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L^1 solutions of non-reflected BSDEs and reflected BSDEs with one and two continuous barriers under general assumptions*

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Abstract

We establish several existence, uniqueness and comparison results for L^1 solutions of non-reflected BSDEs and reflected BSDEs with one and two continuous barriers under the assumptions that the generator g satisfies a one-sided Osgood condition together with a very general growth condition in y, a uniform continuity condition and/or a sub-linear growth condition in z, and a generalized Mokobodzki condition for reflected BSDEs which relates the growth of g and that of the barriers. This generalized Mokobodzki condition is proved to be necessary for existence of L^1 solutions of the reflected BSDEs. We also prove that the L^1 solutions of reflected BSDEs can be approximated by a penalization method and by some sequences of L^1 solutions of reflected BSDEs. The approach is based on a combination between existing methods, their refinement and perfection, but also on some novel ideas and techniques. These results strengthen some existing work on the L^1 solutions of non-reflected BSDEs and reflected BSDEs.

Keywords: reflected backward stochastic differential equation; existence and uniqueness; comparison theorem; stability theorem; L^1 solution.
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1 Introduction

In 1990, Pardoux and Peng [47] first introduced the notion of nonlinear backward stochastic differential equations (BSDEs for short) and established the well known existence and uniqueness result of an L^2 solution for a BSDE with square-integrability data under the assumption that the generator g is uniformly Lipschitz continuous in (y, z). Under the square-integrability assumption on data and the uniformly Lipschitz continuity assumption on generator, El Karoui et al. [10] and Cvitanić and Karatzas [8]

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respectively introduced the notion of nonlinear reflected BSDEs (RBSDEs) with one and two continuous barriers, established the existence and uniqueness of the L^2 solution, and explored that these equations have natural connections with the obstacle problem for PDEs, the optimal stopping problem, the mixed control problem and Dynkin games. Since, the theory of BSDEs and reflected BSDEs has rapidly been developed and applied in many areas. For instance, among others readers are referred to El Karoui et al. [11], El Karoui et al. [12], Jia [36], Peng [48], Peng [49], Peng [50], Peng and Xu [52] and Rosazza Gianin [54] for the applications in mathematical finance, risk measures and the nonlinear expectation theory, Bayraktar and Yao [2], Hamadène and Lepeltier [28], Hamadène et al. [30] and Hu and Tang [33] for the applications in the stochastic control and game theory, and Hamadène and Zhang [31], Hu and Tang [32], Ma and Zhang [43], Pardoux [46], Peng and Xu [51] and Ren and El Otmani [53] for the applications in optimality problems, PDEs and others.

During more than two decades, for the theoretical interests of investigation and interesting applications a lot of works have been devoted to studying the existence and uniqueness of a solution for a non-reflected BSDE and a RBSDE by relaxing the square-integrability assumption on data and the uniformly Lipschitz continuity assumption on generator used in the pioneer papers [47], [10] and [8]. For instance, the uniformly Lipschitz condition of g in (y, z) has been weakened to the monotonicity and general growth condition in y (see assumption (H1) with $\rho(x) = k|x|$ for some constant $k \geq 0$ in Subsection 2.3 of this paper) and the uniform continuity condition in z (see assumption (H2)(i) in Subsection 2.3) in the existence and uniqueness results for L^2 solutions or L^p $(p > 1)$ solutions established respectively in, e.g., Pardoux [46], Briand et al. [3], Briand et al. [5], Jia [35], Jia [36], Chen [7], Fan and Jiang [20], Fan et al. [23] and Fan [14] for non-reflected BSDEs, and Lepeltier et al. [40], Klimsiak [38], Rozkosz and Słomiński [55], Klimsiak [39], Fan [16] and Fan [18] for RBSDEs. And, in the existence and uniqueness results for L^2 solutions or L^p $(p > 1)$ solutions established respectively in Fan and Jiang [22], Fan [13], Fan [14], Fan [16], Yao [58] and Fan [18], the monotonicity condition of g in y was further weakened to the one-sided Osgood condition (see assumption (H1)(i) in Subsection 2.3) and the weak monotonicity condition, which both unify the monotonicity condition, the Mao's non-Lipschitz condition (see Mao [44]) and the usual Osgood condition (see Fan et al. [24]). On the other hand, in the case of concerning only the wellposedness or existence of the L^2 solution or L^p $(p > 1)$ solution, the assumptions required by the generator g have been further relaxed. For example, in Briand et al. [5], Xu [57], Fan [16] and Fan [18], besides the (weak) monotonicity condition in y and the continuity condition in (y, z), a general growth condition in y and a linear growth condition in z (see assumption (HH) with $\alpha = 1$ in Section 4) is the only requirement for the generator g, and in Lepeltier and San Martin [41], Matoussi [45], Hamadène et al. [29] and Jia and Xu [37], the generator g needs only to be continuous and has a linear growth in (y, z) (see assumption (AA) with $\tilde{\alpha} = 1$ in Subsection 2.3).

During the evolution of BSDE theory, many papers have also been interested in the existence and uniqueness of the L^1 solutions for non-reflected BSDEs and RBSDEs with only integrability data. To the best of our knowledge, this problem was first investigated in Peng [48] for BSDEs with positive terminal conditions. In 2003, Briand et al. [3] established an important existence and uniqueness result on the L^1 solutions for BSDEs, where the generator g satisfies the monotonicity and general growth condition in y, the uniformly Lipschitz condition in z and an additional sub-linear growth condition in z (see assumption (H2)(ii)). Recently, this result was successfully extended to the case of reflected BSDEs in Rozkosz and Słomiński [55] and Klimsiak [39] (see also Klimsiak [38] and Bayraktar and Yao [2]). On the other hand, the investigation on the existence and/or uniqueness of the L^1 solutions for non-reflected BSDEs kept going deeper. For example, the monotonicity condition in y of the generator g employed in the existence and uniqueness result of the L^1 solutions in [3] was weakened to the one-sided Osgood condition in Fan [19] and Fan [14], and the uniformly Lipschitz condition in z was also, respectively, weakened to the α-Hölder continuity condition (i.e., $\phi(x) = |x|^\alpha$ in (H2)(i) for some constant $\alpha \in (0,1)$) in Fan and Liu [25] and the uniform continuity condition in z in Fan [14] (but in Fan [14], the generator g needs to be dominated by a deterministic process). And, several existence results on the L^1 solutions of non-reflected BSDEs were also obtained in Briand and Hu [4], Fan [14] and Fan [17], where the generator g does not need to satisfy the uniformly Lipschitz condition or the uniform continuity condition in z. In particular, in Fan [17] the sub-linear growth condition (H2)(ii) employed in [3] was relaxed to assumption (H2')(ii) in Subsection 2.3. We also would like to mention that Hu and Tang [34] and Buckdahn et al. [6] investigated, from a totally new perspective, the existence and uniqueness on the L^1 solutions for non-reflected BSDEs, where the generator g does not need to satisfy the sub-linear growth condition (H2)(ii) or (H2')(ii), but the terminal condition needs to satisfy a stronger integrability condition.

In order to ensure existence of a solution for RBSDEs with two barriers, a Mokobodzki condition (i.e., there exists a quasi-martingale between two barriers) or a certain regularity condition on one of the barriers usually needs to be satisfied as in Cvitanić and Karatzas [8], Bahlali et al. [1] and Peng and Xu [51]. By virtue of the notion of local solutions, these two conditions were replaced with the completely separated condition of the two barriers, which can be more easily verified or checked, in Hamadène and Hassani [26], Hamadène et al. [27], El Asri et al. [9], Bayraktar and Yao [2] and so on. Recently, several generalized Mokobodzki conditions, see (ii) of assumptions (H3), (H3L) and (H3U) in Subsection 2.3 for the case of L^1 solution, were put forward and proved to be sufficient and necessary to ensure the existence of an L^p $(p > 1)$ or L^1 solution for a RBSDE with one or two barriers when the generator g has a general growth in y, see Klimsiak [38], Klimsiak [39], Fan [16] and Fan [18] for more details. Many efforts in this direct can also be found in Lepeltier et al. [40], Xu [56], Xu [57], Rozkosz and Słomiński [55], Li and Shi [42] and references therein.

Enlightened by these works aforementioned, especially by Peng and Xu [51], Klimsiak [38], Bayraktar and Yao [2] and Fan [16], we dedicate this paper to the L^1 solution of non-reflected BSDEs and RBSDEs with one and two continuous barriers under general assumptions on the generator and the data, i.e., (H1), (H2), (H2'), (H3), (H3L), (H3U) and (AA) mentioned above, see Subsection 2.3 again. Our results strengthen some corresponding known works on the L^1 solutions of on-reflected BSDEs and RBSDEs (see Remark 7.6 in Section 7 for more details). Our approach is based on a combination between existing methods, their refinement and perfection, but also on some novel ideas and techniques.

The rest of this paper is organized as follows. Section 2 contains some notations, definitions, assumptions and lemmas which will be used later. Section 3 consists of four subsections, which establish three convergence results respectively with respect to the penalization scheme and the approximation scheme for the L^1 solutions of RBSDEs with one and two barriers under general assumptions, and a general comparison theorem for the L^1 solutions of RBSDEs under assumptions (H1)(i) and (H2). These elementary results will play important roles in the proof of our main results in the subsequent sections. Section 4 is devoted to the L^1 solution of non-reflected BSDEs. In this section, we prove an existence and uniqueness result for the L^1 solution of a BSDE under assumptions (H1) and (H2) (see Theorem 4.2), and an existence result for the minimal and maximal L^1 solutions of a BSDE with generator $g := g^1 + g^2$, where the generator g^1

satisfies assumptions (H1)(i), (H2')(i) and (HH) (resp. (H1) and (H2')), and g^2 satisfies assumption (AA) (see Theorem 4.4 and Corollary 4.5). Section 5 deals with the L^1 solution of RBSDEs with one continuous barrier. By Theorem 5.1 we prove the existence and uniqueness of an L^1 solution for a RBSDE with one lower (resp. upper) barrier under assumptions (H1), (H2) and (H3L) (resp. (H3U)) by the penalization method, and show the sufficient and necessary property of (H3L)(ii) (resp. (H3U)(ii)). And, in Theorem 5.3 we study the same problem, but on existence of a minimal (resp. maximal) L^1 solution for a RBSDE with one lower (resp. upper) barrier and a generator $g := g^1 + g^2$, where the generator g^1 satisfies assumptions (H1) and (H2') and the generator g^2 satisfies assumption (AA). Furthermore, by Theorem 5.4 we show that under the assumptions of Theorem 5.3, the minimal and maximal L^1 solutions for the RBSDE with one lower or upper barrier can be both approximated by a sequence of L^1 solutions for RBSDEs with generators satisfying (H1) and (H2). Section 6 investigates the L^1 solution of RBSDEs with two continuous barriers. By Theorem 6.1 we prove the existence and uniqueness of an L^1 solution for a doubly RBSDE under assumptions (H1), (H2) and (H3) by the penalization method, and show the sufficient and necessary property of (H3)(ii). And, in Theorem 6.3 we study the same problem, but on existence of the minimal and maximal L^1 solutions for a doubly RBSDE with a generator as in Theorem 5.3. Furthermore, by Theorem 6.4 we prove that under the assumptions of Theorem 6.3, the minimal and maximal L^1 solutions for the doubly RBSDE can be both approximated by a sequence of L^1 solutions for doubly RBSDEs with generators satisfying (H1) and (H2). Finally, in Section 7 several examples and remarks are introduced to illustrate further the theoretical results in this paper. And, the proofs of Proposition 3.1 and Proposition 3.3 in Section 3 are detailed in Appendix.

2 Notations, definitions, assumptions and lemmas

2.1 Notations

Let $T > 0$ be a fixed real number and $(\Omega, \mathcal{F}_T, \mathbb{P}; (\mathcal{F}_t)_{t\in[0,T]})$ be a complete filtered probability space carrying a standard d-dimensional Brownian motion $(B_t)_{t\in[0,T]}$ together with the completed σ-algebra filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by $B_\cdot$. Denote by $\mathbb{1}_A$ the indicator function of a set A and by A^c the complement of A. Let $\mathbb{R}_+ := [0, +\infty)$, $a^+ := \max\{a, 0\}$ and $a^- := (-a)^+$ for any real number a, and let $\mathrm{sgn}(x)$ represent the sign of a real number x and $|y|$ the Euclidean norm of $y \in \mathbb{R}^n$ with $n \geq 1$. Furthermore, denote by $\mathcal{S}$ the set of all $(\mathcal{F}_t)$-progressively measurable and continuous real-valued processes $(Y_t)_{t\in[0,T]}$, and for $p > 0$ we denote by $\mathcal{S}^p$ the set of processes $Y_\cdot \in \mathcal{S}$ satisfying

$$\|Y\|_{\mathcal{S}^p} := \left(\mathbb{E}[\sup_{t\in[0,T]} |Y_t|^p]\right)^{1\wedge 1/p} < +\infty.$$

M is the set of all $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^d$-valued processes $(Z_t)_{t\in[0,T]}$ satisfying

$$\mathbb{P}\left(\int_0^T |Z_t|^2 \mathrm{d}t < +\infty\right) = 1,$$

and for $p > 0$, M^p is the set of processes $Z_\cdot \in \mathrm{M}$ satisfying

$$\|Z\|_{\mathrm{M}^p} := \left\{\mathbb{E}\left[\left(\int_0^T |Z_t|^2 \mathrm{d}t\right)^{p/2}\right]\right\}^{1\wedge 1/p} < +\infty.$$

We also use the following spaces with respect to variables and processes defined on $\Omega \times [0, T]$:

- $\mathbb{L}^1(\mathcal{F}_T)$ the set of $\mathcal{F}_T$-measurable real-valued random variables ξ satisfying $\mathbb{E}[|\xi|] < +\infty$;
- $\mathcal{H}$ the set of $(\mathcal{F}_t)$-progressively measurable real-valued processes $X_\cdot$ satisfying

$$\mathbb{P}\left(\int_0^T |X_t|\mathrm{d}t < +\infty\right) = 1;$$

- $\mathcal{H}^1$ the set of processes $X_\cdot \in \mathcal{H}$ satisfying $\|X\|_{\mathcal{H}^1} := \mathbb{E}\left[\int_0^T |X_t|\mathrm{d}t\right] < +\infty$;
- $\mathcal{V}$ the set of $(\mathcal{F}_t)$-progressively measurable and continuous real-valued processes of finite variation;
- $\mathcal{V}^+$ the set of increasing processes $V_\cdot \in \mathcal{V}$ valued 0 at 0;
- $\mathcal{V}^1$ (resp. $\mathcal{V}^{+,1}$) the set of processes $V_\cdot \in \mathcal{V}$ (resp. $\mathcal{V}^+$) satisfying $\mathbb{E}\left[|V|_T\right] < +\infty$.

Here and hereafter, for each $(\mathcal{F}_t)$-stopping time τ valued in $[0,T]$, $|V|_\tau$ represents the random finite variation of $V_\cdot \in \mathcal{V}$ on the stochastic interval $[0,\tau]$. It is clear that $|V|_\tau = V_\tau$ when $V_\cdot \in \mathcal{V}^+$.

For any two processes $K^1_\cdot$ and $K^2_\cdot$ in the space $\mathcal{V}^1$, we say $\mathrm{d}K^1 \perp \mathrm{d}K^2$ means that there exists an $(\mathcal{F}_t)$-progressively measurable set $D \subset \Omega \times [0,T]$ such that

$$\mathbb{E}\left[\int_0^T \mathbb{1}_D(t,\omega)\ \mathrm{d}K^1_t(\omega)\right] = \mathbb{E}\left[\int_0^T \mathbb{1}_{D^c}(t,\omega)\ \mathrm{d}K^2_t(\omega)\right] = 0.$$

And, we say $\mathrm{d}K^1 \le \mathrm{d}K^2$ means that for each $(\mathcal{F}_t)$-progressively measurable set $D \subset \Omega \times [0,T]$,

$$\mathbb{E}\left[\int_0^T \mathbb{1}_D(t,\omega)\ \mathrm{d}K^1_t(\omega)\right] \le \mathbb{E}\left[\int_0^T \mathbb{1}_D(t,\omega)\ \mathrm{d}K^2_t(\omega)\right],$$

i.e., $K^1_t - K^1_s \le K^2_t - K^2_s,\ 0 \le s \le t \le T$.

Finally, we recall that a process $(Y_t)_{t\in[0,T]}$ belongs to the class (D) if the family of variables $\{|Y_\tau| : \tau \text{ is an } (\mathcal{F}_t)\text{-stopping time bounded by } T\}$ is uniformly integrable.

In the rest of this paper, the variable ω in random elements is often omitted and all equalities and inequalities between random variables are understood to hold $\mathbb{P}-a.s.$ without a special illustration.

2.2 Definitions

In this paper, we always assume that $\xi \in \mathbb{L}^1(\mathcal{F}_T)$, $V_\cdot \in \mathcal{V}$, $L_\cdot \in \mathcal{S}$ (or $L_\cdot = -\infty$), $U_\cdot \in \mathcal{S}$ (or $U_\cdot = +\infty$), $L_\cdot \le U_\cdot$, and that a random function, which is usually called a generator,

$$g(\omega,t,y,z) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \longmapsto \mathbb{R}$$

is $(\mathcal{F}_t)$-progressively measurable for each (y,z), and continuous in (y,z) for almost each (ω,t).

We use the following definition for the L^1 solution of non-reflected BSDEs and reflected BSDEs with one and two continuous barriers.

Definition 2.1. By an L^1 solution to BSDE $(\xi, g + \mathrm{d}V)$ we understand a pair $(Y_t,Z_t)_{t\in[0,T]} \in \mathcal{S}^\beta \times \mathrm{M}^\beta$ for each $\beta \in (0,1)$ such that $(Y_t)_{t\in[0,T]}$ belongs to the class (D) and

$$Y_t = \xi + \int_t^T g(s,Y_s,Z_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s - \int_t^T Z_s \cdot \mathrm{d}B_s,\ \ t \in [0,T].$$

By an L^1 solution to $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ we understand a triple $(Y_t, Z_t, K_t)_{t\in[0,T]} \in \mathcal{S}^\beta \times \mathrm{M}^\beta \times \mathcal{V}^{+,1}$ for each $\beta \in (0,1)$ such that $(Y_t)_{t\in[0,T]}$ belongs to the class (D) and

$$\begin{cases} Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s + \int_t^T \mathrm{d}K_s - \int_t^T Z_s \cdot \mathrm{d}B_s, \quad t \in [0,T], \\ L_t \le Y_t, \quad t \in [0,T] \ \text{ and } \int_0^T (Y_t - L_t)\mathrm{d}K_t = 0. \end{cases}$$

By an L^1 solution to $\bar{R}$BSDE $(\xi, g + \mathrm{d}V, U)$ we understand a triple $(Y_t, Z_t, A_t)_{t\in[0,T]} \in \mathcal{S}^\beta \times \mathrm{M}^\beta \times \mathcal{V}^{+,1}$ for each $\beta \in (0,1)$ such that $(Y_t)_{t\in[0,T]}$ belongs to the class (D) and

$$\begin{cases} Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s - \int_t^T \mathrm{d}A_s - \int_t^T Z_s \cdot \mathrm{d}B_s, \quad t \in [0,T], \\ Y_t \le U_t, \quad t \in [0,T] \ \text{ and } \int_0^T (U_t - Y_t)\mathrm{d}A_t = 0. \end{cases}$$

By an L^1 solution to DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ we understand a quadruple $(Y_t, Z_t, K_t, A_t)_{t\in[0,T]} \in \mathcal{S}^\beta \times \mathrm{M}^\beta \times \mathcal{V}^{+,1} \times \mathcal{V}^{+,1}$ for each $\beta \in (0,1)$ such that both $Y_\cdot$ belongs to the class (D), and

$$\begin{cases} Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s + \int_t^T \mathrm{d}K_s - \int_t^T \mathrm{d}A_s - \int_t^T Z_s \cdot \mathrm{d}B_s, \quad t \in [0,T], \\ L_t \le Y_t \le U_t, \quad t \in [0,T], \quad \int_0^T (Y_t - L_t)\mathrm{d}K_t = \int_0^T (U_t - Y_t)\mathrm{d}A_t = 0 \ \text{ and } \ \mathrm{d}K \perp \mathrm{d}A. \end{cases}$$

Furthermore, an L^1 solution $(Y_t, Z_t)_{t\in[0,T]}$ of BSDE $(\xi, g + \mathrm{d}V)$ is called the minimal (resp. maximal) L^1 solution if for any L^1 solution $(Y'_t, Z'_t)_{t\in[0,T]}$ of BSDE $(\xi, g + \mathrm{d}V)$, we have

$$Y_t \le Y'_t, \quad t \in [0,T] \quad (\text{resp. } \ Y_t \ge Y'_t, \quad t \in [0,T]).$$

Similarly, we can define the minimal (resp. maximal) L^1 solution for $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$, $\bar{R}$BSDE $(\xi, g + \mathrm{d}V, U)$ and DRBSDE $(\xi, g + \mathrm{d}V, L, U)$.

2.3 Assumptions

In this paper, we will use the following assumptions with respect to the generator, the terminal condition and the barriers.

(H1) (i) g satisfies the one-sided Osgood condition in y, i.e., there exists a nondecreasing and concave function $\rho(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{\mathrm{d}u}{\rho(u)} = +\infty$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y_1, y_2 \in \mathbb{R}, z \in \mathbb{R}^d$,

$$(g(\omega, t, y_1, z) - g(\omega, t, y_2, z))\mathrm{sgn}(y_1 - y_2) \le \rho(|y_1 - y_2|);$$

(ii) $g(\cdot, 0, 0) \in \mathcal{H}^1$;

(iii) g has a general growth in y, i.e, $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall r > 0$,

$$\psi_\cdot(r) := \sup_{|y|\le r} |g(\cdot, y, 0) - g(\cdot, 0, 0)| \text{ belongs to the space } \mathcal{H}.$$

(H2) (i) g is uniformly continuous in z, i.e., there exists a nondecreasing and continuous function $\phi(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\phi(0) = 0$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d$,

$$|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \le \phi(|z_1 - z_2|);$$

(ii) g has a stronger sub-linear growth in z, i.e., there exist two constants $\gamma \geq 0$ and $\alpha \in (0,1)$ together with a nonnegative process $f_\cdot \in \mathcal{H}^1$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y \in \mathbb{R}, z \in \mathbb{R}^d$,

$$|g(\omega,t,y,z) - g(\omega,t,y,0)| \leq \gamma(f_t(\omega) + |y| + |z|)^\alpha.$$

(H2') (i) g is stronger continuous in (y,z), i.e., $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y \in \mathbb{R},\ g(\omega,t,y,\cdot)$ is continuous, and $g(\omega,t,\cdot,z)$ is continuous uniformly with respect to z;

(ii) g has a sub-linear growth in z, i.e., there exist three constants $\mu, \lambda \geq 0$ and $\alpha \in (0,1)$ together with a nonnegative process $f_\cdot \in \mathcal{H}^1$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y \in \mathbb{R}, z \in \mathbb{R}^d$,

$$|g(\omega,t,y,z) - g(\omega,t,y,0)| \leq f_t(\omega) + \mu|y| + \lambda|z|^\alpha.$$

(H3) (i) $L_\cdot \in \mathcal{S}$ (or $L_\cdot = -\infty$), $U_\cdot \in \mathcal{S}$ (or $U_\cdot = +\infty$), $L_\cdot \leq U_\cdot$, $\xi \in \mathbb{L}^1(\mathcal{F}_T)$ and $L_T \leq \xi \leq U_T$;

(ii) There exists two processes $(C_\cdot, H_\cdot) \in \mathcal{V}^1 \times \mathrm{M}^\beta$ for each $\beta \in (0,1)$ such that

$$X_t := X_0 + \int_0^t \mathrm{d}C_s + \int_0^t H_s \cdot \mathrm{d}B_s, \quad t \in [0,T]$$

belongs to the class (D), $g(\cdot, X_\cdot, 0) \in \mathcal{H}^1$ and $L_t \leq X_t \leq U_t$ for each $t \in [0,T]$.

(H3L) (i) $L_\cdot \in \mathcal{S}$ (or $L_\cdot = -\infty$), $\xi \in \mathbb{L}^1(\mathcal{F}_T)$ and $L_T \leq \xi$;

(ii) There exists two processes $(C_\cdot, H_\cdot) \in \mathcal{V}^1 \times \mathrm{M}^\beta$ for each $\beta \in (0,1)$ such that

$$X_t := X_0 + \int_0^t \mathrm{d}C_s + \int_0^t H_s \cdot \mathrm{d}B_s, \quad t \in [0,T]$$

belongs to the class (D), $g^-(\cdot, X_\cdot, 0) \in \mathcal{H}^1$ and $L_t \leq X_t$ for each $t \in [0,T]$.

(H3U) (i) $U_\cdot \in \mathcal{S}$ (or $U_\cdot = +\infty$), $\xi \in \mathbb{L}^1(\mathcal{F}_T)$ and $\xi \leq U_T$;

(ii) There exists two processes $(C_\cdot, H_\cdot) \in \mathcal{V}^1 \times \mathrm{M}^\beta$ for each $\beta \in (0,1)$ such that

$$X_t := X_0 + \int_0^t \mathrm{d}C_s + \int_0^t H_s \cdot \mathrm{d}B_s, \quad t \in [0,T]$$

belongs to the class (D), $g^+(\cdot, X_\cdot, 0) \in \mathcal{H}^1$ and $X_t \leq U_t$ for each $t \in [0,T]$.

(AA) g has a linear growth in y and a sub-linear growth in z, i.e., there exist three constants $\tilde{\mu}, \tilde{\lambda} \geq 0$ and $\tilde{\alpha} \in (0,1)$ together with a nonnegative process $\tilde{f}_\cdot \in \mathcal{H}^1$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y \in \mathbb{R}, z \in \mathbb{R}^d$,

$$|g(\omega,t,y,z)| \leq \tilde{f}_t(\omega) + \tilde{\mu}|y| + \tilde{\lambda}|z|^{\tilde{\alpha}}.$$

Remark 2.2. It is clear that the assumption (H1)(i) is strictly weaker than both the uniformly Lipschitz condition and the monotonicity condition in y of g employed in Pardoux and Peng [47], El Karoui et al. [10], Cvitanić and Karatzas [8], Briand et al. [3], Xu [57], Klimsiak [38], Rozkosz and Słomiński [55], Bayraktar and Yao [2] and so on, the assumptions (H1)(ii) and (iii) are strictly weaker than the usual linear-growth condition in y of g, and the assumption (H2)(i) is also strictly weaker than both the uniformly Lipschitz condition in z of g used in the references mentioned above and the α-Hölder continuity condition in z of g used in Fan and Liu [25]. In addition, note that the assumption (H2')(ii) is strictly weaker than the assumption (H2)(ii), and that the assumption (H2) will be used in the existence and uniqueness results on non-reflected BSDEs and reflected BSDEs with one and two barriers, and the assumptions (H2') and (AA) will be employed in the existence results. Note also that the assumptions (H3L), (H3U) and (H3) will be, respectively, used for $\underline{R}$BSDEs, $\bar{R}$BSDEs and DRBSDEs, which relate the growth of g and that of the barriers, and which are the so-called generalized Mokobodzki conditions.

Remark 2.3. Since the $\rho(\cdot)$ defined in (H1)(i) is a nondecreasing and concave function defined on $\mathbb{R}_+$ with $\rho(0)=0$, by Lemma 6.1 in Fan and Jiang [22] we know that $\rho(x)/x$ $(x>0)$ is non-increasing and then the function $\rho(\cdot)$ is of linear growth. At the same time, we can choose the continuous modular function of g with respect to the variable z as the $\phi(\cdot)$ defined in (H2)(i), which is also of linear growth. Thus, without loss of generality, we will always assume that there exists a constant $A>0$ such that

$$\forall\, x\in\mathbb{R}_+,\ \ \rho(x)\le A(x+1)\ \text{ and }\ \phi(x)\le A(x+1).$$

In addition, it follows from Proposition 1 in Fan [13] that the concavity condition of the function $\rho(\cdot)$ defined in the assumption (H1)(i) can be replaced with the continuity condition.

2.4 Lemmas

In this subsection, let us introduce several lemmas, which will play an important role later. Firstly, the following a priori estimate comes from Lemma 3.1 in Fan [16].

Lemma 2.4. Let the triple $(\bar{Y}_\cdot,\bar{Z}_\cdot,\bar{V}_\cdot)\in\mathcal{S}\times\mathrm{M}\times\mathcal{V}$ satisfy the following equation:

$$\bar{Y}_t=\bar{Y}_T+\int_t^T \mathrm{d}\bar{V}_s-\int_t^T \bar{Z}_s\cdot \mathrm{d}B_s,\ \ t\in[0,T]. \tag{2.1}$$

We have

(i) For each $p>0$, there exists a constant $C_1>0$ depending only on p such that for each $t\in[0,T]$ and each $(\mathcal{F}_t)$-stopping time τ valued in $[0,T]$,

$$\mathbb{E}\left[\left(\int_{t\wedge\tau}^{\tau}|\bar{Z}_s|^2\mathrm{d}s\right)^{\frac{p}{2}}\middle|\mathcal{F}_t\right]\le C_1\mathbb{E}\left[\sup_{s\in[t,T]}|\bar{Y}_{s\wedge\tau}|^p+\sup_{s\in[t,T]}\left[\left(\int_{s\wedge\tau}^{\tau}\bar{Y}_r\mathrm{d}\bar{V}_r\right)^+\right]^{\frac{p}{2}}\middle|\mathcal{F}_t\right];$$

(ii) If $\bar{Y}_\cdot\in\mathcal{S}^p$ for some $p>1$, then there exists a constant $C_2>0$ depending only on p such that for each $t\in[0,T]$ and each $(\mathcal{F}_t)$-stopping time τ valued in $[0,T]$,

$$\begin{aligned}&\mathbb{E}\left[\sup_{s\in[t,T]}|\bar{Y}_{s\wedge\tau}|^p+\int_{t\wedge\tau}^{\tau}|\bar{Y}_s|^{p-2}\mathbb{1}_{\{|\bar{Y}_s|\neq 0\}}|\bar{Z}_s|^2\mathrm{d}s\middle|\mathcal{F}_t\right]\\ \le\ & C_2\mathbb{E}\left[|\bar{Y}_\tau|^p+\sup_{s\in[t,T]}\left(\int_{s\wedge\tau}^{\tau}|\bar{Y}_r|^{p-1}\mathrm{sgn}(\bar{Y}_r)\mathrm{d}\bar{V}_r\right)^+\middle|\mathcal{F}_t\right].\end{aligned}$$

Secondly, the following observation will be used several times later.

Lemma 2.5. Let the generator g satisfy (H1)(i) and (H2')(ii) (resp. (H2)(ii)), and $(\underline{X}_\cdot,Y_\cdot,\bar{X}_\cdot,Z_\cdot)\in\mathcal{S}\times\mathcal{S}\times\mathcal{S}\times\mathrm{M}$ satisfy $\underline{X}_\cdot\le Y_\cdot\le\bar{X}_\cdot$. Then, $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$,

$$|g(\cdot,Y_\cdot,Z_\cdot)|\le|g(\cdot,\underline{X}_\cdot,0)|+|g(\cdot,\bar{X}_\cdot,0)|+(\mu+A)(|\underline{X}_\cdot|+|\bar{X}_\cdot|)+f_\cdot+A+\lambda|Z_\cdot|^\alpha \tag{2.2}$$

$$(\text{resp. } |g(\cdot,Y_\cdot,Z_\cdot)|\le|g(\cdot,\underline{X}_\cdot,0)|+|g(\cdot,\bar{X}_\cdot,0)|+(\gamma+A)(|\underline{X}_\cdot|+|\bar{X}_\cdot|)+\gamma(1+f_\cdot)+A+\gamma|Z_\cdot|^\alpha).$$

Proof. We only prove the case of (H2'). Another case is similar. Indeed, by (H1)(i) and (H2')(ii) together with $\underline{X}_\cdot\le Y_\cdot\le\bar{X}_\cdot$ and Remark 2.3 we know that $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$,

$$\begin{aligned}g(\cdot,Y_\cdot,Z_\cdot)&\le g(\cdot,Y_\cdot,Z_\cdot)-g(\cdot,\underline{X}_\cdot,Z_\cdot)+|g(\cdot,\underline{X}_\cdot,Z_\cdot)-g(\cdot,\underline{X}_\cdot,0)|+|g(\cdot,\underline{X}_\cdot,0)|\\&\le \rho(|Y_\cdot-\underline{X}_\cdot|)+f_\cdot+\mu|\underline{X}_\cdot|+\lambda|Z_\cdot|^\alpha+|g(\cdot,\underline{X}_\cdot,0)|\\&\le A(|\bar{X}_\cdot-\underline{X}_\cdot|)+A+f_\cdot+\mu|\underline{X}_\cdot|+\lambda|Z_\cdot|^\alpha+|g(\cdot,\underline{X}_\cdot,0)|,\end{aligned}$$

and

$$\begin{array}{rcl} -g(\cdot, Y_\cdot, Z_\cdot) & \leq & g(\cdot, \bar{X}_\cdot, Z_\cdot) - g(\cdot, Y_\cdot, Z_\cdot) + |g(\cdot, \bar{X}_\cdot, Z_\cdot) - g(\cdot, \bar{X}_\cdot, 0)| + |g(\cdot, \bar{X}_\cdot, 0)| \\ & \leq & \rho(|\bar{X}_\cdot - Y_\cdot|) + f_\cdot + \mu|\bar{X}_\cdot| + \lambda|Z_\cdot|^\alpha + |g(\cdot, \bar{X}_\cdot, 0)| \\ & \leq & A(|\bar{X}_\cdot - \underline{X}_\cdot|) + A + f_\cdot + \mu|\bar{X}_\cdot| + \lambda|Z_\cdot|^\alpha + |g(\cdot, \bar{X}_\cdot, 0)|. \end{array}$$

Then, the desired conclusion (2.2) follows immediately. □

Thirdly, the following lemma has a close connection with the generalized Mokobodzki condition, which will be shown in subsequent sections.

Lemma 2.6. Assume that $\xi \in \mathbb{L}^1(\mathcal{F}_T)$, $\bar{V} \in \mathcal{V}^1$, g is a generator and $(Y_\cdot, Z_\cdot)$ is an L^1 solution of BSDE $(\xi, g + \mathrm{d}\bar{V})$. If the generator g satisfies (H1)(i)(ii) and (H2')(ii) (resp. (H2)(ii)), then

$$g(\cdot, Y_\cdot, Z_\cdot) \in \mathcal{H}^1 \ \text{ and } \ g(\cdot, Y_\cdot, 0) \in \mathcal{H}^1. \tag{2.3}$$

Proof. In view of Remark 2.2 we only need to prove the case of (H2')(ii). Indeed, for each positive integer $k \geq 1$, define the following $(\mathcal{F}_t)$-stopping time:

$$\tau_k := \inf\{t \in [0,T] : \int_0^t |Z_s|^2 \mathrm{d}s \geq k\} \wedge T.$$

Note that $\tau_k \to T$ as $k \to +\infty$ due to the fact that $Z_\cdot \in \mathrm{M}$. By Itô-Tanaka's formula we deduce that

$$-\int_0^{\tau_k} \mathrm{sgn}(Y_s) g(s, Y_s, Z_s)\mathrm{d}s \leq |Y_{\tau_k}| - |Y_0| + \int_0^{\tau_k} \mathrm{sgn}(Y_s)\mathrm{d}\bar{V}_s - \int_0^{\tau_k} \mathrm{sgn}(Y_s) Z_s \cdot \mathrm{d}B_s.$$

Then,

$$\begin{array}{rl} & \displaystyle\int_0^{\tau_k} [\rho(|Y_s|) - \mathrm{sgn}(Y_s)(g(s, Y_s, Z_s) - g(s, 0, Z_s))]\mathrm{d}s \\ \leq & \displaystyle |Y_{\tau_k}| + |\bar{V}|_{\tau_k} + \int_0^{\tau_k} (\rho(|Y_s|) + |g(s, 0, Z_s)|)\mathrm{d}s - \int_0^{\tau_k} \mathrm{sgn}(Y_s) Z_s \cdot \mathrm{d}B_s. \end{array}$$

By taking mathematical expectation and letting $k \to \infty$ in the previous inequality, in view of (H1)(i), Levi's lemma and the fact that $Y_\cdot$ belongs to the class (D), we can obtain

$$\begin{array}{rl} & \mathbb{E}\left[\displaystyle\int_0^T |\rho(|Y_s|) - \mathrm{sgn}(Y_s)(g(s, Y_s, Z_s) - g(s, 0, Z_s))|\mathrm{d}s\right] \\ \leq & \mathbb{E}\left[|\xi| + |\bar{V}|_T + \displaystyle\int_0^T (\rho(|Y_s|) + |g(s, 0, Z_s)|)\mathrm{d}s\right]. \end{array} \tag{2.4}$$

Furthermore, noticing that

$$\begin{array}{rl} & \mathbb{E}\left[\displaystyle\int_0^T |g(s, Y_s, Z_s) - g(s, 0, Z_s)|\mathrm{d}s\right] = \mathbb{E}\left[\displaystyle\int_0^T |\mathrm{sgn}(Y_s)(g(s, Y_s, Z_s) - g(s, 0, Z_s))|\mathrm{d}s\right] \\ \leq & \mathbb{E}\left[\displaystyle\int_0^T [|\mathrm{sgn}(Y_s)(g(s, Y_s, Z_s) - g(s, 0, Z_s)) - \rho(|Y_s|)| + \rho(|Y_s|)]\,\mathrm{d}s\right], \end{array}$$

we get that, in view of (2.4), (H2')(ii), (H1)(ii) and Remark 2.3,

$$\begin{aligned}
& \mathbb{E}\left[\int_0^T |g(s,Y_s,Z_s)|\mathrm{d}s\right] \\
\le\ & \mathbb{E}\left[\int_0^T (|g(s,Y_s,Z_s)-g(s,0,Z_s)|+|g(s,0,Z_s)|)\,\mathrm{d}s\right] \\
\le\ & \mathbb{E}\left[|\xi|+|\bar V|_T+2\int_0^T (\rho(|Y_s|)+|g(s,0,Z_s)|)\mathrm{d}s\right] \\
\le\ & \mathbb{E}\left[|\xi|+|\bar V|_T+2\int_0^T (A|Y_s|+A+|g(s,0,0)|+f_s+\lambda|Z_s|^\alpha)\mathrm{d}s\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\int_0^T |g(s,Y_s,0)|\mathrm{d}s\right] &\le \mathbb{E}\left[\int_0^T (|g(s,Y_s,0)-g(s,Y_s,Z_s)|+|g(s,Y_s,Z_s)|)\,\mathrm{d}s\right] \\
&\le \mathbb{E}\left[\int_0^T |g(s,Y_s,Z_s)|\mathrm{d}s\right]+\mathbb{E}\left[\int_0^T (f_s+\mu|Y_s|+\lambda|Z_s|^\alpha)\mathrm{d}s\right].
\end{aligned}$$

Finally, in view of the conditions of Lemma 2.6 together with Hölder's inequality, we get (2.3). □

Finally, a similar argument as in Lemma 3.4 of Fan [16] yields the following two estimates.

Lemma 2.7. Let g be a generator and $(Y_\cdot,Z_\cdot,V_\cdot)\in\mathcal{S}\times\mathrm{M}\times\mathcal{V}$ satisfy the following equation:

$$Y_t=Y_T+\int_t^T g(s,Y_s,Z_s)\mathrm{d}s+\int_t^T \mathrm{d}V_s-\int_t^T Z_s\cdot\mathrm{d}B_s,\ \ t\in[0,T].$$

Assume that there exist two constants $\bar\mu,\bar\lambda>0$ and a nonnegative process $\bar f_\cdot\in\mathcal{H}$ such that

$$\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.,\ \ \mathrm{sgn}(Y_\cdot)g(\cdot,Y_\cdot,Z_\cdot)\le\bar f_\cdot+\bar\mu|Y_\cdot|+\bar\lambda|Z_\cdot|. \tag{2.5}$$

Then for each $p>0$, there exists a nonnegative constant $\bar C$ depending only on $p,\bar\mu,\bar\lambda,T$ such that for each $t\in[0,T]$ and each $(\mathcal{F}_t)$-stopping time τ valued in $[0,T]$, we have

$$\begin{aligned}
& \mathbb{E}\left[\left(\int_{t\wedge\tau}^\tau |Z_s|^2\mathrm{d}s\right)^{\frac{p}{2}}+\left(\int_{t\wedge\tau}^\tau |g(s,Y_s,Z_s)|\mathrm{d}s\right)^p\middle|\mathcal{F}_t\right] \\
\le\ & \bar C\mathbb{E}\left[\sup_{s\in[t,T]}|Y_{s\wedge\tau}|^p+|V|_\tau^p+\left(\int_{t\wedge\tau}^\tau \bar f_s\ \mathrm{d}s\right)^p\middle|\mathcal{F}_t\right].
\end{aligned}$$

Lemma 2.8. Let g be a generator and $(Y_\cdot,Z_\cdot,V_\cdot,K_\cdot)\in\mathcal{S}\times\mathrm{M}\times\mathcal{V}\times\mathcal{V}^+$ satisfy the following equation:

$$Y_t=Y_T+\int_t^T g(s,Y_s,Z_s)\mathrm{d}s+\int_t^T \mathrm{d}V_s+\int_t^T \mathrm{d}K_s-\int_t^T Z_s\cdot\mathrm{d}B_s,\ \ t\in[0,T]$$

or

$$Y_t=Y_T+\int_t^T g(s,Y_s,Z_s)\mathrm{d}s+\int_t^T \mathrm{d}V_s-\int_t^T \mathrm{d}K_s-\int_t^T Z_s\cdot\mathrm{d}B_s,\ \ t\in[0,T].$$

Assume that there exist two constants $\bar{\mu}, \bar{\lambda} > 0$ and a nonnegative process $\bar{f}_\cdot \in \mathcal{H}$ such that

$$\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e., \quad |g(\cdot, Y_\cdot, Z_\cdot)| \leq \bar{f}_\cdot + \bar{\mu}|Y_\cdot| + \bar{\lambda}|Z_\cdot|. \tag{2.6}$$

Then for each $p > 0$, there exists a nonnegative constant $\bar{C}$ depending only on $p, \bar{\mu}, \bar{\lambda}, T$ such that for each $t \in [0, T]$ and each $(\mathcal{F}_t)$-stopping time τ valued in $[0, T]$, we have

$$\begin{aligned} &\mathbb{E}\left[\left(\int_{t\wedge\tau}^{\tau} |Z_s|^2 \mathrm{d}s \right)^{\frac{p}{2}} + |K_\tau - K_{t\wedge\tau}|^p + \left(\int_{t\wedge\tau}^{\tau} |g(s, Y_s, Z_s)| \mathrm{d}s \right)^p \middle| \mathcal{F}_t \right] \\ \leq \quad & \bar{C}\mathbb{E}\left[\sup_{s\in[t,T]} |Y_{s\wedge\tau}|^p + |V|_\tau^p + \left(\int_{t\wedge\tau}^{\tau} \bar{f}_s \, \mathrm{d}s \right)^p \middle| \mathcal{F}_t \right]. \end{aligned}$$

3 Penalization, approximation and comparison theorem

3.1 Penalization for RBSDEs

In this subsection, we prove the following convergence result on the sequence of L^1 solutions of penalized RBSDEs with one continuous barrier.

Proposition 3.1 (Penalization for RBSDEs)**.** Assume that $V_\cdot \in \mathcal{V}^1$, (H3)(i) holds true for $L_\cdot, U_\cdot$ and ξ, and g is a generator. We have

(i) For each $n \geq 1$, let $(Y_\cdot^n, Z_\cdot^n, A_\cdot^n)$ be an L^1 solution of $\bar{R}$BSDE $(\xi, \bar{g}_n + \mathrm{d}V, U)$ with $\bar{g}_n(t, y, z) := g(t, y, z) + n(y - L_t)^-$, i.e.,

$$\begin{cases} Y_t^n = \xi + \displaystyle\int_t^T \bar{g}_n(s, Y_s^n, Z_s^n)\mathrm{d}s + \int_t^T \mathrm{d}V_s - \int_t^T \mathrm{d}A_s^n - \int_t^T Z_s^n \cdot \mathrm{d}B_s, \quad t \in [0, T], \\ Y_t^n \leq U_t, \ t \in [0, T] \ \text{ and } \displaystyle\int_0^T (U_t - Y_t^n)\mathrm{d}A_t^n = 0, \\ K_t^n := n\displaystyle\int_0^t (Y_s^n - L_s)^- \ \mathrm{d}s, \quad t \in [0, T]. \end{cases} \tag{3.1}$$

If for each $n \geq 1$, $Y_\cdot^n \leq Y_\cdot^{n+1} \leq \bar{Y}_\cdot$ with a process $\bar{Y}_\cdot \in \cap_{\beta\in(0,1)}\mathcal{S}^\beta$ of the class (D), $\mathrm{d}A^n \leq \mathrm{d}A^{n+1}$, $K_\cdot^n \leq \bar{K}_\cdot^n \in \mathcal{V}^{+,1}$ with $\sup_{n\geq1} \mathbb{E}[|\bar{K}_T^n|^\beta] < +\infty$ for each $\beta \in (0, 1)$, $\lim\limits_{j\to\infty} \bar{K}_T^{n_j} = \bar{K}_T \in \mathbb{L}^1(\mathcal{F}_T)$ for a subsequence $\{n_j\}$ of $\{n\}$ and $\sup_{n\geq1} \mathbb{E}[|\bar{K}_\tau^n|^2] \leq \mathbb{E}[|\tilde{Y}_\tau|^2]$ for a process $\tilde{Y}_\cdot \in \mathcal{S}$ and each $(\mathcal{F}_t)$-stopping time τ valued in $[0, T]$, and there exist two constants $\bar{\lambda} > 0$, $\alpha \in (0, 1)$ and a nonnegative process $\bar{f}_\cdot \in \mathcal{H}^1$ such that for each $n \geq 1$,

$$\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e., \quad |g(\cdot, Y_\cdot^n, Z_\cdot^n)| \leq \bar{f}_\cdot + \bar{\lambda}|Z_\cdot^n|^\alpha, \tag{3.2}$$

then there exists an L^1 solution $(Y_\cdot, Z_\cdot, K_\cdot, A_\cdot)$ of DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ such that

$$\lim_{n\to\infty} \left(\|Y_\cdot^n - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z_\cdot^n - Z_\cdot\|_{\mathrm{M}^\beta} + \|A_\cdot^n - A_\cdot\|_{\mathcal{S}^1} \right) = 0$$

holds true for each $\beta \in (0, 1)$, and there exists a subsequence $\{K_\cdot^{n_j}\}$ of $\{K_\cdot^n\}$ such that

$$\lim_{j\to\infty} \sup_{t\in[0,T]} |K_t^{n_j} - K_t| = 0.$$

(ii) For each $n \geq 1$, let $(Y_\cdot^n, Z_\cdot^n, K_\cdot^n)$ be an L^1 solution of $\underline{R}$BSDE $(\xi, \underline{g}_n + \mathrm{d}V, L)$ with

$\underline{g}_n(t,y,z) := g(t,y,z) - n(y-U_t)^+$, i.e.,

$$\begin{cases} Y_t^n = \xi + \int_t^T \underline{g}_n(s,Y_s^n,Z_s^n)\mathrm{d}s + \int_t^T \mathrm{d}V_s + \int_t^T \mathrm{d}K_t^n - \int_t^T Z_s^n \cdot \mathrm{d}B_s, \quad t\in[0,T], \\ L_t \le Y_t^n, \ t\in[0,T] \ \text{ and } \int_0^T (Y_t^n - L_t)\mathrm{d}K_t^n = 0, \\ A_t^n := n\int_0^t (Y_s^n - U_s)^+ \,\mathrm{d}s, \quad t\in[0,T]. \end{cases} \tag{3.3}$$

If for each $n\ge 1$, $Y_\cdot^n \ge Y_\cdot^{n+1} \ge \underline{Y}_\cdot$ with a process $\underline{Y}_\cdot \in \cap_{\beta\in(0,1)}\mathcal{S}^\beta$ of the class (D), $\mathrm{d}K^n \le \mathrm{d}K^{n+1}$, $A_\cdot^n \le \bar{A}_\cdot^n \in \mathcal{V}^{+,1}$ with $\sup_{n\ge1}\mathbb{E}[|\bar{A}_T^n|^\beta] < +\infty$ for each $\beta\in(0,1)$, $\lim_{j\to\infty} \bar{A}_T^{n_j} = \bar{A}_T \in \mathbb{L}^1(\mathcal{F}_T)$ for a subsequence $\{n_j\}$ of $\{n\}$ and $\sup_{n\ge1}\mathbb{E}[|\bar{A}_\tau^n|^2] \le \mathbb{E}[|\tilde{Y}_\tau|^2]$ for a process $\tilde{Y}_\cdot \in \mathcal{S}$ and each $(\mathcal{F}_t)$-stopping time τ valued in $[0,T]$, and there exist two constants $\bar\lambda>0$, $\alpha\in(0,1)$ and a nonnegative process $\bar{f}_\cdot \in \mathcal{H}^1$ such that (3.2) holds for each $n\ge1$, then there exists an L^1 solution $(Y_\cdot, Z_\cdot, K_\cdot, A_\cdot)$ of DRBSDE $(\xi, g+\mathrm{d}V, L, U)$ such that

$$\lim_{n\to\infty} \left(\|Y_\cdot^n - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z_\cdot^n - Z_\cdot\|_{\mathrm{M}^\beta} + \|K_\cdot^n - K_\cdot\|_{\mathcal{S}^1}\right) = 0$$

holds true for each $\beta\in(0,1)$, and there exists a subsequence $\{A_\cdot^{n_j}\}$ of $\{A_\cdot^n\}$ such that

$$\lim_{j\to\infty} \sup_{t\in[0,T]} |A_t^{n_j} - A_t| = 0.$$

Proof. We only prove the claim (i). The claim (ii) can be proved in the same way. Now we assume that all the assumptions in (i) are satisfied. Since $Y_\cdot^n$ increases in n, there exists an $(\mathcal{F}_t)$-progressively measurable process $Y_\cdot$ such that $Y_t^n \uparrow Y_t$ for each $t\in[0,T]$. In view of (3.1) and (3.2) with the fact that for each $n\ge1$, $Y_\cdot^1 \le Y_\cdot^n \le \bar{Y}_\cdot$ and $K_T^n \le \bar{K}_T^n$ with $\sup_{n\ge1}\mathbb{E}[|\bar{K}_T^n|^\beta] < +\infty$ for each $\beta\in(0,1)$, by Lemma 2.8 we deduce that for each $\beta\in(0,1)$, there exists a $C_\beta>0$ depending only on $\beta, \bar\lambda, T$ such that

$$\begin{aligned} &\sup_{n\ge1}\mathbb{E}\left[\left(\int_0^T |Z_s^n|^2\mathrm{d}s\right)^{\frac{\beta}{2}} + |A_T^n|^\beta + \left(\int_0^T |g(s,Y_s^n,Z_s^n)|\mathrm{d}s\right)^\beta\right] \\ \le\ & C_\beta\left\{\mathbb{E}\left[\sup_{s\in[0,T]}(|Y_s^1| + |\bar{Y}_s|)^\beta + |V|_T^\beta + \left(\int_0^T \bar{f}_s\,\mathrm{d}s\right)^\beta\right] + \sup_{n\ge1}\mathbb{E}\left[|K_T^n|^\beta\right]\right\} < +\infty. \end{aligned} \tag{3.4}$$

For each positive integer $k\ge1$, we define the following $(\mathcal{F}_t)$-stopping time:

$$\tau_k := \inf\left\{t\ge0:\ |Y_t^1| + |\bar{Y}_t| + |V|_t + \int_0^t \bar{f}_s\,\mathrm{d}s + |\tilde{Y}_t| + L_t^+ \ge k\right\}\wedge T.$$

Then

$$\mathbb{P}\left(\{\omega:\ \exists k_0(\omega)\ge1,\ \forall k\ge k_0(\omega),\ \tau_k(\omega)=T\}\right) = 1. \tag{3.5}$$

Note the fact that $\sup_{n\ge1}\mathbb{E}[|K_{\tau_k}^n|^2] \le \mathbb{E}[|\tilde{Y}_{\tau_k}|^2] \le k^2$ for each $k\ge1$. Again by Lemma 2.8 we deduce that there exists a nonnegative constant $\bar{C}$ depending only on $\bar\lambda, T$ such that for each $k\ge1$,

$$\begin{aligned} &\sup_{n\ge1}\mathbb{E}\left[\int_0^{\tau_k} |Z_s^n|^2\mathrm{d}s + |A_{\tau_k}^n|^2 + \left(\int_0^{\tau_k} |g(s,Y_s^n,Z_s^n)|\mathrm{d}s\right)^2\right] \\ \le\ & \bar{C}\mathbb{E}\left[\sup_{s\in[0,T]}(|Y_{s\wedge\tau_k}^1| + |\bar{Y}_{s\wedge\tau_k}|)^2 + |V|_{\tau_k}^2 + \left(\int_0^{\tau_k}\bar{f}_s\,\mathrm{d}s\right)^2 + |\tilde{Y}_{\tau_k}|^2\right] \le 4\bar{C}k^2. \end{aligned} \tag{3.6}$$

Furthermore, since $\mathrm{d}A^n \leq \mathrm{d}A^{n+1}$, there exists an $(\mathcal{F}_t)$-progressively measurable and increasing process $(A_t)_{t\in[0,T]}$ with $A_0 = 0$ such that $A_t^n \uparrow A_t$ for each $t \in [0,T]$, and for each $j \geq n \geq 1$,

$$0 \leq A_t^j - A_t^n \leq A_T^j - A_T^n, \quad t \in [0,T].$$

Letting first $j \to \infty$, and then taking supremum with respect to t in $[0,T]$, finally letting $n \to \infty$ in the previous inequality yields that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |A_t^n - A_t| = 0, \tag{3.7}$$

which means that $A_\cdot \in \mathcal{V}^+$. On the other hand, note by (3.6) that $\sup_{n\geq 1} \mathbb{E}[|A_{\tau_k}^n|^2] < +\infty$ for each $k \geq 1$. It follows that for each $(\mathcal{F}_t)$-stopping time τ valued in $[0,T]$ and each $k \geq 1$,

$$\lim_{n\to\infty} \mathbb{E}[|A_{\tau\wedge\tau_k}^n - A_{\tau\wedge\tau_k}|] = 0. \tag{3.8}$$

The rest proof is divided into 7 steps, which will be detailed in Appendix.

Step 1. In view of (3.2), (3.5), (3.6) and (3.8), by using a weak convergence argument, Lemma 4.4 of Klimsiak [38] and Lemma A.3 in Bayraktar and Yao [2], we show that $Y_\cdot$ is a càdlàg process.

Step 2. By virtue of the conclusion of the step 1 together with the definition of $K_\cdot^n$ and Dini's theorem, we show that $Y_t \geq L_t$ for each $t \in [0,T]$ and $\lim_{n\to\infty} \sup_{t\in[0,T]} (Y_t^n - L_t)^- = 0$.

Step 3. Making use of (ii) of Lemma 2.4, the definition of $K_\cdot^n$ and $A_\cdot^n$ with (3.1), Hölder's inequality, (3.2), (3.6), the conclusion of the step 2 and Lebesgue's dominated convergence theorem, we show that as $n \to \infty$, the sequence $\{Y_\cdot^n\}$ converges to the process $Y_\cdot$ in the space of $\mathcal{S}^\beta$ for each $\beta \in (0,1)$.

Step 4. Making use of (i) of Lemma 2.4, (3.4) and the conclusion of the step 3, we show that as $n \to \infty$, the sequence $\{Z_\cdot^n\}$ converges to a process $Z_\cdot$ in the space of M^β for each $\beta \in (0,1)$.

Step 5. By virtue of the continuity of g and the conclusions of the steps 3 and 4 together with (3.7), we show that there exists a subsequence $\{K_\cdot^{n_j}\}$ of the sequence $\{K_\cdot^n\}$ which converges uniformly in t to a process $K_\cdot \in \mathcal{V}^{+,1}$ in the sense of almost surely as $j \to \infty$.

Step 6. Utilizing (3.2), (3.7) and Lebesgue's dominated convergence theorem, we show that the sequence $\{A_\cdot^n\}$ converges the process $A_\cdot$ in the space of $\mathcal{S}^1$.

Step 7. Based on all the conclusions of the steps 1–6, we finally show that the $(Y_\cdot, Z_\cdot, K_\cdot, A_\cdot)$ is an L^1 solution of RBSDE $(\xi, g{+}\mathrm{d}V, L, U)$. The proof is then completed. □

3.2 Penalization for BSDEs

In this subsection, we prove the following convergence result on the sequence of L^1 solutions of penalized non-reflected BSDEs.

Proposition 3.2 (Penalization for BSDEs)**.** Assume that $V_\cdot \in \mathcal{V}^1$, (H3)(i) holds true for $L_\cdot, U_\cdot$ and ξ, and g is a generator. We have

(i) Let $(Y_\cdot^n, Z_\cdot^n)$ be an L^1 solution of BSDE $(\xi, \bar{g}_n + \mathrm{d}V)$ with $\bar{g}_n(t,y,z) := g(t,y,z) + n(y - L_t)^-$ for each $n \geq 1$, i.e.,

$$\begin{cases} Y_t^n = \xi + \int_t^T \bar{g}_n(s, Y_s^n, Z_s^n)\mathrm{d}s + \int_t^T \mathrm{d}V_s - \int_t^T Z_s^n \cdot \mathrm{d}B_s, \quad t \in [0,T], \\ K_t^n := n\int_0^t (Y_s^n - L_s)^- \,\mathrm{d}s, \quad t \in [0,T]. \end{cases} \tag{3.9}$$

If for each $n \geq 1$, $Y^n_\cdot \leq Y^{n+1}_\cdot \leq \bar{Y}_\cdot$ with a process $\bar{Y}_\cdot \in \cap_{\beta\in(0,1)}\mathcal{S}^\beta$ of the class (D), and there exist two constants $\bar{\lambda} > 0$, $\alpha \in (0,1)$ and a nonnegative process $\bar{f}_\cdot \in \mathcal{H}^1$ such that (3.2) holds true for each $n \geq 1$, then $\sup_{n\geq 1}\mathbb{E}[|K^n_T|^\beta] < +\infty$ for each $\beta \in (0,1)$, $\sup_{n\geq 1}\mathbb{E}[|K^n_\tau|^2] \leq \mathbb{E}[|\tilde{Y}_\tau|^2]$ for a process $\tilde{Y}_\cdot \in \mathcal{S}$ and each $(\mathcal{F}_t)$-stopping time τ valued in $[0,T]$, there exists an L^1 solution $(Y_\cdot, Z_\cdot, K_\cdot)$ of $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ such that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty}(\|Y^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta}) = 0$$

and there exists a subsequence $\{K^{n_j}_\cdot\}$ of $\{K^n_\cdot\}$ such that

$$\lim_{j\to\infty}\sup_{t\in[0,T]}|K^{n_j}_t - K_t| = 0.$$

(ii) Let $(Y^n_\cdot, Z^n_\cdot)$ be an L^1 solution of BSDE $(\xi, \underline{g}_n + \mathrm{d}V)$ with $\underline{g}_n(t,y,z) := g(t,y,z) - n(y - U_t)^+$ for each $n \geq 1$, i.e.,

$$\begin{cases} Y^n_t = \xi + \int_t^T \underline{g}_n(s, Y^n_s, Z^n_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s - \int_t^T Z^n_s \cdot \mathrm{d}B_s, \quad t \in [0,T], \\ A^n_t := n\int_0^t (Y^n_s - U_s)^+\,\mathrm{d}s, \quad t \in [0,T]. \end{cases} \tag{3.10}$$

If for each $n \geq 1$, $Y^n_\cdot \geq Y^{n+1}_\cdot \geq \underline{Y}_\cdot$ with a process $\underline{Y}_\cdot \in \cap_{\beta\in(0,1)}\mathcal{S}^\beta$ of the class (D), and there exist two constants $\bar{\lambda} > 0$, $\alpha \in (0,1)$ and a nonnegative process $\bar{f}_\cdot \in \mathcal{H}^1$ such that (3.2) holds true for each $n \geq 1$, then $\sup_{n\geq 1}\mathbb{E}[|A^n_T|^\beta] < +\infty$ for each $\beta \in (0,1)$, $\sup_{n\geq 1}\mathbb{E}[|A^n_\tau|^2] \leq \mathbb{E}[|\tilde{Y}_\tau|^2]$ for a process $\tilde{Y}_\cdot \in \mathcal{S}$ and each $(\mathcal{F}_t)$-stopping time τ valued in $[0,T]$, there exists an L^1 solution $(Y_\cdot, Z_\cdot, A_\cdot)$ of $\bar{R}$BSDE $(\xi, g + \mathrm{d}V, U)$ such that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty}(\|Y^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta}) = 0$$

and there exists a subsequence $\{A^{n_j}_\cdot\}$ of $\{A^n_\cdot\}$ such that

$$\lim_{j\to\infty}\sup_{t\in[0,T]}|A^{n_j}_t - A_t| = 0.$$

Proof. We only prove (i), the proof of (ii) is similar. Note first that for each $n \geq 1$, $Y^n_\cdot \leq Y^{n+1}_\cdot \leq \bar{Y}_\cdot \in \cap_{\beta\in(0,1)}\mathcal{S}^\beta$. In view of (3.2), by Lemma 2.8 we can deduce that for each $\beta \in (0,1)$, there exists a nonnegative constant C_β depending only on $\beta, \bar{\lambda}, T$ such that

$$\begin{aligned} &\sup_{n\geq 1}\mathbb{E}\left[\left(\int_0^T |Z^n_s|^2\mathrm{d}s\right)^{\frac{\beta}{2}} + |K^n_T|^\beta + \left(\int_0^T |g(s, Y^n_s, Z^n_s)|\mathrm{d}s\right)^\beta\right] \\ \leq\ & C_\beta\mathbb{E}\left[\sup_{s\in[0,T]}(|Y^1_s| + |\bar{Y}_s|)^\beta + |V|^\beta_T + \left(\int_0^T \bar{f}_s\,\mathrm{d}s\right)^\beta\right] < +\infty, \end{aligned} \tag{3.11}$$

and there also exists a nonnegative constant $\bar{C}$ depending only on $\bar{\lambda}, T$ such that for each $(\mathcal{F}_t)$-stopping time τ valued in $[0,T]$, we have

$$\sup_{n\geq 1}\mathbb{E}\left[\int_0^\tau |Z^n_s|^2\mathrm{d}s + |K^n_\tau|^2\right] \leq \bar{C}\mathbb{E}\left[\sup_{s\in[0,T]}(|Y^1_{s\wedge\tau}| + |\bar{Y}_{s\wedge\tau}|)^2 + |V|^2_\tau + \left(\int_0^\tau \bar{f}_s\,\mathrm{d}s\right)^2\right]. \tag{3.12}$$

For each positive integer $k \geq 1$, define the following $(\mathcal{F}_t)$-stopping time:

$$\tau_k := \inf\left\{t \geq 0 : \ |Y_t^1| + |\bar{Y}_t| + |V|_t + \int_0^t \bar{f}_s \ \mathrm{d}s + L_t^+ \geq k\right\} \wedge T.$$

Then

$$\mathbb{P}\left(\{\omega : \ \exists k_0(\omega) \geq 1, \ \forall k \geq k_0(\omega), \ \tau_k(\omega) = T\}\right) = 1.$$

Thus, by letting $A^n_\cdot \equiv 0$ and $U_\cdot \equiv +\infty$, a same argument as in the proof of the steps 1–5 of Proposition 3.1 yields that there exists a triple $(Y_\cdot, Z_\cdot, K_\cdot) \in \mathcal{S}^\beta \times \mathrm{M}^\beta \times \mathcal{V}^+$ for each $\beta \in (0,1)$ satisfying

$$K_t = Y_0 - Y_t - \int_0^t g(s, Y_s, Z_s)\mathrm{d}s - \int_0^t \mathrm{d}V_s + \int_0^t Z_s \cdot \mathrm{d}B_s.$$

Furthermore, for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} \left(\|Y^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta}\right) = 0$$

and there exists a subsequence $\{K^{n_j}_\cdot\}$ of $\{K^n_\cdot\}$ such that

$$\lim_{j\to\infty} \sup_{t\in[0,T]} |K^{n_j}_t - K_t| = 0.$$

In the sequel, a similar proof to the step 6 of Proposition 3.1 yields that

$$\mathbb{E}[K_T] \leq |Y_0| + \mathbb{E}[|\xi|] + \mathbb{E}[|V|_T] + \|\bar{f}_\cdot\|_{\mathcal{H}^1} + \bar{\lambda} T^{\frac{2-\alpha}{2}} \|Z_\cdot\|_{\mathrm{M}^\alpha} < +\infty,$$

which means that $K_\cdot \in \mathcal{V}^{+,1}$. Finally, similar to the step 7 of Proposition 3.1, it is easy to prove that $(Y_\cdot, Z_\cdot, K_\cdot)$ is an L^1 solution of $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$. The proof is complete. $\square$

3.3 Approximation

In this subsection, we prove the following general approximation result for L^1 solutions of DRBSDEs and both RBSDEs and non-reflected BSDEs as its special cases.

Proposition 3.3 (Approximation)**.** Assume that $V_\cdot \in \mathcal{V}^1$, (H3)(i) holds true for $L_\cdot$, $U_\cdot$ and ξ, g_n is a generator and $(Y^n_\cdot, Z^n_\cdot, K^n_\cdot, A^n_\cdot)$ is an L^1 solution of DRBSDE $(\xi, g_n + \mathrm{d}V, L, U)$ for each $n \geq 1$. If for each $n \geq 1$, $Y^n_\cdot \leq Y^{n+1}_\cdot \leq \bar{Y}_\cdot$, $\mathrm{d}A^n \leq \mathrm{d}A^{n+1} \leq \mathrm{d}\bar{A}$ and $\mathrm{d}K^{n+1} \leq \mathrm{d}K^n \leq \mathrm{d}K^1$ with $\bar{Y}_\cdot \in \cap_{\beta\in(0,1)}\mathcal{S}^\beta$ of the class (D) and $\bar{A} \in \mathcal{V}^{+,1}$ (resp. $\underline{Y}_\cdot \leq Y^{n+1}_\cdot \leq Y^n_\cdot$, $\mathrm{d}A^{n+1} \leq \mathrm{d}A^n \leq \mathrm{d}A^1$ and $\mathrm{d}K^n \leq \mathrm{d}K^{n+1} \leq \mathrm{d}\bar{K}$ with $\underline{Y}_\cdot \in \cap_{\beta\in(0,1)}\mathcal{S}^\beta$ of the class (D) and $\bar{K} \in \mathcal{V}^{+,1}$), g_n tends locally uniformly in (y,z) to a generator g as $n \to \infty$, there exists a constant $\bar{\lambda} > 0$ and a nonnegative process $\tilde{f}_\cdot \in \mathcal{H}^1$ such that for each $n \geq 1$,

$$\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e., \ \ \mathrm{sgn}(Y^n_\cdot) g_n(\cdot, Y^n_\cdot, Z^n_\cdot) \leq \tilde{f}_\cdot + \bar{\lambda}|Z^n_\cdot|, \tag{3.13}$$

and for each $k \geq 1$, there exists a nonnegative process $\bar{f}^k_\cdot \in \mathcal{H}$ such that for each $n \geq 1$,

$$\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e., \ \ |g_n(\cdot, Y^n_\cdot, Z^n_\cdot)\mathbb{1}_{|Y^n_\cdot| \leq k}| \leq \bar{f}^k_\cdot + \bar{\lambda}|Z^n_\cdot|, \tag{3.14}$$

then there exists an L^1 solution $(Y_\cdot, Z_\cdot, K_\cdot, A_\cdot)$ of DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ such that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} \left(\|Y^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta} + \|K^n_\cdot - K_\cdot\|_{\mathcal{S}^1} + \|A^n_\cdot - A_\cdot\|_{\mathcal{S}^1}\right) = 0.$$

Proof. We only prove the case that for each $n \geq 1$, $Y^n_\cdot \leq Y^{n+1}_\cdot \leq \bar{Y}_\cdot$, $\mathrm{d}A^n \leq \mathrm{d}A^{n+1} \leq \mathrm{d}\bar{A}$ and $\mathrm{d}K^{n+1} \leq \mathrm{d}K^n \leq \mathrm{d}K^1$ with $\bar{Y}_\cdot \in \cap_{\beta\in(0,1)}\mathcal{S}^\beta$ of the class (D) and $\bar{A} \in \mathcal{V}^{+,1}$. Another case can be proved in the same way. Firstly, a same argument as that in proving (3.7) together with Lebesgue's dominated convergence theorem yields that there exists an $(\mathcal{F}_t)$-progressively measurable process $(Y_t)_{t\in[0,T]}$ together with $K_\cdot, A_\cdot \in \mathcal{V}^{+,1}$ such that $Y_t^n \uparrow Y_t$ for each $t \in [0,T]$, and

$$\lim_{n\to\infty} (\|K_\cdot^n - K_\cdot\|_{\mathcal{S}^1} + \|A_\cdot^n - A_\cdot\|_{\mathcal{S}^1}) = 0. \tag{3.15}$$

Furthermore, in view of (3.13), by Lemma 2.7 we can deduce that for each $\beta \in (0,1)$, there exists a nonnegative constant C_β depending only on $\beta, \bar{\lambda}, T$ such that

$$\begin{aligned} &\sup_{n\ge 1} \mathbb{E}\left[\left(\int_0^T |Z_s^n|^2 \mathrm{d}s\right)^{\frac{\beta}{2}} + \left(\int_0^T |g_n(s, Y_s^n, Z_s^n)| \mathrm{d}s\right)^{\beta}\right] \\ \le\ & C_\beta \mathbb{E}\left[\sup_{s\in[0,T]} (|Y_s^1| + |\bar{Y}_s|)^\beta + |V|_T^\beta + |K_T^1|^\beta + |\bar{A}_T|^\beta + \left(\int_0^T \tilde{f}_s \, \mathrm{d}s\right)^\beta\right] < +\infty, \end{aligned} \tag{3.16}$$

and there also exists a constant $\bar{C} > 0$ depending only on $\bar{\lambda}, T$ such that for each $(\mathcal{F}_t)$-stopping time τ valued in $[0,T]$, we have

$$\begin{aligned} &\sup_{n\ge 1} \mathbb{E}\left[\int_0^\tau |Z_s^n|^2 \mathrm{d}s\right] \\ \le\ & \bar{C}\mathbb{E}\left[\sup_{s\in[0,T]} (|Y_{s\wedge\tau}^1| + |\bar{Y}_{s\wedge\tau}|)^2 + |V|_\tau^2 + |K_\tau^1|^2 + |\bar{A}_\tau|^2 + \left(\int_0^\tau \tilde{f}_s \, \mathrm{d}s\right)^2\right]. \end{aligned} \tag{3.17}$$

The rest proof is divided into 3 steps, which will be detailed in Appendix.

Step 1. In view of (3.14) and (3.17), making use of the technique of stopping times and (ii) of Lemma 2.4 together with Hölder's inequality and Lebesgue's dominated convergence theorem, we show that as $n \to \infty$, the sequence $\{Y_\cdot^n\}$ converges to the process $Y_\cdot$ in the space of $\mathcal{S}^\beta$ for each $\beta \in (0,1)$.

Step 2. Making use of (i) of Lemma 2.4, (3.16) and the conclusion of the step 1, we show that as $n \to \infty$, the sequence $\{Z_\cdot^n\}$ converges to a process $Z_\cdot$ in the space of M^β for each $\beta \in (0,1)$.

Step 3. By virtue of (3.14), (3.15) and the conclusions of the previous two steps, we show that the $(Y_\cdot, Z_\cdot, K_\cdot, A_\cdot)$ is an L^1 solution of DRBSDE $(\xi, g + \mathrm{d}V, L, U)$. Proposition 3.3 is then proved. ☐

Remark 3.4. Observe that if there exists a constant $\bar{\lambda} > 0$ and a nonnegative process $\bar{f}_\cdot \in \mathcal{H}^1$ such that for each $n \ge 1$,

$$\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e., \quad |g_n(\cdot, Y_\cdot^n, Z_\cdot^n)| \le \bar{f}_\cdot + \bar{\lambda}|Z_\cdot^n|, \tag{3.18}$$

then both (3.13) and (3.14) are satisfied.

3.4 Comparison theorem

We now establish a general comparison theorem for L^1 solutions of RBSDEs with one and two continuous barriers as well as non-reflected BSDEs.

Proposition 3.5 (Comparison Theorem)**.** Assume that $V_\cdot^j \in \mathcal{V}^1$, (H3)(i) holds for $L_\cdot^j, U_\cdot^j$ and ξ^j, g^j is a generator and $(Y_\cdot^j, Z_\cdot^j, K_\cdot^j, A_\cdot^j)$ is an L^1 solution of DRBSDE $(\xi^j, g^j + \mathrm{d}V^j, L^j, U^j)$ for $j = 1, 2$. If $\xi^1 \le \xi^2$, $\mathrm{d}V^1 \le \mathrm{d}V^2$, $L_\cdot^1 \le L_\cdot^2$, $U_\cdot^1 \le U_\cdot^2$, and either

$$\begin{cases} g^1 \text{ satisfies } (H1)(i) \text{ and } (H2); \\ \mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e., \quad \mathbb{1}_{\{Y_t^1 > Y_t^2\}} \left(g^1(t, Y_t^2, Z_t^2) - g^2(t, Y_t^2, Z_t^2)\right) \le 0 \end{cases}$$

or

$$\left\{ \begin{array}{l} g^2 \textit{ satisfies } (H1)(i) \textit{ and } (H2); \\ \mathrm{d}\mathbb{P}\times \mathrm{d}t - a.e., \quad \mathbb{1}_{\{Y_t^1 > Y_t^2\}} \left(g^1(t, Y_t^1, Z_t^1) - g^2(t, Y_t^1, Z_t^1) \right) \le 0 \end{array} \right.$$

is satisfied, then $Y_t^1 \le Y_t^2$ for each $t \in [0, T]$.

Proof. For each positive integer $k \ge 1$, define the following $(\mathcal{F}_t)$-stopping time:

$$\tau_k := \inf\{t \in [0,T] : \int_0^t (|Z_s^1|^2 + |Z_s^2|^2)\mathrm{d}s \ge k\} \wedge T.$$

It follows from Itô-Tanaka's formula that for each $t \in [0, T]$ and $k \ge 1$,

$$\begin{aligned} & (Y^1_{t\wedge\tau_k} - Y^2_{t\wedge\tau_k})^+ \\ \le\ & (Y^1_{\tau_k} - Y^2_{\tau_k})^+ + \int_{t\wedge\tau_k}^{\tau_k} \operatorname{sgn}((Y_s^1 - Y_s^2)^+)(\mathrm{d}V_s^1 - \mathrm{d}V_s^2) \\ & + \int_{t\wedge\tau_k}^{\tau_k} \operatorname{sgn}((Y_s^1 - Y_s^2)^+) \left(g^1(s, Y_s^1, Z_s^1) - g^2(s, Y_s^2, Z_s^2) \right) \mathrm{d}s \\ & + \int_{t\wedge\tau_k}^{\tau_k} \operatorname{sgn}((Y_s^1 - Y_s^2)^+) \left(\mathrm{d}K_s^1 - \mathrm{d}K_s^2 \right) + \int_{t\wedge\tau_k}^{\tau_k} \operatorname{sgn}((Y_s^1 - Y_s^2)^+) \left(\mathrm{d}A_s^2 - \mathrm{d}A_s^1 \right) \\ & + \int_{t\wedge\tau_k}^{\tau_k} \operatorname{sgn}((Y_s^1 - Y_s^2)^+)(Z_s^1 - Z_s^2)\cdot \mathrm{d}B_s. \end{aligned}$$

Since $L_t^1 \le L_t^2 \le Y_t^2$, $L_t^1 \le Y_t^1$, $t \in [0, T]$ and $\int_0^T (Y_s^1 - L_s^1)\mathrm{d}K_s^1 = 0$, we have

$$\begin{aligned} & \int_{t\wedge\tau_k}^{\tau_k} \operatorname{sgn}((Y_s^1 - Y_s^2)^+) \left(\mathrm{d}K_s^1 - \mathrm{d}K_s^2 \right) \\ \le\ & \int_{t\wedge\tau_k}^{\tau_k} \operatorname{sgn}((Y_s^1 - Y_s^2)^+)\mathrm{d}K_s^1 \le \int_{t\wedge\tau_k}^{\tau_k} \operatorname{sgn}((Y_s^1 - L_s^1)^+)\mathrm{d}K_s^1 \\ =\ & \int_{t\wedge\tau_k}^{\tau_k} \mathbb{1}_{\{Y_s^1 > L_s^1\}} |Y_s^1 - L_s^1|^{-1}(Y_s^1 - L_s^1)\mathrm{d}K_s^1 = 0. \end{aligned}$$

Similarly, since $Y_t^1 \le U_t^1 \le U_t^2$, $Y_t^2 \le U_t^2$, $t \in [0, T]$ and $\int_0^T (U_s^2 - Y_s^2)\mathrm{d}A_s^2 = 0$, we have

$$\begin{aligned} & \int_{t\wedge\tau_k}^{\tau_k} \operatorname{sgn}((Y_s^1 - Y_s^2)^+) \left(\mathrm{d}A_s^2 - \mathrm{d}A_s^1 \right) \\ \le\ & \int_{t\wedge\tau_k}^{\tau_k} \operatorname{sgn}((Y_s^1 - Y_s^2)^+)\mathrm{d}A_s^2 \le \int_{t\wedge\tau_k}^{\tau_k} \operatorname{sgn}((U_s^2 - Y_s^2)^+)\mathrm{d}A_s^2 \\ =\ & \int_{t\wedge\tau_k}^{\tau_k} \mathbb{1}_{\{U_s^2 > Y_s^2\}} |U_s^2 - Y_s^2|^{-1}(U_s^2 - Y_s^2)\mathrm{d}A_s^2 = 0. \end{aligned}$$

Thus, noticing that $\mathrm{d}V^1 \le \mathrm{d}V^2$, by virtue of the previous three inequalities we get that

$$\begin{aligned} & (Y^1_{t\wedge\tau_k} - Y^2_{t\wedge\tau_k})^+ \\ \le\ & (Y^1_{\tau_k} - Y^2_{\tau_k})^+ + \int_{t\wedge\tau_k}^{\tau_k} \operatorname{sgn}((Y_s^1 - Y_s^2)^+) \left(g^1(s, Y_s^1, Z_s^1) - g^2(s, Y_s^2, Z_s^2) \right) \mathrm{d}s \\ & + \int_{t\wedge\tau_k}^{\tau_k} \operatorname{sgn}((Y_s^1 - Y_s^2)^+)(Z_s^1 - Z_s^2)\cdot \mathrm{d}B_s, \quad t \in [0, T]. \end{aligned}$$

Finally, in view of the assumptions of g^1 and g^2 together with $\xi^1 \le \xi^2$, the rest proof runs as the proof of Theorem 2.4 and Theorem 2.1 in Fan [14] with $u(t) = v(t) \equiv 1$ and $\lambda(t) \equiv \gamma$, which is omitted. □

Remark 3.6. Observe that in the proof of Proposition 3.5 the following two assumptions are not utilized:

$$\int_0^T (Y_s^2 - L_s^2)\mathrm{d}K_s^2 = 0 \text{ and } \int_0^T (U_s^1 - Y_s^1)\mathrm{d}A_s^1 = 0.$$

By virtue of Proposition 3.5, the following corollary follows immediately.

Corollary 3.7. Assume that $V_\cdot^j \in \mathcal{V}^1$, (H3)(i) holds for $L_\cdot^j, U_\cdot^j$ and ξ^j, g^j is a generator and $(Y_\cdot^j, Z_\cdot^j, K_\cdot^j, A_\cdot^j)$ is an L^1 solution of DRBSDE $(\xi^j, g^j + \mathrm{d}V^j, L^j, U^j)$ for $j = 1,2$. If $\xi^1 \le \xi^2$, $\mathrm{d}V^1 \le \mathrm{d}V^2$, $L_\cdot^1 \le L_\cdot^2$, $U_\cdot^1 \le U_\cdot^2$, g^1 or g^2 satisfies (H1)(i) and (H2), and for each $(y,z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e., \quad g^1(t,y,z) \le g^2(t,y,z),$$

then $Y_t^1 \le Y_t^2$ for each $t \in [0,T]$.

Remark 3.8. From the proof of Proposition 3.5, it is not hard to see that the conclusions in Proposition 3.5 and Corollary 3.7 hold still true when V^j only belongs to $\mathcal{V}$ instead of $\mathcal{V}^1$, and both K^j and A^j only belong to $\mathcal{V}^+$ instead of $\mathcal{V}^{+,1}$ for $j = 1,2$.

Theorem 3.9 (Uniqueness)**.** Let $V_\cdot \in \mathcal{V}^1$, (H3)(i) hold true for $L_\cdot, U_\cdot$ and ξ, and the generator g satisfy assumptions (H1)(i) and (H2). Then DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ admits at most one L^1 solution, i.e, if both $(Y_\cdot, Z_\cdot, K_\cdot, A_\cdot)$ and $(Y_\cdot', Z_\cdot', K_\cdot', A_\cdot')$ are L^1 solutions of DRBSDE $(\xi, g + \mathrm{d}V, L, U)$, then $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$,

$$Y_\cdot = Y_\cdot', \; Z_\cdot = Z_\cdot', \; K_\cdot = K_\cdot' \text{ and } A_\cdot = A_\cdot'.$$

Proof. The conclusion follows from Corollary 3.7, Itô's formula and the Ham-Bananch composition of sign measure. ☐

4 Existence, uniqueness and approximation for L^1 solutions of BSDEs

In this section, we will establish some existence, uniqueness and approximation results on L^1 solutions of BSDEs under general assumptions.

We need the following lemma, which is a direct corollary of Theorem 6.5 in Fan [14].

Lemma 4.1. Let $\xi \in \mathbb{L}^1(\mathcal{F}_T)$ and the generator g satisfy assumptions (H1)(i) and (H2)(i). If g is also bounded, then BSDE (ξ, g) admits a unique L^1 solution.

Let us start with the following existence and uniqueness result.

Theorem 4.2. Let $\xi \in \mathbb{L}^1(\mathcal{F}_T)$, $V_\cdot \in \mathcal{V}^1$ and the generator g satisfy assumptions (H1) and (H2). Then BSDE $(\xi, g + \mathrm{d}V)$ admits a unique L^1 solution.

Proof. The uniqueness part follows immediately from Theorem 3.9 with $L_\cdot \equiv -\infty$ and $U_\cdot = +\infty$. In the sequel, we prove the existence part. Let $\xi \in \mathbb{L}^1(\mathcal{F}_T)$, $V_\cdot \in \mathcal{V}^1$ and g satisfy (H1) and (H2).

We first assume that g is bounded. Note that $V_\cdot \in \mathcal{V}^1$. It follows from Lemma 4.1 that the following BSDE

$$\bar{Y}_s = \xi + V_T + \int_t^T g(s, \bar{Y}_s - V_s, \bar{Z}_s)\mathrm{d}s - \int_t^T \bar{Z}_s \mathrm{d}B_s, \quad t \in [0,T]$$

admits a unique L^1 solution $(\bar{Y}_\cdot, \bar{Z}_\cdot)$. Then the pair $(Y_\cdot, Z_\cdot) := (\bar{Y}_\cdot - V_\cdot, \bar{Z}_\cdot)$ is just the unique L^1 solution of BSDE $(\xi, g + \mathrm{d}V)$.

Now suppose that g is bounded from below. Write $g_n = g \wedge n$. Then g_n is bounded, nondecreasing in n and tends locally uniformly to g as $n \to \infty$, and it is not difficult to check that all g_n satisfy (H1) and (H2) with the same $\rho(\cdot)$, $\psi_\cdot(r)$, $\phi(\cdot)$, γ, $f_\cdot$ and α.

Then by the first step of the proof there exists a unique L^1 solution $(Y^n_\cdot, Z^n_\cdot)$ of BSDE $(\xi, g_n + \mathrm{d}V)$. Furthermore, in view of Remark 2.3, it follows from (H1)(i) and (H2)(ii) of g_n that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $n \geq 1$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$
\begin{aligned}
\operatorname{sgn}(y) g_n(\cdot, y, z) &\leq \operatorname{sgn}(y)(g_n(\cdot, y, z) - g_n(\cdot, 0, z)) + |g_n(\cdot, 0, z) - g_n(\cdot, 0, 0)| + |g_n(\cdot, 0, 0)| \\
&\leq \rho(|y|) + \gamma(f_\cdot + |z|)^\alpha + |g(\cdot, 0, 0)| \\
&\leq A + \gamma(1 + f_\cdot) + |g(\cdot, 0, 0)| + A|y| + \gamma(1 + |z|)^\alpha =: \bar{g}(\cdot, y, z). \qquad (4.1)
\end{aligned}
$$

Note that $\xi \in \mathbb{L}^1(\mathcal{F}_T)$, $V_\cdot \in \mathcal{V}^1$, $f_\cdot \in \mathcal{H}^1$, $g(\cdot, 0, 0) \in \mathcal{H}^1$, and the generator $\bar{g}$ is uniformly Lipschitz in (y, z) and has a sub-linear growth in z. By Theorems 3.9 and 3.11 in Klimsiak [39] we know that BSDE $(|\xi|, \bar{g} + \mathrm{d}|V|)$ admits a unique L^1 solution $(\bar{Y}_\cdot, \bar{Z}_\cdot)$ with $\bar{Y}_\cdot \geq 0$, and BSDE $(-|\xi|, -\bar{g} - \mathrm{d}|V|)$ admits a unique L^1 solution $(\underline{Y}_\cdot, \underline{Z}_\cdot)$ with $\underline{Y}_\cdot \leq 0$. Furthermore, note by (4.1) that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$,

$$
\mathbb{1}_{\{Y^n_t > \bar{Y}_t\}} \left(g_n(t, Y^n_t, Z^n_t) - \bar{g}(t, Y^n_t, Z^n_t) \right) \leq 0
$$

and

$$
\mathbb{1}_{\{\underline{Y}_t > Y^n_t\}} \left(-\bar{g}(t, Y^n_t, Z^n_t) - g_n(t, Y^n_t, Z^n_t) \right) \leq 0.
$$

It follows from Proposition 3.5 and Corollary 3.7 with $L_\cdot = -\infty$ and $U_\cdot = +\infty$ that $\underline{Y}_\cdot \leq Y^n_\cdot \leq Y^{n+1}_\cdot \leq \bar{Y}_\cdot$ for each $n \geq 1$. Thus, by (4.1) we know that (3.13) holds true. In addition, in view of assumptions (H2)(ii) and (H1)(iii), we have for each $n, k \geq 1$,

$$
\begin{aligned}
& |g_n(\cdot, Y^n_\cdot, Z^n_\cdot) \mathbb{1}_{|Y^n_\cdot| \leq k}| \leq |g(\cdot, Y^n_\cdot, Z^n_\cdot)| \mathbb{1}_{|Y^n_\cdot| \leq k} \\
\leq\ & |g(\cdot, Y^n_\cdot, Z^n_\cdot) - g(\cdot, Y^n_\cdot, 0)| \mathbb{1}_{|Y^n_\cdot| \leq k} + |g(\cdot, Y^n_\cdot, 0) - g(\cdot, 0, 0)| \mathbb{1}_{|Y^n_\cdot| \leq k} + |g(\cdot, 0, 0)| \\
\leq\ & \gamma \left(1 + f_\cdot + |Y^n_\cdot| \mathbb{1}_{|Y^n_\cdot| \leq k} + |Z^n_\cdot| \right) + \psi_\cdot(k) + |g(\cdot, 0, 0)| \\
\leq\ & |g(\cdot, 0, 0)| + \gamma \left(1 + f_\cdot + k \right) + \psi_\cdot(k) + \gamma |Z^n_\cdot|.
\end{aligned}
$$

Hence, (3.14) holds also true since $\psi_\cdot(k) \in \mathcal{H}$ and $f_\cdot, g(\cdot, 0, 0) \in \mathcal{H}^1$. Now, we have checked all the conditions in Proposition 3.3 with $L_\cdot = -\infty$, $U_\cdot = +\infty$ and $K^n_\cdot = A^n_\cdot \equiv 0$, and it follows that BSDE $(\xi, g + \mathrm{d}V)$ admits an L^1 solution.

Finally, in the general case, we can approximate g by the sequence g_n, where $g_n := g \vee (-n),\ n \geq 1$. By the previous step there exists a unique L^1 solution $(Y^n_\cdot, Z^n_\cdot)$ of BSDE $(\xi, g_n + \mathrm{d}V)$ for each $n \geq 1$. Repeating arguments in the proof of the previous step yields that $(Y^n_\cdot, Z^n_\cdot)$ converges in $\mathcal{S}^\beta \times \mathrm{M}^\beta$ for each $\beta \in (0, 1)$ to the unique L^1 solution $(Y_\cdot, Z_\cdot)$ of BSDE $(\xi, g + dV)$. □

Next, we will establish a general existence result on the minimal (resp. maximal) L^1 solution of BSDEs. Before that, let us introduce the following weaker assumption (HH) w.r.t. the generator g.

(HH) g has a general growth in y and a sub-linear growth in z, i.e., there exist two constants $\lambda \geq 0$ and $\alpha \in (0, 1)$, a nonnegative process $f_\cdot \in \mathcal{H}^1$ and a nonnegative function $\varphi_\cdot(r) \in \mathbf{S}$ such that

$$
\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.,\ \forall\, y \in \mathbb{R} \text{ and } z \in \mathbb{R}^d, \quad |g(\omega, t, y, z)| \leq f_t(\omega) + \varphi_t(\omega, |y|) + \lambda |z|^\alpha,
$$

here and hereafter, $\mathbf{S}$ denotes the set of nonnegative functions $\varphi_t(\omega, r) : \Omega \times [0, T] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying the following two conditions:

- $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, the function $r \mapsto \varphi_t(\omega, r)$ is increasing and $\varphi_t(\omega, 0) = 0$;
- for each $r \geq 0$, $\varphi_\cdot(\cdot, r) \in \mathcal{H}$.

Remark 4.3. It is not difficult to see that assumption (HH) is strictly weaker than the assumptions (H1)(ii)(iii) and (H2)(ii) (or (H2')(ii)).

Theorem 4.4. Let $\xi \in \mathbb{L}^1(\mathcal{F}_T)$, $V_\cdot \in \mathcal{V}^1$, g^1 satisfy (H1)(i), (H2')(i) and (HH), g^2 satisfy (AA) and the generator $g := g^1 + g^2$. Then BSDE $(\xi, g + \mathrm{d}V)$ admits a minimal (resp. maximal) L^1 solution.

Proof. We only need to prove the case of the minimal solution. The case of the maximal solution can be proved in the same way. Now, we assume that $\xi \in \mathbb{L}^1(\mathcal{F}_T)$, $V_\cdot \in \mathcal{V}^1$, g^1 satisfies (H1)(i), (H2')(i) and (HH) with $\rho(\cdot)$, $f_\cdot$, $\varphi_\cdot(r)$, λ and α, g^2 satisfies (AA) with $\tilde{f}_\cdot$, $\tilde{\mu}$, $\tilde{\lambda}$ and $\tilde{\alpha}$, and the generator $g := g^1 + g^2$.

In view of assumptions of g, it is not very hard to prove that for each $n \geq 1$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, the following function

$$g_n(\omega, t, y, z) := g_n^1(\omega, t, y, z) + g_n^2(\omega, t, y, z)$$

with

$$g_n^1(\omega, t, y, z) := \inf_{u \in \mathbb{R}^d} \left[g^1(\omega, t, y, u) + (n + 2\lambda)|u - z|^\alpha \right] \tag{4.2}$$

and

$$g_n^2(\omega, t, y, z) := \inf_{(u,v) \in \mathbb{R} \times \mathbb{R}^d} \left[g^2(\omega, t, u, v) + (n + 2\tilde{\mu})|u - y| + (n + 2\tilde{\lambda})|v - z|^{\tilde{\alpha}} \right] \tag{4.3}$$

is well defined and $(\mathcal{F}_t)$-progressively measurable, $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, g_n increases in n, is continuous in (y, z), and converges locally uniformly in (y, z) to the generator g as $n \to \infty$, g_n^1 satisfies (H1)(i) with the same $\rho(\cdot)$, (HH) with the same $f_\cdot$, $\varphi_\cdot(r)$, λ and α, (H1)(ii) with $|g_n^1(\cdot, 0, 0)| \leq f_\cdot$, (H1)(iii) with the same $\psi_\cdot(r) := 2f_\cdot + \varphi_\cdot(r)$ and (H2) with $\phi(x) := (n + 2\lambda)|x|^\alpha$, $\gamma := n + 2\lambda$, $f_\cdot :\equiv 0$ and α, g_n^2 satisfies (H1)(i) with $\rho(x) := (n + 2\tilde{\mu})x$, (HH) with the same $\tilde{f}_\cdot$, $\tilde{\mu}r$, $\tilde{\lambda}$ and $\tilde{\alpha}$, (H1)(ii) with $|g_n^2(\cdot, 0, 0)| \leq \tilde{f}_\cdot$, (H1)(iii) with the same $\psi_\cdot(r) := 2\tilde{f}_\cdot + \tilde{\mu}r$ and (H2) with $\phi(x) := (n + 2\tilde{\lambda})|x|^{\tilde{\alpha}}$, $\gamma := n + 2\tilde{\lambda}$, $f_\cdot :\equiv 0$ and $\tilde{\alpha}$. Hence, both (H1) and (H2) are satisfied by the generator g_n for each $n \geq 1$. It then follows from Theorem 4.2 that BSDE $(\xi, g_n + \mathrm{d}V)$ admits a unique L^1 solution $(Y_\cdot^n, Z_\cdot^n)$ for each $n \geq 1$. Furthermore, in view of Remark 2.3, it follows from (H1)(i) and (HH) of g_n^1 together with (HH) of g_n^2 that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $n \geq 1$ and $(y, z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\begin{array}{rcl} \mathrm{sgn}(y) g_n(\cdot, y, z) & \leq & \mathrm{sgn}(y)(g_n^1(\cdot, y, z) - g_n^1(\cdot, 0, z)) + |g_n^1(\cdot, 0, z)| + |g_n^2(\cdot, y, z)| \\ & \leq & \rho(|y|) + f_\cdot + \lambda|z|^\alpha + \tilde{f}_\cdot + \tilde{\mu}|y| + \tilde{\lambda}|z|^{\tilde{\alpha}} \\ & \leq & A + f_\cdot + \tilde{f}_\cdot + (A + \tilde{\mu})|y| + \lambda(1 + |z|)^\alpha + \tilde{\lambda}(1 + |z|)^{\tilde{\alpha}} =: \bar{g}(\cdot, y, z). \end{array} \tag{4.4}$$

In the sequel, in the same way as in the proof of Theorem 4.2, we can deduce that BSDE $(|\xi|, \bar{g} + \mathrm{d}|V|)$ admits a unique L^1 solution $(\bar{Y}_\cdot, \bar{Z}_\cdot)$ with $\bar{Y}_\cdot \geq 0$, BSDE $(-|\xi|, -\bar{g} - \mathrm{d}|V|)$ admits a unique L^1 solution $(\underline{Y}_\cdot, \underline{Z}_\cdot)$ with $\underline{Y}_\cdot \leq 0$, and in view of (4.4) and the fact that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$,

$$\mathbb{1}_{\{Y_t^n > \bar{Y}_t\}} \left(g_n(t, Y_t^n, Z_t^n) - \bar{g}(t, Y_t^n, Z_t^n) \right) \leq 0$$

and

$$\mathbb{1}_{\{\underline{Y}_t > Y_t^n\}} \left(-\bar{g}(t, Y_t^n, Z_t^n) - g_n(t, Y_t^n, Z_t^n) \right) \leq 0,$$

it follows from Proposition 3.5 and Corollary 3.7 that $\underline{Y}_\cdot \leq Y_\cdot^n \leq Y_\cdot^{n+1} \leq \bar{Y}_\cdot$ for each $n \geq 1$. Thus, by (4.4) we deduce that (3.13) holds. In addition, in view of (HH) of g_n^1 and g_n^2, we have for each $n, k \geq 1$,

$$\begin{array}{rcl} |g_n(\cdot, Y_\cdot^n, Z_\cdot^n) \mathbb{1}_{|Y_\cdot^n| \leq k}| & \leq & f_\cdot + \varphi_\cdot(|Y_\cdot^n|) \mathbb{1}_{|Y_\cdot^n| \leq k} + \lambda|Z_\cdot^n|^\alpha + \tilde{f}_\cdot + \tilde{\mu}|Y_\cdot^n| \mathbb{1}_{|Y_\cdot^n| \leq k} + \tilde{\lambda}|Z_\cdot^n|^{\tilde{\alpha}} \\ & \leq & f_\cdot + \tilde{f}_\cdot + \varphi_\cdot(k) + \tilde{\mu}k + \lambda + \tilde{\lambda} + (\lambda + \tilde{\lambda})|Z_\cdot^n|. \end{array} \tag{4.5}$$

Hence, (3.14) holds also true since $\varphi_\cdot(k) \in \mathcal{H}$ and $f_\cdot, \tilde{f}_\cdot \in \mathcal{H}^1$. Now, we have checked all the conditions in Proposition 3.3 with $L_\cdot = -\infty$, $U_\cdot = +\infty$ and $K_\cdot^n = A_\cdot^n \equiv 0$, and it follows that BSDE $(\xi, g + \mathrm{d}V)$ admits an L^1 solution $(Y_\cdot, Z_\cdot)$ such that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} (\|Y_\cdot^n - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z_\cdot^n - Z_\cdot\|_{\mathrm{M}^\beta}) = 0. \tag{4.6}$$

Finally, we show that $(Y_\cdot, Z_\cdot)$ is just the minimal L^1 solution of BSDE $(\xi, g + \mathrm{d}V)$. In fact, if $(Y_\cdot', Z_\cdot')$ is also an L^1 solution of BSDE $(\xi, g + \mathrm{d}V)$, then noticing that $g_n \le g$ and g_n satisfies (H1) and (H2) for each $n \ge 1$, it follows from Corollary 3.7 that that $Y_t^n \le Y_t'$ for each $t \in [0,T]$ and $n \ge 1$. Thus, by (4.6) we know that for each $t \in [0,T]$,

$$Y_t \le Y_t'.$$

Theorem 4.4 is then proved. □

In view of Remark 4.3, the following corollary follows immediately from Theorem 4.4.

Corollary 4.5. Let $\xi \in \mathbb{L}^1(\mathcal{F}_T)$, $V_\cdot \in \mathcal{V}^1$, g^1 satisfy (H1) and (H2'), g^2 satisfy (AA) and the generator $g := g^1 + g^2$. Then BSDE $(\xi, g + \mathrm{d}V)$ admits a minimal (resp. maximal) L^1 solution.

By Corollary 3.7 together with the proof of Theorem 4.4 it is easy to verify that under (H1)(i), (H2')(i) and (HH) (resp. (H1) and (H2')) together with (AA), the comparison theorem for the maximal (resp. minimal) L^1 solutions of the BSDEs holds true. More precisely, we have

Corollary 4.6. Assume that for $j = 1, 2$, $\xi^j \in \mathbb{L}^1(\mathcal{F}_T)$, $V_\cdot^j \in \mathcal{V}^1$, $g^{j,1}$ satisfies (H1)(i), (H2')(i) and (HH) (resp. (H1) and (H2')), $g^{j,2}$ satisfies (AA), $g^j := g^{j,1} + g^{j,2}$ and that $(Y_\cdot^j, Z_\cdot^j)$ is the maximal (resp. minimal) L^1 solution of BSDE $(\xi^j, g^j + \mathrm{d}V^j)$ (recall Theorem 4.4 and Corollary 4.5). If $\xi^1 \le \xi^2$, $\mathrm{d}V^1 \le \mathrm{d}V^2$, and

$$\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.,\ \forall\,(y,z) \in \mathbb{R} \times \mathbb{R}^d,\ \ g^{1,1}(t,y,z) \le g^{2,1}(t,y,z)\ \text{ and }\ g^{1,2}(t,y,z) \le g^{2,2}(t,y,z),$$

then $Y_t^1 \le Y_t^2$ for each $t \in [0,T]$.

Remark 4.7. Observe that either $g^1 \equiv 0$ or $g^2 \equiv 0$ is a special case of the generator $g := g^1 + g^2$ in Theorem 4.4, Corollary 4.5 and Corollary 4.6. Hence, they generalize some known results on the L^1 solution of BSDEs. In addition, by Remark 3.8 we know that the conclusion of Corollary 4.6 is still true when both V^1 and V^2 only belong to $\mathcal{V}$ instead of $\mathcal{V}^1$.

5 Existence, uniqueness and approximation for L^1 solutions of RBSDEs

In this section, we will establish some existence, uniqueness and approximation results on L^1 solutions of RBSDEs with one continuous barrier under general assumptions.

Theorem 5.1. Let $V_\cdot \in \mathcal{V}^1$ and the generator g satisfy assumptions (H1) and (H2).

(i) Assume that (H3L)(i) holds true for $L_\cdot$ and ξ. Then $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ admits an L^1 solution iff (H3L)(ii) is satisfied. Furthermore, if (H3L)(ii) holds also true, then $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ admits a unique L^1 solution $(Y_\cdot, Z_\cdot, K_\cdot)$ such that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} (\|Y_\cdot^n - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z_\cdot^n - Z_\cdot\|_{\mathrm{M}^\beta} + \|K_\cdot^n - K_\cdot\|_{\mathcal{S}^\beta}) = 0, \tag{5.1}$$

where for each $n \ge 1$, $(Y_\cdot^n, Z_\cdot^n)$ is the unique L^1 solution of BSDE $(\xi, \bar{g}_n + \mathrm{d}V)$ with $\bar{g}_n(t,y,z) := g(t,y,z) + n(y - L_t)^-$, i.e., (3.9), (recall Theorem 4.2), and

$$K_t^n := n \int_0^t (Y_s^n - L_s)^- \mathrm{d}s,\ \ t \in [0,T]. \tag{5.2}$$

(ii) Assume that (H3U)(i) holds true for $U.$ and ξ. Then $\bar{R}$BSDE $(\xi, g+\mathrm{d}V, U)$ admits an L^1 solution iff (H3U)(ii) is satisfied. Furthermore, if (H3U)(ii) holds also true, then $\bar{R}$BSDE $(\xi, g+\mathrm{d}V, U)$ admits a unique L^1 solution $(Y., Z., A.)$ such that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} (\|Y^n_\cdot - Y.\|_{\mathcal{S}^\beta} + \|Z^n_\cdot - Z.\|_{\mathrm{M}^\beta} + \|A^n_\cdot - A.\|_{\mathcal{S}^\beta}) = 0, \tag{5.3}$$

where for each $n \ge 1$, $(Y^n_\cdot, Z^n_\cdot)$ is the unique L^1 solution of BSDE $(\xi, \underline{g}_n + \mathrm{d}V)$ with $\underline{g}_n(t,y,z) := g(t,y,z) - n(y-U_t)^+$, i.e., (3.10), (recall Theorem 4.2), and

$$A^n_t := n\int_0^t (Y^n_s - U_s)^+ \mathrm{d}s, \quad t \in [0,T]. \tag{5.4}$$

Proof. We only prove the case of (i), the proof of (ii) is similar. We assume that $V. \in \mathcal{V}^1$, the generator g satisfies (H1) and (H2), and (H3L)(i) holds true for $L.$ and ξ. If $\underline{R}$BSDE $(\xi, g+\mathrm{d}V, L)$ admits an L^1 solution $(Y., Z., K.)$, then from Lemma 2.6 we know that $g(\cdot, Y., Z.) \in \mathcal{H}^1$ and $g(\cdot, Y., 0) \in \mathcal{H}^1$. Thus, (H3L)(ii) is satisfied with

$$(C., H.) := (-\int_0^\cdot g(s, Y_s, Z_s)\mathrm{d}s - V. - K.,\ Z.)$$

and $X. := Y.$. The necessity is proved.

We further assume that (H3L)(ii) holds. The uniqueness of the L^1 solution of $\underline{R}$BSDE $(\xi, g+\mathrm{d}V, L)$ follows from Proposition 3.5. In the sequel, we prove (5.1). For each $n \ge 1$, let $(Y^n_\cdot, Z^n_\cdot)$ be the unique L^1 solution of BSDE $(\xi, \bar{g}_n + \mathrm{d}V)$ with $\bar{g}_n(t,y,z) := g(t,y,z) + n(y-L_t)^-$ and (5.2). We first show that there exists a process $\bar{X}. \in \cap_{\beta\in(0,1)}\mathcal{S}^\beta$ of the class (D) such that for each $n \ge 1$,

$$Y^1_\cdot \le Y^n_\cdot \le Y^{n+1}_\cdot \le \bar{X}.. \tag{5.5}$$

In fact, it follows from (H3L)(ii) that there exists two processes $(C., H.) \in \mathcal{V}^1 \times \mathrm{M}^\beta$ for each $\beta \in (0,1)$ such that

$$X_t = X_T - \int_t^T \mathrm{d}C_s - \int_t^T H_s \cdot \mathrm{d}B_s, \quad t \in [0,T] \tag{5.6}$$

belongs to the class (D), $g^-(\cdot, X., 0) \in \mathcal{H}^1$ and $L_t \le X_t$ for each $t \in [0,T]$. And, by (H2)(ii) together with Hölder's inequality we know that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$,

$$g^-(\cdot, X., H.) \le g^-(\cdot, X., 0) + \gamma(f. + |X.| + H.)^\alpha \ \in \mathcal{H}^1.$$

Then, the equation (5.6) can be rewritten in the form

$$\begin{aligned} X_t &= X_T + \int_t^T g(s, X_s, H_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s - \int_t^T \left(g^+(s, X_s, H_s)\mathrm{d}s + \mathrm{d}C^{0,+}_s + \mathrm{d}V^{0,+}_s\right) \\ &\quad + \int_t^T \left(g^-(s, X_s, H_s)\mathrm{d}s + \mathrm{d}C^{0,-}_s + \mathrm{d}V^{0,-}_s\right) - \int_t^T H_s \cdot \mathrm{d}B_s, \quad t \in [0,T], \end{aligned}$$

where $V. - V_0 = V^{0,+}_\cdot - V^{0,-}_\cdot$ and $C. - C_0 = C^{0,+}_\cdot - C^{0,-}_\cdot$ with $V^{0,+}_\cdot, V^{0,-}_\cdot, C^{0,+}_\cdot, C^{0,-}_\cdot \in \mathcal{V}^{+,1}$. On the other hand, in view of (H1) and (H2), by Theorem 4.2 we know that there exists a unique L^1 solution $(\bar{X}., \bar{Z}.)$ of the BSDE

$$\begin{aligned} \bar{X}_t &= X_T \vee \xi + \int_t^T g(s, \bar{X}_s, \bar{Z}_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s \\ &\quad + \int_t^T \left(g^-(s, X_s, H_s)\mathrm{d}s + \mathrm{d}C^{0,-}_s + \mathrm{d}V^{0,-}_s\right) - \int_t^T \bar{Z}_s \cdot \mathrm{d}B_s, \quad t \in [0,T]. \end{aligned}$$

And, it follows from Proposition 3.5 and Remark 3.8 that $L_t \le X_t \le \bar{X}_t$ for each $t \in [0,T]$. Therefore, for each $n \ge 1$,

$$\begin{aligned}\bar{X}_t &= X_T \vee \xi + \int_t^T g(s, \bar{X}_s, \bar{Z}_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s + n\int_t^T \left(\bar{X}_s - L_s\right)^- \mathrm{d}s \\ &\quad + \int_t^T \left(g^-(s, X_s, H_s)\mathrm{d}s + \mathrm{d}C_s^{0,-} + \mathrm{d}V_s^{0,-}\right) - \int_t^T \bar{Z}_s \cdot \mathrm{d}B_s, \quad t \in [0,T].\end{aligned}$$

Thus, by Corollary 3.7 we know that (5.5) holds true.

In the sequel, in view of assumptions (H1) and (H2), it follows from Lemma 2.6 that $g(\cdot, Y_\cdot^1, 0) \in \mathcal{H}^1$ and $g(\cdot, \bar{X}_\cdot, 0) \in \mathcal{H}^1$, then from Lemma 2.5 together with (5.5) that (3.2) holds true for each $n \ge 1$, with

$$\bar{f}_\cdot := |g(\cdot, Y_\cdot^1, 0)| + |g(\cdot, \bar{X}_\cdot, 0)| + (\gamma + A)(|Y_\cdot^1| + |\bar{X}_\cdot|) + \gamma(1 + f_\cdot) + A \ \in \mathcal{H}^1,$$

$\bar{\lambda} := \gamma$ and α. Thus, we have verified that all conditions in Proposition 3.2 (i) are satisfied, and then it follows that there exists an L^1 solution $(Y_\cdot, Z_\cdot, K_\cdot)$ of $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ such that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} \left(\|Y_\cdot^n - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z_\cdot^n - Z_\cdot\|_{\mathrm{M}^\beta}\right) = 0 \tag{5.7}$$

and there exists a subsequence $\{K_\cdot^{n_j}\}$ of $\{K_\cdot^n\}$ such that

$$\lim_{j\to\infty} \sup_{t\in[0,T]} |K_t^{n_j} - K_t| = 0.$$

Finally, in view of (5.7), in order to prove (5.1) we need only to show that for each $\beta \in (0,1)$

$$\lim_{n\to\infty} \left\| \int_0^\cdot g(s, Y_s^n, Z_s^n)\mathrm{d}s - \int_0^\cdot g(s, Y_s, Z_s)\mathrm{d}s \right\|_{\mathcal{S}^\beta} = 0. \tag{5.8}$$

The proof is similar to that of Theorem 5.8 in Fan [16], but for readers' convenience we list it as follows. In fact, it follows from (H2) (i) that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $n \ge 1$,

$$\begin{aligned} & |g(\cdot, Y_\cdot^n, Z_\cdot^n) - g(\cdot, Y_\cdot, Z_\cdot)| \\ \le\ & |g(\cdot, Y_\cdot^n, Z_\cdot^n) - g(\cdot, Y_\cdot^n, Z_\cdot)| + |g(\cdot, Y_\cdot^n, Z_\cdot) - g(\cdot, Y_\cdot, Z_\cdot)| \\ \le\ & |g(\cdot, Y_\cdot^n, Z_\cdot) - g(\cdot, Y_\cdot, Z_\cdot)| + \phi(|Z_\cdot^n - Z_\cdot|). \end{aligned}$$

Thus, making use of the following basic inequality (see Fan and Jiang [20] for details)

$$\phi(x) \le (m + 2A)x + \phi\left(\frac{2A}{m + 2A}\right), \ \ \forall\, x \ge 0, \ \ \forall m \ge 1$$

together with Hölder's inequality, we get that for each $n, m \ge 1$ and $\beta \in (0,1)$,

$$\begin{aligned} & \left\| \int_0^\cdot g(s, Y_s^n, Z_s^n)\mathrm{d}s - \int_0^\cdot g(s, Y_s, Z_s)\mathrm{d}s \right\|_{\mathcal{S}^\beta} \\ \le\ & \mathbb{E}\left[\left(\int_0^T |g(t, Y_t^n, Z_t^n) - g(t, Y_t, Z_t)|\mathrm{d}t\right)^\beta\right] \\ \le\ & \mathbb{E}\left[\left(\int_0^T |g(t, Y_t^n, Z_t) - g(t, Y_t, Z_t)|\mathrm{d}t\right)^\beta\right] + (m + 2A)^\beta T^{\frac{\beta}{2}} \|Z_\cdot^n - Z_\cdot\|_{\mathrm{M}^\beta} \\ & + \phi^\beta\left(\frac{2A}{m + 2A}\right) T^\beta. \end{aligned} \tag{5.9}$$

Furthermore, in view of assumptions (H1) and (H2) together with (5.5), it follows from Lemma 2.5 and Lemma 2.6 that for each $n \geq 1$,

$$|g(\cdot, Y_\cdot^n, Z_\cdot) - g(\cdot, Y_\cdot, Z_\cdot)| \leq |g(\cdot, Y_\cdot^n, Z_\cdot)| + |g(\cdot, Y_\cdot, Z_\cdot)| \leq \bar{f}_\cdot + \gamma |Z_\cdot|^\alpha + |g(\cdot, Y_\cdot, Z_\cdot)| \in \mathcal{H}^1.$$

Then, Lebesgue's dominated convergence theorem yields that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^T |g(t, Y_t^n, Z_t) - g(t, Y_t, Z_t)| \mathrm{d}t\right)^\beta\right] = 0. \tag{5.10}$$

Thus, letting first $n \to \infty$, and then $m \to \infty$ in (5.9), in view of (5.10), (5.7) and the fact that $\phi(\cdot)$ is continuous and $\phi(0) = 0$, we get (5.8). The proof of Theorem 5.1 is then completed. □

Corollary 5.2. Assume that $\xi^1, \xi^2 \in \mathbb{L}^1(\mathcal{F}_T)$ with $\xi^1 \leq \xi^2$, $V_\cdot^1, V_\cdot^2 \in \mathcal{V}^1$ with $\mathrm{d}V^1 \leq \mathrm{d}V^2$, and both generators g^1 and g^2 satisfy (H1) and (H2) with

$$\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.,\ \forall\, (y,z) \in \mathbb{R} \times \mathbb{R}^d,\ \ g^1(t,y,z) \leq g^2(t,y,z).$$

We have

(i) For $i = 1,2$, let (H3L) hold for ξ^i, $L_\cdot^i$ and $X_\cdot^i$ associated with g^i, and $(Y_\cdot^i, Z_\cdot^i, K_\cdot^i)$ be the unique L^1 solution of $\underline{R}$BSDE $(\xi^i, g^i + \mathrm{d}V^i, L^i)$ (recall Theorem 5.1). If $L_\cdot^1 = L_\cdot^2$, then $dK^1 \geq \mathrm{d}K^2$.

(ii) For $i = 1,2$, let (H3U) hold for ξ^i, $U_\cdot^i$ and $X_\cdot^i$ associated with g^i, and $(Y_\cdot^i, Z_\cdot^i, A_\cdot^i)$ be the unique L^1 solution of $\bar{R}$BSDE $(\xi^i, g^i + \mathrm{d}V^i, U^i)$ (recall Theorem 5.1). If $U_\cdot^1 = U_\cdot^2$, then $dA^1 \leq \mathrm{d}A^2$.

Proof. We only prove (i). The proof is classical, and we list it for readers' convenience. For $n \geq 1$ and $i = 1,2$, by Theorem 4.2 we let $(Y_\cdot^{i,n}, Z_\cdot^{i,n})$ be the unique L^1 solution of the following penalization BSDE:

$$Y_t^{i,n} = \xi^i + \int_t^T g^i(s, Y_s^{i,n}, Z_s^{i,n})\mathrm{d}s + \int_t^T \mathrm{d}V_s^i + \int_t^T \mathrm{d}K_s^{i,n} - \int_t^T Z_s^{i,n} \cdot \mathrm{d}B_s,\ \ t \in [0,T]$$

with

$$K_t^{i,n} := n\int_0^t \left(Y_s^{i,n} - L_s^i\right)^- \mathrm{d}s,\ \ t \in [0,T].$$

In view of the assumptions of Corollary 5.2, it follows from Corollary 3.7 that for each $n \geq 1$, $Y_\cdot^{1,n} \leq Y_\cdot^{2,n}$, and then

$$K_{t_2}^{1,n} - K_{t_1}^{1,n} = n\int_{t_1}^{t_2} \left(Y_s^{1,n} - L_s^1\right)^- \mathrm{d}s \geq n\int_{t_1}^{t_2} \left(Y_s^{2,n} - L_s^2\right)^- \mathrm{d}s = K_{t_2}^{2,n} - K_{t_1}^{2,n}$$

for every $n \geq 1$ and $0 \leq t_1 \leq t_2 \leq T$. Since for each $\beta \in (0,1)$, both $\|K_\cdot^{1,n} - K_\cdot^1\|_{\mathcal{S}^\beta}$ and $\|K_\cdot^{2,n} - K_\cdot^2\|_{\mathcal{S}^\beta}$ converge to zero as $n \to \infty$ by Theorem 5.1, it follows that $K_{t_2}^1 - K_{t_1}^1 \geq K_{t_2}^2 - K_{t_1}^2$ for every $0 \leq t_1 \leq t_2 \leq T$, which proves the desired result. □

Theorem 5.3. Let $V_\cdot \in \mathcal{V}^1$, g^1 satisfy assumptions (H1) and (H2'), g^2 satisfy assumption (AA) and the generator $g := g^1 + g^2$.

(i) Assume that (H3L)(i) holds true for $L_\cdot$ and ξ. Then $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ admits an L^1 solution iff (H3L)(ii) is satisfied for $X_\cdot$, $L_\cdot$ and g (or g^1). Furthermore, if (H3L)(ii) holds also true for $X_\cdot$, $L_\cdot$ and g (or g^1), then $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ admits a minimal L^1 solution (resp. an L^1 solution) $(Y_\cdot, Z_\cdot, K_\cdot)$ such that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} (\|Y^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta}) = 0 \tag{5.11}$$

and there exists a subsequence $\{K^{n_j}_\cdot\}$ of $\{K^n_\cdot\}$ such that

$$\lim_{j\to\infty} \sup_{t\in[0,T]} |K^{n_j}_t - K_t| = 0,$$

where for each $n \ge 1$, $(Y^n_\cdot, Z^n_\cdot)$ is the minimal (resp. maximal) L^1 solution of BSDE $(\xi, \bar{g}_n + \mathrm{d}V)$ with $\bar{g}_n(t,y,z) := g(t,y,z) + n(y - L_t)^-$, i.e., (3.9), (recall Corollary 4.5), and

$$K^n_t := n\int_0^t (Y^n_s - L_s)^- \mathrm{d}s, \quad t \in [0,T]. \tag{5.12}$$

(ii) Assume that (H3U)(i) holds true for $U_\cdot$ and ξ. Then $\bar{R}$BSDE $(\xi, g + \mathrm{d}V, U)$ admits an L^1 solution iff (H3U)(ii) is satisfied for $X_\cdot$, $U_\cdot$ and g (or g^1). Furthermore, if (H3U)(ii) holds also true for $X_\cdot$, $U_\cdot$ and g (or g^1), then $\bar{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ admits a maximal L^1 solution (resp. an L^1 solution) $(Y_\cdot, Z_\cdot, A_\cdot)$ such that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} (\|Y^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta}) = 0 \tag{5.13}$$

and there exists a subsequence $\{A^{n_j}_\cdot\}$ of $\{A^n_\cdot\}$ such that

$$\lim_{j\to\infty} \sup_{t\in[0,T]} |A^{n_j}_t - A_t| = 0,$$

where for each $n \ge 1$, $(Y^n_\cdot, Z^n_\cdot)$ is the maximal (resp. minimal) L^1 solution of BSDE $(\xi, \underline{g}_n + \mathrm{d}V)$ with $\underline{g}_n(t,y,z) := g(t,y,z) - n(y - U_t)^+$, i.e., (3.10), (recall Corollary 4.5), and

$$A^n_t := n\int_0^t (Y^n_s - U_s)^+ \mathrm{d}s, \quad t \in [0,T]. \tag{5.14}$$

Proof. We only prove (i), and (ii) can be proved in the same way. In view of Corollary 4.5, Corollary 4.6, Remark 4.7, Lemma 2.5, Lemma 2.6 and Proposition 3.2, by a similar argument to that in the proof of Theorem 5.1 we can prove that all conclusions in (i) of Theorem 5.3 hold true except for the minimal property of the L^1 solution $(Y_\cdot, Z_\cdot, K_\cdot)$ of $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ when $(Y^n_\cdot, Z^n_\cdot)$ is the minimal L^1 solution of penalized BSDE $(\xi, \bar{g}_n + \mathrm{d}V)$ for each $n \ge 1$. Now, we will show this property.

Indeed, for any L^1 solution $(Y'_\cdot, Z'_\cdot, K'_\cdot)$ of $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$, it is not hard to check that $(Y'_\cdot, Z'_\cdot)$ is an L^1 solution of BSDE $(\xi, \bar{g}_n + \mathrm{d}\bar{V})$ with $\bar{V}_\cdot := V_\cdot + K'_\cdot$ for each $n \ge 1$. Thus, in view of the assumption that $(Y^n_\cdot, Z^n_\cdot)$ is the minimal L^1 solution of penalized BSDE $(\xi, \bar{g}_n + \mathrm{d}V)$ for each $n \ge 1$, Corollary 4.6 yields that for each $n \ge 1$,

$$Y^n_t \le Y'_t, \quad t \in [0,T].$$

Furthermore, since $\lim_{n\to\infty} \|Y^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} = 0$ for each $\beta \in (0,1)$, we know that

$$Y_t \le Y'_t, \quad t \in [0,T],$$

which is the desired result. □

Theorem 5.4. Let $V_\cdot \in \mathcal{V}^1$, g^1 satisfy assumptions (H1) and (H2'), g^2 satisfy assumption (AA) and the generator $g := g^1 + g^2$.

(i) Assume that (H3L) holds true for $L_\cdot$, ξ, $X_\cdot$ and g (or g^1). Then $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ admits a maximal (resp. minimal) L^1 solution $(Y_\cdot, Z_\cdot, K_\cdot)$ such that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} \left(\|Y_\cdot^n - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z_\cdot^n - Z_\cdot\|_{\mathrm{M}^\beta} + \|K_\cdot^n - K_\cdot\|_{\mathcal{S}^1}\right) = 0,$$

where, for each $n \geq 1$, $(Y_\cdot^n, Z_\cdot^n, K_\cdot^n)$ is the unique L^1 solution of $\underline{R}$BSDE $(\xi, g_n + \mathrm{d}V, L)$ with a generator g_n satisfying (H1), (H2) and (H3L) (recall Theorem 5.1(i)).

(ii) Assume that (H3U) holds true for $U_\cdot$, ξ, $X_\cdot$ and g (or g^1). Then $\bar{R}$BSDE $(\xi, g + \mathrm{d}V, U)$ admits a maximal (resp. minimal) L^1 solution $(Y_\cdot, Z_\cdot, A_\cdot)$ such that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} \left(\|Y_\cdot^n - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z_\cdot^n - Z_\cdot\|_{\mathrm{M}^\beta} + \|A_\cdot^n - A_\cdot\|_{\mathcal{S}^1}\right) = 0,$$

where, for each $n \geq 1$, $(Y_\cdot^n, Z_\cdot^n, A_\cdot^n)$ is the unique L^1 solution of $\bar{R}$BSDE $(\xi, g_n + \mathrm{d}V, U)$ with a generator g_n satisfying (H1), (H2) and (H3U) (recall Theorem 5.1(ii)).

Proof. We only prove (i) and consider the case of the maximal L^1 solution. Now, we assume that $V_\cdot \in \mathcal{V}^1$, g^1 satisfies (H1) and (H2') with $\rho(\cdot)$, $\psi_\cdot(r)$, $f_\cdot$, μ, λ and α, g^2 satisfies (AA) with $\tilde{f}_\cdot$, $\tilde{\mu}$, $\tilde{\lambda}$ and $\tilde{\alpha}$, the generator $g := g^1 + g^2$, and (H3L) holds true for $L_\cdot$, ξ, $X_\cdot$ and g (or g^1). In view of assumptions of g, it is not very hard to prove that for each $n \geq 1$ and $(y,z) \in \mathbb{R} \times \mathbb{R}^d$, the following function

$$g_n(\omega, t, y, z) := g_n^1(\omega, t, y, z) + g_n^2(\omega, t, y, z)$$

with

$$g_n^1(\omega, t, y, z) := \sup_{u \in \mathbb{R}^d} \left[g^1(\omega, t, y, u) - (n + 2\lambda)|u - z|^\alpha\right] \tag{5.15}$$

and

$$g_n^2(\omega, t, y, z) := \sup_{(u,v) \in \mathbb{R} \times \mathbb{R}^d} \left[g^2(\omega, t, u, v) - (n + 2\tilde{\mu})|u - y| - (n + 2\tilde{\lambda})|v - z|^{\tilde{\alpha}}\right] \tag{5.16}$$

is well defined and $(\mathcal{F}_t)$-progressively measurable, $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, g_n decreases in n, is continuous in (y,z), and converges locally uniformly in (y,z) to the generator g as $n \to \infty$, g_n satisfies (H1) and (H2), and $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $n \geq 1$ and $(y,z) \in \mathbb{R} \times \mathbb{R}^d$,

$$|g_n^1(\cdot, y, z) - g^1(\cdot, y, 0)| \leq f_\cdot + \mu|y| + \lambda|z|^\alpha, \tag{5.17}$$

and

$$|g_n^2(\cdot, y, z)| \leq \tilde{f}_\cdot + \tilde{\mu}|y| + \tilde{\lambda}|z|^{\tilde{\alpha}}. \tag{5.18}$$

Then, in view of (5.17) and (5.18), we know that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, n \geq 1$ and $(y,z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\begin{array}{rcl} g_n^-(\cdot, y, z) & \leq & (g_n^1(\cdot, y, z))^- + (g_n^2(\cdot, y, z))^- \\ & \leq & (g^1(\cdot, y, 0))^- + f_\cdot + \mu|y| + \lambda|z|^\alpha + \tilde{f}_\cdot + \tilde{\mu}|y| + \tilde{\lambda}|z|^{\tilde{\alpha}}. \end{array} \tag{5.19}$$

Hence, $g_n^-(\cdot, X_\cdot, 0) \in \mathcal{H}^1$ due to $(g^1(\cdot, X_\cdot, 0))^- \in \mathcal{H}^1$, and then (H3L) holds true for $L_\cdot$, ξ, $X_\cdot$ and g_n. It then follows from Theorem 5.1(i) that there exists a unique L^1 solution $(Y_\cdot^n, Z_\cdot^n, K_\cdot^n)$ of $\underline{R}$BSDE $(\xi, g_n + \mathrm{d}V, L)$ for each $n \geq 1$.

In the sequel, let

$$\underline{g}(\cdot, y, z) := g^1(\cdot, y, 0) - (f_\cdot + \tilde{f}_\cdot) - (\mu + \tilde{\mu})|y| - \lambda|z|^\alpha - \tilde{\lambda}|z|^{\tilde{\alpha}} \tag{5.20}$$

and

$$\bar{g}(\cdot, y, z) := g^1(\cdot, y, 0) + (f_\cdot + \tilde{f}_\cdot) + (\mu + \tilde{\mu})|y| + \lambda|z|^\alpha + \tilde{\lambda}|z|^{\tilde{\alpha}}. \tag{5.21}$$

Then by (5.17) and (5.18), $\underline{g} \leq g_n \leq \bar{g}$ for each $n \geq 1$, and both $\underline{g}$ and $\bar{g}$ satisfy (H1) and (H2) with

$$\underline{g}^-(\cdot, X_\cdot, 0) \leq (g^1(\cdot, X_\cdot, 0))^- + (f_\cdot + \tilde{f}_\cdot) + (\mu + \tilde{\mu})|X_\cdot| \in \mathcal{H}^1,$$
$$\bar{g}^-(\cdot, X_\cdot, 0) \leq (g^1(\cdot, X_\cdot, 0))^- \in \mathcal{H}^1.$$

Thus, (H3L) holds also true for $L_\cdot$, ξ, $X_\cdot$, $\underline{g}$ and $\bar{g}$. It then follows from Theorem 5.1 that $\underline{R}$BSDE $(\xi, \underline{g} + \mathrm{d}V, L)$ and $\underline{R}$BSDE $(\xi, \bar{g} + \mathrm{d}V, L)$ admit respectively a unique L^1 solution $(\underline{Y}_\cdot, \underline{Z}_\cdot, \underline{K}_\cdot)$ and $(\bar{Y}_\cdot, \bar{Z}_\cdot, \bar{K}_\cdot)$, and by Corollary 3.7 and Corollary 5.2 we know that for each $n \geq 1$,

$$\underline{Y}_\cdot \leq Y_\cdot^{n+1} \leq Y_\cdot^n \leq \bar{Y}_\cdot \ \text{ and } \ \mathrm{d}\bar{K} \leq \mathrm{d}K^n \leq \mathrm{d}K^{n+1} \leq \mathrm{d}\underline{K}. \tag{5.22}$$

Furthermore, it follows from Lemma 2.6 that $\underline{g}(\cdot, \underline{Y}_\cdot, 0) \in \mathcal{H}^1$ and $\bar{g}(\cdot, \bar{Y}_\cdot, 0) \in \mathcal{H}^1$, and then from (5.20) and (5.21) that $g^1(\cdot, \underline{Y}_\cdot, 0) \in \mathcal{H}^1$ and $g^1(\cdot, \bar{Y}_\cdot, 0) \in \mathcal{H}^1$. And, in view of (5.22) together with assumptions (H1) and (H2') of g^1, it follows from Lemma 2.5 that for each $n \geq 1$,

$$|g^1(\cdot, Y_\cdot^n, Z_\cdot^n)| \leq |g^1(\cdot, \underline{Y}_\cdot, 0)| + |g^1(\cdot, \bar{Y}_\cdot, 0)| + (\mu + A)(|\underline{Y}_\cdot| + |\bar{Y}_\cdot|) + f_\cdot + A + \lambda |Z_\cdot^n|^\alpha. \tag{5.23}$$

Then, by (5.23), (5.18) and (5.22) we can conclude that (3.18) holds true with

$$\bar{f}_\cdot := |g^1(\cdot, \underline{Y}_\cdot, 0)| + |g^1(\cdot, \bar{Y}_\cdot, 0)| + (\mu + A + \tilde{\mu})(|\underline{Y}_\cdot| + |\bar{Y}_\cdot|) + f_\cdot + \tilde{f}_\cdot + A + \lambda + \tilde{\lambda} \ \in \mathcal{H}^1$$

and $\bar{\lambda} := \lambda + \tilde{\lambda}$. Thus, in view of Remark 3.4, we have checked all the conditions in Proposition 3.3 with $U_\cdot = +\infty$ and $A_\cdot^n \equiv 0$, and it follows that $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ admits an L^1 solution $(Y_\cdot, Z_\cdot, K_\cdot)$ such that for each $\beta \in (0,1)$,

$$\lim_{n \to \infty} (\|Y_\cdot^n - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z_\cdot^n - Z_\cdot\|_{\mathrm{M}^\beta} + \|K_\cdot^n - K_\cdot\|_{\mathcal{S}^1}) = 0. \tag{5.24}$$

Finally, we show that $(Y_\cdot, Z_\cdot, K_\cdot)$ is just the maximal L^1 solution of $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$. In fact, if $(Y'_\cdot, Z'_\cdot, K'_\cdot)$ is also an L^1 solution of $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$, then noticing that $g_n \geq g$ and g_n satisfies (H1) and (H2) for each $n \geq 1$, it follows from Corollary 3.7 that $Y_t^n \geq Y'_t$ for each $t \in [0,T]$ and $n \geq 1$. Thus, by (5.24) we know that for each $t \in [0,T]$,

$$Y_t \geq Y'_t.$$

Theorem 5.4 is then proved. □

By Corollary 3.7, Corollary 5.2 and the proof of Theorem 5.4, it is not hard to verify the following comparison result for the minimal (resp. maximal) L^1 solutions of Reflected BSDEs.

Corollary 5.5. Assume that $\xi^1, \xi^2 \in \mathbb{L}^1(\mathcal{F}_T)$ with $\xi^1 \leq \xi^2$, $V_\cdot^1, V_\cdot^2 \in \mathcal{V}^1$ with $\mathrm{d}V^1 \leq \mathrm{d}V^2$, $g^{1,1}$ and $g^{2,1}$ satisfy (H1) and (H2'), $g^{1,2}$ and $g^{2,2}$ satisfy (AA), $g^1 := g^{1,1} + g^{1,2}$ and $g^2 := g^{2,1} + g^{2,2}$ with

$$\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.,\ \forall\, (y,z) \in \mathbb{R} \times \mathbb{R}^d,\ \ g^{1,1}(t,y,z) \leq g^{2,1}(t,y,z) \ \text{ and } \ g^{1,2}(t,y,z) \leq g^{2,2}(t,y,z).$$

We have

(i) For $i = 1, 2$, let (H3L) hold for ξ^i, $L_\cdot^i$ and $X_\cdot^i$ associated with g^i (or $g^{i,1}$), and $(Y_\cdot^i, Z_\cdot^i, K_\cdot^i)$ be the minimal (resp. maximal) L^1 solution of $\underline{R}$BSDE $(\xi^i, g^i + \mathrm{d}V^i, L^i)$ (recall Theorem 5.4). If $L_\cdot^1 \leq L_\cdot^2$, then $Y_t^1 \leq Y_t^2$ for each $t \in [0,T]$, and if $L_\cdot^1 = L_\cdot^2$, then $dK^1 \geq \mathrm{d}K^2$.

(ii) For $i=1,2$, let (H3U) hold for ξ^i, $U^i_\cdot$ and $X^i_\cdot$ associated with g^i (or $g^{i,1}$), and $(Y^i_\cdot, Z^i_\cdot, A^i_\cdot)$ be the minimal (resp. maximal) L^1 solution of $\bar{R}$BSDE $(\xi^i, g^i + \mathrm{d}V^i, U^i)$ (recall Theorem 5.4). If $U^1_\cdot \le U^2_\cdot$, then $Y^1_t \le Y^2_t$ for each $t\in[0,T]$, and if $U^1_\cdot = U^2_\cdot$, then $dA^1 \le \mathrm{d}A^2$.

The following corollary follows immediately from Theorem 5.4.

Corollary 5.6. Let $V_\cdot \in \mathcal{V}^1$, (H3)(i) hold true for $L_\cdot$, $U_\cdot$ and ξ, and the generator g satisfy (AA).

(i) If $L^+_\cdot \in \mathcal{S}^1$, then $\underline{R}$BSDE $(\xi, g+\mathrm{d}V, L)$ admits a minimal (resp. maximal) L^1 solution.

(ii) If $U^-_\cdot \in \mathcal{S}^1$, then $\bar{R}$BSDE $(\xi, g+\mathrm{d}V, U)$ admits a minimal (resp. maximal) L^1 solution.

6 Existence, uniqueness and approximation for L^1 solutions of DRBSDEs

In this section, we will establish some existence, uniqueness and approximation results on L^1 solutions of RBSDEs with two continuous barriers under general assumptions.

Theorem 6.1. Assume that $V_\cdot \in \mathcal{V}^1$, the generator g satisfies assumptions (H1) and (H2), and assumption (H3)(i) holds true for $L_\cdot, U_\cdot$ and ξ. Then, DRBSDE $(\xi, g+\mathrm{d}V, L, U)$ admits an L^1 solution iff (H3)(ii) is satisfied. And, if (H3)(ii) holds also true, then DRBSDE $(\xi, g+\mathrm{d}V, L, U)$ admits a unique L^1 solution $(Y_\cdot, Z_\cdot, K_\cdot, A_\cdot)$. Moreover,

(i) Let $(\underline{Y}^n_\cdot, \underline{Z}^n_\cdot, \underline{A}^n_\cdot)$ be the unique L^1 solution of $\bar{R}$BSDE $(\xi, \bar{g}_n + \mathrm{d}V, U)$ with $\bar{g}_n(t,y,z) := g(t,y,z) + n(y-L_t)^-$ for each $n\ge 1$, i.e.,

$$\begin{cases} \underline{Y}^n_t = \xi + \int_t^T \bar{g}_n(s, \underline{Y}^n_s, \underline{Z}^n_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s - \int_t^T \mathrm{d}\underline{A}^n_s - \int_t^T \underline{Z}^n_s \cdot \mathrm{d}B_s, \quad t\in[0,T], \\ \underline{Y}^n_t \le U_t, \ t\in[0,T] \ \text{ and } \int_0^T (U_t - \underline{Y}^n_t)\mathrm{d}\underline{A}^n_t = 0, \\ \underline{K}^n_t := n\int_0^t (\underline{Y}^n_s - L_s)^- \,\mathrm{d}s, \quad t\in[0,T] \end{cases} \tag{6.1}$$

(Recall Theorem 5.1(ii)). Then, for each $\beta\in(0,1)$,

$$\lim_{n\to\infty} \left(\|\underline{Y}^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} + \|\underline{Z}^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta} + \|\underline{K}^n_\cdot - K_\cdot\|_{\mathcal{S}^\beta} + \|\underline{A}^n_\cdot - A_\cdot\|_{\mathcal{S}^1}\right) = 0. \tag{6.2}$$

(ii) Let $(\bar{Y}^n_\cdot, \bar{Z}^n_\cdot, \bar{K}^n_\cdot)$ be the unique L^1 solution of $\underline{R}$BSDE $(\xi, \underline{g}_n + \mathrm{d}V, L)$ with $\underline{g}_n(t,y,z) := g(t,y,z) - n(y-U_t)^+$ for each $n\ge 1$, i.e.,

$$\begin{cases} \bar{Y}^n_t = \xi + \int_t^T \underline{g}_n(s, \bar{Y}^n_s, \bar{Z}^n_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s + \int_t^T \mathrm{d}\bar{K}^n_s - \int_t^T \bar{Z}^n_s \cdot \mathrm{d}B_s, \quad t\in[0,T], \\ L_t \le \bar{Y}^n_t, \ t\in[0,T] \ \text{ and } \int_0^T (\bar{Y}^n_t - L_t)\mathrm{d}\bar{K}^n_t = 0, \\ \bar{A}^n_t := n\int_0^t (\bar{Y}^n_s - U_s)^+ \,\mathrm{d}s, \quad t\in[0,T] \end{cases} \tag{6.3}$$

(Recall Theorem 5.1(i)). Then, for each $\beta\in(0,1)$,

$$\lim_{n\to\infty} \left(\|\bar{Y}^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} + \|\bar{Z}^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta} + \|\bar{K}^n_\cdot - K_\cdot\|_{\mathcal{S}^1} + \|\bar{A}^n_\cdot - A_\cdot\|_{\mathcal{S}^\beta}\right) = 0. \tag{6.4}$$

(iii) Let $(Y^n_\cdot, Z^n_\cdot)$ be the unique L^1 solution of BSDE $(\xi, g_n + \mathrm{d}V)$ with $g_n(t,y,z) := g(t,y,z) + n(y-L_t)^- - n(y-U_t)^+$ for each $n \geq 1$, i.e.,

$$\begin{cases} Y^n_t = \xi + \int_t^T g_n(s, Y^n_s, Z^n_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s - \int_t^T Z^n_s \cdot \mathrm{d}B_s, \quad t \in [0,T], \\ K^n_t := n\int_0^t (Y^n_s - L_s)^- \,\mathrm{d}s \ \text{ and } \ A^n_t := n\int_0^t (Y^n_s - U_s)^+ \,\mathrm{d}s, \quad t \in [0,T] \end{cases} \tag{6.5}$$

(Recall Theorem 4.2). Then, for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} (\|Y^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta} + \|(K^n_\cdot - A^n_\cdot) - (K_\cdot - A_\cdot)\|_{\mathcal{S}^\beta}) = 0. \tag{6.6}$$

Proof. We assume that $V_\cdot \in \mathcal{V}^1$, the generator g satisfies (H1) and (H2), and (H3)(i) holds true for $L_\cdot$, $U_\cdot$ and ξ. If DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ admits an L^1 solution $(Y_\cdot, Z_\cdot, K_\cdot, A_\cdot)$, then from Lemma 2.6 we know that $g(\cdot, Y_\cdot, Z_\cdot) \in \mathcal{H}^1$ and $g(\cdot, Y_\cdot, 0) \in \mathcal{H}^1$. Thus, (H3)(ii) is satisfied with

$$(C_\cdot, H_\cdot) := (-\int_0^\cdot g(s, Y_s, Z_s)\mathrm{d}s - V_\cdot - K_\cdot + A_\cdot,\ Z_\cdot)$$

and $X_\cdot := Y_\cdot$. The necessity is proved.

We further assume that (H3)(ii) holds. The uniqueness of the L^1 solution of DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ follows from Proposition 3.5. In what follows, it follows from (H3)(ii) that there exists two processes $(C_\cdot, H_\cdot) \in \mathcal{V}^1 \times \mathrm{M}^\beta$ for each $\beta \in (0,1)$ such that

$$X_t = X_T - \int_t^T \mathrm{d}C_s - \int_t^T H_s \cdot \mathrm{d}B_s, \quad t \in [0,T] \tag{6.7}$$

belongs to the class (D), $g(\cdot, X_\cdot, 0) \in \mathcal{H}^1$ and $L_t \leq X_t \leq U_t$ for each $t \in [0,T]$. And, by (H2)(ii) together with Hölder's inequality we know that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $|g(\cdot, X_\cdot, H_\cdot)| \leq |g(\cdot, X_\cdot, 0)| + \gamma(f_\cdot + |X_\cdot| + H_\cdot)^\alpha \ \in \mathcal{H}^1$, and then

$$\check{K}_\cdot := \int_0^\cdot g^-(s, X_s, H_s)\mathrm{d}s + \int_0^\cdot \mathrm{d}C^{0,-}_s + \int_0^\cdot \mathrm{d}V^{0,-}_s \in \mathcal{V}^{+,1}$$

and

$$\check{A}_\cdot := \int_0^\cdot g^+(s, X_s, H_s)\mathrm{d}s + \int_0^\cdot \mathrm{d}C^{0,+}_s + \int_0^\cdot \mathrm{d}V^{0,+}_s \in \mathcal{V}^{+,1},$$

where $V_\cdot - V_0 = V^{0,+}_\cdot - V^{0,-}_\cdot$ and $C_\cdot - C_0 = C^{0,+}_\cdot - C^{0,-}_\cdot$ with $V^{0,+}_\cdot, V^{0,-}_\cdot, C^{0,+}_\cdot, C^{0,-}_\cdot \in \mathcal{V}^{+,1}$. Thus, the equation (6.7) can be rewritten in the form

$$X_t = X_T + \int_t^T g(s, X_s, H_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s + \int_t^T \mathrm{d}\check{K}_s - \int_t^T \mathrm{d}\check{A}_s - \int_t^T H_s \cdot \mathrm{d}B_s, \quad t \in [0,T].$$

Furthermore, in view of (H1) and (H2), by Theorem 4.2 we can let $(\underline{X}_\cdot, \underline{Z}_\cdot)$ be the unique L^1 solution of the following BSDE

$$\underline{X}_t = X_T \wedge \xi + \int_t^T g(s, \underline{X}_s, \underline{Z}_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s - \int_t^T \mathrm{d}\check{A}_s - \int_t^T \underline{Z}_s \cdot \mathrm{d}B_s, \quad t \in [0,T]$$

and $(\bar{X}_\cdot, \bar{Z}_\cdot)$ be the unique L^1 solution of the BSDE

$$\bar{X}_t = X_T \vee \xi + \int_t^T g(s, \bar{X}_s, \bar{Z}_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s + \int_t^T \mathrm{d}\check{K}_s - \int_t^T \bar{Z}_s \cdot \mathrm{d}B_s, \quad t \in [0,T].$$

It follows from Corollary 3.7 that $\underline{X}_\cdot \leq X_\cdot \leq \bar{X}_\cdot$. And, for each $n \geq 1$, by Theorem 4.2 again we can let $(\dot{Y}^n_\cdot, \dot{Z}^n_\cdot)$ and $(\ddot{Y}^n_\cdot, \ddot{Z}^n_\cdot)$ be respectively the unique L^1 solution of the following BSDEs:

$$\dot{Y}^n_t = X_T \wedge \xi + \int_t^T g(s, \dot{Y}^n_s, \dot{Z}^n_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s + n\int_t^T (\dot{Y}^n_s - L_s)^- \mathrm{d}s - \int_t^T \mathrm{d}\check{A}_s - \int_t^T \dot{Z}^n_s \cdot \mathrm{d}B_s$$

and

$$\ddot{Y}^n_t = X_T \vee \xi + \int_t^T g(s, \ddot{Y}^n_s, \ddot{Z}^n_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s + \int_t^T \mathrm{d}\check{K}_s - n\int_t^T (\ddot{Y}^n_s - U_s)^+ \mathrm{d}s - \int_t^T \ddot{Z}^n_s \cdot \mathrm{d}B_s$$

with

$$\dot{K}^n_t := n\int_0^t (\dot{Y}^n_s - L_s)^- \mathrm{d}s \ \text{ and } \ \ddot{A}^n_t := n\int_0^t (\ddot{Y}^n_s - U_s)^+ \mathrm{d}s, \ \ t \in [0,T].$$

In view of $L_\cdot \leq X_\cdot \leq U_\cdot$, it follows from Corollary 3.7 that for each $n \geq 1$,

$$\underline{X}_\cdot \leq \dot{Y}^n_\cdot \leq X_\cdot \leq U_\cdot \ \text{ and } \ L_\cdot \leq X_\cdot \leq \ddot{Y}^n_\cdot \leq \bar{X}_\cdot. \tag{6.8}$$

Note that (H3L) holds true for $L_\cdot$, $X_T \wedge \xi$ and $X_\cdot$, and (H3U) holds true for $U_\cdot$, $X_T \vee \xi$ and $X_\cdot$. In view of (H1) and (H2), it follows from Theorem 5.1 together with Proposition 3.2 that for each $\beta \in (0,1)$,

$$\sup_{n\geq 1}(\mathbb{E}[|\dot{K}^n_T|^\beta] + \mathbb{E}[|\ddot{A}^n_T|^\beta]) < +\infty, \tag{6.9}$$

for a subsequence $\{n_j\}$ of $\{n\}$,

$$\lim_{j\to\infty} \dot{K}^{n_j}_T = \dot{K}_T \in \mathbb{L}^1(\mathcal{F}_T) \ \text{ and } \ \lim_{j\to\infty} \ddot{A}^{n_j}_T = \ddot{A}_T \in \mathbb{L}^1(\mathcal{F}_T), \tag{6.10}$$

and for a process $\tilde{Y}_\cdot \in \mathcal{S}$ and each $(\mathcal{F}_t)$-stopping time τ valued in $[0,T]$,

$$\sup_{n\geq 1}(\mathbb{E}[|\dot{K}^n_\tau|^2] + \mathbb{E}[|\ddot{A}^n_\tau|^2]) \leq \mathbb{E}[|\tilde{Y}_\tau|^2]. \tag{6.11}$$

In the sequel, let $(\underline{Y}^n_\cdot, \underline{Z}^n_\cdot, \underline{A}^n_\cdot)$, $(\bar{Y}^n_\cdot, \bar{Z}^n_\cdot, \bar{K}^n_\cdot)$ and $(Y^n_\cdot, Z^n_\cdot)$ be respectively defined in (i), (ii) and (iii) of Theorem 6.1 for each $n \geq 1$. Firstly, in view of (6.8), it follows from Corollary 3.7 and Corollary 5.2 that for each $n \geq 1$,

$$\underline{Y}^1_\cdot \leq \underline{Y}^n_\cdot \leq \underline{Y}^{n+1}_\cdot \leq \bar{X}_\cdot, \ \ \mathrm{d}\underline{A}^n \leq \mathrm{d}\underline{A}^{n+1} \tag{6.12}$$

and

$$\underline{X}_\cdot \leq \bar{Y}^{n+1}_\cdot \leq \bar{Y}^n_\cdot \leq \bar{Y}^1_\cdot, \ \ \mathrm{d}\bar{K}^n \leq \mathrm{d}\bar{K}^{n+1}. \tag{6.13}$$

It then follows from Lemma 2.5 that for each $n \geq 1$, in view of (H1) and (H2),

$$|g(\cdot, \underline{Y}^n_\cdot, \underline{Z}^n_\cdot)| \leq |g(\cdot, \underline{Y}^1_\cdot, 0)| + |g(\cdot, \bar{X}_\cdot, 0)| + (\gamma + A)(|\underline{Y}^1_\cdot| + |\bar{X}_\cdot|) + \gamma(1 + f_\cdot) + A + \gamma|\underline{Z}^n_\cdot|^\alpha \tag{6.14}$$

and

$$|g(\cdot, \bar{Y}^n_\cdot, \bar{Z}^n_\cdot)| \leq |g(\cdot, \underline{X}_\cdot, 0)| + |g(\cdot, \bar{Y}^1_\cdot, 0)| + (\gamma + A)(|\underline{X}_\cdot| + |\bar{Y}^1_\cdot|) + \gamma(1 + f_\cdot) + A + \gamma|\bar{Z}^n_\cdot|^\alpha \tag{6.15}$$

with, by Lemma 2.6,

$$g(\cdot, \underline{Y}^1_\cdot, 0) \in \mathcal{H}^1, \ \ g(\cdot, \bar{X}_\cdot, 0) \in \mathcal{H}^1, \ \ g(\cdot, \underline{X}_\cdot, 0) \in \mathcal{H}^1 \ \text{ and } \ g(\cdot, \bar{Y}^1_\cdot, 0) \in \mathcal{H}^1. \tag{6.16}$$

And, in view of (6.8), by Proposition 3.5 with Remark 3.6 we deduce that for each $n \geq 1$,

$$\dot{Y}^n_\cdot \leq \underline{Y}^n_\cdot \ \text{ and } \ \bar{Y}^n_\cdot \leq \ddot{Y}^n_\cdot,$$

which means that

$$\underline{K}^n_\cdot = n\int_0^\cdot (\underline{Y}^n_s - L_s)^- \mathrm{d}s \le n\int_0^\cdot (\dot{Y}^n_s - L_s)^- \mathrm{d}s = \dot{K}^n_\cdot \tag{6.17}$$

and

$$\bar{A}^n_\cdot = n\int_0^\cdot (\bar{Y}^n_s - U_s)^+ \mathrm{d}s \le n\int_0^\cdot (\ddot{Y}^n_s - U_s)^+ \mathrm{d}s = \ddot{A}^n_\cdot. \tag{6.18}$$

Thus, in view of (6.12)-(6.18) together with (6.9)-(6.11), all conditions in Proposition 3.1 are satisfied, and it follows that there exists an L^1 solution $(Y_\cdot, Z_\cdot, K_\cdot, A_\cdot)$, indeed a unique L^1 solution, of DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ such that, for each $\beta \in (0,1)$,

$$\begin{aligned}\lim_{n\to\infty} &\left(\|\underline{Y}^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} + \|\underline{Z}^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta} + \|\underline{A}^n_\cdot - A_\cdot\|_{\mathcal{S}^1}\right.\\ &\left.+\|\bar{Y}^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} + \|\bar{Z}^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta} + \|\bar{K}^n_\cdot - K_\cdot\|_{\mathcal{S}^1}\right) = 0,\end{aligned}$$

and there exists a subsequence $\{\underline{K}^{n_j}_\cdot\}$ (resp. $\{\bar{A}^{n_j}_\cdot\}$) of $\{\underline{K}^n_\cdot\}$ (resp. $\{\bar{A}^n_\cdot\}$) such that

$$\lim_{j\to\infty} \sup_{t\in[0,T]} \left(|\underline{K}^{n_j}_t - K_t| + |\bar{A}^{n_j}_t - A_t|\right) = 0.$$

Furthermore, in the same way as in the proof of Theorem 5.1 we can prove that for each $\beta \in (0,1)$,

$$\begin{aligned}\lim_{n\to\infty} &\left(\left\|\int_0^\cdot g(s, \underline{Y}^n_s, \underline{Z}^n_s)\mathrm{d}s - \int_0^\cdot g(s, Y_s, Z_s)\mathrm{d}s\right\|_{\mathcal{S}^\beta}\right.\\ &\left.+\left\|\int_0^\cdot g(s, \bar{Y}^n_s, \bar{Z}^n_s)\mathrm{d}s - \int_0^\cdot g(s, Y_s, Z_s)\mathrm{d}s\right\|_{\mathcal{S}^\beta}\right) = 0.\end{aligned} \tag{6.19}$$

Thus, (6.2) and (6.4) follow immediately.

Finally, in view of the fact that $\underline{Y}^n_\cdot \le U_\cdot$ and $L_\cdot \le \bar{Y}^n_\cdot$ for each $n \ge 1$, it follows from Corollary 3.7 that for each $n \ge 1$,

$$\underline{Y}^1_\cdot \le \underline{Y}^n_\cdot \le Y^n_\cdot \le \bar{Y}^n_\cdot \le \bar{Y}^1_\cdot,$$

which means that, in view of (6.17) and (6.18),

$$K^n_\cdot = n\int_0^\cdot (Y^n_s - L_s)^- \mathrm{d}s \le n\int_0^\cdot (\underline{Y}^n_s - L_s)^- \mathrm{d}s = \underline{K}^n_\cdot \le \dot{K}^n_\cdot \tag{6.20}$$

and

$$A^n_\cdot = n\int_0^\cdot (Y^n_s - U_s)^+ \mathrm{d}s \le n\int_0^\cdot (\bar{Y}^n_s - U_s)^+ \mathrm{d}s = \bar{A}^n_\cdot \le \ddot{A}^n_\cdot. \tag{6.21}$$

Thus, by (6.2) and (6.4) we know that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} \|Y^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} = 0. \tag{6.22}$$

Now, we show the convergence of the sequence $\{Z^n_\cdot\}$. Indeed, for each $n \ge 1$, observe that

$$\begin{aligned}(\bar{Y}_\cdot, \bar{Z}_\cdot, \bar{V}_\cdot) \quad := \quad &(Y^n_\cdot - Y_\cdot, Z^n_\cdot - Z_\cdot,\\ &\int_0^\cdot \left(g(s, Y^n_s, Z^n_s) - g(s, Y_s, Z_s)\right)\mathrm{d}s + (K^n_\cdot - K_\cdot) - (A^n_\cdot - A_\cdot))\end{aligned}$$

satisfies equation (2.1). It follows from (i) of Lemma 2.4 with $t=0$ and $\tau=T$ that there exists a constant $C'>0$ such that for each $n\geq 1$ and $\beta\in(0,1)$,

$$\begin{aligned}\|Z^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta} \quad \leq \quad & C'\mathbb{E}\left[\sup_{t\in[0,T]}|Y^n_t-Y_t|^\beta+\sup_{t\in[0,T]}\left[\left(\int_t^T (Y^n_s-Y_s)\,(\mathrm{d}K^n_s-\mathrm{d}K_s)\right)^+\right]^{\frac{\beta}{2}}\right]\\ & +C'\mathbb{E}\left[\sup_{t\in[0,T]}\left[\left(\int_t^T (Y^n_s-Y_s)\,(\mathrm{d}A_s-\mathrm{d}A^n_s)\right)^+\right]^{\frac{\beta}{2}}\right]\\ & +C'\mathbb{E}\left[\left(\int_0^T |Y^n_s-Y_s|\,|g(s,Y^n_s,Z^n_s)-g(s,Y_s,Z_s)|\,\mathrm{d}s\right)^{\frac{\beta}{2}}\right].\end{aligned}$$

It then follows from the fact of $L_\cdot\leq Y_\cdot\leq U_\cdot$ and the definitions of $K^n_\cdot$ and $A^n_\cdot$ as well as Hölder's inequality that

$$\begin{aligned}& \|Z^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta}\\ \leq \quad & C'\|Y^n_\cdot-Y_\cdot\|_{\mathcal{S}^\beta}+C'\|Y^n_\cdot-Y_\cdot\|^{\frac12}_{\mathcal{S}^\beta}\cdot\left(\left(\mathbb{E}[|K_T|^\beta]\right)^{\frac12}+\left(\mathbb{E}[|A_T|^\beta]\right)^{\frac12}\right)\\ & +C'\|Y^n_\cdot-Y_\cdot\|^{\frac12}_{\mathcal{S}^\beta}\cdot\left(\mathbb{E}\left[\left(\int_0^T (|g(t,Y^n_t,Z^n_t)|+|g(t,Y_t,Z_t)|)\,\mathrm{d}t\right)^\beta\right]\right)^{\frac12}.\end{aligned} \tag{6.23}$$

Thus, in view of (H1) and (H2) of g, it follows from (6.22), (6.23), (6.20), (6.21) and (6.9) together with Lemma 2.7 that for each $\beta\in(0,1)$,

$$\lim_{n\to\infty}\|Z^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta}=0. \tag{6.24}$$

Furthermore, by (6.22) and (6.24), a similar argument to (6.19) yields that for each $\beta\in(0,1)$,

$$\lim_{n\to\infty}\left\|\int_0^\cdot g(s,Y^n_s,Z^n_s)\mathrm{d}s-\int_0^\cdot g(s,Y_s,Z_s)\mathrm{d}s\right\|_{\mathcal{S}^\beta}=0. \tag{6.25}$$

Finally, (6.6) follows from (6.22), (6.24) and (6.25). The proof of Theorem 6.1 is then complete. □

By virtue of (i) of Theorem 6.1, Corollary 3.7 and (ii) of Corollary 5.2, a similar argument to that in Corollary 5.2 yields the following corollary.

Corollary 6.2. Assume that $\xi^1,\xi^2\in\mathbb{L}^1(\mathcal{F}_T)$ with $\xi^1\leq\xi^2$, $V^1_\cdot,V^2_\cdot\in\mathcal{V}^1$ with $\mathrm{d}V^1\leq\mathrm{d}V^2$, and both generators g^1 and g^2 satisfy (H1) and (H2) with

$$\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.,\ \forall\,(y,z)\in\mathbb{R}\times\mathbb{R}^d,\ \ g^1(t,y,z)\leq g^2(t,y,z).$$

For $i=1,2$, let (H3) hold for ξ^i, $L^i_\cdot$, $U^i_\cdot$ and $X^i_\cdot$ associated with g^i, and $(Y^i_\cdot,Z^i_\cdot,K^i_\cdot,A^i_\cdot)$ be the unique L^1 solution of DRBSDE $(\xi^i,g^i+\mathrm{d}V^i,L^i,U^i)$ (recall Theorem 6.1). If $L^1_\cdot=L^2_\cdot$ and $U^1_\cdot=U^2_\cdot$, then

$$dK^1\geq\mathrm{d}K^2\ \text{ and }\ dA^1\leq\mathrm{d}A^2.$$

Theorem 6.3. Let $V_\cdot\in\mathcal{V}^1$, g^1 satisfy assumptions (H1) and (H2'), g^2 satisfy assumption (AA), the generator $g:=g^1+g^2$, and assumption (H3)(i) hold true for $L_\cdot$, $U_\cdot$ and ξ. Then, DRBSDE $(\xi,g+\mathrm{d}V,L,U)$ admits an L^1 solution iff (H3)(ii) is satisfied for $X_\cdot$, $L_\cdot$, $U_\cdot$ and g (or g^1). Moreover, we assume that (H3)(ii) holds also true for $X_\cdot$, $L_\cdot$, $U_\cdot$ and g (or g^1).

(i) For each $n \geq 1$, let $(\underline{Y}^n_\cdot, \underline{Z}^n_\cdot, \underline{A}^n_\cdot)$ be the minimal (resp. maximal) L^1 solution of $\bar{R}$BSDE $(\xi, \bar{g}_n + \mathrm{d}V, U)$ with $\bar{g}_n(t,y,z) := g(t,y,z) + n(y - L_t)^-$ and $\underline{K}^n_\cdot$, i.e., (6.1), (recall Theorem 5.4(ii)). Then, DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ admits a minimal L^1 solution (resp. an L^1 solution) $(\underline{Y}_\cdot, \underline{Z}_\cdot, \underline{K}_\cdot, \underline{A}_\cdot)$ such that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} \left(\|\underline{Y}^n_\cdot - \underline{Y}_\cdot\|_{\mathcal{S}^\beta} + \|\underline{Z}^n_\cdot - \underline{Z}_\cdot\|_{\mathrm{M}^\beta} + \|\underline{A}^n_\cdot - \underline{A}_\cdot\|_{\mathcal{S}^1}\right) = 0,$$

and there exists a subsequence $\{\underline{K}^{n_j}_\cdot\}$ of $\{\underline{K}^n_\cdot\}$ such that

$$\lim_{j\to\infty} \sup_{t\in[0,T]} |\underline{K}^{n_j}_t - \underline{K}_t| = 0.$$

(ii) For each $n \geq 1$, let $(\bar{Y}^n_\cdot, \bar{Z}^n_\cdot, \bar{K}^n_\cdot)$ be the maximal (resp. minimal) L^1 solution of $\underline{R}$BSDE $(\xi, \underline{g}_n + \mathrm{d}V, L)$ with $\underline{g}_n(t,y,z) := g(t,y,z) - n(y - U_t)^+$ and $\bar{A}^n_\cdot$, i.e., (6.3), (recall Theorem 5.4(i)). Then, DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ admits a maximal L^1 solution (resp. an L^1 solution) $(\bar{Y}_\cdot, \bar{Z}_\cdot, \bar{K}_\cdot, \bar{A}_\cdot)$ such that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} \left(\|\bar{Y}^n_\cdot - \bar{Y}_\cdot\|_{\mathcal{S}^\beta} + \|\bar{Z}^n_\cdot - \bar{Z}_\cdot\|_{\mathrm{M}^\beta} + \|\bar{K}^n_\cdot - \bar{K}_\cdot\|_{\mathcal{S}^1}\right) = 0,$$

and there exists a subsequence $\{\bar{A}^{n_j}_\cdot\}$ of $\{\bar{A}^n_\cdot\}$ such that

$$\lim_{j\to\infty} \sup_{t\in[0,T]} |\bar{A}^{n_j}_t - \bar{A}_t| = 0.$$

Proof. We only prove (i), and (ii) can be proved in the same way. In view of Theorem 5.4, Corollary 5.5, Lemma 2.5, Lemma 2.6 and Proposition 3.1, by a similar argument to that in the proof of Theorem 6.1 we can prove that all conclusions in (i) of Theorem 6.3 hold true except for the minimal property of the L^1 solution $(\underline{Y}_\cdot, \underline{Z}_\cdot, \underline{K}_\cdot, \underline{A}_\cdot)$ of DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ when $(\underline{Y}^n_\cdot, \underline{Z}^n_\cdot, \underline{A}^n_\cdot)$ is the minimal L^1 solution of $\bar{R}$BSDE $(\xi, \bar{g}_n + \mathrm{d}V, U)$ for each $n \geq 1$. Now, we will show this property.

Indeed, for any L^1 solution $(Y_\cdot, Z_\cdot, K_\cdot, A_\cdot)$ of DRBSDE $(\xi, g + \mathrm{d}V, L, U)$, it is not hard to check that $(Y_\cdot, Z_\cdot, A_\cdot)$ is an L^1 solution of $\bar{R}$BSDE $(\xi, \bar{g}_n + \mathrm{d}\bar{V}, U)$ with $\bar{V}_\cdot := V_\cdot + K_\cdot$ for each $n \geq 1$. Thus, in view of the assumption that $(\underline{Y}^n_\cdot, \underline{Z}^n_\cdot, \underline{A}^n_\cdot)$ is the minimal L^1 solution of $\bar{R}$BSDE $(\xi, \bar{g}_n + \mathrm{d}V, U)$ for each $n \geq 1$, Corollary 5.5 yields that for each $n \geq 1$,

$$\underline{Y}^n_t \leq Y_t, \quad t \in [0,T].$$

Furthermore, since $\lim_{n\to\infty} \|\underline{Y}^n_\cdot - \underline{Y}_\cdot\|_{\mathcal{S}^\beta} = 0$ for each $\beta \in (0,1)$, we know that

$$\underline{Y}_t \leq Y_t, \quad t \in [0,T],$$

which is the desired result. □

In view of Theorem 6.1, Corollary 6.2 and Proposition 3.3, a similar argument to that in Theorem 5.4 yields the following convergence result, whose proof is omitted.

Theorem 6.4. Let $V_\cdot \in \mathcal{V}^1$, g^1 satisfy assumptions (H1) and (H2'), g^2 satisfy assumption (AA), the generator $g := g^1 + g^2$, and (H3) holds true for $L_\cdot$, $U_\cdot$, ξ, $X_\cdot$ and g (or g^1). Then DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ admits a minimal (resp. maximal) L^1 solution $(Y_\cdot, Z_\cdot, K_\cdot, A_\cdot)$ such that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} \left(\|Y^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} + \|Z^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta} + \|K^n_\cdot - K_\cdot\|_{\mathcal{S}^1} + \|A^n_\cdot - A_\cdot\|_{\mathcal{S}^1}\right) = 0,$$

where for each $n \geq 1$, $(Y^n_\cdot, Z^n_\cdot, K^n_\cdot, A^n_\cdot)$ is the unique L^1 solution of DRBSDE $(\xi, g_n + \mathrm{d}V, L, U)$ with a generator g_n satisfying (H1), (H2) and (H3) (recall Theorem 6.1).

By Corollary 3.7, Corollary 6.2 and the proof of Theorem 6.4, it is not hard to verify the following comparison result for the minimal (resp. maximal) L^1 solutions of DRBSDEs.

Corollary 6.5. Assume that $\xi^1, \xi^2 \in \mathbb{L}^1(\mathcal{F}_T)$ with $\xi^1 \le \xi^2$, $V^1_\cdot, V^2_\cdot \in \mathcal{V}^1$ with $\mathrm{d}V^1 \le \mathrm{d}V^2$, $g^{1,1}$ and $g^{2,1}$ satisfy (H1) and (H2'), $g^{1,2}$ and $g^{2,2}$ satisfy (AA), $g^1 := g^{1,1} + g^{1,2}$ and $g^2 := g^{2,1} + g^{2,2}$ with

$$\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e., \ \forall\, (y,z) \in \mathbb{R} \times \mathbb{R}^d, \ \ g^{1,1}(t,y,z) \le g^{2,1}(t,y,z) \ \text{ and } \ g^{1,2}(t,y,z) \le g^{2,2}(t,y,z).$$

For $i = 1, 2$, let (H3) hold for $L^i_\cdot$, $U^i_\cdot$, ξ^i and $X^i_\cdot$ associated with g^i (or $g^{i,1}$), and $(Y^i_\cdot, Z^i_\cdot, K^i_\cdot, A^i_\cdot)$ be the minimal (resp. maximal) L^1 solution of DRBSDE $(\xi^i, g^i + \mathrm{d}V^i, L^i, U^i)$ (recall Theorem 6.4). If $L^1_\cdot \le L^2_\cdot$ and $U^1_\cdot \le U^2_\cdot$, then $Y^1_t \le Y^2_t$ for each $t \in [0,T]$, and if $L^1_\cdot = L^2_\cdot$ and $U^1_\cdot = U^2_\cdot$, then

$$dK^1 \ge \mathrm{d}K^2 \ \text{ and } \ dA^1 \le \mathrm{d}A^2.$$

7 Examples and remarks

We first introduce several examples which the results of this paper can be applied to. Note that to the best of our knowledge, all conclusions of these examples can not be obtained by any existing results.

Example 7.1. Let the generator $g(\omega,t,y,z) = |y| + \sqrt{|z|}$. Clearly, this g satisfies the uniformly Lipschitz condition in y and the α-Hölder continuity condition in z, and then, in view of Remark 2.2, both assumptions (H1) and (H2) hold true for this g. Then, we have

1) It follows from Theorem 4.2 that for each $\xi \in \mathbb{L}^1(\mathcal{F}_T)$ and $V_\cdot \in \mathcal{V}^1$, BSDE $(\xi, g + \mathrm{d}V)$ admits a unique L^1 solution.
2) It follows from Theorem 5.1 that if (H3L)(i) (resp. (H3U)(i)) is satisfied, $L^+_\cdot \in \mathcal{S}^1$ (resp. $U^-_\cdot \in \mathcal{S}^1$) and $V_\cdot \in \mathcal{V}^1$, then $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ (resp. $\bar{R}$BSDE $(\xi, g + \mathrm{d}V, U)$) admits a unique L^1 solution.
3) It follows from Theorem 6.1 that if (H3) is satisfied and $V_\cdot \in \mathcal{V}^1$, then DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ admits a unique L^1 solution.

Example 7.2. Let the generator g be defined as follows:

$$g(\omega,t,y,z) = h(|y|) + e^{-y|B_t(\omega)|^2} + (e^{-y} \wedge 1) \cdot (\sqrt{|z|} + \sqrt[3]{|z|}) + \frac{1}{\sqrt{t}} 1_{t>0}$$

where, with $\delta > 0$ small enough,

$$h(x) = \begin{cases} -x \ln x & , \quad 0 < x \le \delta; \\ h'(\delta-)(x-\delta) + h(\delta) & , \quad x > \delta; \\ 0 & , \quad \text{other cases.} \end{cases}$$

It is not very hard to verify that this g satisfies assumption (H1) with $\rho(x) = h(x)$, $g(t,0,0) = \frac{1}{\sqrt{t}} 1_{t>0} + 1$, and $\psi_t(\omega, r) = h(\delta) + h'(\delta-)r + e^{r|B_t(\omega)|^2} + 1$, and assumption (H2) with $\phi(x) = \sqrt{|x|} + \sqrt[3]{|x|}$, $\gamma = 2$, $f_t(\omega) \equiv 1$ and $\alpha = 1/2$. We have

1) It follows from Theorem 4.2 that for each $\xi \in \mathbb{L}^1(\mathcal{F}_T)$ and $V_\cdot \in \mathcal{V}^1$, BSDE $(\xi, g + \mathrm{d}V)$ admits a unique L^1 solution.
2) It follows from Theorem 5.1 that if $V_\cdot \in \mathcal{V}^1$ and (H3L) (resp. (H3U)) is satisfied, then $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ (resp. $\bar{R}$BSDE $(\xi, g + \mathrm{d}V, U)$) admits a unique L^1 solution.
3) It follows from Theorem 6.1 that if $V_\cdot \in \mathcal{V}^1$ and (H3) is satisfied, then DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ admits a unique L^1 solution.

Example 7.3. Let the generator $g(\omega,t,y,z)=e^{-y}+\sqrt[3]{|z|\sin|z|}$. It is easy to see that this g satisfies assumptions (H1) and (H2') with $\rho(x)=x$ in view of the fact that it is decreasing in y. However, it can be checked that this generator does not satisfy the assumption (H2)(i). We have

1) It follows from Corollary 4.5 that for each $\xi\in\mathbb{L}^1(\mathcal{F}_T)$ and $V_\cdot\in\mathcal{V}^1$, BSDE $(\xi,g+\mathrm{d}V)$ admits a maximal and a minimal L^1 solutions.
2) It follows from Theorem 5.4 that if $V_\cdot\in\mathcal{V}^1$ and (H3L) (resp. (H3U)) is satisfied, then $\underline{R}$BSDE $(\xi,g+\mathrm{d}V,L)$ (resp. $\bar{R}$BSDE $(\xi,g+\mathrm{d}V,U)$) admits a maximal and a minimal L^1 solutions.
3) It follows from Theorem 6.4 that if $V_\cdot\in\mathcal{V}^1$ and (H3) is satisfied, then DRBSDE $(\xi,g+\mathrm{d}V,L,U)$ admits a maximal and a minimal L^1 solutions.

Example 7.4. Let the generator $g:=g^1+g^2$ with

$$g^1(\omega,t,y,z)=h(|y|)-y^3e^{|B_t(\omega)|^4}-e^y\sin^2|z|+\sqrt{|z|}\cos|z|+\frac{1}{\sqrt[3]{t}}1_{t>0}$$

and

$$g^2(\omega,t,y,z)=\sqrt[3]{|y|}+y\cos y+\sqrt[4]{|y|\cdot|z|}+|B_t(\omega)|,$$

where $h(\cdot)$ is defined in Example 7.2. It is not very hard to verify that g^1 satisfies (H1)(i) with $\rho(x)=h(x)$, (H2')(i) and (HH) with $f_t(\omega)=1+\frac{1}{\sqrt[3]{t}}1_{t>0}+h(\delta)$, $\varphi_t(\omega,r)=h'(\delta-)r+r^3e^{|B_t(\omega)|^4}+e^r-1$, $\lambda=1$ and $\alpha=1/2$, and that g^2 satisfies (AA) with $\tilde{f}_t(\omega)=|B_t(\omega)|+2$, $\tilde{\mu}=3$, $\tilde{\lambda}=1$ and $\tilde{\alpha}=1/2$. We have

1) It follows from Theorem 4.4 that for each $\xi\in\mathbb{L}^1(\mathcal{F}_T)$ and $V_\cdot\in\mathcal{V}^1$, BSDE $(\xi,g+\mathrm{d}V)$ admits a maximal and a minimal L^1 solutions.
2) It follows from Corollary 5.6 that if $V_\cdot\in\mathcal{V}^1$, (H3L)(i) (resp. (H3U)(i)) is satisfied and $L_\cdot^+\in\mathcal{S}^1$ (resp. $U_\cdot^-\in\mathcal{S}^1$), then $\underline{R}$BSDE $(\xi,g^2+\mathrm{d}V,L)$ (resp. $\bar{R}$BSDE $(\xi,g^2+\mathrm{d}V,U)$) admits a maximal and a minimal L^1 solutions.
3) It follows from Theorem 6.4 that if $V_\cdot\in\mathcal{V}^1$ and (H3) is satisfied for g^2, then DRBSDE $(\xi,g^2+\mathrm{d}V,L,U)$ admits a maximal and a minimal L^1 solutions.

We also note that this g^1 satisfies neither assumption (H2) nor assumption (H2')(ii).

Example 7.5. Let the generator $g:=g^1+g^2$ with

$$g^1(\omega,t,y,z)=\bar{h}(|y|)-e^{y|B_t(\omega)|^3}+(e^{-y}\wedge 1)\cdot\sqrt{|z|}\cos|z|+\frac{1}{\sqrt[4]{t}}1_{t>0}$$

and

$$g^2(\omega,t,y,z)=y\cos|z|+\sqrt[3]{|z|}\sin y+\sqrt{1+|y|+|z|}+|B_t(\omega)|^2,$$

where, with $\delta>0$ small enough,

$$\bar{h}(x)=\begin{cases} x|\ln x|\ln|\ln x| & , \quad 0<x\le\delta;\\ \bar{h}'(\delta-)(x-\delta)+\bar{h}(\delta) & , \quad x>\delta;\\ 0 & , \quad \text{other cases.}\end{cases}$$

It is not very hard to verify that g^1 satisfies assumption (H1) with $\rho(x)=\bar{h}(x)$, $g(t,0,0)=\frac{1}{\sqrt[4]{t}}1_{t>0}-1$, and $\psi_t(\omega,r)=\bar{h}(\delta)+\bar{h}'(\delta-)r+e^{r|B_t(\omega)|^3}+1$, and assumption (H2') with $f_t(\omega)\equiv 0$, $\mu=0$, $\lambda=1$ and $\alpha=1/2$, and that g^2 satisfies (AA) with $\tilde{f}_t(\omega)=|B_t(\omega)|^2+2$, $\tilde{\mu}=2$, $\tilde{\lambda}=2$ and $\tilde{\alpha}=1/2$. We have

1) It follows from Corollary 4.5 that for each $\xi\in\mathbb{L}^1(\mathcal{F}_T)$ and $V_\cdot\in\mathcal{V}^1$, BSDE $(\xi,g+\mathrm{d}V)$ admits a maximal and a minimal L^1 solutions.

2) It follows from Theorem 5.4 that if $V_\cdot \in \mathcal{V}^1$ and (H3L) (resp. (H3U)) is satisfied, then $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ (resp. $\bar{R}$BSDE $(\xi, g + \mathrm{d}V, U)$) admits a maximal and a minimal L^1 solutions.
3) It follows from Theorem 6.4 that if $V_\cdot \in \mathcal{V}^1$ and (H3) is satisfied, then DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ admits a maximal and a minimal L^1 solutions.
4) It follows from Corollary 5.6 that if $V_\cdot \in \mathcal{V}^1$, (H3L)(i) (resp. (H3U)(i)) is satisfied and $L_\cdot^+ \in \mathcal{S}^1$ (resp. $U_\cdot^- \in \mathcal{S}^1$), then $\underline{R}$BSDE $(\xi, g^2 + \mathrm{d}V, L)$ (resp. $\bar{R}$BSDE $(\xi, g^2 + \mathrm{d}V, U)$) admits a maximal and a minimal L^1 solutions.

We also note that this g^1 does not satisfy assumption (H2)(i).

Finally, we give the following remark to end this paper.

Remark 7.6. With respect to the work of this paper, we would like to mention the following things.

1) The basic assumptions (H1) and (H2) of the generator g used in this paper are strictly weaker than the corresponding assumptions used in Briand et al. [3], Klimsiak [38], Klimsiak [39], Rozkosz and Słomiński [55] and Bayraktar and Yao [2] for the L^1 solutions, where $\rho(x) = kx$ and $\phi(x) = kx$ for some constant $k \geq 0$. Furthermore, assumption (H2')(ii) is weaker than assumption (H2)(ii), and assumption (HH) is weaker than (H1)(ii)(iii) and (H2')(ii).
2) All of conditions (2.5), (2.6), (3.2), (3.13), (3.14) and (3.18) used respectively in Lemma 2.7, Lemma 2.8, Proposition 3.1, Proposition 3.2, Proposition 3.3 and Remark 3.4 are very general, which is strictly weaker than the usual linear/sub-linear growth condition of g in (y, z). Indeed, when these conditions are satisfied, the generator g can still have a general growth in (y, z), as can be seen in the proof of our main results in Section 4, Section 5 and Section 6.
3) The way by which the comparison theorem (Proposition 3.5) is used in Theorem 4.2 and Theorem 4.4 is interesting in its own right.
4) It is uncertain that the generator g used in Theorem 4.4, Corollary 4.5, Theorem 5.3, Theorem 5.4, Theorem 6.3 and Theorem 6.4 satisfies assumption (H1)(i), as can be seen in Example 7.4 and Example 7.5.
5) Generally speaking, under the assumptions of Theorem 5.3, we do not know whether the maximal L^1 solution of $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ (resp. the minimal L^1 solution of $\bar{R}$BSDE $(\xi, g + \mathrm{d}V, U)$) can be approximated by a sequence of L^1 solutions of BSDEs.
6) Generally speaking, under the assumptions of Theorem 6.3, we do not know whether the maximal (resp. minimal) L^1 solution of DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ can be approximated by a sequence of L^1 solutions of $\bar{R}$BSDEs with upper barrier $U_\cdot$ (resp. $\underline{R}$BSDEs with lower barrier $L_\cdot$). In particular, under the same assumptions we also do not know whether an L^1 solution of DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ can be approximated by a sequence of L^1 solutions of BSDEs in general.
7) The continuity condition of g^2 (resp. g) in (y, z) used in Theorem 4.4, Corollary 4.5, Theorem 5.3, Theorem 5.4, Theorem 6.3 and Theorem 6.4 (resp. Corollary 5.6) can be relaxed to the left-continuity and lower semi-continuity condition in case of the minimal L^1 solution and the right-continuity and upper semi-continuity condition in case of the maximal L^1 solution, with a similar argument as in Fan and Jiang [21], Fan [15] and Fan [18]. The same is in Corollary 4.6, Corollary 5.5 and Corollary 6.5.
8) Since the associated assumptions are more general, the results of this paper strengthen some known corresponding works with respect to the L^1 solutions obtained, for example, in Briand et al. [3], Briand and Hu [4], Fan and Liu [25], Fan [17], Klimsiak [38] and Rozkosz and Słomiński [55].

9) Under assumptions (H1)(i), (H2')(i), (HH) and (H3L) (resp. (H3U)), the existence of an L^1 solution for $\underline{R}$BSDE $(\xi, g + \mathrm{d}V, L)$ (resp. $\bar{R}$BSDE $(\xi, g + \mathrm{d}V, U)$) is still open. And, under assumptions (H1)(i), (H2')(i), (HH) and (H3), the existence of an L^1 solutions for DRBSDE $(\xi, g + \mathrm{d}V, L, U)$ is also open.

A Appendix

In this section, we will supply the details omitted in the proof procedures of Proposition 3.1 and Proposition 3.3.

Complementary of the details for the proof of Proposition 3.1

Now, we will detail the proof of steps 1–7 after the equality (3.8).

Step 1. We show that $Y_\cdot$ is a càdlàg process. Let us first fix a positive integer $k \geq 1$ arbitrarily. Note that $\bar{f}_\cdot \in \mathcal{H}^1$ and $\sup_{n\geq 1} \|Z_\cdot^n \mathbb{1}_{\cdot \leq \tau_k}\|_{\mathrm{M}^2} < +\infty$ by (3.6). It follows from (3.2) that there exists a subsequence $\{g(\cdot, Y_\cdot^{n_j}, Z_\cdot^{n_j})\mathbb{1}_{\cdot \leq \tau_k}\}_{j=1}^{\infty}$ of the sequence $\{g(\cdot, Y_\cdot^n, Z_\cdot^n)\mathbb{1}_{\cdot \leq \tau_k}\}_{n=1}^{\infty}$ which converges weakly to a process ${}^k h_\cdot$ in $\mathcal{H}^1$. Then, for every $(\mathcal{F}_t)$-stopping time τ valued in $[0, T]$, as $j \to \infty$, we have

$$\int_0^\tau \mathbb{1}_{s \leq \tau_k} g(s, Y_s^{n_j}, Z_s^{n_j}) \mathrm{d}s \ \to \ \int_0^\tau {}^k h_s \mathrm{d}s \ \text{ weakly in } \mathbb{L}^1(\mathcal{F}_T). \tag{A.1}$$

Furthermore, since

$$\sup_{n\geq 1} \mathbb{E}\left[\int_0^T |Z_t^n \mathbb{1}_{t\leq \tau_k}|^2 \mathrm{d}t\right] < +\infty,$$

it follows from Lemma 4.4 of Klimsiak [38] that there exists a process ${}^k Z_\cdot \in \mathrm{M}^2$ and a subsequence of the sequence $\{n_j\}_{j=1}^{\infty}$, still denoted by itself, such that for every $(\mathcal{F}_t)$-stopping time τ valued in $[0, T]$,

$$\int_0^\tau \mathbb{1}_{s\leq \tau_k} Z_s^{n_j} \cdot \mathrm{d}B_s \to \int_0^\tau {}^k Z_s \cdot \mathrm{d}B_s \ \text{ weakly in } \mathbb{L}^2(\mathcal{F}_T) \text{ and then in } \mathbb{L}^1(\mathcal{F}_T), \ \text{ as } j \to \infty. \tag{A.2}$$

In the sequel, we define

$${}^k K_t := Y_0 - Y_t - \int_0^t {}^k h_s \mathrm{d}s - \int_0^t \mathrm{d}V_s - \int_0^t \mathrm{d}A_s + \int_0^t {}^k Z_s \cdot \mathrm{d}B_s, \ \ t \in [0, T].$$

Then, for each $(\mathcal{F}_t)$-stopping time τ valued in $[0, T]$, in view of (3.8), (A.1), (A.2) and the fact that $Y_\tau^n \uparrow Y_\tau$ in $\mathbb{L}^1(\mathcal{F}_T)$, we can deduce that the sequence

$$K_{\tau\wedge\tau_k}^{n_j} = Y_0^{n_j} - Y_{\tau\wedge\tau_k}^{n_j} - \int_0^{\tau\wedge\tau_k} g(s, Y_s^{n_j}, Z_s^{n_j}) \mathrm{d}s - \int_0^{\tau\wedge\tau_k} \mathrm{d}V_s - \int_0^{\tau\wedge\tau_k} \mathrm{d}A_s^{n_j} + \int_0^{\tau\wedge\tau_k} Z_s^{n_j} \cdot \mathrm{d}B_s$$

converges weakly to ${}^k K_{\tau\wedge\tau_k}$ in $\mathbb{L}^1(\mathcal{F}_T)$ as $j \to \infty$. Thus, since $K_\cdot^n \in \mathcal{V}^+$ for each $n \geq 1$, we know that

$${}^k K_{\sigma_1\wedge\tau_k} \leq {}^k K_{\sigma_2\wedge\tau_k}$$

for any $(\mathcal{F}_t)$-stopping times $\sigma_1 \leq \sigma_2$ valued in $[0, T]$. Furthermore, in view of the definition of ${}^k K_\cdot$ together with the facts that $V_\cdot \in \mathcal{V}$, $A_\cdot \in \mathcal{V}^+$, $Y_\cdot^n \uparrow Y_\cdot$ and $Y_\cdot^n \in \mathcal{S}$ for each $n \geq 1$, it is not hard to check that ${}^k K_\cdot$ is an optional process with $\mathbb{P} - a.s.$ upper semi-continuous paths. Thus, Lemma A.3 in Bayraktar and Yao [2] yields that ${}^k K_{\cdot\wedge\tau_k}$ is a nondecreasing process, and then it has $\mathbb{P} - a.s.$ right lower semi-continuous paths. Hence, ${}^k K_{\cdot\wedge\tau_k}$ is càdlàg and so is $Y_{\cdot\wedge\tau_k}$ from the definition of ${}^k K_\cdot$. Finally, it follows from (3.5) that $Y_\cdot$ is also a càdlàg process.

Step 2. We show that $Y_t \geq L_t$ for each $t \in [0,T]$ and

$$\lim_{n\to\infty} \sup_{t\in[0,T]} (Y_t^n - L_t)^- = 0. \tag{A.3}$$

In fact, it follows from Fatou's lemma and the definition of $K_\cdot^n$ that for each $\beta \in (0,1)$,

$$\begin{aligned} 0 \leq \mathbb{E}\left[\left(\int_0^T (Y_t - L_t)^- \mathrm{d}t\right)^\beta\right] &\leq \liminf_{n\to\infty} \mathbb{E}\left[\left(\int_0^T (Y_t^n - L_t)^- \mathrm{d}t\right)^\beta\right] \\ &\leq \lim_{n\to\infty} \frac{\sup_{n\geq 1} \mathbb{E}[|K_T^n|^\beta]}{n^\beta} = 0. \end{aligned}$$

Since $Y_\cdot - L_\cdot$ is a càdlàg process, it follows that $(Y_t - L_t)^- = 0$ and hence $Y_t \geq L_t$ for each $t \in [0,T)$. Moreover, $Y_T = Y_T^n = \xi \geq L_T$. Hence

$$(Y_t^n - L_t)^- \downarrow 0$$

for each $t \in [0,T]$ and by Dini's theorem, (A.3) follows.

Step 3. We show the convergence of the sequence $\{Y_\cdot^n\}$ in the space of $\mathcal{S}^\beta$ for each $\beta \in (0,1)$. For each $n, m \geq 1$, observe that

$$\begin{aligned} (\bar{Y}_\cdot, \bar{Z}_\cdot, \bar{V}_\cdot) \;:=\; & (Y_\cdot^n - Y_\cdot^m, Z_\cdot^n - Z_\cdot^m, \\ & \int_0^\cdot (g(s, Y_s^n, Z_s^n) - g(s, Y_s^m, Z_s^m))\,\mathrm{d}s + (K_\cdot^n - K_\cdot^m) - (A_\cdot^n - A_\cdot^m)) \end{aligned} \tag{A.4}$$

satisfies equation (2.1). It then follows from (ii) of Lemma 2.4 with $p = 2$, $t = 0$ and $\tau = \tau_k$ that there exists a constant $C > 0$ such that for each $n, m, k \geq 1$,

$$\begin{aligned} & \mathbb{E}\left[\sup_{t\in[0,T]} |Y_{t\wedge\tau_k}^n - Y_{t\wedge\tau_k}^m|^2\right] \\ \leq\; & C\mathbb{E}\left[|Y_{\tau_k}^n - Y_{\tau_k}^m|^2 + \sup_{t\in[0,T]}\left(\int_{t\wedge\tau_k}^{\tau_k} (Y_s^n - Y_s^m)\,(\mathrm{d}K_s^n - \mathrm{d}K_s^m)\right)^+\right. \\ & \quad + \sup_{t\in[0,T]}\left(\int_{t\wedge\tau_k}^{\tau_k} (Y_s^n - Y_s^m)\,(\mathrm{d}A_s^m - \mathrm{d}A_s^n)\right)^+ \\ & \quad \left. + \int_0^{\tau_k} |Y_s^n - Y_s^m|\,|g(s, Y_s^n, Z_s^n) - g(s, Y_s^m, Z_s^m)|\,\mathrm{d}s\right]. \end{aligned} \tag{A.5}$$

Furthermore, by virtue of the definition of $K_\cdot^n$ and $A_\cdot^n$ with (3.1) we know that for each $t \in [0,T]$,

$$\begin{aligned} & \int_{t\wedge\tau_k}^{\tau_k} (Y_s^n - Y_s^m)\,(\mathrm{d}K_s^n - \mathrm{d}K_s^m) \\ =\; & \int_{t\wedge\tau_k}^{\tau_k} [(Y_s^n - L_s) - (Y_s^m - L_s)]\,\mathrm{d}K_s^n - \int_{t\wedge\tau_k}^{\tau_k} [(Y_s^n - L_s) - (Y_s^m - L_s)]\,\mathrm{d}K_s^m \\ \leq\; & \int_{t\wedge\tau_k}^{\tau_k} (Y_s^m - L_s)^- \mathrm{d}K_s^n + \int_{t\wedge\tau_k}^{\tau_k} (Y_s^n - L_s)^- \mathrm{d}K_s^m \\ \leq\; & \sup_{s\in[0,T]} (Y_{s\wedge\tau_k}^m - L_{s\wedge\tau_k})^- |K_T^n| + \sup_{s\in[0,T]} (Y_{s\wedge\tau_k}^n - L_{s\wedge\tau_k})^- |K_T^m| \end{aligned} \tag{A.6}$$

and

$$
\begin{aligned}
&\int_{t\wedge\tau_k}^{\tau_k} (Y_s^n - Y_s^m)\,(\mathrm{d}A_s^m - \mathrm{d}A_s^n) \\
= &\int_{t\wedge\tau_k}^{\tau_k} [(U_s - Y_s^m) - (U_s - Y_s^n)]\,(\mathrm{d}A_s^m - \mathrm{d}A_s^n) \\
= &-\int_{t\wedge\tau_k}^{\tau_k} (U_s - Y_s^m)\mathrm{d}A_s^n - \int_{t\wedge\tau_k}^{\tau_k} (U_s - Y_s^n)\mathrm{d}A_s^m \leq 0.
\end{aligned}
\tag{A.7}
$$

Combining (3.2), (A.5), (A.6) and (A.7) with Hölder's inequality yields that for each $m, n, k \geq 1$,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} |Y_{t\wedge\tau_k}^n - Y_{t\wedge\tau_k}^m|^2\right] \\
\leq\ & C\mathbb{E}\left[|Y_{\tau_k}^n - Y_{\tau_k}^m|^2 + 2\int_0^{\tau_k} |Y_t^n - Y_t^m|(\bar{f}_t + \bar{\lambda})\mathrm{d}t\right] \\
&+C\left(\mathbb{E}\left[\sup_{t\in[0,T]} \left|(Y_{t\wedge\tau_k}^m - L_{t\wedge\tau_k})^-\right|^2\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[|K_{\tau_k}^n|^2\right]\right)^{\frac{1}{2}} \\
&+C\left(\mathbb{E}\left[\sup_{t\in[0,T]} \left|(Y_{t\wedge\tau_k}^n - L_{t\wedge\tau_k})^-\right|^2\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[|K_{\tau_k}^m|^2\right]\right)^{\frac{1}{2}} \\
&+2C\bar{\lambda}\left(\mathbb{E}\left[\int_0^{\tau_k} |Y_t^n - Y_t^m|^2\mathrm{d}t\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^{\tau_k} (|Z_t^n| + |Z_t^m|)^2\,\mathrm{d}t\right]\right)^{\frac{1}{2}}.
\end{aligned}
\tag{A.8}
$$

Note that $Y_\cdot^n \uparrow Y_\cdot$, $\bar{f}_\cdot \in \mathcal{H}^1$, $|Y_{\cdot\wedge\tau_k}^1| + |\bar{Y}_{\cdot\wedge\tau_k}| + L_{\cdot\wedge\tau_k}^+ \leq k$ and $\sup_{n\geq 1}(\mathbb{E}[|K_{\tau_k}^n|^2] + \|Z_\cdot^n \mathbb{1}_{\cdot\leq\tau_k}\|_{\mathrm{M}^2}) < +\infty$ for each $k \geq 1$ by (3.6). In view of (A.3), from (A.8) and Lebesgue's dominated convergence theorem it follows that for each $k \geq 1$, as $n, m \to \infty$,

$$
\mathbb{E}\left[\sup_{t\in[0,T]} |Y_{t\wedge\tau_k}^n - Y_{t\wedge\tau_k}^m|^2\right] \to 0,
$$

which implies that for each $k \geq 1$, as $n, m \to \infty$,

$$
\sup_{t\in[0,T]} |Y_{t\wedge\tau_k}^n - Y_{t\wedge\tau_k}^m| \to 0 \text{ in probability } \mathbb{P}.
$$

And, by (3.5) and the fact that $Y_\cdot^n \uparrow Y_\cdot$ we know that

$$
\sup_{t\in[0,T]} |Y_t^n - Y_t| \to 0, \quad \text{as } n \to \infty. \tag{A.9}
$$

So, $Y_\cdot$ is a continuous process, and then belongs to the space $\mathcal{S}^\beta$ for each $\beta \in (0, 1)$ and the class (D) due to the fact that both $Y_\cdot^1$ and $\bar{Y}_\cdot$ belong to them as well as $Y_\cdot^1 \leq Y_\cdot^n \leq \bar{Y}_\cdot$. Finally, from (A.9) and Lebesgue's dominated convergence theorem it follows that for each $\beta \in (0, 1)$,

$$
\lim_{n\to\infty} \|Y_\cdot^n - Y_\cdot\|_{\mathcal{S}^\beta} = \lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^n - Y_t|^\beta\right] = 0. \tag{A.10}
$$

Step 4. We show the convergence of the sequence $\{Z_\cdot^n\}$ in the space of M^β for each $\beta \in (0, 1)$. Note that (A.4) solves (2.1). It follows from (i) of Lemma 2.4 with $t = 0$ and $\tau = T$ that there exists a nonnegative constant $C' \geq 0$ such that for each $m, n \geq 1$ and

$\beta \in (0,1)$, we have

$$
\begin{aligned}
& \mathbb{E}\left[\left(\int_0^T |Z_t^n - Z_t^m|^2 \mathrm{d}t\right)^{\frac{\beta}{2}}\right] \\
\leq\ & C'\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^n - Y_t^m|^\beta + \sup_{t\in[0,T]}\left[\left(\int_t^T (Y_s^n - Y_s^m)\left(\mathrm{d}K_s^n - \mathrm{d}K_s^m\right)\right)^+\right]^{\frac{\beta}{2}}\right] \\
& +C'\mathbb{E}\left[\sup_{t\in[0,T]}\left[\left(\int_t^T (Y_s^n - Y_s^m)\left(\mathrm{d}A_s^m - \mathrm{d}A_s^n\right)\right)^+\right]^{\frac{\beta}{2}}\right] \\
& +C'\mathbb{E}\left[\left(\int_0^T |Y_s^n - Y_s^m|\, |g(s,Y_s^n,Z_s^n) - g(s,Y_s^m,Z_s^m)|\,\mathrm{d}s\right)^{\frac{\beta}{2}}\right].
\end{aligned}
$$

Then, it follows from Hölder's inequality together with (A.7) that

$$
\begin{aligned}
& \mathbb{E}\left[\left(\int_0^T |Z_t^n - Z_t^m|^2 \mathrm{d}t\right)^{\frac{\beta}{2}}\right] \\
\leq\ & C'\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^n - Y_t^m|^\beta\right] + C'\left(\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^n - Y_t^m|^\beta\right]\right)^{\frac{1}{2}}\left\{\left(\mathbb{E}\left[|K_T^n|^\beta\right]\right)^{\frac{1}{2}}\right. \\
& \left. + \left(\mathbb{E}\left[|K_T^m|^\beta\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}\left[\left(\int_0^T \left(|g(t,Y_t^n,Z_t^n)| + |g(t,Y_t^m,Z_t^m)|\right)\mathrm{d}t\right)^\beta\right]\right)^{\frac{1}{2}}\right\},
\end{aligned}
$$

from which together with (3.4) and (A.10) yields that there exists a process $(Z_t)_{t\in[0,T]} \in \cap_{\beta\in(0,1)}\mathrm{M}^\beta$ satisfying, for each $\beta \in (0,1)$,

$$
\lim_{n\to\infty} \|Z_\cdot^n - Z_\cdot\|_{\mathrm{M}^\beta} = \lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^T |Z_t^n - Z_t|^2 \mathrm{d}t\right)^{\frac{\beta}{2}}\right] = 0. \tag{A.11}
$$

Step 5. We show the uniform convergence of a subsequence of the sequence $\{K_\cdot^n\}$ in the sense of almost surely. Since g is continuous in (y,z) and satisfies (3.2), by (A.9) and (A.11) we can deduce that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$
\lim_{j\to\infty}\int_0^T |g(t,Y_t^{n_j},Z_t^{n_j}) - g(t,Y_t,Z_t)|\mathrm{d}t = 0,
$$

and then

$$
\lim_{j\to\infty}\sup_{t\in[0,T]}\left|\int_0^t g(t,Y_t^{n_j},Z_t^{n_j})\mathrm{d}t - \int_0^t g(t,Y_t,Z_t)\mathrm{d}t\right| = 0. \tag{A.12}
$$

Thus, combining (3.7), (A.9), (A.11) and (A.12) yields that $\mathbb{P}-a.s.$, for each $t\in[0,T]$,

$$
K_t^{n_j} = Y_0^{n_j} - Y_t^{n_j} - \int_0^t g(s,Y_s^{n_j},Z_s^{n_j})\mathrm{d}s - \int_0^t \mathrm{d}V_s - A_t^{n_j} + \int_0^t Z_s^{n_j}\cdot \mathrm{d}B_s
$$

tends to

$$
K_t := Y_0 - Y_t - \int_0^t g(s,Y_s,Z_s)\mathrm{d}s - \int_0^t \mathrm{d}V_s - A_t + \int_0^t Z_s\cdot \mathrm{d}B_s
$$

as $j \to \infty$ and that

$$\lim_{j\to\infty} \sup_{t\in[0,T]} |K_t^{n_j} - K_t| = 0. \tag{A.13}$$

Hence, $K_\cdot \in \mathcal{V}^+$ due to $K_\cdot^n \in \mathcal{V}^+$ for each $n \ge 1$. Furthermore, note by the assumption that $K_T^n \le \bar{K}_T^n$ for each $n \ge 1$ with $\lim_{k\to\infty} \bar{K}_T^{n_k} = \bar{K}_T \in \mathbb{L}^1(\mathcal{F}_T)$ for a subsequence $\{n_k\}$ of $\{n\}$. It follows that $K_T \in \mathbb{L}^1(\mathcal{F}_T)$ and then $K_\cdot \in \mathcal{V}^{+,1}$.

Step 6. We show that the convergence of the sequence $\{A_\cdot^n\}$ in the space of $\mathcal{S}^1$. Indeed, for each $k \ge 1$, define the following $(\mathcal{F}_t)$-stopping time:

$$\sigma_k := \inf\{t \in [0,T] : \int_0^t |Z_s|^2 \mathrm{d}s \ge k\} \wedge T.$$

It is clear that $\sigma_k \to T$ as $k \to +\infty$ due to the fact that $Z_\cdot \in \mathrm{M}$. For each $k \ge 1$, we have

$$A_{\sigma_k} = Y_{\sigma_k} - Y_0 + \int_0^{\sigma_k} g(s, Y_s, Z_s)\mathrm{d}s + \int_0^{\sigma_k} \mathrm{d}V_s + K_{\sigma_k} - \int_0^{\sigma_k} Z_s \cdot \mathrm{d}B_s,$$

and then

$$\mathbb{E}[A_{\sigma_k}] \le |Y_0| + \mathbb{E}\left[|Y_{\sigma_k}| + \int_0^T |g(s, Y_s, Z_s)|\mathrm{d}s + |V|_T + K_T\right].$$

Letting $k \to \infty$, in view of Fatou's lemma and the fact that $Y_\cdot$ belongs to the class (D), yields that

$$\mathbb{E}[A_T] \le |Y_0| + \mathbb{E}\left[|\xi| + \int_0^T |g(s, Y_s, Z_s)|\mathrm{d}s + |V|_T + K_T\right].$$

Furthermore, in view of (A.12) and (3.2), it follows from Hölder's inequality that

$$\begin{aligned} & \mathbb{E}\left[\int_0^T |g(s, Y_s, Z_s)|\mathrm{d}s\right] \\ = \ & \mathbb{E}\left[\lim_{j\to\infty} \int_0^T |g(t, Y_t^{n_j}, Z_t^{n_j})|\mathrm{d}t\right] \le \mathbb{E}\left[\lim_{j\to\infty} \int_0^T (\bar{f}_t + \bar{\lambda}|Z_t^{n_j}|^\alpha)\mathrm{d}t\right] \\ = \ & \mathbb{E}\left[\int_0^T (\bar{f}_t + \bar{\lambda}|Z_t|^\alpha)\mathrm{d}t\right] \le \|\bar{f}_\cdot\|_{\mathcal{H}^1} + \bar{\lambda}T^{\frac{2-\alpha}{2}} \|Z_\cdot\|_{\mathrm{M}^\alpha} < +\infty. \end{aligned}$$

Thus, we have $\mathbb{E}[A_T] < \infty$ and $A_\cdot \in \mathcal{V}^{+,1}$. Finally, note that $0 \le A_\cdot^n \le A_\cdot$ for each $n \ge 1$. From (3.7) and Lebesgue's dominated convergence theorem it follows that

$$\lim_{n\to\infty} \|A_\cdot^n - A_\cdot\|_{\mathcal{S}^1} = 0. \tag{A.14}$$

Step 7. We show that the $(Y_\cdot, Z_\cdot, K_\cdot, A_\cdot)$ is an L^1 solution of RBSDE $(\xi, g + \mathrm{d}V, L, U)$. In fact, it has been proved that $Y_\cdot$ belongs to the class (D), $(Y_\cdot, Z_\cdot, K_\cdot, A_\cdot) \in \mathcal{S}^\beta \times \mathrm{M}^\beta \times \mathcal{V}^{+,1} \times \mathcal{V}^{+,1}$ for each $\beta \in (0,1)$ and it solves

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s + \int_t^T \mathrm{d}K_s - \int_t^T \mathrm{d}A_s - \int_t^T Z_s \cdot \mathrm{d}B_s, \quad t \in [0,T].$$

By Step 2 we know that $Y_t \ge L_t$ for each $t \in [0,T]$, and then

$$\int_0^T (Y_t - L_t)\mathrm{d}K_t \ge 0.$$

On the other hand, in view of (A.9) and (A.13), it follows from the definition of $K^n_\cdot$ that

$$\int_0^T (Y_t - L_t)\mathrm{d}K_t = \lim_{j\to\infty}\int_0^T (Y_t^{n_j} - L_t)\mathrm{d}K_t^{n_j} \le 0.$$

Consequently, we have

$$\int_0^T (Y_t - L_t)\mathrm{d}K_t = 0.$$

Furthermore, noticing that $Y^n_\cdot \le U_\cdot$ and $\int_0^T (U_t - Y_t^n)\,\mathrm{d}A_t^n = 0$ for each $n \ge 1$, from (A.10) and (A.14) we can deduce that $Y_t \le U_t$ for each $t \in [0,T]$, and

$$\int_0^T (U_t - Y_t)\,\mathrm{d}A_t = \lim_{n\to\infty}\int_0^T (U_t - Y_t^n)\,\mathrm{d}A_t^n = 0.$$

Finally, let us show that $\mathrm{d}K \bot \mathrm{d}A$. In fact, for each $n \ge 1$, we can define the following $(\mathcal{F}_t)$-progressively measurable set

$$D_n := \{(\omega, t) \subset \Omega \times [0,T] : \ Y_t^n(\omega) \ge L_t(\omega)\}.$$

Then, from the definition of $K^n_\cdot$ we know that for each $n \ge 1$,

$$\mathbb{E}\left[\int_0^T \mathbb{1}_{D_n}\mathrm{d}K_t^n\right] = 0,$$

and, in view of $\int_0^T (U_t - Y_t^n)\mathrm{d}A_t^n = 0$,

$$\mathbb{E}\left[\int_0^T \mathbb{1}_{D_n^c}\mathrm{d}A_t^n\right] = \mathbb{E}\left[\int_0^T \mathbb{1}_{\{Y_t^n < L_t \le U_t\}}|U_t - Y_t^n|^{-1}(U_t - Y_t^n)\,\mathrm{d}A_t^n\right] = 0.$$

Thus, noticing that $D_n \subset D_{n+1}$ for each $n \ge 1$ due to $Y^n_\cdot \le Y^{n+1}_\cdot$, by (A.13) and (A.14) we can deduce that

$$\mathbb{E}\left[\int_0^T \mathbb{1}_{\cup D_n}\mathrm{d}K_t\right] = \lim_{j\to\infty}\mathbb{E}\left[\int_0^T \mathbb{1}_{D_{n_j}}\mathrm{d}K_t^{n_j}\right] = 0$$

and

$$\mathbb{E}\left[\int_0^T \mathbb{1}_{\cap D_n^c}\mathrm{d}A_t\right] = \lim_{n\to\infty}\mathbb{E}\left[\int_0^T \mathbb{1}_{D_n^c}\mathrm{d}A_t^n\right] = 0.$$

Hence, $\mathrm{d}K \bot \mathrm{d}A$. Proposition 3.1 is then proved.

Complementary of the details for the proof of Proposition 3.3

Now, we will detail the proof of steps 1–3 after eq. (3.17).

Step 1. We show the convergence of the sequence $\{Y^n_\cdot\}$ in the space of $\mathcal{S}^\beta$ for each $\beta \in (0,1)$. For each positive integer $k, l \ge 1$, we introduce the following two $(\mathcal{F}_t)$-stopping times:

$$\begin{aligned}\sigma_k \quad &:= \quad \inf\left\{t \ge 0 : \ |Y_t^1| + |\bar{Y}_t| + |V|_t + K_t^1 + \bar{A}_t + \int_0^t \tilde{f}_s\ \mathrm{d}s \ge k\right\} \wedge T;\\ \tau_{k,l} \quad &:= \quad \inf\left\{t \ge 0 : \ \int_0^t \bar{f}_s^k\ \mathrm{d}s \ge l\right\} \wedge \sigma_k.\end{aligned}$$

Then we have

$$\mathbb{P}\left(\{\omega : \ \exists k_0(\omega), l_0(\omega) \ge 1, \ \forall k \ge k_0(\omega), \ \forall l \ge l_0(\omega), \ \ \tau_{k,l}(\omega) = T\}\right) = 1. \tag{A.15}$$

For each $n,m\geq 1$, observe that

$$
\begin{aligned}
(\bar{Y}_\cdot,\bar{Z}_\cdot,\bar{V}_\cdot) \quad := \quad & (Y^n_\cdot - Y^m_\cdot, Z^n_\cdot - Z^m_\cdot,\\
& \int_0^\cdot \left(g_n(s,Y^n_s,Z^n_s)-g_m(s,Y^m_s,Z^m_s)\right)\mathrm{d}s + (K^n_\cdot - K^m_\cdot) + (A^m_\cdot - A^n_\cdot))
\end{aligned}
\tag{A.16}
$$

satisfies equation (2.1). It then follows from (ii) of Lemma 2.4 with $p=2$, $t=0$ and $\tau=\tau_{k,l}$ that there exists a constant $C>0$ such that for each $n,m,k,l\geq 1$,

$$
\begin{aligned}
& \mathbb{E}\left[\sup_{t\in[0,T]}|Y^n_{t\wedge\tau_{k,l}}-Y^m_{t\wedge\tau_{k,l}}|^2\right]\\
\leq \quad & C\mathbb{E}\left[|Y^n_{\tau_{k,l}}-Y^m_{\tau_{k,l}}|^2+\sup_{t\in[0,T]}\left(\int_{t\wedge\tau_{k,l}}^{\tau_{k,l}}(Y^n_s-Y^m_s)\left(\mathrm{d}K^n_s-\mathrm{d}K^m_s\right)\right)^+\right.\\
& \qquad +\sup_{t\in[0,T]}\left(\int_{t\wedge\tau_{k,l}}^{\tau_{k,l}}(Y^n_s-Y^m_s)\left(\mathrm{d}A^m_s-\mathrm{d}A^n_s\right)\right)^+\\
& \qquad \left.+\int_0^{\tau_{k,l}}|Y^n_s-Y^m_s|\,|g_n(s,Y^n_s,Z^n_s)-g_m(s,Y^m_s,Z^m_s)|\,\mathrm{d}s\right].
\end{aligned}
\tag{A.17}
$$

Furthermore, note that $L_\cdot\leq Y^n_\cdot\leq U_\cdot$ and that $\int_0^T(Y^n_t-L_t)\mathrm{d}K^n_t=\int_0^T(U_t-Y^n_t)\mathrm{d}A^n_t=0$ for each $n\geq 1$. It follows that for each $t\in[0,T]$ and $k,l,m,n\geq 1$,

$$
\begin{aligned}
& \int_{t\wedge\tau_{k,l}}^{\tau_{k,l}}(Y^n_s-Y^m_s)\left(\mathrm{d}K^n_s-\mathrm{d}K^m_s\right)\\
= \quad & \int_{t\wedge\tau_{k,l}}^{\tau_{k,l}}\left[(Y^n_s-L_s)-(Y^m_s-L_s)\right]\left(\mathrm{d}K^n_s-\mathrm{d}K^m_s\right)\\
= \quad & -\int_{t\wedge\tau_{k,l}}^{\tau_{k,l}}(Y^n_s-L_s)\mathrm{d}K^m_s-\int_{t\wedge\tau_{k,l}}^{\tau_{k,l}}(Y^m_s-L_s)\mathrm{d}K^n_s\leq 0
\end{aligned}
\tag{A.18}
$$

and

$$
\begin{aligned}
& \int_{t\wedge\tau_{k,l}}^{\tau_{k,l}}(Y^n_s-Y^m_s)\left(\mathrm{d}A^m_s-\mathrm{d}A^n_s\right)\\
= \quad & \int_{t\wedge\tau_{k,l}}^{\tau_{k,l}}\left[(U_s-Y^m_s)-(U_s-Y^n_s)\right]\left(\mathrm{d}A^m_s-\mathrm{d}A^n_s\right)\\
= \quad & -\int_{t\wedge\tau_{k,l}}^{\tau_{k,l}}(U_s-Y^m_s)\mathrm{d}A^n_s-\int_{t\wedge\tau_{k,l}}^{\tau_{k,l}}(U_s-Y^n_s)\mathrm{d}A^m_s\leq 0.
\end{aligned}
\tag{A.19}
$$

By the definition of $\tau_{k,l}$ and the fact that $Y^1_\cdot\leq Y^n_\cdot\leq\bar{Y}_\cdot$, we know that $\mathbb{1}_{\cdot\leq\tau_{k,l}}\leq\mathbb{1}_{|Y^n_\cdot|\leq k}$ holds true for each $k,l,n\geq 1$. Then, combining (3.14), (A.17), (A.18), (A.19) and Hölder's inequality yields that

$$
\begin{aligned}
& \mathbb{E}\left[\sup_{t\in[0,T]}|Y^n_{t\wedge\tau_{k,l}}-Y^m_{t\wedge\tau_{k,l}}|^2\right]\\
\leq \quad & C\mathbb{E}\left[|Y^n_{\tau_{k,l}}-Y^m_{\tau_{k,l}}|^2+2\int_0^{\tau_{k,l}}|Y^n_t-Y^m_t|\bar{f}^k_t\mathrm{d}t\right]\\
& +2C\bar{\lambda}\left(\mathbb{E}\left[\int_0^{\tau_{k,l}}|Y^n_t-Y^m_t|^2\mathrm{d}t\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_0^{\tau_{k,l}}(|Z^n_t|+|Z^m_t|)^2\,\mathrm{d}t\right]\right)^{\frac{1}{2}}.
\end{aligned}
\tag{A.20}
$$

Note that $Y^1_\cdot\leq Y^n_\cdot\uparrow Y_\cdot\leq\bar{Y}_\cdot$. By the definition of $\tau_{k,l}$ and (3.17), it follows from (A.20) and Lebesgue's dominated convergence theorem that for each $k,l\geq 1$, as $n,m\to\infty$,

$$
\mathbb{E}\left[\sup_{t\in[0,T]}|Y^n_{t\wedge\tau_{k,l}}-Y^m_{t\wedge\tau_{k,l}}|^2\right]\to 0,
$$

which implies that for each $k, l \geq 1$, as $n, m \to \infty$,

$$\sup_{t\in[0,T]} |Y^n_{t\wedge\tau_{k,l}} - Y^m_{t\wedge\tau_{k,l}}| \to 0 \text{ in probability } \mathbb{P}.$$

And, by (A.15) and the monotonicity of $Y^n_\cdot$ with respect to n we know that

$$\sup_{t\in[0,T]} |Y^n_t - Y_t| \to 0, \quad \text{as } n \to \infty. \tag{A.21}$$

So, $Y_\cdot$ is a continuous process, and then belongs to the space $\mathcal{S}^\beta$ for each $\beta \in (0,1)$ and the class (D) due to the fact that both $Y^1_\cdot$ and $\bar{Y}_\cdot$ belong to them as well as $Y^1_\cdot \leq Y^n_\cdot \leq \bar{Y}_\cdot$. Finally, from (A.21) and Lebesgue's dominated convergence theorem it follows that for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} \|Y^n_\cdot - Y_\cdot\|_{\mathcal{S}^\beta} = \lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} |Y^n_t - Y_t|^\beta\right] = 0. \tag{A.22}$$

Step 2. We show the convergence of the sequence $\{Z^n_\cdot\}$ in the space of M^β for each $\beta \in (0,1)$. Note that (A.16) solves (2.1). It follows from (i) of Lemma 2.4 with $t = 0$ and $\tau = T$ that there exists a nonnegative constant $C' \geq 0$ such that for each $m, n \geq 1$ and $\beta \in (0,1)$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T |Z^n_t - Z^m_t|^2 \mathrm{d}t\right)^{\frac{\beta}{2}}\right] \\
\leq\ & C'\mathbb{E}\left[\sup_{t\in[0,T]} |Y^n_t - Y^m_t|^\beta + \sup_{t\in[0,T]}\left[\left(\int_t^T (Y^n_s - Y^m_s)(\mathrm{d}K^n_s - \mathrm{d}K^m_s)\right)^+\right]^{\frac{\beta}{2}}\right] \\
& + C'\mathbb{E}\left[\sup_{t\in[0,T]}\left[\left(\int_t^T (Y^n_s - Y^m_s)(\mathrm{d}A^m_s - \mathrm{d}A^n_s)\right)^+\right]^{\frac{\beta}{2}}\right] \\
& + C'\mathbb{E}\left[\left(\int_0^T |Y^n_s - Y^m_s|\,|g_n(s, Y^n_s, Z^n_s) - g_m(s, Y^m_s, Z^m_s)|\,\mathrm{d}s\right)^{\frac{\beta}{2}}\right].
\end{aligned}$$

Then, in view of (A.18) and (A.19), it follows from Hölder's inequality that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T |Z^n_t - Z^m_t|^2 \mathrm{d}t\right)^{\frac{\beta}{2}}\right] \\
\leq\ & C'\mathbb{E}\left[\sup_{t\in[0,T]} |Y^n_t - Y^m_t|^\beta\right] + C'\left(\mathbb{E}\left[\sup_{t\in[0,T]} |Y^n_t - Y^m_t|^\beta\right]\right)^{\frac{1}{2}} \\
& \cdot \left(\mathbb{E}\left[\left(\int_0^T (|g_n(t, Y^n_t, Z^n_t)| + |g_m(t, Y^m_t, Z^m_t)|)\,\mathrm{d}t\right)^\beta\right]\right)^{\frac{1}{2}},
\end{aligned}$$

from which together with (A.22) and (3.16) yields that there exists a process $(Z_t)_{t\in[0,T]} \in \cap_{\beta\in(0,1)}\mathrm{M}^\beta$ satisfying, for each $\beta \in (0,1)$,

$$\lim_{n\to\infty} \|Z^n_\cdot - Z_\cdot\|_{\mathrm{M}^\beta} = \lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^T |Z^n_t - Z_t|^2 \mathrm{d}t\right)^{\frac{\beta}{2}}\right] = 0. \tag{A.23}$$

Step 3. We show that the $(Y_\cdot, Z_\cdot, K_\cdot, A_\cdot)$ is an L^1 solution of DRBSDE $(\xi, g + \mathrm{d}V, L, U)$. Since g_n tends locally uniformly in (y, z) to the generator g as $n \to \infty$ and satisfies (3.14), by (A.21) and (A.23) together with (A.15) we can deduce that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\lim_{j\to\infty} \int_0^T |g_{n_j}(t, Y_t^{n_j}, Z_t^{n_j}) - g(t, Y_t, Z_t)|\mathrm{d}t = 0.$$

Then,

$$\lim_{j\to\infty} \sup_{t\in[0,T]} \left| \int_0^t g_{n_j}(t, Y_t^{n_j}, Z_t^{n_j})\mathrm{d}t - \int_0^t g(t, Y_t, Z_t)\mathrm{d}t \right| = 0. \tag{A.24}$$

Combining (3.15), (A.21), (A.23) and (A.24) yields that

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s + \int_t^T \mathrm{d}V_s + \int_t^T \mathrm{d}K_s - \int_t^T \mathrm{d}A_s - \int_t^T Z_s \cdot \mathrm{d}B_s, \quad t \in [0, T].$$

Since $L_t \le Y_t^n \le U_t$ and $Y_t^n \uparrow Y_t$ for each $t \in [0, T]$, we have $L_t \le Y_t \le U_t$ for each $t \in [0, T]$. Furthermore, in view of (A.22) and (3.15), it follows that

$$\int_0^T (Y_t - L_t)\mathrm{d}K_t = \lim_{n\to\infty} \int_0^T (Y_t^n - L_t)\mathrm{d}K_t^n = 0$$

and

$$\int_0^T (U_t - Y_t)\mathrm{d}K_t = \lim_{n\to\infty} \int_0^T (U_t - Y_t^n)\mathrm{d}A_t^n = 0.$$

Finally, let us show that $\mathrm{d}K \bot \mathrm{d}A$. In fact, for each $n \ge 1$, since $\mathrm{d}K^n \bot \mathrm{d}A^n$, there exists an $(\mathcal{F}_t)$-progressively measurable set $D_n \subset \Omega \times [0, T]$ such that

$$\mathbb{E}\left[\int_0^T \mathbb{1}_{D_n}\mathrm{d}K_t^n\right] = \mathbb{E}\left[\int_0^T \mathbb{1}_{D_n^c}\mathrm{d}A_t^n\right] = 0.$$

Then, in view of (3.15) and the fact that $\mathrm{d}K \le \mathrm{d}K^n$ for each $n \ge 1$,

$$0 \le \mathbb{E}\left[\int_0^T \mathbb{1}_{\cup D_n}\mathrm{d}K_t\right] \le \sum_{n=1}^\infty \mathbb{E}\left[\int_0^T \mathbb{1}_{D_n}\mathrm{d}K_t\right] \le \sum_{n=1}^\infty \mathbb{E}\left[\int_0^T \mathbb{1}_{D_n}\mathrm{d}K_t^n\right] = 0$$

and

$$0 \le \mathbb{E}\left[\int_0^T \mathbb{1}_{\cap D_n^c}\mathrm{d}A_t\right] = \lim_{m\to\infty} \mathbb{E}\left[\int_0^T \mathbb{1}_{\cap D_n^c}\mathrm{d}A_t^m\right] \le \lim_{m\to\infty} \mathbb{E}\left[\int_0^T \mathbb{1}_{D_m^c}\mathrm{d}A_t^m\right] = 0.$$

Hence, $\mathrm{d}K \bot \mathrm{d}A$. Proposition 3.3 is then proved.

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