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The stochastic Cauchy problem driven by a cylindrical Lévy process

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Abstract

In this work, we derive sufficient and necessary conditions for the existence of a weak and mild solution of an abstract stochastic Cauchy problem driven by an arbitrary cylindrical Lévy process. Our approach requires to establish a stochastic Fubini result for stochastic integrals with respect to cylindrical Lévy processes. This approach enables us to conclude that the solution process has almost surely scalarly square integrable paths. Further properties of the solution such as the Markov property and stochastic continuity are derived.

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1 Introduction

Cylindrical Lévy processes naturally extend the class of cylindrical Brownian motions, which have been the standard model for random perturbations of partial differential equations for the last 50 years. The general concept of cylindrical Lévy processes in Banach spaces has been recently introduced by Applebaum and Riedle in [3]. However, so far only specific examples of cylindrical Lévy processes have been applied for modelling the driving noise of stochastic partial differential equations.

In this work we consider a linear evolution equation driven by an additive noise, or equivalently a stochastic Cauchy problem, of the form:

$$\mathrm{d}Y(t) = AY(t)\,\mathrm{d}t + B\,\mathrm{d}L(t) \qquad \text{for all } t \in [0,T]. \tag{1.1}$$

Here, L is a cylindrical Lévy process on a separable Hilbert space U, the coefficient A is the generator of a strongly continuous semigroup on a separable Hilbert space V

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and B is a linear, bounded operator from U to V. In this general setting, we present a complete theory for the existence of a mild and weak solution of (1.1) and derive some fundamental properties of the solution and its trajectories.

Only for specific examples of cylindrical Lévy processes L and sometimes under further restrictive assumptions on the generator A, the stochastic Cauchy problem (1.1) has been considered in most of the literature. There, typically one of the following two approaches are exploited: either the considered cylindrical Lévy process L is of such a form that the question of existence of a weak solution reduces to the study of a sequence of infinitely many one-dimensional processes or the underlying Hilbert space V is embedded in a larger space. The first approach is applied for example in the works Brzeźniak et al. [5], Liu and Zhai [16], and Priola and Zabczyk [21]. In these publications, the considered examples of a cylindrical Lévy process L only act along the eigenbasis of the generator A in an independent way. The second approach is utilised for example in the works [6] by Brzeźniak and van Neerven for cylindrical Brownian motion, [7] by Brzeźniak and Zabczyk for a cylindrical Lévy process modelled as a subordinated cylindrical Brownian motion or [20] by Peszat and Zabczyk for a general case. Although this approach is elegant and natural, one typically obtains conditions for the existence of a weak solution in terms of the larger space which per se is not related to the equation under consideration.

The stochastic Cauchy problem (1.1) exhibits a new phenomena which has not been observed in the Gaussian setting, i.e. when L is a cylindrical Brownian motion: the solution may exist as a stochastic process in the underlying Hilbert space V, but its trajectories are highly irregular; see for example Brzeźniak et al. [5], Brzeźniak and Zabczyk [7] and Peszat and Zabczyk [19]. In fact, the only positive results on some analytical regularity of the paths can be found in Liu and Zhai [17] and Peszat and Zabczyk [20]. However, these results are very restrictive and do not cover most of the considered examples of cylindrical Lévy processes.

For establishing the existence of a weak solution, the general noise considered in our work prevents us from following standard arguments as exploited for genuine Lévy processes, attaining values in V. In this case, one can either utilise the Lévy-Itô decomposition together with a stochastic Fubini theorem for the martingale part as it is done by Applebaum in [1] or by Peszat and Zabczyk in [19], or an integration by parts formula as accomplished by Chojnowska-Michalik in [8]. However, since our noise is cylindrical it does not enjoy a Lévy-Itô decomposition in the underlying Hilbert space. Also exploiting an integration by parts formula seems to be excluded as such a formula would indicate certain regularity of the paths. We circumvent these problems by applying a stochastic Fubini theorem but without decomposing the integrator of the stochastic integral.

However, also the stochastic Fubini theorem cannot be derived by standard methods due to the lack of a Lévy-Itô decomposition of the cylindrical Lévy process in the underlying Hilbert space. Even more, most of the results require finite moments of the stochastic integral, which is not guaranteed in our general framework; see Applebaum [1], Da Prato and Zabczyk [9] and Filipović et al. [12]. In our work, we succeed in establishing a stochastic Fubini result by using the observation that the iterated integrals can be considered as the inner product in a space of integrable functions. This observation and its elegant utilisation originates from the work van Neerven and Veraar [26]. Similar as in this work [26], however without having the theory of γ-radonifying operators at hand, we derive by tightness arguments, that the parameterised stochastic integral with respect to a cylindrical Lévy process defines a random variable in a space of integrable functions, which enables us to consider the iterated integrals as an inner product.

Surprisingly, our stochastic Fubini result and its application to the representation of the weak solution of (1.1) immediately yields that the trajectories of the solution are scalarly square integrable. As far as we know, this is the first positive result on an analytical path property of the solution of the stochastic Cauchy problem independent of the driving cylindrical Lévy process. Furthermore, having established the representation of the solution for (1.1) by a stochastic integral, which itself is based on the rich theory of cylindrical measures and cylindrical random variables, enables us to study further properties of the solution and its trajectories. More specifically, we show without any assumptions on the cylindrical Lévy process that the solution process is a Markov process and continuous in probability. For specific examples of cylindrical Lévy processes, these properties were established in [7] and [21]. However, there the arguments are strongly restricted to the specific examples under consideration. We are also able to provide a condition in our general framework which implies the non-existence of a modification of the solution with scalarly càdlàg trajectories, a phenomenon which has often been observed in several publications above cited. In fact, our condition covers all the examples in the literature, where this phenomenon has been observed, and it does not only strengthen the result in a few cases but also allows a geometric interpretation.

Our article begins with Section 2 where we fix most of our notations and introduce cylindrical Lévy processes and the stochastic integrals. In Section 3 we present and establish the stochastic Fubini theorem for stochastic integrals with respect to cylindrical Lévy processes and deterministic integrands. In the following Section 4 we apply the stochastic Fubini theorem to derive the existence of the weak solution of the stochastic Cauchy problem (1.1). In the final Section 5 we present some fundamental properties of the solution.

2 Preliminaries

Let U and V be separable Hilbert spaces with norms $\|\cdot\|$ and orthonormal bases $(e_k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$, respectively. We identify the dual of a Hilbert space with itself. The Borel σ-algebra of U is denoted by $\mathfrak{B}(U)$. The space of Radon probability measures on $\mathfrak{B}(U)$ is denoted by $\mathcal{M}(U)$ and is equipped with the Prokhorov metric. The space of all linear, bounded operators from U to V is denoted by $\mathcal{L}(U,V)$, equipped with the operator norm $\|\cdot\|_{\mathrm{op}}$; its subset of Hilbert-Schmidt operators is denoted by $\mathcal{L}_2(U,V)$, equipped with the norm $\|\cdot\|_{\mathrm{HS}}$. It follows from the standard characterisation of compact sets in Hilbert spaces, that a set $K\subseteq \mathcal{L}_2(U,V)$ is compact if and only if it is closed, bounded and satisfies

$$\lim_{N\to\infty}\sup_{\varphi\in K}\sum_{k=N+1}^{\infty}\left(\|\varphi e_k\|^2+\|\varphi^* h_k\|^2\right)=0. \tag{2.1}$$

The space of continuous functions from $[0,T]$ to U is denoted by $C([0,T];U)$ and it is equipped with the supremum norm $\|\cdot\|_\infty$. The space of all equivalence classes of measurable functions $f\colon \Omega\to U$ on a probability space $(\Omega,\mathcal{F},P)$ is denoted by $L^0_P(\Omega;U)$, and it is equipped with the topology of convergence in probability. The space $L^p_P(\Omega;U)$ for $p\in[1,\infty)$ contains all equivalence classes of measurable functions $f\colon\Omega\to U$ which are p-th integrable, and it is equipped with the usual norm.

For a subset Γ of U, sets of the form

$$C(u_1,...,u_n;B):=\{u\in U:(\langle u,u_1\rangle,...,\langle u,u_n\rangle)\in B\},$$

for $u_1,...,u_n\in\Gamma$ and $B\in\mathfrak{B}(\mathbb{R}^n)$ are called *cylindrical sets with respect to* Γ; the set of all these cylindrical sets is denoted by $\mathcal{Z}(U,\Gamma)$ and it is a σ-algebra if Γ is finite and otherwise an algebra. A function $\mu\colon\mathcal{Z}(U,U)\to[0,\infty]$ is called a *cylindrical measure*, if for each finite subset $\Gamma\subseteq U$ the restriction of μ on the σ-algebra $\mathcal{Z}(U,\Gamma)$ is a measure. A cylindrical measure is called finite if $\mu(U) < \infty$ and a cylindrical probability measure if $\mu(U) = 1$. A *cylindrical random variable* Z in U is a linear and continuous map

$$Z\colon U \to L_P^0(\Omega;\mathbb{R}).$$

Each cylindrical random variable Z defines a cylindrical probability measure λ by

$$\lambda\colon \mathcal{Z}(U,U) \to [0,1], \qquad \lambda(C) = P\big((Zu_1,\dots,Zu_n) \in B\big),$$

for cylindrical sets $C = C(u_1,...,u_n;B)$. The cylindrical probability measure λ is called the *cylindrical distribution* of Z. The characteristic function of a cylindrical random variable Z is defined by

$$\varphi_Z\colon U \to \mathbb{C}, \qquad \varphi_Z(u) = E[\exp(iZu)],$$

and it uniquely determines the cylindrical distribution of Z.

A family $(L(t) : t \geqslant 0)$ of cylindrical random variables is called a *cylindrical process*. It is called a *cylindrical Lévy process* if for all $u_1,...,u_n \in U$ and $n \in \mathbb{N}$, the stochastic process $((L(t)u_1,...,L(t)u_n) : t \geqslant 0)$ is a Lévy process in $\mathbb{R}^n$. This concept is introduced in [3] and it naturally extends the notion of a cylindrical Brownian motion. The characteristic function of $L(t)$ for all $t \geqslant 0$ is given by

$$\varphi_{L(t)}\colon U \to \mathbb{C}, \qquad \varphi_{L(t)}(u) = \exp\big(t\Psi(u)\big),$$

where $\Psi\colon U \to \mathbb{C}$ is called the symbol of L, and is of the form

$$\Psi(u) = ia(u) - \frac{1}{2}\langle Qu,u\rangle + \int_U \left(e^{i\langle u,h\rangle} - 1 - i\langle u,h\rangle \mathbb{1}_{B_\mathbb{R}}(\langle u,h\rangle)\right)\mu(\mathrm{d}h),$$

where $a\colon U \to \mathbb{R}$ is a continuous mapping with $a(0) = 0$, $Q\colon U \to U$ is a positive, symmetric operator and μ is a cylindrical measure on $\mathcal{Z}(U,U)$ satisfying

$$\int_U \big(\langle u,h\rangle^2 \wedge 1\big)\,\mu(\mathrm{d}h) < \infty \qquad \text{for all } u \in U.$$

We call (a,Q,μ) the *(cylindrical) characteristics of* L; see [23].

A function $g\colon [0,T] \to U$ is called *regulated* if g has only discontinuities of the first kind. The space of all regulated functions is denoted by $R([0,T];U)$ and it is a Banach space when equipped with the supremum norm; see [4, Ch. II.1.3] for this and other properties we will use. A function $f\colon [0,T] \to \mathcal{L}(U,V)$ is called *weakly in* $R([0,T];U)$ or *weakly regulated* if $f^*(\cdot)v$ is in $R([0,T];U)$ for each $v \in V$. Without further assumptions, one can define the integral $\int_0^T \mathbb{1}_A(t)f^*(t)v\,\mathrm{d}L(t)$ for all $v \in V$ and $A \in \mathfrak{B}([0,T])$ for a weakly regulated function $f\colon [0,T] \to \mathcal{L}(U,V)$. In this way, one obtains a cylindrical random variable by the prescription

$$Z_A\colon V \to L_P^0(\Omega;\mathbb{R}), \qquad Z_Av = \int_0^T \mathbb{1}_A(t)f^*(t)v\,\mathrm{d}L(t).$$

A much stronger property of f is the following definition of stochastically integrability: a function $f\colon [0,T] \to \mathcal{L}(U,V)$ is called *stochastically integrable with respect to* L if f is weakly in $R([0,T];U)$ and if for each $A \in \mathfrak{B}([0,T])$ there exists a V-valued random variable I_A such that

$$\langle I_A, v\rangle = Z_Av \qquad \text{for all } v \in V.$$

The stochastic integral I_A is also denoted by $\int_A f(s)\,\mathrm{d}L(s) := I_A$. From the very definition it follows that

$$\left\langle \int_A f(s)\,\mathrm{d}L(s), v \right\rangle = \int_A f^*(s)v\,\mathrm{d}L(s) \qquad \text{for all } v \in V. \tag{2.2}$$

Necessary and sufficient conditions for the stochastic integrability of a function are derived in the work [24, Thm. 5.10]. In particular, for Hilbert spaces U and V it states that a function $f\colon [0,T] \to \mathcal{L}(U,V)$, which is weakly in $R([0,T];U)$, is stochastically integrable with respect to a cylindrical Lévy process with characteristics (a,Q,μ) if and only if the following is satisfied:

(1) for every sequence $(v_n)_{n\in\mathbb{N}} \subseteq V$ converging weakly to 0 and $A \in \mathfrak{B}([0,T])$ we have

$$\lim_{n\to\infty} \int_A a(f^*(s)v_n)\,\mathrm{d}s = 0. \tag{2.3}$$

(2) $$\int_0^T \mathrm{tr}\big[f(t)Qf^*(t)\big]\,\mathrm{d}t < \infty; \tag{2.4}$$

(3) $$\lim_{m\to\infty} \sup_{n\geqslant m} \int_0^T \int_U \left(\sum_{k=m}^n \langle f(t)u, h_k\rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\,\mathrm{d}t = 0. \tag{2.5}$$

3 Stochastic Fubini theorem

In this section, we prove a stochastic version of Fubini's theorem, which will play an essential role later. As the cylindrical Lévy process L does not enjoy a Lévy-Itô decomposition in U we cannot exploit standard arguments. We will always denote by (a,Q,μ) the characteristics of L. Let $(S,\mathcal{S},\eta)$ be a finite measure space and $L^2_\eta(S;U)$ the Bochner space.

Theorem 3.1. *Let $g\colon S\times[0,T]\to U$ be a function satisfying the following assumptions:*

(a) *g is $\mathcal{S}\otimes\mathfrak{B}([0,T])$ measurable;*

(b) *the map $t\mapsto g(s,t)$ is regulated for η-almost all $s\in S$;*

(c) *the map $t\mapsto g(\cdot,t)$ belongs to $R([0,T];L^2_\eta(S;U))$.*

Then, P-almost surely, we have

$$\int_S \int_0^T g(s,t)\,\mathrm{d}L(t)\,\eta(\mathrm{d}s) = \int_0^T \int_S g(s,t)\,\eta(\mathrm{d}s)\,\mathrm{d}L(t),$$

and all integrals are well defined; in particular, we have

(1) *the map $t\mapsto \int_S g(s,t)\,\eta(\mathrm{d}s)$ is in $R([0,T];U)$;*

(2) *the process $\left(\int_0^T g(s,t)\,\mathrm{d}L(t) :\ s\in S\right)$ defines a random variable in $L^2_\eta(S;\mathbb{R})$.*

We divide the proof of the theorem into several lemmas. The theory of integration developed in [24] applies to deterministic integrands $\Phi\colon [0,T]\to\mathcal{L}(U,V)$ which are weakly regulated. In this case, the function Φ is integrable if and only if it satisfies the Conditions (2.3), (2.4) and (2.5). The following lemma shows that if Φ is Hilbert-Schmidt valued these conditions are already satisfied, i.e. Φ is stochastically integrable. This is in line with the general integration theory for random integrands developed in [14], where the random integrands are assumed to have cáglád trajectories.

Lemma 3.2. *Every regulated function $\Phi\colon [0,T] \to \mathcal{L}_2(U,V)$ is stochastically integrable with respect to L.*

Proof. From the inequality $\|\Phi^*(t)v\| \leqslant \|\Phi(t)\|_{\mathrm{HS}}\|v\|$ for all $v \in V$ and a Cauchy argument, it follows that the operator Φ is weakly in $R([0,T];U)$. We prove the stochastic integrability of Φ by verifying Conditions (2.3), (2.4) and (2.5). To verify (2.3), let $v_n \to 0$ weakly in V. As the operator $\Phi^*(t)$ is compact for each $t \in [0,T]$, it follows $\Phi^*(t)v_n \to 0$ in the norm topology of U. Since a is continuous and maps bounded sets to bounded sets by Lemma 3.2 in [24], Lebesgue's theorem on dominated convergence implies

$$\int_A a(\Phi^*(t)v_n)\,\mathrm{d}t \to 0 \qquad \text{as } n \to \infty \text{ for each } A \in \mathfrak{B}([0,T]).$$

Since the mapping $t \mapsto \Phi(t)$ is regulated and thus bounded, we obtain

$$\int_0^T \operatorname{tr}\big[\Phi(t)Q\Phi^*(t)\big]\,\mathrm{d}t = \int_0^T \Big\|\Phi(t)Q^{\frac{1}{2}}\Big\|_{\mathrm{HS}}^2\,\mathrm{d}t < \infty,$$

which shows Condition (2.4). To prove Condition (2.5), note that the monotone convergence theorem guarantees

$$\sup_{n\geqslant m}\int_0^T \int_U \left(\sum_{k=m}^n \langle \Phi(t)u, h_k\rangle^2 \wedge 1\right)\mu(\mathrm{d}u)\,\mathrm{d}t = \int_0^T f_m(t)\,\mathrm{d}t, \tag{3.1}$$

where for each $m \in \mathbb{N}$ and $t \in [0,T]$ we define

$$f_m(t) := \sup_{n\geqslant m}\int_U \left(\sum_{k=m}^n \langle \Phi(t)u, h_k\rangle^2 \wedge 1\right)\mu(\mathrm{d}u).$$

Let λ denote the cylindrical distribution of $L(1)$. As $\Phi(t)$ is Hilbert-Schmidt for each fixed $t \in [0,T]$, the image measure $\lambda \circ \Phi^{-1}(t)$ is a genuine infinitely divisible measure with classical Lévy measure $\mu \circ \Phi^{-1}(t)$. Consequently, we can apply the monotone convergence theorem and Lebesgue's theorem on dominated convergence to obtain for each $t \in [0,T]$ that

$$\begin{aligned} f_m(t) &= \sup_{n\geqslant m}\int_V \left(\sum_{k=m}^n \langle v, h_k\rangle^2 \wedge 1\right)(\mu\circ\Phi^{-1}(t))(\mathrm{d}v) \\ &= \int_V \left(\sum_{k=m}^\infty \langle v, h_k\rangle^2 \wedge 1\right)(\mu\circ\Phi^{-1}(t))(\mathrm{d}v) \to 0 \qquad \text{as } m\to\infty. \end{aligned} \tag{3.2}$$

Since the set $K := \overline{\{\Phi(t) : t \in [0,T]\}}$ is a compact subset of $\mathcal{L}_2(U,V)$ by Problem 1 in [10, Ch. VII.6], Proposition 5.3 in [14] implies that the set $\{\lambda\circ\varphi^{-1} : \varphi \in K\}$ is relatively compact in the space of probability measures on $\mathfrak{B}(V)$. Since $\lambda\circ\varphi^{-1}$ is infinitely divisible with Lévy measure $\mu\circ\varphi^{-1}$, Theorem VI.5.3 in [18] implies

$$\sup_{\varphi\in K}\int_{\|v\|\leqslant 1}\|v\|^2(\mu\circ\varphi^{-1})(\mathrm{d}v) < \infty \text{ and } \sup_{\varphi\in K}(\mu\circ\varphi^{-1})(\{v : \|v\|>1\}) < \infty.$$

Consequently, we obtain

$$\sup_{m\in\mathbb{N}}\sup_{t\in[0,T]} f_m(t) \leqslant \sup_{\varphi\in K}\int_{\|v\|\leqslant 1}\|v\|^2(\mu\circ\varphi^{-1})(\mathrm{d}v) + \sup_{\varphi\in K}\int_{\|v\|>1}(\mu\circ\varphi^{-1})(\mathrm{d}v) < \infty. \tag{3.3}$$

The limit (3.2) and the inequality (3.3) enable us to apply Lebesgue's theorem in (3.1), which proves Condition (2.5). ☐

For some $u \in U$ and $v \in V$, we define the operator $u \otimes v \colon U \to V$ by $(u \otimes v)(w) := \langle u, w \rangle v$.

Lemma 3.3. *If $\Phi \colon [0,T] \to \mathcal{L}_2(U,V)$ is a regulated function, then $\sum_{j=1}^{m} e_j \otimes \Phi(\cdot) e_j$ converges to Φ in $R\big([0,T], \mathcal{L}_2(U,V)\big)$ as $m \to \infty$.*

Proof. By Problem 1 in [10, Ch. VII.6], the set $K := \overline{\{\Phi(t) : t \in [0,T]\}}$ is compact in $\mathcal{L}_2(U,V)$. By applying (2.1) we conclude

$$\begin{aligned} \sup_{t\in[0,T]} \left\| \Phi(t) - \sum_{j=1}^{m} e_j \otimes \Phi(t) e_j \right\|_{\mathrm{HS}}^2 &= \sup_{t\in[0,T]} \sum_{i=1}^{\infty} \left\| \Phi(t)e_i - \sum_{j=1}^{m} \langle e_j, e_i \rangle \Phi(t) e_j \right\|^2 \\ &= \sup_{t\in[0,T]} \sum_{i=m+1}^{\infty} \|\Phi(t)e_i\|^2 \\ &\leqslant \sup_{\varphi\in K} \sum_{i=m+1}^{\infty} \|\varphi e_i\|^2 \to 0 \text{ as } m \to \infty, \end{aligned}$$

which completes the proof. □

Lemma 3.4. *For each regulated function $\Phi \colon [0,T] \to \mathcal{L}_2(U,V)$ there exists a sequence of partitions $\{(t_k^n)_{k=0}^{N_n} : n \in \mathbb{N}\}$ of $[0,T]$ with $\max_{0\leqslant k \leqslant N_n - 1} |t_{k+1}^n - t_k^n| \to 0$ as $n \to \infty$ such that the functions*

$$\Phi_{m,n}(t) := \begin{cases} \displaystyle\sum_{j=1}^{m} e_j \otimes \Phi\left(\tfrac{t_k^n + t_{k+1}^n}{2}\right) e_j, & \text{if } t \in (t_k^n, t_{k+1}^n),\ k = 0, \dots, N_n - 1, \\ \displaystyle\sum_{j=1}^{m} e_j \otimes \Phi(t_k^n) e_j, & \text{if } t = t_k^n,\ k = 0, \dots, N_n, \end{cases}$$

satisfy

$$\lim_{m,n\to\infty} \sup_{t\in[0,T]} \|\Phi_{m,n}(t) - \Phi(t)\|_{HS} = 0, \tag{3.4}$$

and

$$\lim_{m,n\to\infty} \int_0^T \Phi_{m,n}(t)\,\mathrm{d}L(t) = \int_0^T \Phi(t)\,\mathrm{d}L(t) \qquad \textit{in probability.} \tag{3.5}$$

Proof. Using [10, 7.6.1], we can construct a sequence $\{(t_k^n)_{k=0}^{N_n} : n \in \mathbb{N}\}$ of partitions of $[0,T]$ such that $\max\limits_{0\leqslant k\leqslant N_n-1} |t_{k+1}^n - t_k^n| \to 0$ and that the functions

$$\Phi_n(t) := \begin{cases} \Phi\left(\tfrac{t_k^n + t_{k+1}^n}{2}\right), & \text{if } t \in (t_k^n, t_{k+1}^n),\ k = 0, \dots, N_n - 1, \\ \Phi(t_k^n), & \text{if } t = t_k^n,\ k = 0, \dots, N_n, \end{cases}$$

satisfy $\sup_{t\in[0,T]} \|\Phi(t) - \Phi_n(t)\|_{\mathrm{HS}} \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ be given. Then there exists some $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have

$$\sup_{t\in[0,T]} \|\Phi(t) - \Phi_n(t)\|_{\mathrm{HS}} \leqslant \frac{\varepsilon}{2}. \tag{3.6}$$

By Lemma 3.3, there exists $M > 0$, such that for all $m \geqslant M$, we have

$$\sup_{t\in[0,T]} \left\| \Phi(t) - \sum_{j=1}^{m} e_j \otimes \Phi(t) e_j \right\|_{\mathrm{HS}} \leqslant \frac{\varepsilon}{2}. \tag{3.7}$$

Using (3.6) and (3.7) we have for all $n \geqslant N$ and $m \geqslant M$,

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\Phi(t) - \Phi_{m,n}(t)\|_{\mathrm{HS}} \\
&\quad\leqslant \sup_{t\in[0,T]} \|\Phi(t) - \Phi_n(t)\|_{\mathrm{HS}} + \sup_{t\in[0,T]} \|\Phi_n(t) - \Phi_{m,n}(t)\|_{\mathrm{HS}} \\
&\quad= \sup_{t\in[0,T]} \|\Phi(t) - \Phi_n(t)\|_{\mathrm{HS}} + \sup_{t\in[0,T]} \Bigg(\sum_{k=0}^{N_n} \mathbb{1}_{\{t_k^n\}}(t) \left\| \Phi(t_k^n) - \sum_{j=1}^m e_j \otimes \Phi(t_k^n) e_j \right\|_{\mathrm{HS}} \\
&\qquad + \sum_{k=0}^{N_n-1} \mathbb{1}_{(t_k^n, t_{k+1}^n)}(t) \left\| \Phi\left(\tfrac{t_k^n + t_{k+1}^n}{2}\right) - \sum_{j=1}^m e_j \otimes \Phi\left(\tfrac{t_k^n + t_{k+1}^n}{2}\right) e_j \right\|_{\mathrm{HS}} \Bigg) \\
&\quad\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which proves (3.4). Let $P_{m,n}$ denote the distribution of $\int_0^T \Phi_{m,n}(t)\,\mathrm{d}L(t)$. For establishing (3.5), it is sufficient by [13, Lemma 2.4] to show:

(i) $\left\langle \int_0^T \Phi_{m,n}(t)\,\mathrm{d}L(t) - \int_0^T \Phi(t)\,\mathrm{d}L(t), v \right\rangle \to 0$ in probability for all $v \in V$;

(ii) $\{P_{m,n} : m, n \in \mathbb{N}\}$ is relatively compact in $\mathcal{M}(V)$.

As $\Phi_{m,n}^*(\cdot)v$ converges uniformly to $\Phi^*(\cdot)v$ for each $v \in V$ due to (3.4), Lemma 5.2 in [24] implies

$$\left\langle \int_0^T \Phi_{m,n}(t)\,\mathrm{d}L(t) - \int_0^T \Phi(t)\,\mathrm{d}L(t), v \right\rangle = \int_0^T \big(\Phi_{m,n}^*(t) - \Phi^*(t)\big)v\,\mathrm{d}L(t) \to 0$$

in probability which establishes (i). To prove (ii), we define the set

$$K_1 := \left\{ \sum_{j=1}^m e_j \otimes \varphi e_j : m \in \mathbb{N} \cup \{\infty\}, \varphi \in K \right\},$$

where $K := \overline{\{\Phi(t) : t \in [0,T]\}}$. Since K is a compact subset of $\mathcal{L}_2(U,V)$, it follows that K_1 is closed and bounded. By applying (2.1) we obtain

$$\begin{aligned}
&\lim_{N\to\infty} \sup_{\psi\in K_1} \sum_{k=N+1}^\infty \big(\|\psi e_k\|^2 + \|\psi^* h_k\|^2 \big) \\
&\quad= \lim_{N\to\infty} \sup_{\varphi\in K} \sup_{m\in\mathbb{N}\cup\{\infty\}} \sum_{k=N+1}^\infty \left(\left\| \sum_{j=1}^m \langle e_j, e_k\rangle \varphi e_j \right\|^2 + \left\| \sum_{j=1}^m \langle \varphi e_j, h_k\rangle e_j \right\|^2 \right) \\
&\quad\leqslant \lim_{N\to\infty} \sup_{\varphi\in K} \sum_{k=N+1}^\infty \big(\|\varphi e_k\|^2 + \|\varphi^* h_k\|^2 \big) = 0,
\end{aligned}$$

which shows that K_1 is a compact subset of $\mathcal{L}_2(U,V)$. Proposition 5.3 in [14] guarantees that the set $\{\lambda \circ \psi^{-1} : \psi \in K_1\}$, is relatively compact in the space of probability measures on $\mathfrak{B}(V)$, where λ is the cylindrical distribution of $L(1)$. Since

$$P_{m,n} = (\lambda \circ (\psi_{m,1}^n)^{-1})^{*(t_1^n - t_0^n)} * \cdots * (\lambda \circ (\psi_{m,N_n-1}^n)^{-1})^{*(t_{N_n}^n - t_{N_n-1}^n)},$$

where $\psi_{m,k}^n := \sum_{j=1}^m e_j \otimes \Phi\left(\frac{t_k^n + t_{k+1}^n}{2}\right) e_j$ and $\psi_{m,k}^n$ is in the compact set K_1 for each $k \in \{0, \dots, N_n\}$, Lemma 5.4 in [14] implies (ii). $\square$

Lemma 3.5. *The map* $J\colon R\big([0,T];L^2_\eta(S;U)\big)\to R\big([0,T];\mathcal{L}_2(U,L^2_\eta(S;\mathbb{R}))\big)$ *with* $J(f)(t)u=\langle u,f(t)(\cdot)\rangle$ *is a well defined isometric isomorphism.*

Proof. For each $t\in[0,T]$ and $f\in R\big([0,T];L^2_\eta(S;U)\big)$, the map $J(f)(t)$ defines a linear and continuous operator from U to $L^2_\eta(S;\mathbb{R})$ and satisfies

$$\|J(f)(t)\|_{\mathrm{HS}}^2=\sum_{j=1}^\infty\int_S\langle e_j,f(t)(s)\rangle^2\,\eta(\mathrm{d}s)=\|f(t)\|^2_{L^2_\eta(S;U)}. \tag{3.8}$$

As $t\mapsto f(t)$ is regulated, the isometry (3.8) shows by a Cauchy argument that $t\mapsto J(f)(t)$ is regulated. Consequently, J is a well defined linear isometry and it is left to show that J is surjective.

For this purpose, let Φ be in $R\big([0,T];\mathcal{L}_2(U,L^2_\eta(S;\mathbb{R}))\big)$. We define

$$f\colon[0,T]\to L^2_\eta(S;U),\qquad f(t)(\cdot):=\sum_{j=1}^\infty\left(\Phi(t)e_j\right)(\cdot)e_j,$$

where the series converges in $L^2_\eta(S;U)$. Since we have that $\|f(t)\|_{L^2_\eta(S;U)}=\|\Phi(t)\|_{\mathcal{L}_2(U,L^2_\eta(S;\mathbb{R}))}$, the function $t\mapsto f(t)$ is regulated and satisfies

$$(J(f)(t))(u)=\sum_{j=1}^\infty\Phi(t)e_j(\cdot)\langle u,e_j\rangle=\sum_{j=1}^\infty\Phi(t)\big(\langle u,e_j\rangle e_j\big)(\cdot)=\Phi(t)\,(u)\,(\cdot),$$

which completes the proof. □

Lemma 3.6. *Let* $g\colon S\times[0,T]\to U$ *be a function such that the map* $t\to g(s,t)$ *is regulated for* η*-almost all* $s\in S$*, and* $\{(t_k^n)_{k=0}^{N_n}:n\in\mathbb{N}\}$ *be a sequence of partitions of* $[0,T]$ *with* $\max_{0\leqslant k\leqslant N_n-1}|t^n_{k+1}-t^n_k|\to 0$*. Then the functions* $g_{m,n}\colon S\times[0,T]\to U$ *defined by*

$$g_{m,n}(s,t):=\begin{cases}\displaystyle\sum_{j=1}^m\left\langle e_j,g\left(s,\tfrac{t_k^n+t_{k+1}^n}{2}\right)\right\rangle e_j & \text{if } t\in(t_k^n,t_{k+1}^n),\,k=0,\dots,N_n-1,\\ \displaystyle\sum_{j=1}^m\langle e_j,g(s,t_k^n)\rangle e_j, & \text{if } t=t_k^n,\,k=0,\dots,N_n,\end{cases}$$

satisfy for η*-almost all* $s\in S$ *that*

$$\|g_{m,n}(s,t)-g(s,t)\|\to 0\quad\text{for Lebesgue-almost all } t\in[0,T].$$

Proof. For each $n\in\mathbb{N}$, define $g_n\colon S\times[0,T]\to U$ by

$$g_n(s,t):=\sum_{k=0}^{N_n-1}\mathbb{1}_{(t_k^n,t_{k+1}^n)}(t)g\left(s,\frac{t_k^n+t_{k+1}^n}{2}\right)+\sum_{k=0}^{N_n-1}\mathbb{1}_{\{t_k^n\}}(t)g(s,t_k^n).$$

Let $s\in S$ be such that $g(s,\cdot)$ is regulated. Then the set $A_s\subseteq[0,T]$ of discontinuities of $g(s,\cdot)$ has Lebesgue measure 0 and for each $t\in A_s^c$ it follows that

$$\lim_{n\to\infty}\|g_n(s,t)-g(s,t)\|=0. \tag{3.9}$$

The set $\overline{\{g(s,t):t\in[0,T]\}}$ is compact in U by Problem 1 in [10, VII.6]. The compactness criterion in Hilbert spaces implies

$$\sup_{t\in[0,T]}\left\|\sum_{j=1}^m\langle g(s,t),e_j\rangle e_j-g(s,t)\right\|^2=\sup_{t\in[0,T]}\sum_{j=m+1}^\infty\langle g(s,t),e_j\rangle^2\to 0\quad\text{as } m\to\infty. \tag{3.10}$$

By using (3.9) and (3.10) we obtain for each $t \in A_s^c$ that

$$\begin{aligned}
&\|g_{m,n}(s,t) - g(s,t)\| \\
&\quad \leqslant \|g_{m,n}(s,t) - g_n(s,t)\| + \|g_n(s,t) - g(s,t)\| \\
&\quad = \sum_{k=0}^{N_n-1} \mathbb{1}_{(t_k^n, t_{k+1}^n)}(t) \left\| \sum_{j=1}^m \left\langle g\left(s, \tfrac{t_k^n + t_{k+1}^n}{2}\right), e_j \right\rangle e_j - g\left(s, \tfrac{t_k^n + t_{k+1}^n}{2}\right) \right\| \\
&\qquad + \sum_{k=0}^{N_n} \mathbb{1}_{\{t_k^n\}}(t) \left\| \sum_{j=1}^m \langle g(s,t_k^n), e_j\rangle e_j - g(s,t_k^n) \right\| + \|g_n(s,t) - g(s,t)\| \\
&\quad \leqslant \sup_{r\in[0,T]} \left\| \sum_{j=1}^m \langle g(s,r), e_j\rangle e_j - g(s,r) \right\| + \|g_n(s,t) - g(s,t)\| \\
&\quad \to 0 \qquad \text{as } m,n \to \infty,
\end{aligned}$$

which completes the proof. □

In the following we extend our definition of the space $L_P^0(\Omega; U)$ to a finite measure space $(A, \mathcal{A}, \sigma)$ and a complete metric space (E, d). In this case, $L_\sigma^0(A; E)$ denotes the space of the equivalence classes of all separably-valued, measurable functions from A to E. As before, the space is an F-space equipped with the metric

$$\rho(f,g) := \int_A \big(d(f(x), g(x)) \wedge 1\big)\, \sigma(\mathrm{d}x). \tag{3.11}$$

Instead of separably-valued, one can equivalently require strong measurability of the functions.

Lemma 3.7. *Let $(A_1, \mathcal{A}_1, \sigma_1)$ and $(A_2, \mathcal{A}_2, \sigma_2)$ be two finite measure spaces and V be a separable Hilbert space. Then*

$$L_{\sigma_1}^0\big(A_1; L_{\sigma_2}^0(A_2; V)\big) \cong L_{\sigma_1\otimes\sigma_2}^0\big(A_1 \times A_2; V\big).$$

In particular, the isomorphism is given such that for each $\mathcal{A}_1$-measurable function $F\colon A_1 \to L_{\sigma_2}^0(A_2; V)$, there corresponds an $\mathcal{A}_1 \otimes \mathcal{A}_2$-measurable function $f\colon A_1 \times A_2 \to V$ such that for σ_1-almost all $x \in A_1$, we have $F(x) = f(x, \cdot)$ in $L_{\sigma_2}^0(A_2; V)$, and conversely.

Proof. The lemma can be proved similarly as Lemma III.11.16 in [11] by replacing L^p-norms for $p \geqslant 1$ by the corresponding metrics as defined in (3.11). □

Proof of Theorem 3.1. Lemma 3.5 guarantees that the mapping

$$\Phi\colon [0,T] \to \mathcal{L}_2(U, L_\eta^2(S; \mathbb{R})), \qquad \Phi(t)u := \langle u, g(\cdot, t)\rangle,$$

is well defined and regulated. Let $\Phi_{m,n}$ denote the functions defined in Lemma 3.4 for $V = L_\eta^2(S; \mathbb{R})$. Lemma 3.4 together with Lemma 3.7 imply, upon passing to a subsequence, that for $(\eta \otimes P)$-almost all $(s, \omega) \in S \times \Omega$ we have

$$\left(\left(\int_0^T \Phi(t)\, \mathrm{d}L(t)\right)(\omega)\right)(s) = \lim_{m,n\to\infty} \left(\left(\int_0^T \Phi_{m,n}(t)\, \mathrm{d}L(t)\right)(\omega)\right)(s). \tag{3.12}$$

For each $h \in L^2_\eta(S;\mathbb{R})$, we obtain by (2.2) that

$$
\begin{aligned}
&\left\langle \int_0^T \Phi_{m,n}(t)\,\mathrm{d}L(t), h \right\rangle_{L^2_\eta(S;\mathbb{R})} \\
&\qquad = \int_0^T \Phi^*_{m,n}(t) h \,\mathrm{d}L(t) \\
&\qquad = \sum_{k=0}^{N_n-1} \big(L(t^n_{k+1}) - L(t^n_k)\big) \left(\sum_{j=1}^m \left\langle e_j, \Phi^*\left(\tfrac{t^n_k + t^n_{k+1}}{2}\right) h \right\rangle e_j \right) \\
&\qquad = \sum_{k=0}^{N_n-1} \sum_{j=1}^m \left\langle \Phi\left(\tfrac{t^n_k + t^n_{k+1}}{2}\right) e_j, h \right\rangle_{L^2_\eta(S;\mathbb{R})} \big(L(t^n_{k+1}) - L(t^n_k)\big)(e_j) \\
&\qquad = \left\langle \sum_{k=0}^{N_n-1} \sum_{j=1}^m \Phi\left(\tfrac{t^n_k + t^n_{k+1}}{2}\right) e_j \big(L(t^n_{k+1}) - L(t^n_k)\big)(e_j), h \right\rangle_{L^2_\eta(S;\mathbb{R})}.
\end{aligned}
$$

Therefore, for η-almost all $s \in S$, we have

$$
\begin{aligned}
&\left(\int_0^T \Phi_{m,n}(t)\,\mathrm{d}L(t) \right)(s) \\
&\qquad = \left(\sum_{k=0}^{N_n-1} \sum_{j=1}^m \Phi\left(\tfrac{t^n_k + t^n_{k+1}}{2}\right) e_j \big(L(t^n_{k+1}) - L(t^n_k)\big)(e_j) \right)(s) \\
&\qquad = \sum_{k=0}^{N_n-1} \sum_{j=1}^m \left(\Phi\left(\tfrac{t^n_k + t^n_{k+1}}{2}\right) e_j \right)(s) \big(L(t^n_{k+1}) - L(t^n_k)\big)(e_j) \\
&\qquad = \sum_{k=0}^{N_n-1} \big(L(t^n_{k+1}) - L(t^n_k)\big) \left(\sum_{j=1}^m \left(\Phi\left(\tfrac{t^n_k + t^n_{k+1}}{2}\right) e_j \right)(s) e_j \right) \\
&\qquad = \int_0^T g_{m,n}(s,t)\,\mathrm{d}L(t), \qquad (3.13)
\end{aligned}
$$

where $g_{m,n}$ denotes the functions defined in Lemma 3.6. By Lemma 5.4 in [24], we have for each $\alpha \in \mathbb{R}$ that

$$
\begin{aligned}
&E\left[\exp\left(i\alpha \int_0^T \big(g_{m,n}(s,t) - g(s,t)\big)\,\mathrm{d}L(t) \right) \right] \\
&\qquad = \exp\left(\int_0^T \Psi\big(\alpha\,(g_{m,n}(s,t) - g(s,t))\big)\,\mathrm{d}t \right), \qquad (3.14)
\end{aligned}
$$

where Ψ denotes the Lévy symbol of L. Note that

$$
\sup_{t\in[0,T]} \|g_{m,n}(s,t) - g(s,t)\|^2 \leqslant 4 \sup_{t\in[0,T]} \|g(s,t)\|^2 < \infty.
$$

Since Ψ is continuous and maps bounded sets to bounded sets according to Lemma 3.2 in [24], it follows by Lebesgue's theorem on dominated convergence and Lemma 3.6 that

$$
\lim_{m,n\to\infty} \int_0^T \Psi\left(\alpha(g_{m,n}(s,t) - g(s,t))\right)\,\mathrm{d}t = 0.
$$

Consequently, we deduce from (3.14) that for η-almost all $s \in S$,

$$\lim_{m,n\to\infty} \int_0^T g_{m,n}(s,t)\,\mathrm{d}L(t) = \int_0^T g(s,t)\,\mathrm{d}L(t) \qquad \text{in probability.} \tag{3.15}$$

Comparing limits in (3.12) and (3.15) by means of (3.13), we obtain for η-almost all $s \in S$, that we have for P-almost all $\omega \in \Omega$:

$$\left(\int_0^T g(s,t)\,\mathrm{d}L(t)\right)(\omega) = \left(\left(\int_0^T \Phi(t)\,\mathrm{d}L(t)\right)(\omega)\right)(s). \tag{3.16}$$

By (3.15) and Lemma 3.7, the left hand side in (3.16) is $\mathcal{S}\otimes\mathcal{F}$ measurable, as well as the right hand side due to (3.12). A further application of Fubini's theorem implies for P-almost all $\omega \in \Omega$ that

$$\left(\int_0^T g(s,t)\,\mathrm{d}L(t)\right)(\omega) = \left(\left(\int_0^T \Phi(t)\,\mathrm{d}L(t)\right)(\omega)\right)(s) \quad \text{for } \eta\text{-a.a. } s \in S.$$

By integrating both sides and denoting by 1 the function in $L^2_\eta(S;\mathbb{R})$ which constantly equals one, we obtain by (2.2) that

$$\begin{aligned}
\int_S \left(\int_0^T g(s,t)\,\mathrm{d}L(t)\right)(\omega)\,\eta(\mathrm{d}s) &= \int_S \left(\left(\int_0^T \Phi(t)\,\mathrm{d}L(t)\right)(\omega)\right)(s)\,\eta(\mathrm{d}s) \\
&= \left\langle \left(\int_0^T \Phi(t)\,\mathrm{d}L(t)\right)(\omega), 1 \right\rangle_{L^2_\eta(S;\mathbb{R})} \\
&= \left(\int_0^T \Phi^*(t)1\,\mathrm{d}L(t)\right)(\omega) \\
&= \left(\int_0^T \int_S g(s,t)\eta(\mathrm{d}s)\,\mathrm{d}L(t)\right)(\omega),
\end{aligned}$$

which completes the proof. □

4 Cauchy problem

We consider the following stochastic Cauchy problem driven by a cylindrical Lévy process L in a separable Hilbert space U:

$$\begin{aligned}
\mathrm{d}Y(t) &= AY(t)\,\mathrm{d}t + B\,\mathrm{d}L(t) \qquad \text{for all } t \in [0,T], \\
Y(0) &= y_0,
\end{aligned} \tag{4.1}$$

where A is a generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a separable Hilbert space V, $B\colon U \to V$ is a linear and continuous operator and the initial condition y_0 is in V.

In the case of L being a cylindrical Brownian motion, the concept of weak solution is defined in [9] and the existence and uniqueness of weak solution is established. Their definition requires weak solutions to have almost surely Bochner integrable paths. In case of Banach spaces, a similar definition is used in [25]. However, as it is known that the solution of (4.1) may exhibit highly irregular paths, the requirement of Bochner integrable paths is too restrictive in our situation. A weaker condition requires only that the paths $t \mapsto \langle Y(t), A^*v\rangle$ are integrable for $v \in \mathcal{D}(A^*)$; see [6], [19] and [27]. We will impose a slightly stronger condition but which is still weaker than Bochner integrability of the paths.

Definition 4.1. *A V-valued stochastic process* $(Y(t) : t \geqslant 0)$ *is called* weakly Bochner regular *if* $t \mapsto \langle Y(t), g(t)\rangle$ *is integrable on* $[0,T]$ *for each* $g \in C([0,T];V)$ *and for every sequence* $(g_n)_{n\in\mathbb{N}} \subseteq C([0,T];V)$ *with* $\|g_n\|_\infty \to 0$ *as* $n \to \infty$*, we have*

$$\int_0^T \langle Y(s), g_n(s)\rangle \,\mathrm{d}s \to 0 \qquad \text{in probability as } n \to \infty.$$

If the stochastic process Y has Bochner integrable paths on $[0,T]$, then Y is also weakly Bochner regular as shown by a simple estimate.

Definition 4.2. *A* V*-valued, progressively measurable stochastic process* $(Y(t) : t \in [0,T])$ *is called a* weak solution *of the stochastic Cauchy problem* (4.1) *if* Y *is weakly Bochner regular and satisfies for every* $v \in \mathcal{D}(A^*)$ *and* $t \in [0,T]$*,* P*-almost surely, that*

$$\langle Y(t), v\rangle = \langle y_0, v\rangle + \int_0^t \langle Y(s), A^*v\rangle \,\mathrm{d}s + L(t)(B^*v). \tag{4.2}$$

Theorem 4.3. *If the mapping* $s \mapsto T(s)B$ *is stochastically integrable on* $[0,T]$ *with respect to* L*, then*

$$Y(t) = T(t)y_0 + \int_0^t T(t-s)B \,\mathrm{d}L(s), \quad t \in [0,T],$$

is a weak solution of the stochastic Cauchy problem (4.1)*.*

Example 4.4. In this and the next example we set $V = U$, $B = \mathrm{Id}$ and assume that there exist $\lambda_k \geqslant 0$ with $\lambda_k \to \infty$ as $k \to \infty$ such that

$$T^*(t)e_k = e^{-\lambda_k t}e_k \qquad \text{for all } t \in [0,T],\ k \in \mathbb{N}. \tag{4.3}$$

In the literature, e.g. [5], [16], [17] and [20], often cylindrical Lévy processes of the following form are considered:

$$L(t)u := \sum_{k=1}^\infty \langle e_k, u\rangle \sigma_k \ell_k(t) \qquad \text{for all } t \in [0,T],\ u \in U, \tag{4.4}$$

where $(\ell_k)_{k\in\mathbb{N}}$ is a sequence of independent, symmetric, real valued Lévy processes with characteristics $(0,0,\mu_k)$ and $(\sigma_k)_{k\in\mathbb{N}}$ is a real valued sequence such that the series in (4.4) converges in $L^0_P(\Omega;\mathbb{R})$. By using (2.5) we obtain that $T(\cdot)$ is stochastically integrable with respect to L if and only if

$$\sum_{k=1}^\infty \int_0^T \int_{\mathbb{R}} \left(e^{-2\lambda_k s}|\sigma_k\beta|^2 \wedge 1\right) \mu_k(\mathrm{d}\beta) \,\mathrm{d}t < \infty; \tag{4.5}$$

see Corollary 6.3 in [24]. For example, if $(\ell_k)_{k\in\mathbb{N}}$ is a family of independent, identically distributed, standardised, symmetric α-stable processes with $\alpha \in (0,2)$, one easily computes that $T(\cdot)$ is stochastically integrable w.r.t. L if and only if

$$\sum_{k=1}^\infty \frac{|\sigma_k|^\alpha}{\lambda_k} < \infty. \tag{4.6}$$

This result on the existence of a weak solution of the stochastic Cauchy problem (4.1) coincides with the result in [21].

Example 4.5. We assume the same setting as in Example 4.4 but model L as the canonical α-stable cylindrical Lévy process for $\alpha \in (0,2)$, i.e. the characteristic function of $L(t)$ is of the form

$$\varphi_{L(t)} : U \to \mathbb{C}, \qquad \varphi_{L(t)}(u) = \exp\left(-t\|u\|^\alpha\right).$$

Obviously, each finite dimensional projection $\big((L(t)u_1,\dots,L(t)u_n)\,:\,t\geqslant 0\big)$ for $u_1,\dots,u_n\in U$ is an α-stable Lévy process in $\mathbb{R}^n$. Using this fact, it is shown in Theorem 4.1 in [22] that a semigroup $(T(t))_{t\geqslant 0}$ satisfying the spectral decomposition (4.3) is stochastically integrable with respect to L if and only if

$$\int_0^T \|T(s)\|_{\mathrm{HS}}^{\alpha}\,\mathrm{d}s<\infty. \tag{4.7}$$

In the work [7], the authors consider the stochastic Cauchy problem in Banach spaces driven by a subordinated cylindrical Brownian motion, a slightly more general noise than the canonical α-stable cylindrical Lévy process. As the approach in [7] relies on embedding the underlying space U in a larger space, the derived conditions are less explicit than (4.7) and only sufficient.

Before proving Theorem 4.3 we establish stochastic continuity of the convolution.

Proposition 4.6. *If the mapping $s\mapsto T(s)B$ is stochastically integrable on $[0,T]$ with respect to L, then*

$$Y_A(t):=\int_0^t T(t-s)B\,\mathrm{d}L(s),$$

defines a stochastically continuous process $(Y_A(t)\,:\,t\in[0,T])$ in V.

Proof. Let P_t denote the probability distribution of $Y_A(t)$. By [13, Lemma 2.4], it is enough to show that

(i) $\big(\langle Y_A(t),v\rangle\,:\,t\in[0,T]\big)$ is stochastically continuous for each $v\in V$;

(ii) $\{P_t\,:\,t\in[0,T]\}$ is relatively compact in $\mathcal{M}(V)$.

Proof of (i): for every $t\in[0,T]$, $v\in V$ and $\varepsilon>0$, we have by (2.2) that

$$\begin{aligned}
&|\langle Y_A(t+\varepsilon),v\rangle-\langle Y_A(t),v\rangle|\\
&\quad=\left|\int_0^{t+\varepsilon} B^*T^*(t+\varepsilon-s)v\,\mathrm{d}L(s)-\int_0^t B^*T^*(t-s)v\,\mathrm{d}L(s)\right|\\
&\quad\leqslant\left|\int_0^t B^*T^*(t-s)(T^*(\varepsilon)v-v)\,\mathrm{d}L(s)\right|+\left|\int_t^{t+\varepsilon} B^*T^*(t+\varepsilon-s)v\,\mathrm{d}L(s)\right|\\
&\quad=:|I_1(\varepsilon)|+|I_2(\varepsilon)|\,.
\end{aligned}\tag{4.8}$$

The random variable $I_1(\varepsilon)$ has the characteristic function $\varphi_{1,\varepsilon}\colon\,\mathbb{R}\to\mathbb{C}$ given by

$$\varphi_{1,\varepsilon}(\beta)=\exp\left(\int_0^t \Psi\big(\beta B^*T^*(s)(T^*(\varepsilon)v-v)\big)\,\mathrm{d}s\right).$$

By using standard properties of the semigroup we obtain

$$\sup_{s\in[0,T]}\|\beta B^*T^*(s)(T^*(\varepsilon)v-v)\|\to 0\qquad\text{as }\varepsilon\to 0,$$

which implies $\varphi_{1,\varepsilon}(\beta)\to 1$ for all $\beta\in\mathbb{R}$ due to Lemma 5.1 in [24]. Thus, $I_1(\varepsilon)$ converges to 0 in probability as $\varepsilon\to 0$. The characteristic function $\varphi_{2,\varepsilon}\colon\,\mathbb{R}\to\mathbb{C}$ of the random variable $I_2(\varepsilon)$ obeys

$$\varphi_{2,\varepsilon}(\beta)=\exp\left(\int_0^{\varepsilon}\Psi(\beta B^*T^*(s)v)\,\mathrm{d}s\right)\to 1\text{ as }\varepsilon\to 0.$$

Consequently, we obtain that $I_2(\varepsilon) \to 0$ in probability. The arguments above show by (4.8) that $\langle Y_A(t+\varepsilon), v\rangle \to \langle Y_A(t), v\rangle$ in probability as $\varepsilon \to 0$. Analogously, we can show that $\langle Y_A(t-\varepsilon), v\rangle \to \langle Y_A(t), v\rangle$ in probability, which yields Property (i).

Proof of (ii): it follows from Lemma 5.4 in [24] that the probability distribution P_t of $Y_A(t)$ is an infinitely divisible probability measure in $\mathcal{M}(V)$ with characteristics (c_t, S_t, θ_t) given for all $v \in V$ by

$$\begin{aligned}
\langle c_t, v\rangle &= \int_0^t a(B^*T^*(s)v)\,\mathrm{d}s + \int_V \langle h, v\rangle \left(\mathbb{1}_{B_V}(h) - \mathbb{1}_{B_{\mathbb{R}}}(\langle h, v\rangle)\right) \theta_t(\mathrm{d}h), \\
\langle S_t v, v\rangle &= \int_0^t \langle B^*T^*(s)v, QB^*T^*(s)v\rangle\,\mathrm{d}s, \\
\theta_t &= (\text{leb} \otimes \mu) \circ \chi_t^{-1} \quad \text{on } \mathcal{Z}(V),
\end{aligned}$$

where $\chi_t \colon [0,t] \times U \to V$ is defined by $\chi_t(s,u) := T(s)Bu$.

Let $\tilde{P}_t$ denote the infinitely divisible probability measure with characteristics $(0, S_t, \theta_t)$. Theorem VI.5.1 in [18] guarantees that the set $\{\tilde{P}_t : t \in [0,T]\}$ is relatively compact if and only if the set $\{\theta_t : t \in [0,T]\}$ restricted to the complement of any neighbourhood of the origin is relatively compact in $\mathcal{M}(V)$ and the operators $T_t \colon V \to V$ defined by

$$\langle T_t v, v\rangle := \langle S_t v, v\rangle + \int_{\|h\| \leqslant 1} \langle v, h\rangle^2\, \theta_t(\mathrm{d}h)$$

satisfy

$$\sup_{t \in [0,T]} \sum_{k=1}^{\infty} \langle T_t h_k, h_k\rangle < \infty \qquad \text{and} \qquad \lim_{N\to\infty} \sup_{t \in [0,T]} \sum_{k=N}^{\infty} \langle T_t h_k, h_k\rangle = 0. \tag{4.9}$$

For a set A in the cylindrical algebra $\mathcal{Z}(V)$ we have

$$\theta_t(A) = \int_0^t \int_U \mathbb{1}_A(T(s)Bu)\mu(\mathrm{d}u)\,\mathrm{d}s \leqslant \int_0^T \int_U \mathbb{1}_A(T(s)Bu)\mu(\mathrm{d}u)\,\mathrm{d}s = \theta_T(A).$$

Since $\mathfrak{B}(V)$ is the sigma algebra generated by $\mathcal{Z}(V)$ and $\mathcal{Z}(V)$ is closed under intersection, we conclude $\theta_t \leqslant \theta_T$ on $\mathfrak{B}(V)$ for all $t \in [0,T]$. Let θ_t^c denote the restriction of θ_t to the complement of a neighbourhood V_1 of the origin. Since θ_T^c is a Radon measure by [15, Prop. 1.1.3], there exists for each $\varepsilon > 0$ a compact set $K \subseteq V_1$ such that $\theta_T^c(K^c) \leqslant \varepsilon$. Consequently, we obtain $\theta_t^c(K^c) \leqslant \theta_T^c(K^c) \leqslant \varepsilon$ for all $t \in [0,T]$, which shows by Prokhorov's theorem that $\{\theta_t : t \in [0,T]\}$ restricted to the complement of any neighbourhood of the origin is relatively compact in $\mathcal{M}(V)$.

The stochastic integrability of $s \mapsto T(s)B$ implies by (2.4) and Lebesgue's theorem that

$$\begin{aligned}
\lim_{N\to\infty} \sup_{t \in [0,T]} \sum_{k=N}^{\infty} \langle S_t h_k, h_k\rangle\,\mathrm{d}s &= \lim_{N\to\infty} \int_0^T \sum_{k=N}^{\infty} \langle T(s)BQB^*T^*(s)h_k, h_k\rangle\,\mathrm{d}s \\
&= 0.
\end{aligned} \tag{4.10}$$

Condition (2.5) of stochastic integrability implies

$$
\begin{aligned}
&\sup_{t\in[0,T]} \sum_{k=N}^{\infty} \int_{\|h\|\leqslant 1} \langle h_k, h\rangle^2 \theta_t(\mathrm{d}h) \\
&\quad\leqslant \sup_{t\in[0,T]} \sup_{m\geqslant N} \int_V \left(\sum_{k=N}^{m} \langle h_k, h\rangle^2 \wedge 1 \right) \theta_t(\mathrm{d}h) \\
&\quad= \sup_{t\in[0,T]} \sup_{m\geqslant N} \int_0^t \int_U \left(\sum_{k=N}^{m} \langle h_k, T(s)Bu\rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\,\mathrm{d}s \\
&\quad= \sup_{m\geqslant N} \int_0^T \int_U \left(\sum_{k=N}^{m} \langle h_k, T(s)Bu\rangle^2 \wedge 1 \right) \mu(\mathrm{d}u)\,\mathrm{d}s \\
&\quad\to 0 \qquad \text{as } N\to\infty.
\end{aligned} \tag{4.11}
$$

The limits (4.10) and (4.11) show that the second condition in (4.9) is satisfied. As the first condition in (4.9) follows analogously, we conclude that $\{\tilde{P}_t : t\in[0,T]\}$ is relatively compact.

Let $\{\tilde{P}_{t_n}\}_{n\in\mathbb{N}}$ be a weakly convergent subsequence. Without any restriction we can assume that there exists $t\in[0,T]$ such that $t_n\to t$. For the characteristic functions $\varphi_{P_{t_n}}$ of P_{t_n} we obtain

$$
\begin{aligned}
&|\varphi_{P_{t_n}}(v) - \varphi_{P_t}(v)| \\
&= \left| \exp\left(\int_0^{t_n} \Psi(B^*T^*(t_n-s)v)\,\mathrm{d}s \right) - \exp\left(\int_0^{t} \Psi(B^*T^*(t-s)v)\,\mathrm{d}s \right) \right| \\
&= \left| \exp\left(\int_t^{t_n} \Psi(B^*T^*(s)v)\,\mathrm{d}s \right) - 1 \right| \left| \exp\left(\int_0^{t} \Psi(B^*T^*(s)v)\,\mathrm{d}s \right) \right|.
\end{aligned} \tag{4.12}
$$

Since Ψ maps bounded sets to bounded sets, we obtain for each $\delta>0$ that

$$
\sup_{\|v\|<\delta} \left| \int_t^{t_n} \Psi(B^*T^*(s)v)\,\mathrm{d}s \right| \to 0 \quad \text{as } n\to\infty,
$$

which implies by (4.12) that

$$
\sup_{\|v\|<\delta} |\varphi_{P_{t_n}}(v) - \varphi_{P_t}(v)| \to 0 \quad \text{as } n\to\infty.
$$

As $\tilde{P}_{t_n} = P_{t_n} * \delta_{-c_{t_n}}$, Theorem 2.3.8 in [15] implies that $\{P_{t_n}\}$ converges weakly, which completes the proof of Property (ii). ☐

Proof of Theorem 4.3. We can assume $y_0=0$ due to linearity. Lemma 6.2 in [24] guarantees that the map $r\mapsto T(s-r)B$ is stochastically integrable on $[0,s]$ for each $s\in(0,T]$. Thus, we can define

$$
Y(s) := \int_0^s T(s-r)B\,\mathrm{d}L(r) \qquad \text{for all } s\in[0,T].
$$

We first show that Y is weakly Bochner regular. Let g be in $C([0,T];V)$ and define

$$
f\colon [0,T]\times[0,T]\to U, \qquad f(s,r) = \mathbb{1}_{[0,s]}(r)B^*T^*(s-r)g(s). \tag{4.13}
$$

By using (2.2) we conclude for all $s\in[0,T]$ that

$$
\langle Y(s), g(s)\rangle = \int_0^s B^*T^*(s-r)g(s)\,\mathrm{d}L(r) = \int_0^T f(s,r)\,\mathrm{d}L(r). \tag{4.14}
$$

For fixed $s \in [0,T]$ the map $r \mapsto f(s,r)$ is regulated. Moreover, by defining $m := \sup_{s\in[0,T]} \|B^*T^*(s)\|_{\rm op}^2$, we obtain for $\varepsilon > 0$ and $r \in [0, T-\varepsilon]$ that

$$\begin{aligned}
&\|f(\cdot, r+\varepsilon) - f(\cdot, r)\|^2_{L^2([0,T];U)} \\
&= \int_0^T \|\mathbb{1}_{[r+\varepsilon,T]}(s)B^*T^*(s-r-\varepsilon)g(s) - \mathbb{1}_{[r,T]}(s)B^*T^*(s-r)g(s)\|^2 \, \mathrm{d}s \\
&= \int_{r+\varepsilon}^T \|B^*T^*(s-r-\varepsilon)(\mathrm{Id} - T^*(\varepsilon))g(s)\|^2 \, \mathrm{d}s \\
&\qquad\qquad\qquad + \int_r^{r+\varepsilon} \|B^*T^*(s-r)g(s)\|^2 \, \mathrm{d}s \\
&\leqslant m \int_0^T \|(\mathrm{Id} - T^*(\varepsilon))g(s)\|^2 \, \mathrm{d}s + \varepsilon m \|g\|_\infty^2 \\
&\to 0 \qquad \text{as } \varepsilon \to 0,
\end{aligned}$$

which shows that the mapping $r \mapsto f(\cdot, r)$ is right continuous. In a similar way, we establish that $r \mapsto f(\cdot, r)$ is left continuous. Thus, we can apply Theorem 3.1 to conclude by using (4.14) that the mapping $s \mapsto \langle Y(s), g(s)\rangle$ is square-integrable on $[0,T]$.

Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $C([0,T];V)$ with $g_n \to 0$. By Lemma 5.4 in [24] and Theorem 3.1, the Lévy symbol of the infinitely divisible random variable $\int_0^T \langle Y(s), g_n(s)\rangle \, \mathrm{d}s$ is given by

$$\Phi_n \colon \mathbb{R} \to \mathbb{C}, \qquad \Phi_n(\beta) = \int_0^T \Psi\left(\int_r^T \beta B^*T^*(s-r)g_n(s) \, \mathrm{d}s \right) \mathrm{d}r,$$

where $\Psi\colon U \to \mathbb{C}$ is the Lévy symbol of L. As Ψ is continuous and maps bounded sets to bounded sets according to Lemma 3.2 and Lemma 5.1 of [24], a repeated application of Lebesgue's theorem implies $\Phi_n(\beta) \to 0$ for every $\beta \in \mathbb{R}$, which proves that Y is weakly Bochner regular.

Taking $T = t$ and $g(s) = A^*v$ for every $s \in [0,t]$ in the definition of f in (4.13), we can apply Theorem 3.1 to obtain for each $v \in \mathcal{D}(A^*)$ that

$$\begin{aligned}
\int_0^t \langle Y(s), A^*v\rangle ds &= \int_0^t \left(\int_0^s B^*T^*(s-r)A^*v \, \mathrm{d}L(r) \right) \mathrm{d}s \\
&= \int_0^t \left(\int_r^t B^*T^*(s-r)A^*v \, \mathrm{d}s \right) \mathrm{d}L(r) \\
&= \int_0^t \left(B^*T^*(t-r)v - B^*T^*(0)v \right) \mathrm{d}L(r) \\
&= \langle Y(t), v\rangle - L(t)(B^*v),
\end{aligned}$$

which shows (4.2). Proposition 4.6 guarantees that the stochastic process $\left(\int_0^t T(t-r)B \, \mathrm{d}L(r) : \, t \in [0,T] \right)$ is stochastically continuous and since it is also adapted, it has a progressively measurable modification by Proposition 3.6 in [9] which completes the proof. □

To prove uniqueness of the solution we follow the same approach as in [9], for which we need the following integration by parts formula.

Lemma 4.7. *If $g\colon [0,T] \to U$ is a function of the form $g(t) = \tau(t)u$ for $u \in U$ and $\tau \in C^1([0,T];\mathbb{R})$, then*

$$\int_0^T g(s) \, \mathrm{d}L(s) = -\int_0^T L(s)(g'(s)) \, \mathrm{d}s + L(T)(g(T)).$$

Proof. For a sequence $\{(t_k^n)_{k=0}^{N_n} : n \in \mathbb{N}\}$ of partitions of the interval $[0,T]$ with $\max_{0\leqslant k\leqslant N_n-1} |t_{k+1}^n - t_k^n| \to 0$ as $n\to\infty$ define the simple functions

$$g_n\colon [0,T]\to U, \qquad g_n(t) := \sum_{k=0}^{N_n-1} g(t_k^n)\mathbb{1}_{[t_k^n,t_{k+1}^n)}(t) + \mathbb{1}_{\{T\}}(t)g(T).$$

As g_n converges to g uniformly on $[0,T]$, Lemma 5.1 of [24] implies

$$\int_0^T g_n(s)\,\mathrm{d}L(s) \to \int_0^T g(s)\,\mathrm{d}L(s) \qquad \text{in probability.} \tag{4.15}$$

On the other hand, P-almost surely we obtain

$$\begin{aligned}\int_0^T g_n(s)\,\mathrm{d}L(s) &= \sum_{k=0}^{N_n-1} \big(L(t_{k+1}^n) - L(t_k^n)\big)\,(\tau(t_k^n)u)\\ &= \sum_{k=0}^{N_n-1} \tau(t_k^n)\,\big(L(t_{k+1}^n) - L(t_k^n)\big)\,(u)\\ &= -\sum_{k=0}^{N_n-1} \big(\tau(t_{k+1}^n) - \tau(t_k^n)\big)L(t_{k+1}^n)(u) + \tau(T)L(T)(u). \end{aligned}\tag{4.16}$$

Applying the mean value theorem, we obtain for some $\xi_k^n \in (t_k^n, t_{k+1}^n)$ that

$$\begin{aligned}&\sum_{k=0}^{N_n-1} \big(\tau(t_{k+1}^n) - \tau(t_k^n)\big)\,L(t_{k+1}^n)(u)\\ &\quad= \sum_{k=0}^{N_n-1} \tau'(\xi_k^n)(t_{k+1}^n - t_k^n)L(t_{k+1}^n)(u) \qquad\qquad (4.17)\\ &\quad= \sum_{k=0}^{N_n-1} \tau'(\xi_k^n)(t_{k+1}^n - t_k^n)L(\xi_k^n)(u) - \sum_{k=0}^{N_n-1} \tau'(\xi_k^n)(t_{k+1}^n - t_k^n)\big(L(\xi_k^n)(u) - L(t_{k+1}^n)(u)\big).\end{aligned}$$

As the map $s\mapsto \tau'(s)L(s)u$ has only countable number of discontinuities, it is Riemann integrable and we obtain

$$\lim_{n\to\infty} \sum_{k=0}^{N_n-1} \tau'(\xi_k^n)(t_{k+1}^n - t_k^n)L(\xi_k^n)(u) = \int_0^T L(s)(u\tau'(s))\,\mathrm{d}s. \tag{4.18}$$

To show that the second term in (4.17) approaches 0 we define

$$M_k^n := \sup_{s\in[t_k^n,t_{k+1}^n]} L(s)u, \qquad\qquad m_k^n := \inf_{s\in[t_k^n,t_{k+1}^n]} L(s)u.$$

Riemann integrability of the map $s\mapsto L(s)u$ implies

$$\begin{aligned}&\left|\sum_{k=0}^{N_n-1} \tau'(\xi_k^n)(t_{k+1}^n - t_k^n)\big(L(\xi_k^n)(u) - L(t_{k+1}^n)(u)\big)\right|\\ &\qquad\leqslant \sum_{k=0}^{N_n-1} |\tau'(\xi_k^n)|\,\big|t_{k+1}^n - t_k^n\big|\,\big|L(\xi_k^n)(u) - L(t_{k+1}^n)(u)\big|\\ &\qquad\leqslant \|\tau'\|_\infty \sum_{k=0}^{N_n-1} \big|t_{k+1}^n - t_k^n\big|\,|M_k^n - m_k^n|\\ &\qquad\to 0 \qquad \text{as}\quad n\to\infty.\end{aligned}\tag{4.19}$$

Taking the limit in (4.16) by applying (4.17), (4.18) and (4.19) and comparing it to the limit in (4.15) completes the proof. □

Theorem 4.8. *If there exists a weak solution Y of the stochastic Cauchy problem* (4.1)*, then the mapping $s \mapsto T(s)B$ is stochastically integrable on $[0,T]$ with respect to L and Y is given by*

$$Y(t) = T(t)y_0 + \int_0^t T(t-s)B \, \mathrm{d}L(s).$$

Proof. We can assume that $y_0 = 0$ due to linearity. For every $v \in \mathcal{D}(A^*)$ and $t \in [0,T]$ we have P-a.s. that

$$\langle Y(t), v \rangle = \int_0^t \langle Y(s), A^* v \rangle \, \mathrm{d}s + L(t)(B^* v). \tag{4.20}$$

Let f be in $C^1([0,T];\mathbb{R})$ and v in $\mathcal{D}(A^*)$. By using (4.20) and applying the integration by parts formula in Lemma 4.7 to $g(\cdot) = f(\cdot)B^*v$ and the classical integration by parts formula for Lebesgue integrals we obtain

$$\begin{aligned}
&\int_0^t f'(s)\langle Y(s), v\rangle \, \mathrm{d}s \\
&\quad = \int_0^t f'(s) \left(\int_0^s \langle Y(r), A^* v\rangle \, \mathrm{d}r \right) \mathrm{d}s + \int_0^t f'(s) L(s)(B^* v) \, \mathrm{d}s \\
&\quad = f(t) \int_0^t \langle Y(s), A^* v \rangle \, \mathrm{d}s - \int_0^t f(s) \langle Y(s), A^* v\rangle \, \mathrm{d}s \\
&\qquad + f(t)L(t)(B^*v) - \int_0^t f(s) B^* v \, \mathrm{d}L(s).
\end{aligned}$$

Rearranging the terms and using (4.20), we obtain by defining $F(\cdot) = f(\cdot)v$ that

$$\langle Y(t), F(t)\rangle = \int_0^t \langle Y(s), F'(s) + A^*F(s)\rangle \, \mathrm{d}s + \int_0^t B^*F(s) \, \mathrm{d}L(s). \tag{4.21}$$

For $v \in \mathcal{D}(A^{*2})$, the function $G := T^*(t-\cdot)v$ is in $C^1([0,t];\mathcal{D}(A^*))$. Due to Lemma 8.4 in [25], we can find a sequence $F_n \in \operatorname{span}\{f(\cdot)w : f \in C^1([0,t];\mathbb{R}),\, w \in \mathcal{D}(A^*)\}$ such that F_n converges to G in $C^1([0,t];\mathcal{D}(A^*))$. Then $F_n' + A^*F_n \to 0$ in $C([0,t];V)$. The weakly Bochner regularity implies for a subsequence that

$$\int_0^t \langle Y(s), F_{n_k}'(s) + A^* F_{n_k}(s)\rangle \, \mathrm{d}s \to 0 \quad P\text{-a.s.}$$

Moreover, since B^*F_n converges to B^*G in $C([0,t];U)$, Lemma 5.2 in [24] implies

$$\int_0^t B^*F_n(s) \, \mathrm{d}L(s) \to \int_0^t B^*G(s) \, \mathrm{d}L(s) \qquad \text{in probability.}$$

Consequently, (4.21) holds for F replaced by G, which gives

$$\langle Y(t), v\rangle = \int_0^t B^*T^*(t-s)v \, \mathrm{d}L(s) \qquad \text{for all } v \in \mathcal{D}(A^{*2}).$$

Since $\mathcal{D}(A^{*2})$ is dense in V, for any $v \in V$, we can find a sequence $\{v_n\}$ in $\mathcal{D}(A^{*2})$ with $v_n \to v$ as $n \to \infty$. Since $B^*T^*(t-\cdot)v_n$ converges to $B^*T^*(t-\cdot)v$ in $C([0,t];U)$ it follows from [24, Lemma 5.2] that

$$\lim_{n\to\infty} \int_0^t B^*T^*(t-s)v_n \, \mathrm{d}L(s) = \int_0^t B^*T^*(t-s)v \, \mathrm{d}L(s) \qquad \text{in probability,}$$

and hence P-a.s.

$$\langle Y(t), v\rangle = \int_0^t B^*T^*(t-s)v \, \mathrm{d}L(s) \qquad \text{for all } v \in V.$$

This establishes the stochastic integrability of $s \mapsto T(s)B$ on $[0,T]$. □

5 Properties of the solution

We begin this section with discussing some path properties of the solution. Various specific examples of the stochastic Cauchy problem (4.1) were observed in the literature in which the solution Y exists but does not have a modification $\tilde{Y}$ with scalarly càdlàg paths; see e.g. [5], [16] and [20]. Even the weaker property that the real valued process $(\langle Y(t), v\rangle : t \in [0,T])$ has a modification with càdlàg paths for each $v \in V$ can be verified only in a few specific examples. However, our stochastic Fubini Theorem 3.1 immediately implies that this real valued stochastic process $(\langle Y(t), v\rangle : t \in [0,T])$ has square-integrable trajectories:

Theorem 5.1. *If $(Y(t) : t \in [0,T])$ is the weak solution of the stochastic Cauchy problem* (4.1)*, then for every $v \in V$, P-a.s.*

$$\int_0^T \langle Y(t), v\rangle^2 \, \mathrm{d}t < \infty.$$

Proof. By choosing $g(s) = v$ for all $s \in [0,T]$ in (4.13), the following arguments in the proof of Theorem 4.3 show that the function

$$f\colon [0,T] \times [0,T] \to U, \qquad f(s,r) = \mathbb{1}_{[0,s]}(r)B^*T^*(s-r)v$$

satisfies the assumption of Theorem 3.1. Consequently, we conclude that the stochastic process $(\langle Y(t), v\rangle : t \in [0,T])$ defines a random variable in $L^2([0,T];\mathbb{R})$ for each $v \in V$. □

Theorem 5.2. *The weak solution $(Y(t) : t \in [0,T])$ of the stochastic Cauchy problem* (4.1) *is stochastically continuous.*

Proof. Follows immediately from Proposition 4.6. □

As mentioned in the introduction, it has been observed for specific examples of a cylindrical Lévy process, that the solution of (4.1) has highly irregular paths in an analytical sense. In our general setting, we state a condition in the result below which implies such highly irregular paths of the solution. This condition does not only allow a geometric interpretation of this phenomena but is also easy to verify in many examples including the ones considered in the literature.

Theorem 5.3. *Assume that an orthonormal basis $(h_k)_{k\in\mathbb{N}}$ of V is in the domain of A^* and let L be a cylindrical Lévy process with cylindrical characteristics (a, Q, μ). If for all $c > 0$*

$$\lim_{n\to\infty} \mu\left(\left\{u \in U : \sum_{k=1}^{n} \langle u, B^*h_k\rangle^2 > c\right\}\right) = \infty, \tag{5.1}$$

then there does not exist any modification $\widetilde{Y}$ of the weak solution Y of (4.1) *such that for each $v \in V$ the stochastic process $(\langle \widetilde{Y}(t), v\rangle : t \in [0,T])$ has càdlàg paths.*

Remark 5.4. Note, that if μ is a genuine Lévy measure then Condition (5.1) cannot be satisfied for any constant $c > 0$. This is due to the fact that in this case, μ is a finite Radon measure on each complement of the origin; see [15].

Example 5.5. (continues Example 4.4). Assume that the cylindrical Lévy process L is given by (4.4) and $B = \mathrm{Id}$ in equation (4.1). The independence of the real valued Lévy processes $(\ell_k)_{k\in\mathbb{N}}$ implies that the cylindrical Lévy measure μ has support only in $\cup_{k=1}^{\infty}\mathrm{span}\{e_k\}$, and thus Condition (5.1) reduces to

$$\sum_{k=1}^{\infty}\mu\Big(\Big\{u\in U:\langle u,h_k\rangle^2>c\Big\}\Big)=\infty,$$

for all $c>0$. For this special case, the conclusion of Theorem 5.3 has already been derived in [20].

For example, if $(\ell_k)_{k\in\mathbb{N}}$ is a family of independent, identically distributed symmetric α-stable Lévy processes, then Condition (5.1) is satisfied for $B = \mathrm{Id}$; see [16].

Example 5.6 (Continues Example 4.5)**.** Let L be the canonical α-stable process, introduced in Example 4.5. By using properties of α-stable distributions in $\mathbb{R}^n$ one calculates for each $n\in\mathbb{N}$ that

$$\mu\left(\left\{u\in U:\sum_{k=1}^{n}\langle u,h_k\rangle^2>c^2\right\}\right)=\frac{1}{c^\alpha c_\alpha}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)},$$

where Γ denotes the Gamma function and c_α is a constant only depending on α. As the right hand side converges to ∞ as $n\to\infty$, Condition (5.1) is satisfied for $B=\mathrm{Id}$; see [22, Theorem 5.1].

Proof of Theorem 5.3 (based on ideas from [16]). For each $n\in\mathbb{N}$ and $t\in[0,T]$ define $L_n(t):=\big(L(t)B^*h_1,\dots,L(t)B^*h_n\big)$ and $Y_n(t):=\big(\langle Y(t),h_1\rangle,\dots,\langle Y(t),h_n\rangle\big)$. It follows from Definition 4.2 of a weak solution that for every $t\in[0,T]$ we have P-a.s.

$$Y_n(t)=Y_n(0)+\int_0^t\big(\langle Y(s),A^*h_1\rangle,\dots,\langle Y(s),A^*h_n\rangle\big)\,ds+L_n(t).$$

Consequently, the n-dimensional processes $(Y_n(t):\,t\in[0,T])$ and $(L_n(t):\,t\in[0,T])$ jump at the same time by the same size, which implies

$$\sup_{t\in[0,T]}|\Delta L_n(t)|^2=\sup_{t\in[0,T]}|\Delta Y_n(t)|^2\leqslant 4\sup_{t\in[0,T]}|Y_n(t)|^2,$$

where $\Delta g(t):=g(t)-g(t-)$ for càdlàg functions $g\colon[0,T]\to\mathbb{R}^n$. It follows that

$$\begin{aligned}
P\left(\sup_{t\in[0,T]}\sum_{k=1}^{\infty}\langle Y(t),h_k\rangle^2<\infty\right)&=\lim_{c\to\infty}P\left(\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}\sum_{k=1}^{n}\langle Y(t),h_k\rangle^2\leqslant\frac{1}{4}c^2\right)\\
&=\lim_{c\to\infty}\lim_{n\to\infty}P\left(\sup_{t\in[0,T]}\sum_{k=1}^{n}\langle Y(t),h_k\rangle^2\leqslant\frac{1}{4}c^2\right)\\
&=\lim_{c\to\infty}\lim_{n\to\infty}P\left(\sup_{t\in[0,T]}|Y_n(t)|^2\leqslant\frac{1}{4}c^2\right)\\
&\leqslant\lim_{c\to\infty}\lim_{n\to\infty}P\left(\sup_{t\in[0,T]}|\Delta L_n(t)|^2\leqslant c^2\right)\\
&=\lim_{c\to\infty}\lim_{n\to\infty}\exp\Big(-T\mu_n\Big(\{\beta\in\mathbb{R}^n:|\beta|>c\}\Big)\Big),
\end{aligned}$$

where μ_n denotes the Lévy measure of the $\mathbb{R}^n$-valued Lévy process L_n. Since $\mu_n=\mu\circ\pi_n^{-1}$ for $\pi_n\colon U\to\mathbb{R}^n$ and $\pi_n u=(\langle u,B^*h_1)\rangle,\dots,\langle u,B^*h_n\rangle)$ due to [3, Th. 2.4], we obtain by

(5.1) that

$$P\left(\sup_{t\in[0,T]}\sum_{k=1}^{\infty}\langle Y(t),h_k\rangle^2<\infty\right)=0,$$

which completes the proof by an application of Theorem 2.3 in [20]. □

We continue to consider mean square continuity of the solution. For this purpose, we naturally require that the cylindrical Lévy process has weak second moments, i.e. $E[|L(1)u|^2]<\infty$ for all $u\in U$. In this case, the cylindrical Lévy process with characteristics (a,Q,μ) can be written as

$$L(t)u=t\langle\tilde a,u\rangle+W(t)u+M(t)u\qquad\text{for all } t\geqslant 0,\ u\in U,$$

where $\tilde a\in U$, W is a cylindrical Brownian motion with covariance operator Q and M is a cylindrical Lévy process independent of W and with characteristics $(a',0,\mu)$. Here $a'\colon U\to\mathbb{R}$ is defined by $a'(u):=-\int_{|\beta|>1}\beta\,(\mu\circ u^{-1})(\mathrm{d}\beta)$ and $\langle\tilde a,u\rangle=a(u)-a'(u)$ for all $u\in U$; see Corollary 3.12 in [3]. It follows for any function $f\in R([0,T];U)$ that

$$\int_0^t f(s)\,\mathrm{d}L(s)=\int_0^t\langle\tilde a,f(s)\rangle\,\mathrm{d}s+\int_0^t f(s)\,\mathrm{d}W(s)+\int_0^t f(s)\,\mathrm{d}M(s). \tag{5.2}$$

Example 5.7. Assume that L has weak second moments. If

$$\int_0^T\|T(s)B\|_{\mathrm{HS}}^2\,\mathrm{d}s<\infty, \tag{5.3}$$

then there exists a weak solution $(Y(t):\,t\in[0,T])$ of the Cauchy problem (4.1) and it satisfies $E[\|Y(t)\|^2]<\infty$ for all $t\in[0,T]$.

Proof. For showing the existence of a solution, we have to establish that $t\mapsto T(t)B$ is stochastically integrable. Conditions (2.3) and (2.4) can be verified similarly as in the proof of Lemma 3.2. Since L has weak second moments, the closed graph theorem guarantees that $L(t)\colon U\to L_P^2(\Omega;\mathbb{R})$ is continuous, which implies

$$C:=\sup_{\|u^*\|\leqslant 1}\int_U\langle u,u^*\rangle^2\mu(\mathrm{d}u)\leqslant\|L(1)\|_{\mathrm{op}}^2<\infty.$$

Consequently, Condition (2.5) is satisfied since

$$\begin{aligned}
&\int_0^T\int_U\left(\sum_{k=m}^{n}\langle u,B^*T^*(s)h_k\rangle^2\wedge 1\right)\mu(\mathrm{d}u)\,\mathrm{d}s\\
&\quad\leqslant\sum_{k=m}^{n}\int_0^T\int_U\langle u,B^*T^*(s)h_k\rangle^2\mu(\mathrm{d}u)\,\mathrm{d}s\\
&\quad=\sum_{k=m}^{n}\int_0^T\int_U\|B^*T^*(s)h_k\|^2\left\langle u,\frac{B^*T^*(s)h_k}{\|B^*T^*(s)h_k\|}\right\rangle^2\mu(\mathrm{d}u)\,\mathrm{d}s\\
&\quad\leqslant\sup_{\|u^*\|\leqslant 1}\int_U\langle u,u^*\rangle^2\,\mu(\mathrm{d}u)\sum_{k=m}^{n}\int_0^T\|B^*T^*(s)h_k\|^2\,\mathrm{d}s\\
&\quad\to 0\qquad\text{as } m,n\to\infty,
\end{aligned} \tag{5.4}$$

where we applied (5.3) in the last line. As the Lévy measure θ_t of the infinitely divisible random variable $Y(t)$ is given by $(\mathrm{leb}\otimes\mu)\circ\chi_t^{-1}$ on $\mathcal{Z}(V)$ where $\chi_t\colon[0,t]\times U\to V$ and

$\chi_t(s,u) = T(s)Bu$, we obtain by a similar calculation as in (5.4) that

$$\int_V \|v\|^2 \, \theta_t(\mathrm{d}v) = \sum_{k=1}^\infty \int_0^t \int_U \langle u, B^*T^*(s)h_k\rangle^2 \mu(\mathrm{d}u)\,\mathrm{d}s \leqslant C \int_0^t \|B^*T^*(s)\|_{\mathrm{HS}}^2 \,\mathrm{d}s < \infty.$$

Consequently, we have $E[\|Y(t)\|^2] < \infty$ for all $t \in [0,T]$. □

Theorem 5.8. *Assume that L has weak second moments. If the weak solution $(Y(t) : t \in [0,T])$ of the stochastic Cauchy problem (4.1) has finite second moments, i.e. $E[\|Y(t)\|^2] < \infty$ for all $t \in [0,T]$, then Y is continuous in mean-square, i.e. $Y \in C([0,T]; L^2_P(\Omega; V))$.*

Proof. Let $\Phi\colon [0,T] \to \mathcal{L}(U,V)$ be a stochastically integrable, regulated function and $\Phi(\cdot)\tilde{a}$ be Pettis integrable. Then we obtain for each $t \in [0,T]$ and $\Psi \in \mathcal{L}(V,V)$ by (5.2) and using the fact that W and M have mean zero and are independent:

$$\begin{aligned}
&E\left[\left\|\int_0^t \Psi\Phi(t-s)\,\mathrm{d}L(s)\right\|^2\right] \\
&= \sum_{k=1}^\infty E\left[\left|\int_0^t \Phi^*(t-s)\Psi^*h_k \,\mathrm{d}L(s)\right|^2\right] \\
&= \sum_{k=1}^\infty \Bigg(E\left[\left|\int_0^t \langle \tilde{a}, \Phi^*(s)\Psi^*h_k\rangle \,\mathrm{d}s\right|^2\right] + \int_0^t \langle Q\Phi^*(s)\Psi^*h_k, \Phi^*(s)\Psi^*h_k\rangle \,\mathrm{d}s \\
&\qquad + \int_0^t \int_U \langle u, \Phi^*(s)\Psi^*h_k\rangle^2 \, \mu(\mathrm{d}u)\,\mathrm{d}s \Bigg) \\
&= \left\|\int_0^t \Psi\Phi(s)\tilde{a}\,\mathrm{d}s\right\|^2 + \int_0^t \left\|\Psi\Phi(s)Q^{1/2}\right\|_{\mathrm{HS}}^2 \mathrm{d}s + \int_V \|\Psi v\|^2 \, \eta_t(\mathrm{d}v),
\end{aligned} \tag{5.5}$$

where η_t is the (genuine) Lévy measure of $\int_0^t \Phi(s)\,\mathrm{d}L(s)$ and is given by $\eta_t = (\mathrm{leb}\otimes\mu)\circ\xi_t^{-1}$ where $\xi_t\colon [0,t]\times U \to V$ is defined by $\xi_t(s,u) = \Phi(s)u$.

We can assume $y_0 = 0$. Theorem 4.8 implies

$$Y(t) = \int_0^t T(t-s)B\,\mathrm{d}L(s) \qquad \text{for all } t \in [0,T].$$

As $Y(t)$ has finite second moments it follows $\int_V \|v\|^2 \, \theta_t(\mathrm{d}v) < \infty$, where θ_t is the (genuine) Lévy measure of $Y(t)$ and is given by $\theta_t = (\mathrm{leb}\otimes\mu)\circ\chi_t^{-1}$ where $\chi_t\colon [0,t]\times U \to V$ is defined by $\chi_t(s,u) = T(s)Bu$. For any $t \in [0,T]$ and $\varepsilon > 0$ we obtain

$$\begin{aligned}
&E[\|Y(t+\varepsilon) - Y(t)\|^2] \\
&= E\left[\left\|\int_0^t (T(t+\varepsilon-s)B - T(t-s)B)\,\mathrm{d}L(s) + \int_t^{t+\varepsilon} T(t+\varepsilon-s)B\,\mathrm{d}L(s)\right\|^2\right] \\
&\leqslant 2E\left[\left\|\int_0^t (T(\varepsilon)-\mathrm{Id})T(t-s)B\,\mathrm{d}L(s)\right\|^2\right] + 2E\left[\left\|\int_t^{t+\varepsilon} T(t+\varepsilon-s)B\,\mathrm{d}L(s)\right\|^2\right].
\end{aligned} \tag{5.6}$$

By applying (5.5) we conclude

$$\begin{aligned}
&E\left[\left\|\int_0^t (T(\varepsilon)-\mathrm{Id})T(t-s)B\,\mathrm{d}L(s)\right\|^2\right] \\
&\leqslant t\int_0^t \|(T(\varepsilon)-\mathrm{Id})T(s)B\tilde{a}\|^2 \,\mathrm{d}s + \int_0^t \left\|(T(\varepsilon)-\mathrm{Id})T(s)BQ^{1/2}\right\|_{\mathrm{HS}}^2 \mathrm{d}s \\
&\qquad + \int_V \|(T(\varepsilon)-\mathrm{Id})v\|^2 \, \theta_t(\mathrm{d}v).
\end{aligned}$$

Applying Lebesgue's theorem to each of the terms above shows

$$E\left[\left\|\int_0^t (T(\varepsilon)-\mathrm{Id})T(t-s)B\,\mathrm{d}L(s)\right\|^2\right]\to 0 \qquad \text{as } \varepsilon\to 0. \tag{5.7}$$

By a similar computation as in (5.5) we obtain for the second term in (5.6) that

$$\begin{aligned}E\left[\left\|\int_t^{t+\varepsilon} T(t+\varepsilon-s)B\,\mathrm{d}L(s)\right\|^2\right] &= \left\|\int_0^\varepsilon T(s)B\tilde{a}\,\mathrm{d}s\right\|^2+\int_0^\varepsilon \left\|T(s)BQ^{1/2}\right\|_{\mathrm{HS}}^2\mathrm{d}s\\ &\quad+\sum_{k=1}^\infty\int_0^T \mathbb{1}_{[0,\varepsilon]}(s)\int_U \langle u,B^*T^*(s)h_k\rangle^2\,\mu(\mathrm{d}u)\,\mathrm{d}s.\end{aligned} \tag{5.8}$$

The first two terms in (5.8) converge to 0 as $\varepsilon\to 0$. Since

$$\sum_{k=1}^\infty\int_0^T \mathbb{1}_{[0,\varepsilon]}(s)\int_U\langle u,B^*T^*(s)h_k\rangle^2\mu(\mathrm{d}u)\,\mathrm{d}s\leqslant\int_V\|v\|^2\,\theta_T(\mathrm{d}v)<\infty,$$

we can apply Lebesgue's theorem to the third term in (5.8) and obtain

$$E\left[\left\|\int_t^{t+\varepsilon} T(t-s)B\,\mathrm{d}L(s)\right\|^2\right]\to 0 \qquad \text{as } \varepsilon\to 0. \tag{5.9}$$

Applying (5.7) and (5.9) to (5.6) shows that Y is mean-square continuous from the right. Analogously, we can prove that Y is mean-square continuous from the left which completes the proof. □

We now prove the flow property and Markov property of the solution of the stochastic Cauchy problem (4.1). For this purpose we assume that $t\mapsto T(t)B$ is stochastically integrable and define for $0\leqslant s\leqslant t\leqslant T$ the mapping

$$\Phi_{s,t}\colon V\times\Omega\to V,\qquad \Phi_{s,t}(v)=T(t-s)v+\int_s^t T(t-r)B\,\mathrm{d}L(r).$$

Theorem 5.9. *Let* $(Y(t):\,t\in[0,T])$ *be the weak solution of* (4.1). *Then we have:*

(a) *the family* $\{\Phi_{s,t}:\,0\leqslant s\leqslant t\leqslant T\}$ *is a stochastic flow, i.e.* $\Phi_{s,s}=\mathrm{Id}$ *and*

$$\Phi_{s,t}\circ\Phi_{r,s}=\Phi_{r,t}\qquad \textit{for all } 0\leqslant r\leqslant s\leqslant t\leqslant T.$$

(b) *the weak solution* $(Y(t):\,t\in[0,T])$ *is a Markov process with respect to the filtration* $(\mathcal{F}_t)_{t\in[0,T]}$ *defined by* $\mathcal{F}_t:=\sigma(\{L(s)u:\,s\in[0,t],\,u\in U\})$.

Proof. (a): we first show that for all $0\leqslant r\leqslant s\leqslant t\leqslant T$ we have

$$T(t-s)\left(\int_r^s T(s-q)B\,\mathrm{d}L(q)\right)=\int_r^s T(t-q)B\,\mathrm{d}L(q). \tag{5.10}$$

For any $v\in V$, we obtain by (2.2)

$$\begin{aligned}\left\langle T(t-s)\left(\int_r^s T(s-q)B\,\mathrm{d}L(q)\right),v\right\rangle &= \left\langle\int_r^s T(s-q)B\,\mathrm{d}L(q),T^*(t-s)v\right\rangle\\ &=\int_r^s B^*T^*(s-q)(T^*(t-s)v)\,\mathrm{d}L(q)\\ &=\int_r^s B^*T^*(t-q)v\,\mathrm{d}L(q)\\ &=\left\langle\int_r^s T(t-q)B\,\mathrm{d}L(q),v\right\rangle,\end{aligned}$$

which shows (5.10). This enables us to conclude as in [1, Prop. 2.1] that for each $v \in V$

$$\Phi_{s,t}(\Phi_{r,s}(v)) = \Phi_{r,t}(v),$$

which completes the proof of (a).

(b): follows by standard arguments (see e.g. Theorem 6.4.2 in [2]). $\square$

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