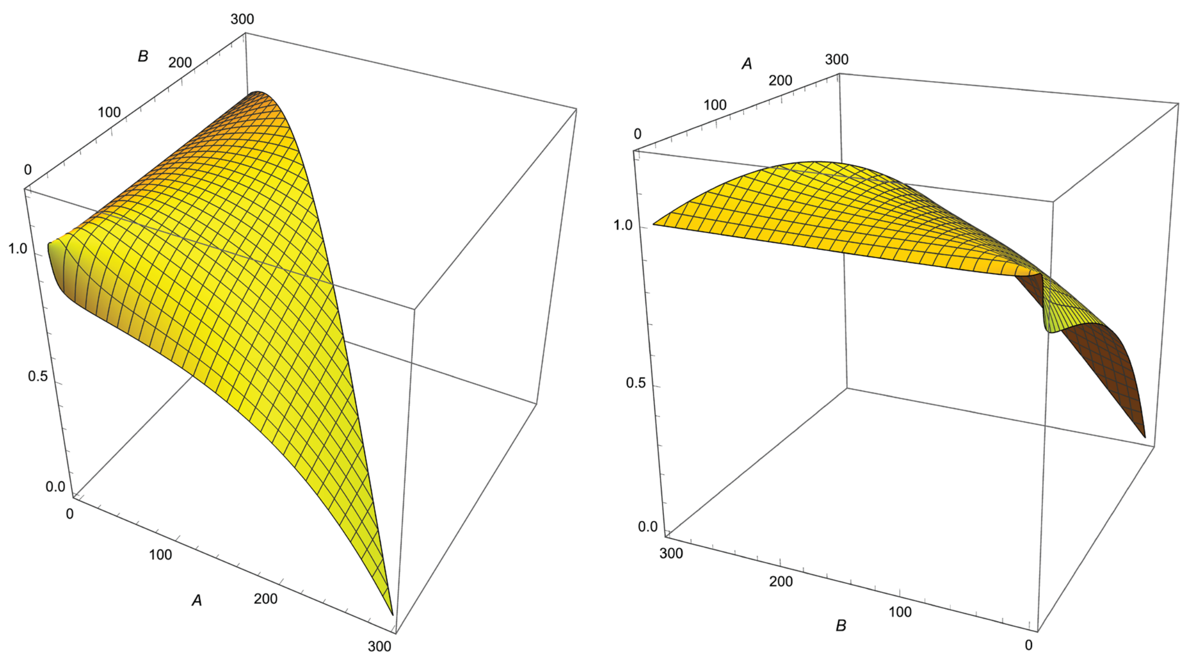

FIG. 4. *A plot of R_{123} defined in (3.2) as a function of (A, B) when $N = 300$. (The domain of R_{123} is restricted to $A, B \geq 1$ with $A + B \leq N - 1$.). The second figure is a rotation of the first that reveals the singularity near* $(1, 1)$.

TABLE 6
The estimated regression coefficients in fitting a sextic polynomial in $x := A/N$ and $y := B/N$ to $P^{\mathrm{GR}}_{A,B,C}(\sigma)/P^{\mathrm{ICM}}_{A,B,C}(\sigma)$, when $1 \le A \le B \le C$ and $A + B + C = N$. Here $N = 300$

	$\sigma = 321$	$\sigma = 312$	$\sigma = 213$
$\hat\beta_{00}$	0.00000716459	−0.00000434510	0.951694
$\hat\beta_{10}$	2.27836	2.27978	7.07267
$\hat\beta_{01}$	2.28007	2.28028	−3.74069
$\hat\beta_{20}$	−2.25895	0.0295644	−85.5467
$\hat\beta_{11}$	−2.24587	−2.30603	−23.1325
$\hat\beta_{02}$	−0.00740617	−2.28407	37.7612
$\hat\beta_{30}$	0.0793010	−0.618044	336.603
$\hat\beta_{21}$	−0.581954	0.285484	646.261
$\hat\beta_{12}$	−0.191389	0.241672	−111.109
$\hat\beta_{03}$	0.0630814	0.0171583	−197.597
$\hat\beta_{40}$	1.48069	0.533492	−557.212
$\hat\beta_{31}$	7.41061	−3.71135	−2327.47
$\hat\beta_{22}$	3.41508	−4.35883	−1723.69
$\hat\beta_{13}$	−5.89118	5.79382	874.263
$\hat\beta_{04}$	−2.67189	2.49997	540.005
$\hat\beta_{50}$	−0.556140	−4.34151	401.498
$\hat\beta_{41}$	−3.73983	−14.9318	2829.27
$\hat\beta_{32}$	−5.73258	14.4755	5167.69
$\hat\beta_{23}$	0.0461851	13.5284	1655.90
$\hat\beta_{14}$	1.48366	−7.28960	−1925.03
$\hat\beta_{05}$	−0.274686	−2.71321	−746.323
$\hat\beta_{60}$	−0.0378116	2.48372	−100.681
$\hat\beta_{51}$	−1.05892	21.5551	−1101.29
$\hat\beta_{42}$	−4.84129	6.48502	−3408.10
$\hat\beta_{33}$	−5.77799	−35.4065	−3643.90
$\hat\beta_{24}$	1.08693	−12.3409	−213.799
$\hat\beta_{15}$	5.19094	5.44466	1420.66
$\hat\beta_{06}$	1.88817	1.07997	410.303

Let $\boldsymbol{Y}$ be the column vector of values of R_{321} (with $N = 300$), indexed by the vectors (A, B, C) (with $1 \le A \le B \le C$ and $A + B + C = N$) ordered lexicographically, let $\boldsymbol{X}$ be the matrix whose rows are indexed as the entries of $\boldsymbol{Y}$, and with row (A, B, C) containing $1, x, y, x^2, xy, y^2, x^3, x^2y, \dots, y^6$, where $x = A/N$ and $y = B/N$. Note that $\boldsymbol{Y}$ has length 7500 and $\boldsymbol{X}$ is 7500 by 28. To quantify the claim that $\boldsymbol{X}'\boldsymbol{X}$ becomes closer to being singular as the degree of the approximating polynomial increases, we note that, with $N = 300$, $\det(\boldsymbol{X}'\boldsymbol{X})$ is 1.68×10^6 for quadratic approximation, 3.90×10^{-29} for quartic, 1.14×10^{-136} for sextic, and 1.10×10^{-415} for octic.

The estimated regression coefficients are

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y},$$

and the values of the fitted polynomial $\hat p_{321}$ are the entries of $\boldsymbol{X}\hat{\boldsymbol{\beta}}$. Table 6 lists the estimated regression coefficients, and the error sum of squares is 7.86×10^{-9} for $\sigma = 321$, 8.95×10^{-9} for $\sigma = 312$, and 0.0177 for $\sigma = 213$. Additional detail is given in the Supplementary Materials (Section 6).

This gives the approximation

$$\begin{aligned} \hat P^{\mathrm{GR}}_{A,B,C}(321) &:= P^{\mathrm{ICM}}_{A,B,C}(321)\,\hat p_{321}\left(\frac{A}{N}, \frac{B}{N}\right) \\ &= \frac{A}{N}\frac{B}{B+C}\,\hat p_{321}\left(\frac{A}{N}, \frac{B}{N}\right), \end{aligned} \tag{3.3}$$

and the cases $\sigma = 312$ and $\sigma = 213$ are treated in the same way. The derivation assumed $N = 300$ throughout. We did the same computation for $N = 200$, and the estimated regression coefficients did not change much, indicating stability. We expect the approximation to be reasonable for other (perhaps much larger) values of N. That is, for general N use the approximation (3.3) in which the function $\hat p_{321}$ is determined by the coefficients in Table 6 computed from the $N = 300$ data. We investigate this in two examples below.

EXAMPLE 3.1. Table 7 compares exact values of $P_{A,B,C}(\sigma)$ with its interpolation approximation $\bar P_{A,B,C}(\sigma)$ and its regression-corrected ICM $\hat P_{A,B,C}(\sigma)$. In the two examples, which are representative, we find that, for $\sigma = 321$ and $\sigma = 312$ (and their "complements" $\sigma = 231$ and $\sigma = 132$), the regression approximation is often accurate to six significant digits (except near the boundary of $\mathscr{X}$). But with $\sigma = 213$ (and $\sigma = 123$) the regression approximation is not as good, perhaps only three

TABLE 7
Two examples comparing the exact value of $P_{A,B,C}(\sigma)$ (to six significant digits) with its interpolation approximation $\bar P_{A,B,C}(\sigma)$ and its regression approximation $\hat P_{A,B,C}(\sigma)$

σ	123	132	213	231	312	321
$P_{23,45,67}(\sigma)$	0.342769	0.264802	0.153527	0.108430	0.0685310	0.0619406
$\bar P_{23,45,67}(\sigma)$	0.342763	0.264801	0.153533	0.108430	0.0685326	0.0619404
$\hat P_{23,45,67}(\sigma)$	0.342744	0.264802	0.153552	0.108430	0.0685311	0.0619404
$P_{10,40,90}(\sigma)$	0.532690	0.268542	0.110167	0.0553389	0.0171721	0.0160897
$\bar P_{10,40,90}(\sigma)$	0.532702	0.268540	0.110155	0.0553369	0.0171744	0.0160917
$\hat P_{10,40,90}(\sigma)$	0.532773	0.268542	0.110084	0.0553389	0.0171721	0.0160897

TABLE 8

Approximations to $P^{\mathrm{GR}}_{A,B,C}(\sigma)$ when $A=169$, $B=301$, and $C=817$. Row (a) uses ICM, row (b) uses Monte Carlo, row (c) uses linear interpolation, row (d) uses linear regression, and row (e) is exact (see the last paragraph of Section 2.2). All figures are rounded to the degree shown

	σ	123	132	213	231	312	321
(a)	$P^{\mathrm{ICM}}_{A,B,C}(\sigma)$	0.406548	0.193791	0.228261	0.095960	0.0400865	0.0353535
(b)	$\widehat{P}^{\mathrm{GR}}_{A,B,C}(\sigma)$	0.419345	0.207492	0.215650	0.106286	0.0261205	0.0251065
(c)	$\bar{P}^{\mathrm{GR}}_{A,B,C}(\sigma)$	0.419603	0.207878	0.215207	0.106174	0.0259991	0.0251395
(d)	$\hat{P}^{\mathrm{GR}}_{A,B,C}(\sigma)$	0.419635	0.207879	0.215175	0.106174	0.0259984	0.0251388
(e)	$P^{\mathrm{GR}}_{A,B,C}(\sigma)$	0.4195973	0.2078788	0.2152123	0.1061744	0.02599843	0.02513876

or four significant digits. In the latter case, we see from Table 6 that the estimated regression coefficients are substantially larger, which is indicative of a poorer fit. On the other hand, the interpolation approximation is typically accurate to four or five decimal places.

EXAMPLE 3.2 (Example 2.3 continued). Recall that, in Example 2.3, we estimated $P_{A,B,C}(\sigma)$ when $A=169$, $B=301$, and $C=817$. We did so using linear interpolation based on the $N=300$ data. Results are restated in Table 8 (row (c)), so that we can compare them with the ICM (row (a)), Monte Carlo (row (b)), and the regression approximation (row (d)), which used (3.3) (and its analogues for $\sigma=312$ and $\sigma=213$) with A, B, and C as above and $N=1287$.

We find that linear interpolation and linear regression match to four or more decimal places, Monte Carlo to three, and ICM to one or two.

4. A CONJECTURE AND MORE THAN THREE PLAYERS

This section treats two further topics, the scaling conjecture and $k\ge 4$ players (in particular, $k=4$).

4.1 Scaling Conjecture

The *scaling conjecture* says, for all $A,B,C\ge 1$, $\sigma\in S_3$, and $n\ge 2$,

$$P_{nA,nB,nC}(\sigma)\doteq P_{A,B,C}(\sigma). \tag{4.1}$$

As noted in Section 1, this is closely related to the result, provable as a consequence of Donsker's theorem, that $\lim_{n\to\infty}P_{nA,nB,nC}(\sigma)$ exists and can be expressed in terms of standard two-dimensional Brownian motion.

To formulate such a theorem, we adopt the setup used by Ferguson (1995). Let Δ be the equilateral triangle with vertices $(-1,0)$, $(1,0)$, and $(0,\sqrt{3})$, and let V_3 be the edge that lies on the x-axis. Let A, B, and C be positive integers and $N:=A+B+C$. Then the barycentric coordinates $(A/N,B/N,C/N)$ correspond to the initial state $\boldsymbol{x}:=((B-A)/N,\sqrt{3}C/N)$.

THEOREM 4.1. *Let $\{\boldsymbol{B}(t),t\ge 0\}$ be standard two-dimensional Brownian motion, and let T_1 be the exit time of $\boldsymbol{x}+\boldsymbol{B}$ from Δ. Then*

$$\begin{aligned}&\lim_{n\to\infty}[P_{nA,nB,nC}(321)+P_{nA,nB,nC}(312)]\\&\quad=P(\boldsymbol{x}+\boldsymbol{B}(T_1)\in V_3).\end{aligned}\tag{4.2}$$

Furthermore,

$$\begin{aligned}&\lim_{n\to\infty}P_{nA,nB,nC}(321)\\&\quad=E\biggl(\frac{|\boldsymbol{x}+\boldsymbol{B}(T_1)-(1,0)|}{2};\boldsymbol{x}+\boldsymbol{B}(T_1)\in V_3\biggr).\end{aligned}\tag{4.3}$$

The integrand in (4.3) is the proportion of the length of the edge V_3 that lies between the exit position $\boldsymbol{x}+\boldsymbol{B}(T_1)$ and the corner $(1,0)$ corresponding to player 2 winning all. This amounts to applying the two-player gambler's ruin formula to the exit position.

Ferguson (1995) (see also Bruss, Louchard and Turner, 2003) and Hajek (1987) used conformal mapping to give complicated expressions for the right-hand sides of (4.2) and (4.3), respectively. It remains to massage their formulas into computable form. In a special case this can easily be done for Ferguson's formula. If the initial state (A,B,C) satisfies $A=B$, or equivalently, if the initial state in barycentric coordinates has the form $(a,a,1-2a)$, then the *Mathematica* function defined in Figure 5 gives the exit probability in (4.2).

Ferguson (1995) showed that standard two-dimensional Brownian motion starting at $(0,\sqrt{3}/2)$ exits the equilateral triangle with vertices $(-1,0)$, $(1,0)$, and $(0,\sqrt{3})$

```
In[1]:= exit[a_] := (q = InverseBetaRegularized[1 - 2 a, 1/2, 1/6];
          y0 = Sqrt[q / (1 - q)];
          theta = ArcTan[(y0^2 - 1) / (2 y0)];
          N[2 (Pi / 2 - theta) / (2 Pi), 18]);
        exit[1/3]
        exit[1/4]
Out[2]= 0.333333333333333333
Out[3]= 0.142154976126000748
```

FIG. 5. *Mathematica code for the right-hand side of (4.2) when the initial state, in barycentric coordinates, is $(a,a,1-2a)$.*

TABLE 9
$P_{A,B,C}(\sigma)$ for $(A, B, C) = (2n, 3n, 5n)$ ($1 \le n \le 15$ and $n = 20, 25, 30$), rounded to 12 significant digits, in support of the scaling conjecture. Here we include only three choices of σ. Results for the others can be deduced from (3.1)

A, B, C	$\sigma = 213$	$\sigma = 312$	$\sigma = 321$
2, 3, 5	0.190419015064	0.0704242611225	0.0662121426098
4, 6, 10	0.190374670083	0.0704067672263	0.0662043067857
6, 9, 15	0.190371967724	0.0704057817695	0.0662038677034
8, 12, 20	0.190371502992	0.0704056143412	0.0662037932082
10, 15, 25	0.190371375036	0.0704055684270	0.0662037727906
12, 18, 30	0.190371328913	0.0704055519070	0.0662037654463
14, 21, 35	0.190371309103	0.0704055448186	0.0662037622955
16, 24, 40	0.190371299477	0.0704055413763	0.0662037607656
18, 27, 45	0.190371294349	0.0704055395436	0.0662037599511
20, 30, 50	0.190371291418	0.0704055384960	0.0662037594855
22, 33, 55	0.190371289645	0.0704055378624	0.0662037592040
24, 36, 60	0.190371288521	0.0704055374610	0.0662037590256
26, 39, 65	0.190371287781	0.0704055371968	0.0662037589082
28, 42, 70	0.190371287279	0.0704055370172	0.0662037588284
30, 45, 75	0.190371286927	0.0704055368917	0.0662037587726
40, 60, 100	0.190371286171	0.0704055366216	0.0662037586526
50, 75, 125	0.190371285964	0.0704055365478	0.0662037586198
60, 90, 150	0.190371285890	0.0704055365213	0.0662037586080

along the x-axis with probability about 0.1421. Now $(0, \sqrt{3}/2)$ has barycentric coordinates $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, so Figure 5 shows that Ferguson's probability, evaluated to 12 decimal places, is 0.142154976126. On the other hand, the corresponding gambler's ruin probability with $N = 300$ is

$$\begin{aligned} &P_{75,75,150}(\text{player 3 goes broke first}) \\ &\quad = P_{75,75,150}(312) + P_{75,75,150}(321) \\ &\quad \doteq 0.142154976161, \end{aligned} \tag{4.4}$$

which we have computed to 18 decimal places, the first ten of which agree with Ferguson's number!

In support of the scaling conjecture we present evidence in Table 9. We have looked at many other examples. Scaling to good approximation seems to hold always.

A second piece of evidence comes from

$$\begin{aligned} P_{i,i,N-2i}(123) &= P_{i,i,N-2i}(213) \\ &= \frac{1}{2}\left(1 - \frac{2i}{N}\right), \quad 1 \le i < N/2. \end{aligned} \tag{4.5}$$

These are *exactly* invariant under scaling. Indeed, they match the ICM.

A third piece of evidence comes from the Brownian motion approximation of the random walk. As we have already seen for $k = 3$, the gambler's ruin walk converges to Brownian motion on the k-simplex (Denisov and Wachtel, 2015). It follows that the first hitting probabilities converge to those of Brownian motion. Finally, the Brownian motion extinction probabilities are scale invariant via properties of Brownian motion.

A fourth piece of evidence comes from the asymptotic approximation (2.4) above. This is (approximately) scale invariant.

The rapid convergence of rescaled probabilities (as seen in Table 9) is surprising. Theorem 4.1 shows that these approach limits expressible in terms of standard two-dimensional Brownian motion. We might denote the limit of $P_{nA,nB,nC}(\sigma)$ as $n \to \infty$ by $P^{\text{BM}}_{A,B,C}(\sigma)$, where the superscript refers to Brownian motion. For example, if $\sigma = 321$, this limit is given by (4.3). Usually, Gaussian approximation of features of random walk converge at rate $1/\sqrt{N}$. The numerics would be explained by the following conjecture, which may be regarded as a more precise version of the scaling conjecture (4.1).

CONJECTURE 4.2.

(a) *For each $A, B, C \ge 1$, $\sigma \in S_3$, and $n \ge 2$,*

$$|P_{nA,nB,nC}(\sigma) - P_{A,B,C}(\sigma)| < 0.0004.$$

(b) *For each $A, B, C \ge 1$ and $\sigma \in S_3$, let $N := A + B + C$. Then*

$$|P_{nA,nB,nC}(\sigma) - P^{\text{BM}}_{A,B,C}(\sigma)| = O\left(\frac{1}{(nN)^4}\right) \quad \text{as } n \to \infty.$$

In the case of (a), we have found differences as large as 0.000383. As for (b), the ten-digit match seen in (4.4) is consistent with this because $1/(300)^4 \doteq 1.235 \times 10^{-10}$.

In practical problems scale invariance and smoothness (so fine details don't matter much) can reduce things to "manageable numbers" within the range of computer calculation.

TABLE 10

$P_{A,B,C,D}(\sigma)$ for $(A,B,C,D)=(n,2n,3n,4n)$ $(1\le n\le 10)$ and four choices of $\sigma\in S_4$, rounded to nine decimal places, in support of the scaling conjecture. Notice that the rate of convergence appears to be slower than for the three-player data in Table 9

A, B, C, D	$\sigma = 1234$	$\sigma = 2143$	$\sigma = 3412$	$\sigma = 4321$
1, 2, 3, 4	0.147755766	0.055231830	0.012087939	0.007499579
2, 4, 6, 8	0.148462055	0.054618468	0.012147611	0.007459339
3, 6, 9, 12	0.148582024	0.054511807	0.012158593	0.007452294
4, 8, 12, 16	0.148621208	0.054476628	0.012162415	0.007449874
5, 10, 15, 20	0.148638685	0.054460859	0.012164179	0.007448762
6, 12, 18, 24	0.148647981	0.054452450	0.012165136	0.007448161
7, 14, 21, 28	0.148653514	0.054447436	0.012165712	0.007447800
8, 16, 24, 32	0.148657074	0.054444206	0.012166086	0.007447565
9, 18, 27, 36	0.148659501	0.054442002	0.012166342	0.007447405
10, 20, 30, 40	0.148661229	0.054440432	0.012166526	0.007447290

4.2 Gambler's Ruin with More than Three Players

The questions above make sense for k players with initial capitals $A_1, A_2, \ldots, A_k$. The exact calculations of Section 2.2 are (potentially) available. We have carried them out to give exact results for $k=4$ and $N := A_1 + A_2 + A_3 + A_4$ as large as 100. The results for $N = 100$ are in the Supplementary Materials (Section 6). By analogy with (4.5),

$$P_{i,i,i,N-3i}(1234) = \frac{1}{6}\left(1-\frac{3i}{N}\right), \quad 1 \le i < N/3.$$

The scaling conjecture for $k=4$, either in the form

$$P_{A',B',C',D'}(\sigma) \doteq P_{A,B,C,D}(\sigma)$$
$$\text{whenever } \frac{A'}{A} = \frac{B'}{B} = \frac{C'}{C} = \frac{D'}{D},$$

or in the equivalent form

$$P_{nA,nB,nC,nD}(\sigma) \doteq P_{A,B,C,D}(\sigma), \quad n \ge 2,$$

seems to hold. Here $A, B, C, D \ge 1$ and $\sigma \in S_4$ are arbitrary. Table 10 gives a few data points. These numbers are consistent with those of Marfil and David (2020). Notice that convergence is slower for four players than for three.

The ICM formula is available for all k. Preliminary investigations (including Table 13) suggest it is just as unreliable as an approximation to $P_{A_1,\ldots,A_k}(\sigma)$ as it is when $k=3$. We have tried interpolation (Section 4.3) but not yet regression. Plots such as Figure 4 are not feasible when $k=4$.

One final point: The constant$/N^3$ results described above for $k=3$ should not stir false hope of similar results for $k=4$. There are reasons to expect that

$$P_{1,1,1,N-3}(4321) \sim \frac{\text{constant}}{N^\kappa}$$

with κ an irrational number. This (heuristically) follows from the connection between gambler's ruin and the "cops and robbers" problem. See Ratzkin and Treibergs (2009). Table 11 gives ten data points, which suggest $\kappa = 5.72\cdots$.

4.3 Linear Interpolation for Four Players

Just as we could interpolate three-player elimination order probabilities with arbitrary N from three known such probabilities with $N = 300$, we can also interpolate four-player elimination order probabilities with arbitrary N from four known such probabilities with $N = 100$.

Given positive integers A, B, C, and D, let $N := A + B + C + D$ and

$$A_0 := A\frac{100}{N}, \qquad B_0 := B\frac{100}{N},$$
$$C_0 := C\frac{100}{N}, \qquad D_0 := D\frac{100}{N}.$$

Typically, these are not integers. Therefore, consider the eight points

$$\boldsymbol{v}_{000} := (\lfloor A_0\rfloor, \lfloor B_0\rfloor, \lfloor C_0\rfloor, 100 - \lfloor A_0\rfloor - \lfloor B_0\rfloor - \lfloor C_0\rfloor),$$
$$\boldsymbol{v}_{001} := (\lfloor A_0\rfloor, \lfloor B_0\rfloor, \lceil C_0\rceil, 100 - \lfloor A_0\rfloor - \lfloor B_0\rfloor - \lceil C_0\rceil),$$
$$\boldsymbol{v}_{010} := (\lfloor A_0\rfloor, \lceil B_0\rceil, \lfloor C_0\rfloor, 100 - \lfloor A_0\rfloor - \lceil B_0\rceil - \lfloor C_0\rfloor),$$
$$\boldsymbol{v}_{100} := (\lceil A_0\rceil, \lfloor B_0\rfloor, \lfloor C_0\rfloor, 100 - \lceil A_0\rceil - \lfloor B_0\rfloor - \lfloor C_0\rfloor),$$

TABLE 11

$P_{1,1,1,N-3}(4321)$ for $N = 10, 20, \ldots, 100$

N	$P_{1,1,1,N-3}(4321)$	N	$P_{1,1,1,N-3}(4321)$
10	$2.61956573 \times 10^{-4}$	60	$9.43556904 \times 10^{-9}$
20	$5.03729359 \times 10^{-6}$	70	$3.90711745 \times 10^{-9}$
30	$4.96691782 \times 10^{-7}$	80	$1.82032195 \times 10^{-9}$
40	$9.58966829 \times 10^{-8}$	90	$9.28008330 \times 10^{-10}$
50	$2.67684672 \times 10^{-8}$	100	$5.07937120 \times 10^{-10}$

TABLE 12
The final four in the 2019 *World Series of Poker Millionaire Maker Event. The big blind was* 1,600,000

Player	Chip count	Big blinds (rounded)	Big blinds ×16 (exact)	Actual payoff
Vincas Tamasauskas	9,700,000	6	97	\$464,375
Lokesh Garg	12,500,000	8	125	\$619,017
John Gorsuch	14,400,000	9	144	\$1,344,930
Kazuki Ikeuchi	183,900,000	115	1839	\$830,783
Totals	220,500,000	138	2205	

$$\boldsymbol{v}_{011} := (\lfloor A_0 \rfloor, \lceil B_0 \rceil, \lceil C_0 \rceil, 100 - \lfloor A_0 \rfloor - \lceil B_0 \rceil - \lceil C_0 \rceil),$$
$$\boldsymbol{v}_{101} := (\lceil A_0 \rceil, \lfloor B_0 \rfloor, \lceil C_0 \rceil, 100 - \lceil A_0 \rceil - \lfloor B_0 \rfloor - \lceil C_0 \rceil),$$
$$\boldsymbol{v}_{110} := (\lceil A_0 \rceil, \lceil B_0 \rceil, \lfloor C_0 \rfloor, 100 - \lceil A_0 \rceil - \lceil B_0 \rceil - \lfloor C_0 \rfloor),$$
$$\boldsymbol{v}_{111} := (\lceil A_0 \rceil, \lceil B_0 \rceil, \lceil C_0 \rceil, 100 - \lceil A_0 \rceil - \lceil B_0 \rceil - \lceil C_0 \rceil),$$

and choose four of them for the purpose of linear interpolation, discarding any whose fourth coordinate is neither $\lfloor D_0 \rfloor$ nor $\lceil D_0 \rceil$. Denote by $\{a\} := a - \lfloor a \rfloor$ the fractional part of a.

If $\{A_0\} + \{B_0\} + \{C_0\} \in (0, 1)$, then we choose $\boldsymbol{v}_{000}$, $\boldsymbol{v}_{001}$, $\boldsymbol{v}_{010}$, and $\boldsymbol{v}_{100}$.

If $\{A_0\} + \{B_0\} + \{C_0\} \in (2, 3)$, then we choose $\boldsymbol{v}_{011}$, $\boldsymbol{v}_{101}$, $\boldsymbol{v}_{110}$, and $\boldsymbol{v}_{111}$.

If $\{A_0\} + \{B_0\} + \{C_0\} \in (1, 2)$, then we choose four of the six points $\boldsymbol{v}_{001}$, $\boldsymbol{v}_{010}$, $\boldsymbol{v}_{100}$, $\boldsymbol{v}_{011}$, $\boldsymbol{v}_{101}$, and $\boldsymbol{v}_{110}$ in such a way that the resulting tetrahedron contains (A_0, B_0, C_0, D_0) in its interior. The choice is not unique.

Let us call these four points (A_i, B_i, C_i, D_i) $(i = 1, 2, 3, 4)$. We can estimate $P_{A,B,C,D}(\sigma)$ by linear interpolation from the four values of $P_{A_i,B_i,C_i,D_i}(\sigma)$ $(i = 1, 2, 3, 4)$. As before, we represent (A_0, B_0, C_0, D_0) in barycentric coordinates. The relevant weights are

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} := \begin{pmatrix} A_1 - A_4 & A_2 - A_4 & A_3 - A_4 \\ B_1 - B_4 & B_2 - B_4 & B_3 - B_4 \\ C_1 - C_4 & C_2 - C_4 & C_3 - C_4 \end{pmatrix}^{-1} \cdot \begin{pmatrix} A_0 - A_4 \\ B_0 - B_4 \\ C_0 - C_4 \end{pmatrix}$$

and $\lambda_4 := 1 - \lambda_1 - \lambda_2 - \lambda_3$, so that

$$(A_0, B_0, C_0, D_0) = \sum_{i=1}^{4} \lambda_i (A_i, B_i, C_i, D_i),$$

and our interpolation estimate is then

$$\bar{P}_{A,B,C,D}(\sigma) := \sum_{i=1}^{4} \lambda_i P_{A_i,B_i,C_i,D_i}(\sigma).$$

If one or more of the weights λ_i is negative, that indicates (A_0, B_0, C_0, D_0) lies outside the resulting tetrahedron, and we must choose the four points differently.

EXAMPLE 4.3. At the final table of the 2019 World Series of Poker Millionaire Maker Event, at the time the fifth-place finisher was eliminated, the remaining four players had chip counts (in units of 100,000, 1/16 of the big blind) equal to $A = 97$, $B = 125$, $C = 144$, and $D = 1839$ (WSOP, 2019b). See Table 12. Thus, $N = 2205$ and A, B, C, and D, multiplied by $100/N$, are $A_0 \doteq 4.40$, $B_0 \doteq 5.67$, $C_0 \doteq 6.53$, and $D_0 \doteq 83.40$. Since $\{A_0\} + \{B_0\} + \{C_0\} \doteq 1.60$, we must choose four of the six vertices $\boldsymbol{v}_2 = (4, 5, 7, 84)$, $\boldsymbol{v}_3 = (4, 6, 6, 84)$, $\boldsymbol{v}_4 = (5, 5, 6, 84)$, $\boldsymbol{v}_5 = (4, 6, 7, 83)$, $\boldsymbol{v}_6 = (5, 5, 7, 83)$, and $\boldsymbol{v}_7 = (5, 6, 6, 83)$. We choose $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, $\boldsymbol{v}_5$, and $\boldsymbol{v}_7$, the four points closest to (A_0, B_0, C_0, D_0). We find that

$$\lambda_1 = \frac{146}{441}, \qquad \lambda_2 = \frac{31}{441},$$
$$\lambda_3 = \frac{88}{441}, \qquad \lambda_4 = \frac{176}{441},$$

and results are shown in Table 13. For the record, the actual elimination order turned out to be $\sigma = 1243$, the seventh most likely result.

5. SUMMARY

In summary, we have discussed six different methods of approximating the gambler's ruin probabilities:

1. Exact computation by Markov chain methods (Section 2.2)
2. Arbitrarily precise computation by Jacobi iteration (Section 2.3)
3. Linear interpolation from exact probabilities (Section 2.4)
4. Monte Carlo methods (Section 2.5)
5. Regression on ICM (Section 3)
6. Approximation by Brownian motion (Section 4.1)

While exact computation using Markov chain methods and arbitrarily precise computation using Jacobi iteration are feasible for $N := A + B + C$ not too large, it seems difficult for N of practical interest. Linear interpolation is our preferred method, using the nearly exact results for $N = 300$. Monte Carlo allows computation for a single

TABLE 13
For $(A, B, C, D) = (97, 125, 144, 1839)$, *row* (a) *gives* $P^{\mathrm{ICM}}_{A,B,C,D}(\sigma)$, *and row* (b) *gives the interpolated approximations* $\bar{P}^{\mathrm{GR}}_{A,B,C,D}(\sigma)$. *Here* $\varepsilon = 10^{-5}$

	$\sigma = 1234$	$\sigma = 1243$	$\sigma = 1324$	$\sigma = 1342$	$\sigma = 1423$	$\sigma = 1432$
	$\sigma = 2134$	$\sigma = 2143$	$\sigma = 2314$	$\sigma = 2341$	$\sigma = 2413$	$\sigma = 2431$
	$\sigma = 3124$	$\sigma = 3142$	$\sigma = 3214$	$\sigma = 3241$	$\sigma = 3412$	$\sigma = 3421$
	$\sigma = 4123$	$\sigma = 4132$	$\sigma = 4213$	$\sigma = 4231$	$\sigma = 4312$	$\sigma = 4321$
(a)	0.184762	0.0328106	0.170195	0.0299478	0.00376238	0.00372801
	0.143375	0.0254611	0.118324	0.0205440	0.00287798	0.00281381
	0.114645	0.0201732	0.102712	0.0178333	0.00245171	0.00241914
	0.000198451	0.000196638	0.000195621	0.000191260	0.000191977	0.000189427
(b)	0.193685	0.0365869	0.177240	0.0338414	0.00127429	0.00127379
	0.139180	0.0264827	0.118035	0.0228907	0.000947236	0.000946428
	0.107017	0.0207477	0.0988571	0.0193277	0.000811526	0.000811141
	$0.751143\,\varepsilon$	$0.750991\,\varepsilon$	$0.750703\,\varepsilon$	$0.750331\,\varepsilon$	$0.750250\,\varepsilon$	$0.750030\,\varepsilon$

A, B, C of interest and is useful for two- or three-digit accuracy. Regression analysis is quite accurate but probably needs an app to to be "real-time useful." Brownian approximation changes the problem into one that requires special function calculations and so probably also needs an app. Finally, the widely used ICM is roughly useful (say for single-digit accuracy), and it can be "done in your head."

6. SUPPLEMENTARY MATERIALS

Supplementary materials include four *Mathematica* programs, four output files, and three regression analyses using *Mathematica*. Here are the details.

1. *Mathematica* program to compute three-player elimination order probabilities in double precision by Markov chain methods (N arbitrary), and output when $N = 200$.
http://www.math.utah.edu/~ethier/3ruin-program.nb,
http://www.math.utah.edu/~ethier/3ruin200-output.
2. *Mathematica* program to compute three-player elimination order probabilities in double precision by iteration (N arbitrary), and output when $N = 300$.
http://www.math.utah.edu/~ethier/3iteration-program.nb,
http://www.math.utah.edu/~ethier/3iteration300-output.
3. *Mathematica* program to compute four-player elimination order probabilities in single precision by Markov chain methods (N arbitrary), and output when $N = 50$.
http://www.math.utah.edu/~ethier/4ruin-program.nb,
http://www.math.utah.edu/~ethier/4ruin50-output.
4. *Mathematica* program to compute four-player elimination order probabilities in single precision by iteration (N arbitrary), and output when $N = 100$.
http://www.math.utah.edu/~ethier/4iteration-program.nb,
http://www.math.utah.edu/~ethier/4iteration100-output.
5. *Mathematica* files containing regression analyses for $\sigma = 321$, 312, and 213.
http://www.math.utah.edu/~ethier/regression321-300.nb,
http://www.math.utah.edu/~ethier/regression312-300.nb,
http://www.math.utah.edu/~ethier/regression213-300.nb.

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