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Sample covariances of random-coefficient AR(1) panel model

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Abstract: The present paper obtains a complete description of the limit distributions of sample covariances in $N \times n$ panel data when N and n jointly increase, possibly at different rate. The panel is formed by N independent samples of length n from random-coefficient AR(1) process with the tail distribution function of the random coefficient regularly varying at the unit root with exponent $\beta > 0$. We show that for $\beta \in (0, 2)$ the sample covariances may display a variety of stable and non-stable limit behaviors with stability parameter depending on β and the mutual increase rate of N and n.

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Contents

1. Introduction

Dynamic panels providing information on a large population of heterogeneous individuals such as households, firms, etc. observed at regular time periods, are often described by simple autoregressive models with random parameters near unity. One of the simplest models for individual evolution is the random-coefficient AR(1) (RCAR(1)) process

$$X(t) = aX(t-1) + \varepsilon(t), \quad t \in \mathbb{Z}, \tag{1.1}$$

with standardized i.i.d. innovations $\{\varepsilon(t), t \in \mathbb{Z}\}$ and a random autoregressive coefficient $a \in [0, 1)$ independent of $\{\varepsilon(t), t \in \mathbb{Z}\}$. Granger [10] observed that in the case when the distribution of a is sufficiently dense near unity the stationary solution of RCAR(1) equation in (1.1) may have long memory in the sense that the sum of its lagged covariances diverges. To be more specific, assume that the random coefficient $a \in [0, 1)$ has a density function of the following form

$$\phi(x) = \psi(x)(1-x)^{\beta-1}, \quad x \in [0, 1), \tag{1.2}$$

where $\beta > 0$ and $\psi(x)$, $x \in [0, 1)$ is a bounded function with $\lim_{x\uparrow 1} \psi(x) =: \psi(1) > 0$. Then for $\beta > 1$ the covariance function of stationary solution of RCAR(1) equation in (1.1) with standardized finite variance innovations decays as $t^{-(\beta-1)}$, viz.,

$$\gamma(t) := \mathrm{E}X(0)X(t) = \mathrm{E}\frac{a^{|t|}}{1-a^2} \sim \frac{\psi(1)}{2}\Gamma(\beta-1)t^{-(\beta-1)}, \quad t \to \infty, \tag{1.3}$$

implying $\sum_{t\in\mathbb{Z}} |\operatorname{Cov}(X(0), X(t))| = \infty$ for $\beta \in (1, 2]$. The same long memory property applies to the contemporaneous aggregate of N independent individual evolutions $\{X_i(t)\}$, $i = 1, \ldots, N$ of (1.1) and the limit Gaussian aggregated process arising when $N \to \infty$. Various properties of the RCAR(1) and more general RCAR equations were studied in Gonçalves and Gouriéroux [9], Zaffaroni [32], Celov et al. [3], Oppenheim and Viano [18], Puplinskaitė and Surgailis [25], Philippe et al. [19] and other works, see Leipus et al. [14] for review.

Statistical inference in the RCAR(1) model was discussed in several works. Leipus et al. [13], Celov et al. [4] discussed nonparametric estimation of the mixing density $\phi(x)$ using empirical covariances of the limit aggregated process. For panel RCAR(1) data, Robinson [29] and Beran et al. [1] discussed parametric estimation of the mixing density. In nonparametric context, Leipus et al. [15] studied estimation of the empirical d.f. of a from panel RCAR(1) observations and derived its asymptotic properties as $N, n \to \infty$, while [16] discussed estimation of β in (1.2) and testing for long memory in the above panel model. For a $N \times n$ panel comprising N samples $\{X_i(t), t = 1, \ldots, n\}$ of length n, $i = 1, \ldots, N$ of independent RCAR(1) processes in (1.1) with mixing distribution in (1.2), Pilipauskaitė and Surgailis [20] studied the asymptotic distribution of the sample mean

$$\bar{X}_{N,n} := \frac{1}{Nn} \sum_{i=1}^{N} \sum_{t=1}^{n} X_i(t) \tag{1.4}$$

as $N, n \to \infty$, possibly at a different rate. [20] showed that for $0 < \beta < 2$ the limit distribution of this statistic depends on whether $N/n^{\beta} \to \infty$ or $N/n^{\beta} \to 0$ in which cases $\bar{X}_{N,n}$ is asymptotically stable with stability parameter depending on β and taking values in the interval $(0, 2]$. See Table 2 below. As shown in [20], under the 'intermediate' scaling $N/n^{\beta} \to c \in (0, \infty)$ the limit distribution of $\bar{X}_{N,n}$ is more complicated and is given by a stochastic integral with respect to a certain Poisson random measure.

The present paper discusses asymptotic distribution of sample covariances (covariance estimators), for all $(t, s) \in \mathbb{Z}^2$,

$$\widehat{\gamma}_{N,n}(t, s) := \frac{1}{Nn} \sum_{1 \le i, i+s \le N} \sum_{1 \le k, k+t \le n} (X_i(k) - \bar{X}_{N,n})(X_{i+s}(k+t) - \bar{X}_{N,n}), \tag{1.5}$$

computed from a similar RCAR(1) panel $\{X_i(t), t = 1, \ldots, n, i = 1, \ldots, N\}$ as in [20], as N, n jointly increase, possibly at a different rate, and the lag $(t, s) \in \mathbb{Z}^2$ is fixed, albeit arbitrary. Particularly, for $(t, s) = (0, 0)$, (1.5) agrees with the sample variance:

$$\widehat{\gamma}_{N,n}(0, 0) = \frac{1}{Nn} \sum_{i=1}^{N} \sum_{k=1}^{n} (X_i(k) - \bar{X}_{N,n})^2. \tag{1.6}$$

The true covariance function $\gamma(t, s) := \mathrm{E} X_i(k) X_{i+s}(k+t)$ of the RCAR(1) panel model with mixing density in (1.2) exists when $\beta > 1$ and is given by

$$\gamma(t, s) = \begin{cases} \gamma(t), & s = 0, \\ 0, & s \ne 0, \end{cases} \tag{1.7}$$

where $\gamma(t)$ is defined in (1.3). Note that $\gamma(t)$ cannot be recovered from a single realization of the nonergodic RCAR(1) process $\{X(t)\}$ in (1.1). However, the covariance function in (1.7) can be consistently estimated from the RCAR(1) $N \times n$ panel when $N, n \to \infty$, together with rates. The limit distribution of the sample covariance may exist even for $0 < \beta < 1$ when the covariance itself is undefined. As it turns out, the limit distribution of $\widehat{\gamma}_{N,n}(t,s)$ depends on the mutual increase rate of N and n, and is also different for temporal, or iso-sectional lags ($s = 0$) and cross-sectional lags ($s \neq 0$). The distinctions between the cases $s = 0$ and $s \neq 0$ are due to the fact that, in the latter case, the statistic in (1.5) involves products $X_i(k)X_{i+s}(k+t)$ of independent processes X_i and X_{i+s}, whereas in the former case, $X_i(k)$ and $X_i(k+t)$ are dependent r.v.s. The main results of this paper are summarized in Table 1 below. Rigorous formulations are given in Sections 3 and 4. For better comparison, Table 2 presents the results of [20] about the sample mean in (1.4) for the same panel model.

Table 1

Limit distribution of sample covariances $\widehat{\gamma}_{N,n}(t,s)$ in (1.5).

a) temporal lags ($s = 0$)

Mutual increase rate of N, n	Parameter region	Limit distribution
$N/n^\beta \to \infty$	$0 < \beta < 2$, $\beta \neq 1$	asymmetric β-stable
$N/n^\beta \to 0$	$0 < \beta < 2$, $\beta \neq 1$	asymmetric β-stable
$N/n^\beta \to c \in (0,\infty)$	$0 < \beta < 2$, $\beta \neq 1$	'intermediate Poisson'
arbitrary	$\beta > 2$	Gaussian

b) cross-sectional lags ($s \neq 0$)

Mutual increase rate of N, n	Parameter region	Limit distribution
$N/n^{2\beta} \to \infty$	$1 < \beta < 3/2$	Gaussian
$N/n^{2\beta} \to \infty$	$1/2 < \beta < 1$	symmetric (2β)-stable
$N/n^{2\beta} \to 0$	$3/4 < \beta < 3/2$	symmetric $(4\beta/3)$-stable
$N/n^{2\beta} \to c \in (0,\infty)$	$3/4 < \beta < 3/2$	'intermediate Poisson'
arbitrary	$\beta > 3/2$	Gaussian

Table 2

Limit distribution of the sample mean $\bar{X}_{N,n}$ in (1.4).

Mutual increase rate of N, n	Parameter region	Limit distribution
$N/n^\beta \to \infty$	$1 < \beta < 2$	Gaussian
$N/n^\beta \to \infty$	$0 < \beta < 1$	symmetric (2β)-stable
$N/n^\beta \to 0$	$0 < \beta < 2$	symmetric β-stable
$N/n^\beta \to c \in (0,\infty)$	$0 < \beta < 2$	'intermediate Poisson'
arbitrary	$\beta > 2$	Gaussian

Remark 1.1. *(i) β-stable limits in Table 1 a) arising when $N/n^\beta \to 0$ and $N/n^\beta \to \infty$ have* different *scale parameters and hence the limit distribution of temporal sample covariances is different in the two cases.*

(ii) 'Intermediate Poisson' limits in Tables 1, 2 refer to infinitely divisible distributions defined through certain stochastic integrals w.r.t. Poisson random measure. A similar terminology was used in [22].

(iii) It follows from our results (see Theorem 4.1 below) that a scaling transition similar as in the case of the sample mean [20] arises in the interval $0 < \beta < 2$ *for temporal sample covariances and product random fields $X_v(u)X_v(u+t)$, $(u,v) \in$*

$\mathbb{Z}^2$ involving temporal lags, with the critical rate $N \sim n^\beta$ separating regimes with different limit distributions. For 'cross-sectional' product fields $X_v(u)X_{v+s}(u+t)$, $(u,v) \in \mathbb{Z}^2$, $s \neq 0$ involving cross-sectional lags, a similar scaling transition occurs in the interval $0 < \beta < 3/2$ with the critical rate $N \sim n^{2\beta}$ between different scaling regimes, see Theorem 3.1. The notion of scaling transition for long-range dependent random fields in $\mathbb{Z}^2$ was discussed in Puplinskaitė and Surgailis [26, 27], Pilipauskaitė and Surgailis [22, 23].

(iv) The limit distributions of cross-sectional sample covariances in the missing intervals $0 < \beta < 1/2$ and $0 < \beta < 3/4$ of Table 1 b) are given in Corollary 3.1 below. They are more complicated and not included in Table 1 b) since the term $Nn(\bar{X}_{N,n})^2$ due to the centering by the sample mean in (1.5) may play the dominating role.

(v) We expect that the asymptotic distribution of sample covariances in the RCAR(1) panel model with common innovations (see [21]) can be analyzed in a similar fashion. Due to the differences between the two models (the common and the idiosyncratic innovation cases), the asymptotic behavior of sample covariances might be quite different in these two cases.

(vi) The results in Table 1 a) are obtained under the finite 4th moment conditions on the innovations, see Theorems 4.1 and 4.2 below. Although the last condition does not guarantee the existence of the 4th moment of the RCAR(1) process, it is crucial for the limit results, including the CLT in the case $\beta > 2$. Scaling transition for sample variances of long-range dependent Gaussian and linear random fields on $\mathbb{Z}^2$ with finite 4th moment was established in Pilipauskaitė and Surgailis [23]. On the other side, Surgailis [31], Horváth and Kokoszka [12] obtained stable limits of sample variances and autocovariances for long memory moving averages with finite 2nd moment and infinite 4th moment. Finally, we mention the important works of Davis and Resnick [6] and Davis and Mikosch [5] on limit theory for sample covariance and correlation functions of moving averages and some nonlinear processes with infinite variance, respectively.

The rest of the paper is organized as follows. Section 2 presents some preliminary facts, including the definition and properties of the intermediate processes appearing in Table 1. Section 3 contains rigorous formulations and the proofs of the asymptotic results for cross-sectional sample covariances (1.5), $s \neq 0$ and the corresponding partial sums processes. Analogous results for temporal sample covariances and partial sums processes are presented in Section 4. Section 4 also contains some applications of these results to estimation of the autocovariance function $\gamma(t)$ in (1.3) from panel data. Some auxiliary proofs are given in Appendix A.

2. Preliminaries

This section contains some preliminary facts which will be used in the following sections.

2.1. Double stochastic integrals and quadratic forms

Let B_i, $i = 1, 2$ be independent standard Brownian motions (BMs) on the real line. Let

$$I_i(f) := \int_{\mathbb{R}} f(s)\mathrm{d}B_i(s), \quad I_{ij}(g) := \int_{\mathbb{R}^2} g(s_1, s_2)\mathrm{d}B_i(s_1)\mathrm{d}B_j(s_2), \quad i, j = 1, 2, \tag{2.1}$$

denote Itô-Wiener stochastic integrals (single and double) w.r.t. B_i, B_j. The integrals in (2.1) are jointly defined for any (non-random) integrands $f \in L^2(\mathbb{R})$, $g \in L^2(\mathbb{R}^2)$; moreover, $\mathrm{E}I_i(f) = \mathrm{E}I_{ij}(g) = 0$ and

$$\mathrm{E}I_i(f)I_{i'}(f') = \begin{cases} 0, & i \neq i', \\ \langle f, f'\rangle, & i = i', \end{cases} \quad f, f' \in L^2(\mathbb{R}), \tag{2.2}$$

$$\mathrm{E}I_i(f)I_{i'j'}(g) = 0, \qquad \forall\, i, i', j', \quad f \in L^2(\mathbb{R}), g \in L^2(\mathbb{R}^2),$$

$$\mathrm{E}I_{ij}(g)I_{i'j'}(g') = \begin{cases} 0, & (i,j) \notin \{(i',j'), (j',i')\}, \\ \langle g, g'\rangle, & (i,j) \in \{(i',j'), (j',i')\},\ i \neq j, \quad g, g' \in L^2(\mathbb{R}^2), \\ 2\langle g, \operatorname{sym} g'\rangle, & i = i' = j = j', \end{cases} \tag{2.3}$$

where $\langle f, f'\rangle = \int_{\mathbb{R}} f(s)f'(s)\mathrm{d}s$ ($\|f\| := \sqrt{\langle f, f\rangle}$), $\langle g, g'\rangle = \int_{\mathbb{R}^2} g(s_1, s_2)g'(s_1, s_2)\mathrm{d}s_1\mathrm{d}s_2$ ($\|g\| := \sqrt{\langle g, g\rangle}$) denote scalar products (norms) in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$, respectively, and sym denotes the symmetrization, see, e.g. ([7], Sections 11.5, 14.3). Note that for $g(s_1, s_2) = f_1(s_1)f_2(s_2)$, $f_i \in L^2(\mathbb{R})$, $i = 1, 2$ we have $I_{ii}(g) = I_i(f_1)I_i(f_2) - \langle f_1, f_2\rangle$, $I_{12}(g) = I_1(f_1)I_2(f_2)$, in particular, $I_{12}(g) =_{\mathrm{d}} \|f_1\|\|f_2\|Z_1Z_2$, where $Z_i \sim N(0, 1)$, $i = 1, 2$ are independent standard normal r.v.s.

Let $\{\varepsilon_i(s),\ s \in \mathbb{Z}\}$, $i = 1, 2$ be independent sequences of standardized i.i.d. r.v.s, $\mathrm{E}\varepsilon_i(s) = 0$, $\mathrm{E}\varepsilon_i(s)\varepsilon_{i'}(s') = 1$ if $(i, s) = (i', s')$, $\mathrm{E}\varepsilon_i(s)\varepsilon_{i'}(s') = 0$ if $(i, s) \neq (i', s')$, $i, i' = 1, 2$, $s, s' \in \mathbb{Z}$. Consider the centered quadratic form

$$Q_{ij}(h) = \sum_{s_1, s_2 \in \mathbb{Z}} h(s_1, s_2)[\varepsilon_i(s_1)\varepsilon_j(s_2) - \mathrm{E}\varepsilon_i(s_1)\varepsilon_j(s_2)], \quad i, j = 1, 2, \tag{2.4}$$

where $h \in L^2(\mathbb{Z}^2)$. For $i = j$ we additionally assume $\mathrm{E}\varepsilon_i^4(0) < \infty$. Then the sum in (2.4) converges in L^2 and

$$\operatorname{Var}(Q_{ij}(h)) \leq (1 + \mathrm{E}\varepsilon_i^4(0)\delta_{ij}) \sum_{s_1, s_2 \in \mathbb{Z}} h^2(s_1, s_2), \tag{2.5}$$

see ([7], (4.5.4)). With any $h \in L^2(\mathbb{Z}^2)$ and any $\alpha_1, \alpha_2 > 0$ we associate its extension to $L^2(\mathbb{R}^2)$, namely,

$$\widetilde{h}^{(\alpha_1,\alpha_2)}(s_1, s_2) := (\alpha_1\alpha_2)^{1/2} h(\lfloor \alpha_1 s_1 \rfloor, \lfloor \alpha_2 s_2 \rfloor), \quad (s_1, s_2) \in \mathbb{R}^2, \tag{2.6}$$

with $\|\widetilde{h}^{(\alpha_1,\alpha_2)}\|^2 = \sum_{s_1, s_2 \in \mathbb{Z}} h^2(s_1, s_2)$. We shall use the following criterion for the convergence in distribution of quadratic forms in (2.4) towards double stochastic integrals (2.1).

Proposition 2.1 ([7], Proposition 11.5.5)**.** *Let $i, j = 1, 2$ and $Q_{ij}(h_{\alpha_1,\alpha_2})$, $\alpha_1, \alpha_2 > 0$ be a family of quadratic forms as in (2.4) with coefficients $h_{\alpha_1,\alpha_2} \in L^2(\mathbb{Z}^2)$. For $i = j$ we additionally assume $\mathrm{E}\varepsilon_i^4(0) < \infty$. Suppose for some $g \in L^2(\mathbb{R}^2)$ we have that*

$$\lim_{\alpha_1,\alpha_2\to\infty} \|\widetilde{h}^{(\alpha_1,\alpha_2)}_{\alpha_1,\alpha_2} - g\| = 0. \tag{2.7}$$

Then $Q_{ij}(h_{\alpha_1,\alpha_2}) \to_{\mathrm{d}} I_{ij}(g)$ $(\alpha_1, \alpha_2 \to \infty)$, where $I_{ij}(g)$ is defined as in (2.1).

2.2. The 'cross-sectional' intermediate process

Let $\mathrm{d}\mathcal{M}_\beta \equiv \mathcal{M}_\beta(\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}B_1, \mathrm{d}B_2)$ denote Poisson random measure on $(\mathbb{R}_+ \times C(\mathbb{R}))^2$ with mean

$$\mathrm{d}\mu_\beta \equiv \mu_\beta(\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}B_1, \mathrm{d}B_2) := \psi(1)^2 (x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 P_B(\mathrm{d}B_1)P_B(\mathrm{d}B_2), \tag{2.8}$$

where $\beta > 0$ is parameter and P_B is the Wiener measure on $C(\mathbb{R})$. Let $\mathrm{d}\widetilde{\mathcal{M}}_\beta := \mathrm{d}\mathcal{M}_\beta - \mathrm{d}\mu_\beta$ be the centered Poisson random measure. We shall often use finiteness of the following integrals:

$$\int_{\mathbb{R}_+^2} \min\big\{1, \frac{1}{x_1x_2(x_1+x_2)}\big\}(x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 < \infty, \quad \forall\, 0 < \beta < 3/2, \tag{2.9}$$

$$\int_{\mathbb{R}_+^2} \min\big\{1, \frac{1}{x_1+x_2}\big\}(x_1x_2)^{\beta-2}\mathrm{d}x_1\mathrm{d}x_2 < \infty, \quad \forall\, 1 < \beta < 3/2, \tag{2.10}$$

see Appendix A. Let

$$\mathcal{Y}_i(u; x) = \int_{-\infty}^u \mathrm{e}^{-x(u-s)}\mathrm{d}B_i(s), \quad u \in \mathbb{R},\ x > 0, \tag{2.11}$$

be a family of stationary Ornstein-Uhlenbeck (O-U) processes subordinated to $B_i = \{B_i(s),\ s \in \mathbb{R}\}$, B_i, $i = 1, 2$ being independent BMs. Let

$$z(\tau; x_1, x_2) := \int_0^\tau \prod_{i=1}^2 \mathcal{Y}_i(u; x_i)\mathrm{d}u, \quad \tau \ge 0, \tag{2.12}$$

be a family of integrated products of independent O-U processes indexed by x_1, $x_2 > 0$. We use the representation of (2.12)

$$z(\tau; x_1, x_2) = \int_{\mathbb{R}^2} \big\{\int_0^\tau \prod_{i=1}^2 \mathrm{e}^{-x_i(u-s_i)}\mathbf{1}(u > s_i)\mathrm{d}u\big\}\mathrm{d}B_1(s_1)\mathrm{d}B_2(s_2) \tag{2.13}$$

as the double Itô-Wiener integral in (2.1). The 'cross-sectional' intermediate process $\mathcal{Z}_\beta$ is defined as stochastic integral w.r.t. the Poisson measure $\mathcal{M}_\beta$, viz.,

$$\mathcal{Z}_\beta(\tau) := \int_{\mathcal{L}_1} z(\tau; x_1, x_2)\mathrm{d}\mathcal{M}_\beta + \int_{\mathcal{L}_1^c} z(\tau; x_1, x_2)\mathrm{d}\widetilde{\mathcal{M}}_\beta, \tag{2.14}$$

where $\mathcal{L}_1^c := (\mathbb{R}_+ \times C(\mathbb{R}))^2 \setminus \mathcal{L}_1$ and

$$\mathcal{L}_1 := \{(x_1, x_2, B_1, B_2) \in (\mathbb{R}_+ \times C(\mathbb{R}))^2 : x_1 x_2 (x_1 + x_2) \le 1\} \tag{2.15}$$

with $\mu_\beta(\mathcal{L}_1) < \infty$. For $1/2 < \beta < 3/2$ the two integrals in (2.14) can be combined in a single one:

$$\mathcal{Z}_\beta(\tau) = \int_{(\mathbb{R}_+ \times C(\mathbb{R}))^2} z(\tau; x_1, x_2) \mathrm{d}\widetilde{\mathcal{M}}_\beta. \tag{2.16}$$

These and other properties of $\mathcal{Z}_\beta$ are stated in the following proposition whose proof is given in the Appendix A. We also refer to [28, 20] for general properties of stochastic integrals w.r.t. Poisson random measure.

Proposition 2.2. *(i) The process $\mathcal{Z}_\beta$ in (2.14) is well-defined for any $0 < \beta < 3/2$. It has stationary increments, infinitely divisible finite-dimensional distributions, and the joint ch.f. given by*

$$\mathrm{E} \exp \big\{ \mathrm{i} \sum_{j=1}^m \theta_j \mathcal{Z}_\beta(\tau_j) \big\} = \exp \big\{ \int_{(\mathbb{R}_+ \times C(\mathbb{R}))^2} (\mathrm{e}^{\mathrm{i} \sum_{j=1}^m \theta_j z(\tau_j; x_1, x_2)} - 1) \mathrm{d}\mu_\beta \big\}, \tag{2.17}$$

where $\theta_j \in \mathbb{R}$, $\tau_j \ge 0$, $j = 1, \ldots, m$, $m \in \mathbb{N}$. Moreover, the distribution of $\mathcal{Z}_\beta$ is symmetric: $\{\mathcal{Z}_\beta(\tau), \tau \ge 0\} =_{\mathrm{fdd}} \{-\mathcal{Z}_\beta(\tau), \tau \ge 0\}$.

(ii) $\mathrm{E}|\mathcal{Z}_\beta(\tau)|^p < \infty$ for $p < 2\beta$ and $\mathrm{E}\mathcal{Z}_\beta(\tau) = 0$ for $1/2 < \beta < 3/2$.

(iii) For $1/2 < \beta < 3/2$, $\mathcal{Z}_\beta$ can be defined as in (2.16). Moreover, if $1 < \beta < 3/2$, then $\mathrm{E}\mathcal{Z}^2_\beta(\tau) < \infty$ and

$$\mathrm{E}\mathcal{Z}_\beta(\tau_1)\mathcal{Z}_\beta(\tau_2) = (\sigma_\infty^2/2)(\tau_1^{2(2-\beta)} + \tau_2^{2(2-\beta)} - |\tau_2 - \tau_1|^{2(2-\beta)}), \ \tau_1, \tau_2 \ge 0, \tag{2.18}$$

where $\sigma_\infty^2 := \psi(1)^2 \Gamma(\beta - 1)^2 / (4(2-\beta)(3-2\beta))$.

(iv) For $1/2 < \beta < 3/2$, the process $\mathcal{Z}_\beta$ has a.s. continuous trajectories.

(v) (Asymptotic self-similarity) As $b \to 0$,

$$b^{\beta-2}\mathcal{Z}_\beta(b\tau) \to_{\mathrm{fdd}} \sigma_\infty B_{2-\beta}(\tau), \quad 1 < \beta < 3/2, \tag{2.19}$$

$$b^{-1}(\log b^{-1})^{-1/(2\beta)}\mathcal{Z}_\beta(b\tau) \to_{\mathrm{fdd}} \tau V_{2\beta}, \qquad 0 < \beta < 1, \tag{2.20}$$

where $\{B_{2-\beta}(\tau), \tau \ge 0\}$ is a fractional Brownian motion with $\mathrm{E}[B_{2-\beta}(\tau)]^2 = \tau^{2(2-\beta)}$, $\tau \ge 0$, $2 - \beta \in (1/2, 1)$, σ_∞^2 is given in (2.18), and $V_{2\beta}$ is a symmetric (2β)-stable r.v. with ch.f. $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta V_{2\beta}} = \mathrm{e}^{-c_\infty |\theta|^{2\beta}}$, $\theta \in \mathbb{R}$, $c_\infty := \psi(1)^2 2^{1-2\beta}\Gamma(\beta + (1/2))\Gamma(1-\beta)/\sqrt{\pi}$. For any $0 < \beta < 3/2$, as $b \to \infty$,

$$b^{-1/2}\mathcal{Z}_\beta(b\tau) \to_{\mathrm{fdd}} \mathcal{A}^{1/2} B(\tau), \tag{2.21}$$

where $\mathcal{A} > 0$ is a $(2\beta/3)$-stable r.v. with Laplace transform $\mathrm{E}\mathrm{e}^{-\theta\mathcal{A}} = \mathrm{e}^{-\sigma_0 \theta^{2\beta/3}}$, $\theta \ge 0$, $\sigma_0 := \psi(1)^2 2^{-2\beta/3}\Gamma(1 - (2\beta/3))\,\mathrm{B}(\beta/3, \beta/3)/(2\beta)$, and $\{B(\tau), \tau \ge 0\}$ is a standard BM, independent of $\mathcal{A}$. Finite-dimensional distributions of the limit process in (2.21) are symmetric $(4\beta/3)$-stable.

2.3. The 'iso-sectional' intermediate process

Let $\mathrm{d}\mathcal{M}^*_\beta \equiv \mathcal{M}^*_\beta(\mathrm{d}x, \mathrm{d}B)$ denote Poisson random measure on $\mathbb{R}_+ \times C(\mathbb{R})$ with mean

$$\mathrm{d}\mu^*_\beta \equiv \mu^*_\beta(\mathrm{d}x, \mathrm{d}B) := \psi(1)x^{\beta-1}\mathrm{d}x P_B(\mathrm{d}B), \tag{2.22}$$

where $0 < \beta < 2$ is parameter and P_B is the Wiener measure on $C(\mathbb{R})$. Let $\mathrm{d}\widetilde{\mathcal{M}}^*_\beta := \mathrm{d}\mathcal{M}^*_\beta - \mathrm{d}\mu^*_\beta$ be the centered Poisson random measure. Let $\mathcal{Y}(\cdot; x) \equiv \mathcal{Y}_1(\cdot; x)$ be the family of O-U processes as in (2.11), and

$$z^*(\tau; x) := \int_0^\tau \mathcal{Y}^2(u; x)\mathrm{d}u, \quad \tau \ge 0,\, x > 0, \tag{2.23}$$

be integrated squared O-U processes. Note that $\mathrm{E}z^*(\tau; x) = \tau \mathrm{E}\mathcal{Y}^2(0; x) = \tau \int_{-\infty}^0 \mathrm{e}^{2xs}\mathrm{d}s = \tau/(2x)$. We will use the representation

$$z^*(\tau; x) = \int_{\mathbb{R}^2} \Big\{ \int_0^\tau \prod_{i=1}^2 \mathrm{e}^{-x(u-s_i)}\mathbf{1}(u > s_i)\mathrm{d}u\Big\}\mathrm{d}B(s_1)\mathrm{d}B(s_2) + \tau/(2x) \tag{2.24}$$

as the double Itô-Wiener integral. The 'iso-sectional' intermediate process $\mathcal{Z}^*_\beta$ is defined for $\beta \in (0, 2)$, $\beta \ne 1$ as stochastic integral w.r.t. the above Poisson measure, viz.,

$$\mathcal{Z}^*_\beta(\tau) := \int_{\mathbb{R}_+ \times C(\mathbb{R})} z^*(\tau; x) \begin{cases} \mathrm{d}\mathcal{M}^*_\beta, & 0 < \beta < 1, \\ \mathrm{d}\widetilde{\mathcal{M}}^*_\beta, & 1 < \beta < 2, \end{cases} \quad \tau \ge 0. \tag{2.25}$$

Proposition 2.3 stating properties of $\mathcal{Z}^*_\beta$ is similar to Proposition 2.2.

Proposition 2.3. *(i) The process $\mathcal{Z}^*_\beta$ in (2.25) is well-defined for any $0 < \beta < 2$, $\beta \ne 1$. It has stationary increments, infinitely divisible finite-dimensional distributions, and the joint ch.f. given by*

$$\begin{aligned} &\mathrm{E}\exp\Big\{\mathrm{i}\sum_{j=1}^m \theta_j \mathcal{Z}^*_\beta(\tau_j)\Big\} \\ &\quad = \exp\Big\{ \int_{\mathbb{R}_+ \times C(\mathbb{R})} (\mathrm{e}^{\mathrm{i}\sum_{j=1}^m \theta_j z^*(\tau_j; x)} - 1 - \mathrm{i}\sum_{j=1}^m \theta_j z^*(\tau_j; x)\mathbf{1}(1 < \beta < 2))\mathrm{d}\mu^*_\beta\Big\}, \end{aligned} \tag{2.26}$$

where $\theta_j \in \mathbb{R}$, $\tau_j \ge 0$, $j = 1, \dots, m$, $m \in \mathbb{N}$.

(ii) $\mathrm{E}|\mathcal{Z}^*_\beta(\tau)|^p < \infty$ for any $0 < p < \beta < 2$, $\beta \ne 1$ and $\mathrm{E}\mathcal{Z}^*_\beta(\tau) = 0$ for $1 < \beta < 2$.

(iii) For $1 < \beta < 2$, the process $\mathcal{Z}^*_\beta$ has a.s. continuous trajectories.

(iv) (Asymptotic self-similarity) For any $0 < \beta < 2$, $\beta \ne 1$,

$$b^{-1}\mathcal{Z}^*_\beta(b\tau) \to_{\mathrm{fdd}} \begin{cases} \tau V^*_\beta & \text{as } b \to 0, \\ \tau V^+_\beta & \text{as } b \to \infty, \end{cases} \tag{2.27}$$

where V^+_β, V^*_β *are a completely asymmetric* β*-stable r.v.s with ch.f.s* $\mathrm{E}e^{\mathrm{i}\theta V^+_\beta} = \exp\{\psi(1)\int_{\mathbb{R}_+}(e^{\mathrm{i}\theta/(2x)} - 1 - \mathrm{i}(\theta/(2x))\mathbf{1}(1<\beta<2))x^{\beta-1}\mathrm{d}x\}$, $\mathrm{E}e^{\mathrm{i}\theta V^*_\beta} = \exp\{\psi(1)\int_{\mathbb{R}_+}\mathrm{E}(e^{\mathrm{i}\theta Z^2/(2x)} - 1 - \mathrm{i}(\theta Z^2/(2x))\mathbf{1}(1<\beta<2))x^{\beta-1}\mathrm{d}x\}$, $\theta \in \mathbb{R}$ *and* $Z \sim N(0,1)$.

2.4. Conditional long-run variance of products of RCAR(1) processes

We use some facts in Proposition 2.4, below, about conditional variance of the partial sums process of the product $Y_{ij}(t) := X_i(t)X_j(t)$ of two RCAR(1) processes. Split $Y_{ij}(t) = Y^+_{ij}(t) + Y^-_{ij}(t)$, where $Y^+_{ij}(t) = \sum_{s_1\wedge s_2\ge 1} a_i^{t-s_1}a_j^{t-s_2}\mathbf{1}(t \ge s_1\vee s_2)\varepsilon_i(s_1)\varepsilon_j(s_2)$, $Y^-_{ij}(t) = \sum_{s_1\wedge s_2\le 0} a_i^{t-s_1}a_j^{t-s_2}\mathbf{1}(t\ge s_1\vee s_2)\varepsilon_i(s_1)\varepsilon_j(s_2)$. For $i = j$ we assume additionally that $\mathrm{E}\varepsilon_i^4(0) < \infty$.

Proposition 2.4. *We have*

$$\mathrm{Var}\big[\sum_{t=1}^n Y_{ij}(t)|a_i,a_j\big] \sim \mathrm{Var}\big[\sum_{t=1}^n Y^+_{ij}(t)|a_i,a_j\big] \sim A_{ij}n, \quad n\to\infty, \tag{2.28}$$

where

$$A_{ij} := \begin{cases} \dfrac{1+a_ia_j}{(1-a_i^2)(1-a_j^2)(1-a_ia_j)}, & i\ne j,\\ \dfrac{1+a_i^2}{1-a_i^2}\big(\dfrac{2}{(1-a_i^2)^2} + \dfrac{\mathrm{cum}_4}{1-a_i^4}\big), & i=j \end{cases} \tag{2.29}$$

with cum_4 *being the 4th cumulant of* $\varepsilon_i(0)$*. Moreover, for any* $n \ge 1$, $i,j \in \mathbb{Z}$, $a_i, a_j \in [0,1)$

$$\mathrm{Var}\big[\sum_{t=1}^n Y_{ij}(t)|a_i,a_j\big] \le \frac{C_{ij}n^2}{(1-a_i)(1-a_j)}\min\big\{1, \frac{1}{n(2-a_i-a_j)}\big\}, \tag{2.30}$$

where $C_{ij} := 4$ $(i\ne j)$, $:= 2(2+|\mathrm{cum}_4|)$ $(i=j)$.

Proof. Let $i \ne j$. We have

$$\mathrm{E}[Y_{ij}(t)Y_{ij}(s)|a_i,a_j] = \mathrm{E}[X_i(t)X_i(s)|a_i]\mathrm{E}[X_j(t)X_j(s)|a_j] = \frac{(a_ia_j)^{|t-s|}}{(1-a_i^2)(1-a_j^2)}$$

and hence

$$J_n(a_i,a_j) := \mathrm{E}\big[\big(\sum_{t=1}^n Y_{ij}(t)\big)^2|a_i,a_j\big] = \frac{n}{(1-a_i^2)(1-a_j^2)}\sum_{t=-n}^n (a_ia_j)^{|t|}\big(1-\frac{|t|}{n}\big). \tag{2.31}$$

Relation (2.31) implies (2.28). It also implies $J_n(a_i,a_j) \le 2n^2/((1-a_i)(1-a_j))$. Note also $1-a_ia_j \ge (1/2)((1-a_i)+(1-a_j))$. Hence and from (2.31) we obtain

$$J_n(a_i,a_j) \le \frac{n}{(1-a_i^2)(1-a_j^2)}\big(1+2\sum_{t=1}^\infty (a_ia_j)^t\big)$$

$$\leq \frac{2n}{(1-a_i)(1-a_j)(1-a_i a_j)} \leq \frac{4n}{(1-a_i)(1-a_j)(2-a_i-a_j)},$$

proving (2.30). The proof of (2.28)–(2.30) for $i = j$ is similar using $\mathrm{Cov}[Y_{ii}(t), Y_{ii}(s)|a_i] = 2(a_i^{|t-s|}/(1-a_i^2))^2 + \mathrm{cum}_4\, a_i^{2|t-s|}/(1-a_i^4)$. □

3. Asymptotic distribution of cross-sectional sample covariances

Theorems 3.1 and 3.2 discuss the asymptotic distribution of partial sums process

$$S_{N,n}^{t,s}(\tau) := \sum_{i=1}^{N} \sum_{u=1}^{\lfloor n\tau \rfloor} X_i(u) X_{i+s}(u+t), \quad \tau \geq 0, \tag{3.1}$$

where t and $s \in \mathbb{Z}$, $s \neq 0$ are fixed and N and n tend to infinity, possibly at a different rate. The asymptotic behavior of sample covariances $\widehat{\gamma}_{N,n}(t,s)$ is discussed in Corollary 3.1. As it turns out, these limit distributions do not depend on t, s which is due to the fact that the sectional processes $\{X_i(t),\, t \in \mathbb{Z}\}$, $i \in \mathbb{Z}$ are independent and stationary.

Theorem 3.1. *Let the mixing distribution satisfy condition (1.2) with $0 < \beta < 3/2$. Let $N, n \to \infty$ so as*

$$\lambda_{N,n} := \frac{N^{1/(2\beta)}}{n} \to \lambda_\infty \in [0, \infty]. \tag{3.2}$$

Then the following statements (i)–(iii) hold for $S_{N,n}^{t,s}(\tau)$, $(t,s) \in \mathbb{Z}^2$, $s \neq 0$ in (3.1) depending on λ_∞ in (3.2).

(i) Let $\lambda_\infty = \infty$. Then

$$n^{-2} \lambda_{N,n}^{-\beta} S_{N,n}^{t,s}(\tau) \to_{\mathrm{fdd}} \sigma_\infty B_{2-\beta}(\tau), \qquad 1 < \beta < 3/2, \tag{3.3}$$

$$n^{-2} \lambda_{N,n}^{-1} (\log \lambda_{N,n})^{-1/(2\beta)} S_{N,n}^{t,s}(\tau) \to_{\mathrm{fdd}} \tau V_{2\beta}, \qquad 0 < \beta < 1, \tag{3.4}$$

where the limit processes are the same as in (2.19), (2.20).

(ii) Let $\lambda_\infty = 0$ and $\mathrm{E}|\varepsilon(0)|^{2p} < \infty$ for some $p > 1$. Then

$$n^{-2} \lambda_{N,n}^{-3/2} S_{N,n}^{t,s}(\tau) \to_{\mathrm{fdd}} \mathcal{A}^{1/2} B(\tau), \tag{3.5}$$

where the limit process is the same as in (2.21).

(iii) Let $0 < \lambda_\infty < \infty$. Then

$$n^{-2} \lambda_{N,n}^{-3/2} S_{N,n}^{t,s}(\tau) \to_{\mathrm{fdd}} \lambda_\infty^{1/2} \mathcal{Z}_\beta(\tau/\lambda_\infty), \tag{3.6}$$

where $\mathcal{Z}_\beta$ is the intermediate process in (2.14).

Theorem 3.2. *Let the mixing distribution satisfy condition* (1.2) *with* $\beta > 3/2$ *and assume* $\mathrm{E}|\varepsilon(0)|^{2p} < \infty$ *for some* $p > 1$*. Then for any* $(t,s) \in \mathbb{Z}^2$*,* $s \neq 0$ *as* $N, n \to \infty$ *in arbitrary way,*

$$n^{-1/2} N^{-1/2} S_{N,n}^{t,s}(\tau) \to_{\mathrm{fdd}} \sigma B(\tau), \quad \sigma^2 := \mathrm{E} A_{12}, \tag{3.7}$$

where A_{12} *is defined in* (2.29)*.*

Remark 3.1. *Our proof of Theorem 3.1 (ii) requires establishing the asymptotic normality of a bilinear form in i.i.d. r.v.s, which has a non-zero diagonal, see the r.h.s. of* (3.52)*. For this purpose, we use the martingale CLT and impose an additional condition of* $\mathrm{E}|\varepsilon(0)|^{2p} < \infty$*,* $p > 1$*. To establish the CLT for quadratic forms with non-zero diagonal, [2] took similar approach and also needed* $2p$ *finite moments. In Theorem 3.2 we also assume* $\mathrm{E}|\varepsilon(0)|^{2p} < \infty$*,* $p > 1$*. However, it can be proved under* $\mathrm{E}\varepsilon^2(0) < \infty$ *applying another technique that is approximation by* m*-dependent r.v.s. Moreover, this result holds if* (1.2) *is replaced by* $\mathrm{E}A_{12} < \infty$*.*

Note that the asymptotic distribution of sample covariances $\widehat{\gamma}_{N,n}(t,s)$ in (1.5) coincides with that of the statistics

$$\widetilde{\gamma}_{N,n}(t,s) := (Nn)^{-1} S_{N,n}^{t,s}(1) - (\bar{X}_{N,n})^2. \tag{3.8}$$

For $s \neq 0$ the limit behavior of the first term on the r.h.s. of (3.8) can be obtained from Theorems 3.1 and 3.2. It turns out that for some values of β, the second term on the r.h.s. can play the dominating role. The limit behavior of $\bar{X}_{N,n}$ was identified in [20] and is given in the following proposition, with some simplifications.

Proposition 3.1. *Let the mixing distribution satisfy condition* (1.2) *with* $\beta > 0$*.*

(i) Let $1 < \beta < 2$ *and* $N/n^\beta \to \infty$*. Then*

$$N^{1/2} n^{(\beta-1)/2} \bar{X}_{N,n} \to_{\mathrm{d}} \bar{\sigma}_\beta Z, \tag{3.9}$$

where $Z \sim N(0,1)$ *and* $\bar{\sigma}_\beta^2 := \psi(1)\Gamma(\beta-1)/((3-\beta)(2-\beta))$*.*

(ii) Let $0 < \beta < 1$ *and* $N/n^\beta \to \infty$*. Then*

$$N^{1-1/(2\beta)} \bar{X}_{N,n} \to_{\mathrm{d}} \bar{V}_{2\beta}, \tag{3.10}$$

where $\bar{V}_{2\beta}$ *is a symmetric* (2β)*-stable r.v. with ch.f.* $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta \bar{V}_{2\beta}} = \mathrm{e}^{-\bar{K}_\beta |\theta|^{2\beta}}$*,* $\theta \in \mathbb{R}$*,* $\bar{K}_\beta := \psi(1) 4^{-\beta} \Gamma(1-\beta)/\beta$*.*

(iii) Let $0 < \beta < 2$ *and* $N/n^\beta \to 0$*. Then*

$$N^{1-1/\beta} n^{1/2} \bar{X}_{N,n} \to_{\mathrm{d}} \bar{W}_\beta, \tag{3.11}$$

where $\bar{W}_\beta$ *is a symmetric* β*-stable r.v. with ch.f.* $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta \bar{W}_\beta} = \mathrm{e}^{-\bar{k}_\beta |\theta|^\beta}$*,* $\theta \in \mathbb{R}$ *and* $\bar{k}_\beta := \psi(1) 2^{-\beta/2} \Gamma(1-\beta/2)/\beta$*.*

(iv) Let $\beta > 2$*. Then as* $N, n \to \infty$ *in arbitrary way,*

$$N^{1/2} n^{1/2} \bar{X}_{N,n} \to_{\mathrm{d}} \bar{\sigma} Z, \tag{3.12}$$

where $Z \sim N(0,1)$ *and* $\bar{\sigma}^2 := \mathrm{E}(1-a)^{-2}$*.*

From Theorems 3.1 and Proposition 3.1 we see that the r.h.s. of (3.8) may exhibit *two* 'bifurcation points' of the limit behavior, viz., as $N \sim n^{2\beta}$ and $N \sim n^{\beta}$. Depending on the value of β the first or the second term may dominate, and the limit behavior of $\widehat{\gamma}_{N,n}(t,s)$ gets more complicated. The following corollary provides this limit without detailing the 'intermediate' situations and also with exception of some particular values of β where *both* terms on the r.h.s. may contribute to the limit. Essentially, the corollary follows by comparing the normalizations in Theorems 3.1 and Proposition 3.1.

Corollary 3.1. *Assume that the mixing distribution satisfies condition* (1.2) *with* $\beta > 0$ *and* $\mathrm{E}|\varepsilon(0)|^{2p} < \infty$ *for some* $p > 1$ *and* $(t,s) \in \mathbb{Z}^2$, $s \neq 0$ *be fixed albeit arbitrary.*

(i) Let $N/n^{2\beta} \to \infty$ *and* $1 < \beta < 3/2$. *Then*

$$N^{1/2} n^{\beta-1} \widehat{\gamma}_{N,n}(t,s) \to_{\mathrm{d}} \sigma_\infty Z,$$

where $Z \sim N(0,1)$ *and* σ_∞ *is the same as in Theorem 3.1 (i).*

(ii) Let $N/n^{2\beta} \to \infty$ *and* $1/2 < \beta < 1$. *Then*

$$N^{1-1/(2\beta)} (\log(N^{1/(2\beta)}/n))^{-1/(2\beta)} \widehat{\gamma}_{N,n}(t,s) \to_{\mathrm{d}} V_{2\beta},$$

where $V_{2\beta}$ *is symmetric* (2β)*-stable r.v. defined in Theorem 3.1 (i).*

(iii) Let $N/n^{2\beta} \to \infty$ *and* $0 < \beta < 1/2$. *Then*

$$N^{2-1/\beta} \widehat{\gamma}_{N,n}(t,s) \to_{\mathrm{d}} -(\bar{V}_{2\beta})^2, \tag{3.13}$$

where $\bar{V}_{2\beta}$ *is a symmetric* (2β)*-stable r.v. defined in Proposition 3.1 (ii).*

(iv) Let $N/n^{2\beta} \to 0$, $N/n^{\beta} \to \infty$ *and* $3/4 < \beta < 3/2$. *Then*

$$N^{1-3/(4\beta)} n^{1/2} \widehat{\gamma}_{N,n}(t,s) \to_{\mathrm{d}} W_{4\beta/3}, \tag{3.14}$$

where $W_{4\beta/3}$ *is a symmetric* $(4\beta/3)$*-stable r.v. with ch.f.* $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta W_{4\beta/3}} = \mathrm{e}^{-(\sigma_0/2^{2\beta/3})|\theta|^{4\beta/3}}$, $\theta \in \mathbb{R}$ *and* σ_0 *is the same constant as in Theorem 3.1 (ii).*

(v) Let $N/n^{2\beta} \to 0$, $1/2 < \beta < 3/4$ *and* $N/n^{2\beta/(4\beta-1)} \to \infty$. *Then the convergence in* (3.14) *holds.*

(vi) Let $N/n^{\beta} \to \infty$, $1/2 < \beta < 3/4$ *and* $N/n^{2\beta/(4\beta-1)} \to 0$. *Then the convergence in* (3.13) *holds.*

(vii) Let $N/n^{2\beta} \to 0$, $N/n^{\beta} \to \infty$ *and* $0 < \beta < 1/2$. *Then the convergence in* (3.13) *holds.*

(viii) Let $N/n^{\beta} \to 0$ *and* $3/4 < \beta < 3/2$. *Then the convergence in* (3.14) *holds.*

(ix) Let $N/n^{\beta} \to 0$, $0 < \beta < 3/4$ *and* $N/n^{2\beta/(5-4\beta)} \to \infty$. *Then*

$$N^{2-2/\beta} \widehat{\gamma}_{N,n}(t,s) \to_{\mathrm{d}} -(\bar{W}_{\beta})^2, \tag{3.15}$$

where $\bar{W}_\beta$ *is a symmetric* β*-stable r.v. defined in Proposition 3.1 (iii).*

(x) Let $0 < \beta < 3/4$ *and* $N/n^{2\beta/(5-4\beta)} \to 0$*. Then the convergence in (3.14) holds.*

(xi) For $3/2 < \beta < 2$*, let* $N/n^\beta \to \lambda_\infty^* \in [0, \infty]$ *and for* $\beta > 2$*, let* $N, n \to \infty$ *in arbitrary way. Then*

$$N^{1/2} n^{1/2} \widehat{\gamma}_{N,n}(t,s) \to_{\mathrm{d}} N(0, \sigma^2), \tag{3.16}$$

where σ^2 *is given as in Theorem 3.2.*

The proof of Theorem 3.1 in cases (i)–(iii) is given Subsections 3.1–3.3. To avoid excessive notation, the discussion is limited to the case $(t,s) = (0,1)$ or the partial sums process $S_{N,n}(\tau) := \sum_{i=1}^N \sum_{t=1}^{\lfloor n\tau \rfloor} X_i(t) X_{i+1}(t)$. Later on we shall extend them to general case (t,s), $s \neq 0$.

Let us give an outline of the proof of Theorem 3.1. Similarly to [20] we use the method of characteristic function combined with 'vertical' Bernstein's blocks, due to the fact that $S_{N,n}$ is not a sum of row-independent summands as in [20]. Write

$$S_{N,n}(\tau) = S_{N,n;q}(\tau) + S^\dagger_{N,n;q}(\tau) + S^\ddagger_{N,n;q}(\tau), \tag{3.17}$$

where the main term

$$S_{N,n;q}(\tau) := \sum_{k=1}^{\tilde{N}_q} Y_{k,n;q}(\tau) \tag{3.18}$$

with

$$Y_{k,n;q}(\tau) = \sum_{(k-1)q < i < kq} \sum_{t=1}^{\lfloor n\tau \rfloor} X_i(t) X_{i+1}(t), \quad 1 \le k \le \tilde{N}_q := \lfloor \frac{N}{q} \rfloor,$$

is a sum of $\tilde{N}_q$ 'large' blocks of size $q-1$ with

$$q \equiv q_{N,n} \to \infty \quad \text{as } N, n \to \infty. \tag{3.19}$$

The convergence rate of $q \in \mathbb{N}$ in (3.19) will be slow enough (e.g. $q = O(\log N)$) and specified later on. The two other terms in the decomposition (3.17),

$$\begin{aligned} S^\dagger_{N,n;q}(\tau) &:= \sum_{k=1}^{\tilde{N}_q} \sum_{t=1}^{\lfloor n\tau \rfloor} X_{kq}(t) X_{kq+1}(t), \\ S^\ddagger_{N,n;q}(\tau) &:= \sum_{q\tilde{N}_q < i \le N} \sum_{t=1}^{\lfloor n\tau \rfloor} X_i(t) X_{i+1}(t), \end{aligned} \tag{3.20}$$

contain respectively $\tilde{N}_q = o(N)$ and $N - q\tilde{N}_q < q = o(N)$ row sums and will be shown to be negligible. More precisely, we show that in each case (i)–(iii) of Theorem 3.1,

$$A_{N,n}^{-1} S_{N,n;q}(\tau) \to_{\mathrm{fdd}} \mathcal{S}_\beta(\tau), \tag{3.21}$$

$$A_{N,n}^{-1} S_{N,n;q}^{\dagger}(\tau) = o_{\mathrm{p}}(1), \quad A_{N,n}^{-1} S_{N,n;q}^{\ddagger}(\tau) = o_{\mathrm{p}}(1), \tag{3.22}$$

where $A_{N,n}$ and $\mathcal{S}_\beta$ denote the normalization and the limit process, respectively, particularly,

$$A_{N,n} := n^2 \begin{cases} \lambda_{N,n}^{\beta}, & \lambda_\infty = \infty, \quad 1 < \beta < 3/2, \\ \lambda_{N,n}(\log \lambda_{N,n})^{1/(2\beta)}, & \lambda_\infty = \infty, \quad 0 < \beta < 1, \\ \lambda_{N,n}^{3/2}, & \lambda_\infty \in [0,\infty), \ 0 < \beta < 3/2. \end{cases} \tag{3.23}$$

Note that *the summands* $Y_{k,n;q}$, $1 \le k \le \tilde{N}_q$ *in* (3.18) *are independent and identically distributed*, and the limit $\mathcal{S}_\beta(\tau)$ is infinitely divisible in cases (i)–(iii) of Theorem 3.1. Hence use of characteristic functions to prove (3.21) is natural. The proofs are limited to one-dimensional convergence at a given $\tau > 0$ since the convergence of general finite-dimensional distributions follows in a similar way. Accordingly, the proof of (3.21) for fixed $\tau > 0$ reduces to

$$\Phi_{N,n;q}(\theta) \to \Phi(\theta) \quad \text{as } N, n \to \infty,\ \lambda_{N,n} \to \lambda_\infty, \quad \forall\, \theta \in \mathbb{R}, \tag{3.24}$$

where

$$\Phi_{N,n;q}(\theta) := \tilde{N}_q \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta A_{N,n}^{-1} Y_{1,n;q}(\tau)} - 1], \quad \Phi(\theta) := \log \mathrm{E} \mathrm{e}^{\mathrm{i}\theta \mathcal{S}_\beta(\tau)}. \tag{3.25}$$

To prove (3.24) write

$$A_{N,n}^{-1} Y_{1,n;q}(\tau) = \sum_{i=1}^{q-1} y_i(\tau), \quad \text{where } y_i(\tau) := A_{N,n}^{-1} \sum_{t=1}^{\lfloor n\tau \rfloor} X_i(t) X_{i+1}(t). \tag{3.26}$$

We use the identity:

$$\prod_{1 \le i < q} (1 + w_i) - 1 = \sum_{1 \le i < q} w_i + \sum_{|D| \ge 2} \prod_{i \in D} w_i, \tag{3.27}$$

where the sum $\sum_{|D|\ge 2}$ is taken over all subsets $D \subset \{1, \ldots, q-1\}$ of cardinality $|D| \ge 2$. Applying (3.27) with $w_i = \mathrm{e}^{\mathrm{i}\theta y_i(\tau)} - 1$ we obtain

$$\Phi_{N,n;q}(\theta) := \tilde{N}_q (q-1)[\mathrm{E}\mathrm{e}^{\mathrm{i}\theta y_1(\tau)} - 1] + \tilde{N}_q \sum_{|D|\ge 2} \mathrm{E} \prod_{i \in D} [\mathrm{e}^{\mathrm{i}\theta y_i(\tau)} - 1]. \tag{3.28}$$

Thus, since $\tilde{N}_q(q-1)/N \to 1$, (3.24) follows from

$$N[\mathrm{E}\mathrm{e}^{\mathrm{i}\theta y_1(\tau)} - 1] \to \Phi(\theta), \tag{3.29}$$

$$N \sum_{|D|\ge 2} \mathrm{E} \prod_{i \in D} [\mathrm{e}^{\mathrm{i}\theta y_i(\tau)} - 1] \to 0. \tag{3.30}$$

Let us explain the main idea of the proof of (3.29). Assuming $\phi(x) = (1-x)^{\beta-1}$ in (1.2) the l.h.s. of (3.29) can be written as

$$N[\mathrm{E}\mathrm{e}^{\mathrm{i}\theta y_1(\tau)} - 1] = N \int_{(0,1]^2} \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta y_1(\tau)} - 1 \big| a_i = 1 - z_i, i = 1,2](z_1 z_2)^{\beta-1} \mathrm{d}z_1 \mathrm{d}z_2$$

$$= \frac{N}{B_{N,n}^{2\beta}} \int_{(0,B_{N,n}]^2} \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta z_{N,n}(\tau;x_1,x_2)} - 1](x_1 x_2)^{\beta-1} \mathrm{d}x_1 \mathrm{d}x_2, \tag{3.31}$$

where

$$z_{N,n}(\tau;x_1,x_2) := \frac{1}{A_{N,n}} \sum_{s_1,s_2\in\mathbb{Z}} \varepsilon_1(s_1)\varepsilon_2(s_2) \sum_{t=1}^{\lfloor n\tau\rfloor} \prod_{i=1}^{2} \big(1 - \frac{x_i}{B_{N,n}}\big)^{t-s_i} \mathbf{1}(t \ge s_i) \tag{3.32}$$

and $B_{N,n} \to \infty$ is a scaling factor of the autoregressive coefficient. In cases (ii) and (iii) of Theorem 3.1 (proof of (3.5) and (3.6)) we choose this scaling factor $B_{N,n} = N^{1/(2\beta)}$ so that $N/B_{N,n}^{2\beta} = 1$ and prove that the integral in (3.31) converges to $\int_{\mathbb{R}_+^2} \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta z(\tau;x_1,x_2)} - 1](x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 = \Phi(\theta)$, where $z(\tau;x_1,x_2)$ is a random process and $\Phi(\theta)$ is the required limit in (3.24). A similar scaling $B_{N,n} = (N\log\lambda_{N,n})^{1/(2\beta)}$ applies in the case $\lambda_\infty = \infty$, $0 < \beta < 1$ (proof of (3.4)) although in this case the factor $N/B_{N,n}^{2\beta} = 1/\log\lambda_{N,n}$ in front of the integral in (3.31) does not trivialize and the proof of the limit in (3.24) is more delicate. On the other hand, in the case of the Gaussian limit (3.3), the choice $B_{N,n} = n$ leads to $N/B_{N,n}^{2\beta} = \lambda_{N,n}^{2\beta} \to \infty$ and (3.31) tends to $-(\theta^2/2)\int_{\mathbb{R}_+^2} \mathrm{E}z^2(\tau;x_1,x_2)(x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 = \Phi(\theta)$ with $z(\tau;x_1,x_2)$ defined in (2.12) as shown in Subsection 3.3 below.

To summarize the above discussion: in each case (i)–(iii) of Theorem 3.1, to prove the limit (3.21) of the main term, it suffices to verify relations (3.29) and (3.30). The proof of the first relation in (3.22) is very similar to (3.21) since $S^{\dagger}_{N,n;q}(\tau)$ is also a sum of i.i.d. r.v.s and the argument of (3.21) applies with small changes. The proof of the second relation in (3.22) seems even simpler. In the proofs we repeatedly use the following inequalities:

$$|\,\mathrm{e}^{\mathrm{i}z} - 1| \le 2 \wedge |z|, \quad |\mathrm{e}^{\mathrm{i}z} - 1 - \mathrm{i}z| \le (2|z|) \wedge (z^2/2), \quad z \in \mathbb{R}. \tag{3.33}$$

3.1. *Proof of Theorem 3.1 (iii): case* $0 < \lambda_\infty < \infty$

Proof of (3.29). For notational brevity, we assume $\lambda_{N,n} = \lambda_\infty = 1$ since the general case as in (3.2) requires unsubstantial changes. Recall from (2.17) that $\Phi(\theta) = \int_{\mathbb{R}_+^2} \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta z(\tau;x_1,x_2)} - 1](x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2$, where $z(\tau;x_1,x_2)$ is the double Itô-Wiener integral in (2.12). Also recall the representation (3.31), (3.32), where $A_{N,n} = n^2$, $B_{N,n} = n$ and $z_{N,n}(\tau;x_1,x_2) = Q_{12}(h_n(\cdot;\tau;x_1,x_2))$ is a quadratic form as in (2.4) with coefficients

$$h_n(s_1,s_2;\tau;x_1,x_2) := \frac{1}{n^2} \sum_{t=1}^{\lfloor n\tau\rfloor} \prod_{i=1}^{2} \big(1 - \frac{x_i}{n}\big)^{t-s_i} \mathbf{1}(t \ge s_i), \quad s_1,s_2 \in \mathbb{Z}. \tag{3.34}$$

By Proposition 2.1, with $\alpha_1 = \alpha_2 = n$, the point-wise convergence

$$\mathrm{E}[\mathrm{e}^{\mathrm{i}\theta z_{N,n}(\tau;x_1,x_2)} - 1] = \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta Q_{12}(h_n(\cdot;\tau;x_1,x_2))} - 1] \to \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta z(\tau;x_1,x_2)} - 1] \tag{3.35}$$

for any fixed $x_1, x_2 \in \mathbb{R}_+$ follows from L_2-convergence of the kernels:

$$\|\widetilde{h}_n(\cdot;\tau;x_1,x_2) - h(\cdot;\tau;x_1,x_2)\| \to 0, \tag{3.36}$$

where

$$\begin{aligned}\widetilde{h}_n(s_1,s_2;\tau;x_1,x_2) &:= nh_n(\lfloor ns_1\rfloor,\lfloor ns_2\rfloor;\tau;x_1,x_2)\\ &= \frac{1}{n}\sum_{t=1}^{\lfloor n\tau\rfloor}\prod_{i=1}^{2}\big(1-\frac{x_i}{n}\big)^{t-\lfloor ns_i\rfloor}\mathbf{1}(t\ge\lfloor ns_i\rfloor)\\ &\to \int_0^\tau \prod_{i=1}^{2}\mathrm{e}^{-x_i(t-s_i)}\mathbf{1}(t>s_i)\mathrm{d}t =: h(s_1,s_2;\tau;x_1,x_2) \end{aligned}\tag{3.37}$$

point-wise for any $x_i > 0$, $s_i \in \mathbb{R}$, $s_i \neq 0$, $i = 1, 2$, $\tau > 0$ fixed. We also use the dominating bound

$$|\widetilde{h}_n(s_1,s_2;\tau;x_1,x_2)| \le Ch(s_1,s_2;2\tau;x_1,x_2), \quad s_1,s_2\in\mathbb{R},\ 0<x_1,x_2<n, \tag{3.38}$$

with $C > 0$ independent of s_i, x_i, $i = 1, 2$ which follows from the definition of $\widetilde{h}_n(\cdot;\tau;x_1,x_2)$ and the inequality $1 - x \le \mathrm{e}^{-x}$, $x > 0$. Since $h(\cdot;2\tau;x_1,x_2) \in L^2(\mathbb{R}^2)$, (3.37), (3.38) and the dominated convergence theorem imply (3.36) and (3.35).

It remains to show the convergence of the corresponding integrals, viz.,

$$\int_{(0,n]^2} \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta z_{N,n}(\tau;x_1,x_2)} - 1](x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 \to \Phi(\theta), \tag{3.39}$$

where $\Phi(\theta) = \int_{\mathbb{R}_+^2} \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta z(\tau;x_1,x_2)} - 1](x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2$. From (3.31) and $\mathrm{E}z_{N,n}(\tau; x_1,x_2) = 0$ we obtain

$$|\mathrm{E}[\mathrm{e}^{\mathrm{i}\theta z_{N,n}(\tau;x_1,x_2)} - 1]| \le C\begin{cases}1, & x_1x_2(x_1+x_2)\le 1,\\ \mathrm{E}z_{N,n}^2(\tau;x_1,x_2), & x_1x_2(x_1+x_2)>1,\end{cases} \tag{3.40}$$

where

$$\begin{aligned}\mathrm{E}z_{N,n}^2(\tau;x_1,x_2) &= \frac{1}{A_{N,n}^2}\mathrm{E}\big[\big(\sum_{t=1}^{\lfloor n\tau\rfloor}Y_{12}(t)\big)^2|a_i = 1-\frac{x_i}{B_{N,n}}, i=1,2\big]\\ &= \frac{1}{n^4}\mathrm{E}\big[\big(\sum_{t=1}^{\lfloor n\tau\rfloor}Y_{12}(t)\big)^2|a_i = 1-\frac{x_i}{n}, i=1,2\big]\\ &\le \frac{C}{n^3\frac{x_1}{n}\frac{x_2}{n}}\min\big\{n,\frac{1}{\frac{x_1+x_2}{n}}\big\} \ = \ \frac{C}{x_1x_2}\min\big\{1,\frac{1}{x_1+x_2}\big\},\end{aligned}\tag{3.41}$$

see (3.32) and the bound in (2.30). In view of inequality (2.9), the dominated convergence theorem applies, proving (3.39) and (3.29).

Proof of (3.30). Choose $q = q_{N,n} = \lfloor \log n \rfloor$. Let $J_q(\theta)$ denote the l.h.s. of (3.30). Using the identity $\sum_{D\subset\{1,\dots,q-1\}:|D|\ge 2}\prod_{i\in D} w_i = \sum_{1\le i<j<q} w_i w_j \prod_{i<k<j}(1+w_k)$ with $w_i = \mathrm{e}^{\mathrm{i}\theta y_i(\tau)} - 1$, see (3.27), we can rewrite $J_q(\theta) = \sum_{1\le i<j<q} T_{ij}(\theta)$, where

$$\begin{aligned} T_{ij}(\theta) := N\mathrm{E}\big[&(\mathrm{e}^{\mathrm{i}\theta y_i(\tau)}-1)(\mathrm{e}^{\mathrm{i}\theta y_j(\tau)}-1)\exp\big\{\mathrm{i}\theta\sum\nolimits_{i<k<j} y_k(\tau)\big\} \\ &\times(\mathbf{1}(a_i<a_{j+1})+\mathbf{1}(a_i>a_{j+1}))\big] = T'_{ij}(\theta)+T''_{ij}(\theta). \end{aligned} \tag{3.42}$$

Since $|J_q(\theta)| \le q^2\max_{1\le i<j<q}|T_{ij}(\theta)| \le (\log n)^2\max_{1\le i<j<q}|T_{ij}(\theta)|$, (3.30) follows from

$$|T_{ij}(\theta)| \le Cn^{-\delta}, \quad \forall\, 1\le i<j, \tag{3.43}$$

with $C,\delta>0$ independent of n. Using $\mathrm{E}[y_i(\tau)|a_k,\varepsilon_j(k),k,j\in\mathbb{Z},j\ne i]=0$ and (3.41) we obtain

$$\begin{aligned} |T'_{ij}(\theta)| &\le CN\mathrm{E}\big[\min\big\{1,\mathrm{E}[y_i^2(\tau)|a_k,k\in\mathbb{Z}]\big\}\mathbf{1}(a_i<a_{j+1})\big] \qquad (3.44)\\ &\le \frac{C}{n^\beta}\int_{(0,n]^3}\min\big\{1,\frac{1}{x_ix_{i+1}(x_i+x_{i+1})}\big\}\mathbf{1}(x_i>x_{j+1}) \\ &\qquad\qquad\times(x_ix_{i+1}x_{j+1})^{\beta-1}\mathrm{d}x_i\mathrm{d}x_{i+1}\mathrm{d}x_{j+1} \\ &= \frac{C}{n^\beta}\int_{(0,n]^2}\min\big\{1,\frac{1}{x_1x_2(x_1+x_2)}\big\}x_1^{2\beta-1}x_2^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 \le \frac{C}{n^\beta}(T'_n+T''_n), \end{aligned}$$

where

$$\begin{aligned} T'_n &:= \int_{0<x_1<x_2<n}\min\big\{1,\frac{1}{x_1x_2^2}\big\}x_1^{2\beta-1}x_2^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 \\ &\le \int_0^1 x_1^{2\beta-1}\mathrm{d}x_1\Big(\int_{x_1}^{x_1^{-1/2}}x_2^{\beta-1}\mathrm{d}x_2 + x_1^{-1}\int_{x_1^{-1/2}}^n x_2^{\beta-3}\mathrm{d}x_2\Big) \\ &\quad+\int_1^n x_1^{2\beta-2}\mathrm{d}x_1\int_{x_1}^n x_2^{\beta-3}\mathrm{d}x_2 \le C\Big(\int_0^1 x_1^{3\beta/2-1}\mathrm{d}x_1+\int_1^n x_1^{3\beta-4}\mathrm{d}x_1\Big) \\ &\le Cn^{3(\beta-1)\vee 0}(1+\mathbf{1}(\beta=1)\log n) \end{aligned}$$

and similarly,

$$\begin{aligned} T''_n &:= \int_{0<x_2<x_1<n}\min\big\{1,\frac{1}{x_1^2x_2}\big\}x_1^{2\beta-1}x_2^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 \\ &= \int_0^1 x_1^{2\beta-1}\mathrm{d}x_1\int_0^{x_1}x_2^{\beta-1}\mathrm{d}x_2 \\ &\quad+\int_1^n x_1^{2\beta-1}\mathrm{d}x_1\Big(\int_0^{x_1^{-2}}x_2^{\beta-1}\mathrm{d}x_2+x_1^{-2}\int_{x_1^{-2}}^{x_1}x_2^{\beta-2}\mathrm{d}x_2\Big) \\ &\le C((\log n)\mathbf{1}(\beta<1)+(\log n)^2\mathbf{1}(\beta=1)+n^{3(\beta-1)}\mathbf{1}(\beta>1)). \end{aligned}$$

Whence, the bound in (3.43) follows for $T'_{ij}(\theta)$ with any $0<\delta<\beta\wedge(3-2\beta)$, for $0<\beta<3/2$. Since $|T''_{ij}(\theta)| \le CN\mathrm{E}[\min\{1,\mathrm{E}[y_j^2(\tau)|a_k,k\in\mathbb{Z}]\}\mathbf{1}(a_{j+1}<a_i)]$ can be symmetrically handled, this proves (3.43) and (3.30).

Proof of (3.22). Since $A_{N,n}^{-1} S_{N,n;q}^{\dagger}(\tau) = \sum_{k=1}^{\tilde{N}_q} y_{kq}(\tau)$ is a sum of $\tilde{N}_q$ i.i.d. r.v.s $y_{kq}(\tau)$, $k = 1, \dots, \tilde{N}_q$, the first relation in (3.22) follows from

$$\tilde{N}_q \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta y_1(\tau)} - 1] \to 0, \quad \forall\, \theta \in \mathbb{R}. \tag{3.45}$$

Clearly, (3.45) is a direct consequence of (3.29) and the fact that $\tilde{N}_q/N \to 0$.

Consider the second relation in (3.22). Let $L_q := N - q\tilde{N}_q$ be the number of summands in $S_{N,n;q}^{\ddagger}(\tau)$. Then $A_{N,n}^{-1} S_{N,n;q}^{\ddagger}(\tau) =_{\mathrm{fdd}} \sum_{i=1}^{L_q} y_i(\tau)$ and

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta A_{N,n}^{-1} S_{N,n;q}^{\ddagger}(\tau)} - 1 = L_q \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta y_1(\tau)} - 1] + \sum_{|D| \ge 2} \mathrm{E} \prod_{i \in D} [\mathrm{e}^{\mathrm{i}\theta y_i(\tau)} - 1], \tag{3.46}$$

where the last sum is taken over all $D \subset \{1, \dots, L_q\}$, $|D| \ge 2$. Since $L_q < q = o(N)$ from (3.29), (3.30) we infer that the r.h.s. of (3.46) vanishes, proving (3.22), and thus completing the proof of Theorem 3.1, case (iii).

3.2. Proof of Theorem 3.1 (ii): case $\lambda_\infty = 0$, or $N = o(n^{2\beta})$

Proof of (3.29). Note the log-ch.f. of the r.h.s. in (3.5) can be written as

$$\begin{aligned} \Phi(\theta) &= \log \mathrm{E}\mathrm{e}^{\mathrm{i}\theta \mathcal{A}^{1/2} B(\tau)} = \log \mathrm{E}\mathrm{e}^{-(\theta^2\tau/2)\mathcal{A}} = -\sigma_0 (\theta^2\tau/2)^{2\beta/3} \\ &= -\psi(1)^2 \int_{\mathbb{R}_+^2} \big(1 - \exp\big\{ - \frac{\theta^2 \tau}{4x_1x_2(x_1+x_2)} \big\}\big) (x_1x_2)^{\beta-1} \mathrm{d}x_1 \mathrm{d}x_2 \end{aligned} \tag{3.47}$$

with $\sigma_0 > 0$ given by the integral

$$\sigma_0 := \frac{\psi(1)^2}{2^{2\beta/3}} \int_{\mathbb{R}_+^2} \big(1 - \exp\big\{ - \frac{1}{x_1x_2(x_1+x_2)} \big\}\big) (x_1x_2)^{\beta-1} \mathrm{d}x_1 \mathrm{d}x_2. \tag{3.48}$$

Relation (3.47) follows by change of variable $x_i \to (\theta^2\tau/4)^{1/3} x_i$, $i = 1, 2$. The convergence of the integral in (3.48) follows from (2.9). The explicit value of σ_0 in (3.48) is given in Proposition 2.2 (v) and computed in the Appendix A. Recall the representation in (3.31), where $B_{N,n} = N^{1/(2\beta)}$, $N/B_{N,n}^{2\beta} = 1$ and

$$\begin{aligned} &z_{N,n}(\tau; x_1, x_2) \\ &\quad = \frac{1}{N^{3/(4\beta)} n^{1/2}} \sum_{s_1, s_2 \in \mathbb{Z}} \varepsilon_1(s_1)\varepsilon_2(s_2) \sum_{t=1}^{\lfloor n\tau \rfloor} \prod_{i=1}^{2} \big(1 - \frac{x_i}{N^{1/(2\beta)}}\big)^{t - s_i} \mathbf{1}(t \ge s_i). \end{aligned} \tag{3.49}$$

Let us prove the (conditional) CLT:

$$z_{N,n}(\tau; x_1, x_2) \to_{\mathrm{fdd}} \frac{B(\tau)}{(2x_1x_2(x_1+x_2))^{1/2}}, \tag{3.50}$$

implying the point-wise convergence

$$\mathrm{E}[1 - \mathrm{e}^{\mathrm{i}\theta z_{N,n}(\tau; x_1, x_2)}] \to 1 - \exp\big\{ - \frac{\theta^2 \tau}{4x_1x_2(x_1+x_2)} \big\} \tag{3.51}$$

of the integrands in (3.31) and (3.48), for any fixed $(x_1, x_2) \in \mathbb{R}_+^2$. As in the rest of the paper, we restrict the proof of (3.50) to one-dimensional convergence, and set $\tau = 1$ for concreteness. Split (3.49) as $z_{N,n}(1; x_1, x_2) = z^+_{N,n}(x_1, x_2) + z^-_{N,n}(x_1, x_2)$, where $z^+_{N,n}(x_1, x_2) := N^{-3/(4\beta)} n^{-1/2} \sum_{s_1,s_2=1}^n \varepsilon_1(s_1)\varepsilon_2(s_2)\cdots$ corresponds to the sum over $1 \le s_1, s_2 \le n$ alone. Thus, we shall prove that

$$z^-_{N,n}(x_1, x_2) = o_{\mathrm{p}}(1) \quad \text{and} \quad z^+_{N,n}(x_1, x_2) \to_{\mathrm{d}} N\big(0, \frac{1}{2x_1 x_2(x_1 + x_2)}\big). \tag{3.52}$$

Arguing as in the proof of (2.30) it is easy to show that

$$\begin{aligned} \mathrm{E}(z^-_{N,n}(x_1, x_2))^2 &\le \frac{C}{N^{3/(2\beta)} n}\big(\frac{x_1 + x_2}{N^{1/(2\beta)}}\big)^{-2}\big\{\big(\frac{x_1}{N^{1/(2\beta)}}\big)^{-2} + \big(\frac{x_2}{N^{1/(2\beta)}}\big)^{-2} \\ &\qquad + \big(\frac{x_1}{N^{1/(2\beta)}}\big)^{-1}\big(\frac{x_2}{N^{1/(2\beta)}}\big)^{-1}\big\} \\ &= C\lambda_{N,n}(x_1 + x_2)^{-2}\{x_1^{-2} + x_2^{-2} + (x_1 x_2)^{-1}\}, \end{aligned}$$

where $\lambda_{N,n} \to 0$, implying the first relation in (3.52). To prove the second relation in (3.52) we use the martingale CLT in Hall and Heyde [11]. (The same approach is used to prove CLT for quadratic forms in [2].) Towards this aim, write $z^+_{N,n}(x_1, x_2)$ as a sum of zero-mean square-integrable martingale difference array

$$z^+_{N,n}(x_1, x_2) = \sum_{k=1}^n Z_k,$$

$$Z_k := \varepsilon_1(k) \sum_{s=1}^{k-1} f(k, s)\, \varepsilon_2(s) + \varepsilon_2(k) \sum_{s=1}^{k-1} f(s, k)\, \varepsilon_1(s) + \varepsilon_1(k)\varepsilon_2(k) f(k, k)$$

w.r.t. the filtration $\mathcal{F}_k$ generated by $\{\varepsilon_i(s), 1 \le s \le k, i = 1, 2\}$, $0 \le k \le n$, where

$$f(s_1, s_2) := \frac{1}{N^{3/(4\beta)} n^{1/2}} \sum_{t=1}^n \prod_{i=1}^2 \big(1 - \frac{x_i}{N^{1/(2\beta)}}\big)^{t - s_i} \mathbf{1}(t \ge s_i), \quad 1 \le s_1, s_2 \le n.$$

Accordingly, the second convergence in (3.52) follows from

$$\sum_{k=1}^n \mathrm{E}[Z_k^2 | \mathcal{F}_{k-1}] \to_{\mathrm{p}} \frac{1}{2x_1 x_2(x_1 + x_2)} \quad \text{and} \quad \sum_{k=1}^n \mathrm{E}[Z_k^2 \mathbf{1}(|Z_k| > \epsilon)] \to 0, \tag{3.53}$$

for any $\epsilon > 0$. Note the conditional variance $v_k^2 := \mathrm{E}[Z_k^2 | \mathcal{F}_{k-1}]$ is equal to

$$v_k^2 = \big(\sum_{s=1}^{k-1} f(k, s)\varepsilon_2(s)\big)^2 + \big(\sum_{s=1}^{k-1} f(s, k)\varepsilon_1(s)\big)^2 + f^2(k, k),$$

where

$$\sum_{k=1}^n \mathrm{E}Z_k^2 = \sum_{k=1}^n \mathrm{E}v_k^2 = \sum_{s_1,s_2=1}^n f^2(s_1, s_2) = \mathrm{E}(z^+_{N,n}(x_1, x_2))^2 \to \frac{1}{2x_1 x_2(x_1 + x_2)} \tag{3.54}$$

is a direct consequence of the asymptotics in (2.28), where $a_i = 1 - x_1/N^{1/(2\beta)}$, $a_j = 1 - x_2/N^{1/(2\beta)}$. Therefore the first relation in (3.53) follows from (3.54) and

$$R_n := \sum_{k=1}^{n} (v_k^2 - \mathrm{E}v_k^2) = o_{\mathrm{p}}(1). \tag{3.55}$$

To show (3.55) we split $R_n = R_n' + R_n''$ into the sum of 'diagonal' and 'off-diagonal' parts, viz.,

$$\begin{aligned} R_n' &:= \sum_{i=1}^{2} \sum_{1 \le s < n} c_i(s)(\varepsilon_i^2(s) - 1), \\ R_n'' &:= \sum_{i=1}^{2} \sum_{1 \le s_1, s_2 < n, s_1 \ne s_2} c_i(s_1, s_2)\varepsilon_i(s_1)\varepsilon_i(s_2), \end{aligned}$$

where

$$\begin{aligned} c_1(s) := \sum_{s < k \le n} f^2(s,k), \quad c_2(s) &:= \sum_{s < k \le n} f^2(k,s), \\ c_1(s_1, s_2) &:= \sum_{s_1 \vee s_2 < k \le n} f(s_1,k) f(s_2,k), \\ c_2(s_1, s_2) &:= \sum_{s_1 \vee s_2 < k \le n} f(k,s_1) f(k,s_2). \end{aligned}$$

Using the elementary bound for $1 \le s_1, s_2 \le n$:

$$\sum_{t=1}^{n} \prod_{i=1}^{2} a_i^{t-s_i} \mathbf{1}(t \ge s_i) \le (a_2^{s_1 - s_2} \mathbf{1}(1 \le s_2 \le s_1) + a_1^{s_2 - s_1} \mathbf{1}(1 \le s_1 \le s_2)) S(a_1, a_2),$$

where $S(a_1, a_2) := \sum_{t=0}^{\infty} (a_1 a_2)^t = (1 - a_1 a_2)^{-1} \le 2(2 - a_1 - a_2)^{-1}$, we obtain

$$|c_i(s)| \le C n^{-1} x_i^{-1} (x_1 + x_2)^{-2}, \quad \sum_{s_1, s_2 = 1}^{n} c_i^2(s_1, s_2) \le C \lambda_{N,n} x_i^{-3} (x_1 + x_2)^{-4} \tag{3.56}$$

for $i = 1, 2$. By (3.56), for $1 < p < 2$ and $x_1, x_2 > 0$ fixed

$$\mathrm{E}|R_n'|^p \le C \sum_{i=1}^{2} \sum_{s=1}^{n-1} |c_i(s)|^p \le C n^{-(p-1)} = o(1), \tag{3.57}$$

$$\mathrm{E}|R_n''|^2 \le \sum_{i=1}^{2} \sum_{s_1, s_2 = 1}^{n} c_i^2(s_1, s_2) \le C \lambda_{N,n} = o(1), \tag{3.58}$$

proving (3.55) and the first relation in (3.53). The proof of the second relation in (3.53) is similar since it reduces to $T_n := \sum_{k=1}^{n} \mathrm{E}[|Z_k|^{2p}] = o(1)$ for the same $1 < p \le 2$, where

$$\mathrm{E}|Z_k|^{2p} \le C\big(\mathrm{E}|\sum_{s=1}^{k-1} f(k,s)\,\varepsilon_2(s)|^{2p} + \mathrm{E}|\sum_{s=1}^{k-1} f(s,k)\,\varepsilon_1(s)|^{2p} + |f(k,k)|^{2p}\big)$$

$$\le C\big((\sum_{s=1}^{k-1} f^2(k,s))^p + (\sum_{s=1}^{k-1} f^2(s,k))^p + |f(k,k)|^{2p}\big)$$

by Rosenthal's inequality, see e.g. ([7], Lemma 2.5.2), and the sum $T_n = O(n^{-(p-1)}) = o(1)$ similarly to (3.57). This proves (3.53), (3.52), and the pointwise convergence in (3.51).

Now we return to the proof of (3.29), whose both sides are written as respective integrals (3.31) and (3.47). Due to the convergence of the integrands (see (3.51)), it suffices to justify the passage to the limit using a dominated convergence theorem argument. The dominating function independent of N, n is obtained from (3.31) and $\mathrm{E} z_{N,n}(\tau; x_1, x_2) = 0$ and from (3.40), (3.41), (2.9) similarly as in the case $\lambda_\infty \in (0,\infty)$ above. This proves (3.29).

Proofs of (3.30) *and* (3.22) are completely analogous to those in the case $\lambda_\infty \in (0,\infty)$ except that we now choose $q = \lfloor \log N \rfloor$ and replace n in (3.43) and elsewhere in the proof of (3.30) and (3.22), case $\lambda_\infty \in (0,\infty)$, by $N^{1/(2\beta)}$. This ends the proof of Theorem 3.1, case (ii).

3.3. *Proof of Theorem 3.1 (i): case $\lambda_\infty = \infty$, or $n = o(N^{1/(2\beta)})$*

Case $1 < \beta < 3/2$. *Proof of* (3.29). In this case, $\Phi(\theta) := -\sigma_\infty^2 \tau^{2(2-\beta)}\theta^2/2$, $B_{N,n} = n$ and $A_{N,n} = n^2 \lambda_{N,n}^\beta = n^{2-\beta} N^{1/2}$. Rewrite the l.h.s. of (3.29) as

$$N[\mathrm{E e}^{\mathrm{i}\theta y_1(\tau)} - 1] = \int_{[0,n)^2} \mathrm{E}\Lambda_{N,n}(\theta;\tau;x_1,x_2)(x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2, \tag{3.59}$$

where

$$\Lambda_{N,n}(\theta;\tau;x_1,x_2) := \lambda_{N,n}^{2\beta}\big[\mathrm{e}^{\mathrm{i}\theta\lambda_{N,n}^{-\beta}\tilde z_{N,n}(\tau;x_1,x_2)} - 1 - \mathrm{i}\theta\lambda_{N,n}^{-\beta}\tilde z_{N,n}(\tau;x_1,x_2)\big]$$

and where $\tilde z_{N,n}(\tau; x_1, x_2)$ is defined as in (3.32) with $A_{N,n}$ replaced by $\tilde A_{N,n} := n^2 = A_{N,n}/\lambda_{N,n}^\beta$. As shown in the proof of Case (iii) (the 'intermediate limit'), for any $x_1, x_2 > 0$

$$\tilde z_{N,n}(\tau;x_1,x_2) \to_{\mathrm{d}} z(\tau;x_1,x_2) \quad \text{and} \quad \mathrm{E}\tilde z_{N,n}^2(\tau;x_1,x_2) \to \mathrm{E}z^2(\tau;x_1,x_2), \tag{3.60}$$

see (3.35), where $z(\tau; x_1, x_2)$ is defined in (2.12) and the last expectation in (3.60) is given in (A.2). Then using Skorohod's representation we extend (3.60) to

$$\tilde z_{N,n}(\tau;x_1,x_2) \to z(\tau;x_1,x_2) \quad \text{a.s.}$$

implying also

$$\Lambda_{N,n}(\theta;\tau;x_1,x_2) \to -(\theta^2/2)z^2(\tau;x_1,x_2) \quad \text{a.s.}$$

Since $|\Lambda_{N,n}(\theta;\tau;x_1,x_2)| \le C\tilde z_{N,n}^2(\tau;x_1,x_2)$ and (3.60) holds, by Pratt's lemma we obtain

$$\mathrm{E}\Lambda_{N,n}(\theta;\tau;x_1,x_2) \to -(\theta^2/2)\mathrm{E}z^2(\tau;x_1,x_2), \quad \forall\,(x_1,x_2) \in \mathbb{R}_+^2. \tag{3.61}$$

Relation (3.29) follows from (3.59), (3.61) and the dominated convergence theorem, using the dominating bound

$$|\mathrm{E}\Lambda_{N,n}(\theta;\tau;x_1,x_2)| \le C\mathrm{E}\tilde z^2_{N,n}(\tau;x_1,x_2) \le \frac{C}{x_1x_2}\min\big\{1,\frac{1}{x_1+x_2}\big\}, \qquad (3.62)$$

see (3.41), and integrability of the dominating function, see (2.10).

Proof of (3.30) is similar to that in case (iii) $0<\lambda_\infty<\infty$ above with $q=\lfloor\log n\rfloor$. It suffices to check the bound (3.43) for $T_{ij}(\theta)=T'_{ij}(\theta)+T''_{ij}(\theta)$ given in (3.42). By the same argument as in (3.44), we obtain $|T'_{ij}(\theta)| \le CN\mathrm{E}[y_i^2(\tau)\mathbf{1}(a_i<a_{j+1})]$. The bound on $\mathrm{E}\tilde z^2_{N,n}(\tau;x_1,x_2)$ in (3.62) further implies

$$\begin{aligned}|T'_{ij}(\theta)| &\le \frac{C}{n^\beta}\int_{(0,n]^3}\frac{1}{x_1x_2}\min\big\{1,\frac{1}{x_1+x_2}\big\}(x_1x_2x_3)^{\beta-1}\mathbf{1}(x_3<x_1)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\ &\le \frac{C}{n^\beta}(T'_n+T''_n),\end{aligned}$$

where

$$\begin{aligned}T'_n &:= \int_0^n \min\big\{1,\frac{1}{x_1}\big\}x_1^{2\beta-2}\mathrm{d}x_1\int_0^{x_1}x_2^{\beta-2}\mathrm{d}x_2\\ &= C\Big(\int_0^1 x_1^{3\beta-3}\mathrm{d}x_1+\int_1^n x_1^{3\beta-4}\mathrm{d}x_1\Big)\le Cn^{3\beta-3}\end{aligned}$$

and

$$\begin{aligned}T''_n &:= \int_0^n \min\big\{1,\frac{1}{x_2}\big\}x_2^{\beta-2}\mathrm{d}x_2\int_0^{x_2}x_1^{2\beta-2}\mathrm{d}x_1\\ &= C\Big(\int_0^1 x_2^{3\beta-3}\mathrm{d}x_2+\int_1^n x_2^{3\beta-4}\mathrm{d}x_2\Big)\le Cn^{3\beta-3}.\end{aligned}$$

Then $|T''_{ij}(\theta)| \le CN\mathrm{E}[y_j^2(\tau)\mathbf{1}(a_i>a_{j+1})]$ can be handled in the same way. Whence, the bound in (3.43) follows with any $0<\delta<3-2\beta$, for $1<\beta<3/2$. This proves (3.30). Proof of (3.22) using $\tilde N_q/N\to 0$ and $L_q=N-q\tilde N_q<q=o(N)$ is completely analogous to that in case (iii) $0<\lambda_\infty<\infty$. This completes the proof of Theorem 3.1, case (i) for $1<\beta<3/2$.

Case $0<\beta<1$. *Proof of* (3.29). In the rest of this proof, write $\lambda\equiv\lambda_{N,n}=N^{1/(2\beta)}/n\to\infty$ for brevity. Also denote $\lambda':=\lambda(\log\lambda)^{1/(2\beta)}$, $\log\lambda'/\log\lambda\to 1$. Let $B_{N,n}:=\lambda' n$, then

$$z_{N,n}(\tau;x_1,x_2) := \frac{1}{\lambda' n^2}\sum_{s_1,s_2\in\mathbb{Z}}\varepsilon_1(s_1)\varepsilon_2(s_2)\sum_{t=1}^{\lfloor n\tau\rfloor}\prod_{i=1}^{2}\big(1-\frac{x_i}{\lambda' n}\big)^{t-s_i}\mathbf{1}(t\ge s_i). \qquad (3.63)$$

Split the r.h.s. of (3.29) as follows:

$$
\begin{aligned}
&N\mathrm{E}[\mathrm{e}^{\mathrm{i}\theta y_1(\tau)} - 1] \\
&\quad = \frac{1}{\log\lambda}\int_{(0,\lambda' n]^2} \big(\mathbf{1}(1 < x_1 + x_2 < \lambda) + \mathbf{1}(x_1 + x_2 > \lambda) + \mathbf{1}(x_1 + x_2 < 1)\big) \\
&\qquad\qquad\qquad \times \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta z_{N,n}(\tau; x_1, x_2)} - 1](x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 \ =: \sum_{i=1}^{3} L_i.
\end{aligned}
$$

Here, L_1 is the main term and $L_i, i = 2, 3$ are remainders. Indeed, $|L_3| = O(1/\log\lambda) = o(1)$. To estimate L_2 we need the bound

$$
\mathrm{E}z^2_{N,n}(\tau; x_1, x_2) \le \frac{C}{x_1x_2}\min\big\{1, \frac{\lambda'}{x_1 + x_2}\big\}, \tag{3.64}
$$

which follows from (2.30) similarly to (3.41). Using (3.64) we obtain

$$
\begin{aligned}
|L_2| &\le \frac{C}{\log\lambda}\int_{x_1+x_2>\lambda}\min\big\{1, \frac{\lambda'}{x_1x_2(x_1 + x_2)}\big\}(x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 \\
&= \frac{C}{\log\lambda}(J'_\lambda + J''_\lambda), \qquad (3.65)
\end{aligned}
$$

where, by change of variables: $x_1 + x_2 = y$, $x_1 = yz$,

$$
\begin{aligned}
J'_\lambda &:= \int_{x_1+x_2>\lambda}\mathbf{1}(x_1x_2(x_1 + x_2) < \lambda')(x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 \\
&= \int_\lambda^\infty\int_0^1 \mathbf{1}(y^3z(1 - z) < \lambda')y^{2\beta-1}(z(1 - z))^{\beta-1}\mathrm{d}z\mathrm{d}y \\
&\le C\int_\lambda^\infty y^{2\beta-1}\mathrm{d}y\int_0^{1/2} z^{\beta-1}\mathbf{1}(y^3z < 2\lambda')\mathrm{d}z \\
&\le C(\lambda')^\beta\int_\lambda^\infty y^{-\beta-1}\mathrm{d}y = C(\log\lambda)^{1/2}
\end{aligned}
$$

since $0 < \beta < 1$. Similarly,

$$
\begin{aligned}
J''_\lambda &:= \lambda'\int_{x_1+x_2>\lambda}\mathbf{1}(x_1x_2(x_1 + x_2) > \lambda')(x_1 + x_2)^{-1}(x_1x_2)^{\beta-2}\mathrm{d}x_1\mathrm{d}x_2 \\
&\le C\lambda'\int_\lambda^\infty y^{2\beta-4}\mathrm{d}y\int_0^{1/2} z^{\beta-2}\mathbf{1}(y^3z > \lambda')\mathrm{d}z \le C(\log\lambda)^{1/2}.
\end{aligned}
$$

This proves $|L_2| = O(1/\log\lambda) = o(1)$.

Consider the main term L_1. Although $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta z_{N,n}(\tau;x_1,x_2)}$ and hence the integrand in L_1 point-wise converge for any $(x_1, x_2) \in \mathbb{R}^2_+$, see below, this fact is not very useful since the contribution to the limit of L_1 from bounded x_i's is negligible due to the presence of the factor $1/\log\lambda \to 0$ in front of this integral. It turns out that the main (non-negligible) contribution to this integral comes from unbounded x_1, x_2 with $x_1/x_2 + x_2/x_1 \to \infty$ and $x_1x_2 \to z \in \mathbb{R}_+$. To see this, by change of variables $y = x_1 + x_2$, $x_1 = yw$ and then $w = z/y^2$ we rewrite

$$
L_1 = \frac{1}{\log\lambda}\int_1^\lambda V_{N,n}(\theta; y)\frac{\mathrm{d}y}{y}, \tag{3.66}
$$

where

$$V_{N,n}(\theta;y) := 2\int_0^{y^2/2} \mathrm{E}\big[\exp\big\{\mathrm{i}\theta z_{N,n}\big(\tau;\frac{z}{y},y\big(1-\frac{z}{y^2}\big)\big)\big\}-1\big]z^{\beta-1}\big(1-\frac{z}{y^2}\big)^{\beta-1}\mathrm{d}z. \tag{3.67}$$

In view of $L_i = o(1)$, $i = 2, 3$ relation (3.29) follows from representation (3.66) and the existence of the limit:

$$\lim_{y\to\infty, y=O(\lambda)} V_{N,n}(\theta;y) = V(\theta) := -k_\infty|\theta|^{2\theta}\tau^{2\beta}, \tag{3.68}$$

where the constant $k_\infty > 0$ is defined below in (3.71). More precisely, (3.68) says that for any $\epsilon > 0$ there exists $K > 0$ such that for any $N, n, y \geq K$ satisfying $y \leq \lambda$, $\lambda \geq K$

$$|V_{N,n}(\theta;y) - V(\theta)| < \epsilon. \tag{3.69}$$

To show that (3.69) implies $L_1 \to V(\theta)$ it suffices to split $L_1 - V(\theta) = (\log\lambda)^{-1} \int_K^\lambda (V_{N,n}(\theta;y) - V(\theta))\frac{\mathrm{d}y}{y} + (\log\lambda)^{-1}\int_1^K (V_{N,n}(\theta;y) - V(\theta))\frac{\mathrm{d}y}{y}$ and use (3.69) together with the fact that $|V_{N,n}(\theta;y)| \leq C$ is bounded uniformly in N, n, y.

To prove (3.69), rewrite $V(\theta)$ of (3.68) as the integral

$$\begin{aligned} V(\theta) &= 2\int_0^\infty z^{\beta-1}\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta\tau Z_1 Z_2/(2\sqrt{z})} - 1)\mathrm{d}z \\ &= -2\mathrm{E}\int_0^\infty z^{\beta-1}(1-\mathrm{e}^{-\theta^2\tau^2 Z_1^2/(8z)})\mathrm{d}z = -k_\infty|\theta|^{2\beta}\tau^{2\beta} \end{aligned} \tag{3.70}$$

with $Z_1, Z_2 \sim N(0,1)$ independent normals and

$$\begin{aligned} k_\infty &= 2\mathrm{E}\int_0^\infty z^{\beta-1}(1-\mathrm{e}^{-Z_1^2/(8z)})\mathrm{d}z = 2^{1-3\beta}\mathrm{E}|Z_1|^{2\beta}\int_0^\infty z^{\beta-1}(1-\mathrm{e}^{-1/z})\mathrm{d}z \\ &= 2^{1-2\beta}\Gamma(\beta+1/2)\Gamma(1-\beta)/(\sqrt{\pi}\beta). \end{aligned} \tag{3.71}$$

Let

$$\begin{aligned} \Lambda_{N,n}(z;y) &:= \mathrm{E}\big[\exp\big\{\mathrm{i}\theta z_{N,n}\big(\tau;\frac{z}{y},y\big(1-\frac{z}{y^2}\big)\big)\big\}-1\big], \\ \Lambda(z) &:= \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta\tau Z_1 Z_2/(2\sqrt{z})} - 1], \end{aligned}$$

denote the corresponding expectations in (3.67), (3.70). Clearly, (3.69) follows from

$$\lim_{y\to\infty, y=O(\lambda)} \Lambda_{N,n}(z;y) = \Lambda(z), \quad \forall\, z > 0, \tag{3.72}$$

and

$$|\Lambda_{N,n}(z;y)| \leq C(1\wedge(1/z)), \quad \forall\, 0 < y < \lambda,\ 0 < z < y^2/2. \tag{3.73}$$

The dominating bound in (3.73) is a consequence of (3.64). To show (3.72) use Proposition 2.1 by writing $z_{N,n}(\tau; z/y, y')$, $y' := y(1 - z/y^2)$ in (3.67) as the quadratic form: $z_{N,n}(\tau; z/y, y') = Q_{12}(h_{\alpha_1,\alpha_2}(\cdot; \tau; z))$ with

$$h_{\alpha_1,\alpha_2}(s_1, s_2; \tau; z) := \sqrt{\frac{y}{zy'}} \frac{1}{\sqrt{\alpha_1\alpha_2}} \frac{1}{n} \sum_{t=1}^{\lfloor n\tau \rfloor} \prod_{i=1}^{2} \big(1 - \frac{1}{\alpha_i}\big)^{t-s_i} \mathbf{1}(t \ge s_i), \ \ s_1, s_2 \in \mathbb{Z}, \tag{3.74}$$
$$\alpha_1 := \lambda' ny/z, \quad \alpha_2 := \lambda' n/y'.$$

If

$$n, \alpha_1, \alpha_2, y, y' \to \infty \ \text{ so that } \ y/y' \to 1 \ \text{ and } \ n = o(\alpha_i),\ i = 1, 2, \tag{3.75}$$

then

$$\begin{aligned}\widetilde{h}^{(\alpha_1,\alpha_2)}_{\alpha_1,\alpha_2}(s_1, s_2; \tau; z) &:= \sqrt{\alpha_1\alpha_2} h_{\alpha_1,\alpha_2}(\lfloor \alpha_1 s_1 \rfloor, \lfloor \alpha_2 s_2 \rfloor; \tau; z)\\ &= \sqrt{\frac{y}{zy'}} \frac{1}{n} \sum_{t=1}^{\lfloor n\tau \rfloor} \prod_{i=1}^{2} \big(1 - \frac{1}{\alpha_i}\big)^{t-\lfloor \alpha_i s_i \rfloor} \mathbf{1}(t \ge \lfloor \alpha_i s_i \rfloor)\\ &\to \frac{\tau}{\sqrt{z}} \prod_{i=1}^{2} \mathrm{e}^{s_i} \mathbf{1}(s_i < 0) =: h(s_1, s_2; \tau; z)\end{aligned} \tag{3.76}$$

point-wise for any $\tau > 0$, $z > 0$, $s_i \in \mathbb{R}$, $s_i \neq 0$, $i = 1, 2$ fixed. Moreover, under the same conditions (3.75), $\|\widetilde{h}^{(\alpha_1,\alpha_2)}_{\alpha_1,\alpha_2}(\cdot; \tau; z) - h(\cdot; \tau; z)\| \to 0$, implying the convergence $Q_{12}(h_{\alpha_1,\alpha_2}(\cdot; \tau; z)) \to_{\mathrm{d}} I_{12}(h(\cdot; \tau; z)) =_{\mathrm{d}} \tau Z_1 Z_2/(2\sqrt{z})$, $Z_i \sim N(0, 1)$, $i = 1, 2$ by Proposition 2.1. Conditions on n, y, y', λ' in (3.75) are obviously satisfied due to $y, y' = O(\lambda) = o(\lambda')$. This proves (3.72) and (3.68), thereby completing the proof of of (3.29).

Proof of (3.30). For $T_{ij}(\theta)$ defined by (3.42) let us prove (3.43). Denote $N'_\lambda := (N \log \lambda)^{1/(2\beta)}$. Similarly to (3.44) we have that

$$\begin{aligned}|T_{ij}(\theta)| \le \frac{C}{N^{1/2}(\log \lambda)^{3/2}} \int_{(0,N'_\lambda]^3} &\min\{1, \mathrm{E} z^2_{N,n}(\tau; x_1, x_2)\} \mathbf{1}(x_3 < x_1)\\ &\times (x_1x_2x_3)^{\beta-1} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3\end{aligned} \tag{3.77}$$

with $z_{N,n}(\tau; x_1, x_2)$ defined by (3.63). Whence using (3.64) similarly as in the proof of case (i) we obtain

$$\begin{aligned}|T_{ij}(\theta)| \le \frac{C}{N^{1/2}(\log \lambda)^{3/2}} \int_{(0,N'_\lambda]^2} &\min\big\{1, \frac{1}{x_1x_2} \min\big\{1, \frac{\lambda'}{x_1 + x_2}\big\}\big\}\\ &\times x_1^{2\beta-1} x_2^{\beta-1} \mathrm{d}x_1 \mathrm{d}x_2 = \frac{C}{N^{1/2}(\log \lambda)^{3/2}} \sum_{i=1}^{3} T_{\lambda,i},\end{aligned}$$

where

$$T_{\lambda,1} := \int_{(0,N'_\lambda]^2} \mathbf{1}(x_1 + x_2 < \lambda') \min\{1, \frac{1}{x_1x_2}\} x_1^{2\beta-1} x_2^{\beta-1} \mathrm{d}x_1 \mathrm{d}x_2,$$
$$T_{\lambda,2} := \int_{(0,N'_\lambda]^2} \mathbf{1}(x_1x_2(x_1 + x_2) < \lambda', x_1 + x_2 > \lambda') x_1^{2\beta-1} x_2^{\beta-1} \mathrm{d}x_1 \mathrm{d}x_2,$$
$$T_{\lambda,3} := \int_{(0,N'_\lambda]^2} \mathbf{1}(x_1x_2(x_1 + x_2) > \lambda', x_1 + x_2 > \lambda') \frac{\lambda'}{x_1 + x_2} x_1^{2\beta-2} x_2^{\beta-2} \mathrm{d}x_1 \mathrm{d}x_2.$$

By changing variables x_1, x_2 as in (3.66)–(3.67) we get $T_{\lambda,1} \le C \int_0^{\lambda'} y^{\beta-1} \mathrm{d}y \le C(\lambda')^\beta$. Also, similarly to the estimation of J'_λ, J''_λ, following (3.65) we obtain $T_{\lambda,2} + T_{\lambda,3} \le C(\lambda')^\beta \int_{\lambda'}^{2N'_\lambda} y^{-1} \mathrm{d}y \le C(\lambda')^\beta \log(N'_\lambda/\lambda')$. Hence, we conclude that

$$|T_{ij}(\theta)| \le \frac{C(\lambda')^\beta \log(N'_\lambda/\lambda')}{N^{1/2}(\log\lambda)^{3/2}} \le \frac{C\log n}{n^\beta \log\lambda},$$

proving (3.43) with any $0 < \delta < \beta$. This proves (3.30). We omit the proof of (3.22) which is completely similar to that in case (iii) and elsewhere. This completes the proof of Theorem 3.1 for $(t,s) = (0,1)$.

Proof of Theorem 3.1 in the general case $(t,s) \in \mathbb{Z}^2$, $s \ge 1$. Similarly to (3.17) we decompose $S^{t,s}_{N,n}(\tau)$ in (3.1) as

$$S^{t,s}_{N,n}(\tau) = S^{t,s}_{N,n;q}(\tau) + S^{t,s;\dagger}_{N,n;q}(\tau) + S^{t,s;\ddagger}_{N,n;q}(\tau), \tag{3.78}$$

where the main term

$$S^{t,s}_{N,n;q}(\tau) := \sum_{k=1}^{\tilde N_q} Y^{t,s}_{k,n;q}(\tau), \quad Y^{t,s}_{k,n;q}(\tau) := \sum_{(k-1)q<i\le kq-s} \sum_{u=1}^{\lfloor n\tau\rfloor} X_i(u)X_{i+s}(u+t) \tag{3.79}$$

is a sum of independent $\tilde N_q = \lfloor N/q\rfloor$ blocks of size $q - s = q_{N,n} - s \to \infty$, and

$$S^{t,s;\dagger}_{N,n;q}(\tau) := \sum_{k=1}^{\tilde N_q} \sum_{kq-s<i\le kq} \sum_{u=1}^{\lfloor n\tau\rfloor} X_i(u)X_{i+s}(u+t),$$
$$S^{t,s;\ddagger}_{N,n;q}(\tau) := \sum_{q\tilde N_q<i\le N} \sum_{u=1}^{\lfloor n\tau\rfloor} X_i(u)X_{i+s}(u+t)$$

are remainder terms. The proof of (3.29)–(3.30) for $A^{-1}_{N,n} Y^{t,s}_{1,n;q}(\tau) = \sum_{i=1}^{q-s} y^{t,s}_i(\tau)$, $y^{t,s}_i(\tau) := A^{-1}_{N,n} \sum_{u=1}^{\lfloor n\tau\rfloor} X_i(u)X_{i+s}(u+t)$ is completely analogous since the distribution of $y^{t,s}_i(\tau)$ does not depend on t and $s \ne 0$. □

3.4. Proof of Theorem 3.2

The proof uses the following result of [23]:

Lemma 3.1 ([23], Lemma 7.1)**.** *Let $\{\xi_{ni},\, 1 \le i \le N_n\}$, $n \ge 1$, be a triangular array of m-dependent r.v.s with zero mean and finite variance. Assume that: (L1) ξ_{ni}, $1 \le i \le N_n$ are identically distributed for any $n \ge 1$, (L2) $\xi_{n1} \to_{\mathrm{d}} \xi$, $\mathrm{E}\xi_{n1}^2 \to \mathrm{E}\xi^2 < \infty$ for some r.v. ξ and (L3) $\mathrm{var}(\sum_{i=1}^{N_n} \xi_{ni}) \sim \sigma^2 N_n$, $\sigma^2 > 0$. Then $N_n^{-1/2} \sum_{i=1}^{N_n} \xi_{ni} \to_{\mathrm{d}} N(0, \sigma^2)$.*

For notational simplicity, we consider only one-dimensional convergence at $\tau > 0$. Let $(Nn)^{-1/2} S_{N,n}^{t,s}(\tau) = N^{-1/2} \sum_{i=1}^{N} \xi_{ni}$, where $\xi_{ni} := n^{-1/2} \times \sum_{u=1}^{\lfloor n\tau \rfloor} X_i(u) X_{i+s}(u+t)$, $1 \le i \le N$ are $|s|$-dependent, identically distributed r.v.s with zero mean and finite variance. Since ξ_{ni}, $1 \le i \le N$ are uncorrelated, it follows that $\mathrm{E}(\sum_{i=1}^{N} \xi_{ni})^2 = N\mathrm{E}\xi_{n1}^2$, where $\xi_{n1} =_{\mathrm{d}} \xi_n := n^{-1/2} \sum_{u=1}^{\lfloor n\tau \rfloor} X_1(u) X_2(u)$. Proposition 2.4 implies $\mathrm{E}[\xi_n^2 | a_1, a_2] \sim \tau A_{12}$, and so $\mathrm{E}\xi_n^2 \sim \tau\sigma^2$, where $\sigma^2 := \mathrm{E}A_{12} < \infty$. It remains to show that $\xi_n \to_{\mathrm{d}} \sqrt{A_{12}} B(\tau)$, where A_{12} is independent of $B(\tau)$. This follows from the martingale CLT similarly to (3.50). By the lemma above, we conclude that $(Nn)^{-1/2} S_{Nn}^{t,s}(\tau) \to_{\mathrm{d}} \sigma B(\tau)$. Theorem 3.2 is proved. $\square$

4. Asymptotic distribution of temporal (iso-sectional) sample covariances

The limit distribution of iso-sectional sample covariances $\widehat{\gamma}_{N,n}(t, 0)$ in (1.5) and the corresponding partial sums process $S_{N,n}^{t,0}(\tau)$ of (3.1) is obtained similarly as in the cross-sectional case, with certain differences which are discussed below. Since the conditional expectation $\mathrm{E}[S_{N,n}^{t,0}(\tau) | a_1, \cdots, a_N] =: T_{N,n}^{t,0}(\tau) \ne 0$, a natural decomposition is

$$S_{N,n}^{t,0}(\tau) = \widetilde{S}_{N,n}^{t,0}(\tau) + T_{N,n}^{t,0}(\tau), \tag{4.1}$$

where $\widetilde{S}_{N,n}^{t,0}(\tau) := S_{N,n}^{t,0}(\tau) - T_{N,n}^{t,0}(\tau)$ is the conditionally centered term with $\mathrm{E}[\widetilde{S}_{N,n}^{t,0}(\tau) | a_1, \cdots, a_N] = 0$, and

$$T_{N,n}^{t,0}(\tau) := \lfloor n\tau \rfloor \sum_{i=1}^{N} a_i^t / (1 - a_i^2), \quad t \ge 0, \tag{4.2}$$

is proportional to a sum of i.i.d. r.v.s $a_i^t/(1 - a_i^2)$, $1 \le i \le N$ with regularly decaying tail distribution function

$$\mathrm{P}\big(a^t/(1-a^2) > x\big) \sim \mathrm{P}(a > 1 - 1/(2x)) \sim c_a x^{-\beta}, \quad x \to \infty, \quad c_a := \psi(1)/(2^\beta \beta),$$

see condition (1.2). Accordingly, the limit distribution of appropriately normalized and centered term $T_{N,n}^{t,0}(\tau)$ does not depend on t and can be found from the classical CLT and turns out to be a $(\beta \wedge 2)$-stable line, under normalization $nN^{1/(\beta\wedge 2)}$ $(\beta \neq 2)$. The other term, $\widetilde{S}^{t,0}_{N,n}(\tau)$, in (4.1), is a sum of mutually independent partial sums processes $Y^{t,0}_{i,n}(\tau) := \sum_{u=1}^{\lfloor n\tau\rfloor}(X_i(u)X_i(u+t) - \mathrm{E}[X_i(u)X_i(u+t)|a_i])$, $1 \le i \le N$ with conditional variance

$$\begin{aligned}
\mathrm{Var}[Y^{t,0}_{i,n}(1)|a_i] &\sim nA^{t,0}_{ii}, \quad n \to \infty, \quad \text{where}\\
A^{t,0}_{ii} &:= \frac{1+a_i^2}{1-a_i^2}\Big(\frac{1+a_i^{2|t|}}{(1-a_i^2)^2} + \frac{a_i^{2|t|}(2|t| + \mathrm{cum}_4)}{1-a_i^4}\Big).
\end{aligned}$$

The proof of the last fact follows similarly to that of (2.29) and is omitted. As $a_i \uparrow 1$, $A^{t,0}_{ii} \sim 1/(2(1-a_i)^3)$ and the limit distribution of $\widetilde{S}^{t,0}_{N,n}(\tau)$ can be shown to exhibit a trichotomy on the interval $0 < \beta < 3$ depending on the limit λ^*_∞ in (4.3). It turns out that for $\beta > 2$ the asymptotically Gaussian term $T^{t,0}_{N,n}(\tau)$ dominates $\widetilde{S}^{t,0}_{N,n}(\tau)$ in all cases of λ^*_∞, while in the interval $0 < \beta < 2$ $T^{t,0}_{N,n}(\tau)$ and $\widetilde{S}^{t,0}_{N,n}(\tau)$ have the same convergence rate. Somewhat surprisingly, the limit distribution of $S^{t,0}_{N,n}(\tau)$ is a β-stable line in both cases $\lambda^*_\infty = \infty$ and $\lambda^*_\infty = 0$ with different scale parameters of the random slope coefficient of this line.

Rigorous description of the above limit results is given in the following Theorems 4.1 and 4.2. The proofs of these theorems are similar and actually simpler than the corresponding Theorems 3.1 and 3.2 dealing with non-horizontal sample covariances, due to the fact that $S^{t,0}_{N,n}(\tau)$ is a sum of row-independent summands contrary to $S^{t,s}_{N,n}(\tau)$, $s \neq 0$. Because of this, we omit some details of the proof of Theorems 4.1 and 4.2. We also omit the more delicate cases $\beta = 1$ and $\beta = 2$ where the limit results may require a change of normalization or additional centering.

Theorem 4.1. *Let the mixing distribution satisfy condition* (1.2) *with* $0 < \beta < 2$, $\beta \neq 1$. *Let* $N, n \to \infty$ *so that*

$$\lambda^*_{N,n} := \frac{N^{1/\beta}}{n} \to \lambda^*_\infty \in [0,\infty]. \tag{4.3}$$

In addition, assume $\mathrm{E}\varepsilon^4(0) < \infty$. *Then the following statements (i)–(iii) hold for* $S^{t,0}_{N,n}(\tau)$, $t \in \mathbb{Z}$ *in* (3.1) *depending on* λ^*_∞ *in* (4.3).

(i) Let $\lambda^*_\infty = \infty$. *Then*

$$n^{-1}N^{-1/\beta}\big(S^{t,0}_{N,n}(\tau) - \mathrm{E}S^{t,0}_{N,n}(\tau)\mathbf{1}(1<\beta<2)\big) \to_{\mathrm{fdd}} \tau V^*_\beta, \tag{4.4}$$

where V^*_β *is a completely asymmetric* β*-stable r.v. with ch.f. in* (4.7) *below.*

(ii) Let $\lambda^*_\infty = 0$. *Then*

$$n^{-1}N^{-1/\beta}\big(S^{t,0}_{N,n}(\tau) - \mathrm{E}S^{t,0}_{N,n}(\tau)\mathbf{1}(1<\beta<2)\big) \to_{\mathrm{fdd}} \tau V^+_\beta, \tag{4.5}$$

where V^+_β *is a completely asymmetric* β*-stable r.v. with ch.f. in* (4.8) *below.*

(iii) Let $0 < \lambda^*_\infty < \infty$*. Then*

$$n^{-1}N^{-1/\beta}\big(S^{t,0}_{N,n}(\tau) - \mathrm{E}S^{t,0}_{N,n}(\tau)\mathbf{1}(1<\beta<2)\big) \to_{\mathrm{fdd}} \lambda^*_\infty \mathcal{Z}^*_\beta(\tau/\lambda^*_\infty), \tag{4.6}$$

where $\mathcal{Z}^*_\beta$ *is the 'diagonal intermediate' process in* (2.25).

Remark 4.1. *The r.v.s* V^*_β *and* V^+_β *in* (4.4) *and* (4.5) *have respective stochastic integral representations*

$$V^*_\beta = \int_{\mathbb{R}_+\times C(\mathbb{R})} \Big\{\int_{-\infty}^0 \mathrm{e}^{xs}\mathrm{d}B(s)\Big\}^2 \mathrm{d}(\mathcal{M}^*_\beta - \mathrm{E}\mathcal{M}^*_\beta\mathbf{1}(1<\beta<2)),$$
$$V^+_\beta = \int_{\mathbb{R}_+\times C(\mathbb{R})} (2x)^{-1}\mathrm{d}(\mathcal{M}^*_\beta - \mathrm{E}\mathcal{M}^*_\beta\mathbf{1}(1<\beta<2))$$

w.r.t. Poisson random measure $\mathcal{M}^*_\beta$ *in* (2.22)*. Note* $\int_{-\infty}^0 \mathrm{e}^{xs}\mathrm{d}B(s) =_{\mathrm{d}} Z/\sqrt{2x}$*,* $Z \sim N(0,1)$*. The fact that both* V^*_β *and* V^+_β *have completely asymmetric* β*-stable distribution follows from their ch.f.s:*

$$\begin{aligned}
\mathrm{E}\mathrm{e}^{\mathrm{i}\theta V^*_\beta} &= \exp\Big\{\psi(1)\int_0^\infty \mathrm{E}\big(\mathrm{e}^{\mathrm{i}\theta Z^2/(2x)} - 1 - \mathrm{i}(\theta Z^2/(2x))\mathbf{1}(1<\beta<2)\big)x^{\beta-1}\mathrm{d}x\Big\}\\
&= \exp\big\{-c^*_\beta|\theta|^\beta(1-\mathrm{i}\,\mathrm{sign}(\theta)\tan(\pi\beta/2))\big\}, && (4.7)\\
\mathrm{E}\mathrm{e}^{\mathrm{i}\theta V^+_\beta} &= \exp\Big\{\psi(1)\int_0^\infty \big(\mathrm{e}^{\mathrm{i}\theta/(2x)} - 1 - \mathrm{i}(\theta/(2x))\mathbf{1}(1<\beta<2)\big)x^{\beta-1}\mathrm{d}x\Big\}\\
&= \exp\big\{-c^+_\beta|\theta|^\beta(1-\mathrm{i}\,\mathrm{sign}(\theta)\tan(\pi\beta/2))\big\}, \quad \theta\in\mathbb{R}, && (4.8)
\end{aligned}$$

where

$$c^+_\beta := \frac{\psi(1)\Gamma(2-\beta)\cos(\pi\beta/2)}{2^\beta\beta(1-\beta)}, \qquad c^*_\beta := c^+_\beta\mathrm{E}|Z|^{2\beta} \tag{4.9}$$

with $\mathrm{E}|Z|^{2\beta} = 2^\beta\Gamma(\beta+1/2)/\sqrt{\pi} \neq 1$ *unless* $\beta = 1$*, implying that* V^*_β *and* V^+_β *have different distributions.*

Theorem 4.2. *Let the mixing distribution satisfy condition* (1.2) *with* $\beta > 2$*. In addition, assume* $\mathrm{E}\varepsilon^4(0) < \infty$*. Then for any* $t\in\mathbb{Z}$*, as* $N, n\to\infty$ *in arbitrary way,*

$$n^{-1}N^{-1/2}(S^{t,0}_{N,n}(\tau) - \mathrm{E}S^{t,0}_{N,n}(\tau)) \to_{\mathrm{fdd}} \tau\sigma^*_t Z, \tag{4.10}$$

where $Z\sim N(0,1)$ *and* $(\sigma^*_t)^2 := \mathrm{Var}(a^{|t|}/(1-a^2))$*.*

Remark 4.2. *If* $\beta < 1$*, then* $\gamma(t,0)$ *is undefined for any* $t\in\mathbb{Z}$*. In the sequel we use the convention* $\gamma(t,0)\mathbf{1}(1<\beta<2) := 0$ *if* $\beta<1$*,* $:= \gamma(t,0)$ *if* $\beta > 1$*.*

Corollary 4.1. *(i) Let the conditions of Theorem 4.1 (i) be satisfied. Then for any* $t\in\mathbb{Z}$

$$N^{1-1/\beta}(\widehat{\gamma}_{N,n}(t,0) - \gamma(t,0)\mathbf{1}(1<\beta<2)) \to_{\mathrm{d}} V^*_\beta.$$

(ii) Let the conditions of Theorem 4.1 (ii) be satisfied. Then for any $t \in \mathbb{Z}$

$$N^{1-1/\beta}(\widehat{\gamma}_{N,n}(t,0) - \gamma(t,0)\mathbf{1}(1<\beta<2)) \to_{\mathrm{d}} V_\beta^+.$$

(iii) Let the conditions of Theorem 4.1 (iii) be satisfied. Then for any $t \in \mathbb{Z}$

$$N^{1-1/\beta}(\widehat{\gamma}_{N,n}(t,0) - \gamma(t,0)\mathbf{1}(1<\beta<2)) \to_{\mathrm{d}} \lambda_\infty^* \mathcal{Z}_\beta^*(1/\lambda_\infty^*).$$

(iv) Let the conditions of Theorem 4.2 be satisfied. Then for any $t \in \mathbb{Z}$

$$N^{1/2}(\widehat{\gamma}_{N,n}(t,0) - \gamma(t,0)) \to_{\mathrm{d}} \sigma_t^* Z, \quad Z \sim N(0,1).$$

Proof of Theorem 4.1. Let $t \ge 0$ and

$$y^{t,0}(\tau) := \frac{1}{nN^{1/\beta}} \sum_{u=1}^{\lfloor n\tau \rfloor} (X(u)X(u+t) - \mathrm{E}X(u)X(u+t)\mathbf{1}(1<\beta<2)). \tag{4.11}$$

It suffices to prove that

$$\Phi_{N,n}^{t,0}(\theta) \to \Phi^*(\theta) \quad \text{as } N,\, n \to \infty,\, \lambda_{N,n}^* \to \lambda_\infty^*, \quad \forall\, \theta \in \mathbb{R}, \tag{4.12}$$

where, using $\mathrm{E}y^{t,0}(\tau)\mathbf{1}(1<\beta<2) = 0$,

$$\Phi_{N,n}^{t,0}(\theta) := N\mathrm{E}[\mathrm{e}^{\mathrm{i}\theta y^{t,0}(\tau)} - 1 - \mathrm{i}\theta y^{t,0}(\tau)\mathbf{1}(1<\beta<2)], \quad \Phi^*(\theta) := \log \mathrm{E}\mathrm{e}^{\mathrm{i}\theta \mathcal{S}_\beta^*(\tau)}, \tag{4.13}$$

and $\mathcal{S}_\beta^*(\tau)$ denotes the limit process in (4.4)–(4.6). Similarly to (3.31),

$$\Phi_{N,n}^{t,0}(\theta) = \psi(1)\int_{(0,1/N^{1/\beta}]} \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta z_{N,n}^{t,0}(\tau;x)} - 1 - \mathrm{i}\theta z_{N,n}^{t,0}(\tau;x)\mathbf{1}(1<\beta<2)]x^{\beta-1}\mathrm{d}x, \tag{4.14}$$

where $z_{N,n}^{t,0}(\tau;x) := y^{t,0}(\tau)|_{a=1-x/N^{1/\beta}}$. Next we decompose $y^{t,0}(\tau) = y^*(\tau) + y^+(\tau)$, where

$$\begin{aligned} y^*(\tau) &:= \frac{1}{nN^{1/\beta}} \sum_{u=1}^{\lfloor n\tau \rfloor} (X(u)X(u+t) - \mathrm{E}[X(u)X(u+t)|a]), \\ y^+(\tau) &:= \frac{\lfloor n\tau \rfloor}{nN^{1/\beta}} (\mathrm{E}[X(0)X(t)|a] - \mathrm{E}[X(0)X(t)\mathbf{1}(1<\beta<2)]) \\ &= \frac{\lfloor n\tau \rfloor}{nN^{1/\beta}} \big(\frac{a^t}{1-a^2} - \mathrm{E}\big[\frac{a^t}{1-a^2}\mathbf{1}(1<\beta<2)\big]\big). \end{aligned}$$

Accordingly, we decompose $z_{N,n}^{t,0}(\tau;x) = z_{N,n}^*(\tau;x) + z_{N,n}^+(\tau;x)$, where

$$\begin{aligned} &z_{N,n}^*(\tau;x) \\ &:= \frac{1}{nN^{1/\beta}} \sum_{s_1,s_2 \in \mathbb{Z}} \overline{\varepsilon(s_1)\varepsilon(s_2)} \sum_{u=1}^{\lfloor n\tau \rfloor} \big(1 - \frac{x}{N^{1/\beta}}\big)^{2u+t-s_1-s_2} \mathbf{1}(u \ge s_1, u+t \ge s_2), \end{aligned} \tag{4.15}$$

$$z^{+}_{N,n}(\tau;x) := \frac{\lfloor n\tau\rfloor}{nN^{1/\beta}}\Big(\frac{(1-\frac{x}{N^{1/\beta}})^t}{1-(1-\frac{x}{N^{1/\beta}})^2} - \mathrm{E}\Big[\frac{a^t}{1-a^2}\mathbf{1}(1<\beta<2)\Big]\Big),$$

where $\overline{\varepsilon(s_1)\varepsilon(s_2)} := \varepsilon(s_1)\varepsilon(s_2) - \mathrm{E}\varepsilon(s_1)\varepsilon(s_2)$.

Proof of (4.12), *case* $0 < \lambda^*_\infty < \infty$. We have

$$\Phi^*(\theta) = \psi(1)\int_0^\infty \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta\lambda^*_\infty z^*(\tau/\lambda^*_\infty;x)} - 1 - \mathrm{i}\theta\lambda^*_\infty z^*(\tau/\lambda^*_\infty;x)\mathbf{1}(1<\beta<2)]x^{\beta-1}\mathrm{d}x, \tag{4.16}$$

where the last expectation is taken w.r.t. the Wiener measure P_B. Similarly as in the proof of (3.29) we prove the point-wise convergence of the integrands in (4.14) and (4.16): for any $x > 0$

$$\begin{aligned}\Lambda^{t,0}_{N,n}(\theta;x) &:= \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta z^{t,0}_{N,n}(\tau;x)} - 1 - \mathrm{i}\theta z^{t,0}_{N,n}(\tau;x)\mathbf{1}(1<\beta<2)] \\ &\to \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta\lambda^*_\infty z^*(\tau/\lambda^*_\infty;x)} - 1 - \mathrm{i}\theta\lambda^*_\infty z^*(\tau/\lambda^*_\infty;x)\mathbf{1}(1<\beta<2)].\end{aligned} \tag{4.17}$$

The proof of (4.17) using Proposition 2.1 is very similar to that of (3.35) and we omit the details. Using (4.17) and the dominated convergence theorem we can prove the convergence of integrals, or (4.12). The application of the dominated convergence theorem is guaranteed by the dominating bound

$$|\Lambda^{t,0}_{N,n}(\theta;x)| \le C(1\wedge(1/x))\{1(0<\beta<1) + (1/x)\mathbf{1}(1<\beta<2)\}, \tag{4.18}$$

which is a consequence of $|z^{+}_{N,n}(\tau;x)| \le C/x$, $\mathrm{E}(z^*_{N,n}(\tau;x))^2 \le Cx^{-2}$, see (2.30). Particularly, for $0<\beta<1$ we get $|\Lambda^{t,0}_{N,n}(\theta;x)| \le 2$ and $|\Lambda^{t,0}_{N,n}(\theta;x)| \le \mathrm{E}(|z^*_{N,n}(\tau;x)| + |z^{+}_{N,n}(\tau;x)|) \le C((\mathrm{E}|z^*_{N,n}(\tau;x)|^2)^{1/2} + (1/x)) \le C/x$, hence (4.18) follows. For $1<\beta<2$ (4.18) follows similarly. This proves (4.12) for $0<\lambda^*_\infty<\infty$.

Proof of (4.12), *case* $\lambda^*_\infty = 0$. In this case

$$\Phi^*(\theta) = \psi(1)\int_{\mathbb{R}_+}[\mathrm{e}^{\mathrm{i}\theta(\tau/(2x))} - 1 - \mathrm{i}\theta(\tau/(2x))\mathbf{1}(1<\beta<2)]x^{\beta-1}\mathrm{d}x,$$

see (4.8). From (2.30) we have $\mathrm{E}(z^*_{N,n}(\tau;x))^2 \le Cx^{-2}\min\{1,\lambda^*_{N,n}/x\} = o(1)$ and hence

$$\Lambda^{t,0}_{N,n}(\theta;x) \to \mathrm{e}^{\mathrm{i}\theta\tau/(2x)} - 1 - \mathrm{i}\theta(\tau/(2x))\mathbf{1}(1<\beta<2)$$

for any $x>0$ similarly as in (4.17). Finally, the use of the dominating bound in (4.18), which is also valid in this case completes the proof of (4.12) for $\lambda^*_\infty = 0$.

Proof of (4.12), *case* $\lambda^*_\infty = \infty$. In this case

$$\Phi^*(\theta) = \psi(1)\int_{\mathbb{R}_+}\mathrm{E}[\mathrm{e}^{\mathrm{i}\theta(\tau Z^2/(2x))} - 1 - \mathrm{i}\theta(\tau Z^2/(2x))\mathbf{1}(1<\beta<2)]x^{\beta-1}\mathrm{d}x, \tag{4.19}$$

see (4.7). Write $z^*_{N,n}(\tau;x)$ in (4.15) as quadratic form: $z^*_{N,n}(\tau;x) = Q_{11}(h(\tau;x;\cdot))$ in (2.4) and apply Proposition 2.1 with $\alpha_1 = \alpha_2 \equiv \alpha := N^{1/\beta}$. Note $\widetilde{h}^{(\alpha,\alpha)}(\tau;x;s_1,s_2) = n^{-1}\sum_{u=1}^{\lfloor n\tau\rfloor}(1-x/N^{1/\beta})^{u-\lfloor N^{1/\beta}s_1\rfloor}(1-x/N^{1/\beta})^{t+u-\lfloor N^{1/\beta}s_2\rfloor}$ $\mathbf{1}(u \ge \lfloor N^{1/\beta}s_1\rfloor, u+t \ge \lfloor N^{1/\beta}s_2\rfloor) \to g(s_1,s_2) := \tau\mathrm{e}^{x(s_1+s_2)}\mathbf{1}(s_1\vee s_2 \le 0)$ point-wise a.e. in $(s_1,s_2) \in \mathbb{R}^2$ and also in $L^2(\mathbb{R}^2)$. Then conclude $z^*_{N,n}(\tau;x) \to_{\mathrm{d}} I_{11}(g) =_{\mathrm{d}} \int_{\mathbb{R}^2} g(s_1,s_2)\mathrm{d}B(s_1)\mathrm{d}B(s_2) =_{\mathrm{d}} \tau\{\ (\int_{-\infty}^0 \mathrm{e}^{sx}\mathrm{d}B(s))^2 - \mathrm{E}(\int_{-\infty}^0 \mathrm{e}^{sx}\mathrm{d}B(s))^2\} =_{\mathrm{d}} \tau(Z^2-1)/(2x)$ for any $x>0$, where $Z \sim N(0,1)$. On the other hand, $z^+_{N,n}(\tau;x) \to \tau/(2x)$ and therefore

$$\Lambda^{t,0}_{N,n}(\theta;x) \to \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta\tau Z^2/(2x)} - 1 - \mathrm{i}\theta(\tau Z^2/(2x))\mathbf{1}(1<\beta<2)]$$

for any $x>0$, proving the point-wise convergence of the integrands in (4.14) and (4.19). The remaining details are similar as in the previous cases and omitted. This ends the proof of Theorem 4.1. □

Proof of Theorem 4.2. Consider the decomposition in (4.1), where $n^{-1}T^{t,0}_{N,n}(\tau) = (\lfloor n\tau\rfloor/n)\sum_{i=1}^N a_i^t/(1-a_i^2)$ is a sum of i.i.d. r.v.s with finite variance $(\sigma^*_t)^2 = \mathrm{Var}(a^{|t|}/(1-a^2))$ and therefore

$$n^{-1}N^{-1/2}(T^{t,0}_{N,n}(\tau) - \mathrm{E}T^{t,0}_{N,n}(\tau)) \to_{\mathrm{fdd}} \tau\sigma^*_t Z$$

holds by the classical CLT as $N,n\to\infty$ in arbitrary way and where $Z \sim N(0,1)$. Hence, the statement of the theorem follows from $\widetilde{S}^{t,0}_{N,n}(1) = o_{\mathrm{p}}(nN^{1/2})$. By Proposition 2.4 (2.30) we have that $\mathrm{Var}(\widetilde{S}^{t,0}_{N,n}(1)) = N\mathrm{E}\,\mathrm{Var}[\sum_{u=1}^n X(u)X(u+t)|a] \le CNn^2\mathrm{E}[(1-a)^{-2}\min\{1,(n(1-a))^{-1}\}]$, where the last expectation vanishes as $n\to\infty$, due to $\mathrm{E}(1-a)^{-2}<\infty$. Theorem 4.2 is proved. □

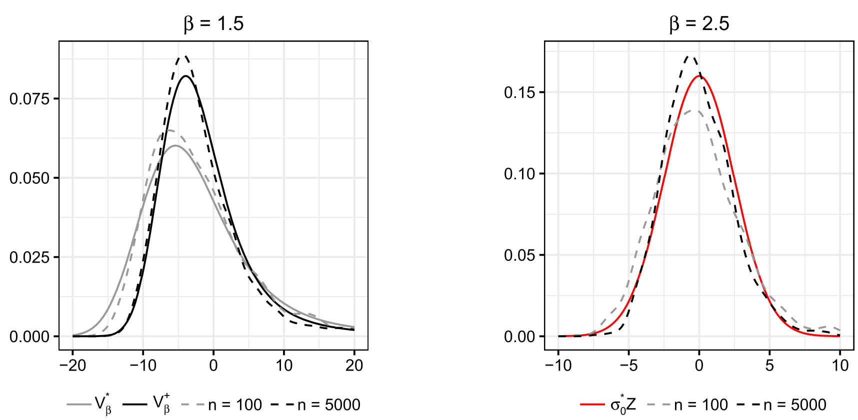

FIG 1. *Density of the limiting random variables in cases [left] (i), (ii), [right] (iv) of Corollary 4.1 for $t=0$ and their kernel density estimates constructed from a random sample of size 1000 from $\widehat{\gamma}_{N,n}(0,0)$ in (1.6) with $N=5000$, $a^2 \sim \mathrm{Beta}(2,\beta)$, $\varepsilon(0) \sim N(0,1)$.*

To illustrate our results, we use $a^2 \sim \mathrm{Beta}(\alpha, \beta)$, $\alpha, \beta > 0$, as in [10]. Then condition (1.2) holds with the same β and we can explicitly compute parameters of the limit distributions in cases (i), (ii), (iv) of Corollary 4.1. Figure 1 shows the density of the corresponding limiting random variables for $\alpha = 2$, $\beta = 1.5$, 2.5 and $t = 0$. We also plot the kernel density estimates constructed using 1000 RCAR(1) panels with $N = 5000$, $n = 100$, 5000, $\varepsilon(0) \sim N(0,1)$. More specifically, we use a random sample of $N^{1/\beta}(\widehat{\gamma}_{N,n}\ (0,0) - \gamma(0,0))$ if $\beta = 1.5$ and $N^{1/2}(\widehat{\gamma}_{N,n}(0,0) - \gamma(0,0))$ if $\beta = 2.5$. On the l.h.s. we can see that the empirical distribution of $\widehat{\gamma}_{N,n}\ (0,0)$ is different for $n = 100$, 5000, whereas on the r.h.s. both kernel density estimates are quite close to the limiting normal density.

In the finite variance case $\beta > 1$, Corollary 4.1 can be used for statistical inference about the covariance $\gamma(t,0) = \gamma(t)$ in (1.3), provided parameters of the limit distributions are consistently estimated. Denote by

$$F^*_{\beta,\psi}(x) := \mathrm{P}(V^*_\beta \le x), \quad F^+_{\beta,\psi}(x) := \mathrm{P}(V^+_\beta \le x), \quad x \in \mathbb{R}, \tag{4.20}$$

the c.d.f.s of the above stable r.v.s, which are uniquely determined by β, $\psi(1) \equiv \psi$ in (1.2), see (4.7)–(4.9). The same is true for the (marginal) distribution $\mathcal{Z}^*_\beta(\tau)$ of the 'diagonal intermediate' process in (2.25). In Corollary 4.2 we suppose the existence of estimators

$$\hat{\beta}_{N,n} = \beta + o_{\mathrm{p}}(1/\log N), \quad \hat{\psi}_{N,n} = \psi + o_{\mathrm{p}}(1), \tag{4.21}$$

$$\hat{\sigma}^2_{N,n,t} = (\sigma^*_t)^2 + o_{\mathrm{p}}(1), \tag{4.22}$$

which is discussed in Remark 4.4 below. Corollary 4.2 omits the 'intermediate' case $\lambda^*_\infty \in (0,\infty)$, partly because in this case the limit distribution is less tractable and depends on λ^*_∞ which is difficult to assess in a finite sample.

Corollary 4.2. *(i) Let the conditions of Theorem 4.1 (i) be satisfied,* $1 < \beta < 2$, *and* $\hat{\beta}_{N,n}$, $\hat{\psi}_{N,n}$ *be estimators as in* (4.21)*. Then for any* $t \in \mathbb{Z}$

$$\sup_{x\in\mathbb{R}} \left|\mathrm{P}\big(N^{1-1/\hat{\beta}_{N,n}}(\widehat{\gamma}_{N,n}(t,0) - \gamma(t)) \le x\big) - F^*_{\hat{\beta}_{N,n},\hat{\psi}_{N,n}}(x)\right| = o_{\mathrm{p}}(1). \tag{4.23}$$

(ii) Let the conditions of Theorem 4.1 (ii) be satisfied, $1 < \beta < 2$, *and* $\hat{\beta}_{N,n}$, $\hat{\psi}_{N,n}$ *be estimators as in* (4.21)*. Then for any* $t \in \mathbb{Z}$

$$\sup_{x\in\mathbb{R}} \left|\mathrm{P}\big(N^{1-1/\hat{\beta}_{N,n}}(\widehat{\gamma}_{N,n}(t,0) - \gamma(t)) \le x\big) - F^+_{\hat{\beta}_{N,n},\hat{\psi}_{N,n}}(x)\right| = o_{\mathrm{p}}(1). \tag{4.24}$$

(iii) Let the conditions of Theorem 4.2 be satisfied, $\beta > 2$, *and* $\hat{\sigma}^2_{N,n,t}$ *be an estimator as in* (4.22)*. Then for any* $t \in \mathbb{Z}$

$$\sup_{x\in\mathbb{R}} \left|\mathrm{P}\big((N/\hat{\sigma}^2_{N,n,t})^{1/2}(\widehat{\gamma}_{N,n}(t,0) - \gamma(t)) \le x\big) - \mathrm{P}(Z \le x)\right| = o_{\mathrm{p}}(1), \tag{4.25}$$

where $Z \sim N(0,1)$.

Proof. Consider (4.23). Write $N^{1-1/\hat{\beta}_{N,n}}(\widehat{\gamma}_{N,n}(t,0)-\gamma(t)) = N^{1-1/\beta}(\widehat{\gamma}_{N,n}(t,0)-\gamma(t))+\xi_{N,n}$, where $\xi_{N,n} := (N^{(1/\beta)-(1/\hat{\beta}_{N,n})}-1)N^{1-1/\beta}(\widehat{\gamma}_{N,n}(t,0)-\gamma(t)) = o_{\mathrm{p}}(1)$ due to (4.21) and Corollary 4.1 (i). Therefore, $\sup_{x\in\mathbb{R}}|\mathrm{P}(N^{1-1/\hat{\beta}_{N,n}}(\widehat{\gamma}_{N,n}(t,0)-\gamma(t)) \le x) - F^*_{\beta,\psi}(x)| \to 0$. Relation $\sup_{x\in\mathbb{R}}|F^*_{\beta,\psi}(x) - F^*_{\hat{\beta}_{N,n},\hat{\psi}_{N,n}}(x)| = o_{\mathrm{p}}(1)$ follows from (4.21) and continuity of the c.d.f. $F^*_{\beta,\psi}$ in β, ψ. This proves (4.23). The proof of (4.24), (4.25) is analogous. □

Remark 4.3. *Using Corollary 4.2 we can construct asymptotic confidence intervals for $\gamma(t)$, as follows. For $\alpha \in (0,1)$ denote by $q_{\beta,\psi}(\alpha)$ the α-quantile of the c.d.f. $F^*_{\beta,\psi}$ in* (4.20)*. Then, since $\alpha = F^*_{\hat{\beta}_{N,n},\hat{\psi}_{N,n}}(q_{\hat{\beta}_{N,n},\hat{\psi}_{N,n}}(\alpha))$ a.s., $\mathrm{P}(N^{1-1/\hat{\beta}_{N,n}}(\widehat{\gamma}_{N,n}(t,0)-\gamma(t)) \le q_{\hat{\beta}_{N,n},\hat{\psi}_{N,n}}(\alpha)) - \alpha = o_{\mathrm{p}}(1)$ follows from* (4.23)*; moreover since the above quantity is non-random, we get that $|\mathrm{P}(N^{1-1/\hat{\beta}_{N,n}}(\widehat{\gamma}_{N,n}(t,0)-\gamma(t)) \le q_{\hat{\beta}_{N,n},\hat{\psi}_{N,n}}(\alpha)) - \alpha| = o(1)$, implying that*

$$\Big[\widehat{\gamma}_{N,n}(t,0) - \frac{q_{\hat{\beta}_{N,n},\hat{\psi}_{N,n}}(1-\alpha/2)}{N^{1-1/\hat{\beta}_{N,n}}},\ \widehat{\gamma}_{N,n}(t,0) - \frac{q_{\hat{\beta}_{N,n},\hat{\psi}_{N,n}}(\alpha/2)}{N^{1-1/\hat{\beta}_{N,n}}}\Big]$$

is the asymptotic confidence interval for $\gamma(t)$, for any confidence level $\alpha \in (0,1)$. Analogous confidence intervals for $\gamma(t)$ can be defined in the case (4.24)*; in the case* (4.25) *they follow in a standard way.*

Remark 4.4. *Estimation of the tail parameter β in the RCAR(1) panel model was studied in [16]. Particularly, [16] developed a modified version $\hat{\beta}_{N,n}$ of the Goldie–Smith estimator in [8] and proved its asymptotic normality, under additional (rather stringent) conditions on the mutual increase rate of N and n. A similar estimator $\hat{\psi}_{N,n}$ can be defined following [8]. We expect that these estimators satisfy the consistency as in* (4.21) *under much weaker assumptions on N, n. Finally, for $t \ge 0$ the estimator $\hat{\sigma}^2_{N,n,t}$ in* (4.22) *can be defined (see the proof in Appendix A) as*

$$\hat{\sigma}^2_{N,n,t} := \frac{1}{N}\sum_{i=1}^{N}\Big(\frac{1}{n}\sum_{k=1}^{n-t}X_i(k)X_i(k+t)\Big)^2 - \Big(\frac{1}{Nn}\sum_{i=1}^{N}\sum_{k=1}^{n-t}X_i(k)X_i(k+t)\Big)^2. \tag{4.26}$$

Remark 4.5. *In general, in the RCAR(1) model the autoregressive coefficient a can take a value from $(-1,1)$. In the latter case if the distribution of a is sufficiently dense at -1, the (unconditional) autocovariance function of the RCAR(1) process oscillates when decaying slowly, which is usually referred to as seasonal long memory. The restriction $a \in [0,1)$ in the present paper (as well as in [23, 16] and some other papers) is basically due to technical reasons. We expect that, under assumption* (1.2)*, most of our results hold in the general case $a \in (-1,1)$ provided the concentration of the mixing distribution near -1 is not too strong, e.g. if $\mathrm{E}(1+a)^{-\beta'} < \infty$ for some $\beta' > \beta$.*

Appendix A

Proof of Proposition 2.2. (i) The existence of $\mathcal{Z}_\beta$ follows from

$$J_\beta := \int_{\mathcal{L}_1^c} |z(\tau; x_1, x_2)|^2 \mathrm{d}\mu_\beta < \infty \tag{A.1}$$

and $\mu_\beta(\mathcal{L}_1) < \infty$. We have $\mu_\beta(\mathcal{L}_1) = \psi(1)^2 \int_{\mathbb{R}_+^2} \mathbf{1}(x_1x_2(x_1+x_2) < 1)(x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 \le C\int_0^\infty x_1^{\beta-1}\mathrm{d}x_1 \int_0^{x_1} \mathbf{1}(x_2 < 1/x_1^2)x_2^{\beta-1}\mathrm{d}x_2 = C(\int_0^1 x_1^{\beta-1}\mathrm{d}x_1 \int_0^{x_1} x_2^{\beta-1}\mathrm{d}x_2 + \int_1^\infty x_1^{\beta-1}\mathrm{d}x_1 \int_0^{1/x_1^2} x_2^{\beta-1}\mathrm{d}x_2) \le C(\int_0^1 x_1^{2\beta-1}\mathrm{d}x_1 + \int_1^\infty x_1^{-\beta-1}\mathrm{d}x_1) < \infty$ since $\beta > 0$.

Consider (A.1). Then

$$J_\beta = C\int_{\mathbb{R}_+^2} \mathbf{1}(x_1x_2(x_1+x_2) > 1)\mathrm{E}|z(\tau; x_1, x_2)|^2 (x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2,$$

where

$$\begin{aligned} \mathrm{E}|z(\tau; x_1, x_2)|^2 &= \int_{(0,\tau]^2} \prod_{i=1}^2 \mathrm{E}[\mathcal{Y}_i(u_1; x_i)\mathcal{Y}_i(u_2; x_i)]\mathrm{d}u_1\mathrm{d}u_2 \\ &= \frac{1}{4x_1x_2}\int_{(0,\tau]^2} \mathrm{e}^{-(x_1+x_2)|u_1-u_2|}\mathrm{d}u_1\mathrm{d}u_2 \\ &\le \frac{C\tau^2}{x_1x_2}\big(1 \wedge \frac{1}{\tau(x_1+x_2)}\big). \end{aligned} \tag{A.2}$$

Hence,

$$\begin{aligned} J_\beta &\le C\int_{\mathbb{R}_+^2} \mathbf{1}(x_1x_2(x_1+x_2) > 1)(x_1+x_2)^{-1}(x_1x_2)^{\beta-2}\mathrm{d}x_1\mathrm{d}x_2 \\ &\le C\int_{\mathbb{R}_+^2} \mathbf{1}(x_2 > x_1,\, x_1x_2^2 > 1)x_1^{\beta-2}x_2^{\beta-3}\mathrm{d}x_1\mathrm{d}x_2 \\ &= C\Big(\int_0^1 x_1^{\beta-2}\mathrm{d}x_1 \int_{x_1^{-1/2}}^\infty x_2^{\beta-3}\mathrm{d}x_2 + \int_1^\infty x_1^{\beta-2}\mathrm{d}x_1 \int_{x_1}^\infty x_2^{\beta-3}\mathrm{d}x_2\Big) < \infty \end{aligned}$$

if $0 < \beta < 3/2$. The remaining facts in (i) are easy and we omit the details.

(ii) Similarly as in ([20], proof of Proposition 3.1 (ii)) it suffices to show for any $0 < p < 2\beta$ that $J_{p,\beta}(\tau) < \infty$, where

$$J_{p,\beta}(\tau) := \begin{cases} \displaystyle\int_{\mathbb{R}_+^2} \mathrm{E}|z(\tau; x_1, x_2)|^p (x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2, & 0 < p \le 2, \\ \displaystyle\int_{\mathbb{R}_+^2} \mathrm{E}\big[|z(\tau; x_1, x_2)|^p \vee |z(\tau; x_1, x_2)|^2\big](x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2, & p > 2. \end{cases} \tag{A.3}$$

Let first $0 < p \le 2$. Using $\mathrm{E}|z(\tau;x_1,x_2)|^p \le (\mathrm{E}|z(\tau;x_1,x_2)|^2)^{p/2}$ and (A.2), we obtain

$$\begin{aligned} J_{p,\beta}(\tau) &\le C\int_{\mathbb{R}_+^2}\Big(\int_{(0,\tau]^2}\mathrm{e}^{-(x_1+x_2)|u_1-u_2|}\mathrm{d}u_1\mathrm{d}u_2\Big)^{p/2}(x_1x_2)^{\beta-1-p/2}\mathrm{d}x_1\mathrm{d}x_2 \\ &=: C\tau^{2(p-\beta)}I_{p,\beta}, \end{aligned} \tag{A.4}$$

where

$$\begin{aligned} I_{p,\beta} &\le \int_{\mathbb{R}_+^2}\big(1\wedge\frac{1}{x_1+x_2}\big)^{p/2}(x_1x_2)^{\beta-1-p/2}\mathrm{d}x_1\mathrm{d}x_2 \\ &\le C\int_0^\infty\int_0^{x_1}\big(1\wedge\frac{1}{x_1}\big)^{p/2}(x_1x_2)^{\beta-1-p/2}\mathrm{d}x_1\mathrm{d}x_2 \\ &= C\int_0^\infty\big(1\wedge\frac{1}{x_1}\big)^{p/2}x_1^{2\beta-p-1}\mathrm{d}x_1 < \infty \end{aligned} \tag{A.5}$$

if $p/2 < \beta < 3p/4$, thus proving (A.3) for $0 < p \le 2$.

Next for $2 < p < 3$ we need the inequality for double Itô-Wiener integrals: for any $p \ge 2$, $g \in L^2(\mathbb{R}^2)$

$$\begin{aligned} \mathrm{E}\Big|\int_{\mathbb{R}^2}g(s_1,s_2)\mathrm{d}B_1(s_1)\mathrm{d}B_2(s_2)\Big|^p &\le C\Big(\mathrm{E}\Big|\int_{\mathbb{R}^2}g(s_1,s_2)\mathrm{d}B_1(s_1)\mathrm{d}B_2(s_2)\Big|^2\Big)^{p/2} \\ &= C\Big(\int_{\mathbb{R}^2}|g(s_1,s_2)|^2\mathrm{d}s_1\mathrm{d}s_2\Big)^{p/2}. \end{aligned} \tag{A.6}$$

Indeed, by using Gaussianity and independence of B_1, B_2 and Minkowski inequality for $I_2(g) := \int_{\mathbb{R}^2}g(s_1,s_2)\mathrm{d}B_1(s_1)\mathrm{d}B_2(s_2)$ we obtain

$$\begin{aligned} \big(\mathrm{E}|I_2(g)|^p\big)^{2/p} &= \big(\mathrm{E}_{B_1}\mathrm{E}_{B_2}[|I_2(g)|^p|B_1]\big)^{2/p} \le C\big(\mathrm{E}_{B_1}\big(\mathrm{E}_{B_2}[|I_2(g)|^2|B_1]\big)^{p/2}\big)^{2/p} \\ &\le C\mathrm{E}_{B_2}\big\{\mathrm{E}_{B_1}[|I_2(g)|^p|B_2]\big\}^{2/p} \le C\mathrm{E}_{B_2}\big\{\big(\mathrm{E}_{B_1}[|I_2(g)|^2|B_2]\big)^{p/2}\big\}^{2/p} \\ &= C\mathrm{E}_{B_2}\mathrm{E}_{B_1}[|I_2(g)|^2|B_2] = C\mathrm{E}|I_2(g)|^2. \end{aligned}$$

Using inequality (A.6) and (A.4), (A.5) we obtain

$$\begin{aligned} J_{p,\beta}(\tau) &\le C\Big(\int_{\mathbb{R}_+^2}\mathrm{E}|z(\tau;x_1,x_2)|^p(x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 \\ &\quad + \int_{\mathbb{R}_+^2}\mathrm{E}|z(\tau;x_1,x_2)|^2(x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2\Big) \le C(I_{p,\beta}(\tau)+I_{2,\beta}(\tau)) < \infty \end{aligned}$$

if $p/2 < \beta < 3p/4$, thus proving (A.3) and part (ii).

(iii) Follows from stationarity of increments of $\mathcal{Z}_\beta$ (part (i)) and $J_{2,\beta}(\tau) = \sigma_\infty^2\tau^{2(2-\beta)}$, where according to (A.2),

$$\sigma_\infty^2 = \int_{\mathbb{R}_+^2}\mathrm{E}z^2(1;x_1,x_2)\mathrm{d}\mu_\beta$$

$$= \frac{\psi(1)^2}{4} \int_{(0,1]^2} \mathrm{d}u_1 \mathrm{d}u_2 \Big(\int_0^\infty \mathrm{e}^{-x|u_1-u_2|} x^{\beta-2} \mathrm{d}x \Big)^2$$
$$= \frac{\psi(1)^2}{4} \Gamma(\beta-1)^2 \int_{(0,1]^2} |u_1-u_2|^{2(1-\beta)} \mathrm{d}u_1 \mathrm{d}u_2 = \frac{\psi(1)^2 \Gamma(\beta-1)^2}{4(2-\beta)(3-2\beta)}.$$

(iv) Follows from stationarity of increments, $\mathrm{E}|\mathcal{Z}_\beta(\tau)|^p \le C J_{p,\beta}(\tau)$, $1 < p \le 2$, where $J_{p,\beta}(\tau)$ is the same as in (A.3), and Kolmogorov's criterion; cf. ([20], proof of Proposition 3.1 (iv)).

(v) The proofs are very similar to those of Theorem 3.1 (i), (ii), hence we omit some details. For notational simplicity, we only prove one-dimensional convergence at $\tau > 0$.

Proof of (2.19). As $b \to 0$, consider

$$\Phi_b(\theta) := \log \mathrm{E} \mathrm{e}^{\mathrm{i}\theta b^{\beta-2} \mathcal{Z}_\beta(b\tau)} = \psi(1)^2 \int_{\mathbb{R}_+^2} \mathrm{E}\Psi(\theta b^{\beta-2} z(b\tau; x_1, x_2))(x_1 x_2)^{\beta-1} \mathrm{d}x_1 \mathrm{d}x_2,$$

where $\Psi(z) := \mathrm{e}^{\mathrm{i}z} - 1 - \mathrm{i}z$, $z \in \mathbb{R}$. Since $b^{-2} z(b\tau; x_1, x_2) =_{\mathrm{d}} z(\tau; bx_1, bx_2)$, rewrite

$$\Phi_b(\theta) = \psi(1)^2 b^{-2\beta} \int_{\mathbb{R}_+^2} \mathrm{E}\Psi(\theta b^\beta z(\tau; x_1, x_2))(x_1 x_2)^{\beta-1} \mathrm{d}x_1 \mathrm{d}x_2,$$

where $b^{-2\beta}\Psi(\theta b^\beta z(\tau; x_1, x_2)) \to -(\theta^2/2) z^2(\tau; x_1, x_2)$ a.s. Note $|b^{-2\beta}\Psi(\theta b^\beta z(\tau; x_1, x_2))| \le (\theta^2/2) z^2(\tau; x_1, x_2)$, where the dominating function satisfies (A.2) and (2.10). Hence, by the dominated convergence theorem,

$$\Phi_b(\theta) \to -(\theta^2/2)\psi(1)^2 \int_{\mathbb{R}_+^2} \mathrm{E} z^2(\tau; x_1, x_2)(x_1 x_2)^{\beta-1} \mathrm{d}x_1 \mathrm{d}x_2 = \log \mathrm{E} \mathrm{e}^{\mathrm{i}\theta\sigma_\infty B_{2-\beta}(\tau)},$$

which finishes the proof.

Proof of (2.20) follows that of Theorem 3.1 (i), case $0 < \beta < 1$. As $b \to 0$, consider

$$\Phi_b(\theta) := \log \mathrm{E} \mathrm{e}^{\mathrm{i}\theta b^{-1} (\log b^{-1})^{-1/(2\beta)} \mathcal{Z}_\beta(b\tau)}$$
$$= \psi(1)^2 (\log b^{-1})^{-1} \int_{\mathbb{R}_+^2} \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta z_b(\tau; x_1, x_2)} - 1](x_1 x_2)^{\beta-1} \mathrm{d}x_1 \mathrm{d}x_2,$$

where

$$z_b(\tau; x_1, x_2) := b^{-1} (\log b^{-1})^{-1/(2\beta)} z\big(b\tau; (\log b^{-1})^{-1/(2\beta)} x_1, (\log b^{-1})^{-1/(2\beta)} x_2\big)$$

satisfies

$$\mathrm{E}|z_b(\tau; x_1, x_2)|^2 \le \frac{C}{x_1 x_2} \big(1 \wedge \frac{b^{-1} (\log b^{-1})^{1/(2\beta)}}{x_1 + x_2}\big), \tag{A.7}$$

see (A.2). Split

$$\Phi_b(\theta) = \psi(1)^2 (\log b^{-1})^{-1} \int_{\mathbb{R}_+^2} (\mathbf{1}(1 < x_1 + x_2 < b^{-1}) + \mathbf{1}(x_1 + x_2 > b^{-1})$$

$$+\mathbf{1}(x_1+x_2<1))\mathrm{E}[\mathrm{e}^{\mathrm{i}\theta z_b(\tau;x_1,x_2)}-1](x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 =: \sum_{i=1}^{3} L_i.$$

Using (A.7), we can show that L_i, $i=2,3$ are remainders. By change of variables: $y=x_1+x_2$, $x_1=yw$ and then $w=z/y^2$, we rewrite the main term

$$\begin{aligned} L_1 &= \frac{1}{\log b^{-1}}\int_1^{b^{-1}} V_b(\theta;y)\frac{\mathrm{d}y}{y}, \\ V_b(\theta;y) &:= 2\psi(1)^2\int_0^{y^2/2}\Lambda_b(z;y)z^{\beta-1}\big(1-\frac{z}{y^2}\big)^{\beta-1}\mathrm{d}z \end{aligned} \tag{A.8}$$

with $\Lambda_b(z;y) := \mathrm{E}[\exp\{\mathrm{i}\theta z_b(\tau;\frac{z}{y},y(1-\frac{z}{y^2}))\}-1]$, which satisfies $|\Lambda_b(z;y)| \le C(1\wedge\frac{1}{z})$ for all $0<\frac{z}{y^2}<\frac{1}{2}$, $0<y<b^{-1}$. Here the dominating bound is a consequence of (A.7). Then

$$L_1 \to \log \mathrm{E}\mathrm{e}^{\mathrm{i}\theta\tau V_{2\beta}} = 2\psi(1)^2\int_0^\infty \Lambda(z)z^{\beta-1}\mathrm{d}z, \tag{A.9}$$

where $\Lambda(z) := \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta\tau Z_1Z_2/(2\sqrt{z})}-1]$ with $Z_i\sim N(0,1)$, $i=1,2$ being independent r.v.s, follows from

$$\lim_{y\to\infty, y=O(b^{-1})}\Lambda_b(z;y) = \Lambda(z), \quad \forall\, z>0, \tag{A.10}$$

for more details we refer the reader to the proof of Theorem 3.1 (i) case $0<\beta<1$. More precisely, (A.10) says that for every $\epsilon>0$ there exists a small $\delta>0$ such that for all $0<b<\delta$, if $\delta^{-1}<y<b^{-1}$, then $|\Lambda_b(z;y)-\Lambda(z)|<\epsilon$. To show (A.10), note $z_b(\tau;\frac{z}{y},y(1-\frac{z}{y^2})) = I_{12}(h_b(\cdot;\tau;z))$ is a double Itô-Wiener stochastic integral w.r.t. independent standard Brownian motions $\{B_i(s),\, s\in\mathbb{R}\}$, $i=1,2$ for

$$\begin{aligned} h_b(s_1,s_2;\tau;z) &:= (\log b^{-1})^{-1/(2\beta)}\int_0^\tau \prod_{i=1}^2 \mathrm{e}^{-\frac{1}{\alpha_i}(bu-s_i)}\mathbf{1}(s_i<bu)\mathrm{d}u, \quad s_1,s_2\in\mathbb{R}, \\ \alpha_1 &:= (\log b^{-1})^{1/(2\beta)}y/z, \quad \alpha_2 := (\log b^{-1})^{1/(2\beta)}/y', \quad y' := y\big(1-\frac{z}{y^2}\big). \end{aligned}$$

We have that $z_b(\tau;\frac{z}{y},y(1-\frac{z}{y^2})) =_{\mathrm{d}} I_{12}(\widetilde{h}_b(\cdot;\tau;z))$, where

$$\begin{aligned} \widetilde{h}_b(s_1,s_2;\tau;z) &:= \sqrt{\alpha_1\alpha_2}h_b(\alpha_1s_1,\alpha_2s_2;\tau;z) \\ &= \sqrt{\frac{y}{zy'}}\int_0^\tau\prod_{i=1}^2 \mathrm{e}^{-\frac{1}{\alpha_i}(bu-\alpha_is_i)}\mathbf{1}(\alpha_is_i<bu)\mathrm{d}u, \quad s_1,s_2\in\mathbb{R}. \end{aligned}$$

If $b\to0$, $y,y'\to\infty$ so that $y/y'\to1$ and $b/\alpha_i\to0$, $i=1,2$, then $\|\widetilde{h}_b(\cdot;\tau;z)-h(\cdot;\tau;z)\|\to0$ with

$$h(s_1,s_2;\tau;z) := \frac{\tau}{\sqrt{z}}\prod_{i=1}^2\mathrm{e}^{s_i}\mathbf{1}(s_i<0), \quad s_1,s_2\in\mathbb{R}, \tag{A.11}$$

implies the convergence $z_b(\tau; \frac{z}{y}, y(1-\frac{z}{y^2})) \to_{\mathrm{d}} I_{12}(h(\cdot;\tau;z)) =_{\mathrm{d}} \tau Z_1 Z_2/2\sqrt{z}$. Conditions on b, y, y' are obviously satisfied due to $y, y' = O(b^{-1}) = o(b^{-1} (\log b^{-1})^{1/(2\beta)})$. This proves (A.10) and (A.9), thereby completing the proof of of (2.20).

Proof of (2.21) follows that of Theorem 3.1 (ii). We will prove that as $b \to \infty$,

$$\begin{aligned}
&\log \mathrm{E}\mathrm{e}^{\mathrm{i}\theta b^{-1/2}\mathcal{Z}_\beta(b\tau)} \\
&= \psi(1)^2 \int_{\mathbb{R}_+^2} \mathrm{E}[\mathrm{e}^{\mathrm{i}\theta b^{-1/2} z(b\tau;x_1,x_2)} - 1](x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 \qquad \text{(A.12)}\\
&\to \psi(1)^2 \int_{\mathbb{R}_+^2} [\mathrm{e}^{-\theta^2\tau/(4x_1x_2(x_1+x_2))} - 1](x_1x_2)^{\beta-1}\mathrm{d}x_1\mathrm{d}x_2 \\
&= \log \mathrm{E}\mathrm{e}^{\mathrm{i}\theta\mathcal{A}^{1/2}B(\tau)}.
\end{aligned}$$

By (A.2), we have that $\mathrm{E}[\exp\{\mathrm{i}\theta b^{-1/2} z(b\tau;x_1,x_2)\} - 1] \le C\min\{1, (x_1x_2(x_1 + x_2))^{-1}\}$. In view of (2.9), the dominated convergence theorem applies if the integrands in (A.12) converge pointwise, i.e. for every $(x_1, x_2) \in \mathbb{R}_+^2$,

$$b^{-1/2} z(b\tau; x_1, x_2) \to_{\mathrm{d}} (2x_1x_2(x_1+x_2))^{-1/2} B(\tau). \qquad \text{(A.13)}$$

To simplify notation, let $\tau = 1$ and all $b \in \mathbb{N}$. Define

$$z_b^+(x_1,x_2) := \int_0^b \int_0^b f(s_1,s_2)\mathrm{d}B_1(s_1)\mathrm{d}B_2(s_2),$$

where

$$f(s_1,s_2) := b^{-1/2} \int_0^b \prod_{i=1}^2 \mathrm{e}^{-x_i(u-s_i)} \mathbf{1}(u > s_i)\mathrm{d}u,$$

and $z_b^-(x_1,x_2) := b^{-1/2} z(b;x_1,x_2) - z_b^+(x_1,x_2)$. Since $\mathrm{E}(z_b^-(x_1,x_2))^2 = O(b^{-1})$ implies $z_b^-(x_1,x_2) = o_{\mathrm{p}}(1)$, we only need to prove that

$$z_b^+(x_1,x_2) \to_{\mathrm{d}} N\big(0, \frac{1}{2x_1x_2(x_1+x_2)}\big) \quad \text{as } b \to \infty. \qquad \text{(A.14)}$$

Write $z_b^+(x_1,x_2) = \sum_{k=1}^b Z_k$ as a sum of a sum of a zero-mean square-integrable martingale difference array

$$\begin{aligned}
Z_k := &\int_{k-1}^k \int_0^{k-1} f(s_1,s_2)\mathrm{d}B_1(s_1)\mathrm{d}B_2(s_2) \\
&+ \int_0^{k-1} \int_{k-1}^k f(s_1,s_2)\mathrm{d}B_1(s_1)\mathrm{d}B_2(s_2) \\
&+ \int_{k-1}^k \int_{k-1}^k f(s_1,s_2)\mathrm{d}B_1(s_1)\mathrm{d}B_2(s_2)
\end{aligned}$$

w.r.t. the filtration $\mathcal{F}_k$ generated by $\{B_i(s), 0 \le s \le k, i = 1, 2\}$, $k = 0, \dots, b$. By the martingale CLT in Hall and Heyde [11], (A.14) then follows from

$$\sum_{k=1}^{b} \mathrm{E}[Z_k^2 | \mathcal{F}_{k-1}] \to_{\mathrm{p}} \frac{1}{2x_1x_2(x_1+x_2)} \quad \text{and} \quad \sum_{k=1}^{b} \mathrm{E}[Z_k^2 \mathbf{1}(|Z_k| > \epsilon)] \to 0, \tag{A.15}$$

for any $\epsilon > 0$. Since $\sum_{k=1}^{b} \mathrm{E}Z_k^2 = \int_0^b \int_0^b f^2(s_1, s_2)\mathrm{d}s_1\mathrm{d}s_2 = \mathrm{E}(z_b^+(x_1, x_2))^2 \to (2x_1x_2(x_1+x_2))^{-1}$, consider $R_b := \sum_{k=1}^{b}(\mathrm{E}[Z_k^2|\mathcal{F}_{k-1}] - \mathrm{E}Z_k^2)$, where

$$\begin{aligned}
\mathrm{E}[Z_k^2|\mathcal{F}_{k-1}] &= \int_{k-1}^{k} \Big(\int_0^{k-1} f(s_1, s_2)\mathrm{d}B_2(s_2)\Big)^2 \mathrm{d}s_1 \\
&\quad + \int_{k-1}^{k} \Big(\int_0^{k-1} f(s_1, s_2)\mathrm{d}B_1(s_1)\Big)^2 \mathrm{d}s_2 \\
&\quad + \int_{k-1}^{k} \int_{k-1}^{k} f^2(s_1, s_2)\mathrm{d}s_1\mathrm{d}s_2.
\end{aligned}$$

By rewriting $R_b =_{\mathrm{d}} \sum_{i=1}^{2} \int_0^b \int_0^b c_i(s_1, s_2)\mathrm{d}B_i(s_1)\mathrm{d}B_i(s_2)$ with $c_1(s_1, s_2) = \int_{\lceil s_1 \vee s_2 \rceil}^{b} f(s_1, s)f(s_2, s)\mathrm{d}s$, $c_2(s_1, s_2) = \int_{\lceil s_1 \vee s_2 \rceil}^{b} f(s, s_1)f(s, s_2)\mathrm{d}s$ and using the elementary bound:

$$\begin{aligned}
f(s_1, s_2) &\le Cb^{-1/2}(\mathrm{e}^{-x_1(s_2-s_1)}\mathbf{1}(s_1 < s_2) \\
&\qquad + \mathrm{e}^{-x_2(s_1-s_2)}\mathbf{1}(s_1 \ge s_2)), \quad 0 \le s_1, s_2 \le b,
\end{aligned} \tag{A.16}$$

we obtain $\mathrm{E}|R_b|^2 = \sum_{i=1}^{2} \int_0^b \int_0^b c_i^2(s_1, s_2)\mathrm{d}s_1\mathrm{d}s_2 = O(b^{-1}) = o(1)$, which proves $R_b = o_{\mathrm{p}}(1)$ and completes the proof of the first relation in (A.15). Finally, using (A.6), (A.16), we obtain $\sum_{k=1}^{b} \mathrm{E}|Z_k|^4 = O(b^{-1}) = o(1)$, which implies the second relation in (A.15) and completes the proof of (A.14).

Proposition 2.2 is proved. □

Proof of Proposition 2.3. (i) Split $\mathcal{Z}_\beta^*(\tau) = \widetilde{\mathcal{Z}}_\beta^*(\tau) + \tau V_\beta^+$ with

$$\begin{aligned}
\widetilde{\mathcal{Z}}_\beta^*(\tau) &:= \int_{\mathbb{R}_+ \times C(\mathbb{R})} \big(z^*(\tau; x) - \frac{\tau}{2x}\big)\mathrm{d}(\mathcal{M}_\beta^* - \mathrm{E}\mathcal{M}_\beta^* \mathbf{1}(1 < \beta < 2)), \\
V_\beta^+ &:= \int_{\mathbb{R}_+ \times C(\mathbb{R})} \frac{1}{2x}\mathrm{d}(\mathcal{M}_\beta^* - \mathrm{E}\mathcal{M}_\beta^* \mathbf{1}(1 < \beta < 2)),
\end{aligned}$$

where $\mathcal{M}_\beta^*$ is a Poisson random measure on $\mathbb{R}_+ \times C(\mathbb{R})$ with mean $\mu_\beta^* = \mathrm{E}\mathcal{M}_\beta^*$ given in (2.22). The existence of V_β^+ follows from $\int_0^\infty \min\{1, x^{-1}\}x^{\beta-1}\mathrm{d}x < \infty$ if $\beta \in (0, 1)$ and $\int_0^\infty \min\{x^{-1}, x^{-2}\}x^{\beta-1}\mathrm{d}x < \infty$ if $\beta \in (1, 2)$. The process $\widetilde{\mathcal{Z}}_\beta^*$ is well-defined if

$$\begin{aligned}
J_{p,\beta}^*(\tau) &:= \int_{\mathbb{R}_+ \times C(\mathbb{R})} |z^*(\tau; x) - \tau/(2x)|^p \mathrm{d}\mu_\beta^* \\
&= C \int_0^\infty \mathrm{E}|z^*(\tau; x) - \tau/(2x)|^p x^{\beta-1}\mathrm{d}x < \infty,
\end{aligned} \tag{A.17}$$

where $0 < p \le 1$ for $\beta \in (0,1)$ and $1 \le p \le 2$ for $\beta \in (1,2)$. We have $\mathrm{E}|z^*(\tau;x) - \tau/(2x)|^p \le (\mathrm{Var}(z^*(\tau;x)))^{p/2}$, where

$$
\begin{aligned}
\mathrm{Var}(z^*(\tau;x)) &= \int_{(0,\tau]^2} \mathrm{Cov}(\mathcal{Y}^2(u_1;x), \mathcal{Y}^2(u_2;x))\mathrm{d}u_1\mathrm{d}u_2 \\
&= 2\int_{(0,\tau]^2}\int_{\mathbb{R}^2} \mathrm{d}s_1\mathrm{d}s_2 \mathrm{e}^{-2x(u_1+u_2-s_1-s_2)}\mathbf{1}(s_1 \vee s_2 < u_1 \wedge u_2)\mathrm{d}u_1\mathrm{d}u_2 \\
&= \frac{1}{2x^2}\int_{(0,\tau]^2} \mathrm{e}^{-2x|u_1-u_2|}\mathrm{d}u_1\mathrm{d}u_2 = \frac{1}{8x^4}(2x\tau - 1 + \mathrm{e}^{-2x\tau}) \\
&\le C\frac{\tau^2}{x^2}\big(1 \wedge \frac{1}{x\tau}\big),
\end{aligned}
\tag{A.18}
$$

hence, $J^*_{p,\beta}(\tau) \le C\tau^{2p-\beta} < \infty$ for $p < \beta < 3p/2$. This completes the proof of part (i).

(ii) $\mathrm{E}|V^+_\beta|^p < \infty$ for $0 < p < \beta$, since V^+_β is a β-stable r.v. Similarly to (A.3), $\mathrm{E}|\widetilde{\mathcal{Z}}^*_\beta(\tau)|^p < \infty$ follows from $J^*_{p,\beta}(\tau) < \infty$ in (A.17), where p is sufficiently close to β and such that $0 < p < \beta < 3p/2$. This proves part (ii).

(iii) Follows from part (ii) by Kolmogorov's criterion, similarly as in the proof of Proposition 2.2.

(iv) For notational simplicity, we only prove one-dimensional convergence at $\tau > 0$. We have $\log \mathrm{E}\exp\{\mathrm{i}\theta b^{-1}\mathcal{Z}_\beta(b\tau)\} = \psi(1)\int_{\mathbb{R}_+} \Lambda_b(x)x^{\beta-1}\mathrm{d}x$, where

$$
\Lambda_b(x) := \mathrm{E}\big[\exp\big\{\mathrm{i}\theta b^{-1}z^*(b\tau;x)\big\} - 1 - \mathrm{i}\theta b^{-1}z^*(b\tau;x)\mathbf{1}(1<\beta<2)\big].
$$

Substituting $\mathrm{E}|z^*(b\tau;x)| \le (\mathrm{E}|z^*(b\tau;x)|^2)^{1/2}$ and $\mathrm{E}|z^*(b\tau;x)|^2 = \mathrm{Var}(z^*(b\tau;x)) + (b\tau/(2x))^2 \le C(b/x)^2$ by (A.18) into

$$
|\Lambda_b(x)| \le C\begin{cases}\min\big\{1, b^{-1}\mathrm{E}|z^*(b\tau;x)|\big\}, & 0<\beta<1, \\ \min\big\{b^{-1}\mathrm{E}|z^*(b\tau;x)|, b^{-2}\mathrm{E}|z^*(b\tau;x)|^2\big\}, & 1<\beta<2,\end{cases}
$$

we obtain the bounds: $|\Lambda_b(x)| \le C\min\{1, x^{-1}\}$ if $0 < \beta < 1$, and $|\Lambda_b(x)| \le C\min\{x^{-1}, x^{-2}\}$ if $1 < \beta < 2$. The result then follows from the dominated convergence theorem once we show that for all $x \in \mathbb{R}_+$,

$$
\Lambda_b(x) \to \begin{cases}\exp\{\mathrm{i}\theta\tau/(2x)\} - 1 - (\mathrm{i}\theta\tau/(2x))\mathbf{1}(1<\beta<2) & \text{as } b\to\infty, \\ \mathrm{E}[\exp\{\mathrm{i}\theta Z^2\tau/(2x)\} - 1 - (\mathrm{i}\theta Z^2\tau/(2x))\mathbf{1}(1<\beta<2)] & \text{as } b \to 0,\end{cases}
\tag{A.19}
$$

where $Z \sim N(0,1)$. Using (A.18), we get $\mathrm{E}|b^{-1}z^*(b\tau;x) - (\tau/(2x))|^2 = b^{-2}\,\mathrm{Var}(z^*(b\tau;x)) \le Cb^{-1} = o(1)$ as $b \to \infty$, which implies the first convergence in (A.19). To prove the second convergence in (A.19), note $Z/\sqrt{2x} =_{\mathrm{d}} \mathcal{Y}(0;x)$. It suffices to show that as $b \to 0$,

$$
\mathrm{E}|b^{-1}z^*(b\tau;x) - \tau\mathcal{Y}^2(0;x)| = \mathrm{E}\Big|\int_0^\tau (\mathcal{Y}^2(bu;x) - \mathcal{Y}^2(0;x))\mathrm{d}u\Big|
$$

$$\leq \int_0^\tau \mathrm{E}|\mathcal{Y}^2(bu;x) - \mathcal{Y}^2(0;x)|\mathrm{d}u = o(1).$$

Factorizing the difference of squares and applying the Cauchy-Schwarz inequality, this follows from

$$\mathrm{E}|\mathcal{Y}(bu;x) - \mathcal{Y}(0;x)|^2 = \int_0^{bu} \mathrm{e}^{-2xs}\mathrm{d}s + \frac{1}{2x}(\mathrm{e}^{-xbu} - 1)^2 \leq Cbu.$$

Proposition 2.3 is proved. □

Calculation of the constant σ_0 in Proposition 2.2 (v). We have

$$\begin{aligned}
\sigma_0 \cdot \frac{2^{2\beta/3}}{\psi(1)^2} &= \int_{\mathbb{R}_+^2} \big(1 - \exp\{-(u_1+u_2)^{-1}(u_1u_2)^{-1}\}\big)(u_1u_2)^{\beta-1}\mathrm{d}u_1\mathrm{d}u_2 \\
&\underset{u_2=u_1v_2}{=} \int_{\mathbb{R}_+^2} \big(1 - \exp\{-u_1^{-3}(1+v_2)^{-1}v_2^{-1}\}\big)u_1^{2\beta-1}v_2^{\beta-1}\mathrm{d}u_1\mathrm{d}v_2 \\
&\underset{u_1=v_1^{-1/3}}{=} \frac{1}{3}\int_{\mathbb{R}_+^2} \big(1 - \exp\{-v_1(1+v_2)^{-1}v_2^{-1}\}\big)v_1^{-2\beta/3-1}v_2^{\beta-1}\mathrm{d}v_1\mathrm{d}v_2 \\
&= \frac{1}{3}\int_{\mathbb{R}_+^2} \Big(\int_0^{1/((1+v_2)v_2)} \mathrm{e}^{-v_1t}\mathrm{d}t\Big)v_1^{-2\beta/3}v_2^{\beta-1}\mathrm{d}v_1\mathrm{d}v_2 \\
&= \frac{\Gamma(1-\frac{2\beta}{3})}{3}\int_0^\infty v_2^{\beta-1}\mathrm{d}v_2 \int_0^{1/((1+v_2)v_2)} t^{2\beta/3-1}\mathrm{d}t \\
&= \frac{\Gamma(1-\frac{2\beta}{3})}{2\beta}\int_0^\infty (1+v_2)^{-2\beta/3}v_2^{\beta/3-1}\mathrm{d}v_2 \\
&\underset{v_2=s^{-1}-1}{=} \frac{\Gamma(1-\frac{2\beta}{3})}{2\beta}\int_0^1 s^{2\beta/3}(s^{-1}-1)^{\beta/3-1}s^{-2}\mathrm{d}s \\
&= \frac{\Gamma(1-\frac{2\beta}{3})\,\mathrm{B}(\frac{\beta}{3},\frac{\beta}{3})}{2\beta}.
\end{aligned}$$

□

Proof of (4.22). By Corollary 4.1 (iv), $\frac{1}{Nn}\sum_{i=1}^N\sum_{k=1}^{n-t} X_i(k)X_i(k+t) \to_{\mathrm{p}} \gamma(t) = \mathrm{E}\frac{a^t}{1-a^2}$. Hence, relation (4.22) for (4.26) follows from

$$\frac{1}{N}\sum_{i=1}^N \Big(\frac{1}{n}\sum_{k=1}^{n-t} X_i(k)X_i(k+t)\Big)^2 \to_{\mathrm{p}} \mathrm{E}\Big(\frac{a^t}{1-a^2}\Big)^2. \tag{A.20}$$

By the LLN, $\frac{1}{N}\sum_{i=1}^N(\frac{a_i^t}{1-a_i^2})^2 \to_{\mathrm{p}} \mathrm{E}(\frac{a^t}{1-a^2})^2$. Therefore by Minkowski's inequality, for (A.20) we only need to show that

$$\frac{1}{N}\sum_{i=1}^N \Big(\frac{1}{n}\sum_{k=1}^{n} X_i(k)X_i(k+t) - \frac{a_i^t}{1-a_i^2}\Big)^2 = o_{\mathrm{p}}(1).$$

By taking expectations this follows from

$$\mathrm{E}\Big(\frac{1}{n}\sum_{k=1}^{n} X_i(k)X_i(k+t) - \frac{a_i^t}{1-a_i^2}\Big)^2 = \frac{1}{n^2}\mathrm{E}\,\mathrm{Var}\Big[\sum_{k=1}^{n} X_i(k)X_i(k+t)\big|a_i\Big] = o(1). \tag{A.21}$$

Using $\mathrm{Cov}[X_i(k)X_i(k+t), X_i(k')X_i(k'+t)|a_i] = \frac{a^{2(|k-k'|+t)}}{1-a^4}\ \mathrm{cum}_4\ + \frac{a^{2|k-k'|}}{(1-a^2)^2} + \frac{a^{2\max\{|k-k'|,t\}}}{(1-a^2)^2}$ and the same bound as in (2.30) we see that the l.h.s. of (A.21) does not exceed $C\mathrm{E}[\frac{1}{(1-a_i)^2}\min\{1, \frac{1}{n(1-a_i)}\}]$ which vanishes as $n\to\infty$ by the dominated convergence theorem, due to $\mathrm{E}\frac{1}{(1-a)^2} < \infty$. □

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