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Sharpness of Lenglart's domination inequality and a sharp monotone version*

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Abstract

We prove that the best so far known constant $c_p = \frac{p^{-p}}{1-p}$, $p \in (0,1)$ of a domination inequality, which originates to Lenglart, is sharp. In particular, we solve an open question posed by Revuz and Yor [12]. Motivated by the application to maximal inequalities, like e.g. the Burkholder-Davis-Gundy inequality, we also study the domination inequality under an additional monotonicity assumption. In this special case, a constant which stays bounded for p near 1 was proven by Pratelli and Lenglart. We provide the sharp constant for this case.

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1 Introduction

In this note, we prove that the best so far known constant c_p of a domination inequality, which originates to Lenglart [6, Corollaire II] (see Theorem 1.1), is sharp. In particular, we solve an open question posed by Revuz and Yor [12, Question IV.1, p.178]. Furthermore, motivated by the method of applying Lenglart's inequality to extend maximal inequalities to small exponents, we study Lenglart's domination inequality under an additional monotonicity assumption: A result by Pratelli [10] and Lenglart [6] implies (under the additional monotonicity assumption) a constant, which is bounded by 2, and hence considerably improves the constant of Lenglart's inequality for p near 1. We provide a sharp constant. The sharpness of our monotone version of Lenglart's inequality is related to a result by Wang [16].

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space satisfying the usual conditions. The following lemma is [8, Lemma 2.2 (ii)]:

Theorem 1.1 (Lenglart's inequality)**.** *Let X and G be non-negative adapted right-continuous processes, and let G be in addition non-decreasing and predictable such that $\mathbb{E}[X_\tau \mid \mathcal{F}_0] \leq \mathbb{E}[G_\tau \mid \mathcal{F}_0] \leq \infty$ for any bounded stopping time τ. Then for all $p \in (0,1)$,*

$$\mathbb{E}\left[\left(\sup_{t\geq 0} X_t\right)^p \,\middle|\, \mathcal{F}_0\right] \leq c_p \mathbb{E}\left[\left(\sup_{t\geq 0} G_t\right)^p \,\middle|\, \mathcal{F}_0\right],$$

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where $c_p := \frac{p^{-p}}{1-p}$.

In the original work by Lenglart [6, Corollaire II], the inequality is proven for $c_p = \frac{2-p}{1-p}$, $p \in (0,1)$. The constant c_p is improved to $\frac{p^{-p}}{1-p}$ by Revuz and Yor in [12, Exercise IV.4.30] for continuous processes X and G. This result is generalized to càdlàg processes by Ren and Shen in [11, Theorem 1] and is extended to a more general setting than [6, Corollaire II] by Mehri and Scheutzow [8, Lemma 2.2 (ii)]. Furthermore, the growth rate of the optimal constant $c_p^{(opt)}$ for càdlàg processes has been studied (see [11, Theorem 2]): It holds that $(c_p^{(opt)})^{1/p} = O(1/p)$ for $p \to 0^+$. We prove (see Theorem 2.1) that $\frac{p^{-p}}{1-p}$ is sharp.

Lenglart's inequality yields a very short proof of the Burkholder-Davis-Gundy inequality for continuous local martingales for small exponents (see e.g. [12, Theorem IV.4.1]): Let $(M_t)_{t\geq 0}$ be a continuous local martingale with $M_0 = 0$. To prove $\mathbb{E}[\langle M,M\rangle_t^{q/2}] \lesssim \mathbb{E}[\sup_{t\geq 0}|M_t|^q]$ for $q \in (0,2)$, take

$$X_t := \langle M,M\rangle_t, \qquad G_t := \sup_{0\leq s\leq t}|M_s|^2.$$

Using that $M_t^2 - \langle M,M\rangle_t$ is a continuous local martingale, we have $\mathbb{E}[X_\tau] \leq \mathbb{E}[G_\tau]$ for any bounded stopping time τ. Applying Lenglart's inequality with $p = q/2$, we obtain

$$\mathbb{E}[\langle M,M\rangle_t^{q/2}] \leq c_{q/2}\mathbb{E}[\sup_{t\geq 0}|M_t|^q].$$

For $q = 1$, this implies $c_{BDG,1} = c_{q/2} = 2\sqrt{2} \approx 2,8284$. The optimal BDG constant can be computed numerically for this case (see Schachermayer and Stebegg [13]) and is $c_{BDG,1}^{(opt)} \approx 1,2727$. A better constant than $c_{q/2}$ can be achieved if we apply the following proposition due to Lenglart [6, Proposition I] and Pratelli [10, Proposition 1.2] instead:

Proposition 1.2 (Lenglart, Pratelli)**.** *Let F be a concave non-decreasing function with $F(0) = 0$ and let $c > 0$ be a constant. Let Y and G be adapted non-negative right-continuous processes starting in 0. Furthermore, let G be non-decreasing and predictable. Assume that $\mathbb{E}[Y_\tau] \leq c\mathbb{E}[G_\tau]$ holds for all finite stopping times τ. Then, for all finite stopping times τ, we have*

$$\mathbb{E}[F(Y_\tau)] \leq (1+c)\mathbb{E}[F(G_\tau)].$$

Let X and G be as in Theorem 1.1. Assume in addition that both processes start in 0. Then Proposition 1.2 implies, choosing $F(x) = x^p$ for some $p \in (0,1)$ and optimizing over c, that

$$\mathbb{E}[X_\tau^p] \leq (1-p)^{-(1-p)}p^{-p}\mathbb{E}[G_\tau^p]. \tag{1.1}$$

Hence, Proposition 1.2 gives $c_{BDG,1} = 2$. We show that the constant of inequality (1.1) can be improved to p^{-p} (see Theorem 2.2 and Remark 2.4), which is sharp. In particular, by the argument described above we now achieve $c_{BDG,1} = \sqrt{2} \approx 1,4142$. For the right-hand side of the BDG inequality $\mathbb{E}[\sup_{t\geq 0}|M_t|] \lesssim \mathbb{E}[\langle M,M\rangle_t^{1/2}]$, the monotone version of Lenglart's inequality does not yield a sharper constant than the normal Lenglart's inequality.

Lenglart's inequality is frequently applied to extrapolate maximal inequalities to smaller exponents (see e.g. [2], [7], [14], [15] and [17]). Furthermore, Lenglart's inequality is a useful tool for proving stochastic Gronwall inequalities (see e.g. [1] and [8]) and more generally studying SDEs (see e.g. [5] and [9]). In many of the application examples listed above, the additional assumption, that X is non-decreasing is satisfied. Hence, instead, Theorem 2.2 could be applied, improving the constant considerably for p near 1.

2 Main results

We assume, unless otherwise stated, that all processes are defined on an underlying filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ which satisfies the usual conditions. The following theorem answers the open question posed by Revuz and Yor [12, Question IV.1, p.178].

Theorem 2.1 (Sharpness of Lenglart's inequality)**.** *For all $p \in (0,1)$, there exist families of continuous processes $X^{(n)} = (X_t^{(n)})_{t\geq 0}$ and $G^{(n)} = (G_t^{(n)})_{t\geq 0}$ (depending on p) which satisfy the assumptions of Theorem 1.1 such that*

$$\frac{p^{-p}}{1-p} = \lim_{n\to\infty} \frac{\mathbb{E}\Big[\big(\sup_{t\geq 0} X_t^{(n)}\big)^p\Big]}{\mathbb{E}\Big[\big(\sup_{t\geq 0} G_t^{(n)}\big)^p\Big]}. \tag{2.1}$$

In particular, the constant $c_p = \frac{p^{-p}}{1-p}$ in Theorem 1.1 is sharp.

As explained in the introduction, the application to maximal inequalities motivates us to consider the following monotone version of Lenglart's inequality. We assume in addition that X is non-decreasing and obtain a considerably improved constant for p near 1.

Theorem 2.2 (Sharp monotone Lenglart's inequality)**.** *Let X and G be non-decreasing non-negative adapted right-continuous processes, and let G be in addition predictable such that $\mathbb{E}[X_\tau \mid \mathcal{F}_0] \leq \mathbb{E}[G_\tau \mid \mathcal{F}_0] \leq \infty$ for any bounded stopping time τ. Then for all $p \in (0,1)$,*

$$\mathbb{E}\left[\left(\sup_{t\geq 0} X_t\right)^p \,\middle|\, \mathcal{F}_0\right] \leq p^{-p}\, \mathbb{E}\left[\left(\sup_{t\geq 0} G_t\right)^p \,\middle|\, \mathcal{F}_0\right]. \tag{2.2}$$

Furthermore, for all $p \in (0,1)$ there exist continuous processes $\tilde{X} = (\tilde{X}_t)_{t\geq 0}$ and $\tilde{G} = (\tilde{G}_t)_{t\geq 0}$, satisfying the assumptions above such that

$$p^{-p} = \lim_{n\to\infty} \frac{\mathbb{E}\Big[\big(\sup_{t\geq 0} \tilde{X}_{t\wedge n}\big)^p\Big]}{\mathbb{E}\Big[\big(\sup_{t\geq 0} \tilde{G}_{t\wedge n}\big)^p\Big]}.$$

In particular, the constant p^{-p} is sharp.

Remark 2.3. Inequality (2.2) is a sharpened special case of Proposition 1.2, its proof is a modification of the proof of [10, Proposition 1.2]. The theorem generalizes a result by Garsia [4, Theorem III.4.4, page 113]. In [16, Theorem 2], Wang proved that [4, Theorem III.4.4, page 113] is sharp. Hence, by translating his result from discrete to continuous time proves sharpness of p^{-p}.

Remark 2.4. Theorem 2.2 can be also applied when X is not non-decreasing. In that case, the theorem implies for any stopping time τ the inequality $\mathbb{E}[X_\tau^p] \leq p^{-p}\, \mathbb{E}[G_\tau^p]$. This can by seen by defining $\hat{X}_t := X_\tau \mathbb{1}_{[\tau,\infty)}(t)$ for all $t \geq 0$ and noting that $(\hat{X}_t)_{t\geq 0}$ and $(G_{t\wedge\tau})_{t\geq 0}$ satisfy the assumptions of Theorem 2.2.

Remark 2.5. In Theorem 2.2, the assumption that G is right-continuous and predictable can be replaced by the assumption that G is left-continuous and adapted.

Remark 2.6. A key part of the proof of Lenglart's inequality is the inequality

$$\mathbb{P}\left(\sup_{t\geq 0} X_t > c \,\middle|\, F_0\right) \leq \frac{1}{c}\mathbb{E}\left[\sup_{t\geq 0} G_t \wedge d \,\middle|\, \mathcal{F}_0\right] + \mathbb{P}\left(\sup_{t\geq 0} G_t \geq d \,\middle|\, \mathcal{F}_0\right)$$

for all $c, d > 0$. If X is non-decreasing, this can be improved to

$$\frac{1}{c}\mathbb{E}\left[\sup_{t\geq 0} X_t \wedge c \,\middle|\, \mathcal{F}_0\right] \leq \frac{1}{c}\mathbb{E}\left[\sup_{t\geq 0} G_t \wedge d \,\middle|\, \mathcal{F}_0\right] + \mathbb{P}\left(\sup_{t\geq 0} G_t \geq d \,\middle|\, \mathcal{F}_0\right),$$

which is used to prove the monotone version of Lenglart’s inequality.

Remark 2.7. If G is not predictable and no further assumptions are made, then there exists no finite constant in inequality (2.2). An example which demonstrates this can be found in [6, Remarque after Corollaire II].

Theorem 1.1, Theorem 2.1, and Theorem 2.2 also hold in discrete time. Here, sharpness of p^{-p} follows immediately from [16, Theorem 2].

Corollary 2.8 (Discrete Lenglart’s inequality)**.** *Let $(X_n)_{n\in\mathbb{N}_0}$ and $(G_n)_{n\in\mathbb{N}_0}$ be non-negative adapted processes, and let G be in addition non-decreasing and predictable such that $\mathbb{E}[X_\tau \mid \mathcal{F}_0] \leq \mathbb{E}[G_\tau \mid \mathcal{F}_0] \leq \infty$ for any bounded stopping time τ. Then for all $p \in (0,1)$,*

$$\mathbb{E}\left[\left(\sup_{n\in\mathbb{N}_0} X_n\right)^p \,\middle|\, \mathcal{F}_0\right] \leq c_p\, \mathbb{E}\left[\left(\sup_{n\in\mathbb{N}_0} G_n\right)^p \,\middle|\, \mathcal{F}_0\right], \tag{2.3}$$

where $c_p := \frac{p^{-p}}{1-p}$ and the constant c_p is sharp.

If we assume in addition, that $(X_n)_{n\in\mathbb{N}_0}$ is non-decreasing, then we have

$$\mathbb{E}\left[\left(\sup_{n\in\mathbb{N}_0} X_n\right)^p \,\middle|\, \mathcal{F}_0\right] \leq p^{-p}\, \mathbb{E}\left[\left(\sup_{n\in\mathbb{N}_0} G_n\right)^p \,\middle|\, \mathcal{F}_0\right] \tag{2.4}$$

and the constant p^{-p} is sharp.

3 Proof of Theorem 2.1

Proof of Theorem 2.1. Choose an arbitrary $p \in (0,1)$ for the remainder of this proof. First, we define non-decreasing processes $\tilde{X} = (\tilde{X}_t)_{t\geq 0}$ and $\tilde{G} = (\tilde{G}_t)_{t\geq 0}$ which satisfy the assumptions of Theorem 1.1, such that

$$p^{-p} = \lim_{n\to\infty} \frac{\mathbb{E}\big[\big(\sup_{t\geq 0} \tilde{X}_{t\wedge n}\big)^p\big]}{\mathbb{E}\big[\big(\sup_{t\geq 0} \tilde{G}_{t\wedge n}\big)^p\big]}.$$

To obtain the extra factor $(1-p)^{-1}$, we modify $\tilde{X}$ and $\tilde{G}$ using an independent Brownian motion: This gives us the families $\{(X_t^{(n)})_{t\geq 0}, n \in \mathbb{N}\}$ and $\{(G_t^{(n)})_{t\geq 0}, n \in \mathbb{N}\}$.

Note that if we have non-negative random variables $X_{RV} := 1$ and G_{RV} with $\mathbb{E}[X_{RV}] = \mathbb{E}[G_{RV}]$, then we obtain $\mathbb{E}[X_{RV}^p] >> \mathbb{E}[G_{RV}^p]$ for example by choosing G_{RV} to be very large on a set with small probability and everywhere else 0. Keeping this in mind, we construct $\tilde{X}$ and $\tilde{G}$ as follows: Let Z be an exponentially distributed random variable on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[Z] = 1$. Set

$$A : [0,\infty) \to [0,\infty), \quad t \mapsto \exp(t/p).$$

Define for all $t \geq 0$

$$\tilde{X}_t := A(Z)\mathbb{1}_{[Z,\infty)}(t), \qquad \tilde{G}_t := \int_0^{t\wedge Z} A(s)\mathrm{d}s.$$

Choose $\tilde{\mathcal{F}}_t := \sigma(\{Z \leq r\} \mid 0 \leq r \leq t)$ for all $t \geq 0$. Observe that $\tilde{X}$ and $\tilde{G}$ are non-decreasing non-negative adapted right-continuous processes, and $\tilde{G}$ is in addition continuous, hence predictable. Furthermore, due to Z being exponentially distributed, $\tilde{G}$ is the compensator of $\tilde{X}$, implying $\mathbb{E}[\tilde{X}_\tau] = \mathbb{E}[\tilde{G}_\tau]$ for all bounded τ.

Now we use the processes $\tilde{X}$ and $\tilde{G}$ to construct the families $\{(X_t^{(n)})_{t\geq 0}, n \in \mathbb{N}\}$ and $\{(G_t^{(n)})_{t\geq 0}, n \in \mathbb{N}\}$: Assume w.l.o.g. that there exists a Brownian motion B on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the smallest filtration satisfying the usual conditions which contains $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ and w.r.t. which B is a Brownian motion. Denote by $g_{n,n+1} : [0,\infty) \to [0,1]$ a continuous non-decreasing function such that

$$g_{n,n+1}(t) = 0 \quad \forall t \leq n, \quad \text{and} \quad g_{n,n+1}(t) = 1 \quad \forall t \geq n+1. \tag{3.1}$$

Define:

$$\begin{aligned}
\tau^{(n)} &:= \inf\{t \geq n+1 \mid \tilde{X}_n + (B_t - B_{n+1})\mathbb{1}_{\{t\geq n+1\}} = 0\},\\
X_t^{(n)} &:= g_{n,n+1}(t)\tilde{X}_n + (B_{t\wedge\tau^{(n)}} - B_{t\wedge(n+1)})\\
G_t^{(n)} &:= \tilde{G}_{t\wedge n}
\end{aligned}$$

The stopping time $\tau^{(n)}$ ensures that $X_t^{(n)}$ is non-negative. By construction, we have for every bounded $(\mathcal{F}_t)_{t\geq 0}$ stopping time τ

$$\mathbb{E}[X_\tau^{(n)}] \leq \mathbb{E}[\tilde{X}_{\tau\wedge n} + B_{\tau\wedge\tau^{(n)}} - B_{\tau\wedge(n+1)}] = \mathbb{E}[\tilde{G}_{\tau\wedge n}] = \mathbb{E}[G_\tau^{(n)}].$$

Hence, $(X_t^{(n)})_{t\geq 0}$ and $(G_t^{(n)})_{t\geq 0}$ are continuous processes that satisfy the assumptions of Theorem 1.1.

It remains to calculate $\mathbb{E}\big[\big(\sup_{t\geq 0} X_t^{(n)}\big)^p\big]$ and $\mathbb{E}\big[\big(\sup_{t\geq 0} G_t^{(n)}\big)^p\big]$, to show that equation (2.1) is satisfied. We have

$$\begin{aligned}
\mathbb{E}[\tilde{X}_t^p] &= \int_0^\infty A(x)^p \mathbb{1}_{\{t\geq x\}} \exp(-x)\mathrm{d}x = t,\\
\mathbb{E}[\tilde{G}_t^p] &= \int_0^\infty \left(\int_0^{t\wedge x} A(s)\mathrm{d}s\right)^p \exp(-x)\mathrm{d}x \leq p^p\,(t+1),
\end{aligned} \tag{3.2}$$

which implies in particular that $\mathbb{E}\big[\big(\sup_{t\geq 0} G_t^{(n)}\big)^p\big] \leq p^p\,(n+1)$.

We calculate $\mathbb{E}\big[\big(\sup_{t\geq 0} X_t^{(n)}\big)^p\big]$ using the independence of Z and B. To this end, let $\tilde{B}$ be some Brownian motion and consider for all $0 \leq x < a^{1/p}$ the stopping times

$$\sigma_x := \inf\{t \geq 0 \mid \tilde{B}_t + x = 0\}, \quad \sigma_{x,a} := \inf\{t \geq 0 \mid \tilde{B}_t + x = a^{1/p}\}.$$

Define the family of random variables $Y_x := \sup_{t\geq 0} \tilde{B}_{t\wedge\sigma_x} + x$, $x \geq 0$. Then $\mathbb{E}[\tilde{B}_{\sigma_x\wedge\sigma_{x,a}}] = 0$ implies $\mathbb{P}[Y_x \geq a^{1/p}] = \mathbb{P}[\sigma_{x,a} < \sigma_x] = xa^{-1/p}$, and hence

$$\mathbb{E}[Y_x^p] = x^p + \int_{x^p}^\infty \mathbb{P}[Y_x \geq a^{1/p}]\mathrm{d}a = x^p + x^p\frac{p}{1-p} = \frac{x^p}{1-p}. \tag{3.3}$$

Hence, we have by (3.2), (3.3) and independence of $(B_t - B_{n+1})_{t\geq n+1}$ and $\mathcal{F}_{n+1}$:

$$\begin{aligned}
\mathbb{E}\big[\big(\sup_{t\geq 0} X_t^{(n)}\big)^p\big] &= \mathbb{E}\big[\mathbb{E}\big[\big(\sup_{t\geq 0} X_t^{(n)}\big)^p \mid \mathcal{F}_{n+1}\big]\big]\\
&= \mathbb{E}\left[\frac{1}{1-p}\big(\tilde{X}_n\big)^p\right]\\
&= \frac{n}{1-p}.
\end{aligned}$$

Therefore, we have:

$$c_p \geq \frac{\mathbb{E}[(\sup_{t\geq 0} X_t^{(n)})^p]}{\mathbb{E}[(\sup_{t\geq 0} G_t^{(n)})^p]} \geq \frac{n}{1-p}\frac{p^{-p}}{n+1},$$

which implies (2.1). □

4 Proof of Theorem 2.2

Remark 4.1. The following proof of inequality (2.2) is a modification of the proof of [10, Proposition 1.2]. Sharpness of the constant can be proven using [16, Theorem 2].

Proof of Theorem 2.2. We first show that p^{-p} is the optimal constant. Sharpness of p^{-p} can be proven by translating [16, Theorem 2] into continuous time. Alternatively, one can use the processes $\tilde{X}$ and $\tilde{G}$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$ from the proof of Theorem 2.1: Equation (3.2) implies, that

$$p^{-p} = \lim_{n\to\infty} p^{-p}\frac{n}{n+1} \leq \lim_{n\to\infty} \frac{\mathbb{E}\Big[\Big(\sup_{t\geq 0}\tilde{X}_{t\wedge n}\Big)^p\Big]}{\mathbb{E}\Big[\Big(\sup_{t\geq 0}\tilde{G}_{t\wedge n}\Big)^p\Big]},$$

and therefore that p^{-p} is sharp.

Now we prove that inequality (2.2) holds true. We may assume w.l.o.g. that $(G_t)_{t\geq 0}$ is bounded (because it is predictable). This implies $\mathbb{E}[\sup_{t\geq 0} X_t] < \infty$. To shorten notation, we define

$$X_\infty := \sup_{t\geq 0} X_t, \qquad G_\infty := \sup_{t\geq 0} G_t. \tag{4.1}$$

We use the following formulas for positive random variables Z (equation (4.3) is a direct consequence of (4.2), alternatively see also [3, Theorem 20.1, p. 38-39]):

$$\mathbb{E}[Z^p \mid \mathcal{F}_0] = \int_0^\infty \mathbb{P}[Z \geq u^{1/p} \mid \mathcal{F}_0]\,\mathrm{d}u, \tag{4.2}$$

$$\mathbb{E}[Z^p \mid \mathcal{F}_0] = p(1-p)\int_0^\infty \mathbb{E}[Z\wedge u \mid \mathcal{F}_0]\,u^{p-2}\mathrm{d}u. \tag{4.3}$$

We will apply (4.3) to X_∞. To estimate $\mathbb{E}[X_\infty \wedge t \mid F_0]$, we fix some $t, \lambda > 0$ and define:

$$\tau := \inf\{s \geq 0 \mid G_s \geq \lambda t\}.$$

Because $(G_t)_{t\geq 0}$ is predictable, there exists a sequence of stopping times $(\tau^{(n)})_{n\in\mathbb{N}}$ that announces τ. Therefore, we have on the set $\{G_0 < \lambda t\}$:

$$\begin{aligned}\mathbb{E}[X_{\tau-} \mid \mathcal{F}_0] &= \lim_{n\to\infty}\mathbb{E}[X_{\tau^{(n)}} \mid \mathcal{F}_0] \leq \lim_{n\to\infty}\mathbb{E}[G_{\tau^{(n)}} \mid \mathcal{F}_0]\\ &\leq \mathbb{E}[G_\infty \wedge \lambda t \mid \mathcal{F}_0] = \lambda\mathbb{E}[(G_\infty\lambda^{-1})\wedge t \mid \mathcal{F}_0].\end{aligned} \tag{4.4}$$

On $\{\tau = \infty\}$ we have $\lim_{n\to\infty} X_{\tau^{(n)}} \wedge t = X_\infty \wedge t$, which implies on the set $\{G_0 < \lambda t\}$:

$$\mathbb{E}[X_\infty \wedge t - X_{\tau_-}\wedge t \mid \mathcal{F}_0] \leq t\mathbb{E}[\mathbb{1}_{\{\tau<+\infty\}} \mid \mathcal{F}_0]. \tag{4.5}$$

Combining inequalities (4.4) and (4.5) gives:

$$\begin{aligned}\mathbb{E}[X_\infty\wedge t \mid \mathcal{F}_0] &\leq t\mathbb{1}_{\{G_0\geq\lambda t\}} + \big(\mathbb{E}[X_{\tau_-} \mid \mathcal{F}_0] + \mathbb{E}[X_\infty\wedge t - X_{\tau_-}\wedge t \mid \mathcal{F}_0]\big)\mathbb{1}_{\{G_0<\lambda t\}}\\ &\leq \lambda\mathbb{E}[(G_\infty\lambda^{-1})\wedge t \mid \mathcal{F}_0] + t\mathbb{P}[G_\infty \geq \lambda t \mid \mathcal{F}_0].\end{aligned} \tag{4.6}$$

Applying (4.3) to X_∞ and inserting (4.6) gives:

$$\begin{aligned}\mathbb{E}[X_\infty^p \mid \mathcal{F}_0] \leq{}& \lambda p(1-p)\int_0^\infty \mathbb{E}[(G_\infty\lambda^{-1})\wedge u \mid \mathcal{F}_0]u^{p-2}\mathrm{d}u\\ &+ p(1-p)\int_0^\infty \mathbb{P}[G_\infty \geq \lambda u \mid \mathcal{F}_0]u^{p-1}\mathrm{d}u.\end{aligned}$$

Applying (4.2) and (4.3) to G_∞ in the previous inequality implies:

$$\begin{aligned}\mathbb{E}[X_\infty^p \mid \mathcal{F}_0] &\le \lambda^{1-p}\mathbb{E}[G_\infty^p \mid \mathcal{F}_0] + (1-p)\int_0^\infty \mathbb{P}[G_\infty \ge \lambda y^{1/p} \mid \mathcal{F}_0]\mathrm{d}y \\ &\le \lambda^{-p}\big(\lambda + 1 - p\big)\mathbb{E}[G_\infty^p \mid \mathcal{F}_0].\end{aligned}$$

Choosing $\lambda = p$ implies the assertion of the theorem. □

5 Proof of Corollary 2.8

Proof of Corollary 2.8. We first prove inequalities (2.3) and (2.4): We turn the processes $(X_n)_{n\in\mathbb{N}_0}$ and $(G_n)_{n\in\mathbb{N}_0}$ into càdlàg processes in continuous time as follows: Set for all $n \in \mathbb{N}_0, t \in [n, n+1)$:

$$X_t := X_n, \qquad G_t := G_n, \qquad \mathcal{F}_t := \mathcal{F}_n.$$

As we can approximate $(G_t)_{t\ge 0}$ by left-continuous adapted processes, it is predictable. Now Theorem 1.1 and Theorem 2.2 immediately imply inequalities (2.3) and (2.4).

The sharpness of p^{-p} follows from [16, Theorem 2]. We show that $\frac{p^{-p}}{1-p}$ is sharp. Let $X^{(n)}$, $G^{(n)}$, A and $(\mathcal{F}_t)_{t\ge 0}$ be as in proof of Theorem 2.1. Fix some arbitrary $N \in \mathbb{N}$. Set for all $k, n \in \mathbb{N}$

$$\begin{aligned} X_0^{(n,N)} &:= X_0^{(n)} & X_k^{(n,N)} &:= X_{k2^{-N}}^{(n)}, \\ G_0^{(n,N)} &:= G_0^{(n)} & G_k^{(n,N)} &:= G_{(k-1)2^{-N}}^{(n)} + \int_{(k-1)2^{-N}\wedge n}^{k2^{-N}\wedge n} A(s)\mathrm{d}s, \\ \mathcal{F}_0^{(n,N)} &:= \mathcal{F}_0 & \mathcal{F}_k^{(n,N)} &:= \mathcal{F}_{k2^{-N}}. \end{aligned}$$

The processes $(X_k^{(n,N)})_{k\in\mathbb{N}_0}$ and $(G_k^{(n,N)})_{k\in\mathbb{N}_0}$ are non-negative and adapted, $(G_k^{(n,N)})_{k\in\mathbb{N}_0}$ is in addition non-decreasing and predictable. Since $G_{k2^{-N}}^{(n)} \le G_k^{(n,N)}$, the processes satisfy the Lenglart domination assumption. Hence, noting that

$$\begin{aligned}\lim_{N\to\infty} \mathbb{E}\left[\left(\sup_{k\in\mathbb{N}_0} X_k^{(n,N)}\right)^p\right] &= \mathbb{E}\left[\left(\sup_{t\ge 0} X_t^{(n)}\right)^p\right], \\ \lim_{N\to\infty} \mathbb{E}\left[\left(\sup_{k\in\mathbb{N}_0} G_k^{(n,N)}\right)^p\right] &= \mathbb{E}\left[\left(\sup_{t\ge 0} G_t^{(n)}\right)^p\right],\end{aligned}$$

implies the assertion of the corollary. □

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