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Gradient estimates and maximal dissipativity for the Kolmogorov operator in Φ_2^4

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Abstract

We consider the transition semigroup P_t of the Φ_2^4 stochastic quantisation on the torus $\mathbb{T}^2$ and prove the following new estimate (Theorem 3.10)

$$|DP_t\varphi(x)\cdot h| \le c\,t^{-\beta}\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^{\gamma},$$

for some α, β, γ, s positive. Thanks to this estimate, we show that cylindrical functions are a core for the corresponding Kolmogorov equation. Some consequences of this fact are discussed in a final remark.

Keywords: stochastic quantization; Kolmogorov operators; gradient estimates; maximal dissipativity.
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1 Introduction

We consider the equation of the Φ_2^4 stochastic quantisation on the torus $\mathbb{T}^2$:

$$\begin{cases} dX = (AX - :X^3:)dt + dW, \\ \\ X(0) = x, \end{cases} \tag{1.1}$$

where

$$Ax = \Delta x - x, \quad D(A) = H^2(\mathbb{T}^2).$$

This equation has been the object of several studies. In particular, in [3] existence and uniqueness of a mild solution to (1.1) has been proved for almost all initial datas x with respect to the Gibbs measure ν defined below; later in [12] well-posedness was proved *for all* x in a negative Besov space. Moreover, in [10] the strong Feller property of the corresponding transition semigroup $(P_t)_{t\ge 0}$ was proved. This latter result is improved in [14] where it is proved that, for $t>0$, P_t maps bounded Borel functions to α-Hölder functions for some $\alpha\in(0,1)$.

The main result of the present paper (Theorem 3.10 below) goes further in this direction: we prove that P_t maps bounded borelian functions to Lipschitz functions and give an estimate of the gradient of $P_t\varphi$. We use a method similar to the one used in [2] and [4] as well as similar estimates as in [14]. In the second part we use this result for showing that cylindrical functions are a core for the infinitesimal generator $\mathcal{K}$ of P_t and present some consequences of this fact.

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2 Notations

We use the classical $L^p = L^p(\mathbb{T}^2)$ spaces for $p \in [1, \infty]$. For $p = 2$, we write $H = L^2$. We use also the L^2 based Sobolev spaces $H^s = H^s(\mathbb{T}^2)$ for $s \in \mathbb{R}$ and the Besov spaces $B^s_{p,q} = B^s_{p,q}(\mathbb{T}^2)$, for $s \in \mathbb{R}$, $p, q \in [1, \infty]$. These are defined in terms of Fourier series and Littlewood–Paley theory (see [1]). Note that $H^s = B^s_{2,2}$. When $p = q = \infty$, we write $B^s_{\infty,\infty} = C^s$ and for $s > 0$ non integer these spaces coincide with the classical Hölder spaces.

Besov spaces are convenient when multiplying two functions when one has negative regularity. Recall that for $p_1,\, p_2 \in [1, \infty]$ and $\alpha_1,\, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 > 0$ we have for $\alpha = \alpha_1 \wedge \alpha_2 \wedge (\alpha_1 + \alpha_2)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\tilde p = p_1 \vee p_2$

$$\|uv\|_{B^{\alpha}_{p,\tilde p}} \le C\|u\|_{B^{\alpha_1}_{p_1,p_1}} \|v\|_{B^{\alpha_2}_{p_2,p_2}}$$

and for any $\kappa > 0$

$$\|uv\|_{B^{\alpha-\kappa}_{p,p}} \le C\|u\|_{B^{\alpha_1}_{p_1,p_1}} \|v\|_{B^{\alpha_2}_{p_2,p_2}}\,.$$

We use the eigenprojectors of A corresponding to the eigenvalues $|k|^2+1 \le N$, $k \in \mathbb{Z}^2$, and denote it by Π_N.

Define the Gaussian measure

$$\mu = \prod_{k\in\mathbb{Z}^2} N\left(0, \frac{1}{1+|k|^2}\right), \text{ on } \mathcal{H} = \mathbb{R}^{\mathbb{Z}^2}.$$

Note that all Besov spaces are embedded in $\mathcal{H}$ thanks to the Fourier series, the injection being simply $u \mapsto (u_k)_{k\in\mathbb{Z}^2}$ where $(u_k)_{k\in\mathbb{Z}^2}$ are the Fourier coefficients of u.

For $x_N \in \Pi_N H$, we set

$$: x_N^2 : = x_N^2 - \rho_N^2, \quad : x_N^3 := x_N^3 - 3\rho_N^2 x_N,$$

where

$$\rho_N = \frac{1}{2\pi}\,\left[\sum_{|k|\le N} \frac{1}{1+|k|^2}\right]^{1/2}.$$

More generally, the n^{th} Wick power of x_N is defined by $: x_N^n := \sqrt{n!}\rho_N^n H_n(\frac{1}{\rho_N} x_N)$ where H_n is the n^{th} Hermite polynomial.

As well known, there exists the limit

$$\lim_{N\to\infty} : (\Pi_N x)^n : =: x^n :, \text{ in } L^r(\mathcal{H}, \mu; B^s_{p,q}), \tag{2.1}$$

for any $r \in [1, \infty),\ p, q \in [1, \infty],\ s < 0$.

Then, we define for $t \in \mathbb{R}$,

$$Z_\infty(t) = \int_{-\infty}^t e^{(t-s)A} dW(s).$$

Since $Z_\infty(t)$ has law μ, we know that $: \Pi_N Z_\infty^n(t) :$ converges to $: Z_\infty^n(t) :$ in $L^r(\mathcal{H}, \mu; B^s_{p,q})$ for any $r \in [1, \infty),\ p, q \in [1, \infty],\ s < 0$, $t \in \mathbb{R}$.

We set moreover $X = Y_\infty + Z_\infty = Y + Z$ where

$$Z(t) = \int_0^t e^{(t-s)A} dW(s) = Z_\infty(t) - e^{At} Z_\infty(0).$$

Clearly:

$$:(\Pi_N X)^3: = (\Pi_N Y_\infty)^3 + 3(\Pi_N Y_\infty)^2 \Pi_N Z_\infty + 3(\Pi_N Y_\infty) : (\Pi_N Z_\infty)^2 : + : (\Pi_N Z_\infty)^3 : .$$

It is therefore natural to consider the following definition for $:X^3:$ in (1.1):

$$:X^3: = Y_\infty^3 + 3Y_\infty^2 Z_\infty + 3Y_\infty : Z_\infty^2 : + : Z_\infty^3 : .$$

We also have

$$:X^3: = Y^3 + 3Y^2 Z + 3Y : Z^2 : + : Z^3 :$$

if we set

$$:Z^3: = -(e^{At} Z_\infty(0))^3 + 3(e^{At} Z_\infty(0))^2 Z_\infty - 3(e^{At} Z_\infty(0)) : Z_\infty^2 : + : Z_\infty^3 : .$$

We will see that Y and Y_∞ have positive regularity whereas Z has the same smoothness as Z_∞. Using the product rules above, it can be seen that all products above make sense.

Finally, we define $Z_N = \Pi_N Z$ and introduce the following approximation of (1.1):

$$\begin{cases} \dfrac{dY_N}{dt} = AY_N - (Y_N^3 + 3Y_N^2 Z_N + 3Y_N : Z_N^2 : + : Z_N^3 :), \\ \\ Y_N(0) = x. \end{cases} \tag{2.2}$$

Equivalently, setting $X_N = Y_N + Z_N$, we may write with a slight abuse of notation

$$:X_N^3: = Y_N^3 + 3Y_N^2 Z_N + 3Y_N : Z_N^2 : + : Z_N^3 :$$

and

$$\begin{cases} dX_N = (AX_N + : X_N^3 :)dt + \Pi_N dW, \\ \\ X_N(0) = x. \end{cases} \tag{2.3}$$

Note that this is not a Galerkin approximation. However, since $:X_N^3: = X_N^3 - 3\rho_N^2 X_N$ and $\Pi_N dW$ is a finite dimensional white noise, it is classical to prove that X_N exists and is unique. Moreover, it is smooth in space when the initial datum is smooth; in this case Y_N is also spatially smooth and has moreover regularity $3/2^-$ in time. All the computations done in the sections below are rigorous when the initial datum is smooth, for initial data in a negative Besov space, they are obtained thanks to a preliminary smoothing of the initial datum.

It is not difficult to prove that (X_N) converges to the unique solution of (1.1) (see [14]) in convenient topologies. For instance, if for some $\alpha > 0$ not too large, $x \in C^{-\alpha}$, the convergence holds in $C([0,T]; C^{-\alpha})$.

Let us define the Gibbs measure ν on $\mathcal{H}$

$$\nu(dx) = \mathcal{Z}^{-1} e^{-2U(x)} \mu(dx), \tag{2.4}$$

where

$$U(x) = \frac{1}{4} \int_H : x^4 : (\xi) d\xi \tag{2.5}$$

and

$$\mathcal{Z} := \int_H e^{-2U(x)} \mu(dx).$$

Then

$$e^{-2U(x)} \in L^r(\mathcal{H}, \mu),$$

for any $r \in [1, \infty)$ thanks to the Nelson inequality (see [13], Chapter V2).

We can write equation (1.1) as

$$\begin{cases} \dfrac{dY}{dt} = AY - Y^3 - 3Y^2Z - 3Y :Z^2: - :Z^3:, \\ \\ Y(0) = x, \end{cases} \tag{2.6}$$

or, in mild form,

$$Y(t) = e^{At}x - \int_0^t e^{A(t-s)}[Y^3(s) + 3Y^2(s)Z(s) + 3Y(s) :Z^2(s): + :Z^3(s):]ds. \tag{2.7}$$

We define the transition semigroups P_t and P_t^N associated to (1.1) and (2.3) respectively. If we try to obtain estimates on the differential of these semigroups with respect to the initial data, we are lead to use Gronwall's lemma and obtain exponentials of the solution which we do not know how to estimate. To remedy this, introduce the following modified semigroup:

$$S_t^N\varphi(x) = \mathbb{E}\left(e^{-K\int_0^t V(X_N(s))ds}\varphi(X_N(t,x))\right) \tag{2.8}$$

where the potential V is given by

$$V(X_N) = \left|:X_N^2:\right|_{H^{-\alpha}}^p = \left|X_N^2 - \rho_N^2\right|_{H^{-\alpha}}^p$$

for some $K > 0$, $p > 0$, $\alpha > 0$ to be chosen. This quantity is well defined for $\varphi \in \mathcal{B}_b(C^{-\alpha})$, the space of borelian bounded functions on $C^{-\alpha}$.

The following relation formally follows from Duhamel formula since $P_N\varphi$ satisfies the Kolmogorov equation associated to (2.3) and $S_N\varphi$ satisfies the same equation with an extra term due to the potential V:

$$P_t^N\varphi(x) = S_t^N\varphi(x) + K\int_0^t S_{t-s}^N(VP_s\varphi)(x)ds. \tag{2.9}$$

It can be proved rigorously by an approximation argument.

3 Estimates

Let us consider the equation:

$$\begin{cases} \dfrac{d\eta^{h,N}}{dt} = (A\eta^{h,N} - 3 :X_N^2: \eta^{h,N}), \\ \\ \eta^{h,N}(0) = h. \end{cases} \tag{3.1}$$

It is classical that $\eta^{h,N} = D_xX_N \cdot h$ is the differential of X_N with respect to the initial data $x \in C^{-\alpha}$ in the direction $h \in C^{-\alpha}$.

In this section, we derive several estimates on η_N and X_N. These hold only on finite time intervals and we restrict to $t \in [0, 1]$.

Lemma 3.1. Let $\alpha < \frac{1}{3}$, $\epsilon > 0$ such that $\alpha+\epsilon < \frac{1}{3}$ and $p \geq 1$, $s > 0$ such that $\alpha+\epsilon+\frac{2}{3p}+\frac{s}{3} < \frac{1}{3}$ then there exists c depending on α, ϵ, p such that for all $t \in (0, 1]$, $x \in C^{-\alpha}$, $h \in C^{-s}$

$$|\eta^{h,N}(t)|_{C^{\alpha+\epsilon}} \leq ct^{-(\alpha+\epsilon+s)/2}|h|_{C^{-s}}\, e^{c\int_0^t \left|:X_N^2(s):\right|_{H^{-\alpha}}^p ds}$$

and

$$|\eta^{h,N}(t)|_{L^\infty} \leq ct^{-s/2}|h|_{C^{-s}}\, e^{c\int_0^t \left|:X_N^2(s):\right|_{H^{-\alpha}}^p ds}.$$

Remark 3.2. The potential V chosen to define the auxiliary semigroup S_t^N has been chosen to compensate the exponential factors appearing in the estimate of Lemma 3.1. There clearly are other possibilities for this potential. In the proof below, we could do the estimates differently and use another Besov norm for $:X_N^2:$. This would change the singularity in time in the integrals and decrease the power p but not yield simpler computations. We chose to use the simple $H^{-\alpha}$ norm because below we need to differentiate V and the differential is easier to write with a norm of a Hilbert space.

Proof. Write (3.1) in mild form

$$\eta^{h,N}(t) = e^{At}h - 3\int_0^t e^{A(t-s)}\left(:X_N^2(s): \eta^{h,N}(s)\right)\, ds.$$

Below we omit the dependence on N. By the smoothing of the heat semigroup e^{At}, the product rule in Besov spaces and $t\in[0,1]$, we have

$$|\eta^h(t)|_{C^{\alpha+\epsilon}} \le ct^{-(\alpha+\epsilon+s)/2}|h|_{C^{-s}} + c\int_0^t |t-r|^{-(\alpha+\epsilon+1/2)}\left|:X^2(r): \eta^h(r)\right|_{H^{-\alpha-\epsilon/2}}\, dr$$

$$\le ct^{-(\alpha+\epsilon+s)/2}|h|_{C^{-s}} + c\int_0^t |t-r|^{-(\alpha+\epsilon+1/2)}\left|:X^2(r):\right|_{H^{-\alpha}}\left|\eta^h(r)\right|_{C^{\alpha+\epsilon}}\, dr.$$

Let $\lambda(t) = t^{(\alpha+\epsilon+s)/2}\left|\eta^h(t)\right|_{C^{\alpha+\epsilon}}$, then since $t\in[0,1]$,

$$\lambda(t) \le c|h|_{C^{-s}} + c\int_0^t |t-r|^{-(\alpha+\epsilon+1/2)}\, r^{-(\alpha+\epsilon+s)/2}\left|:X^2(r):\right|_{H^{-\alpha}}\lambda(r)dr.$$

Let $\frac{1}{p}+\frac{1}{q}=1$ and

$$\frac{(3\alpha+3\epsilon+1+s)q}{2} < 1, \quad i.e.\ \frac{3\alpha+3\epsilon+1+s}{2} < 1-\frac{1}{p}.$$

Then

$$\begin{aligned}
\lambda^p(t) &\le c|h|^p_{C^{-s}} + c\left(\int_0^t |t-r|^{-q(\alpha+\epsilon+1/2)}\, r^{-q(\alpha+\epsilon+s)/2}\, dr\right)^{p/q}\int_0^t \left|:X^2(r):\right|^p_{H^{-\alpha}}\lambda^p(r)dr\\
&\le c|h|^p_{C^{-s}} + c\int_0^t \left|:X^2(r):\right|^p_{H^{-\alpha}}\lambda^p(r)dr.
\end{aligned}$$

The first inequality follows by Gronwall's lemma.

For the second inequality, we write:

$$|\eta^h(t)|_{L^\infty} \le ct^{-s/2}|h|_{C^{-s}} + c\int_0^t |t-r|^{-(\alpha+\epsilon+1/2)}\left|:X^2(r): \eta^h(r)\right|_{H^{-\alpha-\epsilon/2}}\, dr$$

$$\le ct^{-s/2}|h|_{C^{-s}} + c\int_0^t |t-r|^{-(\alpha+\epsilon+1/2)}\left|:X^2(r):\right|_{H^{-\alpha}}\left|\eta^h(r)\right|_{C^{\alpha+\epsilon}}\, dr$$

and with p, q as above:

$$\begin{aligned}
|\eta^h(t)|^p_{L^\infty} &\le ct^{-sp/2}|h|^p_{C^{-s}} + c\int_0^t r^{p(\alpha+\epsilon+s)/2}\left|:X^2(r):\right|^p_{H^{-\alpha}}\left|\eta^h(r)\right|^p_{C^{\alpha+\epsilon}}\, dr\\
&\le ct^{-sp/2}|h|^p_{C^{-s}} + c\int_0^t \left|:X^2(r):\right|^p_{H^{-\alpha}}\, e^{cp\int_0^t r\left|:X_N^2(s):\right|^p_{H^{-\alpha}}\, ds}|h|^p_{C^{-s}}dr\\
&\le c|h|^p_{C^{-s}}\left(t^{-sp/2} + e^{cp\int_0^t\left|:X_N^2(s):\right|^p_{H^{-\alpha}}\, ds}\right).
\end{aligned}$$

The conclusion follows. ☐

We now estimate moments of Y_N solution of (2.2). We estimate Y_N in $L^r(\mathbb{T}^2)$, $r\geq 2$, using similar arguments as in [14] and [12], but in a simplified way since we do not need such refined estimates for large times.

Lemma 3.3. Let $\alpha\in[0,1)$ and $r\geq 2$, there exist $c>0$ depending on α and r such that

$$\frac{1}{r}\,\frac{d}{dt}\,|Y_N|^r_{L^r}+\frac{r-1}{r^2}\,\left|\nabla((Y_N)^{r/2})\right|^2_{L^2}+\frac{1}{2}\,|Y_N|^{r+2}_{L^{r+2}}\leq c\,\mathcal{L}_\alpha(Z_N)^{k_r},$$

where $k_r=\dfrac{r+2}{1-\alpha}$ and

$$\mathcal{L}_\alpha(Z_N)=|Z_N|_{C^{-\alpha}}+|:Z_N^2:|_{C^{-\alpha}}+|:Z_N^3:|_{C^{-\alpha}}+1.$$

Proof. Again we omit the dependence on N. Multiplying equation (2.2) by Y^{r-1} and integrating on $\mathbb{T}^2$ we obtain:

$$\frac{1}{r}\,\frac{d}{dt}\,|Y|^r_{L^r}+\frac{4(r-1)}{r^2}\,\left|\nabla(Y^{r/2})\right|^2_{L^2}+|Y|^{r+2}_{L^{r+2}}=-\int_{\mathbb{T}^2}(3Y^{r+1}\,Z+3Y^r\ :Z^2:\ +Y^{r-1}\ :Z^3:)d\xi.$$

We estimate each of the three terms in the right hand side. By Proposition A.8 and Proposition A.9 in [14], we have

$$3\left|\int_{\mathbb{T}^2}Y^{r+1}\,Z\,d\xi\right|\leq 3|Y^{r+1}|_{B^\alpha_{1,1}}\,|Z|_{C^{-\alpha}}\leq c\left(|Y^{r+1}|^{1-\alpha}_{L^1}\,|\nabla Y^{r+1}|^\alpha_{L^1}+|Y^{r+1}|_{L^1}\right)|Z|_{C^{-\alpha}}.$$

Clearly

$$|Y^{r+1}|_{L^1}=|Y|^{r+1}_{L^{r+1}}\leq c|Y|^{r+1}_{L^{r+2}}.$$

Moreover

$$|\nabla Y^{r+1}|_{L^1}=(r{+}1)|Y^r\nabla Y|_{L^1}\ \leq(r{+}1)|Y^{\frac{r+2}{2}}|_{L^2}\,|Y^{\frac{r-2}{2}}\nabla Y|_{L^2}=\frac{2(r+1)}{r}|Y|^{\frac{r+2}{2}}_{L^{r+2}}\,|\nabla Y^{r/2}|_{L^2}.$$

It follows that

$$\begin{aligned}3\left|\int_{\mathbb{T}^2}Y^{r+1}\,Z\,d\xi\right|&\leq c\left(|Y|^{(1-\alpha)(r+1)+\alpha(r+2)/2}_{L^{r+2}}\,|\nabla Y^{r/2}|^\alpha_{L^2}+|Y|^{r+1}_{L^{r+2}}\right)\,|Z|_{C^{-\alpha}}\\&=c\left(|Y|^{((1-\alpha/2)r+1}_{L^{r+2}}\,|\nabla Y^{r/2}|^\alpha_{L^2}+|Y|^{r+1}_{L^{r+2}}\right)\,|Z|_{C^{-\alpha}}\\&\leq\frac{1}{6}\,|Y|^{r+2}_{L^{r+2}}+\frac{r-1}{r^2}\,|\nabla Y^{r/2}|^2_{L^2}+c\left(|Z|^{p_1}_{C^{-\alpha}}+|Z|^{p_2}_{C^{-\alpha}}\right)\end{aligned}$$

with

$$\frac{(1-\alpha/2)r+1}{r+2}+\frac{\alpha}{2}+\frac{1}{p_1}=1$$

and $p_2=r+2$. Note that since $\alpha\in(0,1)$, p_1 exists and is equal to $\dfrac{r+2}{1-\alpha}$. We proceed similarly for the second one and obtain:

$$3\left|\int_{\mathbb{T}^2}Y^r\ :Z^2:\ d\xi\right|\leq\frac{1}{6}\,|Y|^{r+2}_{L^{r+2}}+\frac{r-1}{r^2}\,|\nabla Y^{r/2}|^2_{L^2}+c\left(|:Z^2:|^{p_3}_{C^{-\alpha}}+|:Z^2:|^{p_4}_{C^{-\alpha}}\right)$$

with

$$\frac{\alpha}{2}+\frac{r(1-\alpha/2)}{r+2}+\frac{1}{p_3}=1$$

and $p_4=\frac{r+2}{2}$ and for the third one

$$\left|\int_{\mathbb{T}^2}Y^{r-1}\ :Z^3:\ d\xi\right|\leq\frac{1}{6}\,|Y|^{r+2}_{L^{r+2}}+\frac{r-1}{r^2}\,|\nabla Y^{r/2}|^2_{L^2}+c\left(|:Z^3:|^{p_5}_{C^{-\alpha}}+|:Z^3:|^{p_6}_{C^{-\alpha}}\right)$$

with

$$\frac{\alpha}{2} + \frac{r(1-\alpha/2)-1}{r+2} + \frac{1}{p_5} = 1$$

and $p_6 = \frac{r+2}{3}$.

Using these three inequalities in the energy equality and noting that $p_1 = \frac{r+2}{1-\alpha} \geq p_i$, $i = 2, ..., 5$ we obtain

$$\frac{1}{2}\frac{d}{dt}|Y|_{L^r}^r + \frac{r-1}{r^2}\left|\nabla(Y^{r/2})\right|_{L^2}^2 + \frac{1}{2}|Y|_{L^{r+2}}^{r+2} \leq c\left(|Z|_{C^{-\alpha}} + |:Z^2:|_{C^{-\alpha}} + |:Z^3:|_{C^{-\alpha}} + 1\right)^{p_1}.$$

This gives the result with $k_r = p_1$. □

Lemma 3.4. Let $\alpha < \frac{1}{3}$, $k \geq 1$ there exists $C_{\alpha,k} > 0$ and $\kappa_{\alpha,k} > 0$ such that

$$\mathbb{E}\left(\sup_{t\in[0,1]}|Y_N(t)|_{C^{-\alpha}}^k\right) \leq C_{\alpha,k}\left(|x|_{C^{-\alpha}} + 1\right)^{\kappa_{\alpha,k}}.$$

Proof. Since

$$|Y_N|_{L^r}^{r+2} \leq c|Y_N|_{L^{r+2}}^{r+2}$$

it follows from Lemma 3.3 that

$$\frac{d}{dt}|Y_N(t)|_{L^r}^r + c|Y_N(t)|_{L^r}^{r+2} \leq c\,\mathcal{L}_\alpha(Z_N(t))^{k_r}$$

and by [14, Lemma 3.8] it follows for $t \in [0,1]$

$$|Y_N(t)|_{L^r} \leq c\max\left(t^{-r/2}, \sup_{t\in[0,1]}\mathcal{L}_\alpha(Z_N(t))^{\frac{r}{1-\alpha}}\right).$$

We choose $r > \frac{2}{\alpha}$ and use the embedding $L^r \subset C^{-\alpha}$ to deduce

$$|Y_N(t)|_{C^{-\alpha}} \leq c\max\left(t^{-r/2}, \sup_{t\in[0,1]}\mathcal{L}_\alpha(Z_N(t))^{\frac{r}{1-\alpha}}\right). \tag{3.2}$$

It remain to obtain a bound for $|Y_N(t)|_{C^{-\alpha}}$ for small time. Write the mild form for the equation satisfied by Y_N:

$$Y_N(t) = e^{At}x - \int_0^t e^{A(t-s)}(Y_N^3(s) + 3Y_N^2(s)Z_N(s) + 3Y_N(s)\,:Z_N^2(s): + \,:Z_N^3(s):)ds$$

Let $\alpha < \frac{1}{3}$, $\beta > \alpha$ such that

$$\gamma := \frac{\alpha+\beta}{2} < \frac{1}{3}$$

and define

$$M(t) = \sup_{s\in[0,t]}(|Y_N(s)|_{C^{-\alpha}} + s^\gamma|Y_N(s)|_{C^{-\beta}}).$$

Then similar computations as in the proof of Theorem 3.3 in [14] give for $t \in [0,1]$:

$$M(t) \leq c_1|x|_{C^{-\alpha}} + c_2 t^{1-3\gamma}M(t)^3 + c_3 t^{1-3\gamma}\sup_{t\in[0,1]}\mathcal{L}_\alpha(Z_N(t))^3,$$

for some constants c_1, c_2 which can be chosen larger than 1.

Therefore if we set

$$t^* = \min\left\{\left(c_3\sup_{t\in[0,1]}\mathcal{L}_\alpha(Z_N(t))^3\right)^{-\frac{1}{1-3\gamma}}, \left(8c_2(c_1|x|_{C^{-\alpha}}+1)^2\right)^{-\frac{1}{1-3\gamma}}, 1\right\}$$

we have

$$\sup_{t\in[0,t^*]} M(t) \le 2(c_1|x|_{C^{-\alpha}}+1). \tag{3.3}$$

It follows from (3.2) and (3.3):

$$\begin{aligned}\sup_{t\in[0,1]}|Y_N(t)|_{C^{-\alpha}} &\le \sup_{t\in[0,t^*]}|Y_N(t)|_{C^{-\alpha}}+\sup_{t\in[t^*,1]}|Y_N(t)|_{C^{-\alpha}}\\ &\le 2(c_1|x|_{C^{-\alpha}}+1)+\max\left((t^*)^{-2/r},\sup_{t\in[0,1]}\mathcal{L}_\alpha(Z_N(t))^{\frac{rk_r}{r+2}}\right)\end{aligned}$$

The result follows taking the expectation and using finiteness of the moments of

$$\sup_{t\in[0,1]}\mathcal{L}_\alpha(Z_N(t)).$$

□

We set

$$\widetilde{Z}_N(t)=e^{At}x+\Pi_N\int_0^t e^{A(t-s)}\,dW(s)=e^{At}x+Z_N(t),$$

and

$$\widetilde{Y}_N(t)=X_N(t)-\widetilde{Z}_N(t).$$

Then $\widetilde{Y}_N$ satisfies the same equation as Y_N, see (2.2), with Z_N replaced by $\widetilde{Z}_N$ and 0 initial datum.

Lemma 3.5. Let $\alpha\in(0,1/9)$, $k\ge 1$ then

$$\mathbb{E}\left(\sup_{t\in[0,1]}|\widetilde{Y}_N|^2_{L^2}+\int_0^t|\nabla\widetilde{Y}_N(s)|^2_{L^2}\,ds\right)^k\le C_{\alpha,k}\left(|x|^{12k/(1-\alpha)}_{C^{-\alpha}}+1\right)$$

for any $t>0$.

Proof. Lemma 3.3 clearly also holds for $\widetilde{Y}_N$ provided Z_N is replaced by $\widetilde{Z}_N$. We use it with $r=2$ and again omit to indicate the dependency on N:

$$\frac12\frac{d}{dt}|\widetilde{Y}|^2_{L^2}+\frac14\left|\nabla\widetilde{Y}\right|^2_{L^2}+\frac12\left|\widetilde{Y}\right|^4_{L^4}\le c\,\mathcal{L}_\alpha(\widetilde{Z})^{k_2},$$

c and $k_2=\frac{4}{1-\alpha}$ are given in Lemma 3.3. We use the product rules in Besov spaces and the smoothing effect of e^{At} to prove:

$$\mathcal{L}_\alpha(\widetilde{Z}(s))\le C_{\alpha,\epsilon}\left((s^{-(2\alpha+\epsilon)}+1)\,|x|^3_{C^{-\alpha}}+1\right)\mathcal{L}_\alpha(Z(s))$$

the result follows by integrating on $[0,T]$ and taking expectation of the k^{th} power. Note that since $\alpha<1/9$, it is possible to choose $\epsilon>0$ small enough $(2\alpha+\epsilon)k_2<1$. □

Lemma 3.6. Let $\alpha\in(0,1/9)$, $k\ge1$, and $\epsilon\in(0,\frac{1-4\alpha}{3})$, there exists $\gamma_{\alpha,\epsilon,k},C_{\alpha,\epsilon,k}$ such that

$$\mathbb{E}\left(\sup_{t\in[0,1]}|\widetilde{Y}_N(t)|^k_{H^{\alpha+\epsilon}}\right)\le C_{\alpha,\epsilon,k}\left(|x|_{C^{-\alpha}}+1\right)^{\gamma_{\alpha,\epsilon,k}}.$$

Proof. We write, omitting again N

$$\widetilde{Y}(t)=\int_0^t e^{A(t-s)}\,(\widetilde{Y}^3(s)+3\widetilde{Y}^2(s)\widetilde{Z}(s)+3\widetilde{Y}(s)\ :\widetilde{Z}^2(s):+\ :\widetilde{Z}^3(s):)\,ds$$

and get

$$|\widetilde{Y}(t)|_{H^{\alpha+\epsilon}} \leq c\int_0^t (t-s)^{-(\alpha+\epsilon/2)} \left(|\widetilde{Y}^3|_{H^{-\alpha}} + |\widetilde{Y}^2\widetilde{Z}|_{H^{-\alpha}} + |\widetilde{Y} :\widetilde{Z}^2 :|_{H^{-\alpha}} + |:\widetilde{Z}^3:|_{H^{-\alpha}}\right) ds.$$

We have

$$\begin{aligned}|\widetilde{Y}^3|_{H^{-\alpha}} &\leq c|\widetilde{Y}^2|_{H^{\alpha+\epsilon}}|\widetilde{Y}|_{C^{-\alpha}} \leq c|\widetilde{Y}|^2_{B^{\alpha+\epsilon}_{4,4}}|\widetilde{Y}|_{C^{-\alpha}} \leq c|\widetilde{Y}|^2_{H^{\alpha+\epsilon+1/2}}|\widetilde{Y}|_{C^{-\alpha}} \\ &\leq c\left(|\widetilde{Y}|^{2(1/2-\alpha-\epsilon)}_{L^2}|\nabla\widetilde{Y}|^{2(1/2+\alpha+\epsilon)}_{L^2} + |\widetilde{Y}|^2_{L^2}\right)|\widetilde{Y}|_{C^{-\alpha}}.\end{aligned}$$

Similarly

$$|\widetilde{Y}^2\widetilde{Z}|_{H^{-\alpha}} \leq c|\widetilde{Y}^2|_{H^{\alpha+\epsilon}}|\widetilde{Z}|_{C^{-\alpha}} \leq c\left(|\widetilde{Y}|^{2(1/2-\alpha-\epsilon)}_{L^2}|\nabla\widetilde{Y}|^{2(1/2+\alpha+\epsilon)}_{L^2} + |\widetilde{Y}|^2_{L^2}\right)|\widetilde{Z}|_{C^{-\alpha}}$$

and

$$|\widetilde{Y} :\widetilde{Z}^2:|_{H^{-\alpha}} \leq c|\widetilde{Y}|_{H^{\alpha+\epsilon}}|:\widetilde{Z}^2:|_{C^{-\alpha}} \leq c\left(|\widetilde{Y}|^{1-(\alpha+\epsilon)}_{L^2}|\nabla\widetilde{Y}|^{\alpha+\epsilon}_{L^2} + |\widetilde{Y}|_{L^2}\right)|:\widetilde{Z}^2:|_{C^{-\alpha}}.$$

Gathering these results gives

$$\begin{aligned}&|\widetilde{Y}(t)|_{H^{\alpha+\epsilon}} \\ &\leq c\int_0^t (t-s)^{-(\alpha+\epsilon/2)}\Big[\left(|\widetilde{Y}|^{2(1/2-\alpha-\epsilon)}_{L^2}|\nabla\widetilde{Y}|^{2(1/2+\alpha+\epsilon)}_{L^2} + |\widetilde{Y}|^2_{L^2}\right)(|\widetilde{Y}|_{C^{-\alpha}} + |\widetilde{Z}|_{C^{-\alpha}}) \\ &\quad + \left(|\widetilde{Y}|^{1-(\alpha+\epsilon)}_{L^2}|\nabla\widetilde{Y}|^{\alpha+\epsilon}_{L^2} + |\widetilde{Y}|_{L^2}\right)\ |:\widetilde{Z}^2:|_{C^{-\alpha}} + |:\widetilde{Z}^3:|_{C^{-\alpha}}\Big]ds =: (a)+(b)+(c).\end{aligned}$$

The first term is bounded thanks to Hölder's inequality

$$\begin{aligned}(a) &= c\int_0^t (t-s)^{-(\alpha+\epsilon/2)}\Big[\left(|\widetilde{Y}|^{2(1/2-\alpha-\epsilon)}_{L^2}|\nabla\widetilde{Y}|^{2(1/2+\alpha+\epsilon)}_{L^2} + |\widetilde{Y}|^2_{L^2}\right)(|\widetilde{Y}|_{C^{-\alpha}} + |\widetilde{Z}|_{C^{-\alpha}})ds \\ &\leq c\left(|\widetilde{Y}|^{2(1/2-\alpha-\epsilon)}_{L^{p_1}(0,t,L^2)}\,(|\nabla\widetilde{Y}|^{2(1/2+\alpha+\epsilon)}_{L^2(0,t,L^2)} + |\widetilde{Y}|_{L^{p_2}(0,t,L^2)}\right) \\ &\quad\times\left(|Y|_{L^{p_3}(0,t,C^{-\alpha})} + |x|_{C^{-\alpha}} + |Z|_{L^{p_3}(0,t,C^{-\alpha})}\right)\end{aligned}$$

with

$$\frac{2(1/2-\alpha-\epsilon)}{p_1} + (1/2+\alpha+\epsilon) + \alpha + \frac{\epsilon}{2} < 1 + \frac{1}{p_3}$$

and

$$\frac{2}{p_2} + \alpha + \frac{\epsilon}{2} + \frac{1}{p_3} < 1.$$

Since $\epsilon < \frac{1-4\alpha}{3}$ and, p_1, p_2, p_3 can be chosen such this is satisfied. Then

$$\begin{aligned}(b) &\leq c\int_0^t (t-s)^{-(\alpha+\epsilon/2)}\left(|\widetilde{Y}|^{1-(\alpha+\epsilon)}_{L^2}|\nabla\widetilde{Y}|^{\alpha+\epsilon}_{L^2} + |\widetilde{Y}|_{L^2}\right) \\ &\times\ \left[s^{-(\alpha+\epsilon/2)}\left(|x|^2_{C^{-\alpha}} + |x|_{C^{-\alpha}}|Z|_{C^{-\alpha}}\right) + |:Z^2:|_{C^{-\alpha}}\right]ds \\ &\leq c\left(|\widetilde{Y}|^{1-(\alpha+\epsilon)}_{L^{p_4}(0,t,L^2)}|\nabla\widetilde{Y}|^{\alpha+\epsilon}_{L^2(0,t,L^2)} + |\widetilde{Y}|_{L^{p_5}(0,t,L^2)}\right) \\ &\times\ \left(|x|^2_{C^{-\alpha}} + |x|_{C^{-\alpha}}|Z|_{L^{p_6}(0,t,C^{-\alpha})} + |:Z^2:|_{L^{p_6}(0,t,C^{-\alpha})}\right)\end{aligned}$$

with

$$\alpha+\frac{\epsilon}{2}+\frac{1-(\alpha+\epsilon)}{p_4}+\frac{\alpha+\epsilon}{2}+\alpha+\frac{\epsilon}{2}+\frac{1}{p_6}<1$$

and

$$2(\alpha+\frac{\epsilon}{2})+\frac{1}{p_5}+\frac{1}{p_6}<1$$

which again is possible since $\epsilon<\frac{1-4\alpha}{3}$. Similarly

$$(c)\le c\int_0^t (t-s)^{-(\alpha+\epsilon/2)}\left(s^{-(2\alpha+\epsilon)}(|x|^3_{C^{-\alpha}}+|x|^2_{C^{-\alpha}}|Z|_{C^{-\alpha}})\right.$$

$$\left.+s^{-(\alpha+\epsilon/2)}|x|_{C^{-\alpha}}|:Z^2:|_{C^{-\alpha}}+|:Z^3:|_{C^{-\alpha}}\right)ds$$

Taking expectation we find thanks to Lemma 3.4, Lemma 3.5, Hölder's inequality and finiteness of the moments of $\sup_{t\in[0,1]}|Z|_{C^{-\alpha}}$, $\sup_{t\in[0,1]}|:Z^2:|_{C^{-\alpha}}$ and $\sup_{t\in[0,1]}|:Z^3:|_{C^{-\alpha}}$:

$$\mathbb{E}\left(\sup_{t\in[0,1]}|\widetilde{Y}(t)|^k_{H^{\alpha+\epsilon}}\right)\le c\left(|x|^{\kappa_{\alpha,\epsilon,k}}_{C^{-\alpha}}+1\right)$$

for some $\kappa_{\alpha,\epsilon,k}$ which could be written explicitly. □

Lemma 3.7. Let $\alpha\in(0,1/9)$, for any $k\ge1$, there exists $\overline{\gamma}_{\alpha,k},C_{\alpha,k}$ such that for $t\in(0,1]$

$$\mathbb{E}\left(|:X_N(t)^2:|^k_{H^{-\alpha}}\,ds\right)\le C_{\alpha,k}t^{-\alpha k}\left(|x|_{C^{-\alpha}}+1\right)^{\overline{\gamma}_{\alpha,k}}.$$

Proof. We have, omitting the dependence in N, for $\epsilon>0$

$$\begin{aligned}|:X(t)^2:|_{H^{-\alpha}}&\le c\left(|Y^2(t)|_{H^{-\alpha}}+|Y(t)|_{H^{\alpha+\epsilon}}|Z(t)|_{C^{-\alpha}}+|:Z(t)^2:|_{C^{-\alpha}}\right)\\&\le c\left(|Y(t)|_{H^{\alpha+\epsilon}}\left(|Y(t)|_{C^{-\alpha}}+|Z(t)|_{C^{-\alpha}}\right)+|:Z(t)^2:|_{C^{-\alpha}}\right)\\&\le c\left(\left(t^{-\alpha}|x|_{C^{-\alpha}}+|\widetilde{Y}(t)|_{H^{\alpha+\epsilon}}\right)\left(|Y(t)|_{C^{-\alpha}}+|Z(t)|_{C^{-\alpha}}\right)+|:Z(t)^2:|_{C^{-\alpha}}\right)\end{aligned}$$

The result follows by Lemma 3.4 and Lemma 3.6 choosing ϵ sufficiently small. □

3.1 Estimates on S^N_t

We have

$$S^N_t\varphi(x)=\mathbb{E}\left(e^{-K\int_0^t V(X_N(s))ds}\varphi(X_N(t,x))\right)$$

where the potential V is given by

$$V(x)=\left|:x^2:\right|^p_{H^{-\alpha}}.$$

Lemma 3.8. Let $\alpha\in(0,\frac{1}{9})$, $s>0$, $p\ge1$ such that $\alpha+\frac{1}{3p}+\frac{s}{3}<\frac{1}{3}$ and $\alpha(2p-1)+s<2$. Then there exists γ such that for any $\varphi\in\mathcal{B}_b(C^{-\alpha})$ we have for $t\in[0,1]$, $x\in C^{-\alpha}$, $h\in H$

$$|DS^N_t\varphi(x)\cdot h|\le c\,t^{-(1+s+2\alpha p)/2}\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^\gamma$$

and

$$|DS^N_t(V\varphi)(x)\cdot h|\le c\,t^{-(1+s+2\alpha p)/2}\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^\gamma.$$

Remark 3.9. $V\varphi$ is not bounded but it is clear from Lemma 3.7 that S^N_t can be extended to borelian functions with polynomial growth.

Proof. We only prove the second inequality, the first one is similar.

We omit to mention the dependence of N. From [8] we know that $S_t^N\varphi$ is differentiable in any direction $h \in H$ and we have the following modified Bismut–Elworthy–Li formula:

$$DS_t(V\varphi)(x)\cdot h = \frac{1}{t}\,\mathbb{E}\left(e^{-K\int_0^t V(X(s,x))ds}V(X(t,x))\varphi(X(t,x))\int_0^t (\eta^h(s), dW(s))\right)$$

$$+2K\mathbb{E}\left(e^{-K\int_0^t V(X(s,x))ds}V(X(t,x))\varphi(X(t,x))\int_0^t (1-\tfrac{s}{t})\,V'(X(s,x))\cdot\eta^h(s)\,ds)\right)$$

$$= (a) + (b).$$

Clearly

$$\begin{aligned}(a) \quad &\le \quad \frac{1}{t}\,\|\varphi\|_0\left[\mathbb{E}\left(V(X(t,x))\right)^2\right]^{1/2}\\ \times \quad &\left[\mathbb{E}\left(e^{-2K\int_0^t V(X(s,x)ds}\left|\int_0^t(\eta^h(s), dW(s))\right|^2\right)\right]^{1/2}.\end{aligned}$$

By Lemma 3.7 we have for K large enough and $t \in (0,1]$

$$\left[\mathbb{E}\left(V(X(t,x))\right)^2\right]^{1/2} \le c\,t^{-\alpha p}(1+|x|_{C^{-\alpha}})^{\overline{\gamma}_{\alpha,2p}/2}.$$

To estimate the second factor, we proceed as in the proof of [3, Lemma 4.1]. Writing Itô's formula for

$$d\left|e^{-K\int_0^t V(X(s,x)ds}\int_0^t(\eta^h(s), dW(s))\right|^2$$

we deduce for K large enough

$$\begin{aligned}&\mathbb{E}\left(e^{-2K\int_0^t V(X(s,x)ds}\left|\int_0^t(\eta^h(s), dW(s))\right|^2\right)^{1/2}\\ &\le \mathbb{E}\left(\int_0^t e^{-2K\int_0^s V(X(r,x)dr}\left|\eta^h(s)\right|^2 ds\right)^{1/2}\\ &\le c\mathbb{E}\left(\int_0^t \sigma^{-s}\,ds\right)^{1/2}\ |h|_{C^{-s}}\end{aligned}$$

thanks to Lemma 3.1 and the embedding $L^\infty \subset L^2$. In Lemma 3.1, we choose $\epsilon > 0$ such that $\alpha + \epsilon + \frac{1}{3p} + \frac{s}{3} < \frac{1}{3}$.

It follows that

$$(a) \le c\,t^{-(1+s)/2-\alpha p}\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^{\gamma_{\alpha,2p}/2}.$$

Similarly, the other term is bounded by

$$(b) \quad \le c\|\varphi\|_0\,\left[\mathbb{E}\left(V(X(t,x))\right)^2\right]^{1/2}\ \mathbb{E}\left(e^{-2K\int_0^t V(X(s,x)ds}\left|\int_0^t V'(X(s,x))\cdot\eta^h(s)\,ds\right|^2\right)^{1/2}$$

We have

$$\begin{aligned}|V'(X(s,x))\cdot\eta^h(s)| \quad &= \quad 2p|(:X^2(s):|_{H^{-\alpha}}^{p-2}\,|(:X^2(s):, X(s)\eta^h(s))|_{H^{-\alpha}}\\ &\le \quad 2p|:X^2(s):|_{H^{-\alpha}}^{p-1}\,|X(s)\eta^h(s)|_{H^{-\alpha}}\\ &\le \quad 2p|:X^2(s):|_{H^{-\alpha}}^{p-1}\,|X(s)|_{H^{-\alpha}}\,|\eta^h(s)|_{C^{\alpha+\epsilon}}.\end{aligned}$$

By Lemma 3.1, Minkowski inequality, Lemma 3.4 and Lemma 3.6, we thus can write for K large enough:

$$\begin{aligned}
&E\left(e^{-2K\int_0^t V(X(s,x)ds}\left|\int_0^t V'(X(s,x))\cdot \eta^h(s)\,ds\right|^2\right)^{1/2}\\
&\le c\mathbb{E}\left(\left(\int_0^t |:X^2(s):|_{H^{-\alpha}}^{p-1}\,|X(s)|_{H^{-\alpha}} s^{-(\alpha+\epsilon+s)/2}\,ds\right)^2\right)^{1/2} |h|_{C^{-s}}\\
&\le c\int_0^t \mathbb{E}\left(\left(|:X^2(s):|_{H^{-\alpha}}^{p-1}\,|X(s)|_{H^{-\alpha}}\right)^2\right)^{1/2} s^{-(\alpha+\epsilon+s)/2}\,ds|h|_{C^{-s}}\\
&\le c\left(\int_0^t s^{-(\alpha+\epsilon+s+2\alpha(p-1))/2}\,ds\right)(1+|x|_{C^{-\alpha}})^{(\overline{\gamma}_{\alpha,2p}/2+\overline{\gamma}_{\alpha,4(p-1)}+\kappa_{\alpha,4})/4}|h|_{C^{-s}}.
\end{aligned}$$

We deduce for $\epsilon>0$ small enough so that $\alpha+\epsilon+s+2\alpha(p-1))<2$:

$$(b)\le c\|\varphi\|_0 t^{-\alpha p}(1+|x|_{C^{-\alpha}})^{(\overline{\gamma}_{\alpha,4(p-1)}+\kappa_{\alpha,4})/4}|h|_{C^{-s}}$$

The conclusion follows when gathering the estimates on (a) and (b). □

3.2 Estimate on P_t^N

Recall that

$$P_t^N\varphi(x)=S_t^N\varphi(x)+K\int_0^t S_{t-s}^N(VP_s^N\varphi)ds.$$

Theorem 3.10. Let $s\in(0,1)$ and $\alpha\in(0,1/9)$ such that $\frac{2\alpha}{1-s}<1-s-3\alpha$. Let $p\ge 1$ such that $\frac{2\alpha}{1-s}<\frac{1}{p}<1-s-3\alpha$, then there exists $\gamma>0$, and $c>0$ depending on s, α and p such that for any $\varphi\in\mathcal{B}_b(C^{-\alpha})$ we have for $t>0$, $x\in C^{-\alpha}$, $h\in H$:

$$|DP_t^N\varphi(x)\cdot h|\le c(t^{-(1+s+2\alpha p)/2}+1)\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^{\gamma}.$$

Remark 3.11. It is easy to find α, s satisfying the assumptions of Theorem 3.10. For instance, choose any $s\in(0,1)$, $\delta\in(0,\frac{1-s}{2})$ and $\alpha<\min\{\frac{\delta(1-s)}{1+3\delta};s\}$. Then

$$\frac{\alpha}{\delta}<1-s-3\alpha.$$

Moreover

$$\frac{\alpha}{\delta}>\frac{2\alpha}{1-s}.$$

It follows that one may choose p such that $\frac{2\alpha}{1-s}<\frac{\alpha}{\delta}<\frac{1}{p}<1-s-3\alpha$ and Theorem 3.10 implies

$$|DP_t^N\varphi(x)\cdot h|\le c(t^{-(1+s+\delta)/2}+1)\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^{\gamma}.$$

Remark 3.12. Since H is dense in C^{-s}, this result shows that $DP_t^N\varphi(x)$ belongs to the dual of C^{-s} and the bound extends to $h\in C^{-s}$.

Proof. Since for $t\ge 1$, $\|P_t\varphi\|_0\le\|\varphi\|_0$, it suffices to prove the result for $t\in(0,1]$ and use the Markov property.

Since $\frac{2\alpha}{1-s}>\frac{2\alpha}{2-s+\alpha}$, we have $\frac{2\alpha}{2-s+\alpha}<\frac{1}{p}<1-s-3\alpha$ and this is equivalent to the assumptions of Lemma 3.8, therefore we may write:

$$\begin{aligned}
|DP_t^N\varphi(x)\cdot h| &\le |DS_t^N\varphi(x)\cdot h|+K\int_0^t |DS_{t-s}^N(VP_s\varphi)(x)|ds\\
&\le c\,t^{-(1+s+2\alpha p)/2}\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^{\gamma}\\
&\quad +K\int_0^t c\,(t-s)^{-(1+s+2\alpha p)/2}\,|h|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}})^{\gamma}\,ds.
\end{aligned}$$

This implies the result since $1+s+2\alpha p<2$. $\square$

Letting $N\to\infty$ in Theorem 3.10, we deduce:

Corollary 3.13. Let $s\in(0,1)$ and $\alpha\in(0,1/9)$ such that $\frac{2\alpha}{1-s}<1-s-3\alpha$. Let $p\geq 1$ such that $\frac{2\alpha}{1-s}<\frac{1}{p}<1-s-3\alpha$, then there exists $\gamma>0$, and $c>0$ depending on s, α and p such that for any $\varphi\in\mathcal{B}_b(H)$ we have for $t>0$, x, $y\in C^{-\alpha}$:

$$|P_t\varphi(x)-P_t\varphi(y)|\leq c(t^{-(1+s+2\alpha p)/2}+1)\,|x-y|_{C^{-s}}\|\varphi\|_0\,(1+|x|_{C^{-\alpha}}+|y|_{C^{-\alpha}})^{\gamma}.$$

4 The Kolmogorov operator

Given $N\in\mathbb{N}$, we denote by $\mathcal{F}_NC_b^\infty$ the set of all functions $\varphi:H\to\mathbb{R}$ of the form

$$\varphi(x)=g(x_j,\,|j|\leq N), \tag{4.1}$$

where $g:\mathbb{R}^{(2N+1)^2}\to\mathbb{R},\ (\xi_j,\,|j|\leq N)\to g(\xi_j,\,|j|\leq N)$ is of class C_b^∞. Moreover, we set

$$\mathcal{F}C_b^\infty=\bigcap_{N=1}^{\infty}\mathcal{F}_NC_b^\infty.$$

Note that if $\varphi\in\mathcal{F}_NC_b^\infty$ is of the form (4.1) we have

$$D\varphi(x)=\sum_{|k|\leq N}D_kg(\xi_j,\,|j|\leq N)e_k \tag{4.2}$$

and

$$D^2\varphi(x)=\sum_{|k|\leq N,|h|\leq N}D_hD_kg(\xi_j,\,|j|\leq N)\,e_h\otimes e_k. \tag{4.3}$$

Let us introduce the Kolmogorov operator on $\mathcal{F}C_b^\infty$, setting

$$\mathcal{K}\varphi:=\frac{1}{2}\,\mathrm{Tr}\,[D^2\varphi]+\langle Ax-:x^3:,D\varphi\rangle,\quad \varphi\in\mathcal{F}C_b^\infty. \tag{4.4}$$

This definition is meaningful because if φ is given by (4.2) we have

$$\langle :x^3:,D\varphi\rangle=\sum_{|k|\leq N}\langle D_kg(\xi_j,\,|j|\leq N)e_k,:x^3:\rangle.$$

We set

$$\rho(x)=Z^{-1}e^{-2U(x)},$$

so that

$$D\log\rho(x)=-2DU(x)=-2:x^3:. \tag{4.5}$$

We also consider the approximate measure ν_N in H

$$\nu_N(dx)=Z_N^{-1}e^{-U_N(x)}\mu_N(dx), \tag{4.6}$$

where

$$U_N(x)=\frac{1}{4}\int_H :x_N^4:(\xi)d\xi \tag{4.7}$$

and

$$Z_N:=\int_H e^{-U_N(x)}\mu_N(dx).$$

Note that

$$\mathcal{K}(\varphi^2)=2\varphi\mathcal{K}\varphi+|D\varphi|^2,\quad \forall\,\varphi\in\mathcal{F}_NC_b^\infty. \tag{4.8}$$

Integrating with respect to ν over $\mathcal{H}$, yields

$$\int_{\mathcal{H}} \mathcal{K}\varphi\,\varphi\,d\nu = -\frac{1}{2}\int_{\mathcal{H}} |D\varphi|^2 d\nu, \quad \forall\,\varphi \in \mathcal{F}C_b^\infty. \tag{4.9}$$

We also introduce an approximating operator $\mathcal{K}_N$ on $\mathcal{F}C_b^\infty$ setting, for $\varphi(x) = g(x_j,\,|j| \le M)$

$$\begin{aligned}\mathcal{K}_N\varphi &:= \frac{1}{2}\,\mathrm{Tr}\,[\Pi_N\,D^2\varphi] + \langle Ax - :x_N^3:, D\varphi\rangle\\ &= \frac{1}{2}\sum_{|h|\le N\wedge M} D_h^2 g(\xi_j,\,|j|\le M) - \sum_{|h|\le M}(\alpha_h - \rho_N)x_h D_h g(\xi_j,\,|j|\le M)\\ &- \sum_{|h|\le M}\ \sum_{|j_1|,|j_2|,|j_3|\le M} x_{j_1}x_{j_2}x_{j_3}\delta_{h,j_1+j_2+j_3} D_h g(\xi_j,\,|j|\le M),\end{aligned} \tag{4.10}$$

where $\alpha_h = (1+|h|^2)$.

Similarly as above, we have

$$\int_{\mathcal{H}} \mathcal{K}_N\varphi\,\varphi\,d\nu_N = -\frac{1}{2}\int_{\mathcal{H}} |D\varphi|^2 d\nu_N, \quad \forall\,\varphi \in \mathcal{F}_N C_b^\infty. \tag{4.11}$$

4.1 m-Dissipativity of $\mathcal{K}$

Recall that $\mathcal{K}$ is m-dissipative if it is dissipative and, for all $\lambda > 0$, $(\lambda - \mathcal{K})$ is surjective. In this section we use Lumer–Philips Theorem to prove that $\mathcal{K}$ has an m-dissipative extension and provide a core for this extension. This has several applications. We explain one of these.

Fix $N_0 \in \mathbb{N}$ and let $f \in \mathcal{F}_{N_0}C_b^\infty$. For $N \in \mathbb{N}$ we consider the solution φ_N of the approximating equation

$$\lambda\varphi_N - \frac{1}{2}\,\mathrm{Tr}\,[D^2\Pi_N\varphi_N] + \langle Ax_N - :x_N^3:, D\varphi_N\rangle = f. \tag{4.12}$$

Then, taking into account (4.11) we find the estimate

$$\int_H |D\varphi_N|^2\,d\nu_N \le \frac{2}{\lambda}\int_H f^2\,d\nu_N. \tag{4.13}$$

Theorem 4.1. $\mathcal{K}$ is extendible to an m-dissipative operator in $L^1(\mathcal{H},\nu)$ whose extension we denote by $\mathcal{K}^{(1)}$. Moreover, $\mathcal{F}C_b^\infty$ is a core for $\mathcal{K}^{(1)}$.

Proof. We have only to show

$$\lim_{N\to\infty}\int_{\mathcal{H}} \langle :x_N^3: - :x^3:, D\varphi_N\rangle d\nu = 0. \tag{4.14}$$

This implies that the range of $\lambda - \mathcal{K}$ is dense and we apply the Lumer–Phillips theorem in some Besov spaces (see [11]).

We take α, s and p satisfying the assumptions of Theorem 3.10 and write

$$|DP_t^N f(x)\cdot h| \le c\,t^{-(1+s+\alpha p)/2} + 1)\,|h|_{C^{-s}}\|f\|_0\,(1+|x|_{C^{-\alpha}})^\gamma, \quad t \ge 0,$$

for some $\alpha < \frac{1}{8}$, $\epsilon > 0$ and c, γ depending on s, α, ϵ. Moreover, we can choose $1+\alpha+\epsilon+s < 2$.

We write

$$\varphi_N(x) = \int_0^\infty e^{-\lambda t} P_t^N f(x) ds$$

and deduce

$$|D\varphi_N(x)\cdot h|\le c|h|_{C^{-s}}\,\|f\|_0(1+|x|_{C^{-\alpha}})^\gamma.$$

It then suffices to write

$$\left|\int_{\mathcal H}\langle :x_N^3:-:x^3:\rangle D\varphi_N\right|\,d\nu\le c\int_{\mathcal H}|:x_N^3:-:x^3:|_{C^{-s}}\,\|f\|_0(1+|x|_{C^{-\alpha}})^\gamma\,d\nu$$

By the Hölder inequality this converges to 0. □

Remark 4.2. Arguing as in [4] one can show that the closure $\mathcal K^{(p)}$ of $\mathcal K$ in $L^p(\mathcal H,\nu)$, $p\ge 1$ is m-dissipative. Moreover the gradient operator

$$D:\mathcal FC_b^1\subset L^p(\mathcal H,\nu)\to L^p(\mathcal H,\nu;H)$$

is closable. We denote by $\overline{D}_p$ its closure and by $W^{1,p}(\mathcal H,\nu)$ its domain.

Finally,

$$D(\mathcal K^{(p)})\subset W^{1,2}(\mathcal H,\nu)$$

and it results

$$\int_{\mathcal H}\mathcal K^{(2)}\varphi\,\varphi\,d\nu=-\frac12\int_{\mathcal H}|D\varphi|^2d\nu,\quad\forall\,\varphi\in D(\mathcal K^{(2)}).\tag{4.15}$$

Moreover since, as easily checked

$$\int_{\mathcal H}\mathcal K^{(2)}\varphi\,\psi\,d\nu=-\frac12\int_{\mathcal H}\langle D\varphi,D\psi\rangle d\nu,\quad\forall\,\varphi,\,\psi\in\mathcal F\,C_b^2(H),$$

it results

$$\int_{\mathcal H}\mathcal K^{(2)}\varphi\,\psi\,d\nu=-\frac12\int_{\mathcal H}\langle D\varphi,D\psi\rangle d\nu,\quad\forall\,\varphi,\,\psi\in D(\mathcal K^{(2)}),\tag{4.16}$$

so that $\mathcal K^{(2)}$ is *symmetric*.

Finally, let us recall the classical integration by parts formula for the Gaussian measure μ

$$\int_{\mathcal H}\langle D\varphi,z\rangle\,\psi\,d\mu=-\int_{\mathcal H}\langle D\psi,z\rangle\,\varphi\,d\mu+\int_{\mathcal H}W_z\,\psi\,\varphi\,d\mu,\quad h\in\mathbb Z^2,\ \psi,\,\varphi\,\in\mathcal FC_b^1(\mathcal H),\tag{4.17}$$

where

$$W_z(x)=\langle Q^{-1/2}x,z\rangle=\sum_{k\in\mathbb Z^2}\sqrt{1+|k|^2}\,z_kx_k,\quad\phi\in\mathcal H$$

and Q is the covariance of μ. Since $\nu(dx)=\rho(x)\mu(dx)$, setting $\psi=\rho$ in (4.17) yields

$$\int_{\mathcal H}\langle D\varphi,z\rangle\,\rho\,d\mu=-\int_{\mathcal H}\langle D\rho,z\rangle\,\varphi\,d\mu+\int_{\mathcal H}W_z\,\rho\,\varphi\,d\mu,\quad h\in\mathbb Z^2,\,\varphi\,\in\mathcal FC_b^1(\mathcal H),$$

which is equivalent to

$$\int_{\mathcal H}\langle D\varphi,z\rangle\,d\nu=-\int_{\mathcal H}\langle D\log\rho,z\rangle\,\varphi\,d\nu+\int_{\mathcal H}W_z\,\varphi\,d\nu,\quad h\in\mathbb Z^2,\,\varphi\,\in\mathcal FC_b^1(\mathcal H).$$

Since $\langle D\log\rho,z\rangle=-\langle :x^3:,z\rangle$ we have

$$\int_{\mathcal H}\langle D\varphi,z\rangle\,d\nu=\int_{\mathcal H}\langle :x^3:,z\rangle\,\varphi\,d\nu+\int_{\mathcal H}W_z\,\varphi\,d\nu,\quad h\in\mathbb Z^2,\ \varphi\,\in\mathcal FC_b^1(\mathcal H).$$

Therefore, for any $p>1$ and any $z\in\mathbb Z^2$ there is $C_p>0$ such that

$$\left|\int_{\mathcal H}\langle D\varphi,z\rangle\,d\nu\right|\le C_p\|\varphi\|_{L^p(\mathcal H,\nu)}\,|z|.\tag{4.18}$$

This inequality is useful to develop a geometrical theory for the measure ν, see [6], [5] and [7].

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