Electronic Journal of Probability

Electron. J. Probab. **25** (2020), article no. 153, 1–24.
ISSN: 1083-6489 https://doi.org/10.1214/20-EJP553

The potential function and ladder heights of a recurrent random walk on $\mathbb{Z}$ with infinite variance

Kohei Uchiyama*

Abstract

We consider a recurrent random walk of i.i.d. increments on the one-dimensional integer lattice and obtain a formula relating the hitting distribution of a half-line with the potential function, $a(x)$, of the random walk. Applying it, we derive an asymptotic estimate of $a(x)$ and thereby a criterion for $a(x)$ to be bounded on a half-line. The application is also made to estimate some hitting probabilities as well as to derive asymptotic behaviour for large times of the walk conditioned never to visit the origin.

Keywords: recurrent random walk; ladder height; potential function; infinite variance; first hitting time.
MSC2020 subject classifications: Primary 60G50, Secondary 60J45.
Submitted to EJP on October 17, 2019, final version accepted on November 11, 2020.

1 Introduction

Let $S_n = S_0 + X_1 + \cdots + X_n$ be a random walk on $\mathbb{Z}$ where the starting position S_0 is an unspecified integer and the increments $X_1, X_2, \dots$ are independent and identically distributed random variables defined on some probability space $(\Omega, \mathcal{F}, P)$ and taking values in $\mathbb{Z}$. Let X be a random variable having the same law as X_1. We suppose throughout the paper that

the walk S_n is *recurrent* and *irreducible* (as a Markov chain on $\mathbb{Z}$).

For a subset B of the whole real line $\mathbb{R}$ such that $B \cap \mathbb{Z} \neq \emptyset$, put $\sigma_B = \inf\{n \geq 1 : S_n \in B\}$, the first entrance time of the walk into B. Let Z be the first strictly ascending ladder height that is defined by

$$Z = S_{\sigma_{[S_0+1,\infty)}} - S_0.$$

We also define $\hat{Z} = S_{\sigma_{(-\infty, S_0-1]}} - S_0$, the first strictly descending ladder height. Because of recurrence of the walk Z is a proper random variable whose distribution is concentrated on positive integers $x = 1, 2, \dots$ and similarly for $-\hat{Z}$. Let E indicate the integration by

*Department of Mathematics, Tokyo Institute of Technology, Japan.
E-mail: spa79ze9@voice.ocn.ne.jp

P as usual. If $\sigma^2 := EX^2 < \infty$, then $EZ < \infty$, whereas if $\sigma^2 = \infty$, either $EZ = \infty$ or $E|\hat{Z}| = -\infty$ (cf. [19, Section 17], [3, Theorem 8.4.7]).

Denote by P_x the probability of the random walk with $S_0 = x$ and E_x the expectation by P_x. Put $p^n(x) = P_0[S_n = x]$, $p(x) = p^1(x)$ and define

$$a(x) = \sum_{n=0}^{\infty} \big[p^n(0) - p^n(-x)\big];$$

the series on the RHS is convergent (cf. Spitzer [19, P28.8]). The function $a(x)$, called potential function, plays a central role in the potential theory of recurrent random walks. (This is true for two dimensional walks but here we restrict our discussion below to the one dimensional walks). Spitzer [18] established fundamental facts concerning $a(x)$—its existence, positivity, asymptotic behaviour etc.—and based on them Kesten and Spitzer [12] obtained certain ratio limit theorems for the distributions of the hitting times and sojourn times of a finite set and the transition probabilities of the walk stopped as it hits the set, which were refined by Kesten [10] under mild additional assumptions. An excellent exposition for the principal contents of [18], [12], [10] is given in Chapter 7 of Spitzer's book [19]; extensions to non-lattice random walks are obtained by Ornstein [14], Port and Stone [15], Stone [20]. Kesten [11] conjectured that the series that defines $a(x)$ converges absolutely and provided certain mild sufficient conditions for the absolute convergence.

According to Theorems 6a and 7 of [10]

$$\lim_{n\to\infty} \frac{P_x[S_n = y, S_k \neq 0 \;\text{ for }\; 1 \leq k < n]}{P_0[S_n = 0, S_k \neq 0 \;\text{ for}\, 1 \leq k < n]} = a^\dagger(x)a^\dagger(-y) + \frac{xy}{\sigma^4} \tag{1.1}$$

($x, y \in \mathbb{Z}$). Here (and in the sequel) $a^\dagger(0) = 1$ and $= a(x)$ for $x \neq 0$ and $1/\infty$ is understood to be zero. The asymptotic estimates valid uniformly in x and y of the ratio under the limit above are studied by the present author in [21], [24] in case $\sigma^2 < \infty$ and in [25] for the stable walks with exponent $1 < \alpha < 2$. The denominator of the ratio in (1.1), which equals the probability that the walk starting at zero returns to zero at n for the first time, are estimated with some exact asymptotics in these articles.

The following basic properties of $a(x)$ are found in [19]:

$$a(x+1) - a(x) \to \pm 1/\sigma^2 \quad \text{as} \quad x \to \pm\infty \tag{1.2}$$

and

$$a(x) - \frac{x}{\sigma^2} \begin{cases} = 0 \;\text{ for all }\; x > 0 \quad \text{if} \quad P[X \leq -2] = 0, \\ > 0 \;\text{ for all }\; x > 0 \quad \text{otherwise} \end{cases} \tag{1.3}$$

(the strict positivity in the second case of (1.3) is implicit in [19] if $\sigma^2 < \infty$; see e.g., [21, Eq. (2.9)]). When $\sigma^2 < \infty$ (1.2) entails the exact asymptotics $a(x) \sim |x|/\sigma^2$, whereas in case $\sigma^2 = \infty$ it gives only $a(x) = o(|x|)$ and sharper asymptotic estimates are desired. For the stable walks exact results are given in [1] for the case $1 < \alpha < 2$ apart from an extreme case and in [26, Section 8.1.1] for all the cases $1 \leq \alpha \leq 2$ under some natural side conditions.

In the general case of $\sigma^2 = \infty$ there seems to have been no results of asymptotic estimates of $a(x)$ other than those mentioned above. Very recently the present author gave some relevant results. Let $\sigma^2 = \infty$ and $E|X| < \infty$ and put

$$m_-(x) = \int_0^x dy \int_y^\infty P[X < -u]du, \quad m_+(x) = \int_0^x dy \int_y^\infty P[X > u]du \tag{1.4}$$

and $m(x) = m_-(x) + m_+(x)$. It is shown in [26] that $a(x) + a(-x) \geq C_* x/m(x)$ ($x > 0$) with a universal constant $C_* > 0$; the upper bound is also given so that

$$a(x) + a(-x) \asymp x/m(x) \tag{1.5}$$

under a reasonable side condition, which is satisfied if e.g., $\limsup_{x\to\infty} xm'(x)/m(x) < 1$ or $m_+(x)/m(x) \to 0$. (Here $b_x \asymp c_x$ means that b_x/c_x is bounded away from zero and infinity.)

In this paper we shall show, supposing $\sigma^2 = \infty$, that

$$\begin{cases} a(x)/V_{\rm ds}(x) \to 1/EZ \text{ and } E_x[a(S_{\sigma(-\infty,0]})]/a(x) \to 0 \quad (x\to\infty) & \text{if } \ EZ<\infty, \\ \liminf\limits_{x\to+\infty} a(x)/V_{\rm ds}(x) = 0 \text{ and } a^\dagger(x) = E_x[a(S_{\sigma(-\infty,0]})] \ (x\in\mathbb{Z}) & \text{otherwise,} \end{cases} \tag{1.6}$$

where $V_{\rm ds}$ denotes the renewal function for the weakly descending ladder process, and that there exists $\lim_{x\to\infty} a(-x) \le \infty$, and

$$\begin{array}{ll} 0 < \lim\limits_{x\to\infty} a(-x) < \infty & \text{if } \begin{cases} E|X|<\infty \quad \text{and} \\ \int_0^\infty [t/m_-(t)]^2 P[X>t]dt < \infty, \end{cases} \\ \liminf\limits_{x\to\infty} a(-x) = \infty & \text{otherwise,} \end{array} \tag{1.7}$$

provided $P[X \ge 2] > 0$. In (1.6) the $\liminf$ may be expected to be replaced by $\lim$ (see Remark 2.3(e)). Note that if $E|X| = \infty$, then $EX_+ = EX_- = \infty$ because of the assumed recurrence of the walk. (Here $X_+ = \max\{X, 0\}$ and $X_- = X_+ - X$.) Applying (1.6), we derive asymptotic estimates of some hitting probabilities as well as asymptotic behaviour for large times of the random walk conditioned never to visit the origin. As an intelligible manifestation of the significance of the condition $EZ < \infty$ in the sample path behavior of the walk, we shall observe that $EZ < \infty$ if and only if the walk conditioned never to visit the origin approaches the positive infinity with probability one (Section 7).

The main results (Theorems 2.1 and 2.4) of the present paper are derived from those given in Spitzer's book (that are stated in Section 3 of the present paper) independently of those of [26] in which the proof is solely based on the Fourier integral representation of $a(x)$. In the proof of our main results, we could apply those from [26] whose usage, however, we avoid in order not to cause any suspicion of circular arguments, some of our results (1.6) being used in [26]. For the sake of comparison, we include the case of finite variance when all the results are known or easily derived from known ones.

2 Statements of results

Let S_n be the random walk specified in Introduction and $Z, \hat{Z}, \sigma_B, m_\pm(x)$ and $a(x)$ be as given there. In order to state the results of the paper we further bring in the following notation. Put

$$T = \sigma_{(-\infty,0]} = \inf\{n \ge 1 : S_n \le 0\}$$

(where $(\alpha, \beta]$ denotes the interval $\alpha < x \le \beta$ as usual) and define

$$H^x_{(-\infty,0]}(y) = P_x[S_T = y], \tag{2.1}$$

the hitting distribution of $(-\infty, 0]$ for the walk starting at $x \in \mathbb{Z}$. Likewise let H^x_B be the hitting distribution of a non-empty set $B \subset \mathbb{R}$. [Thus $H^1_{(-\infty,0]}(y) = P[\hat{Z} = y-1], y \le 0$ and $H^{-1}_{[0,\infty)}(y) = P[Z = y+1], y \ge 0$.] There exists $\lim_{x\to\infty} H^x_{(-\infty,0]}(y)$, which we denote by $H^{+\infty}_{(-\infty,0]}(y)$ and similarly for $H^{-\infty}_{[0,\infty)}$. $H^{-\infty}_{[0,\infty)}$ is a probability distribution if $EZ < \infty$ and vanishes identically otherwise [19, P24.7]. Let $V_{\rm ds}(x), x = 0, 1, 2, \ldots$, be the renewal function of the weak descending ladder-height process (see (3.5) or Appendix). For our present purpose it is convenient to bring in the function f_r, the shift of $V_{\rm ds}$ to the right by 1, namely

$$f_r(x) = V_{\rm ds}(x-1) \qquad (x \ge 1).$$

According to [19, P19.5, E27.3], [8, Section XII.3] f_r is a positive harmonic function on $[1, \infty)$, i.e., a positive solution of the equation $f_r(x) = E_x[f_r(S_1); S_1 \geq 1]$, which may be written as

$$f_r(x) = \sum_{y=1}^{\infty} f_r(y)p(y-x), \quad x \geq 1, \tag{2.2}$$

and the solution is unique apart from a constant factor; it turns out that the distribution of Z is expressed as

$$P[Z > x] = \sum_{y=1}^{\infty} f_r(y)p(y+x) \quad (x \geq 0), \tag{2.3}$$

(see Theorem A and (3.10) in Section 3 for more details). Define for any non-negative function $\varphi(y)$, $y \leq 0$,

$$H^x_{(-\infty,0]}\{\varphi\} = E_x[\varphi(S_T)] = \sum_{y \leq 0} H^x_{(-\infty,0]}(y)\varphi(y) \leq \infty.$$

For a set $B \subset \mathbb{R}$ such that $B \cap \mathbb{Z} \neq \emptyset$ let $g_B(x, y)$ denote the Green function of the walk killed as it hits B:

$$g_B(x, y) = E_x\Big[\sum_{0 \leq n < \sigma_B} \delta(S_n, y)\Big] \qquad (x, y \in \mathbb{Z}), \tag{2.4}$$

where $\delta(x, y) = 1$ if $x = y$ and $= 0$ otherwise. This definition is different from that in [19], where the corresponding one agrees with our $g_B(x, y)$ if $x \notin B$, but vanishes if $x \in B$ whereas according to our definition

$$g_B(x, y) = \sum_{z \notin B} p(z-x)g_B(z, y) + \delta(x, y) \quad \text{for} \quad x \in B, y \in \mathbb{Z}$$

(valid also for $x \notin B$); in particular $g_B(x, y) = \delta(x, y)$ whenever $y \in B$. This relation shows that $g_B(x, y)$ equals the hitting distribution of B by the dual (or time-reversed) walk started at y which fact is expressed as

$$g_B(x, y) = P_{-y}[S_{\bar{\sigma}_{-B}} = -x] \quad \text{for} \quad x \in B.$$

Here $-B = \{-z : z \in B\}$ and $\bar{\sigma}_B = \sigma_B$ if $S_0 \notin B$ and $\bar{\sigma}_B = 0$ otherwise.

In case $B = (-\infty, 0]$, $g_B(x, y), x, y \in B$ is expressed explicitly by means of the renewal functions of ascending and descending ladder height processes (cf. Theorem A in Section 3), by which it follows immediately that there exists $\lim_{y\to\infty} g_{(-\infty,0]}(x, y)$ which is denoted by $g_{(-\infty,0]}(x, \infty)$ and given by

$$g_{(-\infty,0]}(x, \infty) = \begin{cases} f_r(x)/EZ & x > 0, \\ H^{-\infty}_{[0,\infty)}(-x) & x \leq 0; \end{cases} \tag{2.5}$$

if $EZ = \infty$, the RHS vanishes so that $g_{(-\infty,0]}(x, \infty) = 0$ for all x (cf. (3.7)).

Theorem 2.1. (i) *For all $x, y \in \mathbb{Z}$,*

$$g_{(-\infty,0]}(x, y) + a(x-y) - H^x_{(-\infty,0]}\{a(\cdot - y)\} = Ag_{(-\infty,0]}(x, \infty), \tag{2.6}$$

where

$$A = \begin{cases} 1/2 & \textit{if} \quad \sigma^2 < \infty, \\ 1 & \textit{if} \quad \sigma^2 = \infty. \end{cases}$$

(ii) *If $EZ < \infty$, then as $x \to \infty$, $a(x)/f_r(x) \to A/EZ$ and $a(-x)/a(x) \to 0$, and*

$$\sum_{x=0}^{\infty} a(-x)P[\,|X| > x] < \infty.$$

(iii) *If $EZ = \infty$, then $\liminf_{x\to\infty} a(x)/f_r(x) = 0$.*

It is natural to extend $f_r(x)$ to a function on $\mathbb{Z}$ by means of (2.2) (so as to make (2.2) valid for all $x \in \mathbb{Z}$), or what amounts to the same thing (in view of (2.3)),

$$f_r(x) = P[Z > -x] \qquad \text{for} \quad x \le 0. \tag{2.7}$$

Since $f_r(0) = 1 < V_{\rm ds}(0) = f_r(1)$, f_r is increasing. According to this *extension* of f_r together with the identity

$$H^{-\infty}_{[0,\infty)}(x) = P[Z > x]/EZ \quad (x \ge 0) \tag{2.8}$$

(cf. [8, (XI.4.10)] or the remark following (3.10)) relation (2.5) is expressed simply as

$$g_{(-\infty,0]}(x,\infty) = f_r(x)/EZ \qquad (x \in \mathbb{Z}). \tag{2.9}$$

For any integer k and any non-negative φ,

$$H^x_{(-\infty,-k]}\{\varphi\} = H^{x+k}_{(-\infty,0]}\{\varphi(\cdot - k)\} \quad \text{and} \quad g_{(-\infty,0]}(x,z) = g_{(-\infty,-k]}(x-k, z-k)$$

so that (2.6) is rephrased as

$$H^x_{(-\infty,-k]}\{a(\cdot - y\} = a(x-y) + g_{(-\infty,-k]}(x,y) - Af_r(x+k)/EZ. \tag{2.10}$$

The following corollary will be often useful in application of Theorem 2.1. For brevity of expression we write

$$a^\dagger(x) = a(x) + \delta(x,0)$$

so that $g_{(-\infty,0]}(x,y) + a(x-y) = a^\dagger(x-y)$ for $y \le 0$.

Corollary 2.2. (i) *Suppose $EZ < \infty$. Then $H^x_{(-\infty,0]}\{a\}/a(x) \to 0$ as $x \to \infty$ and*

$$H^x_{(-\infty,0]}\{a(\cdot - y)\} = a^\dagger(x-y) - Af_r(x)/EZ \quad \textit{for} \quad x \in \mathbb{Z},\, y \le 0. \tag{2.11}$$

(ii) *If $EZ = \infty$, then $H^x_{(-\infty,0]}\{a(\cdot - y)\} = a^\dagger(x-y)$ for $x \in \mathbb{Z}$, $y \le 0$.*

Remark 2.3. (a) The statement $a(-x)/a(x) \to 0$ in Theorem 2.1 follows under the weaker condition $m_+(x)/m(x) \to 0$ $(x \to \infty)$ (cf. [26, Theorem 4]). By Corollary 2.2(i) $a(x) \sim Af_r(x)/EZ$, which is shown in [21] in case $\sigma^2 < \infty$ and generalized in [26] as $a(x) \sim f_r(x)/\int_0^x P[Z > t]dt$ under $m_+(x)/m(x) \to 0$ and $\sigma^2 = \infty$. We include proofs of these parts of Theorem 2.1 which are much simpler than the proofs in [26]—although the latter do not depend on our Theorem 2.1.

(b) By (2.3) it follows that

$$\sum_{x=1}^{\infty} f_r(x) P[X \ge x] = EZ. \tag{2.12}$$

This together with Theorem 2.1(i) shows that if $EZ < \infty$, then

$$\sum_{x=1}^{\infty} [a(x) + a(-x)] P[X > x] < \infty, \tag{2.13}$$

which may be effectively used to derive Chow's criterion for $EZ < \infty$ in a quite different way from [4] (see Remark 3.7 for more details).

(c) The process $M_n := a(S_{n\wedge T})$ is a non-negative martingale under P_x, $x \ne 0$, in particular $a(x) = E_x M_n$. Clearly $M_\infty = a(S_T)$ a.s., so that $H^x_{(-\infty,0]}\{a\} = E_x M_\infty$. Hence Corollary 2.2 implies that (M_n) is uniformly integrable (so that $a(x) = E_x M_\infty$) if and only if $EZ = \infty$.

(d) As another application of Theorem 2.1 we shall consider the random walk conditioned never to visit the origin and observe that the conditional walk distinguishes $+\infty$ and $-\infty$ if and only if either EZ or $E\hat{Z}$ is finite, although its Martin compactification does not whenever $\sigma^2 = \infty$ (see Section 7).

(e) Let $EZ = \infty$ and consider asymptotic behaviour of $a(x)/f_r(x)$ as $x \to \infty$. Theorem 2.1(iii) tells merely $\liminf\ a(x)/f_r(x) = 0$. It however seems to be true quite generally that $\lim a(x)/f_r(x) = 0$. Actually if $E|X| < \infty$ and $\sigma_-^2 := E[X^2; X < 0] < \infty$ (in addition), one has $E|\hat{Z}| < \infty$ by virtue of the dual of (2.12), so that $f_r(x) \sim x/E|\hat{Z}|$, hence

$$[a(x) + a(-x)]/f_r(x) \longrightarrow 0 \qquad (x \to \infty). \tag{2.14}$$

In case $E|X| < \infty = \sigma_-^2$, one has $f_r(x) \gg x/m_-(x)$ (cf. Lemma 3.6), and if (1.5) is applicable, this entails (2.14). If X belongs to the domain of attraction of a stable law with exponent $1 \le \alpha \le 2$ and skewness parameter $-1 \le \beta \le 1$, then unless $\alpha - 1 = \beta = 0$, it follows that $E|X| < \infty$ and $a(x) + a(-x) \sim \kappa x/m(x)$ for a positive constant $\kappa = \kappa_{\alpha,p}$ (cf. [26, Section 8.1.1]), hence (2.14) holds; also in case $\alpha = 1$ and $\beta = 0$, if $P[S_n > 0]$ is further supposed to be convergent as $n \to \infty$, (2.14) holds (proved in Remark 5.2).

The first part of the following theorem provides asymptotic estimates of $a(x)$ as $|x| \to \infty$, and its third part an answer to the open question stated at the very end of Spitzer's book [19] (see Remark 2.5(d) below).

Theorem 2.4. (i) *If $EZ < \infty$ and $\sigma^2 = \infty$, then*

$$1 \le \liminf_{x\to\infty} \frac{a(x)m_-(x)}{x} \le \limsup_{x\to\infty} \frac{a(x)m_-(x)}{x} \le 2; \tag{2.15}$$

and

$$\lim_{x\to\infty} \frac{1}{a(-x)} \sum_{z=1}^{\infty} P[z < Z \le z + x]a(z) = EZ. \tag{2.16}$$

(ii) *If $EZ = \infty$, then $\lim_{x\to\infty} a(-x) = \infty$.*

(iii) *Suppose $\sigma^2 = \infty = EZ$. Then for some constant $C > 1$*

$$C^{-1}a(-x) \le \sum_{w=1}^{x}\sum_{z=1}^{\infty} p(w+z)\Big[\frac{z}{m_-(z)}\Big]^2 \le Ca(-x) \quad (x \ge 1) \tag{2.17}$$

[with all the members vanishing if $P[X \ge 2] = 0$]; and there exists $\lim_{x\to\infty} a(-x) \le \infty$ where the limit is finite if and only if

$$\int_1^{\infty} \frac{t^2}{m_-^2(t)} P[X > t]dt < \infty \tag{2.18}$$

and if this is the case, $\lim_{x\to\infty} a(-x) = H^{-\infty}_{[0,\infty)}\{a\}$.

Remark 2.5. (a) (2.16) entails that $a(-x)$ is asymptotically increasing as $x \to \infty$ if $EZ < \infty$. Such monotonicity of $a(-x)$ however is verified under $a(-x)/a(x) \to 0$ ($x \to \infty$) in [26, Corollary 40] by a quite different approach.

(b) In case $\sigma^2 < \infty$ (2.18) holds whenever $E[X_+^3] < \infty$ which condition is equivalent to $C^- := \lim_{x\to\infty}[a(-x) - x/\sigma^2] < \infty$ [21, Section 2.1], while (2.18) is possibly true even under the condition that for some $\delta > 0$, $P[X > x] > x^{-1}(\log x)^{-1-\delta}$ for all sufficiently large x.

(c) Condition (2.18) implies $EZ < \infty$, the latter being equivalent to the integrability condition $\int_1^\infty tP[X > t]dt/m_-(t) < \infty$ (see [4], [22, Section 2.4]).

(d) Since $\lim_{|x|\to\infty} a(x) = \infty$ if $E|X| = \infty$ entailing $EX_+ = EX_- = \infty$ (because of the recurrence assumption), Theorem 2.4(iii) gives an exact criterion for the trichotomy of $\lim_{|x|\to\infty} a(x) = \infty$, $M^- = \lim_{x\to-\infty} a(x) < \infty$, $M^+ = \lim_{x\to\infty} a(x) < \infty$. (This trichotomy itself is stated at the end of [19].)

Corollary 2.6. *Suppose $\sigma^2 = \infty$. There exists $M^\pm := \lim_{x\to\pm\infty} a(x) \le \infty$ where $M^- = 0$ if and only if $P[X \ge 2] = 0$ and in order that $M^- < \infty$ each of the following conditions are necessary and sufficient.*

(i) $\sum_{z=1}^\infty P[X > z]\big([a(z)]^2 + a(-z)\big) < \infty$.
(ii) $\sum_{z=1}^\infty P[Z > z]a(z) < \infty$ *and* $P[X \le -2] > 0$.
(iii) $\sum_{z=1}^\infty H^x_{[0,\infty)}(z)a(z)$ *is bounded for* $x < 0$ *and* $E|\hat Z| = \infty$.

Proof. The existence of the limit and the condition for $M^- = 0$ follows immediately from Theorem 2.4. Each of conditions (i) and (ii) implies $EZ < \infty$ (see Remark 3.7(b) for (i) and note $M^+ > 0$ under (ii)). The assertion of the corollary then follows from Theorems 2.1 and 2.4 and the identity $H^{-\infty}_{[0,\infty)}(z) = P[Z > z]/EZ$. □

For $y \in \mathbb{Z}$ write σ_y for $\sigma_{\{y\}}$. The results (i) and (ii) given below are taken from Sections 7.3 and 7.5 of [26].

(i) If $m_+(x)/m(x) \to 0$ ($x \to \infty$), then uniformly for $0 \le x \le R$, as $R \to \infty$

$$P_x[\sigma_{[R,\infty)} < T] \sim P_x[\sigma_R < T] \sim \frac{f_r(x)}{f_r(R)}, \text{ and} \tag{2.19}$$

(ii) If $m_+(x)/m(x)$ converges to 0 or to 1 as $x \to \infty$, then uniformly for $0 \le x \le R$, as $R \to \infty$

$$P_x[\sigma_{[R,\infty)} < \sigma_0] \sim P_x[\sigma_R < \sigma_0] \ \text{ and } \ P_x[T < \sigma_R] \sim P_x[\sigma_0 < \sigma_R]. \tag{2.20}$$

(The second relation of (2.20) is the dual of the first.)

These results are supplemented by the following propositions.

Proposition 2.7. *If $P[X \ge 2] > 0$, then for $x \in \mathbb{Z}$,*

$$\lim_{R\to\infty} \frac{P_x[\sigma_R < T\,]}{P_x[\sigma_R < \sigma_0]} = \begin{cases} Af_r(x)/[a^\dagger(x)EZ] & \text{if } \ EZ < \infty, \\ 0 & \text{if } \ EZ = \infty; \end{cases} \tag{2.21}$$

where in case $EZ < \infty$ the convergence in (2.21) is uniform for $0 \le x \le R$.

Proposition 2.8. *If $EZ < \infty$, then for $x \ge 0$, as $R - x \to \infty$,*

$$P_x\big[\sigma_0 < \sigma_{[R,\infty)}\big] \sim P_x\big[T < \sigma_{[R,\infty)}\big] \sim \frac{f_r(R) - f_r(x)}{f_r(R)} \le \frac{f_r(R-x)}{f_r(R)}. \tag{2.22}$$

The second equivalence in (2.22) follows from (2.19) if x ranges over a set depending on R in which $f_r(x) = O(f_r(R) - f_r(x))$ but does not otherwise. If $P[X \ge 2] = 0$, then $P_x[\sigma_R < T\,] = P_x[\sigma_R < \sigma_0]$ and $Af_r(x)/EZ = a^\dagger(x)$ for all $x \in \mathbb{Z}$, where all the terms vanish for $x \le -1$, and the formula (2.21) (necessarily the first case) is still reasonable. The dual statement of (2.21) for the case $EZ < \infty$ may be written as

$$\frac{P_x[\sigma_0 < \sigma_{[R,\infty)}\,]}{P_x[\sigma_0 < \sigma_R]} \sim \frac{Af_l(R-x)}{a^\dagger(-R+x)E|\hat Z|} \quad \text{uniformly for } \ 0 \le x \le R \ \text{ if } \ 1 < E|\hat Z| < \infty.$$

The corresponding one for (2.22) will be stated as Lemma 6.4 in Section 6.

The formula (2.21) says that if $EZ < \infty$, then $P_x[\sigma_R < T \,|\, \sigma_R < \sigma_0]$, being equal to the ratio on the LHS, approaches unity as x becomes large independently of how R is large, while if $EZ = \infty$ this is not the case: this conditional probability tends to zero as $R \to \infty$, in other words, for R large enough the walk—even if it is conditioned on $\sigma_R < \sigma_0$—reaches R only after entering the negative half line with overwhelming probability as far as its starting position x is fixed. If $m_+/m \to 0$ and $\ell_+(x) = \int_0^x P[Z > t]dt$, then $P_x[\sigma_R < \sigma_0] \sim a^\dagger(x)/a(R)$ (see Lemma 6.1), ℓ_+ is slowly varying and $f_r(x) \sim a(x)\ell_+(x)$ [26, Lemma 46], so that by (2.19) it follows that as $x \to \infty$ under $x < R$

$$P_x\big[\sigma_R < T \,\big|\, \sigma_R < \sigma_0\big] \sim \ell_+(x)/\ell_+(R). \tag{2.23}$$

This shows that for each $\varepsilon > 0$, the ratio above approaches 1 as $R \to \infty$ uniformly for $x > \varepsilon R$. The same holds true if $E|\hat{Z}| < \infty$ at least under some regularity condition on the tails of F but can fail in general (see Remark 6.6 of Section 6).

The rest of the paper is organized as follows. In Section 3 we collect fundamental facts used in this paper about f_r, $a(x)$, $g_{(-\infty,0]}$ etc. given in Spitzer [19] and advance several lemmas that are directly derived from them. The proofs of Theorems 2.1 and 2.4 are given in Sections 4 and 5, respectively. The proofs of Propositions 2.7 and 2.8 are given in Section 6. In Section 7 we briefly study large time behaviour of the walk conditioned never to visit the origin. In Section 8 (Appendix) we present a few facts about strictly and weakly ascending ladder height variables.

3 Preliminary lemmas

In this section we collect fundamental results of the recurrent random walks on $\mathbb{Z}$ given in Spitzer's book [19] and then derive some consequences of them that are used in the later sections.

For $B \subset \mathbb{Z}$ we have defined the first hitting time by $\sigma_B = \inf\{n \geq 1 : S_n \in B\}$. For a point $x \in \mathbb{Z}$ write σ_x for $\sigma_{\{x\}}$. For typographical reason we sometimes write σB for σ_B.

Let $u_{\rm as}(x)$, $x = 0, 1, 2, \ldots$ be the renewal sequence of the strictly ascending ladder variables, namely $u_{\rm as}(0) = 1$ and

$$u_{\rm as}(x) = \sum_{n=1}^{\infty} P[Z_1 + \cdots + Z_n = x] \qquad x \geq 1; \tag{3.1}$$

and similarly $v_{\rm ds}(x)$, $x = 0, 1, 2, \ldots$ denotes the renewal sequence of the *weak descending* ladder variables, which may be given by $v_{\rm ds}(0) = 1/c$ and

$$v_{\rm ds}(x) = \frac{1}{c}\sum_{n=1}^{\infty} P[\hat{Z}_1 + \cdots + \hat{Z}_n = -x] \qquad x \geq 1, \tag{3.2}$$

where

$$c = \exp\left[-\sum_{k=1}^{\infty}\frac{1}{k}p^k(0)\right] = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\big|1 - E[e^{itX}]\big|dt\right].$$

(See Appendix for (3.2) as well as for the probabilistic meaning of the constant c.) Owing to the renewal theorem [8], there exist limits

$$u_{\rm as}(\infty) := \lim_{x\to\infty} u_{\rm as}(x) = 1/EZ \qquad \text{and} \qquad v_{\rm ds}(\infty) := \lim_{x\to\infty} v_{\rm ds}(x) = 1/c\,E[-\hat{Z}]\,. \tag{3.3}$$

The Green function $g_B(x, y)$ ($x, y \in \mathbb{Z}$) defined in (2.4) may be written as:

$$g_B(x, y) = \sum_{n=0}^{\infty} P_x[S_n = y, n < \sigma_B].$$

The following theorem follows from the propositions P18.8, P19.3, P19.5 of [19]. For two real numbers s and t write $s \wedge t = \min\{s, t\}$ and $s \vee t = \max\{s, t\}$.

Theorem A. (i) $u_{\rm as}(\infty)v_{\rm ds}(\infty) = 1/cEZE|\hat Z| = 2/\sigma^2$.

(ii) $g_{(-\infty,0]}(x,y) = \sum_{z=1}^{x\wedge y} v_{\rm ds}(x-z)u_{\rm as}(y-z)$ $(x,y>0)$; and

$g_{[0,+\infty)}(x,y) = g_{(-\infty,0]}(-y,-x) = \sum_{z=1}^{|x|\wedge|y|} u_{\rm as}(|x|-z)v_{\rm ds}(|y|-z)$ $(x,y<0)$.

The formulae in Theorem A will often be used in combination with the following representation of the hitting distribution $H^x_{(-\infty,0]}(y)$ of $(-\infty,0]$:

$$H^x_{(-\infty,0]}(y) = \sum_{z=1}^{\infty} g_{(-\infty,0]}(x,z)p(y-z) \qquad (x>0, y\le 0), \tag{3.4}$$

and analogous one for $H^x_{[0,\infty)}$ (see (5.2) for another representation). The function f_r may be written as

$$f_r(x) = v_{\rm ds}(0)+\cdots+v_{\rm ds}(x-1) \qquad (x\ge 1), \tag{3.5}$$

and its dual as $f_l(x) = c^{-1}\left[u_{\rm as}(0)+\cdots+u_{\rm as}(x-1)\right]$ $(x\ge 1)$.

By Theorem A(ii) and $u_{\rm as}(y)\le 1$ it follows that

$$g_{(-\infty,0]}(x,y) \le \begin{cases} f_r(x) & \text{if } x\le y, \\ f_r(x)-f_r(x-y) & \text{if } x>y. \end{cases} \tag{3.6}$$

Let $x\to-\infty$ in $H^x_{[0,\infty)}(y) = \sum_{w=1}^{\infty} g_{[0,+\infty)}(x,-w)p(y+w)$. Noting $g_{[0,+\infty)}(x,-w) = g_{(-\infty,0]}(w,-x) \to f_r(w)/EZ$ and $\sum_{w=1}^{\infty} f_r(w)p(y+w)<\infty$, we then find that

$$H^{-\infty}_{[0,\infty)}(y) := \lim_{x\to-\infty} H^x_{[0,\infty)}(y) = \frac{1}{EZ}\sum_{w=1}^{\infty} f_r(w)p(y+w). \tag{3.7}$$

It also follows that $H^x_{[0,\infty)}(y)\le \sum_{w=1}^{\infty} f_r(w)p(y+w)$, so that

$$H^x_{[0,\infty)}(y) \le (EZ)H^{-\infty}_{[0,\infty)}(y) \quad \text{for all} \quad x\le 0<y \quad \text{if } EZ<\infty. \tag{3.8}$$

In particular the three conditions (a) $EZ=\infty$; (b) $u_{\rm as}(\infty)=0$; (c) $H^{-\infty}_{[0,\infty)}(\cdot)\equiv 0$ are equivalent to one another. Since $g_{[1,\infty)}(0,-y) = g_{(-\infty,0]}(y+1,1) = v_{\rm ds}(y)$ we have for $k>0$

$$P[Z=k] = \sum_{y=0}^{\infty} g_{[1,\infty)}(0,-y)p(k+y) = \sum_{y=0}^{\infty} v_{\rm ds}(y)p(k+y), \tag{3.9}$$

and, by summation by parts,

$$P[Z>x] = \sum_{y=0}^{\infty} v_{\rm ds}(y)P[X>x+y] = \sum_{y=1}^{\infty} f_r(y)p(x+y) \qquad (x\ge 0). \tag{3.10}$$

Note that the last equality together with (3.7) yields (2.8) (i.e., $H^{-\infty}_{[0,\infty)}(x) = P[Z>x]/EZ$).

The next theorem also is taken from Spitzer [19, T28.1, T29.1, P30.2, P30.3].

Theorem B. The series $\sum_{n=0}^{\infty}[p^n(0)-p^n(-x)]$ converges for each $x\in\mathbb{Z}$ and if $a(x)$ denotes the sum, then the following relations hold.

$$g_{\{0\}}(x,y) = a^\dagger(x)+a(-y)-a(x-y) \qquad (x,y\in\mathbb{Z}), \tag{3.11}$$

$$a(x+y)\le a(x)+a(y) \quad \text{and} \quad a^\dagger(x)+a(-x)\ge 1 \qquad (x,y\in\mathbb{Z}), \tag{3.12}$$

$$\sum_{z=-\infty}^{\infty} p(z-x)a(z-y) = a^\dagger(x-y), \tag{3.13}$$

$$\lim_{x\to\pm\infty}[a(x+1)-a(x)] = \pm 1/\sigma^2 \quad \text{and} \quad \lim_{x\to\infty}[a(x)+a(-x)] = \infty. \tag{3.14}$$

If the walk is left-continuous (i.e. $P[X\le -2]=0$), then $a(x)=x/\sigma^2$ for $x>0$; analogously $a(x)=-x/\sigma^2$ for $x<0$ for right-continuous walks; except for left- or right-continuous walks with infinite variance $a(x)>0$ for all $x\ne 0$.

[(3.11) with $x = 0$ and the second inequality of (3.12), not given in [19], follows from (3.13) and $g_{\{0\}}(x, x) \geq 1$, respectively.]

We put

$$\bar{a}(x) = \frac{1}{2}[a(x) + a(-x)].$$

By (3.11) it follows that $g_{\{0\}}(y, y) = 2\bar{a}(y) + \delta(0, y) > 0$ and that

$$P_x[\sigma_y < \sigma_0] = \frac{g_{\{0\}}(x,y)}{g_{\{0\}}(y,y)} = \frac{a^\dagger(x) + a(-y) - a(x-y)}{2\bar{a}(y)} \qquad (x, y \in \mathbb{Z}, y \notin \{x, 0\}). \tag{3.15}$$

The equation (3.13) states that $a(x)$ is harmonic on $x \neq 0$, which together with $a(0) = 0$ entails that the process $M_n := a(S_{\sigma_\xi \wedge n} - \xi)$ is a martingale, provided that $S_0 \neq \xi \in \mathbb{Z}$ a.s. Let B be a non-empty subset of $\mathbb{Z}$. Using the optional sampling theorem and Fatou's lemma we observe first that $a(x-\xi) = \lim_{n\to\infty} E_x[M_n] \geq E_x[a(S_{\sigma_\xi \wedge \sigma_B} - \xi)]$ valid whenever $x \neq \xi$, and then, by using (3.13) again, that if $\xi \in B$,

$$E_\xi[a(S_{\sigma_B} - \xi)] = \sum_{y\in B} p(y-\xi)a(y-\xi) + \sum_{z\notin B} p(z-\xi)E_z[a(S_{\sigma_\xi\wedge\sigma_B} - \xi)] \leq 1, \tag{3.16}$$

so that

$$E_x[a(S_{\sigma_B} - \xi)] \leq a^\dagger(x-\xi) \quad \text{for} \quad \xi \in B, x \in \mathbb{Z}, \tag{3.17}$$

in particular

$$a(y)P_x[\sigma_y < \sigma_0] = E_x[a(S_{\sigma_0\wedge\sigma_y})] \leq a^\dagger(x) \qquad (x, y \in \mathbb{Z}). \tag{3.18}$$

In the rest of this section we prove several lemmas that are derived more or less directly from the results presented above.

Lemma 3.1. *Let $\sigma^2 = \infty$. Then there exists $\lim_{x\to\infty} a(x)$ $(\leq \infty)$ which is zero if and only if the random walk is left-continuous.*

By the last statement of Theorem B this lemma shows that $\inf_{x\geq 1} a(x) > 0$ unless the random walk is left continuous.

Proof. Let $\sigma^2 = \infty$. The relations (3.18) and (3.15) (with x and y interchanged) yield

$$a(y) \geq \frac{a(x)}{a(x) + a(-x)}\big[a(y) + a(-x) - a(y-x)\big] \qquad (x \neq 0). \tag{3.19}$$

On using (3.14) it then follows that

$$\liminf_{y\to\infty} a(y) \geq \frac{a(x)a(-x)}{a(x) + a(-x)} \quad \text{for all} \quad x \neq 0. \tag{3.20}$$

If $\limsup_{x\to\infty} a(x) < \infty$, then $\lim_{x\to\infty} a(-x) = \infty$ in view of (3.14) and the inequality (3.20) gives $\liminf a(x) \geq \limsup a(x)$ so that $\lim a(x)$ exists. If this limit is zero, then the RHS of (3.20) must be zero for all $x > 0$, which is possible only if the walk is left-continuous.

Now suppose $\limsup_{x\to\infty} a(x) = \infty$ and put $M = \liminf_{x\to\infty} a(x) (\leq \infty)$. Contrary to what is to be shown let $M < \infty$. Then one can choose R such that $a(x) + a(-x) > 4M + 6$ for $x > R$. In view of (3.14) there must exist $x_1 > R$ such that $2M + 2 \leq a(x_1) < 2M + 3$, which entails $a(-x_1) > 2M + 3$. Combined with (3.20) these lead to the absurdity

$$M \geq \frac{a(x_1)a(-x_1)}{a(x_1) + a(-x_1)} \geq \frac{a(x_1)}{2} \geq M + 1.$$

Hence M must be infinite. □

Lemma 3.2. *For all* $x, y \in \mathbb{Z}$,

$$-\frac{a(y)}{a(-y)}a(x) \le a(x+y) - a(y) \le a(x) \quad \textit{if} \ a(-y) \ne 0. \tag{3.21}$$

Proof. From (3.15) and (3.18) we have

$$\frac{a(x) + a(y) - a(x+y)}{a(y) + a(-y)} \le \frac{a(x)}{a(-y)} \quad (a(-y) \ne 0)$$

[in (3.19) take $-y$ and x in place of x and y respectively; note that the case $x = 0$ or $-y = x$ is obvious], which, after simple rearrangements, becomes the left-hand inequality of (3.21). The right-hand one is the same as $g_{\{0\}}(x, -y) \ge 0$. □

Put $g(x, y) = g_{\{0\}}(x, y) - \delta(x, 0)$, or explicitly $g(x, y) = a(x) + a(-y) - a(x - y)$.

Lemma 3.3. *If B is a proper subset of $\mathbb{Z}$ such that $0 \in B$, then*

$$g_{\{0\}}(x, y) = g_B(x, y) + E_x[g(S_{\sigma_B}, y)] \quad (x, y \in \mathbb{Z}). \tag{3.22}$$

Proof. Let $\Lambda_B(y)$ be the number of visits to y in the time interval $\{1, 2, \ldots, \sigma_B - 1\}$:

$$\Lambda_B(y) = \sharp\{n \ge 1 : S_n = y, n < \sigma_B\}.$$

Then $g_{\{0\}}(x, y) = \delta(x, y) + E_x[\Lambda_{\{0\}}(y)]$ and similarly for $g_B(x, y)$, and (3.22) can be written as

$$E_x[\Lambda_{\{0\}}(y)] = E_x[\Lambda_B(y)] + E_x[g(S_{\sigma_B}, y)], \tag{3.23}$$

provided that $0 \in B$ which entails $\sigma_B \le \sigma_0$ a.s. Recall that $g(0, y) = 0$ and for $z \ne 0$, $g(z, y) = g_{\{0\}}(x, y)$, the expected number of visits to y before hitting 0. If $y \notin B$, then by the strong Markov property the above equality follows immediately. It therefore suffices to show (3.23) for $y \in B$.

Let $y \in B$, when one always has $\Lambda_B(y) = 0$ a.s. For $x \notin B$, (3.23) then follows immediately. For $x \in B$, one observes that

$$E_x[g(S_{\sigma_B}, y)] = \sum_{z \in B} p(z - x)g(z, y) + \sum_{z \notin B} p(z - x)E_z[g(S_{\sigma_B}, y)],$$

but (3.23) with x replaced by $z \notin B$ is valid so that $E_z[g(S_{\sigma_B}, y)] = E_z[\Lambda_{\{0\}}(y)]$, of which the RHS equals $g(z, y)$ for $z \ne 0$ (hence for $z \notin B$). Thus

$$E_x[g(S_{\sigma_B}, y)] = \sum_{z \ne 0} p(z - x)g(z, y),$$

which shows (3.23), for the last sum equals $E_x[\Lambda_{\{0\}}(y)]$. This finishes the proof. □

Lemma 3.4. *If the walk is not left-continuous and $k^+ := \sup_{x \ge 1} a(-x)/a(x)$, then*

$$0 \le g_{\{0\}}(x, y) - g_{(-\infty, 0]}(x, y) \le (1 + k^+)a(-y) \quad (x, y \in \mathbb{Z}). \tag{3.24}$$

If the walk is not right-continuous and $k^- := \sup_{x \le -1} a(-x)/a(x)$, then

$$0 \le g_{\{0\}}(x, y) - g_{(-\infty, 0]}(x, y) \le (1 + k^-)a^\dagger(x) \quad (x.y \in \mathbb{Z}). \tag{3.25}$$

Proof. Take $B = (-\infty, 0]$ in (3.22) and use the inequality $a(z) - a(z - y) \le \big[a(z)/a(-z)\big]a(-y)$ ($z \le -1$) that follows from (3.21) to see that the difference on the middle member of (3.24) is not larger than

$$E[g(S^x_{\sigma(-\infty, 0]}, y)] \le (1 + k^+)a(-y),$$

hence the right-hand inequality of (3.24). The left-hand one is trivial.

The right-hand inequality of (3.25) is also derived from (3.22) but this time we use the inequality $g(z,y) \le 2\bar{a}(z)$ to have

$$E_x[g(S_T, y)] \le 2E_x[\bar{a}(S_T)] = 2H^x_{(-\infty,0]}\{\bar{a}\}. \tag{3.26}$$

For $z \le 0$, $2\bar{a}(z) \le (1+k^-)a(z)$ by the definition of k^-, while $H^x_{(-\infty,0]}\{a\} \le a^\dagger(x)$ as a special case of (3.17). Hence $2H^x_{(-\infty,0]}\{\bar{a}\} \le (1+k^-)a^\dagger(x)$, showing (3.25). □

In [26], Lemma 3.4 plays a significant role for the proof of (2.19). In this article we apply it only to obtain the next result.

Lemma 3.5. *If either $a(-x)/a(x) \to 0$ or $a(x)/a(-x) \to 0$ as $x \to \infty$ (with the understanding that $a(x) > 0$ ($a(-x) > 0$) for $x > 0$ in the former (latter) case), then*

$$\lim_{x\to\infty} \frac{g_{(-\infty,0]}(x,x)}{g_{\{0\}}(x,x)} = 1; \tag{3.27}$$

and

$$\begin{cases} \lim_{x\to\infty} f_r(x)/2\bar{a}(x) = EZ \le \infty & \text{if } \ a(-x)/a(x) \to 0, \\ \lim_{x\to\infty} f_l(x)/2\bar{a}(x) = -E\hat{Z} \le \infty & \text{if } \ a(x)/a(-x) \to 0. \end{cases} \tag{3.28}$$

The identities in (3.27) and (3.28) are valid whenever $\sigma^2 < \infty$.

Proof. Under the assumption of the lemma it follows from Lemma 3.4 that $g_{[0,\infty)}(x,x) = g_{\{0\}}(x,x) + o(\bar{a}(x)) = 2\bar{a}(x)\{1+o(1)\}$. This verifies (3.27), which entails (3.28) in view of Theorem A(ii). In case $\sigma^2 < \infty$ use the explicit asymptotic forms of $v_{\rm ds}$, $u_{\rm as}$ and $\bar{a}$ to deduce (3.27) from Theorems A and B; then observe $g_{(-\infty,0]}(x,x) \sim f_r(x)/EZ$ to see $f_r(x)/EZ \sim 2\bar{a}(x)$, the first case of (3.28). The second one is similar. □

Lemma 3.6. *If $\sigma_-^2 := E[X_-^2] = \infty > EX_-$, then*

(i) $\int_0^x P[\hat{Z} < -t]dt\big/m_-(x) \ \longrightarrow\ 1/cEZ$ *as* $x \to \infty$*; and*

(ii) $EZ \le \liminf_{x\to\infty} f_r(x)m_-(x)\big/x \le \limsup_{x\to\infty} f_r(x)m_-(x)\big/x \le 2EZ.$

If $\sigma_-^2 < \infty$, then $\liminf$ *and* $\limsup$ *in* (ii) *coincide and equal* $m_-(+\infty)/cE|\hat{Z}| \in (0,\infty)$.

Proof. As a dual relation of (3.10) we have for $t \ge 0$

$$P[\hat{Z} < -t] = v_{\rm ds}(0)\sum_{y=0}^{\infty} u_{\rm as}(y)P[X < -t-y]. \tag{3.29}$$

Let $\sigma_-^2 = \infty > EX_-$. Note that $m_-(x) = \int_0^x dt \int_0^\infty P[X < -t-y]dy$. Replacing $u_{\rm as}(y)$ by $u_{\rm as}(\infty) + o(1)$ in (3.29) and recalling $v_{\rm ds}(0)u_{\rm as}(\infty) = 1/cEZ$ we then infer that

$$\frac{1}{m_-(x)}\sum_{k=0}^{x} P[\hat{Z} < -k] = \frac{v_{\rm ds}(0)}{m_-(x)}\sum_{k=0}^{x}\sum_{y=0}^{\infty} u_{\rm as}(y)P[X < -k-y] = \frac{1}{cEZ} + o(1).$$

Thus (i) is verified. Noting that $cf_r(x+1)$ is the renewal function for the variable $-\hat{Z}$ we use the first inequality of Lemma 1 of Erickson [7] which may read

$$1 \le \frac{cf_r(x+1)}{x}\int_0^x P[\hat{Z} < -t]dt \le 2;$$

combining this with (i) we can readily deduce (ii). The last assertion is obvious, for $m_-(\infty) < \infty$ if $\sigma_-^2 < \infty$ and $f_r(x)/x \to 1/cE|\hat{Z}|$. □

Remark 3.7. By (3.10) and Lemma 3.6(ii) one infers that $\int_0^\infty P[X>t]tdt/m_-(t)<\infty$ if $EZ<\infty$—the necessity part of the Chow's criterion for $EZ<\infty$ (this half of it is also proved by Doney [6]). Combined with (2.13) this shows that if $EZ<\infty$, then both of the following summability conditions hold

$$(\sharp)\ \ \sum_{x=1}^{\infty}\bar a(x)P[X>x]<\infty \quad \text{ and } \quad (\flat)\ \ \int_1^\infty \frac{xP[X>x]}{m_-(x)}dx<\infty.$$

The converse as well as the implication $(\flat)\Rightarrow(\sharp)$ is proved in [22, Lemma 4.1, Lemma 2.9, Proposition 2.1(i)]. Thus the equivalence of $(\flat)$ and $EZ<\infty$ follows. It also holds that $(\sharp)$ and $(\flat)$ are equivalent [22, Corollary 4.1], [26, Eq. (1.3)] as (1.5) may suggest.

Lemma 3.8. *Suppose $EZ<\infty$ and $\sigma^2=\infty$. Then $\lim_{x\to\infty}a(-x)/a(x)=0$.*

Proof. This lemma follows from Theorem 4 of [26] as mentioned previously. The direct proof is easy and given as follows. Let $\bar\sigma_x=\sigma_x$ if $S_0\neq x$ and $=0$ otherwise. Then, applying (3.14) and (3.7) under the assumption of the lemma leads to

$$\frac{a(-x)}{a(x)+a(-x)}=\lim_{z\to-\infty}P_z[\sigma_x<\sigma_0]=\sum_{y=1}^{\infty}H^{-\infty}_{[0,\infty)}(y)P_y[\bar\sigma_x<\sigma_0] \qquad (x>0).$$

As $x\to\infty$ the last sum approaches zero and hence $a(-x)/a(x)\to 0$. □

Lemma 3.9. *If $EZ<\infty$, then*

$$\sum_{y=-\infty}^{0}a(y)P[X<y]<\infty \quad \textit{and} \quad \lim_{x\to+\infty}\frac{H^x_{(-\infty,0]}\{a\}}{f_r(x)}=0.$$

Proof. By Theorem A(ii) $g_{(-\infty,0]}(1,z)=v_{\rm ds}(0)u_{\rm as}(z-1)$ for $z\geq 1$. Suppose $EZ<\infty$. Then $u_{\rm as}(\infty)>0$ and for $y\leq 0$,

$$H^1_{(-\infty,0]}(y)=v_{\rm ds}(0)\sum_{z=1}^{\infty}u_{\rm as}(z-1)p(y-z)\asymp P[X<y],$$

and hence the first assertion follows, for $H^1_{(-\infty,0]}\{a\}<\infty$ by virtue of (3.17). Since $H^x_{(-\infty,0]}(y)$ is less than $f_r(x)P[X<y]$ and $H^x_{(-\infty,0]}(y)/f_r(x)\to 0$ as $x\to\infty$ for each $y\leq 0$, by dominated convergence $H^x_{(-\infty,0]}\{a\}/f_r(x)\to 0$, as desired. □

In view of the following lemma we can define for any non-empty subset B of $\mathbb{Z}$ the function $u_B(x)$, $x\in\mathbb{Z}$ by

$$u_B(x)=g_B(x,y)+a(x-y)-H^x_B\{a(\cdot-y)\}. \tag{3.30}$$

Lemma 3.10. *For each $x\in\mathbb{Z}$ the RHS of (3.30) is independent of $y\in\mathbb{Z}$, and u_B defined therein is non-negative, represented by $u_B(x)=a^\dagger(x-\xi)-H^x_B\{a(\cdot-\xi)\}$ for any $\xi\in B$ and harmonic on $\mathbb{Z}\setminus B$ in the sense that for each $\xi\in B$ fixed,*

$$\sum_{z\notin B}p(z-x)u_B(z)=u_B(x) \quad \textit{for} \quad x\in\mathbb{Z}. \tag{3.31}$$

Identity (3.30) and hence what are advanced below hold true for every recurrent random walk irreducible on $\mathbb{Z}$. The analogous result holds for the two-dimensional recurrent random walks to which the same proof applies.

Proof. In the proof of Lemma 2.9 of [23] it is shown that for each $x \in \mathbb{Z}$ fixed, the RHS of (3.30) is a dual-harmonic function of $y \in \mathbb{Z}$ (i.e., harmonic with respect to the dual transition function $\hat{P}(x,y) := p(x-y)$). In view of the uniqueness theorem of non-negative harmonic function [19, P13.1], [9, Proposition 6-3] the first assertion of the lemma accordingly follows if we show that it is bounded below. To this end it suffices to see that for all $x, y \in \mathbb{Z}$ and $\xi \in B$,

$$H_B^x\{a(\cdot - y)\} \le a^\dagger(x-\xi) + a(\xi - y)$$

(since $a(x-y) - a(\xi - y)$ is a bounded function of y), which however is immediate from the subadditivity $a(\cdot - y) \le a(\cdot - \xi) + a(\xi - y)$ and the inequality (3.17). Taking y from B in (3.30) it follows that

$$u_B(x) = a^\dagger(x-\xi) - H_B^x\{a(\cdot - \xi)\} \qquad \text{for } \xi \in B \tag{3.32}$$

and by the inequality (3.17) we see $u_B \ge 0$. Noting that $\sum_{w\notin B} p(w-x) H_B^w\{a(\cdot - \xi)\} = H_B^x\{a(\cdot-\xi)\} - \sum_{z\in B} p(z-x)\{a(z-\xi)\}$ and using (3.13) one deduces

$$\sum_{w\notin B} p(w-x)\big[a(w-\xi) - H_B^w\{a(\cdot-\xi)\}\big] = a^\dagger(x-\xi) - H_B^x\{a(\cdot-\xi)\}, \tag{3.33}$$

which shows (3.31), for $a(w-\xi)$ can be replaced by $a^\dagger(w-\xi)$ because of the identity $\sum_{w\notin B} p(w-x)\delta(w,\xi) = 0$. □

Remark 3.11. (a) The independence of the RHS of (3.30) from y also follows from Lemma 3.3. Indeed, if $0 \in B$ then we have (3.22) which becomes (3.30) with $u_B(x) = a^\dagger(x) - H_B^x\{a\}$ after a simple rearrangement of terms. For the case $0 \notin B$, pick any $\xi \in B$, consider (3.22) for $B' = B - \xi$ (shift by ξ) in place of B and replace x, y by $x - \xi$ and $y - \xi$. Conversely Lemma 3.3 follows immediately from the first half of Lemma 3.10.

(b) For a positive integer R let $\tau_R = \sigma_{\mathbb{Z}\setminus(-R,R)}$. Then

$$a^\dagger(x-\xi) = E_x[a(S_{\tau_R\wedge\sigma_B} - \xi)] \qquad (x \in \mathbb{Z}, \xi \in B), \tag{3.34}$$

and the function u_B defined in (3.30) is given by

$$u_B(x) = \lim_{R\to\infty} E_x\big[a(S_{\tau_R} - \xi); \tau_R < \sigma_B\big] \qquad (x \in \mathbb{Z}, \xi \in B); \tag{3.35}$$

in particular the limit appearing in (3.35) is independent of the choice of ξ. These formulae are verified as follows. For $x \ne \xi$, $M_n := a(S_{n\wedge\sigma_B} - \xi)$ being a non-negative martingale under P_x that is uniformly bounded on $n < \tau_R$, one obtains the identity $a(x-\xi) = E_x M_{\tau_R}$. As for the case $x = \xi$ suppose that $\xi = 0 \in B$ for simplicity so that $M_{\tau_R} = a(S_{\tau_R\wedge\sigma_B})$. Then

$$E_0 M_{\tau_R} = \sum_{x\in B \text{ or } |x|\ge R} p(x)a(x) + \sum_{x\notin B, |x|<R} p(x) E_x M_{\tau_R} = \sum p(x)a(x) = 1.$$

Thus one has (3.34). For the proof of (3.35) write it as

$$a^\dagger(x-\xi) = E_x\big[a(S_{\tau_R} - \xi); \tau_R < \sigma_B\big] + E_x\big[a(S_{\sigma_B} - \xi); \tau_R \ge \sigma_B\big]. \tag{3.36}$$

On passing to the limit as $R \to \infty$ the equality (3.35) then comes out in view of (3.32), the last expectation converging to $H_B^x\{a(\cdot - \xi)\} = a^\dagger(x-\xi) - u_B(x)$.

When X is of finite range, the identity (3.35) (restricted to $x \notin B$) is shown in the proof of Proposition 4.6.3 of [13] (in a different way from ours).

Let $\hat{H}^x_B$ stand for the hitting distribution of a set B for the dual (time-reversed) walk, in other words $\hat{H}^x_B(y) = H^{-x}_{-B}(-y)$ ($-B = \{-z : z \in B\}$). Then $H^x_B(x) = \hat{H}^x_B(x)$ and

$$g_B(x,y) = \hat{H}^y_B(x)\mathbf{1}_{\mathbb{Z}\setminus B}(y) + \delta(x,y) \quad \text{for } \ x \in B, y \in \mathbb{Z}, \tag{3.37}$$

where $\mathbf{1}_B$ is the indicator function of B.

Lemma 3.12. *Let $\hat{u}_B$ be the dual of u_B: $\hat{u}_B(x) = a^\dagger(\xi - x) - \hat{H}^x_B\{a(\xi - \cdot)\}$ ($\xi \in B$). Then*

$$H^x_B(y)\mathbf{1}_{\mathbb{Z}\setminus B}(x) = \hat{u}_B(y) + \sum_{z\in B} a(x-z)H^z_B(y) - a^\dagger(x-y) \quad \textit{for } \ x \in \mathbb{Z}, y \in B; \tag{3.38}$$

and

$$1 - H^x_B(\xi) = [1 - H^\xi_B(\xi)]a^\dagger(x-\xi) + \sum_{z\in B\setminus\{\xi\}} \big[a(\xi - z) - a(x-z)\big]H^z_B(\xi) \qquad \textit{for } \ \xi \in B, x \notin B \setminus \{\xi\}. \tag{3.39}$$

Proof. Let $x \in B$, substitute the expression of $g_B(x,y)$ given in (3.37) into (3.30), rewrite the resulting identity in terms of the dual objects and interchange x and y, use the equality $\hat{H}^y_B(z) = H^z_B(y)$ for $y, z \in B$, and you obtain (3.38).

For the proof of (3.39) we have only to consider $x \neq \xi$, (3.39) being obviously true for $x = \xi$. Let $\xi \in B$ and subtract the equality (3.38) with $x \notin B, y = \xi$ from that with $x = y = \xi$. Then one finds

$$-H^x_B(\xi) = -a(x-\xi)H^\xi_B(\xi) + \sum_{z\in B\setminus\{\xi\}} \big[a(\xi - z) - a(x-z)\big]H^z_B(\xi) - 1 + a^\dagger(x-\xi),$$

which after a simple transposition of terms becomes (3.39). □

If B is finite, $\hat{u}_B(y) = \frac{1}{2}\lim_{x\to\infty}\big[H^{-x}_B(y) + H^x_B(y)\big]$ and (3.38) is given in [19, P30.1].

The next lemma is a consequence of the second relation of Lemma 3.12.

Lemma 3.13. *Let $\sigma^2 = \infty$. If B_n, $n = 1, 2, \dots$ are non-empty subsets of $\mathbb{Z}$ such that $\min\{|z| : z \in B_n\} \to \infty$ as $n \to \infty$ and $P_0[\sigma_{-B_n} < \sigma_0]/P_0[\sigma_{B_n} < \sigma_0]$ is bounded, then*

$$\lim_{n\to\infty} \frac{P_x[\sigma_{B_n} < \sigma_0]}{P_0[\sigma_{B_n} < \sigma_0]} = a^\dagger(x). \tag{3.40}$$

Proof. If $z, 0 \in B$, then $H^z_B(0) = H^0_{-B}(-z)$, and hence $\sum_{z\in B\setminus\{0\}} H^z_B(0) = 1 - H^0_{-B}(0) = P_0[\sigma_{-B} < \sigma_0]$. Applying this as well as identity (3.39) with $\xi = 0$, $B = B_n \cup \{0\}$ one deduces that

$$\frac{P_x[\sigma_{B_n} < \sigma_0]}{P_0[\sigma_{B_n} < \sigma_0]} = \frac{1 - H^x_B(0)}{1 - H^0_B(0)} = a^\dagger(x) + \frac{\sum_{z\in B_n}[a(-z) - a(x-z)]H^z_{B_n\cup\{0\}}(0)}{P_0[\sigma_{B_n} < \sigma_0]} \tag{3.41}$$

of which the last ratio tends to zero for each x under the condition imposed on B_n in the lemma, for $a(-z) - a(x-z) \to 0$ ($-z \in B_n$) if $\sigma^2 = \infty$. □

4 Proof of Theorem 2.1

Lemma 4.1. *Suppose $EZ = \infty$. Then $a^\dagger(x) = H^x_{(-\infty,0]}\{a\}$.*

Proof. We consider only the case $x > 0$, the asserted formula for $x \le 0$ being deduced from that for $x > 0$ by using (3.13) as in (3.16).

The proof is based on the fact that the function $h(x) := a^\dagger(x) - H^x_{(-\infty,0]}\{a\}$ is non-negative and harmonic on $x > 0$ (according to Lemma 3.10). In view of the uniqueness of harmonic function it suffices to show

$$\liminf_{x\to\infty} \frac{a(x)}{f_r(x)} = 0. \tag{4.1}$$

We have extended f_r to a function on $\mathbb{Z}$, denoted also by f_r, by (2.7), namely $f_r(x) = P[Z > -x]$ $(x \le 0)$. Accordingly, by (3.10) we have $f_r(x) = \sum_{y=1}^{\infty} p(y-x) f_r(y)$ for all $x \in \mathbb{Z}$.

By the assumption of the lemma the walk is not right-continuous. Hence by Lemma 3.1 $\inf_{x<0} a(x) > 0$, so that for some constant C

$$f_r(x) \le C a(x) \quad \text{for} \;\; x < 0. \tag{4.2}$$

Define the operators P and P^- by

$$Pf(x) = \sum_{y \in \mathbb{Z}} p(y-x) f(y) \quad \text{and} \quad P^- f(x) = \sum_{y \le 0} p(y-x) f(y) \qquad (x \in \mathbb{Z}),$$

respectively. Put $G_n(x,y) = p^n(y-x) + \cdots + p(y-x) + \delta(x,y)$ and let G_n also denote the corresponding operator. We may suppose $\inf_{x>0} a(x) > 0$, otherwise $a(x)$ vanishing for all $x > 0$ so that (4.1) is plainly evident. Owing to (4.2) relation (4.1) then follows if we can show

$$\lim_{n\to\infty} \frac{P^n a^\dagger(0)}{P^n f_r(0)} = 0, \tag{4.3}$$

for if (4.1) does not hold, $a^\dagger = a + \delta(\cdot, 0)$ must dominate a positive multiple of f_r so that (4.3) is impossible.

From the identity $Pa = a + \delta(\cdot, 0)$ one deduces by induction that

$$P^n a^\dagger(x) = a(x) + G_n(x, 0). \tag{4.4}$$

On the other hand one obtains that $Pf_r = f_r + P^- f_r$ and by induction again $P^n f_r(x) = f_r(x) + G_{n-1} P^- f_r(x)$, which can be rewritten as

$$P^n f_r(x) = f_r(x) \mathbf{1}_{[1,\infty)}(x) + \sum_{y \le 0} G_n(x,y) f_r(y). \tag{4.5}$$

Since $\sum_{y \le 0} f_r(y) = \sum_{y \ge 0} P[Z > y] = \infty$ due to the assumption of the lemma, for any $K > 0$ one can choose a positive integer M so that $\sum_{-M \le y \le 0} f_r(y) \ge K$; and hence

$$P^n f_r(0) \ge K \min_{0 \le z \le M} G_n(0, -z) \ge 2^{-1} K G_n(0,0)$$

if n is large enough, for the recurrence of the walk implies $\lim_{n\to\infty} G_n(z,0)/G_n(0,0) = 1$ (cf. [19, P2.6]). Combined with (4.4) the inequality derived above implies that $P^n f_r(0) \ge \frac{1}{2} K P^n a^\dagger(0)$ for all sufficiently large n and we can conclude the required relation (4.3). ☐

Proof of Theorem 2.1. By Lemma 3.10 the formula of (i) follows if we verify its special case $y = 0$. Note that $g_{(-\infty,0]}(x,0) = \delta(x,0)$ and by (2.5) $g_{(-\infty]}(\cdot, \infty)$ vanishes if $EZ = \infty$. It is then obvious that if $EZ = \infty$, (i) and (iii) follow from Lemma 4.1 and (4.1), respectively. Let $EZ < \infty$. Then by Lemma 3.10 the difference $a^\dagger(x) - H^x_{(-\infty,0]}\{a\}$ is non-negative and harmonic on $x > 0$, so that it is a constant multiple of $f_r(x)$. The constant factor is determined by using Lemmas 3.5, 3.8, and 3.9. (Note that $\bar{a}(x) \sim a(x)$ if $\sigma^2 < \infty$.) By these the second assertion (ii) also follows. ☐

5 Proof of Theorem 2.4

Proof of (i). The first half of (i) of Theorem 2.4 follows from Lemma 3.6 and Corollary 2.2. Suppose that $EZ < \infty$ and $\sigma^2 = \infty$. The formula (2.16), what is asserted in the second half, may be written as

$$a(-x) \sim \frac{1}{EZ} \sum_{k=1}^{\infty} P\big[k < Z \le x + k\big] a(k) \quad (x \to \infty). \tag{5.1}$$

The hitting distribution of the half line $[0,\infty)$ for the random walk started at a negative site $-x$ agrees with that for the ascending ladder height process started at $-x$. Since $G(x',x'') := u_{\rm as}(x''-x')$ ($x' \le x''$) is the Green function of the ladder height process it accordingly follows that

$$H^{-x}_{[0,\infty)}(k) = \sum_{y=1}^{x} u_{\rm as}(x-y)P[Z=k+y] = \sum_{w=0}^{x-1} u_{\rm as}(w)P[Z=k+x-w]. \tag{5.2}$$

Since for each w, $\sum_{k=0}^\infty P[Z=k+x-w]a(k) \to 0$ as $x\to\infty$ and $u_{\rm as}(w)\to 1/EZ$ ($w\to\infty$), we can conclude (5.1) owing to Corollary 2.2(ii) that gives the identity $H^{-x}_{[0,\infty)}\{a\} = a(-x)$.

Proof of (ii). If $\limsup_{x\to-\infty} a(x) < \infty$, then $H^{x}_{(-\infty,0]}\{a\}$ is bounded, so that EZ cannot be finite, for otherwise by Corollary 2.2(ii) $a(x)+a(-x)$ must be bounded which is impossible in view of (3.14). This shows (ii) by virtue of Lemma 3.1.

Proof of (iii). Let $\sigma^2=\infty$ and $EZ<\infty$. Then $E|\hat Z|=\infty$ so that $a(-x) = H^{-x}_{[0,\infty)}\{a\}$ according to Corollary 2.2 and by the first equality of (5.2) we have

$$a(-x) = \sum_{y=1}^{x} u_{\rm as}(x-y)b(y), \tag{5.3}$$

where

$$b(y) = \sum_{k=1}^{\infty} P[Z=y+k]a(k). \tag{5.4}$$

Since $b(y)\to 0$ under $EZ<\infty$, (5.3) yields

$$a(-x) \sim \frac{1}{EZ}\sum_{y=1}^{x} b(y). \tag{5.5}$$

By (3.9)

$$b(y) = \sum_{k=1}^{\infty}\sum_{w=0}^{\infty} v_{\rm ds}(w)p(y+k+w)a(k), \tag{5.6}$$

which by a change of variables one can rewrite as

$$b(y) = \sum_{j=1}^{\infty} p(y+j)\sum_{w=0}^{j-1} v_{\rm ds}(w)a(j-w).$$

Since f_r is sub-additive (see (8.1)) and $a(x)EZ \sim f_r(x) = v_{\rm ds}(0)+\cdots+v_{\rm ds}(x-1)$, the inner sum is bounded from below and above by positive multiples of $[f_r(j)]^2$, so that $b(y) \asymp \sum_{j=1}^\infty p(y+j)[f_r(j)]^2$. Now substitution into (5.5) yields

$$a(-x) \asymp \sum_{y=1}^{x}\sum_{j=1}^{\infty} p(y+j)[f_r(j)]^2.$$

In view of (2.15) (or Lemma 3.6(ii)) one can replace $f_r(j)$ by $j/m_-(j)$, showing (2.17), the desired asymptotics of $a(-x)$. The rest of (iii) is readily ascertained to be true by (2.17), Lemma 4.1 and (3.8). □

Lemma 5.1. *If $EZ=\infty$ and $a(-x)$ is almost (namely, $a(-y)\ge \delta a(-x)$ if $y>x\ge x_0$ with some $\delta>0$ and $x_0>0$), then*

$$(*) \qquad a(x)/f_r(x)\to 0 \quad \text{as} \quad x\to\infty.$$

Proof. If $E|\hat{Z}| < \infty$, then $f_r(x) \sim Cx$ and the assertion of the lemma is obvious. Let $E|\hat{Z}| = EZ = \infty$ and we show that if $a(x)$ is almost increasing, then $a(-x)/f_l(x) \to 0$, which by duality amounts to the same as the assertion of the lemma. For $x \geq 1$, $a(-x) = H^{-x}_{(\infty,0]}\{a\}$ so that we have (5.3) and for the present purpose it suffices to show that $b(y) \to 0$. Rewrite the expression of $b(y)$ in (5.6) as

$$b(y) = \sum_{w=0}^{\infty} v_{\rm ds}(w) \sum_{k=1}^{\infty} a(k)p(y+k+w).$$

Now suppose that $a(x)$ is almost increasing. Then

$$\sum_{k=0}^{\infty} a(k)p(y+k+w) \leq C \sum_{k=y}^{\infty} a(k)p(k+w) + \sum_{k=1}^{x_1} a(k)p(y+k+w) \tag{5.7}$$

for some x_1. On noting $\sum_{w=0}^{\infty} v_{\rm ds}(w) \sum_{k=1}^{\infty} a(k)p(k+w) = b(0) = Ea(Z) < \infty$ according to (5.4), the dominated convergence therefore concludes that $b(y) \to 0$, as desired. ☐

Remark 5.2. We apply Lemma 5.1 to verify the last assertion of Remark 2.3(e). Let X belong to the domain of attraction of a Cauchy distribution (a stable law with exponent $\alpha = 1$ and skewness parameter $\beta = 0$). Then $\bar{a}(x)$ is dominated by a slowly varying function [26, Remark 62(ii)]. Suppose $\lim P[S_n > 0] = \rho$ in addition. Then $f_r(x)$ is regularly varying with index $1 - \rho$ (cf. [16] for $\rho < 1$ and [27] for $\rho = 1$), so that it plainly follows that $\bar{a}(x)/f_r(x) \to 0$ if $\rho < 1$. Let $\rho = 1$. Then $a(x) \sim a(-x) \sim \int_{x_0}^{x} F(-s)\{A^2(s)\}^{-1}ds$ for some $x_0 > 0$, where $A(s) = \int_0^s \big[1 - F(t) - F(-t)\big]dt$ (cf. [26, Theorem 7]), in particular $a(-x)$ is almost increasing. Thus Lemma 5.1 verifies $(*)$, hence $\bar{a}(x)/f_r(x) \to 0$. We also know that $f_r(x) \sim 1/\int_x^{\infty} \big[F(-s)/\ell^*(s)\big]ds$, where $\ell^*(s) = \int_0^s P[Z > t]dt$ [27, Lemma 3.1]). However, it is not clear whether $(*)$ can be deduced directly from these asymptotic relations.

6 Proofs of Propositions 2.7 and 2.8

We shall employ the following identities:

$$P_x[\sigma_R < T] = \frac{g_{(-\infty,0]}(x,R)}{g_{(-\infty,0]}(R,R)}; \quad P_x[\sigma_R < \sigma_0] = \frac{a^{\dagger}(x) + a(-R) - a(x-R)}{2\bar{a}(R)} \tag{6.1}$$

($R = 1, 2, \ldots,\ x \neq R$). If either $a(-x)/a(x) \to 0$ or $a(x)/a(-x) \to 0$ ($x \to \infty$), then by Lemma 3.5 $g_{(-\infty,0]}(R,R) \sim 2\bar{a}(R)$ so that

$$\frac{P_x[\sigma_R < T\,]}{P_x[\sigma_R < \sigma_0]} \sim \frac{g_{(-\infty,0]}(x,R)}{g_{\{0\}}(x,R)}. \tag{6.2}$$

According to [26, Theorem 4] $a(\mp x)/a(\pm x) \to 0$ if $m_{\pm}(x)/m(x) \to 0$.

Lemma 6.1. *If $a(-x)/a(x) \to 0$ ($x \to \infty$), then for each $M \geq 1$, uniformly for $-M < z < y$*

$$a(-y) - a(z-y) = o(a(z) \vee 1) \qquad (y \to \infty);$$

in particular as $R \to \infty$

$$P_x[\sigma_R < \sigma_0] \sim a^{\dagger}(x)/a(R) \quad \textit{uniformly for} \quad -M < x < R.$$

Proof. This—verified readily by using Lemma 3.2—is contained in Lemma 37(ii) of [26]. ☐

6.1 Proof of Proposition 2.7

The first case of (2.21) follows from the second equivalence in (2.19) together with Lemma 6.1. Without recourse to (2.19) it may be verified as follows. If $EZ < \infty$, by the expression of $g_{(-\infty,0]}(x,y)$ in Theorem A(ii) it follows that uniformly for $x < R$ as $R \to \infty$

$$g_{(-\infty,0]}(x,R) \sim f_r(x)/EZ \tag{6.3}$$

and the relation (2.21) asserted in Proposition 2.7 follows from (6.2) and Lemma 6.1; the uniformity of the convergence is assured by $\lim f_r(x)/a(x) = EZ$.

The case $EZ = \infty$ of (2.21) is essentially contained in the next lemma.

Lemma 6.2. *If* $\sigma^2 = \infty$, $\lim_{R\to\infty} 2\bar{a}(R)P_x[T < \sigma_R < \sigma_0] = H^x_{(-\infty,0]}\{a\}$ $(x \in \mathbb{Z})$.

Proof. First of all it is pointed out that we may suppose $P[X \geq 2] > 0$ so that for all x, $P_x[\sigma_R < \sigma_0] > 0$, for otherwise $H^x_{(-\infty,0]}\{a\}$ vanishes and the result is obvious. Now observe that the equality $P_x[T < \sigma_R < \sigma_0] = P_x[\sigma_R < \sigma_0] - P_x[\sigma_R < T\,]$ together with (6.1) yields

$$2\bar{a}(R)P_x[T < \sigma_R < \sigma_0] = \big[a^\dagger(x) + a(-R) - a(x-R)\big]\left(1 - \frac{P_x[\sigma_R < T\,]}{P_x[\sigma_R < \sigma_0]}\right) \tag{6.4}$$

on the one hand, and since $P_x[T < \sigma_R < \sigma_0] = \sum_{y<0} P_x[S_T = y, T < \sigma_R]P_y[\sigma_R < \sigma_0]$,

$$2\bar{a}(R)P_x[T < \sigma_R < \sigma_0] = \sum_{y<0} P_x[S_T = y, T < \sigma_R]\big[a(y) + a(-R) - a(y-R)\big] \tag{6.5}$$

on the other hand.

Suppose $EZ < \infty$ so that the first case of Proposition 2.7 is applicable. Then under $\sigma^2 = \infty$ the RHS of (6.4) converges to $a^\dagger(x) - Af_r(x)/EZ$, hence one has the identity of the lemma in view of Corollary 2.2(i).

By letting $R \to \infty$ in the identities (6.4) and (6.5), with the help of Fatou's lemma for the infinite series on the RHS of (6.5), we obtain

$$\begin{aligned} H^x_{(-\infty,0]}\{a\} \quad &\leq \quad \liminf_{R\to\infty} 2\bar{a}(R)P_x[T < \sigma_R < \sigma_0] \\ &= \quad a^\dagger(x) - a^\dagger(x) \limsup_{R\to\infty} \frac{P_x[\sigma_R < T\,]}{P_x[\sigma_R < \sigma_0]} \leq a^\dagger(x), \end{aligned} \tag{6.6}$$

provided $\sigma^2 = \infty$. If $EZ = \infty$, then the two extreme members in (6.6) must coincide owing to Corollary 2.2(ii), entailing that the two inequalities above are the equality, of which the latter means that the $\limsup$ vanishes—showing the relation of Proposition 2.7. We can interchange the $\liminf$ and the $\limsup$ in (6.6), which gives the equality of the lemma. □

Remark 6.3. If $\sigma^2 < \infty$, we have $P_x[\sigma_R < T\,]/P_x[\sigma_R < \sigma_0] \sim g_{(-\infty,0]}(x,R)/g_{\{0\}}(x,R)$ uniformly for $x \in \mathbb{Z}$. By (6.4) and (6.3) it therefore follows that as $R \to \infty$

$$2\bar{a}(R)P_x[T < \sigma_R < \sigma_0] = g_{\{0\}}(x,R) - \frac{f_r(x)}{EZ}\{1 + o(1)\} \;\longrightarrow\; a^\dagger(x) + \frac{x}{\sigma^2} - \frac{f_r(x)}{EZ}$$

$(x \in \mathbb{Z})$, where the equality is uniform for $x < R$ (but the convergence is not).

6.2 Proof of Proposition 2.8

We prove the dual assertion that follows.

Lemma 6.4. *If $E|\hat{Z}| < \infty$, then, uniformly for $x \in \mathbb{Z}$, as $R \to \infty$*

$$P_x\big[\sigma_{[R,\infty)} < T\big] = P_x\big[\sigma_R < T\big]\{1 + o(1)\}$$

and as $x \wedge R \to \infty$ under $x < R$

$$P_x\big[\sigma_{[R,\infty)} < T\big] \sim \frac{f_l(R) - f_l(R-x)}{f_l(R)} \le \frac{f_l(x)}{f_l(R)}. \tag{6.7}$$

Proof. Let $\sigma^2 = \infty$. Then the summability of $\hat{Z}$ implies $a(R)/a(-R) \to 0$ as well as the tightness of the family $\{H^y_{(-\infty,0]} : y > 0\}$, which together imply that for each z the probability

$$P_{R-z}[T < \sigma_R] = P_{R-z}[\sigma_0 < \sigma_R] + P_{R-z}[T < \sigma_R < \sigma_0]$$

tends to zero (with z fixed)—use (6.1) for the first term on the RHS; note that the second term is less than $\sum_{y<0} H^{R-z}_{(-\infty,0]}(y)P_y[\sigma_R < \sigma_0]$. Hence

$$\sup_{y>R} P_y\big[T < \sigma_R\big] = \sup_{y'>0} \sum_{z>0} H^{y'}_{(-\infty,0]}(-z) P_{R-z}\big[\tilde{T} < \sigma_R\big] \longrightarrow 0 \quad \text{as} \quad R \to \infty,$$

where $\tilde{T} = \inf\{n \ge 0 : S_n \le 0\}$, and hence the ratio

$$\frac{P_x[\sigma_{[R,\infty)} < T] - P_x[\sigma_R < T]}{P_x[\sigma_{[R,\infty)} < T]} = \sum_{y>R} P_x\big[S_{\sigma_{[R\infty)}} = y \,\big|\, \sigma_{[R,\infty)} < T\big] P_y\big[T < \sigma_R\big]$$

tends to zero uniformly for $x > 0$, which shows the first half of the lemma.

For the proof of the second half we derive the asymptotic form of $P_x[\sigma_R < T]$ by using (6.1). Let $E|\hat{Z}| < \infty$. Then one obtains $g_{(-\infty,0]}(R,R) \sim f_l(R)/E|\hat{Z}|$ because of the dual of (6.3), so that

$$P_x\big[\sigma_{[R,\infty)} < T\big] \sim P_x\big[\sigma_R < T\big] = \frac{\sum_{k=0}^{x-1} v_{\rm ds}(k) u_{\rm as}(R-x+k)}{f_l(R)} \big(E|\hat{Z}| + o(1)\big).$$

In order to verify (6.7) it suffices to see that for each $K > 0$, $\sum_{k=0}^{K} u_{\rm as}(y+k)$ divided by $\sum_{k=0}^{x-1} u_{\rm as}(y+k)$ tends to zero as $x \to \infty$ uniformly for $y \ge 0$. By Lemma 6.5 below it follows that for each $k \le K$ and $j \ge k$, $u_{\rm as}(y+j) \ge u_{\rm as}(y+k)u_{\rm as}(j-k)$ so that

$$\sum_{j=0}^{x-1} u_{\rm as}(y+j) \ge u_{\rm as}(y+k)[u_{\rm as}(0) + \cdots + u_{\rm as}(x-k-1)]$$

and hence the ratio in question is dominated by $K/f_l(x-K)$, which tends to zero as required. The inequality in (6.7) follows by the sub-additivity of f_l (cf. (8.1)). □

Lemma 6.5. *For all integers $x, y \ge 0$, $u_{\rm as}(x+y) \ge u_{\rm as}(x)u_{\rm as}(y)$.*

Proof. The ratio $u_{\rm as}(x+y)/u_{\rm as}(x)$ is not less than the conditional probability that $x+y$ is an ascending ladder point given so is x, but this conditional probability equals $u_{\rm as}(y)$, showing the inequality of the lemma. □

Remark 6.6. Let $E|\hat{Z}| < \infty$ and $\sigma^2 = \infty$. Then by Corollary 2.2 $\bar{a}(R) \sim f_l(R)/E|\hat{Z}|$, so that by (6.2) and Lemma 6.4 it holds that whenever $x < R$ and $x \to \infty$

$$\frac{P_x[\sigma_R < T]}{P_x\big[\sigma_R < \sigma_0\big]} \sim \frac{[f_l(R) - f_l(R-x)]/E|\hat{Z}|}{a(x) + a(-R) - a(x-R)}.$$

For any $0 < \varepsilon < 1/2$, the RHS can be shown to approach unity uniformly for $\varepsilon R < x < R$ if it is further supposed that $1 - F(x)$ is regularly varying with index $-\alpha < -1$ and $F(-x)/[1 - F(x)] \to 0$ (cf. [26, Section 8.1.1])—if $\alpha = 1$, it may approach zero uniformly for $0 < x < (1-\varepsilon_R)R$ for an appropriate F and $\varepsilon_R > 0$ decreasing to 0 [this actually takes place if, e.g., $P[X = x] = x^{-2}(\log x)^{-\lambda}\{1+O(x^{-1})\}$, $F(-x) = x^{-1}(\log x)^{-\lambda-\delta}\{1+O(x^{-1})\}$ (as $x \to \infty$) and $\varepsilon_R \sim \exp\{-(\log R)^{\varepsilon}\}$ with $\lambda > 1$, $0 < \delta < 1$ and $0 < \varepsilon < \delta \wedge \frac{1}{2}$—proof is quite involved even in such a particular case].

7 The random walk conditioned on $\sigma_0 = \infty$

Write $\tilde{P}_x$ for $P_x[\,\cdot\,|\sigma_0 = \infty]$ ($x \in \mathbb{Z}$), the probability law of the conditional process S_n given that it never visits the origin. It is defined as a limit law of $P_x[\cdot|\sigma_0 > k]$ as $k \to \infty$. If $\sigma^2 = \infty$, suppose $P[X \le -2]P[X \ge 2] > 0$ so that $a^\dagger(x) > 0$ for all x. [If $P[X \le -2]P[X \ge 2] = 0$, this conditioning forces the walk to stay either the positive or negative half line once it get into there, yielding the process represented by the harmonic transform by means of f_r or f_l according as the starting site is positive or negative, respectively.] The conditional process is Markovian with state space $\mathbb{Z} \setminus \{0\}$ and the n-step transition law given by

$$\frac{1}{a(x)} q^n(x,y)a(y) \quad (x,y \ne 0) \quad \text{where} \quad q^n(x,y) = P_x[S_n = y, n < \sigma_0]. \tag{7.1}$$

Indeed, for $n < k$

$$P_x[\,S_n = y\,|\,\sigma_0 > k] = q^n(x,y)\frac{P_y[\sigma_0 > k-n]}{P_x[\sigma_0 > k]}, \tag{7.2}$$

and as $k \to \infty$, $P_y[\sigma_0 > k]/P_x[\sigma_0 > k] \to a(y)/a(x)$ while $P_y[\sigma_0 = k-n]/P_x[\sigma_0 > k] \to 0$ for each n (see [19, T32.1, T32.2]) so that the ratio in the RHS of (7.2) converges to $a(y)/a(x)$, showing (7.1). Let $B(R) = \mathbb{Z} \setminus (-R, R)$. Then by Lemma 3.13

$$\lim_{R\to\infty} \frac{P_x[\sigma_{B(R)} < \sigma_0]}{P_0[\sigma_{B(R)} < \sigma_0]} = a^\dagger(x)$$

(in case $\sigma^2 < \infty$ see (3.41) or (7.9) below), from which one can easily deduce that

$$\text{the conditional law } \ P_x[\,\cdot\,|\,\sigma_{B(R)} < \sigma_0] \ \text{ converges to } \ \tilde{P}_x \ \text{ as } \ R \to \infty \tag{7.3}$$

in the sense of convergence of finite dimensional distributions.

It follows from (7.1) that

$$\tilde{H}^x_{(-\infty,0]}(y) := \tilde{P}_x[\,S_T = y] = \frac{1}{a^\dagger(x)} H^x_{(-\infty,0]}(y)a(y) \qquad (x \in \mathbb{Z}, y < 0). \tag{7.4}$$

Therefore by Corollary 2.2

$$\tilde{P}_x[\,T < \infty] = 1 - \frac{Af_r(x)}{E[Z]a^\dagger(x)} \qquad (x \in \mathbb{Z}). \tag{7.5}$$

[Recall $f_r(x) = P[Z > -x]$ for $x \le 0$.] (7.1) also shows $\sum_n \tilde{P}_x[S_n = y] < \infty$. Hence

$$\tilde{P}_x[\,|S_n| \to \infty \ \text{ as } \ n \to \infty] = 1. \tag{7.6}$$

In fact we have the following: in case $\sigma^2 = \infty$, for every $x \in \mathbb{Z}$

$$\begin{aligned} &\text{(a)} \quad \tilde{P}_x[\,\lim S_n = +\infty] = 1 && \text{if} \quad EZ < \infty, \\ &\text{(b)} \quad \tilde{P}_x[\,\limsup S_n = +\infty \text{ and } \liminf S_n = -\infty] = 1 && \text{if } EZ = -E\hat{Z} = \infty; \end{aligned} \tag{7.7}$$

and in case $\sigma^2 < \infty$, either $\lim S_n = +\infty$ or $\lim S_n = -\infty$ with $\tilde{P}_x$-probability one and

$$\tilde{P}_x[\lim S_n = +\infty] = \frac{a^\dagger(x) + \sigma^{-2}x}{2a^\dagger(x)} \qquad (x \in \mathbb{Z}). \tag{7.8}$$

The two identities in (7.7) are readily deduced from (7.5) (or rather (7.4)) and its dual relation as well as (7.6) by virtue of Corollary 2.2. It is also noted that for each $M > 1$

$$\begin{aligned}
&\tilde{P}_x[\sigma_R < \infty] \to 1 \quad \text{as } R \to \infty \text{ uniformly for } x \in (-M, R) \quad \text{if } \ a(-x)/a(x) \to 0; \text{and} \\
&\tilde{P}_x[\sigma_R < T] \to \begin{cases} 1 & \text{as } x \to \infty \text{ uniformly for } R > x \qquad \text{if } \ EZ < \infty, \\ 0 & \text{as } R \to \infty \text{ for each } x \in \mathbb{Z} \qquad\quad\, \text{if } \ EZ = \infty, \end{cases}
\end{aligned}$$

which together in particular show (a). Here the first relation follows from Lemma 6.1 in view of $\tilde{P}_x[\sigma_R < \infty] = P_x[\sigma_R < \sigma_0]a(R)/a^\dagger(x)$ and the second from Proposition 2.7, which shows that if $EZ < \infty$, then uniformly for $0 \le x < R$,

$$\tilde{P}_x[\sigma_R < T] = P_x[\sigma_R < T]a(R)/a^\dagger(x) \sim Af_r(x)/[a^\dagger(x)EZ] \quad (R \to \infty).$$

The formula (7.8) is obtained by applying a theorem from the theory of Martin boundary (see [17, Theorem III29.2]: the Martin kernel $\kappa(\cdot, \pm)$ relative to a reference point $\xi \in \mathbb{Z} \setminus \{0\}$ is given by $[a(\cdot) \pm \sigma^{-2}\cdot]/[a(\xi) + \sigma^{-2}\xi]$). The conditional process $(\tilde{P}_x)_{x\neq 0}$ is a harmonic transform of the walk with absorption at the origin whose Martin boundary contains exactly two extremal harmonic functions h_+ and h_- given by $h_\pm(x) = \lim_{y\to\pm\infty} g_{\{0\}}(x,y)/\sum_{z\neq 0} p(z)g_{\{0\}}(z,y) = a(x) \pm \sigma^{-2}x$ $(x \neq 0)$, provided $\sigma^2 < \infty$. It is noticed that if $\sigma^2 = \infty$, there is only one harmonic function, hence a unique Martin boundary point: $\lim_{|y|\to\infty} g_{\{0\}}(\cdot, y)/g_{\{0\}}(\cdot, \xi) = a(\cdot)/a(\xi)$, so that two geometric boundary points $+\infty$ and $-\infty$ are not distinguished in the Martin boundary whereas the walk itself discerns them provided that either EZ or $E\hat{Z}$ is finite.

The RHS of (7.8) equals the limit as $R \to \infty$ of $P_x[S_{\sigma[R,\infty)} > 0 \,|\, \sigma_{B(R)} < \sigma_0]$ (the probability of the walk exiting the interval $(-R, R)$ from the upper boundary)—as is shown by (7.9) below, and prompted by this fact we here provide a direct proof of (7.8) that is based on (7.3).

Suppose $\sigma^2 < \infty$. Using (3.15) one infers first $P_x[\sigma_{[R,\infty)} \vee \sigma_{(-\infty,-R]} < \sigma_0] = o(1/R)$ and then as $R \to \infty$

$$P_x[\sigma_{[R,\infty)} < \sigma_0] \sim \frac{a^\dagger(x) + \sigma^{-2}x}{2\bar{a}(R)} \quad \text{and} \quad P_x[\sigma_{B(R)} < \sigma_0] \sim \frac{a^\dagger(x)}{\bar{a}(R)}. \tag{7.9}$$

By (7.6) the identity (7.8) follows if we can verify

$$\lim_{M\to\infty} \tilde{P}_x[\sigma_{[M,\infty)} < \sigma_{(-\infty,-M]}] = \lim_{R\to\infty} P_x\big[S_{\sigma[R,\infty)} > 0 \,\big|\, \sigma_{B(R)} < \sigma_0\big], \tag{7.10}$$

the LHS being equal to $\tilde{P}_x[S_n \to \infty]$. For verification of (7.10) let $\tau_M^- = \sigma_{(-\infty,-M]}$ and $\tau_M^+ = \sigma_{[M,\infty)}$. Then as $M \to \infty$

$$\lim_{R\to\infty} P_x\big[\tau_M^+ < \tau_M^-, \tau_R^- < \tau_R^+ \,\big|\, \sigma_{B(R)} < \sigma_0\big] \le \tilde{P}_x[\tau_M^+ < \tau_M^- < \infty] \to 0. \tag{7.11}$$

Similarly $\lim_{R\to\infty} P_x\big[\tau_M^- < \tau_M^+, \tau_R^+ < \tau_R^- \,\big|\, \sigma_{B(R)} < \sigma_0\big] \to 0$ as $M \to \infty$, which entails that

$$\lim_{M\to\infty}\lim_{R\to\infty} P_x\big[\tau_M^+ < \tau_M^-, \tau_R^+ < \tau_R^- \,\big|\, \sigma_{B(R)} < \sigma_0\big] = \lim_{R\to\infty} P_x[\tau_R^+ < \tau_R^- \,|\, \sigma_{B(R)} < \sigma_0].$$

This together with (7.11) shows (7.10).

8 Appendix

Put $Z' = S_{\sigma[S_0,\infty)} - S_0$, the weak ascending ladder height. The renewal functions for the strictly and weakly ascending ladder height processes are defined by $U_{\rm as}(x) = 1 + \sum_{k=1}^{\infty} P[Z_1 + \cdots + Z_k \le x]$ and $V_{\rm as}(x) = 1 + \sum_{k=1}^{\infty} P[Z'_1 + \cdots + Z'_k \le x]$ $(x = 0, 1, 2, \ldots)$. Here (Z_n) and (Z'_n) are i.i.d. copies of Z and Z', respectively. It follows [8, Section XII.1] that $P[Z' \le x] = P[Z' = 0] + P[Z' > 0]P[Z \le x]$ and

$$V_{\rm as}(x) = U_{\rm as}(x)/P[Z' > 0].$$

Let $\tau = \sigma_{[1,\infty)}$, $\tau' = \sigma_{[0,\infty)}$ and $c(t) = e^{-\sum_1^\infty k^{-1}t^k p^k(0)}$ $(t \ge 0)$. Then $S_{\tau'} \overset{\rm law}{=} Z'$ and $S_\tau \overset{\rm law}{=} Z$ under P_0 and $1 - E_0[t^{\tau'} z^{S_{\tau'}}] = c(t)(1 - E_0[t^\tau z^{S_\tau}])$ for $0 \le t < 1, 0 < |z| < 1$ ([19, Proposition 17.5], [8, Section XVIII.3]), so that on letting $z \downarrow 0$ and $t \uparrow 1$ in this order

$$P[Z' > 0] = 1/V_{\rm as}(0) = c(1) = c.$$

For $x = 1, 2, \ldots$, put $\tau(x) = \inf\{n \ge 1 : Z_1 + \cdots + Z_n \ge x\}$, the first epoch when the ladder height process enters $[x, \infty)$. Then $P[\tau(x) > n] = P[Z_1 + \cdots + Z_n \le x - 1]$ $(n \ge 1)$ and $P[\tau(x) > 0] = 1$, and hence

$$U_{\rm as}(x-1) = E\tau(x) \quad (x = 1, 2, \ldots), \tag{8.1}$$

which especially shows that $f_l(x)$, which equals $c\,U_{\rm as}(x-1)$, $x \ge 1$, is sub-additive.

References

[1] B. Belkin, An invariance principle for conditioned recurrent random walk attracted to a stable law, Zeit. Wharsch. Verw. Gebiete **21** (1972), 45–64. MR-0309200

[2] N.H. Bingham, G.M. Goldie and J.L. Teugels, Regular variation, Cambridge Univ. Press, Cambridge, 1989. MR-1015093

[3] K.L. Chung, A Course in Probability Theory, 2nd ed. Academic Press (1970). MR-0346858

[4] Y.S. Chow, On the moments of ladder height variables, Adv. Appl. Math. **7** (1986), 46–54. MR-0834219

[5] R.A. Doney, On the existence of the mean ladder height for random walk, Z. Wahrsch. Verw. Gebiete **59** (1982), 373–382. MR-0721633

[6] R.A. Doney, One-sided local large deviation and renewal theorems in the case of infinite mean, Probab. Theory Rel. Fields **107** (1997), 451–465. MR-1440141

[7] K.B. Erickson, The strong law of large numbers when the mean is undefined, TAMS. **185** (1973), 371–381. MR-0336806

[8] W. Feller, An Introduction to Probability Theory and Its Applications, vol. 2, 2nd edn. John Wiley and Sons, Inc., NY (1971). MR-0270403

[9] J.G. Kemeny, J.L. Snell and A.W. Knapp, Denumerable markov chains, D. Van Nostrand, Co, Inc., Princeton, NJ (1966). MR-0207042

[10] H. Kesten, Ratio limit theorems II, Journal d'Analyse Math. **11** (1963), 323–379. MR-0163365

[11] H. Kesten, The 1971 Rietz Lecture: Sums of independent random variables—without moment conditions, Ann. Math. Statist. **43** (1972), 701–732. MR-0301786

[12] H. Kesten and F. Spitzer, Ratio limit theorems I, Journal d'Analyse Math. **11** (1963), 285–322. MR-0162279

[13] G.F. Lawler and V. Limic, Random walk: a modern introduction, Cambridge Univ. Press (2010). MR-2677157

[14] D. Ornstein, Random walks I. Trans Amer. Math Soc. **138** (1969), 1–43. MR-0238399

[15] S.C. Port and C.J. Stone, Hitting time and hitting places for non-lattice recurrent random walks, J. Math. and Mech. **17** (1967), 35–58. MR-0215375

[16] B.A Rogozin, On the distribution of the first ladder moment and height and fluctuations of a random walk, Theory Probab. Appl. **16** (1971), 575–595. MR-0290473

[17] L.C.G. Rogers and D. Williams, Diffusions, Markov processes and martingales, vol. 1, 2nd ed. Cambridge university press (2000). MR-1780932

[18] F. Spitzer, Hitting probabilities, J. of Math. and Mech. **11** (1962), 593–614. MR-0139219

[19] F. Spitzer, Principles of Random Walks, Van Nostrand, Princeton (1964). MR-0171290

[20] C.J. Stone, On the potential operator for one-dimensional recurrent random walks, Trans. Am. Math. Soc. **136** (1969), 413–426. MR-0238398

[21] K. Uchiyama, One dimensional lattice random walks with absorption at a point/on a half line. J. Math. Soc. Japan **63** (2011), 675–713. MR-2793114

[22] K. Uchiyama, A note on summability of ladder heights and the distributions of ladder epochs for random walks, Stoch. Proc. Appl. **121** (2011), 1938–1961. MR-2819235

[23] K. Uchiyama, Asymptotic behaviour of a random walk killed on a finite set, Potential Anal. **44**, (2016), 497–541. MR-3636595

[24] K. Uchiyama, One dimensional random walks killed on a finite set, Stoch. Proc. Appl. **127** (2017), 2864–2899. MR-3682117

[25] K. Uchiyama, Asymptotically stable random walks of index $1 < \alpha < 2$ killed on a finite set. Stoch. Proc. Appl. **129** (2019), 5151–5199. MR-4025702

[26] K. Uchiyama, Estimates of Potential functions of random walks on $\mathbb{Z}$ with zero mean and infinite variance and their applications (2019). Preprint: available at: http://arxiv.org/abs/1802.09832v4.

[27] K. Uchiyama, A note on the exit problem from an interval for random walks oscillating on $\mathbb{Z}$ with infinite variance (2019). Preprint. http://arxiv.org/abs/1908.00303.