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PROPAGATION OF CHAOS FOR STOCHASTIC SPATIALLY STRUCTURED NEURONAL NETWORKS WITH DELAY DRIVEN BY JUMP DIFFUSIONS

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Spatially structured neural networks driven by jump diffusion noise with monotone coefficients, fully path dependent delay and with a disorder parameter are considered. Well-posedness for the associated McKean–Vlasov equation and a corresponding propagation of chaos result in the infinite population limit are proven. Our existence result for the McKean–Vlasov equation is based on the Euler approximation that is applied to this type of equation for the first time.

1. Introduction and main results. The purpose of this paper is to prove a unified existence and uniqueness result for the McKean–Vlasov equation and a propagation of chaos result for a spatially structured coupled neural network of neural oscillators in the large population limit. Our mathematical framework covers all relevant modeling issues of networks of point neurons. In particular, we incorporate noise terms, both in the local dynamics of the neurons as well as in the synaptic transmission, in order to account for channel noise and synaptic noise in the neural dynamics. We also consider general delay terms modeling finite and variable propagation speed of neural signals. In order to cover all conductance-based neural oscillators and all types of delays being widely accepted in computational neuroscience, we consider stochastic delay differential equations with merely monotone coefficients. As stochastic forcing terms for the local dynamics and the synaptic transmission between individual neurons, we allow jump diffusions given as the independent sum of Brownian motions and Poisson processes.

We also incorporate spatial structure into our networks to take into account morphological properties of brain tissues. With a view toward spatial continuum limits, we consider the positions of the neurons as discrete subsets in a bounded subset $\Gamma \subset \mathbb{R}^k$, and introduce spatial dependence in the network dynamics in terms of an additional space parameter. We will then be in particular interested in the dynamical properties of the network in the infinite population limit, where the spatial distribution of the neurons is given in terms of a general Borel-measure on Γ.

With a view toward modeling brain networks, consisting of subpopulations of neurons, we also introduce a measurable partition of $\Gamma := \bigcup_{1\le\alpha\le P} \Gamma_\alpha$ and consider Γ_α as a given subpopulation.

In order to incorporate variability in the neurons, and henceforth the associated neural dynamics, we finally introduce disorder in terms of a random parameter ω'.

The above mentioned modeling issues lead to the following system of coupled delay-differential equations:

$$\mathrm{d}X_t^{r,\mathcal{A}_N} = f\big(t, r, X_{t^-}^{r,\mathcal{A}_N}, \omega'\big)\,\mathrm{d}t + g\big(t, r, X_{t^-}^{r,\mathcal{A}_N}, \omega'\big)\,\mathrm{d}W_t^r$$

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$$
\begin{aligned}
&+\int_U h\big(t,r,X_{t^-}^{r,\mathcal{A}_N},\omega',\xi\big)\tilde{N}^r(\mathrm{d}t,\mathrm{d}\xi)\\
&+\sum_{\alpha=1}^{P}\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}}\sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha}\theta\big(t,r,\tilde r,X_{t^-}^{r,\mathcal{A}_N},X_{(t-\tau)^-:t^-}^{\tilde r,\mathcal{A}_N},\omega'\big)\,\mathrm{d}t\\
&+\sum_{\alpha=1}^{P}\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}}\sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha}\beta\big(t,r,\tilde r,X_{t^-}^{r,\mathcal{A}_N},X_{(t-\tau)^-:t^-}^{\tilde r,\mathcal{A}_N},\omega'\big)\,\mathrm{d}B_t^{r,\alpha}\\
&+\sum_{\alpha=1}^{P}\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}}\int_U\sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha}\eta\big(t,r,\tilde r,X_{t^-}^{r,\mathcal{A}_N},\\
&\qquad X_{(t-\tau)^-:t^-}^{\tilde r,\mathcal{A}_N},\omega',\xi\big)\tilde{N}^{r,\alpha}(\mathrm{d}t,\mathrm{d}\xi),\\
X_t^{r,\mathcal{A}_N}&=z_t^r,\quad t\in[-\tau,0],r\in\mathcal{A}_N
\end{aligned}
\tag{1.1}
$$

for the time-evolution of the network.

Here, P is the number of different subpopulations, placed in space at disjoint measurable regions Γ_α, $1\le\alpha\le P$, such that $\Gamma:=\bigcup_{1\le\alpha\le P}\Gamma_\alpha$ is a bounded subset in $\mathbb{R}^k$. We suppose that the total number of neurons is N and $\mathcal{A}_N\subset\Gamma$ denotes the position of neurons. In the equation (1.1), the weight of neurons in subpopulation α is $1/\mathcal{S}_{\mathcal{A}_N,\alpha}$ with $\mathcal{S}_{\mathcal{A}_N,\alpha}\neq 0$. Finally, $X_t^{r,\mathcal{A}_N}\in\mathbb{R}^d$ denotes the state of neuron at position $r\in\mathcal{A}_N$ and at time $t\ge 0$.

The disorder ω' is an element of a second probability space $(\Omega',\mathcal{F}',\mathbb{P}')$. We will denote with $\mathcal{E}$ the expectation with respect to $\mathbb{P}'$.

$\tau>0$ is a fixed deterministic delay in the synaptic transmission between different neurons. For any path y_t defined on some subinterval $I\subset\mathbb{R}$, we also use the following notation $y_{(t-\tau):t}$ for the path segment y_s, $s\in[t-\tau,t]$, for any t with $[t-\tau,t]\subset I$. W^r and $B^{r,\alpha}$, $r\in\mathcal{A}_N$, $1\le\alpha\le P$, are independent Brownian motions, respectively, in $\mathbb{R}^m$ and $\mathbb{R}^n$. N^r and $N^{r,\alpha}$, $r\in\mathcal{A}_N$, $1\le\alpha\le P$, are independent time homogeneous Poisson measures on $[0,\infty)\times U$ with intensity measure $\mathrm{d}t\otimes\nu$, where $(U,\mathcal{U},\nu)$ is an arbitrary σ-finite measure space. $\tilde N^r=N^r-\mathrm{d}t\otimes\nu$ and $\tilde N^{r,\alpha}=N^{r,\alpha}-\mathrm{d}t\otimes\nu$, $r\in\mathcal{A}_N$, $1\le\alpha\le P$, denote the associated compensated Poisson martingale measures.

The coefficients

$$
\begin{aligned}
f,g&:[0,\infty[\times\Gamma\times\mathbb{R}^d\times\Omega'\to\mathbb{R}^d,\mathbb{R}^{d\times m},\\
h&:[0,\infty[\times\Gamma\times\mathbb{R}^d\times\Omega'\times U\to\mathbb{R}^d,\\
\theta,\beta&:[0,\infty[\times\Gamma\times\Gamma\times\mathbb{R}^d\times\text{Càglàd}\big([-\tau,0];\mathbb{R}^d\big)\times\Omega'\to\mathbb{R}^d,\mathbb{R}^{d\times n},\\
\eta&:[0,\infty[\times\Gamma\times\Gamma\times\mathbb{R}^d\times\text{Càglàd}\big([-\tau,0];\mathbb{R}^d\big)\times\Omega'\times U\to\mathbb{R}^d,
\end{aligned}
$$

are jointly measurable w.r.t. all variables and continuous w.r.t. $x\in\mathbb{R}^d$. Here, the space Càdlàg$([-\tau,0];\mathbb{R}^d)$ is endowed with the supremum norm. We will specify appropriate monotonicity and growth conditions on the coefficients below (see Hypothesis 1.1).

The initial conditions z^r for $r\in\Gamma_\alpha$ are independent and identically distributed copies of $\hat z^\alpha$ with $\hat z^\alpha\in L^2(\Omega,\mathbb{P};\text{Càdlàg}([-\tau,0];\mathbb{R}^d))$. The space Càglàd$([-\tau,0];\mathbb{R}^d)$ being again endowed with the supremum norm. We assume that $\{W^r\}_{r\in\mathcal{A}_N}$, $\{B^{r,\alpha}\}_{\substack{r\in\mathcal{A}_N\\1\le\alpha\le P}}$, $\{N^r\}_{r\in\mathcal{A}_N}$, $\{N^{r,\alpha}\}_{\substack{r\in\mathcal{A}_N\\1\le\alpha\le P}}$ and the initial conditions $\{z^r\}_{r\in\mathcal{A}_N}$ are independent.

The reason for assuming the coefficients θ, β and η as being defined on cáglád processes stems from our choice of an Euler approximation method (A.2) for the solution of (1.1). We would like to mention that it is also possible to consider a càdlàg version based on a different Euler approximation scheme.

We will prove in Theorem 1.5 below for this general class of neural networks existence and uniqueness of a strong solution of the associated McKean–Vlasov equation, as well as a propagation of chaos result in Section 1.2 under suitable assumptions on the spatial distribution of neurons.

So far, the literature already contains a considerable amount of results concerning mean-field limits of interacting stochastic differential equations and also extensions to the delay case. However, a unified theory including all the above mentioned features of our model, monotonicity of the coefficients, general delay, jump diffusion forcing terms and spatial structure, is not available.

Indeed, results under global Lipschitz assumptions have been obtained in [7, 10, 11], extensions to locally Lipschitz, respectively, merely one-sided Lipschitz coefficients, have been obtained in [3] (with clarification note [4]) and [6, 9]. The clarification [4] refers to an erroneous management of hitting times used in [3] to localize the problem of existence of the McKean–Vlasov equations under local Lipschitz assumptions. Unfortunately, the same problem arises in the paper [9]. Our approach in the present manuscript is different from the approach in all these references, since we apply the Euler approximation to the construction of a solution. To the best of our knowledge, this is the first time in the existing literature this technique has been applied to McKean–Vlasov equations.

As already mentioned, delay terms form an important modeling issue in the neuroscience applications and, therefore, also have been considered in the literature, for example, point delays in [7, 9–11] and other references. Continuum limits of spatially structured biological neural networks and various types of random delay and/or random locations of the single neurons have been considered in [6] and also in [10, 11].

Note that in all the previous mentioned results, the noise is assumed to be diffusive only. The only existing result on McKean–Vlasov limits of biological neural networks driven by Lévy noise is the recent paper [2] for local dynamics under global Lipschitz assumptions or local dynamics of gradient type.

Let us next specify the precise assumptions on the coefficients of (1.1) that we assume for the well-posedness of associated McKean–Vlasov equations.

HYPOTHESIS 1.1. *There exist a probability measure λ on $[-\tau, 0]$ and nonnegative measurable functions $K_t(\omega')$, $L_t(\omega')$, $\bar{K}_t(\omega')$, $\bar{L}_t(\omega')$ and $\tilde{K}_t(R, \omega')$ in $L^1_{\text{loc}}([0, \infty[, dt)$, for all $R > 0$ and all $\omega' \in \Omega'$, such that the following conditions concerning local dynamics, synaptic transmissions and disorder hold*:

- *Assumptions concerning local dynamics*:

$$\begin{aligned}
\text{(H1)}\quad & 2\langle x - \tilde{x}, f(t, r, x, \omega') - f(t, r, \tilde{x}, \omega')\rangle \\
& \quad + |g(t, r, x, \omega') - g(t, r, \tilde{x}, \omega')|^2 \\
& \quad + \int_U |h(t, r, x, \omega', \xi) - h(t, r, \tilde{x}, \omega', \xi)|^2 \nu(\mathrm{d}\xi) \\
& \le L_t(\omega')|x - \tilde{x}|^2,
\end{aligned}$$

$$\begin{aligned}
\text{(H2)}\quad & 2\langle x, f(t, r, x, \omega')\rangle + |g(t, r, x, \omega')|^2 + \int_U |h(t, r, x, \omega', \xi)|^2 \nu(\mathrm{d}\xi) \\
& \le K_t(\omega')(1 + |x|^2),
\end{aligned}$$

$$\text{(H3)}\quad \sup_{|x| \le R} \Big[|f(t, r, x, \omega')| + |g(t, r, x, \omega')|^2$$

$$
\begin{aligned}
&+ \int_U |h(t,r,x,\omega',\xi)|^2 \nu(\mathrm{d}\xi)\Big] \\
&\le \tilde{K}_t(R,\omega').
\end{aligned}
$$

- *Assumptions concerning synaptic transmissions*:

$$
\begin{aligned}
\text{(H4)}\quad & \sum_{\Theta\in\{\theta,\beta\}} |\Theta(t,r,r',x,y_{-\tau:0},\omega') - \Theta(t,r,r',\tilde{x},\tilde{y}_{-\tau:0},\omega')|^2 \\
&\quad + \int_U |\eta(t,r,r',x,y_{-\tau:0},\omega',\xi) - \eta(t,r,r',\tilde{x},\tilde{y}_{-\tau:0},\omega',\xi)|^2 \nu(\mathrm{d}\xi) \\
&\quad \le \bar{L}_t(\omega')\Big[|x-\tilde{x}|^2 + \int_{-\tau}^0 \big(|y_s-\tilde{y}_s|^2 + \mathbf{1}_{\{s<0\}}|y_{s+}-\tilde{y}_{s+}|^2\big)\lambda(\mathrm{d}s)\Big], \\
\text{(H5)}\quad & \sum_{\Theta\in\{\theta,\beta\}} |\Theta(t,r,r',x,y_{-\tau:0},\omega')|^2 + \int_U |\eta(t,r,r',x,y_{-\tau:0},\omega',\xi)|^2 \nu(\mathrm{d}\xi) \\
&\quad \le \bar{K}_t(\omega')\Big[1 + |x|^2 + \int_{-\tau}^0 \big(|y_s|^2 + \mathbf{1}_{\{s<0\}}|y_{s+}|^2\big)\lambda(\mathrm{d}s)\Big],
\end{aligned}
$$

- *Assumption referring to the disorder*:
  (H6) *for all $T>0$ the following expectation value is finite*:

$$
\begin{aligned}
&\mathcal{E}\Big\{\exp\int_0^T \Big[2L_s(\omega') + \bar{L}_s(\omega')\Big(P + 6\sup_{N\in\mathbb{N}} \sum_{\alpha=1}^P \frac{(\#\mathcal{A}_N\cap\Gamma_\alpha)^2}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}}\Big) \\
&\qquad + K_s(\omega') + 3P\bar{K}_s(\omega')\Big]\mathrm{d}s\Big\} < \infty.
\end{aligned}
$$

PROPOSITION 1.2. *Under Hypothesis* 1.1, *equation* (1.1) *has a unique strong solution.*

The proof of this proposition is rather standard. However, we could not find a reference in the literature that covers our setting completely. Therefore, we have incorporated a general existence and uniqueness result for stochastic delay differential equations with monotone coefficients driven by jump diffusions in the Appendix A.

EXAMPLE 1.3 (FitzHugh–Nagumo model with electrical synapses and simple maximum conductance variation). Let us briefly discuss as an important example for networks of conductance-based point neuron models a network of FitzHugh–Nagumo neurons. In this model, two variables, the voltage variable V having a cubic nonlinearity and a slower recovery variable w describe the state of each neuron, that is reduced to one single point. We consider external current acting on the neuron placed at $r\in\Gamma$ with $\mathrm{d}I^r_{\mathrm{ext}} = \lambda^r_1\,\mathrm{d}t + \lambda^r_2\,\mathrm{d}W^r_t$. To account for the time for the signal of the presynaptic neuron to travel down the axon, we incorporate delay in the presynaptic voltage, so that the current $I^{r,\tilde{r}}_t$ of the presynaptic neuron at position $\tilde{r}$ acting on the neuron at position r at time t is given by the following differential equation $\mathrm{d}I^{r,\tilde{r}}_t = (V^{r,\mathcal{A}_N}_{t-} - V^{\tilde{r},\mathcal{A}_N}_{t-\tau})\,\mathrm{d}J^{r,\tilde{r}}_t$ where $J^{r,\tilde{r}}_t$ denotes the maximum conductance which we assume to be given as the following jump diffusion equation:

$$
\mathrm{d}J^{r,\tilde{r}}_t = \frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}}\Big[A^{r,\tilde{r}}_1\,\mathrm{d}t + A^{r,\tilde{r}}_2\,\mathrm{d}B^{r,\alpha}_t + \int_U \bar{\eta}^{r,\tilde{r}}(\xi)\tilde{N}^{r,\alpha}(\mathrm{d}t,\mathrm{d}\xi)\Big].
$$

The network equation in this example is then given as

$$
\begin{cases}
\mathrm{d}V_t^{r,\mathcal{A}_N} \\
\quad = \Big(-\frac{1}{3}(V_{t^-}^{r,\mathcal{A}_N})^3 + V_{t^-}^{r,\mathcal{A}_N} - w_{t^-}^{r,\mathcal{A}_N} + \lambda_1^r\Big)\,\mathrm{d}t + \lambda_2^r\,\mathrm{d}W_t^r \\
\qquad - \sum_{\alpha=1}^P \frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}} \sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha} (V_{t^-}^{r,\mathcal{A}_N} - V_{t-\tau}^{\tilde r,\mathcal{A}_N}) A_1^{r,\tilde r}\,\mathrm{d}t \\
\qquad - \sum_{\alpha=1}^P \frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}} \sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha} (V_{t^-}^{r,\mathcal{A}_N} - V_{t-\tau}^{\tilde r,\mathcal{A}_N}) A_2^{r,\tilde r}\,\mathrm{d}B_t^{r,\alpha} \\
\qquad - \sum_{\alpha=1}^P \frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}} \sum_{\tilde r\in\mathcal{A}_N\cap\Gamma_\alpha} \int_U (V_{t^-}^{r,\mathcal{A}_N} - V_{t-\tau}^{\tilde r,\mathcal{A}_N}) \bar\eta^{r,\tilde r}(\xi)\tilde N^{r,\alpha}(\mathrm{d}t,\mathrm{d}\xi), \\
\mathrm{d}w_t^{r,\mathcal{A}_N} = \lambda_3^r(V_t^{r,\mathcal{A}_N} + \lambda_4^r - \lambda_5^r w_{t^-}^{r,\mathcal{A}_N})\,\mathrm{d}t
\end{cases}
\tag{1.2}
$$

(see, e.g., [3]). Here, the measurable functions $A_i^{r,\tilde r}$, $1 \le i \le 2$ and $\lambda_i^r, 1 \le i \le 2$ are real valued and $\lambda_i^r, 3 \le i \le 5$ are positive. The measurable function $(r,\tilde r) \mapsto \bar\eta^{r,\tilde r}$ takes values in $L^2(U,\mathcal{U},\nu)$.

To obtain a continuum limit for (1.2), we now assume the existence of a finite Borel measure $\mathcal{R}$ on Γ with the following property: For every $\varepsilon > 0$, there exists a (finite) partition $\{\Gamma_\alpha^{m,\varepsilon}, 1 \le m \le M_\alpha^{(\varepsilon)}\}$ of Γ_α such that

$$
\lim_{N\to\infty} \frac{\#\mathcal{A}_N \cap \Gamma_\alpha^{m,\varepsilon}}{\mathcal{S}_{\mathcal{A}_N,\alpha}} = \mathcal{R}(\Gamma_\alpha^{m,\varepsilon}), \quad 1 \le m \le M_\alpha^{(\varepsilon)}
\tag{1.3}
$$

and $\lim_{N\to\infty}\mathcal{S}_{\mathcal{A}_N,\alpha} = \infty$, $1 \le \alpha \le P$ and for every $r,\tilde r \in \Gamma_\alpha^{m,\varepsilon}$ and $r',\tilde r' \in \Gamma_{\alpha'}^{m',\varepsilon}$,

$$
|A_i^{r,r'} - A_i^{\tilde r,\tilde r'}| < \varepsilon, \quad 1 \le i \le 2; \qquad |\lambda_i^r - \lambda_i^{\tilde r}| < \varepsilon, \quad 1 \le i \le 5; \quad \text{and}
$$
$$
\int_U |\bar\eta^{r,r'}(\xi) - \bar\eta^{\tilde r,\tilde r'}(\xi)|^2 \nu(\mathrm{d}\xi) < \varepsilon.
$$

Under these assumptions, the solution of (1.2) converges to the solution of the following McKean–Vlasov equation:

$$
\begin{cases}
\mathrm{d}V_t^r = \Big(-\frac{1}{3}(V_{t^-}^r)^3 + V_{t^-}^r - w_{t^-}^r + \lambda_1^r\Big)\,\mathrm{d}t + \lambda_2^r\,\mathrm{d}W_t^r \\
\qquad - \sum_{\alpha=1}^P \int_{\Gamma_\alpha} (V_{t^-}^r - V_{t-\tau}^{\tilde r}) A_1^{r,\tilde r}\mathcal{R}(\mathrm{d}\tilde r)\,\mathrm{d}t \\
\qquad - \sum_{\alpha=1}^P \int_{\Gamma_\alpha} (V_{t^-}^r - V_{t-\tau}^{\tilde r}) A_2^{r,\tilde r}\mathcal{R}(\mathrm{d}\tilde r)\,\mathrm{d}B_t^{r,\alpha} \\
\qquad - \sum_{\alpha=1}^P \int_U \int_{\Gamma_\alpha} (V_{t^-}^r - V_{t-\tau}^{\tilde r}) \bar\eta^{r,\tilde r}(\xi)\mathcal{R}(\mathrm{d}\tilde r)\tilde N^{r,\alpha}(\mathrm{d}t,\mathrm{d}\xi), \\
\mathrm{d}w_t^r = \lambda_3^r(V_t^r + \lambda_4^r - \lambda_5^r w_{t^-}^r)\,\mathrm{d}t.
\end{cases}
\tag{1.4}
$$

1.1. *Well-posedness of the McKean–Vlasov equation.* For a given finite Borel measure $\mathcal{R}$ on Γ, specifying the spatial (unnormalized) distribution of neurons, the McKean–Vlasov equation for the infinite population limit of the network (1.1) is given by the following equation:

$$
\mathrm{d}\bar X_t^r = f(t,r,\bar X_{t^-}^r,\omega')\,\mathrm{d}t + g(t,r,\bar X_{t^-}^r,\omega')\,\mathrm{d}W_t^r
$$

$$
\begin{aligned}
&+\int_U h(t,r,\bar{X}^r_{t-},\omega',\xi)\tilde{N}^r(\mathrm{d}t,\mathrm{d}\xi)\\
&+\sum_{\alpha=1}^{P}\int_{\Gamma_\alpha}\tilde{\mathbb{E}}[\theta(t,r,r',\bar{X}^r_{t-},\hat{X}^{r'}_{(t-\tau)^-:t^-},\omega')]\mathcal{R}(\mathrm{d}r')\,\mathrm{d}t\\
&+\sum_{\alpha=1}^{P}\int_{\Gamma_\alpha}\tilde{\mathbb{E}}[\beta(t,r,r',\bar{X}^r_{t-},\hat{X}^{r'}_{(t-\tau)^-:t^-},\omega')]\mathcal{R}(\mathrm{d}r')\,\mathrm{d}B^{r,\alpha}_t\\
&+\sum_{\alpha=1}^{P}\int_U\int_{\Gamma_\alpha}\tilde{\mathbb{E}}[\eta(t,r,r',\bar{X}^r_{t-},\hat{X}^{r'}_{(t-\tau)^-:t^-},\omega',\xi)]\mathcal{R}(\mathrm{d}r')\tilde{N}^{r,\alpha}(\mathrm{d}t,\mathrm{d}\xi),
\end{aligned}
\tag{1.5}
$$

$$\bar{X}^r_t = z^r_t, \quad t\in[-\tau,0]$$

for independent Brownian motions W^r, $B^{r,\alpha}$ and independent compensated Poisson measures $\tilde{N}^r$ and $\tilde{N}^{r,\alpha}$. Here, $\hat{X}$ is an independent copy of X, the solution of

$$
\begin{aligned}
\mathrm{d}X^r_t &= f(t,r,X^r_{t-},\omega')\,\mathrm{d}t + g(t,r,X^r_{t-},\omega')\,\mathrm{d}W_t\\
&\quad+\int_U h(t,r,X^r_{t-},\omega',\xi)\tilde{N}(\mathrm{d}t,\mathrm{d}\xi)\\
&\quad+\sum_{\alpha=1}^{P}\int_{\Gamma_\alpha}\tilde{\mathbb{E}}[\theta(t,r,r',X^r_{t-},\tilde{X}^{r'}_{(t-\tau)^-:t^-},\omega')]\mathcal{R}(\mathrm{d}r')\,\mathrm{d}t\\
&\quad+\sum_{\alpha=1}^{P}\int_{\Gamma_\alpha}\tilde{\mathbb{E}}[\beta(t,r,r',X^r_{t-},\tilde{X}^{r'}_{(t-\tau)^-:t^-},\omega')]\mathcal{R}(\mathrm{d}r')\,\mathrm{d}B^{\alpha}_t\\
&\quad+\sum_{\alpha=1}^{P}\int_{\Gamma_\alpha}\int_U\tilde{\mathbb{E}}[\eta(t,r,r',X^r_{t-},\tilde{X}^{r'}_{(t-\tau)^-:t^-},\omega',\xi)]\mathcal{R}(\mathrm{d}r')\tilde{N}^{\alpha}(\mathrm{d}t,\mathrm{d}\xi),
\end{aligned}
\tag{1.6}
$$

$$X^r_t = \hat{z}^\zeta_t, r\in\Gamma_\zeta, \quad \zeta=1,\dots,P, t\in[-\tau,0],$$

where $\tilde{X}=\hat{X}$ is a copy of X, defined on another probability space, say $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and $\tilde{\mathbb{E}}$ denotes expectation with respect to $\tilde{\mathbb{P}}$. To avoid unnecessary notation, we assume $\mathcal{R}(\Gamma_\alpha)=1$, for all $1\le\alpha\le P$.

Note that a solution to (1.6) requires in particular the measurability of $\tilde{X}$ w.r.t. r', since otherwise the integrals of the expectation values w.r.t. $\tilde{\mathbb{P}}$ are not well defined. In Theorem 1.5 below, we will therefore prove existence and uniqueness of a strong solution of equation (1.6) that is also measurable w.r.t. r. Note that this implies in particular the existence of an independent measurable copy $\tilde{X}$ or $\hat{X}$ of X. Therefore, the integrals w.r.t. $\mathcal{R}(\mathrm{d}r')$ in (1.5) and (1.6) are well defined. Since for each $r\in\Gamma$, $\bar{X}^r$ and X^r have the same law, so that $\hat{X}^r$ can be assumed to be a copy of $\bar{X}^r$, too.

In the reference [11], a representation of the McKean–Vlasov equation was given in terms of a continuum family of independent Brownian motions, so called spatially chaotic. In the statement of Theorem 1.8, we only use the solution of equation (1.5) for r in the countable set $\bigcup_{N\in\mathbb{N}}\mathcal{A}_N$ and for $\hat{X}$ being an independent copy of X, the solution of equation (1.6). In this sense, our solution coincides with the solution constructed in [11], but we had circumvented measurability issues of $\bar{X}^r$ w.r.t. r.

LEMMA 1.4. *Assume Hypothesis* 1.1. *For every measurable strong solution X of equation* (1.6) *which belongs to $L^\infty([-\tau,T],\mathrm{d}t;L^2(\Omega\times\Gamma,\mathbb{P}\otimes\mathcal{R};\mathbb{R}^d))$ for $\mathbb{P}'$-almost all $\omega'\in\Omega'$,* *we have*

$$\sup_{s\in[-\tau,t]} \mathbb{E}\big[|X_s^r|^2\big] \le C_1(t,\omega'), \quad r\in\Gamma, t\le T,$$

where

$$C_1(t,\omega') := \Big(\sup_{\substack{u\in[-\tau,0]\\ 1\le\zeta\le P}} \mathbb{E}\big|\hat{z}^\zeta(u)\big|^2 + 1\Big) \times \exp\Big(\int_0^t \big(K_s(\omega') + 3P\bar{K}_s(\omega') + P\big)\,\mathrm{d}s\Big). \tag{1.7}$$

PROOF. Let $\tau_{n,r} := \inf\{t\ge 0; |X_t^r| > n\}$ and

$$\psi_t(\omega') := \exp\Big(-\int_0^t \big(K_s(\omega') + P\bar{K}_s(\omega') + P\big)\,\mathrm{d}s\Big).$$

Using Itô's formula, we get for $r\in\Gamma_\zeta$ that

$$\begin{aligned}
&\psi_{t\wedge\tau_{n,r}}(\omega')\big|X_{t\wedge\tau_{n,r}}^r\big|^2\\
&\quad= \big|\hat{z}^\zeta(0)\big|^2\\
&\qquad+\int_0^{t\wedge\tau_{n,r}} \psi_s(\omega')\big[2\big\langle X_{s-}^r, f(s,r,X_{s-}^r,\omega')\big\rangle + \big|g(s,r,X_{s-}^r,\omega')\big|^2\big]\,\mathrm{d}s\\
&\qquad+\int_0^{t\wedge\tau_{n,r}} \psi_s(\omega')\int_U \big|h(s,r,X_{s-}^r,\omega',\xi)\big|^2 N(\mathrm{d}s,\mathrm{d}\xi)\\
&\qquad+\int_0^{t\wedge\tau_{n,r}} 2\psi_s(\omega')\sum_{\alpha=1}^P\Big\langle X_{s-}^r, \tilde{\mathbb{E}}\int_{\Gamma_\alpha}\theta\big(s,r,r',X_{s-}^r,\tilde{X}^{r'}_{(s-\tau)^-:s^-},\omega'\big)\mathcal{R}(\mathrm{d}r')\Big\rangle\,\mathrm{d}s\\
&\qquad+\int_0^{t\wedge\tau_{n,r}} \psi_s(\omega')\sum_{\alpha=1}^P\Big|\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\beta\big(s,r,r',X_{s-}^r,\tilde{X}^{r'}_{(s-\tau)^-:s^-},\omega'\big)\mathcal{R}(\mathrm{d}r')\Big|^2\,\mathrm{d}s\\
&\qquad+\int_0^{t\wedge\tau_{n,r}} \psi_s(\omega')\sum_{\alpha=1}^P\int_U\Big|\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\eta\big(s,r,r',X_{s-}^r,\tilde{X}^{r'}_{(s-\tau)^-:s^-},\omega'\big)\mathcal{R}(\mathrm{d}r')\Big|^2 N^\alpha(\mathrm{d}s,\mathrm{d}\xi)\\
&\qquad-\int_0^{t\wedge\tau_{n,r}} \psi_s(\omega')\big(K_s(\omega') + P\bar{K}_s(\omega') + P\big)\big|X_{s-}^r\big|^2\,\mathrm{d}s + M_{t\wedge\tau_{n,r}},
\end{aligned}$$

where $M_{t\wedge\tau_{n,r}}$ is a martingale starting from zero. Taking expectation and using (H2) and (H5), we get

$$\begin{aligned}
&\mathbb{E}\big(\psi_{t\wedge\tau_{n,r}}(\omega')\big|X_{t\wedge\tau_{n,r}}^r\big|^2\big)\\
&\quad\le \mathbb{E}\big|\hat{z}^\zeta(0)\big|^2 + \mathbb{E}\int_0^{t\wedge\tau_{n,r}}\psi_s(\omega')\Big[K_s(\omega') + P\bar{K}_s(\omega')\\
&\qquad+\bar{K}_s(\omega')\tilde{\mathbb{E}}\int_\Gamma\int_{-\tau}^0\big(\big|\tilde{X}^{r'}_{(s+u)-}\big|^2 + \mathbf{1}_{\{u<0\}}\big|\tilde{X}^{r'}_{s+u}\big|^2\big)\lambda(\mathrm{d}u)\mathcal{R}(\mathrm{d}r')\Big]\,\mathrm{d}s\\
&\quad\le \mathbb{E}\big|\hat{z}^\zeta(0)\big|^2 + 1 + \mathbb{E}\int_0^{t\wedge\tau_{n,r}} 2\psi_s(\omega')\bar{K}_s(\omega')\sup_{u\in[-\tau,s]}\tilde{\mathbb{E}}\int_\Gamma\big|\tilde{X}_u^{r'}\big|^2\mathcal{R}(\mathrm{d}r')\,\mathrm{d}s
\end{aligned}$$

and with Fatou's lemma it follows that

$$\psi_t(\omega')\mathbb{E}|X_t^r|^2 \le \liminf_{n\to\infty}\mathbb{E}\big(\psi_{t\wedge\tau_{n,r}}(\omega')\big|X_{t\wedge\tau_n}^r\big|^2\big)$$

$$
\begin{aligned}
&\le \mathbb{E}\big|\hat{z}^{\zeta}(0)\big|^2 + 1 \\
&\qquad + \mathbb{E}\int_0^t 2\psi_s(\omega')\bar{K}_s(\omega') \sup_{u\in[-\tau,s]} \tilde{\mathbb{E}}\int_\Gamma \big|\tilde{X}_u^{r'}\big|^2 \mathcal{R}(\mathrm{d}r')\,\mathrm{d}s.
\end{aligned}
\tag{1.8}
$$

Integrating w.r.t. $\mathcal{R}$ and using Gronwall's lemma, we obtain that

$$
\begin{aligned}
&\psi_t(\omega') \sup_{u\in[-\tau,t]} \mathbb{E}\int_\Gamma \big|X_u^r\big|^2 \mathcal{R}(\mathrm{d}r) \\
&\quad \le \Big(\sup_{u\in[-\tau,0]} \mathbb{E}\sum_{\zeta=1}^{P}\big|\hat{z}^{\zeta}(u)\big|^2 + P\Big) \exp\int_0^t \big(2P\bar{K}_s(\omega')\big)\,\mathrm{d}s.
\end{aligned}
\tag{1.9}
$$

By substituting (1.9) in (1.8), we get

$$
\psi_t(\omega')\mathbb{E}\big|X_t^r\big|^2 \le \Big(\sup_{\substack{u\in[-\tau,0]\\ 1\le\zeta\le P}} \mathbb{E}\big|\hat{z}^{\zeta}(u)\big|^2 + 1\Big) \exp\int_0^t \big(2P\bar{K}_s(\omega')\big)\,\mathrm{d}s.
$$

Therefore,

$$
\begin{aligned}
\sup_{s\in[-\tau,t]} \mathbb{E}\big|X_s^r\big|^2 &\le \Big(\sup_{\substack{u\in[-\tau,0]\\ 1\le\zeta\le P}} \mathbb{E}\big|\hat{z}^{\zeta}(u)\big|^2 + 1\Big) \exp\int_0^t \big(K_s(\omega') + 3P\bar{K}_s(\omega') + P\big)\,\mathrm{d}s \\
&= C_1(t,\omega'). \qquad \square
\end{aligned}
$$

THEOREM 1.5. *Equation* (1.6) *has a unique strong solution* $X(\omega') \in L^\infty([-\tau,T],\mathrm{d}t; L^2(\Omega\times\Gamma,\mathbb{P}\otimes\mathcal{R};\mathbb{R}^d))$ *on* $[-\tau,T]$ *for any* $T>0$ *and* $\mathbb{P}'$*-almost every* $\omega'\in\Omega'$ *which is in particular measurable w.r.t.* (r,ω').

The proof of the theorem is postponed to Section 2.

1.2. *Propagation of chaos.* We are now going to state a convergence result for the solution of the network equations (1.1) to the solution of the McKean–Vlasov equation (1.5) in the infinite population limit. To this end, we first have to specify a condition on the spatial density of the approximating network populations and a statement concerning the dependence of X_t^r w.r.t. the spatial parameter r. To this end, consider the following hypothesis:

HYPOTHESIS 1.6. *The coefficients of the network equation* (1.1) *satisfy Hypothesis* 1.1. *In addition, we assume the existence of a finite Borel measure* $\mathcal{R}$ *on* Γ *with the following property: For every* $\varepsilon>0$, *there exists a (finite) partition* $\{\Gamma_\alpha^{m,\varepsilon}, 1\le m\le M_\alpha^{(\varepsilon)}\}$ *of* Γ_α *such that*

$$
\lim_{N\to\infty} \frac{\#(\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon})}{\mathcal{S}_{\mathcal{A}_N,\alpha}} = \mathcal{R}\big(\Gamma_\alpha^{m,\varepsilon}\big), \qquad 1\le m\le M_\alpha^{(\varepsilon)}
\tag{1.10}
$$

and also $\lim_{N\to\infty}\mathcal{S}_{\mathcal{A}_N,\alpha}=\infty$, $1\le\alpha\le P$.

W.r.t. this partition (H1) *and* (H4) *of Hypothesis* 1.1 *are then replaced by the following stronger assumptions*:

(H1') $\forall r,\tilde{r}\in\Gamma_\alpha^{m,\varepsilon}$,

$$
\begin{aligned}
&2\big\langle x-y, f(t,r,x,\omega') - f(t,\tilde{r},\tilde{x},\omega')\big\rangle + \big|g(t,r,x,\omega') - g(t,\tilde{r},\tilde{x},\omega')\big|^2 \\
&\qquad + \int_U \big|h(t,r,x,\omega',\xi) - h(t,\tilde{r},\tilde{x},\omega',\xi)\big|^2 \nu(\mathrm{d}\xi) \\
&\quad \le L_t(\omega')\big[|x-\tilde{x}|^2 + \varepsilon\big(1+|x|^2\big)\big],
\end{aligned}
$$

(H4') $\forall r, \tilde{r} \in \Gamma_\alpha^{m,\varepsilon}$, $\forall r', \tilde{r}' \in \Gamma_{\alpha'}^{m',\varepsilon}$,

$$\begin{aligned}
&\sum_{\Theta\in\{\theta,\beta\}} \big|\Theta(t,r,r',x,y_{-\tau:0},\omega') - \Theta(t,\tilde{r},\tilde{r}',\tilde{x},\tilde{y}_{-\tau:0},\omega')\big|^2 \\
&\qquad + \int_U \big|\eta(t,r,r',x,y_{-\tau:0},\omega',\xi) - \eta(t,\tilde{r},\tilde{r}',\tilde{x},\tilde{y}_{-\tau:0},\omega',\xi)\big|^2 \nu(\mathrm{d}\xi) \\
&\quad \le \bar{L}_t(\omega')\bigg[|x-\tilde{x}|^2 + \int_{-\tau}^0 [|y_s - \tilde{y}_s|^2 + \mathbf{1}_{\{s<0\}}|y_{s^+} - \tilde{y}_{s^+}|^2]\lambda(\mathrm{d}s) \\
&\qquad + \varepsilon\bigg(1 + |x|^2 + \int_{-\tau}^0 [|y_s|^2 + \mathbf{1}_{\{s<0\}}|y_{s^+}|^2]\lambda(\mathrm{d}s)\bigg)\bigg].
\end{aligned}$$

LEMMA 1.7. *Under hypothesis* 1.6, *the solution* $X(\omega') \in L^\infty([-\tau,T],\mathrm{d}t; L^2(\Omega\times\Gamma, \mathbb{P}\otimes\mathcal{R};\mathbb{R}^d))$, $\omega'\in\Omega'$, *of equation* (1.6) *satisfies*

$$\mathbb{E}|X_t^r - X_t^{\tilde{r}}|^2 \le C_2(t,\omega')\varepsilon \quad \forall r,\tilde{r}\in\Gamma_\alpha^{m,\varepsilon}, t\le T,$$

where

$$C_2(t,\omega') := \exp\bigg[\int_0^t (L_s(\omega') + P\bar{L}_s(\omega') + P)\,\mathrm{d}s\bigg](1 + 3C_1(t,\omega')).$$

PROOF. Let $\psi_t(\omega') := \exp[-\int_0^t (L_s(\omega') + P\bar{L}_s(\omega') + P)\,\mathrm{d}s]$. To shorten notation, let $u := (s, r, X_{s-}^r, \omega')$, $\tilde{u} := (s,\tilde{r}, X_{s-}^{\tilde{r}}, \omega')$, $v := (s, r, r', X_{s-}^r, \tilde{X}^{r'}_{(s-\tau)^-:s^-}, \omega')$ and $\tilde{v} := (s,\tilde{r},r', X_{s-}^{\tilde{r}}, \tilde{X}^{r'}_{(s-\tau)^-:s^-}, \omega')$. We then have

$$\begin{aligned}
&\psi_t(\omega')\mathbb{E}|X_t^r - X_t^{\tilde{r}}|^2 \\
&\quad\le \mathbb{E}\int_0^t \psi_s(\omega')\Big[2\langle X_{s-}^r - X_{s-}^{\tilde{r}}, f(u) - f(\tilde{u})\rangle + |g(u)-g(\tilde{u})|^2 \\
&\qquad + \int_U |h(u,\xi) - h(\tilde{u},\xi)|^2\nu(\mathrm{d}\xi) \\
&\qquad - (L_s(\omega') + P\bar{L}_s(\omega') + P)|X_{s-}^r - X_{s-}^{\tilde{r}}|^2\Big]\,\mathrm{d}s \\
&\qquad + \sum_{\alpha=1}^P \mathbb{E}\int_0^t \psi_s(\omega')\bigg[2\Big\langle X_{s-}^r - X_{s-}^{\tilde{r}}, \tilde{\mathbb{E}}\int_{\Gamma_\alpha}(\theta(v)-\theta(\tilde{v}))\mathcal{R}(\mathrm{d}r')\Big\rangle \\
&\qquad + \bigg|\tilde{\mathbb{E}}\int_{\Gamma_\alpha}(\beta(v)-\beta(\tilde{v}))\mathcal{R}(\mathrm{d}r')\bigg|^2 \\
&\qquad + \int_U\bigg|\tilde{\mathbb{E}}\int_{\Gamma_\alpha}(\eta(v,\xi)-\eta(\tilde{v},\xi))\mathcal{R}(\mathrm{d}r')\bigg|^2\nu(\mathrm{d}\xi)\bigg]\,\mathrm{d}s \\
&\quad\le \mathbb{E}\int_0^t \psi_s(\omega')\varepsilon\bigg[L_s(\omega')(1 + |X_{s-}^r|^2) + \bar{L}_s(\omega')\Big(P + P|X_{s-}^r|^2 \\
&\qquad + \tilde{\mathbb{E}}\int_\Gamma\int_{-\tau}^0 [|\tilde{X}^{r'}_{(s+u)^-}|^2 + \mathbf{1}_{\{u<0\}}|\tilde{X}^{r'}_{s+u}|^2]\lambda(\mathrm{d}u)\mathcal{R}(\mathrm{d}r')\Big)\bigg]\,\mathrm{d}s \\
&\quad\le \varepsilon\int_0^t \psi_s(\omega')(P\bar{L}_s(\omega') + L_s(\omega'))(1 + 3C_1(s,\omega'))\,\mathrm{d}s \\
&\quad\le \varepsilon(1 + 3C_1(t,\omega')).
\end{aligned}$$

Dividing by $\psi_t(\omega')$, we get the desired result. $\square$

The following theorem now is our second main result.

THEOREM 1.8. *Under Hypothesis* 1.6 *and the chaotic initial condition assumption* (*i.e.*, *the initial conditions* z^r *for* $r \in \Gamma_\alpha \cap (\bigcup_{N\in\mathbb{N}} \mathcal{A}_N)$ *are independent and identically distributed copies of* $\hat{z}^\alpha$ *with* $\hat{z}^\alpha \in L^2(\Omega, \mathbb{P}; Càdlàg([-\tau, 0]; \mathbb{R}^d))$), *the solution* $(X_t^{r,\mathcal{A}_N}, -\tau \le t \le T)$ *of the network equation* (1.1) *converges in the space* $L^2(\Omega', L^\infty([0,T], L^2(\Omega, \mathbb{E})))$ *toward the process* $(\bar{X}_t^r, -\tau \le t \le T)$ *which is the solution of the mean-field equation* (1.5), *that is*,

$$\lim_{N\to\infty} \mathcal{E} \sup_{\substack{t\in[-\tau,T]\\ r\in\mathcal{A}_N}} \mathbb{E}\left|X_t^{r,\mathcal{A}_N} - \bar{X}_t^r\right|^2 = 0.$$

The proof of the theorem is given in Section 3.

2. Proof of Theorem 1.5.

PROOF. *Existence*: Fix $n \in \mathbb{N}$ and define $\kappa(n,t) := \frac{k\tau}{n}$ for $t \in]\frac{k\tau}{n}, \frac{(k+1)\tau}{n}]$. We then define the process $X^{n,r}$ inductively as follows: Let $X_t^{n,r} := \hat{z}_t^\zeta$ for $t \in [-\tau, 0]$ and $r \in \Gamma_\zeta$. Given that $X_t^{n,r}$ is defined for $t \le \frac{k\tau}{n}$ and for all $r \in \Gamma$ we extend $X_t^{n,r}$ for $t \in]\frac{k\tau}{n}, \frac{(k+1)\tau}{n}]$ as the unique strong solution of

$$\begin{aligned} X_t^{n,r} = {} & X_{\frac{k\tau}{n}}^{n,r} + \int_{\frac{k\tau}{n}}^t f\bigl(s, r, X_{s^-}^{n,r}, \omega'\bigr)\,\mathrm{d}s + \int_{\frac{k\tau}{n}}^t g\bigl(s, r, X_{s^-}^{n,r}, \omega'\bigr)\,\mathrm{d}W_s \\ & + \int_{\frac{k\tau}{n}}^t \int_U h\bigl(s, r, X_{s^-}^{n,r}, \omega', \xi\bigr)\tilde{N}(\mathrm{d}s, \mathrm{d}\xi) \\ & + \sum_{\alpha=1}^P \int_{\frac{k\tau}{n}}^t \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \theta\bigl(s, r, r', X_{s^-}^{n,r}, Y_{\kappa(n,(s-\tau):s)}^{n,r'}, \omega'\bigr)\mathcal{R}\bigl(\mathrm{d}r'\bigr)\,\mathrm{d}s \\ & + \sum_{\alpha=1}^P \int_{\frac{k\tau}{n}}^t \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \beta\bigl(s, r, r', X_{s^-}^{n,r}, Y_{\kappa(n,(s-\tau):s)}^{n,r'}, \omega'\bigr)\mathcal{R}\bigl(\mathrm{d}r'\bigr)\,\mathrm{d}B_s^\alpha \\ & + \sum_{\alpha=1}^P \int_{\frac{k\tau}{n}}^t \int_U \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \eta\bigl(s, r, r', X_{s^-}^{n,r}, Y_{\kappa(n,(s-\tau):s)}^{n,r'}, \omega', \xi\bigr)\mathcal{R}\bigl(\mathrm{d}r'\bigr)\tilde{N}^\alpha(\mathrm{d}s, \mathrm{d}\xi), \end{aligned} \tag{2.1}$$

which exists and is measurable w.r.t. $(t, r, \omega, \omega') \in [\frac{k\tau}{n}, \frac{(k+1)\tau}{n}] \times \Gamma \times \Omega \times \Omega'$ and satisfies

$$1 + \sup_{u\le t} \mathbb{E}\left|X_u^{n,r}\right|^2 \le \Bigl(1 + \sup_{u\le\frac{k\tau}{n}} \mathbb{E}\left|X_u^{n,r}\right|^2\Bigr) \exp\biggl(\int_{k\tau/n}^t \bigl(K_s(\omega') + 3P\bar{K}_s(\omega')\bigr)\,\mathrm{d}s\biggr)$$

according to Theorem A.2. Then for all $n \in \mathbb{N}$ and all $r \in \Gamma$,

$$\sup_{t\in[-\tau,T]} \mathbb{E}\left|X_t^{n,r}\right|^2 \le C_1\bigl(T, \omega'\bigr). \tag{2.2}$$

Here, $Y^{n,r'}$ is an independent copy of $X^{n,r'}$ on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and the expectation $\tilde{\mathbb{E}}$ is taken with respect to $\tilde{\mathbb{P}}$ and

$$Y_{\kappa(n,(s-\tau):s)}^{n,r'}(u) := Y_{\kappa(n,s+u)}^{n,r'}, \quad u \in [-\tau, 0].$$

We also use in the following the notation

$$Y_{\kappa(n,t^+)}^{n,r} := \lim_{s\searrow t} Y_{\kappa(n,s)}^{n,r}.$$

For convenience, we assume that $Y^{n,r}$ is obtained similar to $X^{n,r}$ using independent copies

$$(\{\tilde{z}^\alpha\}_{1\le\alpha\le P}, \tilde{W}, \{\tilde{B}^\alpha\}_{1\le\alpha\le P}, \bar{N}, \{\bar{N}^\alpha\}_{1\le\alpha\le P})$$

of $(\{z^\alpha\}_{1\le\alpha\le P}, W, \{B^\alpha\}_{1\le\alpha\le P}, N, \{N^\alpha\}_{1\le\alpha\le P})$. Note that $X^{n,r}$ is càdlàg, whereas the process $X^{n,r}_{\kappa(n,t)}$, $t \ge -\tau$, is càglàd. It is easy to see, using induction w.r.t. to k, that $X^{(n)}_t$, $t \in]\frac{k\tau}{n}, \frac{(k+1)}{n}]$ is a.s. locally bounded and that the stochastic integrals are well defined and local martingales up to time $+\infty$.

Let us next define the remainder

$$p^{n,r}_t = X^{n,r}_{\kappa(n,t)} - X^{n,r}_{t^-}, \qquad q^{n,r}_t = Y^{n,r}_{\kappa(n,t)} - Y^{n,r}_{t^-}, \qquad t \in [-\tau, T].$$

We can then write

$$\begin{aligned}
X^{n,r}_t &= \hat{z}^\zeta(0) + \int_0^t f(s, r, X^{n,r}_{s^-}, \omega')\,\mathrm{d}s + \int_0^t g(s, r, X^{n,r}_{s^-}, \omega')\,\mathrm{d}W_s \\
&\quad + \int_0^t \int_U h(s, r, X^{n,r}_{s^-}, \omega', \xi)\tilde{N}(\mathrm{d}s, \mathrm{d}\xi) \\
&\quad + \sum_{\alpha=1}^P \int_0^t \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \theta(s, r, r', X^{n,r}_{s^-}, Y^{n,r'}_{(s-\tau)^-:s^-} + q^{n,r'}_{(s-\tau):s}, \omega')\mathcal{R}(\mathrm{d}r')\,\mathrm{d}s \\
&\quad + \sum_{\alpha=1}^P \int_0^t \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \beta(s, r, r', X^{n,r}_{s^-}, Y^{n,r'}_{(s-\tau)^-:s^-} + q^{n,r'}_{(s-\tau):s}, \omega')\mathcal{R}(\mathrm{d}r')\,\mathrm{d}B^\alpha_s \\
&\quad + \sum_{\alpha=1}^P \int_0^t \int_U \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \eta(s, r, r', X^{n,r}_{s^-}, Y^{n,r'}_{(s-\tau)^-:s^-} + q^{n,r'}_{(s-\tau):s}, \\
&\quad \omega', \xi)\mathcal{R}(\mathrm{d}r')\tilde{N}^\alpha(\mathrm{d}s, \mathrm{d}\xi).
\end{aligned} \tag{2.3}$$

In the next step, let us define the stopping times

$$\tau^{n,r}_R := \inf\left\{t \ge 0 : |X^{n,r}_t| > \frac{R}{3}\right\}$$

for given $R > 0$. Then

$$|p^{n,r}_t| \le \frac{2R}{3}, \qquad |X^{n,r}_{t^-}|, |X^{n,r}_{\kappa(n,t)}| \le \frac{R}{3}, \quad t \in (0, \tau^{n,r}_R].$$

We now prove the following properties which complete the existence proof:

(i) For all $t > -\tau$, $\mathbf{1}_{(-\tau,\tau^{n,r}_R]}(t)|p^{n,r}_t| \to 0$ in probability as $n \to \infty$.
(ii) For any stopping time $\tau^* \le T \wedge \tau^{n,r}_R$, we have $\mathbb{E}|X^{n,r}_{\tau^*}|^2 \le C(T, \omega')$.
(iii) $\lim_{R\to\infty} \limsup_{n\to\infty} \mathbb{P}\{\tau^{n,r}_R < T\} = 0$.
(iv) $\forall\varepsilon > 0$, $\lim_{n,m\to\infty} \mathbb{P}\{\sup_{t\in[0,T]}|X^{n,r}_t - X^{m,r}_t| > \varepsilon\} = 0$.
(v) $\exists X : \forall\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}\{\sup_{t\in[0,T]}|X^{n,r}_t - X^r_t| > \varepsilon\} = 0$ and X is a strong solution of equation (1.6).

Proof of (i). Since $\hat{z}^\zeta$ is càdlàg w.r.t. time, it follows that $\mathbf{1}_{(-\tau,0]}(t)|p^{n,r}_t| \to 0$ almost surely. To shorten notation again, let

$$u^n_s := (s, r, X^{n,r}_{s^-}, \omega') \quad \text{and} \quad v^n_s := (s, r, r', X^{n,r}_{s^-}, Y^{n,r'}_{\kappa(n,(s-\tau):s)}, \omega').$$

Using (2.1) and Hypothesis 1.1, we have

$$
\begin{aligned}
&\mathbb{P}\{|p_t^{n,r}| \geq \varepsilon, 0 < t \leq \tau_R^{n,r}\} \\
&\quad \leq \mathbb{P}\bigg\{\int_{\kappa(n,t)}^t \Big(\sup_{|x|\leq R} |f(s,r,x,\omega')| \\
&\qquad + P\sqrt{\bar{K}_s(\omega')\big(1+R^2+2C_1(t,\omega')\big)}\Big)\,\mathrm{d}s \geq \varepsilon/5\bigg\} \\
&\qquad + \mathbb{P}\bigg\{\bigg|\int_{\kappa(n,t)}^t g(u_s^n)\,\mathrm{d}W_s\bigg| \geq \varepsilon/5, t \leq \tau_R^{n,r}\bigg\} \\
&\qquad + \mathbb{P}\bigg\{\bigg|\int_{\kappa(n,t)}^{t^-}\int_U h(u_s^n,\xi)\tilde{N}(\mathrm{d}s,\mathrm{d}\xi)\bigg| \geq \varepsilon/5, t \leq \tau_R^{n,r}\bigg\} \\
&\qquad + \sum_{\alpha=1}^P \mathbb{P}\bigg\{\bigg|\int_{\kappa(n,t)}^t \tilde{\mathbb{E}}\int_{\Gamma_\alpha} \beta(v_s^n)\mathcal{R}(\mathrm{d}r')\mathrm{d}B_s^\alpha\bigg| \geq \frac{\varepsilon}{5P}, t \leq \tau_R^{n,r}\bigg\} \\
&\qquad + \sum_{\alpha=1}^P \mathbb{P}\bigg\{\bigg|\int_{\kappa(n,t)}^{t^-}\int_U \tilde{\mathbb{E}}\int_{\Gamma_\alpha} \eta(v_s^n,\xi)\mathcal{R}(\mathrm{d}r')\tilde{N}^\alpha(\mathrm{d}s,\mathrm{d}\xi)\bigg| \geq \frac{\varepsilon}{5P}, t \leq \tau_R^{n,r}\bigg\},
\end{aligned}
$$

which can be further estimated from above by

$$
\begin{aligned}
&\leq \mathbb{P}\bigg\{\int_{\kappa(n,t)}^t \Big(\tilde{K}_s(R,\omega') + P\sqrt{\bar{K}_s(\omega')\big(1+R^2+2C_1(t,\omega')\big)}\Big)\,\mathrm{d}s \geq \varepsilon/5\bigg\} \\
&\quad + \frac{25}{\varepsilon^2}\mathbb{E}\bigg(\bigg|\int_{\kappa(n,t)}^t \mathbf{1}_{\{s\leq\tau_R^{n,r}\}} g(u_s^n)\,\mathrm{d}W_s\bigg|^2\bigg) \\
&\quad + \frac{25}{\varepsilon^2}\mathbb{E}\bigg(\bigg|\int_{\kappa(n,t)}^{t^-}\int_U \mathbf{1}_{\{s\leq\tau_R^{n,r}\}} h(u_s^n,\xi)\tilde{N}(\mathrm{d}s,\mathrm{d}\xi)\bigg|^2\bigg) \\
&\quad + \frac{25P^2}{\varepsilon^2}\sum_{\alpha=1}^P \mathbb{E}\bigg(\bigg|\int_{\kappa(n,t)}^t \tilde{\mathbb{E}}\int_{\Gamma_\alpha} \mathbf{1}_{\{s\leq\tau_R^{n,r}\}} \beta(v_s^n)\mathcal{R}(\mathrm{d}r')\,\mathrm{d}B_s^\alpha\bigg|^2\bigg) \\
&\quad + \frac{25P^2}{\varepsilon^2}\sum_{\alpha=1}^P \mathbb{E}\bigg(\bigg|\int_{\kappa(n,t)}^{t^-}\int_U \tilde{\mathbb{E}}\int_{\Gamma_\alpha} \mathbf{1}_{\{s\leq\tau_R^{n,r}\}} \eta(v_s^n,\xi)\mathcal{R}(\mathrm{d}r')\tilde{N}^\alpha(\mathrm{d}s,\mathrm{d}\xi)\bigg|^2\bigg)
\end{aligned}
$$

and finally by

$$
\begin{aligned}
&\leq \mathbb{P}\bigg\{\int_{\kappa(n,t)}^t \Big(\tilde{K}_s(R,\omega') + P\sqrt{\bar{K}_s(\omega')\big(1+R^2+2C_1(t,\omega')\big)}\Big)\,\mathrm{d}s \geq \varepsilon/5\bigg\} \\
&\quad + \frac{25}{\varepsilon^2}\int_{\kappa(n,t)}^t \tilde{K}_s(R,\omega')\,\mathrm{d}s + \frac{25P^3}{\varepsilon^2}\int_{\kappa(n,t)}^t \bar{K}_s(\omega')\big(1+R^2+2C_1(t,\omega')\big)\,\mathrm{d}s.
\end{aligned}
$$

So

$$
\limsup_{n\to\infty} \mathbb{P}\{|p_t^{n,r}| \geq \varepsilon, -\tau < t \leq \tau_R^{n,r}\} = 0,
$$

which implies (i).

Proof of (ii). Let τ^* be a stopping time such that $\tau^* \leq T \wedge \tau_R^{n,r}$. Similar to the proof of the corresponding statement in Theorem A.2, we have that

$$
\mathbb{E}|X_{\tau^*}^{n,r}|^2 \leq \mathbb{E}|\hat{z}_0^\zeta|^2 + \mathbb{E}\int_0^{\tau^*}\Big[2\langle X_{s^-}^{n,r}, f(s,r,X_{s^-}^{n,r},\omega')\rangle
$$

$$
\begin{aligned}
&\quad + \big|g\big(s, r, X^{n,r}_{s-}, \omega'\big)\big|^2 + \int_U \big|h\big(s, r, X^{n,r}_{s-}, \omega', \xi\big)\big|^2 \nu(\mathrm{d}\xi)\Big]\,\mathrm{d}s \\
&\quad + \sum_{\alpha=1}^{P} \mathbb{E}\int_0^{\tau^*} \tilde{\mathbb{E}}\int_{\Gamma_\alpha}\Big[2\big\langle X^{n,r}_{s-}, \theta\big(s, r, r', X^{n,r}_{s-}, Y^{n,r'}_{\kappa(n,(s-\tau):s)}, \omega'\big)\big\rangle \\
&\quad + \big|\beta\big(s, r, r', X^{n,r}_{s-}, Y^{n,r'}_{\kappa(n,(s-\tau):s)}, \omega'\big)\big|^2 \\
&\quad + \int_U \big|\eta\big(s, r, r', X^{n,r}_{s-}, Y^{n,r'}_{\kappa(n,(s-\tau):s)}, \omega', \xi\big)\big|^2 \nu(\mathrm{d}\xi)\Big]\mathcal{R}\big(\mathrm{d}r'\big)\,\mathrm{d}s \\
&\le \mathbb{E}\big|\hat{z}^{\zeta}_0\big|^2 + \mathbb{E}\int_0^T \Big[\big(K_s(\omega') + P\bar{K}_s(\omega') + P\big)\big(1 + \big|X^{n,r}_{s-}\big|^2\big) \\
&\quad + \bar{K}_s(\omega')\tilde{\mathbb{E}}\int_\Gamma \int_{-\tau}^0 \big[\big|Y^{n,r'}_{\kappa(n,s+u)}\big|^2 \\
&\quad + \mathbf{1}_{\{u<0\}}\big|Y^{n,r'}_{\kappa(n,(s+u)^+)}\big|^2\big]\lambda(\mathrm{d}u)\mathcal{R}\big(\mathrm{d}r'\big)\Big]\,\mathrm{d}s \\
&\le \mathbb{E}\big|\hat{z}^{\zeta}_0\big|^2 + \big(1 + C_1(T, \omega')\big)\int_0^T \big(K_s(\omega') + 3P\bar{K}_s(\omega') + P\big)\,\mathrm{d}s \\
&\le C\big(T, \omega'\big).
\end{aligned}
$$

Here, we have used that for $t \in (0, T]$

$$
\mathbb{E}\big|X^{n,r}_{t-}\big|^2 \le \liminf_{\delta\downarrow 0} \mathbb{E}\big|X^{n,r}_{t-\delta}\big|^2 \le \sup_{s\in[-\tau,T]} \mathbb{E}\big|X^{n,r}_{s}\big|^2 \le C_1\big(T, \omega'\big).
$$

Proof of (iii). Let

$$
\tau^* = T \wedge \tau^{n,r}_R \wedge \inf\big\{t \ge 0 : \big|X^{n,r}_t\big| \ge a\big\}
$$

in (ii), we get

$$
\mathbb{P}\Big\{\sup_{t\in[0,T\wedge\tau^{n,r}_R]} \big|X^{n,r}_t\big| \ge a\Big\} \le \frac{1}{a^2}\mathbb{E}\big|X^{n,r}_{\tau^*}\big|^2 \le \frac{C(T, \omega')}{a^2}.
$$

So

$$
\begin{aligned}
&\limsup_{R\to\infty}\limsup_{n\to\infty} \mathbb{P}\Big\{\sup_{t\in[0,\tau^{n,r}_R]} \big|X^{n,r}_t\big| \ge \frac{R}{4};\ \tau^{n,r}_R \le T\Big\} \\
&\qquad \le \limsup_{R\to\infty}\limsup_{n\to\infty} \mathbb{P}\Big\{\sup_{t\in[0,T\wedge\tau^{n,r}_R]} \big|X^{n,r}_t\big| \ge \frac{R}{4}\Big\} \\
&\qquad = \limsup_{R\to\infty}\limsup_{n\to\infty}\Big(\frac{16C(T, \omega')}{R^2}\Big) = 0.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\limsup_{R\to\infty}\limsup_{n\to\infty} \mathbb{P}\big\{\tau^{n,r}_R \le T\big\} \\
&\quad \le \limsup_{R\to\infty}\limsup_{n\to\infty} \mathbb{P}\Big\{\sup_{t\in[0,\tau^{n,r}_R]} \big|X^{n,r}_t\big| \ge \frac{R}{4};\ \tau^{n,r}_R \le T\Big\} = 0,
\end{aligned}
$$

which completes the proof of (iii).

Proof of (iv). Let $\tau_R^{n,m,r} := T \wedge \tau_R^{n,r} \wedge \tau_R^{m,r}$ and let

$$\tilde{\tau}_R^{n,r} := \inf\left\{t \geq 0 : |Y_t^{n,r}| > \frac{R}{3}\right\}, \qquad \tilde{\tau}_R^{n,m,r} := T \wedge \tilde{\tau}_R^{n,r} \wedge \tilde{\tau}_R^{m,r}.$$

To shorten notation again, let

$$u_s^n := (s, r, X_{s-}^{n,r}, \omega') \quad \text{and} \quad v_s^n := (s, r, r', X_{s-}^{n,r}, Y_{\kappa(n,(s-\tau):s)}^{n,r'}, \omega').$$

Using Itô's formula, we have for any stopping time $\bar{\tau} \leq t \wedge \tau_R^{n,m,r}$, $t \in [0, T]$, that

$$\begin{aligned}
&\mathbb{E}|X_{\bar{\tau}}^{n,r} - X_{\bar{\tau}}^{m,r}|^2 \\
&\quad\leq \mathbb{E}\int_0^{\bar{\tau}} 2\langle X_{s-}^{n,r} - X_{s-}^{m,r}, f(u_s^n) - f(u_s^m)\rangle \,\mathrm{d}s \\
&\qquad + \mathbb{E}\int_0^{\bar{\tau}} |g(u_s^n) - g(u_s^m)|^2 \,\mathrm{d}s + \mathbb{E}\int_0^{\bar{\tau}}\int_U |h(u_s^n, \xi) - h(u_s^m, \xi)|^2 \nu(\mathrm{d}\xi)\,\mathrm{d}s \\
&\qquad + \sum_{\alpha=1}^P \mathbb{E}\int_0^{\bar{\tau}} 2\left\langle X_{s-}^{n,r} - X_{s-}^{m,r}, \tilde{\mathbb{E}}\int_{\Gamma_\alpha}[\theta(v_s^n) - \theta(v_s^m)]\mathcal{R}(\mathrm{d}r')\right\rangle \mathrm{d}s \\
&\qquad + \sum_{\alpha=1}^P \mathbb{E}\int_0^{\bar{\tau}} \left|\tilde{\mathbb{E}}\int_{\Gamma_\alpha}[\beta(v_s^n) - \beta(v_s^m)]\mathcal{R}(\mathrm{d}r')\right|^2 \mathrm{d}s \\
&\qquad + \sum_{\alpha=1}^P \mathbb{E}\int_0^{\bar{\tau}}\int_U \left|\tilde{\mathbb{E}}\int_{\Gamma_\alpha}[\eta(v_s^n, \xi) - \eta(v_s^m, \xi)]\mathcal{R}(\mathrm{d}r')\right|^2 \nu(\mathrm{d}\xi)\,\mathrm{d}s.
\end{aligned}$$

Hypothesis 1.1 implies

$$\begin{aligned}
&\mathbb{E}|X_{\bar{\tau}}^{n,r} - X_{\bar{\tau}}^{m,r}|^2 \\
&\quad\leq \int_0^t (L_s(\omega') + P + P\bar{L}_s(\omega')) \sup_{u\in[0,s]} \mathbb{E}|X_{u\wedge\bar{\tau}}^{n,r} - X_{u\wedge\bar{\tau}}^{m,r}|^2 \,\mathrm{d}s \\
&\qquad + \sum_{\alpha=1}^P \mathbb{E}\int_0^{\bar{\tau}} 2\bar{L}_s(\omega')\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\biggl(\mathbf{1}_{\{s\leq\tilde{\tau}_R^{n,m,r'}\}}\int_{-\tau}^0 [|Y_{\kappa(n,s+u)}^{n,r'} - Y_{\kappa(m,s+u)}^{m,r'}|^2 \\
&\qquad + \mathbf{1}_{\{u<0\}}|Y_{\kappa(n,(s+u)^+)}^{n,r'} - Y_{\kappa(m,(s+u)^+)}^{m,r'}|^2]\lambda(\mathrm{d}u)\biggr)\mathcal{R}(\mathrm{d}r')\,\mathrm{d}s \\
&\qquad + \sum_{\alpha=1}^P \mathbb{E}\int_0^{\bar{\tau}}\biggl[4\bar{K}_s(\omega')\tilde{\mathbb{E}}\biggl(\int_{\Gamma_\alpha}\mathbf{1}_{\{s>\tilde{\tau}_R^{n,m,r'}\}}\mathcal{R}(\mathrm{d}r')\biggr) \\
&\qquad \times\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\biggl(2 + |X_{s-}^{n,r}|^2 + |X_{s-}^{m,r}|^2 + \int_{-\tau}^0 [|Y_{\kappa(n,s+u)}^{n,r'}|^2 + |Y_{\kappa(m,s+u)}^{m,r'}|^2 \\
&\qquad + \mathbf{1}_{\{u<0\}}(|Y_{\kappa(n,(s+u)^+)}^{n,r'}|^2 + |Y_{\kappa(m,(s+u)^+)}^{m,r'}|^2)]\lambda(\mathrm{d}u)\biggr)\mathcal{R}(\mathrm{d}r')\biggr]\,\mathrm{d}s,
\end{aligned}$$

where we separate the two cases $\{s > \tilde{\tau}_R^{n,m,r'}\}$ and $\{s \leq \tilde{\tau}_R^{n,m,r'}\}$ in order to apply Gronwall's inequality to the difference $\mathbb{E}|X_{s\wedge\bar{\tau}}^{n,r} - X_{s\wedge\bar{\tau}}^{m,r}|^2$. Using (ii), we obtain that

$$\begin{aligned}
&\mathbb{E}|X_{\bar{\tau}}^{n,r} - X_{\bar{\tau}}^{m,r}|^2 \\
&\quad\leq \int_0^t \biggl[6\bar{L}_s(\omega')\mathbb{E}\int_\Gamma\int_{-\tau}^0 \mathbf{1}_{[0,\tau_R^{n,m,r'}]}(s)[|p_{s+u}^{n,r'}|^2 + |p_{s+u}^{m,r'}|^2
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \mathbf{1}_{\{u<0\}}\big(|p^{n,r'}_{(s+u)^+}|^2 + |p^{m,r'}_{(s+u)^+}|^2\big)\big]\lambda(\mathrm{d}u)\mathcal{R}(\mathrm{d}r')\Big]\,\mathrm{d}s \\
&\qquad + 8\big(1+3C_1(T,\omega')\big)\int_0^t \bar{K}_s(\omega')\int_\Gamma \mathbb{P}\{s>\tau_R^{n,m,r'}\}\mathcal{R}(\mathrm{d}r')\,\mathrm{d}s \\
&\qquad + \int_0^t \Big[\big(L_s(\omega')+13P\bar{L}_s(\omega')+P\big) \qquad (2.4)\\
&\qquad \times \sup_{r'\in\Gamma}\sup_{u\in[0,s]} \mathbb{E}\big|X^{n,r'}_{u\wedge\tau_R^{n,m,r'}} - X^{m,r'}_{u\wedge\tau_R^{n,m,r'}}\big|^2\Big]\,\mathrm{d}s \\
&= I^{n,m}_{R,T}(\omega') \\
&\qquad + \int_0^t \Big[\big(L_s(\omega')+13P\bar{L}_s(\omega')+P\big) \\
&\qquad \times \sup_{r'\in\Gamma}\sup_{u\in[0,s]} \mathbb{E}\big|X^{n,r'}_{u\wedge\tau_R^{n,m,r'}} - X^{m,r'}_{u\wedge\tau_R^{n,m,r'}}\big|^2\Big]\,\mathrm{d}s.
\end{aligned}
$$

Choosing $\bar{\tau} = t\wedge\tau_R^{n,m,r}$ for $t\in[0,T]$, we obtain by Gronwall's inequality

$$
\sup_{r\in\Gamma}\sup_{t\in[0,T]} \mathbb{E}\big|X^{n,r}_{t\wedge\tau_R^{n,m,r}} - X^{m,r}_{t\wedge\tau_R^{n,m,r}}\big|^2 \le C(T,\omega')I^{n,m}_{R,T}(\omega'). \tag{2.5}
$$

Inserting this bound in the left-hand side of (2.4) implies

$$
\mathbb{E}\big|X^{n,r}_{\bar\tau} - X^{m,r}_{\bar\tau}\big|^2 \le C(T,\omega')I^{n,m}_{R,T}(\omega').
$$

Note that $C(T,\omega')$ may differ from a line to another line but always $T\mapsto C(T,\omega')$ is an increasing function. By setting

$$
\bar\tau := \tau_R^{n,m,r}\wedge\inf\{t\ge 0: |X^{n,r}_t - X^{m,r}_t|\ge\varepsilon\},
$$

we have

$$
\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\in[0,T]}|X^{n,r}_t - X^{m,r}_t|\ge\varepsilon\Big\} \\
&\quad\le \mathbb{P}\{T>\tau_R^{n,r}\} + \mathbb{P}\{T>\tau_R^{m,r}\} + \mathbb{P}\Big\{\sup_{t\in[0,\tau_R^{n,m,r}]}|X^{n,r}_t - X^{m,r}_t|\ge\varepsilon\Big\} \\
&\quad\le \mathbb{P}\{T>\tau_R^{n,r}\} + \mathbb{P}\{T>\tau_R^{m,r}\} + \frac{1}{\varepsilon^2}\mathbb{E}|X^{n,r}_{\bar\tau} - X^{m,r}_{\bar\tau}|^2 \\
&\quad\le \mathbb{P}\{T>\tau_R^{n,r}\} + \mathbb{P}\{T>\tau_R^{m,r}\} + \frac{C(T,\omega')I^{n,m}_{R,T}}{\varepsilon^2}.
\end{aligned}
$$

From (i) and (iii), one can obtain that

$$
\lim_{R\to\infty}\limsup_{n,m\to\infty} I^{n,m}_{R,T} = 0
$$

and so

$$
\begin{aligned}
&\limsup_{n,m\to\infty}\mathbb{P}\Big\{\sup_{t\in[0,T]}|X^{n,r}_t - X^{m,r}_t|\ge\varepsilon\Big\} \\
&\quad\le \lim_{R\to\infty}\limsup_{n,m\to\infty}\Big[\mathbb{P}\{T>\tau_R^{n,r}\} + \mathbb{P}\{T>\tau_R^{m,r}\} + \frac{C(T,\omega')I^{n,m}_{R,T}}{\varepsilon^2}\Big] = 0.
\end{aligned}
$$

So (iv) is obtained.

Proof of (v). Since the space $L^2(\Omega, \text{Càdlàg}([-\tau, T], E))$ is complete with respect to the topology of convergence in probability, (iv) yields that there exist $X^r, Y^r \in L^2(\Omega, \text{Càdlàg}([-\tau, T], E))$ such that

$$\lim_{n\to\infty} \mathbb{P}\Big\{\sup_{t\in[0,T]} |X_t^{n,r} - X_t^r| \geq \varepsilon\Big\} = 0, \qquad \lim_{n\to\infty} \tilde{\mathbb{P}}\Big\{\sup_{t\in[0,T]} |Y_t^{n,r} - Y_t^r| \geq \varepsilon\Big\} = 0.$$

We next have to show that all terms of equation (2.3) for a subsequence of $n \in \mathbb{N}$ converge almost surely to the terms of equation (1.6). We have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\Big\{\sup_{t\in[0,T]} |Y^{n,r}_{\kappa(n,t)} - Y^r_{t-}| \geq \varepsilon\Big\} \\
&\quad \leq \lim_{n\to\infty} \mathbb{P}\Big\{\sup_{t\in[0,T]} |Y^{n,r}_{\kappa(n,t)} - Y^r_{\kappa(n,t)}| \geq \varepsilon/2\Big\} \\
&\qquad + \lim_{n\to\infty} \mathbb{P}\Big\{\sup_{t\in[0,T]} |Y^{r}_{\kappa(n,t)} - Y^r_{t-}| \geq \varepsilon/2\Big\} = 0.
\end{aligned}$$

So there exists a subsequence, say $\{n_l\}_{l\in\mathbb{N}}$, such that, as $l \to \infty$,

$$\sup_{t\in[0,T]} \big[|X_{t-}^{n_l,r} - X_{t-}^r| + |Y^{n_l,r}_{\kappa(n_l,t)} - Y^r_{t-}|\big] \to 0, \quad \mathbb{P}\otimes\tilde{\mathbb{P}}\text{-a.s.} \tag{2.6}$$

for all (r, ω') in a subset D_0 of $\Gamma \times \Omega'$ of full $\mathcal{R}\otimes\mathbb{P}'$-measure. Now let us define

$$S_r(t) := \sup_{l\in\mathbb{N}} |X_{t-}^{n_l,r}|, \qquad \tilde{S}_r(t) := \sup_{l\in\mathbb{N}} |Y_{t-}^{n_l,r}|.$$

Then

$$\sup_{t\in[0,T]} S_r(t) < \infty, \quad \mathbb{P}\text{-a.s.}, \qquad \sup_{t\in[0,T]} \tilde{S}_r(t) < \infty, \quad \tilde{\mathbb{P}}\text{-a.s.}$$

for all $(r, \omega') \in D_0$. So by (H5) and inequality (2.2), for $(r, \omega') \in D_0$,

$$\int_0^t \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \big|\theta\big(s, r, r', X_{s-}^{n_l,r}, Y^{n_l,r'}_{\kappa(n_l,(s-\tau):s)}, \omega'\big)\big|^2 \mathcal{R}(\mathrm{d}r')\,\mathrm{d}s < \infty, \quad \mathbb{P}\text{-a.s.}$$

Using continuity of θ and $L^1([0, T]\times\tilde{\Omega}\times\Gamma, \mathrm{d}t\otimes\tilde{\mathbb{P}}\otimes\mathcal{R})$-uniform integrability of

$$\big(s, \tilde{\omega}, r'\big) \mapsto \theta\big(s, r, r', X_{s-}^{n_l,r}, Y^{n_l,r'}_{\kappa(n_l,(s-\tau):s)}, \omega'\big)$$

for all $(r, \omega') \in D_0$, we obtain that

$$\begin{aligned}
&\lim_{l\to\infty} \int_0^t \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \theta\big(s, r, r', X_{s-}^{n_l,r}, Y^{n_l,r'}_{\kappa(n_l,(s-\tau):s)}, \omega'\big)\mathcal{R}(\mathrm{d}r')\,\mathrm{d}s \\
&\quad = \int_0^t \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \theta\big(s, r, r', X_{s-}^{r}, Y^{r'}_{(s-\tau)^-:s^-}, \omega'\big)\mathcal{R}(\mathrm{d}r')\,\mathrm{d}s
\end{aligned}$$

$\mathbb{P}$-almost surely for $\mathcal{R}\otimes\mathbb{P}'$-almost all (r, ω'). Let

$$\tau_{r,R} := \inf\{t \geq 0 : S_r(t) > R\} \wedge T, \qquad \tilde{\tau}_{r,R} := \inf\{t \geq 0 : \tilde{S}_r(t) > R\} \wedge T.$$

To shorten notation again, let

$$v_s^n := \big(s, r, r', X_{s-}^{n,r}, Y^{n,r'}_{\kappa(n,(s-\tau):s)}, \omega'\big) \quad \text{and} \quad v_s := \big(s, r, r', X_{s-}^{r}, Y^{r'}_{(s-\tau)^-:s^-)}, \omega'\big).$$

For all $t \in [0, T]$ and for all $(r, \omega') \in D_0$, we have

$$\begin{aligned}
&\mathbb{E}\left|\int_0^{t\wedge\tau_{r,R}} \tilde{\mathbb{E}}\int_{\Gamma_\alpha} [\beta(v_s^{n_l}) - \beta(v)]\mathcal{R}(\mathrm{d}r')\,\mathrm{d}B_s^\alpha\right|^2 \\
&\quad\le 2\mathbb{E}\int_0^t \tilde{\mathbb{E}}\int_{\Gamma_\alpha} \mathbf{1}_{\{s\le\tilde{\tau}_{r,R}\wedge\tau_{r,R}\}}|\beta(v_s^{n_l}) - \beta(v)|^2\mathcal{R}(\mathrm{d}r')\,\mathrm{d}s \\
&\qquad + 4\mathbb{E}\int_0^t \mathbf{1}_{\{s\le\tau_{r,R}\}}\bar{K}_s(\omega')(2 + |X_{s-}^{n_l,r}|^2 + |X_{s-}^r|^2 \\
&\qquad + 4C_1(s,\omega')) \times \tilde{\mathbb{E}}\int_{\Gamma_\alpha} \mathbf{1}_{\{s>\tilde{\tau}_{r,R}\}}\mathcal{R}(\mathrm{d}r')\,\mathrm{d}s.
\end{aligned}$$

So

$$\begin{aligned}
&\mathbb{P}\left\{\left|\int_0^t \tilde{\mathbb{E}}\int_{\Gamma_\alpha} [\beta(v_s^{n_l}) - \beta(v_s)]\mathcal{R}(\mathrm{d}r')\,\mathrm{d}B_s^\alpha\right| > \varepsilon\right\} \\
&\quad\le \frac{1}{\varepsilon^2}\mathbb{E}\left|\int_0^{t\wedge\tau_{r,R}} \tilde{\mathbb{E}}\int_{\Gamma_\alpha} [\beta(v_s^{n_l}) - \beta(v_s)]\mathcal{R}(\mathrm{d}r')\,\mathrm{d}B_s^\alpha\right|^2 + \mathbb{P}\{t > \tau_{r,R}\} \\
&\quad\le \frac{2}{\varepsilon^2}\mathbb{E}\int_0^t \tilde{\mathbb{E}}\int_{\Gamma_\alpha} \mathbf{1}_{\{s\le\tilde{\tau}_{r,R}\wedge\tau_{r,R}\}}|\beta(v_s^{n_l}) - \beta(v_s)|^2\mathcal{R}(\mathrm{d}r')\,\mathrm{d}s \\
&\qquad + \left[\frac{8}{\varepsilon^2}\int_0^t \bar{K}_s(\omega')(1 + 3C_1(s,\omega'))\int_{\Gamma_\alpha} \mathbb{P}\{s > \tilde{\tau}_{r,R}\}\mathcal{R}(\mathrm{d}r')\,\mathrm{d}s + \mathbb{P}\{t > \tau_{r,R}\}\right].
\end{aligned}$$

For given $\delta > 0$, we can now find R sufficiently large, such that the second term on the right-hand side is less than δ. Taking the limit $l \to \infty$ now implies that

$$\lim_{l\to\infty}\mathbb{P}\left\{\left|\int_0^t \tilde{\mathbb{E}}\int_{\Gamma_\alpha} [\beta(v_s^{n_l}) - \beta(v_s)]\mathcal{R}(\mathrm{d}r')\,\mathrm{d}B_s^\alpha\right| > \varepsilon\right\} \le \delta.$$

Therefore,

$$\int_0^t \tilde{\mathbb{E}}\int_{\Gamma_\alpha} \beta(v_s^{n_l})\mathcal{R}(\mathrm{d}r')\,\mathrm{d}B_s^\alpha \to \int_0^t \tilde{\mathbb{E}}\int_{\Gamma_\alpha} \beta(v_s)\mathcal{R}(\mathrm{d}r')\,\mathrm{d}B_s^\alpha$$

in probability. The same argument implies

$$\int_0^t\int_U \tilde{\mathbb{E}}\int_{\Gamma_\alpha} \eta(v_s^{n_l},\xi)\mathcal{R}(\mathrm{d}r')\tilde{N}^\alpha(\mathrm{d}s,\mathrm{d}\xi) \to \int_0^t\int_U \tilde{\mathbb{E}}\int_{\Gamma_\alpha} \eta(v_s,\xi)\mathcal{R}(\mathrm{d}r')\tilde{N}^\alpha(\mathrm{d}s,\mathrm{d}\xi)$$

in probability, and for some further subsequence n_{l_k} the above convergences are $\mathbb{P}$-a.s. The convergence of the terms concerning the local dynamics in (2.3) to the respective terms of (1.6) follow from dominated convergence for the stopped solution (using $\tau_{r,R}$) and (H3). Therefore, X is a solution of equation (1.6) for $\mathcal{R} \times \mathbb{P}'$-almost all (r, ω').

Uniqueness. Let X and Y be two strong solutions of equation (1.6). To shorten the notation again, let $u_s^r = (s, r, X_{s-}^r, \omega')$, $u_s^{r,r'} = (s, r, r', X_{s-}^r, \tilde{X}_{(s-\tau)^-:s^-}^{r'}, \omega')$, $v_s^r = (s, r, Y_{s-}^r, \omega')$ and $v_s^{r,r'} = (s, r, r', Y_{s-}^r, \tilde{Y}_{(s-\tau)^-:s^-}^{r'}, \omega')$. We then have

$$\begin{aligned}
&|X_t^r - Y_t^r|^2 \\
&\quad= M_t + \int_0^t [2\langle X_{s-}^r - Y_{s-}^r, f(u_s^r) - f(v_s^r)\rangle + |g(u_s^r) - g(v_s^r)|^2]\,\mathrm{d}s \\
&\qquad + \int_0^t\int_U |h(u_s^r,\xi) - h(v_s^r,\xi)|^2 N(\mathrm{d}s,\mathrm{d}\xi)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\alpha=1}^{P} \int_0^t \biggl[2\biggl\langle X_{s-}^r - Y_{s-}^r, \int_{\Gamma_\alpha} \tilde{\mathbb{E}}[\theta(u_s^{r,r'}) - \theta(v_s^{r,r'})] \mathcal{R}(\mathrm{d}r') \biggr\rangle \\
&+ \biggl| \int_{\Gamma_\alpha} \tilde{\mathbb{E}}[\beta(u_s^{r,r'}) - \beta(v_s^{r,r'})] \mathcal{R}(\mathrm{d}r') \biggr|^2 \biggr] \mathrm{d}s \\
&+ \sum_{\alpha=1}^{P} \int_0^t \int_U \biggl| \int_{\Gamma_\alpha} \tilde{\mathbb{E}}[\eta(u_s^{r,r'}, \xi) - \eta(v_s^{r,r'}, \xi)] \mathcal{R}(\mathrm{d}r') \biggr|^2 N^\alpha(\mathrm{d}s, \mathrm{d}\xi),
\end{aligned}
$$

where $(\tilde{X}, \tilde{Y})$ are independent copies of (X, Y) and M_t is a local martingale w.r.t. some localizing sequence σ_n, $n \geq 1$, of stopping times. Using Fatou's lemma and Hypothesis 1.1, we then have

$$
\begin{aligned}
&\mathbb{E}|X_t^r - Y_t^r|^2 \\
&\quad \leq \liminf_{l \to \infty} \mathbb{E}|X_{t \wedge \sigma_l}^r - Y_{t \wedge \sigma_l}^r|^2 \\
&\quad \leq \mathbb{E} \int_0^t (L_s(\omega') + P\bar{L}_s(\omega') + P)|X_{s-}^r - Y_{s-}^r|^2 \,\mathrm{d}s \\
&\qquad + \int_0^t \bar{L}_s(\omega') \tilde{\mathbb{E}} \int_\Gamma \int_{-\tau}^0 [|\tilde{X}_{(s+u)^-}^{r'} - \tilde{Y}_{(s+u)^-}^{r'}|^2 \\
&\qquad + \mathbf{1}_{\{u<0\}} |\tilde{X}_{s+u}^{r'} - \tilde{Y}_{s+u}^{r'}|^2] \lambda(\mathrm{d}u) \mathcal{R}(\mathrm{d}r') \,\mathrm{d}s
\end{aligned}
$$

so

$$
\begin{aligned}
&\sup_{s \leq t} \mathbb{E} \int_\Gamma |X_s^r - Y_s^r|^2 \mathcal{R}(\mathrm{d}r) \\
&\quad \leq \int_0^t (L_s(\omega') + 3P\bar{L}_s(\omega') + P) \sup_{u \leq s} \mathbb{E} \int_\Gamma |X_u^r - Y_u^r|^2 \mathcal{R}(\mathrm{d}r) \,\mathrm{d}s.
\end{aligned}
$$

By Gronwall's lemma and Lemma 1.4, we have

$$
\sup_{s \leq T} \mathbb{E} \int_\Gamma |X_s^r - Y_s^r|^2 \mathcal{R}(\mathrm{d}r) = 0,
$$

which proves uniqueness. $\square$

3. Proof of Theorem 1.8.

PROOF. Let us first introduce the notation

$$
⨏_A \psi(r') \mathcal{R}(\mathrm{d}r') := \begin{cases} \dfrac{1}{\mathcal{R}(A)} \displaystyle\int_A \psi(r') \mathcal{R}(\mathrm{d}r'), & \mathcal{R}(A) \neq 0, \\ 0, & \mathcal{R}(A) = 0. \end{cases}
$$

To shorten the notation again, let $u_s^r = (s, r, X_{s^-}^{r,\mathcal{A}_N}, \omega')$, $u_s^{r,\tilde{r}} = (s, r, \tilde{r}, X_{s^-}^{r,\mathcal{A}_N}, X_{(s-\tau)^-:s^-}^{\tilde{r},\mathcal{A}_N}, \omega')$, $v_s^r = (s, r, \bar{X}_{s^-}^r, \omega')$ and $v_s^{r,r'} = (s, r, r', \bar{X}_{s^-}^r, \tilde{X}_{(s-\tau)^-:s^-}^{r'}, \omega')$. Then

$$
\begin{aligned}
&|X_t^{r,\mathcal{A}_N} - \bar{X}_t^r|^2 \\
&\quad = M_t + \int_0^t [2\langle X_{s^-}^{r,\mathcal{A}_N} - \bar{X}_{s^-}^r, f(u_s^r) - f(v_s^r)\rangle \\
&\qquad + |g(u_s^r) - g(v_s^r)|^2] \,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \int_U \big|h(u_s^r,\xi)-h(v_s^r,\xi)\big|^2 N^r(\mathrm{d}s,\mathrm{d}\xi)\\
&+\sum_{\alpha=1}^P \int_0^t 2\Big\langle X_{s^-}^{r,\mathcal{A}_N}-\bar{X}_{s^-}^r, \frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} \theta\big(u_s^{r,\tilde{r}}\big)-\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\theta\big(v_s^{r,r'}\big)\mathcal{R}\big(\mathrm{d}r'\big)\Big\rangle \mathrm{d}s\\
&+\sum_{\alpha=1}^P \int_0^t \bigg|\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} \beta\big(u_s^{r,\tilde{r}}\big)-\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\beta\big(v_s^{r,r'}\big)\mathcal{R}\big(\mathrm{d}r'\big)\bigg|^2 \mathrm{d}s\\
&+\sum_{\alpha=1}^P \int_0^t \int_U \bigg|\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} \eta\big(u_s^{r,\tilde{r}},\xi\big)-\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\eta\big(v_s^{r,r'},\xi\big)\mathcal{R}\big(\mathrm{d}r'\big)\bigg|^2 N^{r,\alpha}(\mathrm{d}s,\mathrm{d}\xi),
\end{aligned}
$$

where M_t is a local martingale up to time T starting from zero with localizing sequence σ_n, $n \geq 1$, of stopping times. Taking expectation, using Fatou's lemma and Hypothesis 1.1, we obtain that

$$
\begin{aligned}
&\mathbb{E}\big|X_t^{r,\mathcal{A}_N}-\bar{X}_t^r\big|^2\\
&\quad\leq \liminf_{n\to\infty}\mathbb{E}\big|X_{t\wedge\sigma_n}^{r,\mathcal{A}_N}-\bar{X}_{t\wedge\sigma_n}^r\big|^2\\
&\quad\leq \int_0^t L_s(\omega')\mathbb{E}\big|X_{s^-}^{r,\mathcal{A}_N}-\bar{X}_{s^-}^r\big|^2\,\mathrm{d}s\\
&\qquad+P\int_0^t \mathbb{E}\big|X_{s^-}^{r,\mathcal{A}_N}-\bar{X}_{s^-}^r\big|^2\,\mathrm{d}s\\
&\qquad+\sum_{\alpha=1}^P \int_0^t \mathbb{E}\bigg|\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} \theta\big(u_s^{r,\tilde{r}}\big)-\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\theta\big(v_s^{r,r'}\big)\mathcal{R}\big(\mathrm{d}r'\big)\bigg|^2 \mathrm{d}s\\
&\qquad+\sum_{\alpha=1}^P \int_0^t \mathbb{E}\bigg|\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} \beta\big(u_s^{r,\tilde{r}}\big)-\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\beta\big(v_s^{r,r'}\big)\mathcal{R}\big(\mathrm{d}r'\big)\bigg|^2 \mathrm{d}s\\
&\qquad+\sum_{\alpha=1}^P \int_0^t \int_U \mathbb{E}\bigg|\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} \eta\big(u_s^{r,\tilde{r}},\xi\big)-\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\eta\big(v_s^{r,r'},\xi\big)\mathcal{R}\big(\mathrm{d}r'\big)\bigg|^2 \nu(\mathrm{d}\xi)\,\mathrm{d}s.
\end{aligned}
\tag{3.1}
$$

Note that

$$
\begin{aligned}
&\sum_{\alpha=1}^P \mathbb{E}\int_0^t \bigg|\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} \theta\big(u_s^{r,\tilde{r}}\big)-\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\theta\big(v_s^{r,r'}\big)\mathcal{R}\big(\mathrm{d}r'\big)\bigg|^2 \mathrm{d}s\\
&\quad\leq 2\sum_{\alpha=1}^P \mathbb{E}\int_0^t \bigg|\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} \big(\theta\big(u_s^{r,\tilde{r}}\big)-\theta\big(s,r,\tilde{r},\bar{X}_{s-}^r,\bar{X}_{(s-\tau)^-:s^-}^{\tilde{r}},\omega'\big)\big)\bigg|^2 \mathrm{d}s\\
&\qquad+2\sum_{\alpha=1}^P \mathbb{E}\int_0^t \bigg|\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} \theta\big(s,r,\tilde{r},\bar{X}_{s-}^r,\bar{X}_{(s-\tau)^-:s^-}^{\tilde{r}},\omega'\big)\\
&\qquad-\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\theta\big(v_s^{r,r'}\big)\mathcal{R}\big(\mathrm{d}r'\big)\bigg|^2 \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\le 2\mathbb{E}\int_0^t \bar{L}_s(\omega')\frac{\#\mathcal{A}_N\cap\Gamma_\alpha}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}}\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha}\Big[|X^{r,\mathcal{A}_N}_{s^-}-\bar{X}^r_{s^-}|^2\\
&\quad+\int_{-\tau}^0\big(|X^{\tilde{r},\mathcal{A}_N}_{(s+u)^-}-\bar{X}^{\tilde{r}}_{(s+u)^-}|^2+\mathbf{1}_{\{u<0\}}|X^{\tilde{r},\mathcal{A}_N}_{s+u}-\bar{X}^{\tilde{r}}_{s+u}|^2\big)\lambda(\mathrm{d}u)\Big]\,\mathrm{d}s+2\sum_{\alpha=1}^P I^\theta_\alpha
\end{aligned}
$$

with

$$
\begin{aligned}
I^\theta_\alpha := \mathbb{E}\int_0^t\Big|&\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}}\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha}\theta\big(s,r,\tilde{r},\bar{X}^r_{s^-},\bar{X}^{\tilde{r}}_{(s-\tau)^-:s^-},\omega'\big)\\
&-\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\theta\big(v^{r,r'}_s\big)\mathcal{R}(\mathrm{d}r')\Big|^2\,\mathrm{d}s.
\end{aligned}
$$

The remaining terms on the right-hand side of (3.1) can be estimated from above similarly so that (3.1) now yields the following estimate:

$$
\begin{aligned}
&\mathbb{E}|X^{r,\mathcal{A}_N}_t-\bar{X}^r_t|^2\\
&\quad\le\int_0^t(L_s(\omega')+P)\mathbb{E}|X^{r,\mathcal{A}_N}_{s^-}-\bar{X}^r_{s^-}|^2\,\mathrm{d}s+2\sum_{\Theta\in\{\theta,\beta\}}\sum_{\alpha=1}^P I^\Theta_\alpha+2\sum_{\alpha=1}^P I^\eta_\alpha\\
&\qquad+2\sum_{\alpha=1}^P\int_0^t\bar{L}_s(\omega')\frac{\#\mathcal{A}_N\cap\Gamma_\alpha}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}}\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha}\mathbb{E}\Big[|X^{r,\mathcal{A}_N}_{s^-}-\bar{X}^r_{s^-}|^2\\
&\qquad+\int_{-\tau}^0\big(|X^{\tilde{r},\mathcal{A}_N}_{(s+u)^-}-\bar{X}^{\tilde{r}}_{(s+u)^-}|^2+\mathbf{1}_{\{u<0\}}|X^{\tilde{r},\mathcal{A}_N}_{s+u}-\bar{X}^{\tilde{r}}_{s+u}|^2\big)\lambda(\mathrm{d}u)\Big]\,\mathrm{d}s,
\end{aligned}
\tag{3.2}
$$

where

$$
\begin{aligned}
I^\Theta_\alpha&=\mathbb{E}\int_0^t\Big|\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}}\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha}\Theta\big(s,r,\tilde{r},\bar{X}^r_{s^-},\bar{X}^{\tilde{r}}_{(s-\tau)^-:s^-},\omega'\big)\\
&\qquad-\tilde{\mathbb{E}}\int_{\Gamma_\alpha}\Theta\big(v^{r,r'}_s\big)\mathcal{R}(\mathrm{d}r')\Big|^2\,\mathrm{d}s\\
&\le 3\mathbb{E}\int_0^t\Big|\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}}\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha}\big[\Theta\big(s,r,\tilde{r},\bar{X}^r_{s^-},\bar{X}^{\tilde{r}}_{(s-\tau)^-:s^-},\omega'\big)-\tilde{\mathbb{E}}\Theta\big(v^{r,\tilde{r}}_s\big)\big]\Big|^2\,\mathrm{d}s\\
&\quad+3\mathbb{E}\int_0^t\Big|\sum_{m=1}^{M^{(\varepsilon)}_\alpha}\frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}}\sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma^{m,\varepsilon}_\alpha}\tilde{\mathbb{E}}\Big[\Theta\big(v^{r,\tilde{r}}_s\big)-\int_{\Gamma^{m,\varepsilon}_\alpha}\Theta\big(v^{r,r'}_s\big)\mathcal{R}(\mathrm{d}r')\Big]\Big|^2\,\mathrm{d}s\\
&\quad+3\mathbb{E}\int_0^t\Big|\sum_{m=1}^{M^{(\varepsilon)}_\alpha}\Big(\frac{\#\mathcal{A}_N\cap\Gamma^{m,\varepsilon}_\alpha}{\mathcal{S}_{\mathcal{A}_N,\alpha}}-\mathcal{R}\big(\Gamma^{m,\varepsilon}_\alpha\big)\Big)\tilde{\mathbb{E}}\int_{\Gamma^{m,\varepsilon}_\alpha}\Theta\big(v^{r,r'}_s\big)\mathcal{R}(\mathrm{d}r')\Big|^2\,\mathrm{d}s\\
&=I+\mathit{II}+\mathit{III}\quad\text{say.}
\end{aligned}
$$

We can now further estimate the integrals I–III from above as follows:

$$
\begin{aligned}
I=\frac{3}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}}\sum_{\tilde{r}_1,\tilde{r}_2\in\mathcal{A}_N\cap\Gamma_\alpha}\int_0^t&\mathbb{E}\operatorname{tr}\big[\big(\Theta\big(s,r,\tilde{r}_1,\bar{X}^r_{s^-},\bar{X}^{\tilde{r}_1}_{(s-\tau)^-:s^-},\omega'\big)-\tilde{\mathbb{E}}\Theta\big(v^{r,\tilde{r}_1}_s\big)\big)^T\\
&\times\big(\Theta\big(s,r,\tilde{r}_2,\bar{X}^r_{s^-},\bar{X}^{\tilde{r}_2}_{(s-\tau)^-:s^-},\omega'\big)-\tilde{\mathbb{E}}\Theta\big(v^{r,\tilde{r}_2}_s\big)\big)\big]\,\mathrm{d}s
\end{aligned}
\tag{3.3}
$$

$$= \frac{3}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} \int_0^t \mathbb{E}\big[\big|\Theta(s,r,\tilde{r},\bar{X}^r_{s-},\bar{X}^{\tilde{r}}_{(s-\tau)^-:s^-},\omega') - \tilde{\mathbb{E}}\Theta(v_s^{r,\tilde{r}})\big|^2\big]\,\mathrm{d}s$$

since for distinct $\tilde{r}_1$ and $\tilde{r}_2$, if $\tilde{r}_1 \neq r$ or $\tilde{r}_2 \neq r$ then $\bar{X}^{\tilde{r}_1}$ or $\bar{X}^{\tilde{r}_2}$ are independent of each other and also independent of $\bar{X}^r$ and, therefore, for arbitrary ith component of Θ,

$$\begin{aligned}
&\mathbb{E}\big[\big(\Theta_i(s,r,\tilde{r}_1,\bar{X}^r_{s-},\bar{X}^{\tilde{r}_1}_{(s-\tau)^-:s^-},\omega') - \tilde{\mathbb{E}}\Theta_i(v_s^{r,\tilde{r}_1})\big) \\
&\qquad \times \big(\Theta_i(s,r,\tilde{r}_2,\bar{X}^r_{s-},\bar{X}^{\tilde{r}_2}_{(s-\tau)^-:s^-},\omega') - \tilde{\mathbb{E}}\Theta_i(v_s^{r,\tilde{r}_2})\big) \mid \bar{X}^r\big] \\
&\quad = \mathbb{E}\big[\big(\Theta_i(s,r,\tilde{r}_1,\bar{X}^r_{s-},\bar{X}^{\tilde{r}_1}_{(s-\tau)^-:s^-},\omega') - \tilde{\mathbb{E}}\Theta_i(v_s^{r,\tilde{r}_1})\big) \mid \bar{X}^r\big] \\
&\qquad \times \mathbb{E}\big[\big(\Theta_i(s,r,\tilde{r}_2,\bar{X}^r_{s-},\bar{X}^{\tilde{r}_2}_{(s-\tau)^-:s^-},\omega') - \tilde{\mathbb{E}}\Theta_i(v_s^{r,\tilde{r}_2})\big) \mid \bar{X}^r\big] \\
&\quad = 0.
\end{aligned}$$

Using Lemma 1.4, we can then further estimate the right-hand side of (3.3) from above by

$$\begin{aligned}
&\frac{3}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} \int_0^t \mathbb{E}\big[\big|\Theta(s,r,\tilde{r},\bar{X}^r_{s-},\bar{X}^{\tilde{r}}_{(s-\tau)^-:s^-},\omega') - \tilde{\mathbb{E}}\Theta(v_s^{r,\tilde{r}})\big|^2\big]\,\mathrm{d}s \\
&\quad \le \frac{12}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha} \int_0^t \bar{K}_s(\omega')\bigg[1 + \mathbb{E}|\bar{X}^r_{s-}|^2 \\
&\qquad + \int_{-\tau}^0 \mathbb{E}|\bar{X}^r_{(s+u)^-}|^2 + \mathbf{1}_{\{u<0\}}\mathbb{E}|\bar{X}^r_{s+u}|^2\lambda(du)\bigg]\,\mathrm{d}s \\
&\quad \le \frac{\#\mathcal{A}_N\cap\Gamma_\alpha}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} \int_0^t 12\bar{K}_s(\omega')\big(1 + 3C_1(s,\omega')\big)\,\mathrm{d}s.
\end{aligned}$$

The next term can be estimated from above as follows:

$$\begin{aligned}
II &\le 3\int_0^t \sum_{m=1}^{M_\alpha^{(\varepsilon)}} \sum_{\tilde{r}\in\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}} \bar{L}_s(\omega')\frac{\#\mathcal{A}_N\cap\Gamma_\alpha}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} \\
&\quad \times \bigg[\tilde{\mathbb{E}} \fint_{\Gamma_\alpha^{m,\varepsilon}} \int_{-\tau}^0 \big[|\tilde{X}^{\tilde{r}}_{(s+u)^-} - \tilde{X}^{r'}_{(s+u)^-}|^2 + \mathbf{1}_{\{u<0\}}|\tilde{X}^{\tilde{r}}_{s+u} - \tilde{X}^{r'}_{s+u}|^2\big]\lambda(\mathrm{d}u)\mathcal{R}(\mathrm{d}r') \\
&\quad + \varepsilon\bigg(1 + \mathbb{E}|\bar{X}^r_{s-}|^2 + \tilde{\mathbb{E}}\int_{-\tau}^0 \big[|\tilde{X}^{\tilde{r}}_{(s+u)^-}|^2 + \mathbf{1}_{\{u<0\}}|\tilde{X}^{\tilde{r}}_{s+u}|^2\big]\lambda(\mathrm{d}u)\bigg)\bigg]\,\mathrm{d}s \\
&\le 3\varepsilon\frac{(\#\mathcal{A}_N\cap\Gamma_\alpha)^2}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} \int_0^t \bar{L}_s(\omega')\big(1 + 3C_1(s,\omega') + C_2(s,\omega')\big)\,\mathrm{d}s
\end{aligned}$$

using Lemma 1.4 and Lemma 1.7. Finally,

$$\begin{aligned}
III &\le 3M_\alpha^{(\varepsilon)} \sum_{m=1}^{M_\alpha^{(\varepsilon)}} \bigg(\frac{\#\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}}{\mathcal{S}_{\mathcal{A}_N,\alpha}} - \mathcal{R}(\Gamma_\alpha^{m,\varepsilon})\bigg)^2 \mathbb{E}\int_0^t \bigg|\tilde{\mathbb{E}}\fint_{\Gamma_\alpha^{m,\varepsilon}} \Theta(v_s^{r,r'})\mathcal{R}(\mathrm{d}r')\bigg|^2\,\mathrm{d}s \\
&\le 3M_\alpha^{(\varepsilon)} \sum_{m=1}^{M_\alpha^{(\varepsilon)}} \bigg(\frac{\#\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}}{\mathcal{S}_{\mathcal{A}_N,\alpha}} - \mathcal{R}(\Gamma_\alpha^{m,\varepsilon})\bigg)^2 \mathbb{E}\int_0^t \bar{K}_s(\omega')\bigg(1 + |\bar{X}^r_{s-}|^2 \\
&\quad + \tilde{\mathbb{E}}\fint_{\Gamma_\alpha^{m,\varepsilon}} \int_{-\tau}^0 \big[|\tilde{X}^{r'}_{(s+u)^-}|^2 + \mathbf{1}_{\{u<0\}}|\tilde{X}^{r'}_{s+u}|^2\big]\lambda(\mathrm{d}u)\mathcal{R}(\mathrm{d}r')\bigg)\,\mathrm{d}s
\end{aligned}$$

$$\leq 3M_\alpha^{(\varepsilon)} \sum_{m=1}^{M_\alpha^{(\varepsilon)}} \left(\frac{\#\mathcal{A}_N \cap \Gamma_\alpha^{m,\varepsilon}}{\mathcal{S}_{\mathcal{A}_N,\alpha}} - \mathcal{R}(\Gamma_\alpha^{m,\varepsilon}) \right)^2 \int_0^t \bar{K}_s(\omega')(1 + 3C_1(s,\omega'))\,\mathrm{d}s$$

using Lemma 1.4. Summing up the above estimates we now obtain that

$$I_\alpha^\Theta \leq 6\left(\frac{\#\mathcal{A}_N \cap \Gamma_\alpha}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} + \varepsilon \frac{(\#\mathcal{A}_N \cap \Gamma_\alpha)^2}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} + M_\alpha^{(\varepsilon)} \sum_{m=1}^{M_\alpha^{(\varepsilon)}} \left(\frac{\#(\mathcal{A}_N \cap \Gamma_\alpha^{m,\varepsilon})}{\mathcal{S}_{\mathcal{A}_N,\alpha}} - \mathcal{R}(\Gamma_\alpha^{m,\varepsilon}) \right)^2 \right) C_2(t,\omega').$$

Similar arguments imply that

$$\begin{aligned} I_\alpha^\eta &= \int_0^t \int_U \left| \frac{1}{\mathcal{S}_{\mathcal{A}_N,\alpha}} \sum_{\tilde{r} \in \mathcal{A}_N \cap \Gamma_\alpha} \eta(s, r, \tilde{r}, \bar{X}^r_{s^-}, \bar{X}^{\tilde{r}}_{(s-\tau)^- : s^-}, \omega', \xi) \right. \\ &\qquad \left. - \tilde{\mathbb{E}} \int_{\Gamma_\alpha} \eta(v_s^{r,r'}, \xi) \mathcal{R}(\mathrm{d}r') \right|^2 \nu(\mathrm{d}\xi)\,\mathrm{d}s \\ &\leq 6\left(\frac{\#\mathcal{A}_N \cap \Gamma_\alpha}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} + \varepsilon \frac{(\#\mathcal{A}_N \cap \Gamma_\alpha)^2}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} \right. \\ &\qquad \left. + M_\alpha^{(\varepsilon)} \sum_{m=1}^{M_\alpha^{(\varepsilon)}} \left(\frac{\#\mathcal{A}_N \cap \Gamma_\alpha^{m,\varepsilon}}{\mathcal{S}_{\mathcal{A}_N,\alpha}} - \mathcal{R}(\Gamma_\alpha^{m,\varepsilon}) \right)^2 \right) C_2(t,\omega'). \end{aligned}$$

So

$$\begin{aligned} &\mathbb{E}|X_t^{r,\mathcal{A}_N} - \bar{X}_t^r|^2 \\ &\quad \leq \int_0^t \left(L_s(\omega') + 6\bar{L}_s(\omega') \sum_{\alpha=1}^P \frac{(\#\mathcal{A}_N \cap \Gamma_\alpha)^2}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} + P \right) \\ &\qquad \times \max_{r \in \mathcal{A}_N} \sup_{u \in [0,s]} \mathbb{E}|X_u^{r,\mathcal{A}_N} - \bar{X}_u^r|^2\,\mathrm{d}s \\ &\qquad + 36 \sum_{\alpha=1}^P \left(\frac{\#\mathcal{A}_N \cap \Gamma_\alpha}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} + \varepsilon \frac{(\#\mathcal{A}_N \cap \Gamma_\alpha)^2}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} \right. \\ &\qquad \left. + M_\alpha^{(\varepsilon)} \sum_{m=1}^{M_\alpha^{(\varepsilon)}} \left(\frac{\#\mathcal{A}_N \cap \Gamma_\alpha^{m,\varepsilon}}{\mathcal{S}_{\mathcal{A}_N,\alpha}} - \mathcal{R}(\Gamma_\alpha^{m,\varepsilon}) \right)^2 \right) C_2(t,\omega'). \end{aligned}$$

Hence Gronwall's lemma implies

$$\begin{aligned} &\sup_{\substack{s \in [0,T] \\ r \in \mathcal{A}_N}} \mathbb{E}|X_s^{r,\mathcal{A}_N} - \bar{X}_s^r|^2 \\ &\quad \leq 36 \sum_{\alpha=1}^P \left(\frac{\#\mathcal{A}_N \cap \Gamma_\alpha}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} + \varepsilon \frac{(\#\mathcal{A}_N \cap \Gamma_\alpha)^2}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}} \right. \\ &\qquad \left. + M_\alpha^{(\varepsilon)} \sum_{m=1}^{M_\alpha^{(\varepsilon)}} \left(\frac{\#\mathcal{A}_N \cap \Gamma_\alpha^{m,\varepsilon}}{\mathcal{S}_{\mathcal{A}_N,\alpha}} - \mathcal{R}(\Gamma_\alpha^{m,\varepsilon}) \right)^2 \right) \end{aligned}$$

$$
\begin{aligned}
&\times C_2(T,\omega')\exp\Biggl[\int_0^T\Biggl(L_s(\omega')+6\bar{L}_s(\omega')\sum_{\alpha=1}^{P}\frac{(\#\mathcal{A}_N\cap\Gamma_\alpha)^2}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}}+P\Biggr)\,\mathrm{d}s\Biggr]\\
&\le 144\sum_{\alpha=1}^{P}\Biggl(\frac{\#\mathcal{A}_N\cap\Gamma_\alpha}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}}+\varepsilon\frac{(\#\mathcal{A}_N\cap\Gamma_\alpha)^2}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}}\\
&\quad+M_\alpha^{(\varepsilon)}\sum_{m=1}^{M_\alpha^{(\varepsilon)}}\biggl(\frac{\#\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}}{\mathcal{S}_{\mathcal{A}_N,\alpha}}-\mathcal{R}(\Gamma_\alpha^{m,\varepsilon})\biggr)^2\Biggr)\\
&\quad\times\exp\Biggl[\int_0^T\Biggl(2L_s(\omega')+\bar{L}_s(\omega')\Biggl(6\sum_{\alpha=1}^{P}\frac{(\#\mathcal{A}_N\cap\Gamma_\alpha)^2}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}}+P\Biggr)\\
&\quad+K_s(\omega')+3P\bar{K}_s(\omega')+3P\Biggr)\,\mathrm{d}s\Biggr].
\end{aligned}
$$

Now by integrating with respect to ω', we get for some finite constant $C(T)$ that

$$
\begin{aligned}
\mathcal{E}\Bigl[\sup_{\substack{s\in[0,T]\\ r\in\mathcal{A}_N}}\mathbb{E}\bigl|X_s^{r,\mathcal{A}_N}-\bar{X}_s^r\bigr|^2\Bigr]\le C(T)\sum_{\alpha=1}^{P}\Biggl(&\frac{\#\mathcal{A}_N\cap\Gamma_\alpha}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}}+\varepsilon\frac{(\#\mathcal{A}_N\cap\Gamma_\alpha)^2}{\mathcal{S}^2_{\mathcal{A}_N,\alpha}}\\
&+M_\alpha^{(\varepsilon)}\sum_{m=1}^{M_\alpha^{(\varepsilon)}}\biggl(\frac{\#\mathcal{A}_N\cap\Gamma_\alpha^{m,\varepsilon}}{\mathcal{S}_{\mathcal{A}_N,\alpha}}-\mathcal{R}(\Gamma_\alpha^{m,\varepsilon})\biggr)^2\Biggr).
\end{aligned}
$$

Hence

$$
\lim_{N\to\infty}\mathcal{E}\Bigl[\sup_{\substack{s\in[0,T]\\ r\in\mathcal{A}_N}}\mathbb{E}\bigl|X_s^{r,\mathcal{A}_N}-\bar{X}_s^r\bigr|^2\Bigr]\le PC(T)\varepsilon,
$$

where ε is arbitrary and, therefore,

$$
\lim_{N\to\infty}\mathcal{E}\Bigl[\sup_{\substack{s\in[0,T]\\ r\in\mathcal{A}_N}}\mathbb{E}\bigl|X_s^{r,\mathcal{A}_N}-\bar{X}_s^r\bigr|^2\Bigr]=0.
$$

□

APPENDIX A: WELL-POSEDNESS FOR SDES WITH PATH-DEPENDENT DELAY DRIVEN BY JUMP DIFFUSIONS

The purpose of this Appendix is to provide a general existence and uniqueness result on strong solutions of stochastic delay differential equations with monotone coefficients driven by jump diffusions, that in particular covers the assumptions on the network equations (1.1). For further reference, we formulate our results under more general assumptions on the coefficients.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with filtration $(\mathcal{F}_t)_{t\ge 0}$ satisfying the usual conditions. Let $(\Omega',\mathcal{F}',\mathbb{P}')$ be a probability space of parameters ω'. Consider the following stochastic delay differential equation:

$$
\begin{aligned}
\mathrm{d}X_t^{\omega'}&=f\bigl(t,\omega,X^{\omega'}_{(t-\tau)^-:t^-},\omega'\bigr)\,\mathrm{d}t+g\bigl(t,\omega,X^{\omega'}_{(t-\tau)^-:t^-},\omega'\bigr)\,\mathrm{d}W_t\\
&\quad+\int_U h\bigl(t,\omega,X^{\omega'}_{(t-\tau)^-:t^-},\omega',\xi\bigr)\tilde{N}(\mathrm{d}t,\mathrm{d}\xi)\qquad\text{(A.1)}\\
X_t^{\omega'}&=z_t^{\omega'},\quad t\in[-\tau,0].
\end{aligned}
$$

Here, W is a standard Brownian motion in $\mathbb{R}^m$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $(W_t - W_s)_{t\geq s}$ is independent of $\mathcal{F}_s$, $s \geq 0$. N is a time homogeneous Poisson measure on $[0,\infty) \times U$ with intensity measure $\mathrm{d}t \otimes \nu$, where $(U, \mathcal{U}, \nu)$ is an arbitrary σ-finite measure space. N is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, and $N(A)$ is independent of $\mathcal{F}_s$, $s \geq 0$, for every measurable subset $A \subseteq (s,\infty) \times U$. Finally, denote with $\tilde{N} := N - \mathrm{d}t \otimes \nu$ the compensated Poisson measure associated with N. The initial condition $z^{\omega'}_{-\tau:0}$ belongs to $L^2(\Omega, \mathbb{P}; \text{Càdlàg}([-\tau,0];\mathbb{R}^d))$ and is measurable w.r.t $(t,\omega,\omega') \in [-\tau,0] \times \Omega \times \Omega'$. Recall that we consider the space Càdlàg$([-\tau,0];\mathbb{R}^d)$ as well as Càglàd$([-\tau,0];\mathbb{R}^d)$, to be endowed with the supremum norm. Finally, we assume that W, N and $z^{\omega'}_{-\tau:0}$ are independent. The coefficients

$$\begin{aligned} f, g : \big([0,\infty) \times \Omega \times \text{Càglàd}\big([-\tau,0];\mathbb{R}^d\big) \times \Omega', \\ \mathcal{BF} \otimes \mathcal{B}\big(\text{Càglàd}\big([-\tau,0];\mathbb{R}^d\big)\big) \otimes \mathcal{F}'\big) \\ \to \big(\mathbb{R}^d, \mathcal{B}\big(\mathbb{R}^d\big)\big), \big(\mathbb{R}^{d\times m}, \mathcal{B}\big(\mathbb{R}^{d\times m}\big)\big) \end{aligned}$$

are progressively measurable and

$$\begin{aligned} h : \big([0,\infty) \times \Omega \times \text{Càglàd}\big([-\tau,0];\mathbb{R}^d\big) \times \Omega' \times U, \\ \mathcal{P} \otimes \mathcal{B}\big(\text{Càglàd}\big([-\tau,0];\mathbb{R}^d\big)\big) \otimes \mathcal{F}' \otimes \mathcal{U}\big) \to \big(\mathbb{R}^d, \mathcal{B}\big(\mathbb{R}^d\big)\big) \end{aligned}$$

is predictable. Here, $\mathcal{BF}$ and $\mathcal{P}$ are the σ-field of progressively measurable sets and predictable sets on $[0,\infty) \times \Omega$, respectively.

The following monotonicity and growth conditions are assumed.

HYPOTHESIS A.1. *There exist a probability measure λ on $[-\tau,0]$ and nonnegative measurable functions $K_t(\omega')$, $L_t(R,\omega')$ and $\tilde{K}_t(R,\omega')$ in $L^1_{\text{loc}}([0,\infty[, dt)$, for all $R > 0$ and all $\omega' \in \Omega'$, such that the following conditions hold*:

(C1) *for $|x|_{L^\infty}, |y|_{L^\infty} \leq R$,*

$$\begin{aligned} &2\big\langle x_0 - y_0, f\big(t,\omega,x_{-\tau:0},\omega'\big) - f\big(t,\omega,y_{-\tau:0},\omega'\big)\big\rangle \\ &\quad + \big|g\big(t,\omega,x_{-\tau:0},\omega'\big) - g\big(t,\omega,y_{-\tau:0},\omega'\big)\big|^2 \\ &\quad + \int_U \big|h\big(t,\omega,x_{-\tau:0},\omega',\xi\big) - h\big(t,\omega,y_{-\tau:0},\omega',\xi\big)\big|^2 \nu(\mathrm{d}\xi) \\ &\leq L_t\big(R,\omega'\big) \int_{-\tau}^0 \big[|x_s - y_s|^2 + \mathbf{1}_{\{s<0\}}|x_{s+} - y_{s+}|^2\big] \lambda(\mathrm{d}s), \end{aligned}$$

(C2)

$$\begin{aligned} &2\big\langle x_0, f\big(t,\omega,x_{-\tau:0},\omega'\big)\big\rangle + \big|g\big(t,\omega,x_{-\tau:0},\omega'\big)\big|^2 \\ &\quad + \int_U \big|h\big(t,\omega,x_{-\tau:0},\omega',\xi\big)\big|^2 \nu(\mathrm{d}\xi) \\ &\leq K_t\big(\omega'\big)\bigg(1 + \int_{-\tau}^0 \big[|x_s|^2 + \mathbf{1}_{\{s<0\}}|x_{s+}|^2\big] \lambda(\mathrm{d}s)\bigg) \end{aligned}$$

(C3) $x_{-\tau:0} \mapsto f(t,\omega,x_{-\tau:0},\omega')$ *as a function from Càglàd$([-\tau,0];\mathbb{R}^d)$ to $\mathbb{R}^d$ is continuous.*

(C4)

$$\begin{aligned}\sup_{|x|_{L^\infty}\le R}\Big[&\big|f\big(t,\omega,x_{-\tau:0},\omega'\big)\big|+\big|g\big(t,\omega,x_{-\tau:0},\omega'\big)\big|^2\\&+\int_U\big|h\big(t,\omega,x_{-\tau:0},\omega',\xi\big)\big|^2\nu(\mathrm{d}\xi)\Big]\\\le{}&\tilde K_t(R,\omega').\end{aligned}$$

We are going to prove existence and uniqueness of a strong solution using the Euler method. To this end, let us introduce for $n\in\mathbb{N}$ the Euler approximation

$$\begin{aligned}X_t^{n,\omega'}={}&X_{\frac{k\tau}{n}}^{n,\omega'}+\int_{\frac{k\tau}{n}}^t f\big(s,\omega,X^{n,\omega'}_{\kappa(n,(s-\tau):s)},\omega'\big)\,\mathrm{d}s\\&+\int_{\frac{k\tau}{n}}^t g\big(s,\omega,X^{n,\omega'}_{\kappa(n,(s-\tau):s)},\omega'\big)\,\mathrm{d}W_s\\&+\int_{\frac{k\tau}{n}}^t\int_U h\big(s,\omega,X^{n,\omega'}_{\kappa(n,(s-\tau):s)},\omega',\xi\big)\tilde N(\mathrm{d}s,\mathrm{d}\xi),\\&t\in\Big]\frac{k\tau}{n},\frac{(k+1)\tau}{n}\Big]\end{aligned}\tag{A.2}$$

to the solution of (A.1). Here, $\kappa(n,t):=\frac{k\tau}{n}$ for $t\in]\frac{k\tau}{n},\frac{(k+1)\tau}{n}]$. The process $X^{n,\omega'}$ can be constructed inductively as follows: $X_t^{n,\omega'}:=z_t^{\omega'}$ for $t\in[-\tau,0]$, and given $X_t^{n,\omega'}$ is defined for $t\le\frac{k\tau}{n}$ we can extend $X_t^{n,\omega'}$ for $t\in]\frac{k\tau}{n},\frac{(k+1)\tau}{n}]$ using (A.2). Note that $X^{n,\omega'}$ is càdlàg, whereas the process $X^{n,\omega'}_{\kappa(n,t)}$, $t\ge-\tau$, is càglàd. It is easy to see, using induction w.r.t. to k, that $X_t^{n,\omega'}$, $t\in]\frac{k\tau}{n},\frac{(k+1)\tau}{n}]$ and is measurable w.r.t. (t,ω,ω') and is a.s. locally bounded and that the stochastic integrals are well defined.

THEOREM A.2. *Under Hypothesis A.1, equation (A.1) has a unique strong solution $(X_t^{\omega'})_{t\ge0}$, and for $\mathbb{P}'$-almost all $\omega'\in\Omega'$, $(X_t^{n,\omega'})_{t\ge0}$ converges to $(X_t^{\omega'})_{t\ge0}$ locally uniformly in probability, that is, for all $T>0$ and $\mathbb{P}'$-almost all $\omega'\in\Omega'$,*

$$\lim_{n\to\infty}\mathbb{P}\Big\{\sup_{t\in[0,T]}\big|X_t^{n,\omega'}-X_t^{\omega'}\big|>\varepsilon\Big\}=0\quad\forall\varepsilon>0$$

and X is measurable w.r.t. $(t,\omega,\omega')\in[-\tau,\infty[\times\Omega\times\Omega'$ and satisfies

$$\begin{aligned}&1+2\mathbb{E}\big|X_t^{\omega'}\big|^2\\&\quad\le\Big(1+2\sup_{u\in[-\tau,0]}\mathbb{E}\big|z_u^{\omega'}\big|^2\Big)\cdot\exp\Big(\int_0^t 2K_s(\omega')\,\mathrm{d}s\Big),\qquad t\ge0.\end{aligned}\tag{A.3}$$

REMARK A.3. To the best of our knowledge, the above theorem is new in this full generality. Although the idea of the proof is based on previous works, in particular [5], [12]. The most far reaching existence and uniqueness results for stochastic delay differential equations driven by jump diffusions have been obtained in [1], [8] and [13] for locally Lipschitz continuous coefficients and in [14] under slightly more general assumptions. Existence and uniqueness of stochastic delay differential with merely monotone coefficients driven by diffusive noise have been obtained in [12].

PROOF OF THEOREM A.2. First, we prove that every strong solution X to equation (A.1) satisfies the moment estimate (A.3). To this end, consider the stopping time $\bar{\sigma}_{R,\omega'} := \mathbf{1}_{\{R>|z^{\omega'}|_\infty\}} \cdot \inf\{t \geq 0 : |X_t^{\omega'}| > R\}$. By Itô's formula, (C2) and (C4), we have

$$
\begin{aligned}
&\mathbb{E}|X^{\omega'}_{t\wedge\bar{\sigma}_{R,\omega'}}|^2 \\
&\quad = \mathbb{E}|z_0^{\omega'}|^2 + \mathbb{E}\int_0^{t\wedge\bar{\sigma}_{R,\omega'}} \Big[2\langle X^{\omega'}_{s-}, f\big(s,\omega,X^{\omega'}_{(s-\tau)^-:s^-},\omega'\big)\rangle \\
&\qquad + \big|g\big(s,\omega,X^{\omega'}_{(s-\tau)^-:s^-},\omega'\big)\big|^2 \\
&\qquad + \int_U \big|h\big(s,\omega,X^{\omega'}_{(s-\tau)^-:s^-},\omega',\xi\big)\big|^2\nu(\mathrm{d}\xi)\Big]\,\mathrm{d}s \\
&\quad \leq \mathbb{E}|z_0^{\omega'}|^2 \\
&\qquad + \mathbb{E}\int_0^{t\wedge\bar{\sigma}_{R,\omega'}} K_s(\omega')\Big(1+\int_{-\tau}^0 \big[|X^{\omega'}_{(s+u)^-}|^2 + \mathbf{1}_{\{u<0\}}|X^{\omega'}_{s+u}|^2\big]\lambda(\mathrm{d}u)\Big)\,\mathrm{d}s \\
&\quad \leq \mathbb{E}|z_0^{\omega'}|^2 + \int_0^t K_s(\omega')\Big(1+2\sup_{u\in[-\tau,s]}\mathbb{E}|X^{\omega'}_{u\wedge\bar{\sigma}_{R,\omega'}}|^2\Big)\,\mathrm{d}s.
\end{aligned}
$$

Therefore, by Gronwall's lemma and subsequently Fatou's lemma,

$$
\begin{aligned}
1+2\mathbb{E}|X_t^{\omega'}|^2 &\leq 1+2\liminf_{R\to\infty}\mathbb{E}|X^{\omega'}_{t\wedge\bar{\sigma}_{R\omega'}}|^2 \\
&\leq \Big(1+2\sup_{u\in[-\tau,0]}\mathbb{E}|z_u^{\omega'}|^2\Big)\exp\Big(\int_0^t 2K_s(\omega')\,\mathrm{d}s\Big).
\end{aligned}
$$

Existence. Let us define the remainder

$$
p_t^{n,\omega'} = X^{n,\omega'}_{\kappa(n,t)} - X^{n,\omega'}_{t^-}, \quad t\in(-\tau,\infty).
$$

We can then write

$$
\begin{aligned}
X_t^{n,\omega'} &= z_0^{\omega'} + \int_0^t f\big(s,\omega,X^{n,\omega'}_{(s-\tau)^-:s^-} + p^{n,\omega'}_{(s-\tau):s},\omega'\big)\,\mathrm{d}s \\
&\quad + \int_0^t g\big(s,\omega,X^{n,\omega'}_{(s-\tau)^-:s^-} + p^{n,\omega'}_{(s-\tau):s},\omega'\big)\,\mathrm{d}W_s \\
&\quad + \int_0^t\int_U h\big(s,\omega,X^{n,\omega'}_{(s-\tau)^-:s^-} + p^{n,\omega'}_{(s-\tau):s},\omega',\xi\big)\tilde{N}(\mathrm{d}s,\mathrm{d}\xi).
\end{aligned} \tag{A.4}
$$

In the next step, define the stopping times

$$
\tau_R^{n,\omega'} := \mathbf{1}_{\{3|z^{\omega'}|_\infty<R\}}\cdot\inf\Big\{t\geq 0 : |X_t^{n,\omega'}| > \frac{R}{3}\Big\}
$$

for given $R>0$. Then

$$
|p_t^{n,\omega'}| \leq \frac{2R}{3}, |X^{n,\omega'}_{t^-}| \leq \frac{R}{3}, \quad t\in\big(0,\tau_R^{n,\omega'}\big].
$$

For $R > 3|z^{\omega'}|_\infty$, the above inequalities extend to all $t\in(-\tau,\tau_R^{n,\omega'}]$ due to the right continuity of $X_t^{n,\omega'}$.

For the proof of existence, however, we will need a control of $|X^{n,\omega'}_{t\wedge\tau_R^{n,\omega'}}|$ in the mean square. To this end, note that for $R > 3|z^{\omega'}|_\infty$ the stochastic integrals

$$M^{n,\omega'}_{t\wedge\tau_R^{n,\omega'}} := 2\int_0^{t\wedge\tau_R^{n,\omega'}} \langle X^{n,\omega'}_{s-}, g\big(s,\omega, X^{n,\omega'}_{\kappa(n,(s-\tau):s)}, \omega'\big)\,\mathrm{d}W_s$$
$$+ 2\int_0^{t\wedge\tau_R^{n,\omega'}} \int_U \langle X^{n,\omega'}_{s-}, h\big(s,\omega, X^{n,\omega'}_{\kappa(n,(s-\tau):s)}, \omega', \xi\big)\tilde{N}(ds,d\xi), \quad t \geq 0$$

are well defined and square-integrable centered martingales. It follows that

$$\sup_R \tau_R^{n,\omega'} = \lim_{R\to\infty} \tau_R^{n,\omega'} = +\infty,$$

hence the stochastic integral $M_t := M^{n,\omega'}_{t\wedge\tau_R^{n,\omega'}}$, $t \leq \tau_R^{n,\omega'}$, is a local martingale up to time $+\infty$ with localizing sequence σ_m, $m \geq 1$, say.

From now on, we will fix $T > 0$ and prove the following properties which complete the proof of existence on $[0,T]$, and hence on $t \geq 0$, since T was arbitrary:

(i) For every $t \geq 0$ and $\omega' \in \Omega'$, $\mathbf{1}_{(-\tau,\tau_R^{n,\omega'}]}(t)p_t^{n,\omega'} \to 0$ in probability as $n \to \infty$.
(ii) For any stopping time $\tau^* \leq T \wedge \tau_R^{n,\omega'}$, R as in (i), we have $\mathbb{E}|X^{n,\omega'}_{\tau^*}|^2 \leq C(T,\omega')(1 + I^{n,\omega'}_{T,R})$, for some upper bound $I^{n,\omega'}_{T,R}$ satisfying $\lim_{n\to\infty} I^{n,\omega'}_{T,R} = 0$.
(iii) $\lim_{R\to\infty}\limsup_{n\to\infty}\mathbb{P}\{\tau_R^{n,\omega'} < T\} = 0$.
(iv) $\forall\varepsilon > 0$, $\lim_{n,m\to\infty}\mathbb{P}\{\sup_{t\in[0,T]}|X_t^{n,\omega'} - X_t^{m,\omega'}| > \varepsilon\} = 0$.
(v) $\exists X : \forall\varepsilon > 0$, $\lim_{n\to\infty}\mathbb{P}\{\sup_{t\in[0,T]}|X_t^{n,\omega'} - X_t^{\omega'}| > \varepsilon\} = 0$ and X is a strong solution of equation (A.1) on $[0,T]$.

Proof of (i). Since $z^{\omega'}$ is càdlàg w.r.t. time, $\mathbf{1}_{(-\tau,0]}(t)p_t^{n,\omega'} \to 0$ almost surely. Using (A.2) and Hypothesis A.1, we have for every $t > 0$

$$\begin{aligned}
&\mathbb{P}\{|p_t^{n,\omega'}| \geq \varepsilon, 0 < t \leq \tau_R^{n,\omega'}\} \\
&\quad\leq \mathbb{P}\left\{\int_{\kappa(n,t)}^t \sup_{|x|_{L^\infty}\leq R}\big|f\big(s,\omega,x_{-\tau:0},\omega'\big)\big|\,\mathrm{d}s \geq \varepsilon/3\right\} \\
&\qquad + \mathbb{P}\left\{\left|\int_{\kappa(n,t)}^t g\big(s,\omega,X^{n,\omega'}_{\kappa(n,(s-\tau):s)},\omega'\big)\,\mathrm{d}W_s\right| \geq \varepsilon/3, t \leq \tau_R^{n,\omega'}\right\} \\
&\qquad + \mathbb{P}\left\{\left|\int_{\kappa(n,t)}^{t^-}\int_U h\big(s,\omega,X^{n,\omega'}_{\kappa(n,(s-\tau):s)},\omega',\xi\big)\tilde{N}(\mathrm{d}s,\mathrm{d}\xi)\right| \geq \varepsilon/3, t \leq \tau_R^{n,\omega'}\right\} \\
&\quad\leq \mathbb{P}\left\{\int_{\kappa(n,t)}^t \tilde{K}_s(R,\omega')\,\mathrm{d}s \geq \varepsilon/3\right\} \\
&\qquad + \frac{9}{\varepsilon^2}\left(\mathbb{E}\left|\int_{\kappa(n,t)}^t \mathbf{1}_{\{s\leq\tau_R^{n,\omega'}\}} g\big(s,\omega,X^{n,\omega'}_{\kappa(n,(s-\tau):s)},\omega'\big)\,\mathrm{d}W_s\right|^2\right) \\
&\qquad + \frac{9}{\varepsilon^2}\mathbb{E}\left(\left|\int_{\kappa(n,t)}^{t^-}\int_U \mathbf{1}_{\{s\leq\tau_R^{n,\omega'}\}} h\big(s,\omega,X^{n,\omega'}_{\kappa(n,(s-\tau):s)},\omega',\xi\big)\tilde{N}(\mathrm{d}s,\mathrm{d}\xi)\right|^2\right)
\end{aligned}$$

$$\leq \frac{3}{\varepsilon}\int_{\kappa(n,t)}^{t}\tilde{K}_s(R,\omega')\,\mathrm{d}s + \frac{9}{\varepsilon^2}\mathbb{E}\left(\int_{\kappa(n,t)\wedge\tau_R^{n,\omega'}}^{t\wedge\tau_R^{n,\omega'}}\tilde{K}_s(R,\omega')\,\mathrm{d}s\right)$$
$$\leq \left(\frac{3}{\varepsilon}+\frac{9}{\varepsilon^2}\right)\int_{\kappa(n,t)}^{t}\tilde{K}_s(R,\omega')\,\mathrm{d}s,$$

so

$$\limsup_{n\to\infty}\mathbb{P}\{|p_t^{n,\omega'}|\geq\varepsilon, -\tau<t\leq\tau_R^{n,\omega'}\}=0,$$

which implies (i).

Proof of (ii). Using Itô's formula equation (A.4) implies that

$$\begin{aligned}|X_t^{n,\omega'}|^2 &= |z_0^{\omega'}|^2 + \int_0^t \big[2\langle X_{s-}^{n,\omega'}, f(s,\omega,X_{\kappa(n,(s-\tau):s)}^{n,\omega'},\omega')\rangle \\ &\quad + |g(s,\omega,X_{\kappa(n,(s-\tau):s)}^{n,\omega'},\omega')|^2\big]\,\mathrm{d}s \\ &\quad + \int_0^t\int_U |h(s,\omega,X_{\kappa(n,(s-\tau):s)}^{n,\omega'},\omega',\xi)|^2 N(\mathrm{d}s,\mathrm{d}\xi) + M_t.\end{aligned}$$

For any stopping time $\tau^*\leq t\wedge\tau_R^{n,\omega'}$, we then have

$$\begin{aligned}&\mathbb{E}|X_{\tau^*\wedge\sigma_m}^{n,\omega'}|^2 \\ &\leq \mathbb{E}|z_0^{\omega'}|^2 + \mathbb{E}\int_0^{\tau^*\wedge\sigma_m}\Big[2\langle X_{s-}^{n,\omega'}, f(s,\omega,X_{\kappa(n,(s-\tau):s)}^{n,\omega'},\omega')\rangle \\ &\quad + |g(s,\omega,X_{\kappa(n,(s-\tau):s)}^{n,\omega'},\omega')|^2 \\ &\quad + \int_U |h(s,\omega,X_{\kappa(n,(s-\tau):s)}^{n,\omega'},\omega',\xi)|^2\nu(\mathrm{d}\xi)\Big]\,\mathrm{d}s \\ &\leq \mathbb{E}|z_0^{\omega'}|^2 + \mathbb{E}\int_0^{\tau^*\wedge\sigma_m} 2\langle p_{s-}^{n,\omega'}, f(s,\omega,X_{\kappa(n,(s-\tau):s)}^{n,\omega'},\omega')\rangle\,\mathrm{d}s \\ &\quad + \mathbb{E}\int_0^{\tau^*\wedge\sigma_m} K_s(\omega')\Big(1+\int_{-\tau}^0\big(|X_{(s+u)^-}^{n,\omega'}+p_{s+u}^{n,\omega'}|^2 \\ &\quad + \mathbf{1}_{\{u<0\}}|X_{s+u}^{n,\omega'}+p_{(s+u)^+}^{n,\omega'}|^2\big)\lambda(\mathrm{d}u)\Big)\,\mathrm{d}s \\ &\leq C(T,\omega') \\ &\quad + \underbrace{\mathbb{E}\int_0^T\Big[2\tilde{K}_s(R,\omega')|p_s^{n,\omega'}| + 2K_s(\omega')\int_{-\tau}^0\big(|p_{s+u}^{n,\omega'}|^2+\mathbf{1}_{\{u<0\}}|p_{(s+u)^+}^{n,\omega'}|^2\big)\lambda(\mathrm{d}u)\Big]\mathbf{1}_{\{s\leq T\wedge\tau_R^{n,\omega'}\}}\,\mathrm{d}s}_{=:I_{T,R}^{n,\omega'}} \\ &\quad + \int_0^t 4K_s(\omega')\sup_{u\in[0,s]}\mathbb{E}|X_{u\wedge\tau^*}^{n,\omega'}|^2\,\mathrm{d}s.\end{aligned}$$

Taking the limit $m\to\infty$ yields by Fatou's lemma that

$$\mathbb{E}|X_{\tau^*}^{n,\omega'}|^2 \leq C(T,\omega') + I_{T,R}^{n,\omega'} + \int_0^t 4K_s(\omega')\sup_{u\in[0,s]}\mathbb{E}|X_{u\wedge\tau^*}^{n,\omega'}|^2\,\mathrm{d}s \tag{A.5}$$

with

$$\lim_{n\to\infty} I_{T,R}^{n,\omega'} = 0.$$

Setting $\tau^* := t \wedge \tau_R^{n,\omega'}$, $t \in [0, T]$, and using Gronwall's inequality, we obtain that

$$\sup_{t\in[0,T]} \mathbb{E}|X_{t\wedge\tau_R^{n,\omega'}}^{n,\omega'}|^2 \le C(T,\omega')(1 + I_{T,R}^{n,\omega'})$$

and for any stopping time $\tau^* \le T \wedge \tau_R^{n,\omega'}$ (A.5) now implies that

$$\mathbb{E}|X_{\tau^*}^{n,\omega'}|^2 \le C(T,\omega')(1 + I_{T,R}^{n,\omega'}).$$

Note that $C(T, \omega')$ may differ from line to line, but always can be chosen to be an increasing function.

Proof of (iii). Let

$$\tau^* = T \wedge \tau_R^{n,\omega'} \wedge \inf\{t \ge 0 : |X_t^{n,\omega'}| \ge a\}.$$

(ii) then implies that

$$\mathbb{P}\Big\{\sup_{t\in[0,T\wedge\tau_R^{n,\omega'}]} |X_t^{n,\omega'}| \ge a\Big\} \le \frac{1}{a^2}\mathbb{E}|X_{\tau^*}^{n,\omega'}|^2 \le \frac{C(T,\omega')}{a^2}(1 + I_{T,R}^{n,\omega'})$$

for any $a > 0$. In particular,

$$\begin{aligned}
&\limsup_{R\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big\{\sup_{t\in[0,\tau_R^{n,\omega'}]} |X_t^{n,\omega'}| \ge \frac{R}{4}; \tau_R^{n,\omega'} < T\Big\}\\
&\quad\le \limsup_{R\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big\{\sup_{t\in[0,T\wedge\tau_R^{n,\omega'}]} |X_t^{n,\omega'}| \ge \frac{R}{4}\Big\}\\
&\quad\le \limsup_{R\to\infty}\limsup_{n\to\infty}\frac{16C(T,\omega')(1 + I_{T,R}^{n,\omega'})}{R^2} = 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\limsup_{R\to\infty}\limsup_{n\to\infty}\mathbb{P}\{\tau_R^{n,\omega'} < T\}\\
&\quad\le \limsup_{R\to\infty}\limsup_{n\to\infty}\Big[\mathbb{P}\Big\{\sup_{t\in[0,\tau_R^{n,\omega'}]} |X_t^{n,\omega'}| \ge \frac{R}{4}; \tau_R^{n,\omega'} < T\Big\}\\
&\qquad + \mathbb{P}\{3|z|_\infty > R\}\Big] = 0,
\end{aligned}$$

which completes the proof of (iii).

Proof of (iv). Let $\tau_R^{n,m,\omega'} := T \wedge \tau_R^{n,\omega'} \wedge \tau_R^{m,\omega'}$. Using Itô's formula, we have for any stopping time $\bar\tau \le t \wedge \tau_R^{n,m,\omega'}$ that

$$\begin{aligned}
&\mathbb{E}|X_{\bar\tau}^{n,\omega'} - X_{\bar\tau}^{m,\omega'}|^2\\
&\quad\le \mathbb{E}\int_0^{\bar\tau} 2\langle X_{s-}^{n,\omega'} - X_{s-}^{m,\omega'}, f(s,\omega, X_{\kappa(n,(s-\tau):s)}^{n,\omega'}, \omega')\\
&\qquad - f(s,\omega, X_{\kappa(n,(s-\tau):s)}^{m,\omega'}, \omega')\rangle \,\mathrm{d}s
\end{aligned}$$

$$+\mathbb{E}\int_0^{\bar{\tau}} \big|g\big(s,\omega,X^{m,\omega'}_{\kappa(n,(s-\tau):s)},\omega'\big)-g\big(s,\omega,X^{m,\omega'}_{\kappa(n,(s-\tau):s)},\omega'\big)\big|^2\,\mathrm{d}s$$

$$+\mathbb{E}\int_0^{\bar{\tau}}\int_U \big|h\big(s,\omega,X^{m,\omega'}_{\kappa(n,(s-\tau):s)},\omega',\xi\big)-h\big(s,\omega,X^{m,\omega'}_{\kappa(n,(s-\tau):s)},\omega',\xi\big)\big|^2\nu(\mathrm{d}\xi)\,\mathrm{d}s.$$

Hypothesis A.1 implies

$$\begin{aligned}
&\mathbb{E}\big|X^{n,\omega'}_{\bar{\tau}}-X^{m,\omega'}_{\bar{\tau}}\big|^2\\
&\quad\le \mathbb{E}\int_0^{\bar{\tau}} 2\big\langle p^{n,\omega'}_{s-}-p^{m,\omega'}_{s-}, f\big(s,\omega,X^{n,\omega'}_{\kappa(n,(s-\tau):s)},\omega'\big)\\
&\qquad - f\big(s,\omega,X^{m,\omega'}_{\kappa(n,(s-\tau):s)},\omega'\big)\big\rangle\,\mathrm{d}s\\
&\qquad + \mathbb{E}\int_0^{\bar{\tau}} L_s(R,\omega')\int_{-\tau}^0\big(\big|X^{n,\omega'}_{\kappa(n,s+u)}-X^{m,\omega'}_{\kappa(n,s+u)}\big|^2\\
&\qquad + \mathbf{1}_{\{u<0\}}\big|X^{n,\omega'}_{\kappa(n,(s+u)^+)}-X^{m,\omega'}_{\kappa(n,(s+u)^+)}\big|^2\big)\lambda(\mathrm{d}u)\mathrm{d}s\\
&\quad\le \mathbb{E}\bigg(\int_0^T \mathbf{1}_{[0,\tau_R^{n,m,\omega'}]}(s)\bigg[4\tilde{K}_s(R,\omega')\big(\big|p^{n,\omega'}_s\big|+\big|p^{m,\omega'}_s\big|\big)\\
&\qquad + 3L_s(R,\omega')\int_{-\tau}^0\big(\big|p^{n,\omega'}_{s+u}\big|^2+\big|p^{m,\omega'}_{s+u}\big|^2\\
&\qquad + \mathbf{1}_{\{u<0\}}\big(\big|p^{n,\omega'}_{(s+u)^+}\big|^2+\big|p^{m,\omega'}_{(s+u)^+}\big|^2\big)\big)\lambda(\mathrm{d}u)\bigg]\,\mathrm{d}s\bigg)\\
&\qquad + \int_0^t 6L_s(R,\omega')\sup_{u\in[0,s]}\mathbb{E}\big|X^{n,\omega'}_{u\wedge\tau_R^{n,m,\omega'}}-X^{m,\omega'}_{u\wedge\tau_R^{n,m,\omega'}}\big|^2\,\mathrm{d}s\\
&\quad= I^{n,m,\omega'}_{R,T}+\int_0^t 6L_s(R,\omega')\sup_{u\in[0,s]}\mathbb{E}\big|X^{n,\omega'}_{u\wedge\tau_R^{n,m,\omega'}}-X^{m,\omega'}_{u\wedge\tau_R^{n,m,\omega'}}\big|^2\,\mathrm{d}s.
\end{aligned}\tag{A.6}$$

By setting $\bar{\tau}:=t\wedge\tau_R^{n,m,\omega'}$, $t\in[0,T]$, and using Gronwall's inequality, we get

$$\sup_{t\in[0,T]}\mathbb{E}\big|X^{n,\omega'}_{t\wedge\tau_R^{n,m,\omega'}}-X^{m,\omega'}_{t\wedge\tau_R^{n,m,\omega'}}\big|^2\le\exp\bigg(\int_0^T 6L_s(R,\omega')\,\mathrm{d}s\bigg)I^{n,m,\omega'}_{R,T}.\tag{A.7}$$

Substituting this bound into the right-hand side of (A.6) implies

$$\mathbb{E}\big|X^{n,\omega'}_{\bar{\tau}}-X^{m,\omega'}_{\bar{\tau}}\big|^2\le C(T,R,\omega')I^{n,m,\omega'}_{R,T}.$$

By setting

$$\bar{\tau}:=\tau_R^{n,m,\omega'}\wedge\inf\big\{t\ge 0:\big|X^{n,\omega'}_t-X^{m,\omega'}_t\big|\ge\varepsilon\big\},$$

we have

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{t\in[0,T]}\big|X^{n,\omega'}_t-X^{m,\omega'}_t\big|\ge\varepsilon\Big\}\\
&\quad\le\mathbb{P}\big\{T>\tau_R^{n,\omega'}\big\}+\mathbb{P}\big\{T>\tau_R^{m,\omega'}\big\}+\mathbb{P}\Big\{\sup_{t\in[0,\tau_R^{n,m,\omega'}]}\big|X^{n,\omega'}_t-X^{m,\omega'}_t\big|\ge\varepsilon\Big\}\\
&\quad\le\mathbb{P}\big\{T>\tau_R^{n,\omega'}\big\}+\mathbb{P}\big\{T>\tau_R^{m,\omega'}\big\}+\frac{1}{\varepsilon^2}\mathbb{E}\big|X^{n,\omega'}_{\bar{\tau}}-X^{m,\omega'}_{\bar{\tau}}\big|^2\\
&\quad\le\mathbb{P}\big\{T>\tau_R^{n,\omega'}\big\}+\mathbb{P}\big\{T>\tau_R^{m,\omega'}\big\}+\frac{C(T,R,\omega')}{\varepsilon^2}I^{n,m,\omega'}_{R,T}.
\end{aligned}$$

(i) and dominated convergence now implies that

$$\limsup_{n,m\to\infty} I_{R,T}^{n,m,\omega'} = 0$$

and using (iii), we get

$$\begin{aligned}
&\limsup_{n,m\to\infty} \mathbb{P}\Big\{\sup_{t\in[0,T]} \big|X_t^{n,\omega'} - X_t^{m,\omega'}\big| \ge \varepsilon\Big\} \\
&\quad\le \lim_{R\to\infty}\limsup_{n,m\to\infty}\bigg[\mathbb{P}\{T > \tau_R^{n,\omega'}\} + \mathbb{P}\{T > \tau_R^{m,\omega'}\} + \frac{C(T,R,\omega')}{\varepsilon^2} I_{R,T}^{n,m}\bigg] = 0.
\end{aligned}$$

So (iv) is obtained.

Proof of (v). Since the space $L^2(\Omega, \text{Càdlàg}([-\tau, T], \mathbb{R}^d))$ is complete w.r.t. convergence in probability, (iv) yields that there exists $X^{\omega'} \in L^2(\Omega, \text{Càdlàg}([-\tau, T], \mathbb{R}^d))$ such that

$$\lim_{n\to\infty} \mathbb{P}\Big\{\sup_{t\in[0,T]} \big|X_t^{n,\omega'} - X_t^{\omega'}\big| \ge \varepsilon\Big\} = 0.$$

We have to show that all terms of equation (A.4) for a subsequence of $n \in \mathbb{N}$ converge almost surely to the terms of equation (A.1). We have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\Big\{\sup_{t\in[0,T]} \big|X_{\kappa(n,t)}^{n,\omega'} - X_{t-}^{\omega'}\big| \ge \varepsilon\Big\} \\
&\quad\le \lim_{n\to\infty} \mathbb{P}\Big\{\sup_{t\in[0,T]} \big|X_{\kappa(n,t)}^{n,\omega'} - X_{\kappa(n,t)}^{\omega'}\big| \ge \varepsilon/2\Big\} \\
&\qquad + \lim_{n\to\infty} \mathbb{P}\Big\{\sup_{t\in[0,T]} \big|X_{\kappa(n,t)}^{\omega'} - X_{t-}^{\omega'}\big| \ge \varepsilon/2\Big\} = 0.
\end{aligned}$$

We can find a subsequence, say $\{n_l\}_{l\in\mathbb{N}}$, such that as $l \to \infty$,

$$\sup_{t\in[0,T]} \big|X_{\kappa(n_l,t)}^{n_l,\omega'} - X_{t-}^{\omega'}\big| \to 0, \quad \mathbb{P}\text{-a.s.}$$

for all ω' in a subset Ω_0' of full $\mathbb{P}'$-measure. Now let us define

$$S(t,\omega') := \sup_{l\in\mathbb{N}} \big|X_{\kappa(n_l,t)}^{n_l,\omega'}\big|,$$

then

$$\sup_{t\in[0,T]} S(t,\omega') < \infty, \quad \mathbb{P}\text{-a.s.}$$

for all ω' in Ω_0'. So by using (C3), (C4) and dominated convergence, we obtain for all ω' in Ω_0' that

$$\lim_{l\to\infty} \int_0^t f\big(s,\omega, X_{\kappa(n_l,(s-\tau):s)}^{n_l,\omega'}, \omega'\big)\,\mathrm{d}s = \int_0^t f\big(s,\omega, X_{(s-\tau)^-:s^-}^{\omega'}, \omega'\big)\,\mathrm{d}s, \quad \mathbb{P}\text{-a.s.}$$

Let $\tau(R) := \inf\{t \ge 0 : S(t) > R\} \wedge T$. For all $t \in [0,T]$ and all $\omega' \in \Omega_0'$, we have by dominated convergence that

$$\begin{aligned}
&\lim_{l\to\infty} \mathbb{E}\bigg|\int_0^{t\wedge\tau(R)} \big[g\big(s,\omega, X_{\kappa(n_l,(s-\tau):s)}^{n_l,\omega'}, \omega'\big) - g\big(s,\omega, X_{(s-\tau)^-:s^-}^{\omega'}, \omega'\big)\big]\,\mathrm{d}W_s\bigg|^2 \\
&\quad= \lim_{l\to\infty} \mathbb{E}\int_0^t \mathbf{1}_{\{s\le\tau(R)\}} \big|g\big(s,\omega, X_{\kappa(n_l,(s-\tau):s)}^{n_l,\omega'}, \omega'\big) - g\big(s,\omega, X_{(s-\tau)^-:s^-}^{\omega'}, \omega'\big)\big|^2\,\mathrm{d}s \\
&\quad= 0.
\end{aligned}$$

So

$$\begin{aligned}
&\mathbb{P}\biggl\{\biggl|\int_0^t \bigl[g\bigl(s,\omega,X^{n_l,\omega'}_{\kappa(n_l,(s-\tau):s)},\omega'\bigr)-g\bigl(s,\omega,X^{\omega'}_{(s-\tau)^-:s^-},\omega'\bigr)\bigr]\,\mathrm{d}W_s\biggr|>\varepsilon\biggr\}\\
&\quad\le \mathbb{P}\biggl\{\biggl|\int_0^{t\wedge\tau(R)} \bigl[g\bigl(s,\omega,X^{n_l,\omega'}_{\kappa(n_l,(s-\tau):s)},\omega'\bigr)-g\bigl(s,\omega,X^{\omega'}_{(s-\tau)^-:s^-},\omega'\bigr)\bigr]\,\mathrm{d}W_s\biggr|>\varepsilon\biggr\}\\
&\qquad+\mathbb{P}\bigl\{t>\tau(R)\bigr\}.
\end{aligned}$$

Fix sufficiently large R such that the second term on the right-hand side is less than $\delta>0$, then taking the limit $l\to\infty$ implies

$$\lim_{l\to\infty}\mathbb{P}\biggl\{\biggl|\int_0^t \bigl[g\bigl(s,\omega,X^{n_l,\omega'}_{\kappa(n_l,(s-\tau):s)},\omega'\bigr)-g\bigl(s,\omega,X^{\omega'}_{(s-\tau)^-:s^-},\omega'\bigr)\bigr]\,\mathrm{d}W_s\biggr|>\varepsilon\biggr\}\le\delta,$$

where $\delta>0$ is arbitrary. Therefore,

$$\begin{aligned}
&\int_0^t g\bigl(s,\omega,X^{n_l,\omega'}_{\kappa(n_l,(s-\tau):s)},\omega'\bigr)\,\mathrm{d}W_s\\
&\qquad\to\int_0^t g\bigl(s,\omega,X^{\omega'}_{(s-\tau)^-:s^-},\omega'\bigr)\,\mathrm{d}W_s\quad\text{in probability.}
\end{aligned}$$

The same argument implies

$$\begin{aligned}
&\int_0^t\int_U h\bigl(s,\omega,X^{n_l,\omega'}_{\kappa(n_l,(s-\tau):s)},\omega',\xi\bigr)\tilde{N}(\mathrm{d}s,\mathrm{d}\xi)\\
&\qquad\to\int_0^t\int_U h\bigl(s,\omega,X^{\omega'}_{(s-\tau)^-:s^-},\omega',\xi\bigr)\tilde{N}(\mathrm{d}s,\mathrm{d}\xi)\quad\text{in probability}
\end{aligned}$$

and for some subsequence n_{l_k} the above convergences are $\mathbb{P}$-a.s. Therefore, X is a solution of equation (A.1) on $[0,T]$. Since $X^{n,\omega'}_t$ is measurable w.r.t. $(t,\omega,\omega')\in[-\tau,\infty[\times\Omega\times\Omega'$, $X^{\omega'}_t$ is also measurable.

Uniqueness. Let X and Y be two solutions of equation (A.1) and define

$$\tau\bigl(R,\omega'\bigr):=\inf\bigl\{t\ge 0;\,\bigl|X^{\omega'}_t\bigr|>R\text{ or }\bigl|Y^{\omega'}_t\bigr|>R\bigr\}.$$

We have

$$\begin{aligned}
&\mathbb{E}\bigl|X^{\omega'}_{t\wedge\tau(R,\omega')}-Y^{\omega'}_{t\wedge\tau(R,\omega')}\bigr|^2\\
&\quad=\mathbb{E}\int_0^{t\wedge\tau(R,\omega')}\bigl[2\bigl\langle X^{\omega'}_{s^-}-Y^{\omega'}_{s^-},f\bigl(s,\omega,X^{\omega'}_{(s-\tau)^-:s^-},\omega'\bigr)-f\bigl(s,\omega,Y^{\omega'}_{(s-\tau)^-:s^-},\omega'\bigr)\bigr\rangle\\
&\qquad+\bigl|g\bigl(s,\omega,X^{\omega'}_{(s-\tau)^-:s^-},\omega'\bigr)-g\bigl(s,\omega,Y^{\omega'}_{(s-\tau)^-:s^-},\omega'\bigr)\bigr|^2\bigr]\,\mathrm{d}s\\
&\qquad+\mathbb{E}\int_0^{t\wedge\tau(R,\omega')}\int_U\bigl|h\bigl(s,\omega,X^{\omega'}_{(s-\tau)^-:s^-},\omega',\xi\bigr)\\
&\qquad-h\bigl(s,\omega,Y^{\omega'}_{(s-\tau)^-:s^-},\omega',\xi\bigr)\bigr|^2\nu(\mathrm{d}\xi)\,\mathrm{d}s\\
&\quad\le\mathbb{E}\int_0^{t\wedge\tau(R,\omega')}L_s\bigl(R,\omega'\bigr)\int_{-\tau}^0\bigl(\bigl|X^{\omega'}_{(s+u)^-}-Y^{\omega'}_{(s+u)^-}\bigr|^2\\
&\qquad+\mathbf{1}_{\{u<0\}}\bigl|X^{\omega'}_{s+u}-Y^{\omega'}_{s+u}\bigr|^2\bigr)\lambda(\mathrm{d}u)\,\mathrm{d}s
\end{aligned}$$

so that

$$\sup_{s\le t}\mathbb{E}\bigl|X^{\omega'}_{s\wedge\tau(R)}-Y^{\omega'}_{s\wedge\tau(R)}\bigr|^2\le\int_0^t 2L_s\bigl(R,\omega'\bigr)\sup_{u\le s}\mathbb{E}\bigl|X^{\omega'}_{u\wedge\tau(R)}-Y^{\omega'}_{u\wedge\tau(R)}\bigr|^2\,\mathrm{d}s.$$

Gronwall's lemma now implies that

$$\sup_{s \le T} \mathbb{E}\big|X^{\omega'}_{s\wedge\tau(R)} - Y^{\omega'}_{s\wedge\tau(R)}\big|^2 = 0,$$

so

$$\mathbb{P}\{X^{\omega'}_s = Y^{\omega'}_s\} = \lim_{R\to\infty}\big[\mathbb{P}\{X^{\omega'}_s = Y^{\omega'}_s, s \le \tau(R)\} + \mathbb{P}\{s > \tau(R)\}\big] = 1$$

and the uniqueness is proved. □

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