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The phase diagram of the complex branching Brownian motion energy model*

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Abstract

Branching Brownian motion (BBM) is a convenient representative of the class of log-correlated random fields. Motivated by the conjectured criticality of the log-correlated fields, we take the viewpoint of statistical physics on the BBM: We consider the partition function of the field of energies given by the "positions" of the particles of the *complex-valued* BBM. In such a *complex BBM energy model*, we allow for arbitrary correlations between the real and imaginary parts of the energies. We identify the fluctuations of the partition function. As a consequence, we get the full phase diagram of the log-partition function. It turns out that the phase diagram is the same as for the field of independent energies, i.e., Derrida's random energy model (REM). Yet, the fluctuations are different from those of the REM in all phases. All results are shown *for any correlation* between the real and imaginary parts of the random energy.

Keywords: Gaussian processes; branching Brownian motion; logarithmic correlations; random energy model; phase diagram; central limit theorem; random variance; martingale convergence.
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1 Introduction

Random energy models (REM) suggested by Derrida [12, 13] turned out to be an instructive playground in the studies of strongly correlated random systems, see, e.g., the recent reviews [36, 26, 8]. In this paper, we focus on the *complex-valued branching Brownian Motion (BBM) energy model* and show that this model lies exactly at the

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borderline of the complex REM universality class. This means that the *phase diagram* of the model is the same as in the complex REM, cf. Derrida [14] and [24]. However, the fluctuations of the partition function of this model are already influenced by the strong correlations and differ from those of the REM in *all phases* of the model, as we show in this work and in [19].

The motivation to consider the complex-valued setup is multi-fold:

1. **Critical phenomena.** Lee and Yang [31] observed that *phase transitions* (=analyticity breaking of the log-partition function) occur at critical points due to the accumulation of *complex zeros* of the partition function (viewed as a function of the external field) around the critical points on the real line, as the size of the system tends to infinity (=thermodynamic limit).
2. **Quantum physics and interference phenomena.** The formalism of quantum physics is based on the sums (and integrals) of *complex exponentials*. This naturally leads to cancellations between the magnitudes of the summands in the partition function. This is a manifestation of the interference phenomenon, see, e.g., Dobrinevski et al. [16].
3. **Random matrix theory and the Riemann zeta function.** The Riemann zeta function is a central object of analytic number theory. Striking relationships between statistical physics of random energy models and randomized versions of the zeta function and characteristic polynomials of random matrices were conjectured by Fyodorov et al. [17].

1.1 Branching Brownian motion

BBM viewed as a random energy model plays a special rôle. It turns out that BBM has correlations which are exactly at the borderline between the regime of weak correlations (REM universality class[1]) and the one of strong correlations[2].

Before stating our results, let us briefly recall the construction of a BBM. Consider a canonical continuous branching process: a *continuous time Galton-Watson* (GW) process [5]. It starts with a single particle located at the origin at time zero. After an exponential time of parameter one, this particle splits into $k \in \mathbb{Z}_+$ particles according to some probability distribution $(p_k)_{k\geq 0}$ on $\mathbb{Z}_+$. Then, each of the new-born particles splits independently at independent exponential (parameter 1) times again according to the same $(p_k)_{k\geq 0}$, and so on. We assume that $\sum_{k=1}^{\infty} p_k = 1$.[3] In addition, we assume that $\sum_{k=1}^{\infty} kp_k = 2$ (i.e., the expected number of children per particle equals two). Besides, we impose the finite second moment assumption:

$$K := \sum_{k=1}^{\infty} k(k-1)p_k < \infty. \tag{1.1}$$

We assume that at time $t = 0$, the GW process starts with just one particle.

For given $t \geq 0$, we label the particles of the process as $i_1(t), \dots, i_{n(t)}(t)$, where $n(t)$ is the total number of particles at time t. Note that under the above assumptions, we have $\mathbb{E}[n(t)] = \mathrm{e}^t$. For $s \leq t$, we denote by $i_k(s,t)$ the unique ancestor of particle $i_k(t)$ at time s. In general, there will be several indices k, l such that $i_k(s,t) = i_l(s,t)$. For $s, r \leq t$, define the time of the most recent common ancestor of particles $i_k(r,t)$ and $i_l(s,t)$ as[4]

$$d(i_k(r,t), i_l(s,t)) := \sup\{u \leq s \wedge r \colon i_k(u,t) = i_l(u,t)\}. \tag{1.2}$$

[1] = the same phase diagram as for the field of independent random energies.

[2] = different phase diagram comparing to the REM one, due to the strictly larger leading order of the minimal energy than the one for the independent field of random energies.

[3] This implies that $p_0 = 0$, so none of the particles ever dies.

[4] Note that $d(\cdot,\cdot)$ is not the distance to the most recent common ancestor of the particles but rather the *overlap* between the particle trajectories.

For $t \geq 0$, the collection of all ancestors naturally induces the random tree

$$\mathbb{T}_t := \{i_k(s,t) \colon 0 \leq s \leq t, 1 \leq k \leq n(t)\} \tag{1.3}$$

called the *GW tree up to time* t. We denote by $\mathcal{F}^{\mathbb{T}_t}$ the σ-algebra generated by the GW process up to time t.

In addition to the genealogical structure, the particles get a *position* in $\mathbb{R}$. Specifically, the first particle starts at the origin at time zero and performs Brownian motion until the first time when the GW process branches. After branching, each new-born particle independently performs Brownian motion (started at the branching location) until their respective next branching times, and so on. We denote the positions of the $n(t)$ particles at time $t \geq 0$ by $x_1(t), \ldots, x_{n(t)}(t)$.

We define BBM as a family of Gaussian processes,

$$x_t := \{x_1(s,t), \ldots, x_{n(t)}(s,t) \colon s \leq t\} \tag{1.4}$$

indexed by time horizon $t \geq 0$. Note that conditionally on the underlying GW tree these Gaussian processes have the following covariance

$$\mathbb{E}\left[x_k(s,t)x_l(r,t) \mid \mathcal{F}^{\mathbb{T}_t}\right] = d(i_k(s,t), i_l(r,t)), \quad s,r \in [0,t], \quad k,l \leq n(t). \tag{1.5}$$

In what follows, to lighten the notation, we will simply write $x_i(s) := x_i(s,t)$, $i \leq n(t)$, $s \leq t$ hoping that this will not cause confusion about the parameter $t \geq 0$.

1.2 A model of complex-valued random energies

In this section, we introduce the *complex BBM random energy model*.

Let $\rho \in [-1,1]$. For any $t \in \mathbb{R}_+$, let $X(t) := (x_k(t))_{k \leq n(t)}$ and $Y(t) := (y_k(t))_{k \leq n(t)}$ be two BBMs with the same underlying GW tree such that, for $k \leq n(t)$,

$$\mathrm{Cov}(x_k(t), y_k(t)) = \rho t. \tag{1.6}$$

In what follows, to lighten the notation, we sometimes drop the dependence of quantities of interest on ρ. Note that

$$Y(t) \overset{\mathrm{D}}{=} \rho X(t) + \sqrt{1-\rho^2}\, Z(t), \tag{1.7}$$

where "$\overset{\mathrm{D}}{=}$" denotes equality in distribution and $Z(t) := (z_i(t))_{i \leq n(t)}$ is a branching Brownian motion independent from $X(t)$ and with the same underlying GW process. Representation (1.7) allows us to handle arbitrary correlations by decomposing the process Y into a part independent from X and a fully correlated one.

We define the *partition function* for the complex BBM energy model with correlation ρ at inverse temperature $\beta := \sigma + i\tau \in \mathbb{C}$ by

$$\mathcal{X}_{\beta,\rho}(t) := \sum_{k=1}^{n(t)} \mathrm{e}^{\sigma x_k(t) + i\tau y_k(t)}. \tag{1.8}$$

1.3 Notation

By $\mathcal{L}[\cdot]$, $\mathcal{L}[\cdot \mid \cdot]$, and $\Longrightarrow$ or wlim, we denote the law, conditional law, and weak convergence respectively. By $\mathcal{N}(0,s^2)$, $s^2 > 0$, we denote the centred complex isotropic Gaussian distribution with density

$$\mathbb{C} \ni z \mapsto \frac{\mathrm{e}^{-|z/s|^2}}{\pi s^2} \in \mathbb{R}_+ \tag{1.9}$$

w.r.t. the Lebesgue measure on $\mathbb{C}$.

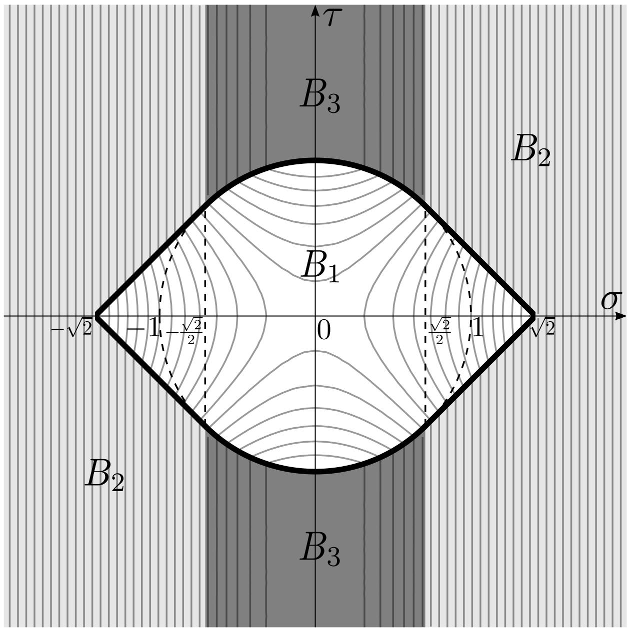

Figure 1: Phase diagram of the REM and the BBM energy model. The grey curves are the level lines of the limiting log-partition function, cf. (1.11). This paper deals with phases B_1 and B_3 and the boundaries. For a treatment of phase B_2, see [19].

1.4 Main results

Let us specify the three domains depicted on Figure 1 analytically:

$$\begin{aligned} B_1 := \mathbb{C} \setminus \overline{B_2 \cup B_3}, \quad B_2 &:= \{\sigma + i\tau \in \mathbb{C} \colon 2\sigma^2 > 1, |\sigma| + |\tau| > \sqrt{2}\}, \\ B_3 &:= \{\sigma + i\tau \in \mathbb{C} \colon 2\sigma^2 < 1, \sigma^2 + \tau^2 > 1\}. \end{aligned} \tag{1.10}$$

Remark 1.1. Some of our results will be stated under the *binary branching* assumption (i.e., $p_k = 0$ for all $k > 2$). Existence of all moments of the offspring distribution would also suffice for all our results and will not require essential changes in the proofs.

Our first result states that the complex BBM energy model indeed has the phase diagram depicted on Figure 1.

Theorem 1.2 (Phase diagram)**.** *For any $\rho \in [-1, 1]$, and any $\beta \in \mathbb{C}$, the complex BBM energy model with binary branching has the same log-partition function and the phase diagram (cf.,* Figure 1*) as the complex REM, i.e.,*

$$\lim_{t \uparrow \infty} \frac{1}{t} \log \mathcal{X}_{\beta,\rho}(t) = \begin{cases} 1 + \frac{1}{2}(\sigma^2 - \tau^2), & \beta \in \overline{B_1}, \\ \sqrt{2}|\sigma|, & \beta \in \overline{B_2}, \\ \frac{1}{2} + \sigma^2, & \beta \in \overline{B_3} \end{cases} \tag{1.11}$$

in probability.

See Section 5 for a proof.

Remark 1.3. 1. For a deterministic regular weighted tree (= directed polymer on a tree), under the assumption of no correlations between the real and imaginary parts of the complex random energies (i.e., case $\rho = 0$), formula (1.11) was obtained by Derrida et al. [15]. Our derivation of Theorem 1.2 is based on the detailed information on the *fluctuations* of the partition function (1.8) which we provide in Section 1.5. The proof in [15] is more direct and does not reach the (CLT) precision which is provided in the following section. Moreover, the arguments in [15] seem to crucially rely on the assumption $\rho = 0$.

2. It is natural to expect that the convergence in (1.11) also holds in L^1, see [24, Theorem 2.15] for a related result for the REM.

1.5 A class of martingales

In the centre of our analysis are the following martingales

$$\mathcal{M}_{\sigma,\tau}(t) := \mathrm{e}^{-t\left(1+2i\rho\sigma\tau+\frac{\sigma^2-\tau^2}{2}\right)}\mathcal{X}_{\beta,\rho}(t) = \sum_{k=1}^{n(t)} \mathrm{e}^{-t\left(1+2i\rho\sigma\tau+\frac{\sigma^2-\tau^2}{2}\right)}\mathrm{e}^{\sigma x_k(t)+i\tau y_k(t)}. \tag{1.12}$$

We denote by $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ the natural filtration associated to $(\mathcal{M}_{\sigma,\tau}(t))_{t\in\mathbb{R}_+}$.

Note that, for $\beta = \sigma \in [0, 1/\sqrt{2})$, $\mathcal{M}_{\sigma,0}(t)$ coincides with the McKean martingale introduced in [9], where it was proven that these martingales converge almost surely and in L^1 to a non-degenerate limit.

The next theorem states that for $\beta \in B_1$ the martingales $\mathcal{M}_{\sigma,\tau}(t)$ are in L^p for some $p > 1$.

Theorem 1.4 (L^p martingale convergence in B_1)**.** *For $\beta = \sigma + i\tau$ with $\beta \in B_1, |\beta| \geq 1$, and any $\rho \in [-1, 1]$, $\mathcal{M}_{\sigma,\tau}(t)$ is a martingale with expectation 1 and it is in L^p for $p \leq \frac{\sqrt{2}}{\sigma}$. Hence, the limit*

$$\lim_{t\uparrow\infty} \mathcal{M}_{\sigma,\tau}(t) =: \mathcal{M}_{\sigma,\tau} \tag{1.13}$$

exists a.s., in L^1, and is non-degenerate.

See Section 2 for a proof.

Remark 1.5. For $|\beta| < 1$, and any $\rho \in [-1, 1]$, it has been proven in [19, Proposition A.1] that $\mathcal{M}_{\sigma,\tau}(t)$ is L^2-bounded.

On the boundary $B_{1,2}$ between phases B_1 and B_2, i.e., on the set

$$B_{1,2} := \overline{B_1} \cap \overline{B_2} = \{\sigma + i\tau \in \mathbb{C} \colon |\sigma| > 1/\sqrt{2}, |\sigma| + |\tau| = \sqrt{2}\} \tag{1.14}$$

a similar result still holds.

Theorem 1.6 (L^p martingale convergence on $B_{1,2}$)**.** *For $\beta \in B_{1,2}$ and any $\rho \in [-1, 1]$, we have that $\mathcal{M}_\beta(t)$ is a L^p-bounded martingale, for any $p < \sqrt{2}/\sigma$ with expectation 1. Hence, the limit*

$$\lim_{t\uparrow\infty} \mathcal{M}_{\sigma,\tau}(t) =: \mathcal{M}_{\sigma,\tau} \tag{1.15}$$

exists a.s. in L^1, and is non-degenerate.

See Section 4.4 for a proof.

Remark 1.7. Similar result for $\rho = 0$ has been obtained for the complex Gaussian multiplicative chaos in [29, Theorem 3.11].

Remark 1.8 (Smoothing transforms)**.** Note that the martingales $\mathcal{M}_{\sigma,\tau}(t)$ satisfy a recursive equation of the form

$$\mathcal{L}\left[\mathcal{M}_{\sigma,\tau}(t+r)\right] = \mathcal{L}\Big[\sum_{k=1}^{n(r)} a_k(r)\mathcal{M}^{(k)}_{\sigma,\tau}(t)\Big], \tag{1.16}$$

where $\mathcal{M}^{(k)}_{\sigma,\tau}(t-r)$ are i.i.d. copies of $\mathcal{M}_{\sigma,\tau}(t)$ and $a_k(r) \in \mathbb{C}$ are some complex weights independent from $\mathcal{M}^{(k)}_{\sigma,\tau}(t-r), k \in \mathbb{N}$. If a limit $\mathcal{M}_{\sigma,\tau}$ as $t \uparrow \infty$ of $\mathcal{M}_{\sigma,\tau}(t+r)$ exists, then it would have to satisfy the equation

$$\mathcal{L}\left[\mathcal{M}_{\sigma,\tau}\right] = \mathcal{L}\Big[\sum_{k=1}^{n(r)} a_k(r)\mathcal{M}^{(k)}_{\sigma,\tau}\Big], \tag{1.17}$$

where $\mathcal{M}^{(k)}_{\sigma,\tau}$ are i.i.d. copies of $\mathcal{M}_{\sigma,\tau}$. This type of equation is called *complex smoothing transform*. We refer to Meiners and Mentemeier [35] and Kolesko and Meiners [27] for more details.

1.6 Conditional central limit theorems

The following three results cover the whole strip $|\sigma| < 1/\sqrt{2}$ and basically are "central limit theorems" (CLTs) with random variance.

Theorem 1.9 (CLT with random variance for $|\sigma| < 1/\sqrt{2}$, $\beta \in B_1$)**.** *Let $\beta = \sigma + i\tau$ with $|\sigma| < 1/\sqrt{2}$ and $\rho \in [-1,1]$. For $\beta \in B_1$,*

$$\operatorname*{wlim}_{r\uparrow\infty}\operatorname*{wlim}_{t\uparrow\infty}\mathcal{L}\left[\frac{\mathcal{M}_{\sigma,\tau}(t+r)-\mathcal{M}_{\sigma,\tau}(r)}{\mathrm{e}^{r(1-\sigma^2-\tau^2)}}\;\Big|\;\mathcal{F}_r\right] = \mathcal{N}\left(0, C_1\mathcal{M}_{2\sigma,0}\right), \tag{1.18}$$

where $C_1 > 0$ is some constant.

See Section 2 for a proof.

Remark 1.10.

1. The scaling on the l.h.s. of (1.18) does not depend on ρ.
2. The appearance of the random variance in Theorem 1.9 (and in the subsequent ones) is in sharp contrast with the REM [24] and generalized REM [25], where CLTs with deterministic variance hold for β in the strip $|\sigma| < 1/\sqrt{2}$.
3. For $\beta \in \mathbb{R}$, a result resembling Theorem 1.9 was obtained by Iksanov and Kabluchko in [20].
4. For a logarithmically correlated field of complex-valued random energies on a Euclidean space without correlations between the real and imaginary parts of the energy (i.e., case $\rho = 0$), a similar result was shown by Lacoin et al. [29, Theorem 3.1].

Theorem 1.11 (CLT with random variance in B_3)**.** *For $\beta \in B_3$, $\rho \in [-1,1]$ and binary branching,*

$$\mathcal{L}\left[\frac{\mathcal{X}_{\beta,\rho}(t)}{\mathrm{e}^{t(1/2+\sigma^2)}}\;\Big|\;\mathcal{M}_{2\sigma,0}\right] \underset{t\uparrow\infty}{\Longrightarrow} \mathcal{N}\left(0, C_2\mathcal{M}_{2\sigma,0}\right), \tag{1.19}$$

where $C_2 > 0$ is some constant.

See Section 3.3 for a proof.

Remark 1.12. In case $\rho = 0$, a similar result has been obtained by Lacoin et al. [29, Theorem 4.2].

A similar result also holds on the boundary between phases B_1 and B_3, i.e., on the set

$$B_{1,3} := \overline{B_1}\cap\overline{B_3} = \{\sigma + i\tau \in \mathbb{C} \colon \sigma^2+\tau^2 = 1, |\sigma| < 1/\sqrt{2}\}. \tag{1.20}$$

Theorem 1.13 (CLT with random variance on $B_{1,3}$)**.** *For $\beta \in B_{1,3}$, $\rho \in [-1,1]$, and binary branching,*

$$\mathcal{L}\left[\frac{\mathcal{X}_{\beta,\rho}(t)}{\sqrt{t}\mathrm{e}^{t(1/2+\sigma^2)}}\;\Big|\;\mathcal{M}_{2\sigma,0}\right] \underset{t\uparrow\infty}{\Longrightarrow} \mathcal{N}\left(0, C_3\mathcal{M}_{2\sigma,0}\right), \tag{1.21}$$

where $C_3 > 0$ is some constant.

See Section 4.1 for a proof.

Remark 1.14. For $\rho = 0$, a similar result for Gaussian multiplicative chaos was obtained by Lacoin et al. [29, Theorem 4.2].

Recall that the behaviour of the partition function at $\beta = \sqrt{2}$ is determined by the martingale $\mathcal{M}_{1,0}(t)$, which is related to another martingale – the so-called *derivative martingale* $\mathcal{Z}(t)$:

$$\mathcal{Z}(t) := \sum_{i=1}^{n(t)} (\sqrt{2}t - x_k(t)) \mathrm{e}^{-\sqrt{2}(\sqrt{2}t - x_k(t))}. \tag{1.22}$$

Lalley and Sellke proved in [30] that $\mathcal{Z}(t)$ converges a.s. as $t \to \infty$ to a non-trivial limit $\mathcal{Z}$ which is a positive and a.s. finite random variable.

At the boundary,

$$B_{2,3} := \overline{B_2} \cap \overline{B_3} = \Big\{ \sigma + i\tau \in \mathbb{C} \colon |\sigma| = 1/\sqrt{2}, |\tau| \geq 1/\sqrt{2} \Big\}, \tag{1.23}$$

including the *triple point*

$$\beta_{1,2,3} := \overline{B_1} \cap \overline{B_2} \cap \overline{B_3} = (1+i)/\sqrt{2}, \tag{1.24}$$

after appropriate rescaling, we have the following CLT with random variance.

Theorem 1.15 (CLT with random variance for $|\sigma| = 1/\sqrt{2}$)**.** *Let $\beta = \sigma + i\tau$ with $|\sigma| = 1/\sqrt{2}$ and $\rho \in [-1,1]$ and assume binary branching. Then:*

(i) For $\tau > 1/\sqrt{2}$,

$$\operatorname*{wlim}_{r\uparrow\infty} \operatorname*{wlim}_{t\uparrow\infty} \mathcal{L}\left[r^{1/4} \cdot \frac{\mathcal{X}_{\beta,\rho}(t+r)}{\mathrm{e}^{(t+r)(1/2+\sigma^2)}} \;\Big|\; \mathcal{F}_r \right] = \mathcal{N}\left(0, C_2 \sqrt{\frac{2}{\pi}} \mathcal{Z}\right). \tag{1.25}$$

(ii) For $\tau = 1/\sqrt{2}$,

$$\operatorname*{wlim}_{r\uparrow\infty} \operatorname*{wlim}_{t\uparrow\infty} \mathcal{L}\left[\frac{r^{1/4}}{\sqrt{t}} \cdot \frac{\mathcal{X}_{\beta,\rho}(t+r)}{\mathrm{e}^{(t+r)(1/2+\sigma^2)}} \;\Big|\; \mathcal{F}_r \right] = \mathcal{N}\left(0, C_3 \sqrt{\frac{2}{\pi}} \mathcal{Z}\right). \tag{1.26}$$

See Section 4.3 for a proof.

Remark 1.16. For $\rho = 0$, a similar result for Gaussian multiplicative chaos was obtained by Lacoin et al. [29, Theorem 4.3].

1.7 Related research

Several models of of complex-valued random energy landscapes were considered in the literature. We group them according to the strength of correlations.

Independent energies. *Complex random energy model* with independent Gaussian energies has been suggested and analysed using heuristic arguments in the seminal work of Derrida [14]. In [24], the results of [14] were confirmed rigorously via the probabilistic analysis of *fluctuations* of the partition function. As a consequence of the fluctuation results, it was shown in [24] that the limiting log-partition function is given by (1.11) and does not depend on the *correlation parameter* ρ, cf. (1.6).

Logarithmic correlations. The BBM energy model is a particularly transparent representative for a whole class of models with the so-called logarithmic correlation strength.

In [15], Derrida et al. considered a landscape of complex-valued random energies attached to the leaves of a deterministic regular tree of fixed depth, as the depth tends to infinity. Similarly to the locations of the BBM particles, the energies of the leaves are generated as a sum of the independent complex-valued weights collected along the path connecting the root to a leaf. This can be seen as a *mean-field model of directed polymers with random complex weights on the regular tree.* For this model, under the assumption $\rho = 0$, the authors of [15] showed that the very same formula (1.11) holds for the directed polymer without resorting to the more informative analysis of fluctuations of the partition function.

Barral et al. [6, 7] studied *complex Gaussian multiplicative cascades* on the unit interval. These works cover Phase I (cf. Fig. 1) via a martingale convergence result. In Phase II, the authors show tightness of the properly rescaled partition function. The model is constructed using a dyadic embedding of the binary tree into the unit interval. This makes the model closely related to that of [15].

On Euclidean spaces in higher dimensions ($d \geq 2$), under the assumption $\rho = 0$, a random energy model on Euclidean spaces with logarithmic (w.r.t. the Euclidean distance) correlations was studied by Lacoin et al. [29]. In [29], for this *Gaussian multiplicative chaos*, the same phase diagram as on Figure 1 was identified. However, only Phases I and III were treated in [29]. Maduale et al. [32] studied the complex cascade model on a regular binary tree closely related to the models of [15, 6, 7]. On the boundary between Phases I and II, [32] provides a modulus of continuity estimate for the chaos. For a review on Gaussian multiplicative chaos, we refer to Rhodes and Vargas [37]. Purely imaginary multiplicative chaos was studied by Junnila et al. [23].

Phase II was studied for the *complex branching BBM energy model* [33] in the case $\rho = 0$. The case $\rho \neq 0$ was treated by the present authors in [19].

As mentioned in the remark below Theorem 1.6, the branching structure of the BBM implies complex distributional equations (1.17), which are referred to as *complex smoothing transform.* A detailed study on how solutions to such equations with complex weights look like was recently done by Meiners and Mentemeier [35], see also the recent paper by Kolesko and Meiners [27]. The case of real-valued scalar weights was treated by Alsmeier and Meiners [2] and by Iksanov and Meiners [22].

Fluctuations of the so-called additive (Biggins') martingale (which is nothing else as the partition function) for a supercritical branching random walk were studied for complex temperatures by Iksanov et al. [21]. In the real-valued case, fluctuations of the derivative martingale for the BBM (cf., (1.22)) were identified by Maillard and Pain [34].

Hairer and Shen [18] studied the *dynamical sine-Gordon model* – a non-linear parabolic SPDE in two spatial dimensions subject to additive space-time white noise. In [18], it is shown that the corresponding Hairer's regularity structure is related to the complex multiplicative Gaussian chaos from [29].

Striking conjectures on the relationships of the log-correlated (complex) random energy models with *characteristic polynomials of random matrices*, and the *Riemann zeta function* were formulated by Fyodorov et al. [17]. Some of these conjectures have been attacked in the mathematics literature, see, e.g., Arguin et al. [3, 4], and Saksman and Webb [38].

General correlations. An approximation of the case of general correlations is the so-called Generalized REM (GREM), see Derrida [13], and Bovier and Kurkova [10, 11]. Fluctuations of the *complex-valued GREM* were identified in [25], where an explicit formula generalizing (1.11) for the log-partition function was derived as a consequence. This formula reveals a much richer phase diagram than the one of the complex REM (cf., Figure 1).

1.8 Organization of the rest of the paper

The remainder of the paper is organized as follows. In Section 2, we prove Theorems 1.4 and 1.9 concerning the behaviour of the partition function in Phase B_1. In Section 3, we treat Phase B_3. We start with a second moment computation which is then (in the next subsection) generalised to a constrained higher moment computation. Finally, in Section 3.3, we prove Theorem 1.11. The boundaries $B_{1,3}, B_{2,3}$ (Theorems 1.13 and 1.15) are proved in Section 4. Section 5 contains the proof of Theorem 1.2.

2 Proof of results for phase B_1

We start by proving the martingale convergence of $\mathcal{M}_{\sigma,\tau}(t)$.

Proof of Theorem 1.4. One readily checks that $\mathcal{M}_{\sigma,\tau}(t)$ is a martingale with expectation 1. Next, we compute the $\frac{\sqrt{2}}{\sigma}$-moment of $\mathcal{M}_{\beta}(t)$. To do this, first consider

$$\mathbb{E}\Big[\Big|\sum_{k=1}^{n(t)} \mathrm{e}^{\sigma x_k(t)+i\tau y_k(t)}\Big|^{\frac{\sqrt{2}}{\sigma}}\Big] = \mathbb{E}\Big[\Big|\sum_{k=1}^{n(t)} \mathrm{e}^{(\sigma+i\rho\tau)x_k(t)+i\sqrt{1-\rho^2}\tau z_k(t)}\Big|^{\frac{\sqrt{2}}{\sigma}}\Big], \tag{2.1}$$

where we used Representation (1.7). The right-hand side of (2.1) is equal to

$$\begin{aligned}&\mathbb{E}\Big[\Big(\Big|\sum_{k=1}^{n(t)} \mathrm{e}^{(\sigma+i\rho\tau)x_k(t)+i\sqrt{1-\rho^2}\tau z_k(t)}\Big|^2\Big)^{\frac{1}{\sqrt{2}\sigma}}\Big] \\ &\qquad = \mathbb{E}\Big[\Big(\sum_{k,j=1}^{n(t)} \mathrm{e}^{\sigma(x_k(t)+x_j(t))+i\rho\tau(x_k(t)-x_j(t))+i\sqrt{1-\rho^2}\tau(z_k(t)-z_j(t))}\Big)^{\frac{1}{\sqrt{2}\sigma}}\Big].\end{aligned} \tag{2.2}$$

By Jensen's inequality for the conditional expectations, and because $1/\sqrt{2}\sigma < 1$, for $\sigma > 1/\sqrt{2}$, (2.2) is bounded from above by

$$\begin{aligned}&\mathbb{E}\Big[\Big(\sum_{k,j=1}^{n(t)} \mathrm{e}^{\sigma(x_k(t)+x_j(t))+i\rho\tau(x_k(t)-x_j(t))}\mathbb{E}\Big[\mathrm{e}^{\sqrt{1-\rho^2}\tau(z_k(t)-z_j(t))}\Big]\Big)^{\frac{1}{\sqrt{2}\sigma}}\Big] \\ &\qquad = \mathbb{E}\Big[\Big(\sum_{k,j=1}^{n(t)} \mathrm{e}^{\sigma(x_k(t)+x_j(t))+i\rho\tau(x_k(t)-x_j(t))}\mathrm{e}^{-(1-\rho^2)\tau^2(t-q_{k,j})}\Big)^{\frac{1}{\sqrt{2}\sigma}}\Big],\end{aligned} \tag{2.3}$$

where we set

$$q_{k,j} := d(x_k(t), x_j(t)). \tag{2.4}$$

Then, we can bound (2.2) from above by

$$\mathbb{E}\Big[\Big(\sum_{l=1}^{\lfloor t\rfloor}\sum_{k,j=1}^{n(t)} \mathbb{1}_{q(k,j)\in[l,l+1)}\mathrm{e}^{2\sigma(x_k(t)+x_j(t))+i\rho\tau(x_k(t)-x_j(t))}\mathrm{e}^{-(1-\rho^2)\tau^2(t-q_{k,j})}\Big)^{\frac{1}{\sqrt{2}\sigma}}\Big] \tag{2.5}$$

Denote by $\sigma_1^{(l)},\dots,\sigma_p^{(l)} \in (0,t)$ all branching times (before time t), and by $n_1^{(l)},\dots,n_p^{(l)} \in \mathbb{N}$ the associated labels of particles which branched. We rewrite (2.5) as

$$\begin{aligned}&\mathbb{E}\Big[\Big(\sum_{l=1}^{\lfloor t\rfloor}\sum_{q=1}^{p}\mathrm{e}^{2\sigma x_{n_q^{(l)}}(\sigma_q^{(l)})} \\ &\qquad\times \sum_{\substack{k,j:\\ q(k,j)=\sigma_q^{(l)}}} \mathrm{e}^{\sigma(x_k(t)+x_j(t)-2x_{n_q^{(l)}}(\sigma_q^{(l)}))+i\rho\tau(x_k(t)-x_j(t))}\mathrm{e}^{-(1-\rho^2)\tau^2(t-q_{k,j})}\Big)^{\frac{1}{\sqrt{2}\sigma}}\Big].\end{aligned} \tag{2.6}$$

By the branching property,

$$\begin{aligned}\mathcal{L}\Big[x_k(t)-x_j(t)\Big] &= \mathcal{L}\Big[x^{(1)}_{k'}(t-q_{k,j})-x^{(2)}_{j'}(t-q_{k,j})\Big],\\ \mathcal{L}\Big[x_k(t)+x_j(t)-2x_{n_q^{(l)}}(\sigma_q^{(l)})\Big] &= \mathcal{L}\Big[x^{(1)}_{k'}(t-q_{k,j})+x^{(2)}_{j'}(t-q_{k,j})\Big],\end{aligned} \tag{2.7}$$

where k' and j' are the labels of two BBM particles at time $t-q_{k,j}$ from two independent copies $X^{(1)}(\cdot)$ and $X^{(2)}(\cdot)$ of a BBM. Using (2.7), we rewrite (2.6) as

$$\begin{aligned}&\mathbb{E}\Big[\Big(\sum_{l=1}^{\lfloor t\rfloor}\sum_{q=1}^{p}\mathrm{e}^{2\sigma x_{n_q^{(l)}}(\sigma_q^{(l)})-(1-\rho^2)\tau^2(t-q_{k,j})}\\ &\times\sum_{\substack{k'\le n^{(1)}(t-q_{k,j}),\\ j'\le n^{(2)}(t-q_{k,j})}}\mathrm{e}^{i\rho\tau\left(x^{(1)}_{k'}(t-q_{k,j})-x^{(2)}_{j'}(t-q_{k,j})\right)}+\mathrm{e}^{i\rho\tau\left(x^{(2)}_{j'}(t-q_{k,j})-x^{(1)}_{k'}(t-q_{k,j})\right)}\Big)^{\frac{1}{\sqrt{2}\sigma}}\Big].\end{aligned} \tag{2.8}$$

In what follows, we denote by $\mathbb{E}[\cdot\mid\sigma_q^{(l)}]$ the conditional expectation given $\sigma_q^{(l)}$. Noting that the term in (2.8) inside of the expectation is nonnegative, as it is a power of a conditional expectation of an absolute value, we use again Jensen since $1/\sqrt{2}\sigma<1$, for $\sigma>1/\sqrt{2}$, we bound (2.8) from above by

$$\begin{aligned}&\mathbb{E}\Big[\Big(\sum_{l=1}^{\lfloor t\rfloor}\sum_{q=1}^{p}\mathrm{e}^{2\sigma x_{n_q^{(l)}}(\sigma_q^{(l)})-(1-\rho^2)\tau^2(t-\sigma_q^{(l)})}\times\\ &\mathbb{E}\Big[\sum_{\substack{k'\le n^{(1)}(t-\sigma_q^{(l)}),\\ j'\le n^{(2)}(t-\sigma_q^{(l)})}}\mathrm{e}^{i\rho\tau\left(x^{(1)}_{k'}(t-\sigma_q^{(l)})-x^{(2)}_{j'}(t-\sigma_q^{(l)})\right)}+\mathrm{e}^{i\rho\tau\left(x^{(2)}_{j'}(t-q_{k,j})-x^{(1)}_{k'}(t-q_{k,j})\right)}\Bigm|\sigma_q^{(l)}\Big]\Big)^{\frac{1}{\sqrt{2}\sigma}}\Big].\end{aligned} \tag{2.9}$$

Recall (1.1). Calculating the inner expectations in (2.9), gives

$$\begin{aligned}&\mathbb{E}\Big[\sum_{\substack{k'\le n^{(1)}(t-\sigma_q^{(l)}),\\ j'\le n^{(2)}(t-\sigma_q^{(l)})}}\mathrm{e}^{i\rho\tau\left(x^{(1)}_{k'}(t-\sigma_q^{(l)})-x^{(2)}_{j'}(t-\sigma_q^{(l)})\right)}+\mathrm{e}^{i\rho\tau\left(x^{(2)}_{j'}(t-q_{k,j})-x^{(1)}_{k'}(t-q_{k,j})\right)}\Bigm|\sigma_q^{(l)}\Big]\\ &=K\mathrm{e}^{2(t-\sigma_q^{(l)})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{d}y'\,\mathrm{e}^{(\sigma+i\tau)y+(\sigma-i\tau)y'}\mathrm{e}^{-\frac{y^2+y'^2}{2(t-\sigma_q^{(l)})}}\frac{1}{2\pi(t-\sigma_q^{(l)})}\\ &=K\mathrm{e}^{(\sigma^2-\rho^2\tau^2)(t-\sigma_q^{(l)})+2(t-\sigma_q^{(l)})}\end{aligned} \tag{2.10}$$

by completing the square. Hence, (2.9) is equal to

$$\mathbb{E}\Big[\Big(\sum_{l=1}^{\lfloor t\rfloor}\sum_{q=1}^{p}\mathrm{e}^{2\sigma x_{n_q^{(l)}}(\sigma_q^{(l)})-(1-\rho^2)\tau^2(t-\sigma_q^{(l)})}K\mathrm{e}^{(\sigma^2-\rho^2\tau^2)(t-\sigma_q^{(l)})+2(t-\sigma_q^{(l)})}\Big)^{\frac{1}{\sqrt{2}\sigma}}\Big]. \tag{2.11}$$

Using again Jensen's inequality ($\sum(\dots)^{\sqrt{2}\sigma}\le(\sum(\dots))^{\sqrt{2}\sigma}$ since $\sqrt{2}\sigma>1$), we bound (2.11) from above by

$$\mathbb{E}\Big[\sum_{l=1}^{\lfloor t\rfloor}\sum_{q=1}^{p}\mathrm{e}^{\sqrt{2}x_{n_q^{(l)}}(\sigma_q^{(l)})}K^{1/\sqrt{2}\sigma}\mathrm{e}^{\frac{(\sigma^2-\tau^2+2)}{\sqrt{2}\sigma}(t-\sigma_q^{(l)})}\Big]. \tag{2.12}$$

Using the many-to-one formula, we obtain that (2.12) is equal to

$$K^{1/\sqrt{2}\sigma}\int_0^t\mathrm{d}q\;\mathrm{e}^{q}\mathrm{e}^{\frac{(\sigma^2-\tau^2+2)(t-q)}{\sqrt{2}\sigma}}\int_{-\infty}^{\infty}\mathrm{d}x\;\mathrm{e}^{\sqrt{2}x-\frac{x^2}{2q}}\frac{1}{\sqrt{2\pi q}}$$

$$= K^{1/\sqrt{2}\sigma} \int_0^t \mathrm{d}q \, \mathrm{e}^{\frac{(\sigma^2-\tau^2+2)(t-q)}{\sqrt{2}\sigma}} \mathrm{e}^{2q}, \tag{2.13}$$

by computing the Gaussian integral. Using (2.13) and noticing that the normalization factor in (1.12) is equal to $\mathrm{e}^{\frac{-2t-(\sigma^2-\tau^2)t}{\sqrt{2}\sigma}}$, we bound the $\frac{\sqrt{2}}{\sigma}$-moment of $\mathcal{M}_{\sigma,\tau}(t)$ by

$$K \int_0^t \mathrm{d}q \, \mathrm{e}^{\frac{(\sigma^2-\tau^2+2)(t-q)-2t-(\sigma^2-\tau^2)t}{\sqrt{2}\sigma}} \mathrm{e}^{2q} = K \int_0^t \mathrm{d}q \, \mathrm{e}^{\frac{(\tau^2-(\sigma-\sqrt{2})^2)q}{\sqrt{2}\sigma}}. \tag{2.14}$$

For $|\tau| + |\sigma| < \sqrt{2}$, the right-hand side of (2.14) is uniformly bounded by some constant C. Since $\mathcal{M}_{\sigma,\tau}(t)$ is bounded in L^p for some $p > 1$, the a.s. limit exists and the convergence also holds in L^1. Moreover, $\mathbb{E}[\mathcal{M}_{\sigma,\tau}(t)] = 1$ and hence the limit is non-degenerate. □

Next, we turn to proving the central limit theorem for $\sigma < 1/\sqrt{2}$.

Proof of Theorem 1.9. We start with the proof of (1.18). Let

$$a_k(r) := \mathrm{e}^{-r\left(1+\frac{\sigma^2}{2}-\tau^2\right)} \mathrm{e}^{\sigma x_k(r)+i\tau y_k(r)}. \tag{2.15}$$

Then, we can rewrite $\mathcal{M}_{\sigma,\tau}(t)$ as

$$\mathcal{M}_{\sigma,\tau}(t+r) = \sum_{k=1}^{n(r)} a_k(r) \mathcal{M}^{(k)}_{\sigma,\tau}(t), \tag{2.16}$$

where $\mathcal{M}^{(k)}_{\sigma,\tau}(t)$ are i.i.d. copies of $\mathcal{M}_{\sigma,\tau}(t)$. Hence, conditional on $\mathcal{F}_r$, $\mathcal{M}_{\sigma,\tau}(t)$ can be written as a sum of independent random variables. To prove a CLT, we want to use the two-dimensional Lindeberg-Feller condition (conditional on $\mathcal{F}_r$). First, we take the limit $t \uparrow \infty$. For $\sigma < 1/\sqrt{2}$ and $\beta \in B_1$, we have $\sigma^2 + \tau^2 < 1$. Then, by [19, Proposition A.1], $\mathcal{M}^{(k)}_{\sigma,\tau}(t)$ is L^2-bounded and

$$\lim_{t\uparrow\infty} \mathbb{E}\left[\left|\mathcal{M}^{(k)}_{\sigma,\tau}(t)\right|^2\right] = C_1 \leq \infty \tag{2.17}$$

Hence, the a.s. limit $\mathcal{M}_{\sigma,\tau}$ exists in L^2 and as $t \uparrow \infty$ the right-hand side of (2.16) converges a.s. to

$$\mathcal{M}_{\sigma,\tau} = \sum_{k=1}^{n(r)} a_k(r) \mathcal{M}^{(k)}_{\sigma,\tau}, \tag{2.18}$$

where $\mathcal{M}^{(k)}_{\sigma,\tau}$ are i.i.d. copies of $\mathcal{M}_{\sigma,\tau}$. To compute the variance of (2.18), consider

$$\sum_{k=1}^{n(r)} \mathbb{E}\left[\left|a_k(r)\mathcal{M}^{(k)}_{\sigma,\tau}\right|^2 \;\middle|\; \mathcal{F}_r\right]. \tag{2.19}$$

Eq. (2.19) is equal to

$$\sum_{k=1}^{n(r)} |a_k(r)|^2 \mathbb{E}\left[\left|\mathcal{M}^{(k)}_{\sigma,\tau}\right|^2\right] = C_1 \sum_{k=1}^{n(r)} |a_k(r)|^2, \tag{2.20}$$

by (2.17). Now,

$$C_1 \sum_{k=1}^{n(r)} |a_k(r)|^2 = C_1 \sum_{k=1}^{n(r)} \mathrm{e}^{2\sigma x_k(r)-2r\left(1+\frac{\sigma^2}{2}-\frac{\tau^2}{2}\right)} = C_1 \mathcal{M}_{2\sigma,0}(r) \mathrm{e}^{-r\left(1-(\sigma^2+\tau^2)\right)}. \tag{2.21}$$

Eq. (2.21) together with the extra rescaling in (1.18) yields

$$C_1 \mathrm{e}^{(1-\sigma^2-\tau^2)r} \sum_{k=1}^{n(r)} |a_k(r)|^2 = C_1 \mathcal{M}_{2\sigma,0}(r), \tag{2.22}$$

which converges to $C_1 \mathcal{M}_{2\sigma,0}$ a.s. as $r \uparrow \infty$.

It remains to check the Lindeberg-Feller condition. We set

$$b_k(r) := a_k(r) \mathrm{e}^{-(1-\sigma^2-\tau^2)r}. \tag{2.23}$$

Let $\varepsilon > 0$ and consider

$$\begin{aligned} \frac{1}{C_1 \mathcal{M}_{2\sigma,0}(r)} \sum_{i=1}^{n(r)} \mathbb{E}\Big[\Big| b_k(r) \left(\mathcal{M}_{\sigma,\tau}^{(k)} - 1 \right) \Big|^2 & \\ \times \mathbb{1}\Big\{ |b_k(r) \left(\mathcal{M}_{\sigma,\tau}^{(k)} - 1 \right) | > \varepsilon \sqrt{C_1 \mathcal{M}_{2\sigma,0}(r)} \Big\} \ \Big| \ \mathcal{F}_r \Big]. \end{aligned} \tag{2.24}$$

We rewrite (2.24) as

$$\begin{aligned} \frac{1}{C_1 \mathcal{M}_{2\sigma,0}(r)} \sum_{i=1}^{n(r)} b_k(r) \bar{b}_k(r) \mathbb{E}\Big[\Big| \left(\mathcal{M}_{\sigma,\tau}^{(k)} - 1 \right) \Big|^2 & \\ \times \mathbb{1}\{ | \left(\mathcal{M}_{\sigma,\tau}^{(k)} - 1 \right) |^2 > \varepsilon^2 |b_k(r)|^{-2} C_1 \mathcal{M}_{2\sigma,0}(r) \} \ \Big| \ \mathcal{F}_r \Big]. \end{aligned} \tag{2.25}$$

We consider the expectation in (2.25) for a fixed k, i.e.,

$$\mathbb{E}\left[\Big| \left(\mathcal{M}_{\sigma,\tau}^{(k)} - 1 \right) \Big|^2 \mathbb{1}\{ | \left(\mathcal{M}_{\sigma,\tau}^{(k)} - 1 \right) |^2 > \varepsilon^2 |b_k(r)|^{-2} C_1^2 \mathcal{M}_{2\sigma,0}(r) \} \ \Big| \ \mathcal{F}_r \right]. \tag{2.26}$$

Using again that by [19, Proposition A.1]

$$\mathbb{E}\left[\Big| \left(\mathcal{M}_{\sigma,\tau}^{(k)} - 1 \right) \Big|^2 \right] = C_1 < \infty, \tag{2.27}$$

we have that (2.26) converges to zero as $r \uparrow \infty$ if

$$|b_k(r)|^{-2} C_1 \mathcal{M}_{2\sigma,0}(r) \underset{r\uparrow\infty}{\longrightarrow} \infty. \tag{2.28}$$

Observe that $\mathcal{M}_{2\sigma,0}(r)$ is a L^2-bounded martingale with mean one, if $\sigma < 1/\sqrt{2}$. Hence, it converges a.s. and in L^1. Consider

$$|b_k(r)|^{-2} = \mathrm{e}^{-2\sigma x_k(r) + 2\left(\frac{1}{2}+\sigma^2\right)r}, \tag{2.29}$$

since $x_k(r) < \sqrt{2}r$ a.s. (by Lalley-Selke argument in [30]). On this event, we have

$$|b_k(r)|^{-2} \geq \mathrm{e}^{(-2\sqrt{2}\sigma + 1 + 2\sigma^2)r} = \mathrm{e}^{(1-\sqrt{2}\sigma)^2 r}, \tag{2.30}$$

which converges to infinity as $r \uparrow \infty$. Hence, (2.28) holds a.s. □

3 Proof of CLT for phase B_3

In this section, we deal with phase B_3 and prove Theorem 1.11.

3.1 Second moment computations

We start by controlling the second moment of

$$N_{\sigma,\tau}(t) := \frac{\mathcal{X}_{\beta,\rho}(t)}{\mathrm{e}^{t(1/2+\sigma^2)}} \tag{3.1}$$

in Phase B_3, and its appropriately rescaled version

$$\hat{N}_{\sigma,\tau}(t) := t^{-1/2} N_{\sigma,\tau}(t) \tag{3.2}$$

on the boundary $B_{1,3}$.

Lemma 3.1. *It holds:*

(i) For $\beta \in B_3$ or $\beta \in B_{2,3} \setminus \{(1+i)/\sqrt{2}\}$ and any $\rho \in [-1,1]$,

$$\lim_{t\uparrow\infty} \mathbb{E}\left[|N_{\sigma,\tau}(t)|^2\right] = C_2 < \infty, \tag{3.3}$$

for some positive constant $0 < C_2 < \infty$[5] *.*

(ii) For $\beta \in B_{1,3}$ or $\beta = \frac{1}{\sqrt{2}}(1+i)$ and any $\rho \in [-1,1]$,

$$\lim_{t\uparrow\infty} \mathbb{E}\left[|\hat{N}_{\sigma,\tau}(t)|^2\right] = C_3 < \infty, \tag{3.4}$$

for some positive constant $0 < C_3 < \infty$.

Proof. **(i)** We have

$$\mathbb{E}\Big[|N_{\sigma,\tau}(t)|^2\Big] = \mathrm{e}^{-2t\left(1/2+\sigma^2\right)} \mathbb{E}\Big[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\sigma(x_k(t)+x_l(t))+i\tau(y_k(t)-y_l(t))}\Big]. \tag{3.5}$$

Using Representation (1.7), we rewrite the right-hand side of (3.5) as

$$\mathrm{e}^{-2t\left(1/2+\sigma^2\right)} \mathbb{E}\Big[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\bar{\lambda} x_l(t)+\lambda x_k(t)+i\tau\sqrt{1-\rho^2}(z_k(t)-z_l(t))}\Big], \tag{3.6}$$

where $\lambda = \sigma + i\rho\tau$ and $(z_k(t))_{k\le n(t)}$ are the particles of a BBM on $\mathbb{T}_t$ that is independent from $X(t)$. By conditioning on $\mathcal{F}^{\mathbb{T}_t}$, we have that (3.6) is equal to

$$\mathrm{e}^{-2t\left(1/2+\sigma^2\right)} \mathbb{E}\Big[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\bar{\lambda} x_l(t)+\lambda x_k(t)} \mathrm{e}^{-(1-\rho^2)\tau^2(t-d(x_k(t),x_l(t)))}\Big]. \tag{3.7}$$

The expectation in (3.7) is equal to

$$\begin{aligned} &K \int_0^t \mathrm{d}q\, \mathrm{e}^{2t-q-(1-\rho^2)\tau^2(t-q)} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{\sqrt{2\pi(t-q)}} \\ &\quad \times \int_{-\infty}^{\infty} \frac{\mathrm{d}y'}{\sqrt{2\pi(t-q)}} \mathrm{e}^{2\sigma x+\sigma(y+y')+i\tau\rho(y-y')} \mathrm{e}^{-\frac{y^2+y'^2}{2(t-q)}} \mathrm{e}^{-x^2/2q}. \end{aligned} \tag{3.8}$$

Computing first the integrals with respect to y and y', we get that (3.8) is equal to

$$K \int_0^t \mathrm{d}q\, \mathrm{e}^{2t-q-(1-\rho^2)\tau^2(t-q)+(\sigma^2-\rho^2\tau^2)(t-q)} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi q}} \mathrm{e}^{2\sigma x} \mathrm{e}^{-x^2/2q}$$

[5] C_2 depends on σ and τ but not on ρ. We do not make this dependence explicit in our notation.

$$= K \int_0^t \mathrm{d}q \; \mathrm{e}^{2t-q-\tau^2(t-q)+\sigma^2(t-q)} \mathrm{e}^{2\sigma^2 q}. \tag{3.9}$$

Plugging (3.9) back into (3.7), we get that (3.7) is equal to

$$\begin{aligned} &\mathrm{e}^{-2t\left(1/2+\sigma^2\right)} K \int_0^t \mathrm{d}q \; \mathrm{e}^{2t-q-\tau^2(t-q)+\sigma^2(t-q)} \mathrm{e}^{2\sigma^2 q} \\ &= K \int_0^t \mathrm{d}q \; \mathrm{e}^{(t-q)(1-\tau^2-\sigma^2)} = K \int_0^t \mathrm{d}q' \; \mathrm{e}^{q'(1-\tau^2-\sigma^2)} \\ &= \frac{K}{1-\tau^2-\sigma^2} \left(\mathrm{e}^{t(1-\tau^2-\sigma^2)} - 1 \right). \end{aligned} \tag{3.10}$$

As $t \uparrow \infty$, the term in (3.10) converges to $\frac{K}{\tau^2+\sigma^2-1}$, which we call C_2 from now on.
(ii) Proceeding as in Part (i), we get that

$$\mathbb{E}\left[|\hat{N}_{\sigma,\tau}(t)|^2 \right] = t^{-1} \mathrm{e}^{-2t\left(1/2+\sigma^2\right)} \mathbb{E}\Big[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\bar{\lambda} x_l(t) + \lambda x_k(t)} \mathrm{e}^{-(1-\rho^2)\tau^2(t-d(x_k(t),x_l(t)))} \Big]. \tag{3.11}$$

Plugging (3.9) into (3.11), we get that (3.11) is equal to

$$K t^{-1} \int_0^t \mathrm{d}q \; \mathrm{e}^{(t-q)(1-\tau^2-\sigma^2)} = K t^{-1} \int_0^t \mathrm{d}q = K, \tag{3.12}$$

since $\sigma^2 + \tau^2 = 1$ in $B_{1,3}$.

□

3.2 Constrained moments computation in B_3

In this section, we continue our preparations for the proof of Theorem 1.11. These consist of computing constrained moments. The following two Lemmata ensure that we can introduce the desired constraint.

Lemma 3.2. *Let $\beta \in B_3$. Then for all $\varepsilon > 0$ and $\delta > 0$, uniformly for all t large enough, there exists A_0 such that for all $A > A_0$*

$$\mathbb{P}\Big\{ \Big| \sum_{k=1}^{n(t)} \mathrm{e}^{\sigma x_k(t) + i\tau y_k(t) - \left(\frac{1}{2}+\sigma^2\right)t} \mathbb{1}\{x_k(t) > 2\sigma t + A\sqrt{t}\} \Big| > \delta \Big\} < \varepsilon. \tag{3.13}$$

Proof. Using a second moment Chebyshev inequality, we bound the probability in (3.13) from above by

$$\mathrm{e}^{-2t\left(1/2+\sigma^2\right)} \mathbb{E}\Big[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\sigma(x_k(t)+x_l(t)) + i\tau(y_k(t)-y_l(t))} \mathbb{1}\{x_k(t), x_l(t) > 2\sigma t + A\sqrt{t}\} \Big]. \tag{3.14}$$

Continuing as in the proof of Lemma 3.1, we rewrite (3.14) as

$$\mathrm{e}^{-2t\left(1/2+\sigma^2\right)} \mathbb{E}\Big[\mathrm{e}^{-(1-\rho^2)\tau^2(t-d(x_k(t),x_l(t)))} \sum_{k,l=1}^{n(t)} \mathrm{e}^{\bar{\lambda} x_l(t) + \lambda x_k(t)} \mathbb{1}\{x_k(t), x_l(t) > 2\sigma t + A\sqrt{t}\} \Big]. \tag{3.15}$$

We rewrite the expectation in (3.15) as

$$K \int_0^t \mathrm{d}q \; \mathrm{e}^{2t-q-(1-\rho^2)\tau^2(t-q)} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi q}} \int_{2\sigma t + A\sqrt{t} - x}^{\infty} \frac{\mathrm{d}y}{\sqrt{2\pi(t-q)}}$$

$$\times \int_{2\sigma t+A\sqrt{t}-x}^{\infty} \frac{\mathrm{d}y'}{\sqrt{2\pi(t-q)}} \mathrm{e}^{2\sigma x+A\sqrt{t}+\sigma(y+y')+i\tau\rho(y-y')} \mathrm{e}^{-\frac{y^2+y'^2}{2(t-q)}} \mathrm{e}^{-x^2/2q}. \quad (3.16)$$

Let $0<r<t$. By splitting the domain of integration with respect to q into two parts $\{0 \le q \le t-r\} \cup \{t-r<q\le t\}$ we write (3.16) as $(\mathrm{I})+(\mathrm{II})$. We can upper bound (I) by (3.8).

$$K \int_0^{t-r} \mathrm{d}q\, \mathrm{e}^{2t-q-(1-\rho^2)\tau^2(t-q)} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi q}} \int_{2\sigma t+A\sqrt{t}-x}^{\infty} \frac{\mathrm{d}y}{\sqrt{2\pi(t-q)}}$$
$$\times \int_{2\sigma t+A\sqrt{t}-x}^{\infty} \frac{\mathrm{d}y'}{\sqrt{2\pi(t-q)}} \mathrm{e}^{2\sigma x+\sigma(y+y')+i\tau\rho(y-y')} \mathrm{e}^{-\frac{y^2+y'^2}{2(t-q)}} \mathrm{e}^{-x^2/2q}. \quad (3.17)$$

Observe that apart from the different domain of integration (with respect to q) (3.17) coincides with (3.8). Performing the same manipulations as after (3.8) in the proof of Lemma 3.1 we can upper bound (3.17) by

$$\mathrm{e}^{-2t\left(1/2+\sigma^2\right)} K \int_0^{t-r} \mathrm{d}q\, \mathrm{e}^{2t-q-\tau^2(t-q)+\sigma^2(t-q)} \mathrm{e}^{2\sigma^2 q}, \quad (3.18)$$

which can be made smaller then $\epsilon/2$ by choosing r sufficiently large. Hence, it remains to consider Term (II), respectively the integration domain $q \in [t-r,t]$ in (3.16). For this part the constrained $x+y>2\sigma t+A\sqrt{t}$ plays an important role. First, observe that for $q \ge t-r$, $\mathbb{P}(y>r)<\mathrm{e}^{-r/2}$. To have $x+y>2\sigma t+A\sqrt{t}$ on the event $\{y<r\}$, it must hold that $x>2\sigma t+A\sqrt{t}-r$. Hence, we can upper bound (II) by

$$e^{-r/2}+K \int_{t-r}^{t} \mathrm{d}q\, \mathrm{e}^{2t-q-(1-\rho^2)\tau^2(t-q)} \int_{-2\sigma t+A\sqrt{t}-r}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi q}} \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{\sqrt{2\pi(t-q)}}$$
$$\times \int_{-\infty}^{-\infty} \frac{\mathrm{d}y'}{\sqrt{2\pi(t-q)}} \mathrm{e}^{2\sigma x+A\sqrt{t}+\sigma(y+y')+i\tau\rho(y-y')} \mathrm{e}^{-\frac{y^2+y'^2}{2(t-q)}} \mathrm{e}^{-x^2/2q}. \quad (3.19)$$

Computing the integral with respect to y and y' in (3.19) we can upper boung (3.19) by

$$e^{-r/2}+K \int_{t-r}^{t} \mathrm{d}q\, \mathrm{e}^{2t-q-(1-\rho^2)\tau^2(t-q)+(\sigma^2-\rho^2\tau^2)(t-q)} \int_{-2\sigma t+A\sqrt{t}-r}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi q}} \mathrm{e}^{2\sigma x} \mathrm{e}^{-x^2/2q}. \quad (3.20)$$

Using the Gaussian tail asymptotics to upper bound the integral with respect to x we see that (3.20) can be made smaller than $\varepsilon/2$ by choosing r and A sufficiently large. ☐

Lemma 3.3. *Let $\beta \in B_3$, $\rho \in [-1,1]$ and $\gamma > \frac{1}{2}$. Let $A>0$. Then, for all $\varepsilon>0$ and $d>0$, there exists $r_0>0$ such that, for all $r>r_0$, uniformly for all t sufficiently large,*

$$\mathbb{P}\Big\{\Big|\sum_{k=1}^{n(t)} \mathrm{e}^{\sigma x_k(t)+i\tau y_k(t)-\left(\frac{1}{2}+\sigma^2\right)t}$$
$$\times \mathbb{1}\{x_k(t)<2\sigma t+A\sqrt{t}, \exists s \in [r,t]\colon x_k(s)>2\sigma s+s^\gamma\}\Big|>\delta\Big\}<\varepsilon. \quad (3.21)$$

Proof. We use again a second moment bound. Similarly to the proof of Lemma 3.2, we bound the probability in (3.21) from above by

$$\mathrm{e}^{-2t\left(1/2+\sigma^2\right)} \mathbb{E}\Big[\sum_{k,l=1}^{n(t)} \mathrm{e}^{-(1-\rho^2)\tau^2(t-d(x_k(t),x_l(t)))} \mathrm{e}^{\bar{\lambda}x_l(t)+\lambda x_k(t)}$$
$$\times \mathbb{1}\{x_k(t), x_l(t)<2\sigma t+A\sqrt{t}, \exists s \in [r,t]\colon x_k(s)>2\sigma s+s^\gamma,$$

$$\exists s' \in [r,t]\colon x_l(s') > 2\sigma s' + (s')^\gamma\}\Big]. \quad (3.22)$$

By only keeping track of the path event for one of the particles, we get that (3.22) is bounded from above by

$$\mathrm{e}^{-2t\left(1/2+\sigma^2\right)}\mathbb{E}\Big[\sum_{k,l=1}^{n(t)} \mathrm{e}^{-(1-\rho^2)\tau^2(t-d(x_k(t),x_l(t)))}\mathrm{e}^{\bar{\lambda}x_l(t)+\lambda x_k(t)}$$
$$\times \mathbb{1}\{x_k(t), x_l(t) < 2\sigma t + A\sqrt{t}\ \exists s \in [r,t]\colon x_k(s) > 2\sigma s + s^\gamma\}\Big]. \quad (3.23)$$

We rewrite (3.23) as

$$K\int_0^t \mathrm{d}q\ \mathrm{e}^{2t-q}\mathrm{e}^{2t-q-(1-\rho^2)\tau^2(t-q)}\mathbb{E}\left[\mathrm{e}^{\bar{\lambda}x_1(t)+\lambda(x_1(q)+x_2(t-q))}\right.$$
$$\left.\times\mathbb{1}\{x_1(t), x_1(q)+x_2(t-q) < 2\sigma t + A\sqrt{t}, \exists s \in [r,t]\colon x_1(s) > 2\sigma s + s^\gamma\}\right], \quad (3.24)$$

where $x_1(\cdot)$ is a standard Brownian motion and $x_2(t-q)$ is an independent $\mathcal{N}(0,t-q)$ distributed random variable. As we are looking for an upper bound we can drop the truncation with respect to $x_1(q)+x_2(t-q)$ and then calculate of the expectation in (3.24) with respect to $x_2(t-q)$. Hence, (3.24) is upper bounded by

$$K\int_0^t \mathrm{d}q\ \mathrm{e}^{2t-q}\mathrm{e}^{-\frac{\lambda^2}{2(t-q)}}\mathrm{e}^{2t-q-(1-\rho^2)\tau^2(t-q)}\mathbb{E}\left[\mathrm{e}^{\bar{\lambda}x_1(t)+\lambda x_1(q)}\right.$$
$$\left.\times\ \mathbb{1}\{x_1(t) < 2\sigma t + A\sqrt{t}, \exists s \in [r,t]\colon x_1(s) > 2\sigma s + s^\gamma\}\right]. \quad (3.25)$$

As in the proof of Lemma 3.2, we can first choose r_1 large enough such that the above integral from 0 to $t-r_1$ is bounded by $\varepsilon/3$. Moreover, $x_1(t) = x_1(q) + \tilde{x}(t-q)$, where $\tilde{x}(t-q)$ is normal distributed with mean zero and variance $t-q$ that is independent from $x_1(s)$ for $s \le q$. Then, for all $R > R_2$,

$$\mathbb{P}\{|\tilde{x}(t-q)| > R\} < \frac{\varepsilon}{3}. \quad (3.26)$$

Observe that the intersection of the event $\{\tilde{x}(t-q) > R\}$ and the event in the indicator in (3.25) is contained in the event

$$\{x_1(t) < 2\sigma t + A\sqrt{t}, \exists s \in [r,t]\colon x_1(s) > 2\sigma s + s^\gamma, \tilde{x}(t-q) > R\}$$
$$\subset \left\{\exists s\colon x_1(s) - \frac{s}{q}x_1(q) < s^\gamma - \frac{(A\sqrt{t}-R)s}{q}\right\}. \quad (3.27)$$

Using that $x_1(s) - \frac{s}{q}x_1(q) = \xi(s)$ is a Brownian bridge that is independent from $x_1(q)$ and also from $\tilde{x}(t-q)$, we bound (3.25) from above by

$$K\int_0^t \mathrm{d}q\ \mathrm{e}^{2t-q}\mathrm{e}^{-\frac{\lambda^2}{2(t-q)}}\mathrm{e}^{2t-q-(1-\rho^2)\tau^2(t-q)}\mathbb{E}\left[\mathrm{e}^{\bar{\lambda}x_1(t)+\lambda x_1(q)}\right]$$
$$\times\ \mathbb{P}\left\{\exists s \in [r,t-r]\colon \xi(s) > s^\gamma - \frac{(A\sqrt{t}-R)s}{q}\right\}. \quad (3.28)$$

By the same computations as in (3.8) and (3.9), we can bound (3.28) from above by

$$C_2\mathbb{P}\left\{\exists s \in [r,t-R-r]\colon \xi(s) > s^\gamma - \frac{(A\sqrt{t}-R)s}{t-R}\right\}. \quad (3.29)$$

It is a well known fact for Brownian bridges (see, e.g., [9, Lemma 2.3] for a precise statement) that by choosing r sufficiently large (3.29) can be made as small as we want. This finishes the proof of Lemma 3.3. □

Define

$$N^{c,A}_{\sigma,\tau}(t) := \sum_{k=1}^{n(t)} \mathrm{e}^{-t(1/2+\sigma^2)} \mathrm{e}^{\sigma x_k(t)+i\tau y_k(t)} \\ \times \mathbb{1}\{x_k(t) < 2\sigma t + A\sqrt{t}, \forall s \in [r,t] \colon x_k(s) \le 2\sigma s + s^\gamma\}. \quad (3.30)$$

The following lemma provides the asymptotics for all moments of (3.30) in the $t \to \infty$ limit.

Lemma 3.4 (Moment asymptotics)**.** *Consider a branching Brownian motion with binary splitting. For $\beta \in B_3$, for any $A > 0$*

$$\lim_{t\to\infty} \mathbb{E}\left[\left|N^{c,A}_{\sigma,\tau}(t)\right|^2\right] = C_{2,A}, \quad (3.31)$$

with $\lim_{A\to\infty} C_{2,A} = C_2$ and, for $k \in \mathbb{N}$, we have

$$\lim_{r\uparrow\infty} \lim_{t\to\infty} \mathbb{E}\left[\left|N^{c,A}_{\sigma,\tau}(t)\right|^{2k} \mid \mathcal{F}_r\right] = k!(C_{2,A}\mathcal{M}_{2\sigma,0})^k \quad \text{a.s. and in } L^1. \quad (3.32)$$

Moreover, for $k' < k$,

$$\lim_{r\uparrow\infty} \lim_{t\to\infty} \mathbb{E}\left[N^{c,A}_{\sigma,\tau}(t)^k \overline{N^{c,A}_{\sigma,\tau}(t)}^{k'} \mid \mathcal{F}_r\right] = 0 \quad \text{a.s. and in } L^1. \quad (3.33)$$

Proof. We proceed by induction over $k \in \mathbb{N}$. For $k = 1$, we observe that

$$1 = \mathbb{1}\{x_k(t) > 2\sigma t + A\sqrt{t}\} + \mathbb{1}\{x_k(t) < 2\sigma t + A\sqrt{t}, \exists s \in [r,t] \colon x_k(s) \le 2\sigma s + s^\gamma\} \\ + \mathbb{1}\{x_k(t) < 2\sigma t + A\sqrt{t}, \forall s \in [r,t] \colon x_k(s) \le 2\sigma s + s^\gamma\}. \quad (3.34)$$

Plugging this decomposition of unity into (3.5) we can bound $\mathbb{E}\left[\left|N_{\sigma,\tau}(t)\right|^2\right]$ by

$$\begin{aligned} &\mathrm{e}^{-2t\left(1/2+\sigma^2\right)} \mathbb{E}\Big[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\sigma(x_k(t)+x_l(t))+i\tau(y_k(t)-y_l(t))} \Big(\mathbb{1}_{\{x_k(t),x_l(t)>2\sigma t+A\sqrt{t}\}} \\ &\quad + 2\mathbb{1}\{x_k(t) > 2\sigma t + A\sqrt{t}, x_l(t) < 2\sigma t + A\sqrt{t}\} \\ &\quad + \mathbb{1}\{x_k(t), x_l(t) < 2\sigma t + A\sqrt{t}, \exists s \in [r,t] \colon x_k(s) \le 2\sigma s + s^\gamma\} \\ &\quad + 2\mathbb{1}\{x_k(t) < 2\sigma t + A\sqrt{t}, \exists s \in [r,t] \colon x_k(s) \le 2\sigma s + s^\gamma\} \\ &\qquad \times \mathbb{1}\{x_l(t) < 2\sigma t + A\sqrt{t}, \forall s \in [r,t] \colon x_l(s) \le 2\sigma s + s^\gamma\}\Big)\Big] \\ &\quad + \mathbb{E}\left[\left|N^{c,A}_{\sigma,\tau}(t)\right|^2\right] \\ &\quad =: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) + (\mathrm{V}). \end{aligned} \quad (3.35)$$

Note that Terms (I) and (III) can be made arbitrarily small by increasing A, resp. r by computations as in the proofs of Lemmas 3.2 and 3.3, respectively.

Term (II) can be treated as (I) as for the bounds in (3.17) and below it suffices that one of the two particle positions is $> \sqrt{2}\sigma t + A\sqrt{t}$.

To control Term (IV), we upper bound it by

$$\mathrm{e}^{-2t\left(1/2+\sigma^2\right)} \mathbb{E}\Big[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\sigma(x_k(t)+x_l(t))+i\tau(y_k(t)-y_l(t))} 2\mathbb{1}\{x_k(t) < 2\sigma t + A\sqrt{t}, \exists s \in [r,t] \colon \\ x_k(s) \le 2\sigma s + s^\gamma, x_l(t) < 2\sigma t + A\sqrt{t}\}\Big]. \quad (3.36)$$

Observe that (3.36) coincides with (3.23). Following the argument after (3.23), we see that (3.23) can be made arbitrarily small by increasing r.

Combining the bounds on the Terms (I), (II), (III) and (IV), the claim follows from Lemma 3.1.

To bound the $2k$-moment, we rewrite (3.32) as

$$\frac{1}{2}\mathbb{E}\Big[\sum_{l_1,\ldots,l_{2k}\le n(t)}\left(\prod_{j=1}^{2k}\mathrm{e}^{-t(1/2+\sigma^2)}\mathrm{e}^{\sigma x_{l_j}(t)+i\tau y_{l_j}(t)}+\prod_{j=1}^{2k}\mathrm{e}^{-t(1/2+\sigma^2)}\mathrm{e}^{\sigma x_{l_j}(t)-i\tau y_{l_j}(t)}\right)\\ \times\mathbb{1}\{x_{l_j}(t)<2\sigma t+A\sqrt{t},\forall s\in[r,t]\colon x_{l_j}(s)\le 2\sigma s+s^\gamma\}\Big], \tag{3.37}$$

by grouping each summand together with its complex conjugate. For $l_1,\ldots,l_{2k}\le n(t)$, we can find a matching using the following algorithm:

1. Choose the two labels j,j' such that $d(x_{l_j},x_{l_{j'}})$ is maximal. Call them l_1 and $l_{\sigma(1)}$ from know on.
2. Delete them.
3. Pick l_j in the remaining set and match it with the remaining $l_{j'}$ such that $d(x_{l_j},x_{l_{j'}})$ is maximal. Iterate.

We refer to the above algorithm as *'optimal matching'*. The pairs obtained in this way we denote by $(l_1,l_{\sigma(1)}),\ldots,(l_k,l_{\sigma(k)})$. We rewrite (3.37) as

$$\frac{1}{2}\mathbb{E}\Big[\sum_{l_2,\ldots,l_k\le n(t)}\Big(\prod_{j=2}^{k}\mathrm{e}^{-t(1+2\sigma^2)}\mathrm{e}^{\sigma\left(x_{l_j}(t)+x_{l_{\sigma(j)}}(t)\right)+i\tau\left(y_{l_j}(t)+y_{l_{\sigma(j)}}(t)\right)}\\ \times\mathrm{e}^{-t(1+2\sigma^2)}\mathrm{e}^{\sigma(x_{l_1}(t)+x_{l_{\sigma(1)}}(t))+i\tau(y_{l_1}(t)-y_{l_{\sigma(1)}}(t))}+\\ \prod_{j=2}^{k}\mathrm{e}^{-t(1+2\sigma^2)}\mathrm{e}^{\sigma\left(x_{l_j}(t)+x_{l_{\sigma(j)}}(t)\right)-i\tau\left(y_{l_j}(t)+y_{l_{\sigma(j)}}(t)\right)}\\ \times\mathrm{e}^{-t(1+2\sigma^2)}\mathrm{e}^{\sigma(x_{l_1}(t)+x_{l_{\sigma(1)}}(t))-i\tau(y_{l_1}(t)-y_{l_{\sigma(1)}}(t))}\Big)\\ \times\mathbb{1}\{x_{l_{\sigma(j)}}(t),x_{l_j}(t)<2\sigma t+A\sqrt{t},\forall s\in[r,t]\colon x_{l_{\sigma(j)}}(s),x_{l_j}(s)\le 2\sigma s+s^\gamma\}\\ \times\mathbb{1}\{x_{l_{\sigma(1)}}(t),x_{l_1}(t)<2\sigma t+A\sqrt{t},\forall s\in[r,t]\colon x_{l_{\sigma(1)}}(s),x_{l_1}(s)\le 2\sigma s+s^\gamma\}\Big]. \tag{3.38}$$

Using (1.7), we can rewrite for $j\in\{1,\sigma(1)\}$

$$y_{l_j}(t)=\rho y_{l_j}(t)+\sqrt{1-\rho^2}z_{l_j}(t), \tag{3.39}$$

where $(z_k(t))_{k\le n(t)}$ are particles of a BBM on the same Galton-Watson tree as $(x_k(t))_{k\le n(t)}$ but independent from it. Observe that using the requirement that $d(x_{l_1},x_{l_{\sigma_1}})$ is chosen maximal, we have

$$i\tau(y_{l_1}(t)-y_{l_{\sigma(1)}}(t))=i\sqrt{1-\rho^2}\tau\left(z_1(t-d(x_{l_1}(t),x_{l_{\sigma_1}}(t)))-z_2(t-d(x_{l_1}(t),x_{l_{\sigma_1}}(t))\right)\\ +i\tau\rho\left(x_{l_1}(t)-x_{l_{\sigma(1)}}(t)\right), \tag{3.40}$$

where z_1,z_2 are two independent $\mathcal{N}(0,(t-d(x_{l_1}(t),x_{l_{\sigma_1}}(t))))$-distributed random variables. Plugging (3.40) into (3.38) and computing the expectation with respect to z_1,z_2 of the first summand (noting that the second is just its complex conjugate), we obtain

$$\begin{aligned}\frac{1}{2}\mathbb{E}\Big[&\sum_{l_2,\ldots,l_k\le n(t)}\prod_{j=2}^{k}\mathrm{e}^{-t(1+2\sigma^2)}\exp\big(\sigma(x_{l_j}(t)+x_{l_{\sigma(j)}}(t))+i\tau(y_{l_j}(t)+y_{l_{\sigma(j)}}(t))\big)\\&\times\mathbb{1}\{x_{l_{\sigma(j)}}(t),x_{l_j}(t)<2\sigma t+A\sqrt{t},\forall s\in[r,t]\colon x_{l_{\sigma(j)}}(s),x_{l_j}(s)\le 2\sigma s+s^\gamma\}\\&\times\mathrm{e}^{-t(1+2\sigma^2)-\tau^2(1-\rho^2)\left(t-d(x_{l_1},x_{l_{\sigma(1)}})\right)}\mathrm{e}^{(\sigma+i\tau\rho)x_{l_1}(t)+(\sigma-i\tau\rho)x_{l_{\sigma(1)}}}\\&\times\mathbb{1}\{x_{l_{\sigma(1)}}(t),x_{l_1}(t)<2\sigma t+A\sqrt{t},\forall s\in[r,t]\colon x_{l_{\sigma(1)}}(s),x_{l_1}(s)\le 2\sigma s+s^\gamma\}\Big].\end{aligned}\tag{3.41}$$

We decompose

$$\begin{aligned}x_{l_{\sigma(1)}}(t)&=x_{l_1}d(x_{l_1},x_{l_{\sigma(1)}})+x^{(1)}(t-d(x_{l_1},x_{l_{\sigma(1)}}));\\x_{l_1}(t)&=x_{l_1}d(x_{l_1},x_{l_{\sigma(1)}})+x^{(2)}(t-d(x_{l_1},x_{l_{\sigma(1)}})),\end{aligned}\tag{3.42}$$

where $x^{(1)},x^{(2)}$ are two independent $\mathcal{N}(0,t-d(x_{l_1},x_{l_{\sigma(1)}}))$-distributed random variables. By Step one of our matching procedure, we can plug (3.41) into (3.42) and compute the expectation with respect to $x^{(1)}$ and $x^{(2)}$, we obtain that (3.41) is bounded from above by[6]

$$\begin{aligned}\frac{1}{2}\mathbb{E}\Big[&\sum_{l_2,\ldots,l_k\le n(t)}\prod_{j=2}^{k}\mathrm{e}^{-t(1+2\sigma^2)}\mathrm{e}^{\sigma\left(x_{l_j}(t)+x_{l_{\sigma(j)}}(t)\right)+i\tau\left(y_{l_j}(t)+y_{l_{\sigma(j)}}(t)\right)}\\&\times\mathbb{1}\{x_{l_{\sigma(j)}}(t),x_{l_j}(t)<2\sigma t+A\sqrt{t},\forall s\in[r,t]\colon x_{l_{\sigma(j)}}(s),x_{l_j}(s)\le 2\sigma s+s^\gamma\}\\&\times\mathrm{e}^{-t(1+2\sigma^2)-\tau^2\left(t-d(x_{l_1},x_{l_{\sigma(1)}})\right)+\sigma^2\left(t-d(x_{l_1},x_{l_{\sigma(1)}})\right)}\mathrm{e}^{2\sigma x_{l_1}d(x_{l_1},x_{l_{\sigma(1)}})}\\&\times\mathbb{1}\{\forall s\in[r,d(x_{l_1},x_{l_{\sigma(1)}})]\colon x_{l_{\sigma(1)}}(s),x_{l_1}(s)\le 2\sigma s+s^\gamma\}\Big].\end{aligned}\tag{3.43}$$

We now introduce the event

$$\mathcal{A}_r=\Big\{\exists s\in[r,d(x_{l_1},x_{l_{\sigma(1)}})],\exists j\in\{2,\ldots,k,\sigma(2),\ldots,\sigma(k)\}\colon d(x_{l_1},x_{l_j})=s\Big\}.\tag{3.44}$$

We can rewrite (3.43) as

$$\mathbb{E}\left[\ldots\times\mathbb{1}_{\mathcal{A}_r}\right]+\mathbb{E}\left[\ldots\times\mathbb{1}_{\mathcal{A}_r^c}\right]=:J_{\mathcal{A}_r}+J_{\mathcal{A}_r^c}.\tag{3.45}$$

We will prove that the first summand is of a smaller order than the second one. We need the following lemma.

Lemma 3.5. *Let x,y be $\mathcal{N}(0,q)$ distributed random variables. Then, for any $m_1,m_2\ge 1$ and constant $C>0$,*

$$\begin{aligned}&\mathbb{E}\left[\left(\mathrm{e}^{(m_1+2)\sigma x+i\tau m_2 x}+\mathrm{e}^{(m_1+2)\sigma x-i\tau m_2 x}\right)\mathbb{1}\{x<2\sigma q+Cq^\gamma\}\right]\\&\underset{q\to\infty}{=}o\Big(\mathrm{e}^{2\sigma q}\mathbb{E}\left[\left(\mathrm{e}^{m_1\sigma x+i\tau m_2 x}+\mathrm{e}^{m_1\sigma x-i\tau m_2 x}\right)\mathbb{1}\{x<2\sigma q+Cq^\gamma\}\right]\\&\qquad\times\mathbb{E}\left[\mathrm{e}^{2\sigma y}\mathbb{1}\{y<2\sigma q+Cq^\gamma\}\right]\Big),\end{aligned}\tag{3.46}$$

and similarly

$$\begin{aligned}&\mathbb{E}\left[\left(\mathrm{e}^{(m_1+1)\sigma x+i\tau(m_2+1)x}+\mathrm{e}^{(m_1+1)\sigma x-i\tau(m_2+1)x}\right)\mathbb{1}\{x<2\sigma q+Cq^\gamma\}\right]\\&\underset{q\to\infty}{=}o\Big(\mathrm{e}^{2\sigma q}\mathbb{E}\left[\left(\mathrm{e}^{m_1\sigma x+i\tau m_2 x}+\mathrm{e}^{m_1\sigma x-i\tau m_2 x}\right)\mathbb{1}\{x<2\sigma q+Cq^\gamma\}\right]\\&\qquad\times\mathbb{E}\left[\left(\mathrm{e}^{(\sigma+i\tau)y}+\mathrm{e}^{(\sigma-i\tau)y}\right)\mathbb{1}\{y<2\sigma q+Cq^\gamma\}\right]\Big).\end{aligned}\tag{3.47}$$

[6] A corresponding lower bound also holds due to the second moment computation in Lemma 3.4.

Proof. The l.h.s. in (3.46) is equal to

$$\int_{-\infty}^{2\sigma q+Cq^\gamma} \frac{\mathrm{d}x}{\sqrt{2\pi q}}\, \mathrm{e}^{(m_1+2)\sigma x+i\tau m_2 x}\mathrm{e}^{-\frac{x^2}{2q}} + \int_{-\infty}^{2\sigma q+Cq^\gamma} \frac{\mathrm{d}x}{\sqrt{2\pi q}}\, \mathrm{e}^{(m_1+2)\sigma x-i\tau m_2 x}\mathrm{e}^{-\frac{x^2}{2q}}. \quad (3.48)$$

Making a change of variable $y=(m_1+2)\sigma q+i\tau m_2 q+x$ in the first summand in (3.48) and $y=(m_1+2)\sigma q-i\tau m_2 q+x$ in the second summand in (3.48), we obtain that (3.48) equals to

$$\mathrm{e}^{((m_1+2)\sigma+i\tau m_2)^2 q/2}\int_{-\infty}^{-m_1\sigma q-i\tau m_2 q+Cq^\gamma} \frac{\mathrm{d}y}{\sqrt{2\pi q}}\, \mathrm{e}^{-y^2/2q}$$
$$+\mathrm{e}^{((m_1+2)\sigma-i\tau m_2)^2 q/2}\int_{-\infty}^{-m_1\sigma q+i\tau m_2 q+Cq^\gamma} \frac{\mathrm{d}y}{\sqrt{2\pi q}}\, \mathrm{e}^{-y^2/2q}. \quad (3.49)$$

For $m_1\geq 1$, by the Gaussian tail asymptotics (as given in [24, Lemma 3.5]), (3.49) is bounded from above by

$$C\mathrm{e}^{2m_1\sigma^2 q+2\sigma q+m_2^2\tau^2 q}\mathrm{e}^{m_1\sigma Cq^\gamma}, \quad (3.50)$$

for some positive constant C. The expectation on the right hand side of (3.46) is equal to

$$\int_{-\infty}^{2\sigma q+Cq^\gamma} \frac{\mathrm{d}x}{\sqrt{2\pi q}}\, \mathrm{e}^{m_1\sigma x+i\tau m_2 x}\mathrm{e}^{-\frac{x^2}{2q}} \int_{-\infty}^{2\sigma q+Cq^\gamma} \frac{\mathrm{d}y}{\sqrt{2\pi q}}\, \mathrm{e}^{2\sigma y}\mathrm{e}^{-\frac{y^2}{2q}}. \quad (3.51)$$

If $m_1>2$, (3.51) is by [24, Lemma 3.5] asymptotically equal to

$$\frac{1}{\sqrt{2\pi(m_1-2)q}}\mathrm{e}^{2m_1\sigma^2 q-2\sigma^2 q+m_2^2\tau^2 q}\mathrm{e}^{2\sigma^2 q}\mathrm{e}^{(m_1-2)\sigma Cq^\gamma}. \quad (3.52)$$

Comparing (3.52) with (3.50) yields the claim of Lemma (3.46). For $m_1=1$ or $m_2=1$, we bound the integral in (3.51) by $\mathrm{e}^{(m_1\sigma+i\tau m_2)^2 q/2+2\sigma^2 q/2}+\mathrm{e}^{(m_1\sigma-i\tau m_2)^2 q/2+2\sigma^2 q/2}$.

The proof of (3.47) follows along the same lines. □

We continue the proof of Lemma 3.4. Consider $J_{\mathcal{A}_r}$. Consider the skeleton generated by the leaves $l_1,l_{\sigma(1)},\ldots,l_k,l_{\sigma(k)}$ of the Galton-Watson tree. By path$(\cdot)$ we denote the unique path (= sequence of edges) leading from the given leaf "$\cdot$" to the root of the tree. To each edge in the Galton-Watson tree, we associate the following number

$$m(e):=\sum_{j\in\{1,\sigma(1),\ldots,k,\sigma(k)\}}\mathbb{1}_{e\subset\text{path}(l_j)}. \quad (3.53)$$

For $k,j\in[n(t)]$, define (cf. Fig. 2)

$$\mathrm{length}(x_k(t),x_j(t)):=d(x_1(t),x_k(t))-d(x_1(t),x_j(t)),\quad t\in\mathbb{R}_+. \quad (3.54)$$

Lemma 3.6. *Consider the path of $x_{l_1}(t)$. There exists l_{j*} which satisfies the following conditions*

(i) m *is constant between* $d(x_1(t),x_{j^*-1}(t))$ *and* $d(x_1(t),x_{j^*}(t))$ *and, moreover,*

$$\mathrm{length}(x_{l_{i-1}},x_{l_i})>2r. \quad (3.55)$$

(ii) $\sum_{i=1}^{j*-1}\mathrm{length}(x_{l_{i-1}},x_{l_i})<(\mathrm{length}(x_{l_{j^*-1}},x_{l_{j^*}}))^\gamma$, *where* length *is defined in* (3.54).

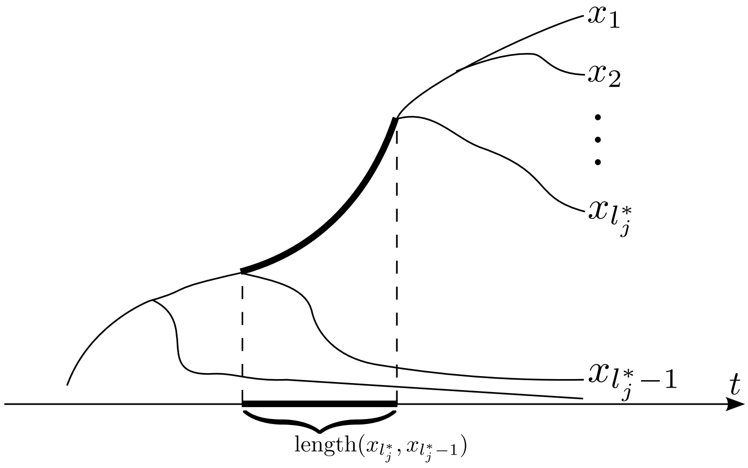

Figure 2: Illustration of the notion of $\mathrm{length}(\cdot,\cdot)$ as defined in (3.54)

Proof. Such a l_{j^*} exists for all $t > t_0(r)$ because there are at most $2k-2$ points, where m it is allowed to change. Hence, there must be a time interval of length $> 2r$ (for t large enough) during which m does not change its value. Observe that if

$$\sum_{i=1}^{j*-1} \mathrm{length}(x_{l_{i-1}}, x_{l_i}) > (2r)^\gamma, \tag{3.56}$$

then only Condition (ii) on $\mathrm{length}(x_{l_{j^*-1}}, x_{l_{j^*}})$ needs to be checked. Assume that $l_1, \dots, l_j$ all do not satisfy (ii). Then,

$$\sum_{i=1}^{j-1} \mathrm{length}(x_{l_{i-1}}, x_{l_i}) \leq Cr^{\left(\frac{1}{\gamma}\right)^j}. \tag{3.57}$$

As $i < 2k-2$ and the total time is equal to t, there must exist j such that

$$\mathrm{length}(x_{l_{j-1}}, x_{l_j}) > Cr^{\left(\frac{1}{\gamma}\right)^j}, \quad \text{for } t > t_0(r), \tag{3.58}$$

where $t_0(r)$ is sufficiently large. □

We call the value of m on the path of $x_{l_1}(t)$ between $d(x_1(t), x_{j^*-1}(t))$ and $d(x_1(t), x_{j^*}(t))$ m^*. Let us use the shortcut $R = d(x_1(t), x_{j^*-1}(t))$ and let

$$\ell = \ell(j^*, t) := \mathrm{length}(x_{l_{j^*-1}}(t), x_{l_{j^*}}(t)). \tag{3.59}$$

Then, on the time interval $(R, R+l)$, m takes the value $m*$. Moreover, up to time R the minimal particle is a.s. $> -\sqrt{2}R$. Hence,

$$x_{l_{j*}}(R+\ell) - x_{l_{j*}}(R) < x_{l_{j*}}(R+\ell) + \sqrt{2}R. \tag{3.60}$$

Since we compute an expectation conditional on $x_{l_{j*}}(R+\ell) < 2\sigma(R+\ell) + (R+\ell)^\gamma$, we obtain on this event

$$x_{l_{j*}}(R+\ell) - x_{l_{j*}}(R) < 2\sigma(R+\ell) + (R+\ell)^\gamma + \sqrt{2}R. \tag{3.61}$$

Due to our choice of j^*, we have $2\sigma R + \sqrt{2}R < C'(\ell)^\gamma$ for some positive constant C'. By taking the expectation with respect to $x_{l_{j*}}(R+\ell) - x_{l_{j*}}(R)$ only, we can extract from $J_{\mathcal{A}} + \overline{J}_{\mathcal{A}}$ the factor

$$\mathbb{E}\Big[\Big(\mathrm{e}^{(m^*\sigma + i\tau m')x_{l_{j*}}(R+\ell) - x_{l_{j*}}(R)} + \mathrm{e}^{(m^*\sigma - i\tau m')x_{l_{j*}}(R+\ell) - x_{l_{j*}}(R)} \Big)$$

$$\times \mathbb{1}\{x_{l_{j*}}(R+\ell) - x_{l_{j*}}(R) < 2\sigma\ell + (C'+1)(\ell)^\gamma\}\Big]. \tag{3.62}$$

By Lemma 3.5, (3.62) is

$$\begin{aligned} o\Big(&\mathrm{e}^{2\sigma\ell}\mathbb{E}\left[\mathrm{e}^{((m^*-2)\sigma+i\tau m')x_{l_{j*}}(R+\ell)-x_{l_{j*}}(R)}\mathbb{1}\{x_{l_{j*}}(R+\ell)-x_{l_{j*}}(R)<2\sigma\ell+(C'+1)(\ell)^\gamma\}\right]\\ &\times\mathbb{E}\left[\mathrm{e}^{2\sigma(x_{l_{j*}}(R+\ell)-x_{l_{j*}}(R))}\mathbb{1}\{x_{l_{j*}}(R+\ell)-x_{l_{j*}}(R)<2\sigma\ell+(C'+1)(\ell)^\gamma\}\right]\Big), \end{aligned} \tag{3.63}$$

for l large (which by Assumption (i) on l corresponds to r large). Note that the quantity, inside the brackets in (3.63), corresponds to the same expectation but where in the underlying tree l_1, l_{σ_1} branched off before time R.

Iteratively, that leads to

$$J_{\mathcal{A}_r} + \overline{J}_{\mathcal{A}_r} \underset{t,r\to\infty}{=} o(J_{\mathcal{A}_r^c} + \overline{J}_{\mathcal{A}_r^c}). \tag{3.64}$$

Since k was chosen arbitrary, we know that the main contribution to the $2k$-th moment comes from the term where $l_1, \dots, l_k$ have split before time r for r large enough. We condition on $\mathcal{F}_r$ and compute:

$$\begin{aligned} &\frac{1}{2}\mathbb{E}\Big[\sum_{l_1,l_2,\dots,l_k\le n(t)}\Big(\prod_{j=2}^k \mathrm{e}^{-t(1+2\sigma^2)}\mathrm{e}^{\sigma(x_{l_j}(t)+x_{l_{\sigma(j)}})+i\tau(y_{l_j}(t)-y_{l_{\sigma(j)}}(t))}\\ &\qquad\qquad + \prod_{j=2}^k \mathrm{e}^{-t(1+2\sigma^2)}\mathrm{e}^{\sigma(x_{l_j}(t)+x_{l_{\sigma(j)}})-i\tau(y_{l_j}(t)-y_{l_{\sigma(j)}}(t))}\Big)\\ &\quad\times \mathbb{1}\{x_{l_{\sigma(j)}}(t), x_{l_j}(t) < 2\sigma t + A\sqrt{t}, \forall s\in[r,t]\colon x_{l_{\sigma(j)}}, x_{l_j} \le 2\sigma s + s^\gamma\}\\ &\quad\times \mathbb{1}\{\sup_{j,j'\le k} d(l_j,l_{j'})<r\}\ \Big|\ \mathcal{F}_r\Big]\\ &= \mathbb{E}\Big[\sum_{l_1,l_2,\dots,l_k\le n(t)}\prod_{j=2}^k b_{l_j}(r)\bar{b}_{l_{\sigma(j)}}(r)\mathbb{E}\left[\Big(\big((N^{\gamma,A}_{\sigma,\tau}(t-r)\big)^{(j)}\Big)^2\right]\ \Big|\ \mathcal{F}_r\Big], \end{aligned} \tag{3.65}$$

where $b_{l_j}(r)$ is defined in (2.23) and $\big(N^{\gamma,A}_{\sigma,\tau}(t-r)\big)^{(j)}$ are i.i.d. copies of $N^{\gamma,A}_{\sigma,\tau}(t-r)$. By our second moment computations (Case $k=1$), as mentioned at the beginning of this proof,

$$\lim_{t\to\infty}\mathbb{E}\left[\Big(\big((N^{\gamma,A}_{\sigma,\tau}(t-r)\big)^{(j)}\Big)^2\right] = C_{2,A}. \tag{3.66}$$

Moreover, by invariance under permutation (in the labelling procedure),

$$\sum_{l_1,l_2,\dots,l_k\le n(t)}\prod_{j=2}^k b_{l_j}(r)\bar{b}_{l_{\sigma(j)}} = k!\Big(\sum_{k=1}^{n(r)}\mathrm{e}^{2\sigma x_k(r)-(1+2\sigma^2)r}\Big)^k. \tag{3.67}$$

Observe that $\sum_{k=1}^{n(r)}\mathrm{e}^{2\sigma x_k(r)-(1+2\sigma^2)r} = \mathcal{M}_{2\sigma,0}(r)$ which converges almost surely to $\mathcal{M}_{2\sigma,0}$. This proves(3.32).

The case $k' < k$ follows similarly. Take an optimal matching (according to the procedure described below (3.37)) of the first k' particles. The other particles will not be matched. Take one l_1 that has not been matched. Along its path, we can again find the first macroscopic piece on which $m(\cdot)$ is constant. Applying Lemma 3.5, we get that the contribution is the largest if $\max_{j\in 1,\dots,k',1,\dots,k} d(l_1,l_j) < R$, for R large enough. Observe that,

$$\mathbb{E}\Big[\sum_{k=1}^{n(t)}\mathrm{e}^{\sigma x_k(t)+i\tau z_k(t)-(\frac{1}{2}+\sigma)t}\ \Big|\ \mathcal{F}_R\Big] = \sum_{k=1}^{n(R)}\mathrm{e}^{\sigma x_k(R)+i\tau z_k(R)-(\frac{1}{2}+\sigma)t}\mathrm{e}^{(\sigma^2-\tau^2+1-2\sigma^2+i2\tau\sigma)(t-R)/2}. \tag{3.68}$$

Since in B_3 it holds that $1-\sigma^2-\tau^2<0$, the summands the r.h.s. of (3.68) converge to zero as $t\uparrow\infty$. This together with the argument in the even case implies Lemma 3.4. $\square$

3.3 Proof of Theorem 1.11

Proof of Theorem 1.11. Recall that the even (resp., odd) moments of the complex isotropic distribution $\mathcal{N}(0, C_{2,A}\mathcal{M}_{2\sigma,0})$ coincide with the r.h.s. of (3.32) (resp., (3.33)). By Lemma 3.4, conditionally on $\mathcal{F}_r$, the moments of $N^{c,A}_{\sigma,\tau}(t)$ converge to the moments of a $\mathcal{N}(0, C_{2,A}\mathcal{M}_{2\sigma,0})$ a.s. as $t\uparrow\infty$ and then $r\uparrow\infty$. Since the normal distribution is uniquely characterised by its moments, this implies convergence in distribution. Moreover, by Lemma 3.2 and Lemma 3.3,

$$\operatorname*{wlim}_{A\uparrow\infty}\operatorname*{wlim}_{t\uparrow\infty}\mathcal{L}\left[N_{\sigma,\tau}(t)-N^{c,A}_{\sigma,\tau}(t)\right]=\delta_0, \tag{3.69}$$

and $\lim_{A\to\infty}C_{2,A}=C_2$. The claim of Theorem 1.11 follows. $\square$

4 The boundaries

In this section, we provide the proofs of Theorems 1.6, 1.13 and Proposition 1.15 describing the limiting fluctuations of the partition function on all boundaries between the phases, i.e., on the 1D manifolds $B_{1,2}=\overline{B_1}\cap\overline{B_2}$, $B_{1,3}=\overline{B_1}\cap\overline{B_3}$, and $B_{2,3}=\overline{B_2}\cap\overline{B_3}$.

4.1 The boundary between phases B_1 and B_3

Proof of Theorem 1.13. The proof of Theorem 1.13 works as in phase B_3. Observe first that

$$\mathbb{E}\left[\frac{\mathcal{X}_{\beta,\rho}(t)}{\sqrt{t}\mathrm{e}^{t(1/2+\sigma^2)}}\right]=\frac{1}{\sqrt{t}}. \tag{4.1}$$

Moreover, let

$$\hat{N}_{\sigma,\tau}(t):=t^{-1/2}N_{\sigma,\tau}(t)\quad\text{and}\quad\hat{N}^{c,A}_{\sigma,\tau}(t):=t^{-1/2}N^{c,A}_{\sigma,\tau}(t). \tag{4.2}$$

By Lemma 3.1 (ii),

$$\lim_{t\uparrow\infty}\mathbb{E}\left[|\hat{N}_{\sigma,\tau}(t)|^2\right]=C_3. \tag{4.3}$$

Now, we need the following.

Lemma 4.1. *For $\beta\in B_3$,*

$$\lim_{t\to\infty}\mathbb{E}\left[|\hat{N}^{c,A}_{\sigma,\tau}(t)|^2\right]=C_{3,A}, \tag{4.4}$$

with $\lim_{A\uparrow\infty}C_{3,A}=C_3$ and, for $k\in\mathbb{N}$, we have

$$\lim_{A\uparrow\infty}\lim_{r\uparrow\infty}\lim_{t\to\infty}\mathbb{E}\left[|\hat{N}^{c,A}_{\sigma,\tau}(t)|^{2k}\;\middle|\;\mathcal{F}_r\right]=k!(C_3\mathbb{M}_{2\sigma,0})^k\quad\textit{a.s. and in } L^1. \tag{4.5}$$

Moreover, for $k'<k$,

$$\lim_{A\uparrow\infty}\lim_{r\uparrow\infty}\lim_{t\to\infty}\mathbb{E}\left[\hat{N}^{c,A}_{\sigma,\tau}(t)^k\,\overline{N^{c,A}_{\sigma,\tau}(t)}^{k'}\;\middle|\;\mathcal{F}_r\right]=0\quad\textit{a.s. and in } L^1. \tag{4.6}$$

Proof. The proof of Lemma 4.1 is a rerun of the proof of Lemma 3.4. $\square$

The claim of Theorem 1.13 follows with the very same arguments as the proof of Theorem 1.9. $\square$

4.2 Real critical point $\beta = \sqrt{2}$

For $|\sigma| = 1/\sqrt{2}$, the following scaling of the martingale $\mathcal{M}_{1,0}(t)$ plays an important role

$$\mathbb{M}^{\mathrm{SH}}_{1,0}(t) := \sqrt{t} \sum_{i=1}^{n(t)} \mathrm{e}^{-\sqrt{2}(\sqrt{2}t - x_k(t))}. \tag{4.7}$$

$\mathbb{M}^{\mathrm{SH}}_{1,0}(t)$ is called *critical additive martingale* and the rescaling appearing in the r.h.s. of (4.7) is referred to as *Seneta-Heyde scaling*. The limiting behaviour of $\mathbb{M}^{\mathrm{SH}}_{1,0}$ in the setting of branching random walks has been first analysed in [1]. An alternative proof is given in [28]. As $t \to \infty$, (4.7) converges in probability to a limiting random variable $\mathbb{M}^{\mathrm{SH}}_{1,0}$.

Lemma 4.2. *Denote $\mathbb{M}^{SH}_{1,0}(t) := \sqrt{t} \sum_{i=1}^{n(t)} \mathrm{e}^{-\sqrt{2}(\sqrt{2} - x_k(t))}$ and $\mathbb{M}^{SH}_{1,0} := \left(\frac{2}{\pi}\right)^{1/2} \mathcal{Z}$, where $\mathcal{Z}$ is the limit of the derivative martingale, cf.* (1.22)*. Then, for $\beta = \sqrt{2}$, the following convergence holds in probability*

$$\mathbb{M}^{\mathrm{SH}}_{1,0}(t) \xrightarrow[t\to\infty]{\mathbb{P}} \mathbb{M}^{\mathrm{SH}}_{1,0}. \tag{4.8}$$

Proof. The proof is just an adaptation of the result for the branching random walk (see [28, Section 6.5]). □

4.3 The boundary between phases B_2 and B_3; and the triple point $\beta = (1+i)/\sqrt{2}$

In this section, we prove the convergence of the moments of the rescaled partition function on the boundary between phases B_2 and B_3 to the moments of a Gaussian random variable with random variance in probability which is the content of Theorem 1.15.

Proof of Theorem 1.15. **(i)** The proof of Theorem 1.15 (i) is a modification of the proof of Theorem 1.9 (ii) in the following way.

Lemma 4.3. *For β with $\sigma = \frac{1}{\sqrt{2}}$, $\rho \in [-1, 1]$ and binary branching*

$$\lim_{t\to\infty} \mathbb{E}\left[\left|N^{c,A}_{\sigma,\tau}(t)\right|^2\right] = C_{2,A}, \tag{4.9}$$

and, for $k \in \mathbb{N}$, we have

$$\lim_{r\uparrow\infty} \lim_{t\to\infty} r^{\frac{2k}{4}} \mathbb{E}\left[\left|N^{c,A}_{\sigma,\tau}(t)\right|^{2k} \mid \mathcal{F}_r\right] = k!(C_{2,A}\mathbb{M}^{\mathrm{SH}}_{1,0})^k \quad \textit{in probability,} \tag{4.10}$$

where $\mathbb{M}^{\mathrm{SH}}_{1,0}$ is the martingale defined in (4.8)*. Moreover, for $k' < k$,*

$$\lim_{r\uparrow\infty} \lim_{t\to\infty} \mathbb{E}\left[N^{c,A}_{\sigma,\tau}(t)^k \overline{N^{c,A}_{\sigma,\tau}(t)}^{k'} \mid \mathcal{F}_r\right] = 0 \quad \textit{in probability.} \tag{4.11}$$

Proof. The proof is a rerun of the proof of Lemma 3.4 with the only difference that the martingale $\mathbb{M}^{\mathrm{SH}}_{1,0}$ only converges in probability towards $\mathbb{M}^{\mathrm{SH}}_{1,0}$ as $t \uparrow \infty$ and that there an additional factor $r^{1/4}$ needed. □

Since (4.10) and (4.11) only hold in probability, using the same method as in the proof of Theorem 1.9, we get the corresponding weak convergence result.

(ii) For the triple point, the argument is similar to **(i)** but with the moments as given in Lemma 4.1 with $\mathbb{M}_{2\sigma,0}$ replaced by $\mathbb{M}^{\mathrm{SH}}_{1,0}$. □

4.4 The boundary between phases B_1 and B_2

In this section, we prove Theorem 1.6.

Proof of Theorem 1.6. For $\beta \in \bar{B}_1 \cap \bar{B}_2 \setminus \{\beta = \sqrt{2}, \beta = \frac{1}{\sqrt{2}}(1+i)\}$, consider in the same way as in the proof of Theorem 1.4 the $\frac{\sqrt{2}}{\gamma}$-moment for some $\gamma > \sigma$ and $\sqrt{2}\gamma > 1$. Then, a rerun of the computation starting from (2.1) up to (2.14) bounds the $\frac{\sqrt{2}}{\gamma}$-moment from above by

$$\begin{aligned} &\int_0^t \mathrm{d}q \, \mathrm{e}^{\frac{(\sigma^2-\tau^2+2)(t-q)-2t-(\sigma^2-\tau^2)t}{\sqrt{2}\sigma}} \mathrm{e}^{\frac{\gamma^2}{\sigma^2}q+q} \\ &= \int_0^t \mathrm{d}q \, \mathrm{e}^{\frac{\left(\tau^2-(\sigma-\sqrt{2})^2+\left(\frac{\gamma^2}{\sigma^2}-1\right)\right)q}{\sqrt{2}\sigma}} \\ &= \int_0^t \mathrm{d}q \, \mathrm{e}^{\frac{\left(\frac{\gamma^2}{\sigma^2}-1\right)q}{\sqrt{2}\sigma}}, \end{aligned} \tag{4.12}$$

since $|\tau| + |\sigma| = \sqrt{2}$. The r.h.s. of (4.12) is uniformly bounded by a constant. Hence, $\mathcal{M}_{\sigma,\tau}(t)$ is in L^p for some $p > 1$. Hence, it converges a.s. and in L^1. The limit is non-degenerate because $\mathbb{E}[\mathcal{M}_{\sigma,\tau}(t)] = 1$ and Theorem 1.6 follows. □

5 Proof of Theorem 1.2

In this section, as a consequence of the fluctuation results of the previous sections, we derive the phase diagram shown on Fig. 1.

Proof of Theorem 1.2. Convergence in probability for $\beta \in B_1$ and B_3 in (1.11) follows from Theorems 1.4 and 1.9 (ii) by [24, Lemma 3.9 (1)]. Convergence for the glassy phase $\beta \in \overline{B_2}$ was shown in [19]. For the boundaries between all three phases, the formula (1.11) follows from the continuity of the limiting log-partition function. □

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