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Talagrand concentration inequalities for stochastic heat-type equations under uniform distance*

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Abstract

In this paper, we established a quadratic transportation cost inequality under the uniform/maximum norm for solutions of stochastic heat equations driven by multiplicative space-time white noise. The proof is based on a new inequality we obtained for the moments of the stochastic convolution with respect to space-time white noise, which is of independent interest. The solutions of such stochastic partial differential equations are typically not semimartingales on the state space.

Keywords: stochastic partial differential equations; stochastic heat equations; transportation cost inequalities; concentration of measure; moment estimates for stochastic convolutions.
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1 Introduction

Let (X, d) be a metric space equipped with the Borel σ-field $\mathcal{B}$. Let μ, ν be two Borel probability measures on the metric space (X, d). The L^p-Wasserstein distance between μ and ν is defined as

$$W_p(\nu, \mu) := \left[\inf \iint_{X\times X} d(x, y)^p \, \pi(\mathrm{d}x, \mathrm{d}y)\right]^{\frac{1}{p}},$$

where the infimum is taken over all probability measures π on the product space $X \times X$ with marginals μ and ν. Recall that the Kullback information (or relative entropy) of ν with respect to μ is defined by

$$H(\nu|\mu) := \int_X \log\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) \mathrm{d}\nu,$$

if ν is absolutely continuous with respect to μ, and $+\infty$ otherwise.

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Definition 1.1. *We say that the measure μ satisfies the L^p-transportation cost inequality if there exists a constant $C > 0$ such that for all probability measures ν,*

$$W_p(\nu, \mu) \leq \sqrt{2CH(\nu|\mu)}. \tag{1.1}$$

The case $p = 2$ is referred to as the quadratic transportation cost inequality.

Transportation cost inequalities have close connections with other functional inequalities, e.g. Poincare inequalities, logarithmic Sobolev inequalities, and they also imply the concentration of measure phenomenon.

For a measurable subset $A \subset X$ and $r > 0$, we denote by A_r the r-neighborhood of A, namely $A_r = \{x : d(x, A) < r\}$. We say that μ has normal concentration (or Gaussian tail estimates) on (X, d) if there are constants $C, c > 0$ such that for every $r > 0$ and every Borel subset A with $\mu(A) \geq \frac{1}{2}$,

$$1 - \mu(A_r) \leq Ce^{-cr^2}. \tag{1.2}$$

The fact that the L^1-transportation cost inequality implies normal concentration was obtained in [16, 17] by Marton and in [23, 24, 25] by Talagrand. An elegant, simple proof of this fact is also contained in the book [14]. The connection of the quadratic transportation cost inequality with other functional inequalities was studied in [19] by Otto and Villani (see also [14]). For other related interesting works, we refer the reader to [2], [9], [15], [21], [22].

The concentration of measure phenomenon has wide applications, e.g. to stochastic finance (see [13]), statistics (see [18]) and the analysis of randomized algorithms (see [5]).

Remark 1.2. The transportation cost inequalities and the concentration of measure phenomenon depend on the underlying topology of the associated metric space. The stronger the topology, the stronger the concentration.

In the past decades, many people established quadratic transportation cost inequalities for various kinds of interesting measures. Let us mention several papers which are relevant to our work. The transportation cost inequalities for stochastic differential equations were obtained by H. Djellout, A. Guillin and L. Wu in [4]. The measure concentration for multidimensional diffusion processes with reflecting boundary conditions was considered by S. Pal in [20]. The quadratic transportation cost inequalities for stochastic partial differential equations (SPDEs) driven by Gaussian noise which is white in time and colored in space were obtained by A. S. Ustunel in [26]. We particularly like to mention the papers [1] by Boufoussi and Hajji, and [12] by D. Khoshnevisan and A. Sarantsev, which are the starting point of our work. In the article [1], the authors obtained the quadratic transportation cost inequality under the L^2-distance for stochastic heat equations. In [12], the authors established the quadratic transportation cost inequality for more general stochastic partial differential equations (SPDEs) under both the L^2-distance and the uniform distance. However, under the uniform distance they only obtained the quadratic transportation cost inequality for SPDEs driven by additive space-time white noise. The difficulty is how to get good moment estimates under the uniform norm for the stochastic convolution with respect to the white noise.

As is well known, one of the essential differences between SPDEs driven by colored noise and SPDEs driven by space-time white noise is that the solution of the later is not a semimartingale and therefore in particular Ito formula could not be used.

The aim of this paper is to prove that under the uniform distance the quadratic transportation cost inequality holds for stochastic heat equations driven by multiplicative space-time white noise. Our new contribution is the p-th moment inequalities under the uniform norm we obtained for the stochastic convolution with respect to the space-time white noise, which is of independent interest. The significance of the inequality is to allow the order p of the moment to be any positive number, not just for sufficiently large ones as in the literature. These new estimates allow us to establish the quadratic transportation cost inequality under the uniform norm for the case of multiplicative space-time withe noise.

The rest of the paper is organized as follows. In Section 2, we recall the setup for stochastic heat equations and state the main result of the paper. In Section 3 we establish the new moment estimates for stochastic convolutions with respect to the space-time white noise under the uniform norm. Section 4 is devoted to the proof of the main result. Two auxiliary results are given in the Appendix.

2 Statement of the main result

In this section, we will recall the setup for the stochastic heat equations driven by space-time white noise and state the main result of the paper. Consider the following equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + b(u(t,x)) + \sigma(u(t,x))\dfrac{\partial^2 W}{\partial t \partial x}(t,x), \quad x \in (0,1), \\ u(t,0) = u(t,1) = 0, \qquad t > 0, \\ u(0,x) = u_0(x), \quad x \in (0,1), \end{cases} \tag{2.1}$$

where $u_0 \in C_0([0,1])$, $\frac{\partial^2 W}{\partial t \partial x}(t,x)$ is a space-time white noise on some filtrated probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, here $\mathcal{F}_t, t \geq 0$ are the argmented filtration generated by the Brownian sheet $\{W(t,x); (t,x) \in [0,\infty) \times [0,1]\}$. The coefficients $b(\cdot), \sigma(\cdot) : \mathbb{R} \to \mathbb{R}$ are deterministic measurable functions. We say that an adapted, continuous random field $\{u(t,x) : (t,x) \in \mathbb{R}_+ \times [0,1]\}$ is a solution to the stochastic partial differential equation (SPDE) (2.1) if for $t \geq 0$ and any $\phi \in C_0^2([0,1])$,

$$\begin{aligned} \int_0^1 u(t,x)\phi(x)\,\mathrm{d}x &= \int_0^1 u_0(x)\phi(x)\,\mathrm{d}x + \frac{1}{2}\int_0^t \mathrm{d}s \int_0^1 u(s,x)\phi''(x)\,\mathrm{d}x \\ &+ \int_0^t \mathrm{d}s \int_0^1 b(u(s,x))\phi(x)\,\mathrm{d}x + \int_0^t\int_0^1 \sigma(u(s,x))\phi(x)\,W(\mathrm{d}s,\mathrm{d}x), \quad \mathbb{P}-a.s. \end{aligned} \tag{2.2}$$

It was shown in [27] that u is a solution to SPDE (2.1) if and only if for $t \geq 0$, u satisfies the following integral equation

$$\begin{aligned} u(t,x) =& P_t u_0(x) + \int_0^t \int_0^1 p_{t-s}(x,y) b(u(s,y))\,\mathrm{d}s\mathrm{d}y \\ &+ \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(u(s,y))\,W(\mathrm{d}s,\mathrm{d}y), \quad \mathbb{P}-a.s., \end{aligned} \tag{2.3}$$

where $P_t, t \geq 0$ and $p_t(x,y)$ are the corresponding semigroup and the heat kernel associated with the operator $\frac{1}{2}\frac{\partial^2}{\partial x^2}$ equipped with the Dirichlet boundary condition on the interval $[0,1]$.

Introduce the hypotheses

(H.1) There exists a constant L_b such that for all $x, y \in \mathbb{R}$,

$$|b(x) - b(y)| \leq L_b |x - y|. \tag{2.4}$$

(H.2) There exist constants K_σ and L_σ such that for all $x, y \in \mathbb{R}$,

$$\begin{aligned} |\sigma(x)| &\leq K_\sigma, \\ |\sigma(x) - \sigma(y)| &\leq L_\sigma |x - y|. \end{aligned} \tag{2.5}$$

It is well known (see [27]) that under the hypotheses (H.1) and (H.2), SPDE (2.1) admits a unique random field solution $u(t,x)$. In fact, for the existence and uniqueness the diffusion coefficient $\sigma(\cdot)$ needs not to be bounded, the stronger assumption (H.2) is needed for proving the transportation cost inequality. To state the precise result, let us introduce some constants.

For $\epsilon > 0$, define

$$C_{T,2,\epsilon} = \inf_{q>10} \frac{2}{q-2} q^{-\frac{q}{2}} \epsilon^{1-\frac{q}{2}} \left(q-2+qC_{T,q}\right)^{\frac{q}{2}}, \tag{2.6}$$

where the constant $C_{T,q}$ has the following bound:

$$C_{T,q} < q^{\frac{q}{2}} T^{\frac{q}{4}-\frac{3}{2}} \left(\frac{2}{\pi}\right)^q \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{q}{2}+1} \left(\frac{6q-8}{q-10}\right)^{\frac{3q}{2}-2}. \tag{2.7}$$

Define

$$C = K_\sigma^2 \inf_{0<\epsilon<\frac{1}{3L_\sigma^2}} \left\{ \frac{3}{1-3\epsilon L_\sigma^2} \sqrt{\frac{2T}{\pi}} \exp\left(\frac{3L_b^2 T}{1-3\epsilon L_\sigma^2} \sqrt{\frac{2T}{\pi}} + \frac{3C_{T,2,\epsilon} L_\sigma^2 T}{1-3\epsilon L_\sigma^2} \right) \right\}. \tag{2.8}$$

Here is the main result.

Theorem 2.1. *Suppose the hypotheses (H.1) and (H.2) hold. Then the law of the solution $u(\cdot,\cdot)$ of SPDE (2.1) satisfies the quadratic transportation cost inequality with the constant C given by (2.8) on the space $C([0,T]\times[0,1])$ (with the uniform distance).*

Remark 2.2. Our methods can be applied to study SPDEs with the Laplacian operator replaced by some pseudo differential operators. One needs only some nice heat-kernel estimates. An example of the kind of operators that can be studied is the fractional Laplacian $\mathcal{L} = -(-\Delta)^\alpha$, where $\alpha \in (1,2)$. The heat kernel estimates for this operator can be found in the paper [3].

3 Moment estimates for stochastic convolution under the uniform norm

In this section, we will establish some moment estimates for the stochastic convolution against space-time white noise. Of particular interest are the estimates of the moments of lower order p. These bounds will play a crucial role in the proof of the main result in next section. We start with the estimate for large order p.

Proposition 3.1. *Let $\{\sigma(s,y) : (s,y) \in \mathbb{R}_+ \times [0,1]\}$ be a random field such that the stochastic integral against space time white noise is well defined. Then for any $T>0$, $p>10$, there exists a constant $C_{T,p}>0$ such that*

$$\begin{aligned} &\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]} \left| \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(s,y)\, W(\mathrm{d}s,\mathrm{d}y) \right|^p \right] \\ \le & C_{T,p} \int_0^T \sup_{y\in[0,1]} \mathbb{E}\left|\sigma(s,y)\right|^p \mathrm{d}s. \end{aligned} \tag{3.1}$$

Remark 3.2. The constant $C_{T,p}$ in (3.1) can be bounded as

$$C_{T,p} < p^{\frac{p}{2}} T^{\frac{p}{4}-\frac{3}{2}} \left(\frac{2}{\pi}\right)^p \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{p}{2}+1} \left(\frac{6p-8}{p-10}\right)^{\frac{3p}{2}-2}. \tag{3.2}$$

Proof. Obviously, we can assume that the right hand side of (3.1) is finite. We employ the factorization method (see e.g. [8]). Choose α such that $\frac{3}{2p} < \alpha < \frac{1}{4} - \frac{1}{p}$. This is possible because $p > 10$. Let

$$(J_\alpha \sigma)(s,y) := \int_0^s \int_0^1 (s-r)^{-\alpha} p_{s-r}(y,z)\sigma(r,z)\, W(\mathrm{d}r, \mathrm{d}z), \tag{3.3}$$

$$(J^{\alpha-1} f)(t,x) := \frac{\sin \pi\alpha}{\pi} \int_0^t \int_0^1 (t-s)^{\alpha-1} p_{t-s}(x,y) f(s,y)\, \mathrm{d}s\mathrm{d}y. \tag{3.4}$$

By the stochastic Fubini theorem (see Theorem 2.6 in [27]), for any $(t,x) \in \mathbb{R}_+ \times [0,1]$,

$$\int_0^t \int_0^1 p_{t-s}(x,y)\sigma(s,y)\, W(\mathrm{d}s, \mathrm{d}y) = J^{\alpha-1}(J_\alpha \sigma)(t,x). \tag{3.5}$$

Therefore

$$\begin{aligned} &\sup_{(t,x)\in[0,T]\times[0,1]} \left| \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(s,y)\, W(\mathrm{d}s, \mathrm{d}y) \right| \\ =& \sup_{(t,x)\in[0,T]\times[0,1]} \left| J^{\alpha-1}(J_\alpha\sigma)(t,x) \right|, \quad \mathbb{P}-a.s.. \end{aligned} \tag{3.6}$$

Recall the well-known Nash-Aronson estimate (see e.g. [7])

$$0 \le p_t(x,y) \le \frac{1}{\sqrt{2\pi t}} \exp^{-\frac{(x-y)^2}{2t}}, \quad \forall\, x,y \in [0,1]. \tag{3.7}$$

A straightforward calculation gives

$$\int_0^1 p_t(x,y)\, \mathrm{d}y < 1, \tag{3.8}$$

$$\int_0^1 p_t(x,y)^2\, \mathrm{d}y \le \sup_{y\in[0,1]} p_t(x,y) \times \int_0^1 p_t(x,y)\, \mathrm{d}y \le C_2 t^{-\frac{1}{2}}, \quad C_2 := \frac{1}{\sqrt{2\pi}}. \tag{3.9}$$

By Höler's inequality, (3.8) and (3.9), we have

$$\begin{aligned} &\mathbb{E} \sup_{(t,x)\in[0,T]\times[0,1]} \left| \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(s,y)\, W(\mathrm{d}s, \mathrm{d}y) \right|^p \\ =&\mathbb{E} \sup_{(t,x)\in[0,T]\times[0,1]} \left| \frac{\sin \pi\alpha}{\pi} \int_0^t \int_0^1 (t-s)^{\alpha-1} p_{t-s}(x,y) J_\alpha\sigma(s,y)\, \mathrm{d}s\mathrm{d}y \right|^p \\ \le& \left| \frac{\sin \pi\alpha}{\pi} \right|^p \mathbb{E} \sup_{(t,x)\in[0,T]\times[0,1]} \left\{ \int_0^t (t-s)^{\alpha-1} \right. \\ &\qquad \left. \times \left(\int_0^1 p_{t-s}(x,y) |J_\alpha\sigma(s,y)|\, \mathrm{d}y \right) \mathrm{d}s \right\}^p \\ \le& \left| \frac{\sin \pi\alpha}{\pi} \right|^p \mathbb{E} \sup_{(t,x)\in[0,T]\times[0,1]} \left\{ \int_0^t (t-s)^{\alpha-1} \right. \\ &\qquad \left. \times \left(\int_0^1 p_{t-s}(x,y) |J_\alpha\sigma(s,y)|^{\frac{p}{2}}\, \mathrm{d}y \right)^{\frac{2}{p}} \mathrm{d}s \right\}^p \\ \le& \left| \frac{\sin \pi\alpha}{\pi} \right|^p \mathbb{E} \sup_{(t,x)\in[0,T]\times[0,1]} \left\{ \int_0^t (t-s)^{\alpha-1} \right. \\ &\qquad \left. \times \left(\int_0^1 p_{t-s}(x,y)^2\, \mathrm{d}y \right)^{\frac{1}{2}\times\frac{2}{p}} \left(\int_0^1 |J_\alpha\sigma(s,y)|^p\, \mathrm{d}y \right)^{\frac{1}{2}\times\frac{2}{p}} \mathrm{d}s \right\}^p \end{aligned}$$

$$\begin{aligned}
&\le \left|\frac{\sin\pi\alpha}{\pi}\right|^p C_2\mathbb{E}\sup_{t\in[0,T]}\left\{\int_0^t (t-s)^{\alpha-1-\frac{1}{2p}}\left(\int_0^1 |J_\alpha\sigma(s,y)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}\mathrm{d}s\right\}^p \\
&\le \left|\frac{\sin\pi\alpha}{\pi}\right|^p C_2\mathbb{E}\sup_{t\in[0,T]}\left[\left(\int_0^t (t-s)^{(\alpha-1-\frac{1}{2p})\frac{p}{p-1}}\,\mathrm{d}s\right)^{\frac{p-1}{p}\times p}\right. \\
&\qquad\qquad \left.\times\left(\int_0^t\int_0^1 |J_\alpha\sigma(s,y)|^p\,\mathrm{d}y\mathrm{d}s\right)^{\frac{1}{p}\times p}\right] \\
&\le \left|\frac{\sin\pi\alpha}{\pi}\right|^p C_2\times\left(\int_0^T s^{(\alpha-1-\frac{1}{2p})\frac{p}{p-1}}\,\mathrm{d}s\right)^{p-1}\times\int_0^T\int_0^1\mathbb{E}|J_\alpha\sigma(s,y)|^p\,\mathrm{d}y\mathrm{d}s \\
&\le C'_{T,p}\sup_{(s,y)\in[0,T]\times[0,1]}\mathbb{E}\left|\int_0^s\int_0^1 (s-r)^{-\alpha}p_{s-r}(y,z)\sigma(r,z)\,W(\mathrm{d}r,\mathrm{d}z)\right|^p, \qquad (3.10)
\end{aligned}$$

where we have used the condition $\alpha>\frac{3}{2p}$, so that

$$\begin{aligned}
C'_{T,p,\alpha} &= \left|\frac{\sin\pi\alpha}{\pi}\right|^p C_2\times\left(\int_0^T s^{(\alpha-1-\frac{1}{2p})\frac{p}{p-1}}\,\mathrm{d}s\right)^{p-1}\times T \\
&= \left|\frac{\sin\pi\alpha}{\pi}\right|^p C_2\left(\frac{p-1}{\alpha p-\frac{3}{2}}\right)^{p-1} T^{\alpha p-\frac{1}{2}}<\infty. \qquad (3.11)
\end{aligned}$$

For any fixed $(s,y)\in[0,T]\times[0,1]$, let

$$Z_t := \int_0^t\int_0^1 (s-r)^{-\alpha}p_{s-r}(y,z)\sigma(r,z)\,W(\mathrm{d}r,\mathrm{d}z),\quad t\in[0,s].$$

Then it is easy to see that $\{Z_t\}_{t\in[0,s]}$ is a real-valued martingale (on the interval $[0,s]$). Applying the Bukrholder-Davis-Gundy inequality (see Proposition 4.4 in [11] and also [27]), we have for $t\in[0,s]$,

$$\begin{aligned}
\mathbb{E}|Z_t|^p &\le (4p)^{\frac{p}{2}}\mathbb{E}\langle Z\rangle_t^{\frac{p}{2}} \\
&= (4p)^{\frac{p}{2}}\mathbb{E}\left(\int_0^t\int_0^1 (s-r)^{-2\alpha}p_{s-r}(y,z)^2\sigma(r,z)^2\,\mathrm{d}r\mathrm{d}z\right)^{\frac{p}{2}}. \qquad (3.12)
\end{aligned}$$

Taking $\frac{2}{p}$-th power on both sides of the above inequality, using (3.9) and Hölder's inequality, we get

$$\begin{aligned}
&\left\|\int_0^s\int_0^1 (s-r)^{-\alpha}p_{s-r}(y,z)\sigma(r,z)\,W(\mathrm{d}r,\mathrm{d}z)\right\|_{L^p(\Omega)}^2 \\
&= \|Z_s\|_{L^p(\Omega)}^2 \\
&\le 4p\left\|\int_0^s\int_0^1 (s-r)^{-2\alpha}p_{s-r}(y,z)^2\sigma(r,z)^2\,\mathrm{d}r\mathrm{d}z\right\|_{L^{\frac{p}{2}}(\Omega)} \\
&\le 4p\int_0^s\int_0^1 (s-r)^{-2\alpha}p_{s-r}(y,z)^2\,\|\sigma(r,z)\|_{L^p(\Omega)}^2\,\mathrm{d}r\mathrm{d}z \\
&\le 4p\int_0^s (s-r)^{-2\alpha}\left(\int_0^1 p_{s-r}(y,z)^2\,\mathrm{d}z\right)\sup_{z\in[0,1]}\|\sigma(r,z)\|_{L^p(\Omega)}^2\,\mathrm{d}r \\
&\le 4C_2p\int_0^s (s-r)^{-2\alpha-\frac{1}{2}}\sup_{z\in[0,1]}\|\sigma(r,z)\|_{L^p(\Omega)}^2\,\mathrm{d}r \\
&\le 4C_2p\left(\int_0^s (s-r)^{(-2\alpha-\frac{1}{2})\times\frac{p}{p-2}}\,\mathrm{d}r\right)^{\frac{p-2}{p}}\times\left(\int_0^s\sup_{z\in[0,1]}\|\sigma(r,z)\|_{L^p(\Omega)}^p\,\mathrm{d}r\right)^{\frac{2}{p}}. \qquad (3.13)
\end{aligned}$$

Therefore we take $\frac{p}{2}$-th power on both sides of the above inequality to obtain

$$\begin{aligned}&\sup_{(s,y)\in[0,T]\times[0,1]}\mathbb{E}\left|\int_0^s\int_0^1(s-r)^{-\alpha}p_{s-r}(y,z)\sigma(r,z)\,W(\mathrm{d}r,\mathrm{d}z)\right|^p\\ \le& C''_{T,p}\times\int_0^T\sup_{z\in[0,1]}\mathbb{E}\left|\sigma(r,z)\right|^p\,\mathrm{d}r,\end{aligned}\tag{3.14}$$

where the condition $\alpha<\frac{1}{4}-\frac{1}{p}$ was used to see that

$$\begin{aligned}C''_{T,p,\alpha}=&(4C_2p)^{\frac{p}{2}}\times\left(\int_0^T r^{(-2\alpha-\frac{1}{2})\times\frac{p}{p-2}}\,\mathrm{d}r\right)^{\frac{p-2}{2}}\\ =&(4C_2p)^{\frac{p}{2}}\times\left(\frac{p-2}{\frac{p}{2}-2-2\alpha p}\right)^{\frac{p-2}{2}}T^{\frac{p}{4}-1-\alpha p}<\infty.\end{aligned}\tag{3.15}$$

Combining (3.10) with (3.14), we obtain

$$\begin{aligned}&\mathbb{E}\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right|^p\\ \le& C_{T,p}\int_0^T\sup_{z\in[0,1]}\mathbb{E}\left|\sigma(r,z)\right|^p\,\mathrm{d}r,\end{aligned}\tag{3.16}$$

where

$$C_{T,p}=\min_{\frac{3}{2p}<\alpha<\frac{1}{4}-\frac{1}{p}}C'_{T,p,\alpha}\times C''_{T,p,\alpha}.\tag{3.17}$$

In view of (3.11), (3.15) and (3.9), a straightforward calculation leads to

$$C_{T,p}<p^{\frac{p}{2}}T^{\frac{p}{4}-\frac{3}{2}}\left(\frac{2}{\pi}\right)^p\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{p}{2}+1}\left(\frac{6p-8}{p-10}\right)^{\frac{3p}{2}-2}.\tag{3.18}$$

This completes the proof of the estimate (3.1). ∎

Lemma 3.3. *Let $\sigma(s,y)$ be as in Proposition 3.1, then for any $T>0$, $p>10$, $\lambda>0$, there exists a constant $C_{T,p}>0$ such that*

$$\begin{aligned}&\mathbb{P}\left(\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right|>\lambda\right)\\ \le&\mathbb{P}\left(\int_0^T\sup_{y\in[0,1]}\left|\sigma(s,y)\right|^p\,\mathrm{d}s>\lambda^p\right)\\ &+\frac{C_{T,p}}{\lambda^p}\mathbb{E}\min\left\{\lambda^p,\int_0^T\sup_{y\in[0,1]}\left|\sigma(s,y)\right|^p\,\mathrm{d}s\right\}.\end{aligned}\tag{3.19}$$

Here the constant $C_{T,p}$ is the same as the constant $C_{T,p}$ in (3.1).

Proof. For any $\lambda>0$, define

$$\Omega_\lambda:=\left\{\omega\in\Omega:\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^p\,\mathrm{d}s\le\lambda^p\right\}.\tag{3.20}$$

By Chebyshev's inequality, we have

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right|>\lambda\right)\\
\le&\mathbb{P}(\Omega\backslash\Omega_\lambda)+\mathbb{P}\left(\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right|\mathbb{1}_{\Omega_\lambda}>\lambda\right)\\
\le&\mathbb{P}(\Omega\backslash\Omega_\lambda)+\frac{1}{\lambda^p}\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\mathbb{1}_{\Omega_\lambda}\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right|^p\right].
\end{aligned}
\tag{3.21}
$$

Now, we introduce the random field

$$
\widetilde{\sigma}(s,y):=\sigma(s,y)\mathbb{1}_{\left\{\omega\in\Omega:\,\int_0^s\sup_{y\in[0,1]}|\sigma(r,y)|^p\,\mathrm{d}r\le\lambda^p\right\}}. \tag{3.22}
$$

Note that the stochastic integral of $\widetilde{\sigma}(\cdot,\cdot)$ with respect to the space time white noise is well defined. Since for any $\omega\in\Omega_\lambda$,

$$
\int_0^t\int_0^1|\sigma(s,y)-\widetilde{\sigma}(s,y)|^2\,\mathrm{d}s\mathrm{d}y=0,\quad\forall\, t\in[0,T], \tag{3.23}
$$

by the local property of the stochastic integral (see Lemma A.1 in Appendix),

$$
\begin{aligned}
&\mathbb{1}_{\Omega_\lambda}\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\\
=&\mathbb{1}_{\Omega_\lambda}\int_0^t\int_0^1 p_{t-s}(x,y)\widetilde{\sigma}(s,y)\,W(\mathrm{d}s,\mathrm{d}y),\quad \mathbb{P}-a.s..
\end{aligned}
\tag{3.24}
$$

Hence using the bound (3.1), we get

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\mathbb{1}_{\Omega_\lambda}\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right|^p\right]\\
=&\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\mathbb{1}_{\Omega_\lambda}\int_0^t\int_0^1 p_{t-s}(x,y)\widetilde{\sigma}(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right|^p\right]\\
\le&\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\widetilde{\sigma}(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right|^p\right]\\
\le&C_{T,p}\mathbb{E}\int_0^T\sup_{y\in[0,1]}|\widetilde{\sigma}(s,y)|^p\,\mathrm{d}s\\
\le&C_{T,p}\mathbb{E}\min\left\{\lambda^p,\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^p\,\mathrm{d}s\right\}.
\end{aligned}
\tag{3.25}
$$

Combining (3.21) with (3.25), we obtain (3.19). ■

Proposition 3.4. *Let $\{\sigma(s,y):(s,y)\in\mathbb{R}_+\times[0,1]\}$ be a random field such that the stochastic integral against space time white noise is well defined. Then the following two estimates hold:*

(i) for any $T>0$, $0<p\le 10$, $q>10$, there exists a constant $C_{T,p,q}$ such that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right|^p\right]\\
\le&C_{T,p,q}\mathbb{E}\left[\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^q\,\mathrm{d}s\right]^{\frac{p}{q}}.
\end{aligned}
\tag{3.26}
$$

(ii) For any $T>0$, $0<p\le 10$, $\epsilon>0$, there exists a constant $C_{T,p,\epsilon}$ such that

$$\begin{aligned}&\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right|^p\right]\\ \le&\epsilon\mathbb{E}\left[\sup_{(s,y)\in[0,T]\times[0,1]}|\sigma(s,y)|^p\right]+C_{T,p,\epsilon}E\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^p\,\mathrm{d}s.\end{aligned}\tag{3.27}$$

Remark 3.5. The significance of the estimates (3.26) and (3.27) is that they allow p to be small, which is crucial for the proof of the transportation cost inequality in the next section.

Proof. The estimate (3.26) can be easily derived from (3.19) and Lemma A.2 in Appendix as follows:

$$\begin{aligned}&\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right|^p\right]\\ =&\int_0^\infty p\lambda^{p-1}\mathbb{P}\left(\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right|>\lambda\right)\mathrm{d}\lambda\\ \le&\int_0^\infty p\lambda^{p-1}\mathbb{P}\left(\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^q\,\mathrm{d}s>\lambda^q\right)\mathrm{d}\lambda\\ &+C_{T,q}\int_0^\infty p\lambda^{p-1-q}\mathbb{E}\min\left\{\lambda^q,\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^q\,\mathrm{d}s\right\}\mathrm{d}\lambda\\ =&C_{T,p,q}\mathbb{E}\left[\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^q\,\mathrm{d}s\right]^{\frac{p}{q}},\end{aligned}\tag{3.28}$$

where

$$C_{T,p,q}:=1+C_{T,q}\frac{q}{q-p},\tag{3.29}$$

and the constant $C_{T,q}$ is defined in (3.17).

Let us now prove the assertion (ii) in Proposition 3.4. From (3.26) it follows that for any $q>10$,

$$\begin{aligned}&\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,y)\,W(ds,dy)\right|^p\right]\\ \le&C_{T,p,q}\mathbb{E}\left[\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^q\,ds\right]^{\frac{p}{q}}\\ \le&C_{T,p,q}\mathbb{E}\left[\sup_{(s,y)\in[0,T]\times[0,1]}|\sigma(s,y)|^{q-p}\times\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^p\,ds\right]^{\frac{p}{q}}\\ =&C_{T,p,q}\mathbb{E}\left[\sup_{(s,y)\in[0,T]\times[0,1]}|\sigma(s,y)|^{\frac{(q-p)p}{q}}\times\left(\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^p\,ds\right)^{\frac{p}{q}}\right]\\ \le&\epsilon\mathbb{E}\left[\sup_{(s,y)\in[0,T]\times[0,1]}|\sigma(s,y)|^p\right]+C_{T,p,q}\times C_{T,p,q,\epsilon}\mathbb{E}\int_0^T\sup_{y\in[0,1]}|\sigma(s,y)|^p\,ds,\end{aligned}\tag{3.30}$$

where we have used the following Young inequality

$$
\begin{aligned}
ab \le & \frac{\epsilon}{C_{T,p,q}}\, a^{\frac{q}{q-p}} + C_{T,p,q,\epsilon}\, b^{\frac{q}{p}}, \\
C_{T,p,q,\epsilon} := & p\left(\frac{q-p}{\epsilon/C_{T,p,q}}\right)^{\frac{q-p}{p}} q^{-\frac{q}{p}}.
\end{aligned} \tag{3.31}
$$

Set

$$
C_{T,p,\epsilon} := \inf_{q>10} C_{T,p,q} \times C_{T,p,q,\epsilon}. \tag{3.32}
$$

Combining (3.29) and (3.31) gives

$$
C_{T,p,\epsilon} = \inf_{q>10} \frac{p}{q-p} q^{-\frac{q}{p}} \epsilon^{1-\frac{q}{p}} \left(q - p + qC_{T,q}\right)^{\frac{q}{p}}, \tag{3.33}
$$

where the constant $C_{T,q}$ is bounded by the right hand side of (3.2) with p replaced by q. Now, (3.27) follows from (3.30) with the constant $C_{T,p,\epsilon}$ defined above. ■

4 Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. Let μ be the law of the random field solution $u(\cdot,\cdot)$ of SPDE (2.1), viewed as a probability measure on $C([0,T]\times[0,1])$. First we recall a lemma proved in [12] describing the probability measures ν that are absolutely continuous with respect to μ.

Let $\nu \ll \mu$ on $C([0,T]\times[0,1])$. Define a new probability measure $\mathbb{Q}$ on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ by

$$
\mathrm{d}\mathbb{Q} := \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(u)\,\mathrm{d}\mathbb{P}. \tag{4.1}
$$

Denote the Radon-Nikodym derivative restricted on $\mathcal{F}_t$ by

$$
M_t := \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}, \quad t\in[0,T].
$$

Then $M_t, t\in[0,T]$ forms a $\mathbb{P}$-martingale. The following result was proved in [12].

Lemma 4.1. *There exists an adapted random field* $h = \{h(s,x),(s,x)\in[0,T]\times[0,1]\}$ *such that* $\mathbb{Q}-a.s.$ *for all* $t\in[0,T]$*,*

$$
\int_0^t\int_0^1 h^2(s,x)\,\mathrm{d}s\mathrm{d}x < \infty
$$

and $\widetilde{W} : [0,T]\times[0,1]\to\mathbb{R}$ *defined by*

$$
\widetilde{W}(t,x) := W(t,x) - \int_0^t\int_0^x h(s,y)\,\mathrm{d}s\mathrm{d}y, \tag{4.2}
$$

is a Brownian sheet under the measure $\mathbb{Q}$*. Moreover,*

$$
M_t = \exp\left(\int_0^t\int_0^1 h(s,x)\,W(\mathrm{d}s,\mathrm{d}x) - \frac{1}{2}\int_0^t\int_0^1 h^2(s,x)\,\mathrm{d}s\mathrm{d}x\right), \quad \mathbb{Q}-a.s., \tag{4.3}
$$

and

$$
H(\nu|\mu) = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T\int_0^1 h^2(s,x)\,\mathrm{d}s\mathrm{d}x\right], \tag{4.4}
$$

where $\mathbb{E}^{\mathbb{Q}}$ *stands for the expectation under the measure* $\mathbb{Q}$*.*

Proof of Theorem 2.1. Take $\nu \ll \mu$ on $C([0,T]\times[0,1])$. Define the corresponding measure $\mathbb{Q}$ by (4.1). Let $h(t,x)$ be the corresponding random field appeared in Lemma 4.1. Then the solution $u(t,x)$ of equation (2.1) satisfies the following SPDE under the measure $\mathbb{Q}$,

$$\begin{aligned} u(t,x) =& P_t u_0(x) + \int_0^t \int_0^1 p_{t-s}(x,y) b(u(s,y))\,\mathrm{d}s\mathrm{d}y \\ &+ \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(u(s,y))\,\widetilde{W}(\mathrm{d}s,\mathrm{d}y) \\ &+ \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(u(s,y))h(s,y)\,\mathrm{d}s\mathrm{d}y. \end{aligned} \tag{4.5}$$

Consider the solution of the following SPDE:

$$\begin{aligned} v(t,x) =& P_t u_0(x) + \int_0^t \int_0^1 p_{t-s}(x,y) b(v(s,y))\,\mathrm{d}s\mathrm{d}y \\ &+ \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(v(s,y))\,\widetilde{W}(\mathrm{d}s,\mathrm{d}y). \end{aligned} \tag{4.6}$$

By Lemma 4.1 it follows that under the measure $\mathbb{Q}$, the law of (v,u) forms a coupling of (μ,ν). Therefore by the definition of the Wasserstein distance,

$$W_2(\nu,\mu)^2 \le \mathbb{E}^{\mathbb{Q}}\left[\sup_{(t,x)\in[0,T]\times[0,1]} |u(t,x)-v(t,x)|^2\right].$$

In view of (4.4), to prove the quadratic transportation cost inequality

$$W_2(\nu,\mu) \le \sqrt{2CH(\nu|\mu)}, \tag{4.7}$$

it is sufficient to show that

$$\mathbb{E}^{\mathbb{Q}}\left[\sup_{(t,x)\in[0,T]\times[0,1]} |v(t,x)-u(t,x)|^2\right] \le C\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \int_0^1 h^2(s,y)\,\mathrm{d}s\mathrm{d}y\right] \tag{4.8}$$

when the right hand side of (4.8) is finite. For simplicity, in the sequel we still denote $\mathbb{E}^{\mathbb{Q}}$ by the symbol $\mathbb{E}$. From (4.6) and (4.5) it follows that

$$\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]} |v(t,x)-u(t,x)|^2\right] \le 3(I+II+III), \tag{4.9}$$

where

$$\begin{aligned} I :=& \mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]} \left|\int_0^t \int_0^1 p_{t-s}(x,y)\big[b(v(s,y))-b(u(s,y))\big]\,\mathrm{d}s\mathrm{d}y\right|^2\right], \\ II :=& \mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]} \left|\int_0^t \int_0^1 p_{t-s}(x,y)\big[\sigma(v(s,y))-\sigma(u(s,y))\big]\,\widetilde{W}(\mathrm{d}s,\mathrm{d}y)\right|^2\right], \\ III :=& \mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]} \left|\int_0^t \int_0^1 p_{t-s}(x,y)\sigma(u(s,y))h(s,y)\,\mathrm{d}s\mathrm{d}y\right|^2\right]. \end{aligned}$$

By Holder's inequality and (3.9), the term I can be estimated as follows:

$$I \le L_b^2\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]} \left|\int_0^t \int_0^1 p_{t-s}(x,y)|v(s,y)-u(s,y)|\,\mathrm{d}s\mathrm{d}y\right|^2\right]$$

$$
\begin{aligned}
&\le L_b^2\mathbb{E}\Bigg\{\sup_{(t,x)\in[0,T]\times[0,1]}\Bigg[\left(\int_0^t\int_0^1 p_{t-s}(x,y)^2\,\mathrm{d}s\mathrm{d}y\right)\\
&\qquad\qquad\times\left(\int_0^t\int_0^1 |v(s,y)-u(s,y)|^2\,\mathrm{d}s\mathrm{d}y\right)\Bigg]\Bigg\}\\
&\le\sqrt{\frac{2T}{\pi}}L_b^2\mathbb{E}\int_0^T\int_0^1 |v(s,y)-u(s,y)|^2\,\mathrm{d}s\mathrm{d}y\\
&\le\sqrt{\frac{2T}{\pi}}L_b^2\int_0^T\mathbb{E}\left[\sup_{(r,y)\in[0,s]\times[0,1]}|v(r,y)-u(r,y)|^2\right]\mathrm{d}s. \qquad (4.10)
\end{aligned}
$$

For the term II, applying the estimate (3.27) we obtain that for any $\epsilon>0$,

$$
\begin{aligned}
II \le&\epsilon\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}|\sigma(v(t,x))-\sigma(u(t,x))|^2\right]\\
&+C_{T,2,\epsilon}\mathbb{E}\int_0^T\sup_{y\in[0,1]}|\sigma(v(s,y))-\sigma(u(s,y))|^2\,\mathrm{d}s\\
\le&\epsilon L_\sigma^2\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}|v(t,x)-u(t,x)|^2\right]\\
&+C_{T,2,\epsilon}L_\sigma^2\int_0^T\mathbb{E}\left[\sup_{(r,y)\in[0,s]\times[0,1]}|v(r,y)-u(r,y)|^2\right]\mathrm{d}s. \qquad (4.11)
\end{aligned}
$$

The term III can be bounded as follows:

$$
\begin{aligned}
III \le&K_\sigma^2\mathbb{E}\Bigg\{\sup_{(t,x)\in[0,T]\times[0,1]}\Bigg[\left(\int_0^t\int_0^1 p_{t-s}(x,y)^2\,\mathrm{d}s\mathrm{d}y\right)\\
&\qquad\times\left(\int_0^t\int_0^1 h^2(s,y)\,\mathrm{d}s\mathrm{d}y\right)\Bigg]\Bigg\}\\
\le&\sqrt{\frac{2T}{\pi}}K_\sigma^2\mathbb{E}\left[\int_0^T\int_0^1 h^2(s,y)\,\mathrm{d}s\mathrm{d}y\right]. \qquad (4.12)
\end{aligned}
$$

Set

$$
Y(t):=\mathbb{E}\left[\sup_{(s,x)\in[0,t]\times[0,1]}|v(s,x)-u(s,x)|^2\right]. \qquad (4.13)
$$

Putting (4.9)-(4.12) together, we obtain

$$
\begin{aligned}
Y(T)\le&3\sqrt{\frac{2T}{\pi}}L_b^2\int_0^T Y(s)\,\mathrm{d}s+3\epsilon L_\sigma^2 Y(T)+3C_{T,2,\epsilon}L_\sigma^2\int_0^T Y(s)\,\mathrm{d}s\\
&+3\sqrt{\frac{2T}{\pi}}K_\sigma^2\mathbb{E}\left[\int_0^T\int_0^1 h^2(s,y)\,\mathrm{d}s\mathrm{d}y\right]. \qquad (4.14)
\end{aligned}
$$

Recall that (see e.g. Theorem 3.13 in [6])

$$
\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}|u(t,x)|^2\right]<\infty, \qquad (4.15)
$$

$$
\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}|v(t,x)|^2\right]<\infty. \qquad (4.16)
$$

Hence $Y(T) < \infty$ for any $T > 0$. Taking any $\epsilon < \frac{1}{3L_\sigma^2}$, we deduce from (4.14) that

$$\begin{aligned} Y(T) \le & \frac{3L_b^2}{1-3\epsilon L_\sigma^2}\sqrt{\frac{2T}{\pi}}\int_0^T Y(s)\,\mathrm{d}s + \frac{3C_{T,2,\epsilon}L_\sigma^2}{1-3\epsilon L_\sigma^2}\int_0^T Y(s)\,\mathrm{d}s \\ & + \frac{3K_\sigma^2}{1-3\epsilon L_\sigma^2}\sqrt{\frac{2T}{\pi}}\mathbb{E}\left[\int_0^T\int_0^1 h^2(s,y)\,\mathrm{d}s\mathrm{d}y\right]. \end{aligned} \tag{4.17}$$

Clearly, (4.17) still holds if we replace T with any $t \in [0,T]$. Applying Gronwall's inequality, we obtain

$$\begin{aligned} Y(T) \le & K_\sigma^2 \inf_{0<\epsilon<\frac{1}{3L_\sigma^2}} \left\{ \frac{3}{1-3\epsilon L_\sigma^2}\sqrt{\frac{2T}{\pi}} \exp\left(\frac{3L_b^2 T}{1-3\epsilon L_\sigma^2}\sqrt{\frac{2T}{\pi}} + \frac{3C_{T,2,\epsilon}L_\sigma^2 T}{1-3\epsilon L_\sigma^2}\right)\right\} \\ & \times \mathbb{E}\left[\int_0^T\int_0^1 h^2(s,y)\,\mathrm{d}s\mathrm{d}y\right]. \end{aligned} \tag{4.18}$$

This proves (4.8) with the constant C to be

$$C = K_\sigma^2 \inf_{0<\epsilon<\frac{1}{3L_\sigma^2}} \left\{ \frac{3}{1-3\epsilon L_\sigma^2}\sqrt{\frac{2T}{\pi}} \exp\left(\frac{3L_b^2 T}{1-3\epsilon L_\sigma^2}\sqrt{\frac{2T}{\pi}} + \frac{3C_{T,2,\epsilon}L_\sigma^2 T}{1-3\epsilon L_\sigma^2}\right)\right\}, \tag{4.19}$$

where, according to (3.33) and (3.2),

$$C_{T,2,\epsilon} = \inf_{q>10} \frac{2}{q-2} q^{-\frac{q}{2}} \epsilon^{1-\frac{q}{2}} \left(q-2+qC_{T,q}\right)^{\frac{q}{2}}, \tag{4.20}$$

and the constant $C_{T,q}$ is bounded by

$$C_{T,q} < q^{\frac{q}{2}} T^{\frac{q}{4}-\frac{3}{2}} \left(\frac{2}{\pi}\right)^q \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{q}{2}+1} \left(\frac{6q-8}{q-10}\right)^{\frac{3q}{2}-2}. \tag{4.21}$$

Hence the proof of Theorem 2.1 is complete. ■

A Appendix

The following local property of the Walsh stochastic integral against space-time white noise is similar to that of the Ito integral.

Lemma A.1. *Let $\{\sigma(t,x) : (t,x) \in [0,T]\times[0,1]\}$ be a random field such that the stochastic integral against space time white noise is well defined. Let $\Omega_0 \subset \Omega$ be a measurable subset such that for a.s. $\omega \in \Omega_0$,*

$$\int_0^T\int_0^1 |\sigma(t,x)|^2\,\mathrm{d}t\mathrm{d}x = 0. \tag{A.1}$$

Then for a.s. $\omega \in \Omega_0$,

$$\int_0^T\int_0^1 \sigma(t,x)\,W(\mathrm{d}t,\mathrm{d}x) = 0. \tag{A.2}$$

Proof. The local property can be similarly proved as that of Ito integral. We only outline the proof here. Firstly, we note that the local property obviously holds when $\sigma(\cdot,\cdot)$ is a simple process. When $\sigma(t,x)$ is a bounded, continuous random field, we can prove the local property through an approximation of σ by a sequence of simple processes. For the general random field $\sigma(\cdot,\cdot)$, the local property can be proved by further two approximations, first by bounded random fields and then by continuous random fields. ■

Lemma A.2. *Let* $X \geq 0$ *be a random variable, then for any* $0 < p < q$*,*

$$\mathbb{E}X^p = \int_0^\infty p x^{p-1} \mathbb{P}(X > x)\,\mathrm{d}x, \tag{A.3}$$

$$\int_0^\infty \frac{\mathbb{E}\min\{x^q, X\}}{x^q} p x^{p-1}\,\mathrm{d}x = \frac{q}{q-p}\mathbb{E}\left[X^{\frac{p}{q}}\right]. \tag{A.4}$$

Proof. (A.3) and (A.4) can be easily proved by the Fubini theorem. (A.4) is similar to Lemma 2 in [10], for completeness, we provide the proof here.

$$\begin{aligned}\int_0^\infty \frac{\mathbb{E}\min\{x^q, X\}}{x^q} p x^{p-1}\,\mathrm{d}x =& \mathbb{E}\int_0^{X^{\frac{1}{q}}} p x^{p-1}\,\mathrm{d}x + \mathbb{E}\left[X \int_{X^{\frac{1}{q}}}^\infty p x^{p-1-q}\,\mathrm{d}x\right] \\ =& \mathbb{E}\left[X^{\frac{p}{q}}\right] - \frac{p}{p-q}\mathbb{E}\left[X\left(X^{\frac{1}{q}}\right)^{p-q}\right] \\ =& \frac{q}{q-p}\mathbb{E}\left[X^{\frac{p}{q}}\right]. \quad \blacksquare\end{aligned} \tag{A.5}$$

References

[1] Boufoussi, B. and Hajji, S.: Transportation inequalities for stochastic heat equations. *Statist. Probab. Lett.* **139**, (2018), 75–83. MR-3802186

[2] Cattiaux, P. and Guillin, A.: Semi Log-Concave Markov Diffusions. *Séminaire de Probabilités XLVI*, Springer, Cham, 2014. 231–292 pp. MR-3330820

[3] Chen, Z.-Q., Kim, P. and Song, R.: Heat kernel estimates for the Dirichlet fractional Laplacian. *J. Eur. Math. Soc.* **12**, (2010), no. 5, 1307–1329. MR-2677618

[4] Djellout, H., Guillin, A. and Wu, L.: Transportation cost-information inequalities and applications to random dynamical systems and diffusions. *Ann. Probab.* **32**, (2004), no. 3B, 2702–2732. MR-2078555

[5] Dubhashi, D. P. and Panconesi, A.: Concentration of measure for the analysis of randomized algorithms. *Cambridge University Press*, Cambridge, 2009. xvi+196 pp. MR-2547432

[6] Dalang, R. C., Khoshnevisan, D. and Zhang, T.: Global solutions to stochastic reaction diffusion equations with super-linear drift and multiplicative noise. *Ann. Probab.* **47**, (2019), no. 1, 519–559. MR-3909975

[7] Dalang, R. C., Khoshnevisan, D., Mueller, C., Nualart, D. and Xiao, Y.: A minicourse on Stochastic Partial Differential Equations. *Lecture Notes in Mathematics* **1962**, Springer-Verlag, Berlin, 2009. xii+216 pp. MR-1500166

[8] Da Prato, G. and Zabczyk, J.: Stochastic equations in infinite dimensions. *Encyclopedia of Mathematics and its Applications* **152**, second edition, Cambridge University Press, Cambridge, 2014. xviii+493 pp. MR-3236753

[9] Gozlan, N., Roberto, C. and Samson, P.-M.: Characterization of Talagrand's transport-entropy inequalities in metric spaces. *Ann. Probab.* **41**, (2013), no. 5, 3112–3139. MR-3127876

[10] Ichikawa, A.: Some inequalities for martingales and stochastic convolutions. *Stochastic Anal. Appl.* **4**, (1986), no. 3, 329–339. MR-0857085

[11] Khoshnevisan, D.: Analysis of stochastic partial differential equations. *CBMS Regional Conference Series in Mathematics* **119**, American Mathematical Society, Providence, RI, 2014. viii+116 pp. MR-3222416

[12] Khoshnevisan, D. and Sarantsev, A.: Talagrand concentration inequalities for stochastic partial differential equations, arXiv:1709.07098v3. MR-3222416

[13] Lacker, D.: Liquidity, risk measures, and concentration of measure. *Math. Oper. Res.* **43**, (2018), no. 3, 813–837. MR-3846074

[14] Ledoux, M.: The concentration of measure phenomenon. *Mathematical Surveys and Monographs* **89**, American Mathematical Society, Providence, RI, 2001. x+181 pp. MR-1849347

[15] Li, Y. and Wang, X.: Transportation inequalities for stochastic wave equation, arXiv:1811.06385

[16] Marton, K.: Bounding $\bar{d}$-distance by informational divergence: a method to prove measure concentration. *Ann. Probab.* **24**, (1996), no. 2, 857–866. MR-1404531

[17] Marton, K.: A measure concentration inequality for contracting Markov chains. *Geom. Funct. Anal.* **6**, (1996), no. 3, 556–571. MR-1392329

[18] Massart, P.: Concentration inequalities and model selection. *Lecture Notes in Mathematics* **1896**, Springer, Berlin, 2007. xiv+337 pp. MR-2319879

[19] Otto, F. and Villani, C.: Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.* **173**, (2000), no. 2, 361–400. MR-1760620

[20] Pal, S.: Concentration for multidimensional diffusions and their boundary local times. *Probab. Theory Related Fields* **154**, (2012), no. 1-2, 225–254. MR-2981423

[21] Pal, S. and Sarantsev, A.: A Note on Transportation Cost Inequalities for Diffusions with Reflections. *Electron. Commun. Probab.* **24**, (2019), no. 21, 1–11. MR-3940196

[22] Pal, S. and Shkolnikov, M.: Concentration of measure for Brownian particle systems interacting through their ranks. *Ann. Appl. Probab.* **24**, (2014), no. 4, 1482–1508. MR-3211002

[23] Talagrand, M.: Concentration of measure and isoperimetric inequalities in product spaces. *Inst. Hautes Études Sci. Publ. Math.* **81**, (1995), 73–205. MR-1361756

[24] Talagrand, M.: Transportation cost for Gaussian and other product measures. *Geom. Funct. Anal.* **6**, (1996), no. 3, 587–600. MR-1392331

[25] Talagrand, M.: New concentration inequalities in product spaces. *Invent. Math.* **126**, (1996), no. 3, 505–563. MR-1419006

[26] Üstünel, A. S.: Transportation cost inequalities for diffusions under uniform distance. *Stochastic Analysis and Related Topics*, Springer, Heidelberg, 2012. 203–214 pp.

[27] Walsh, J. B.: An introduction to stochastic partial differential equations. *École d'été de probabilités de Saint-Flour, XIV—1984*, Springer, Berlin, 1986. 265–439 pp. MR-0876085

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