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Estimates of Dirichlet heat kernels for subordinate Brownian motions*

Panki Kim† Ante Mimica‡

Abstract

In this paper, we discuss estimates of transition densities of subordinate Brownian motions in open subsets of Euclidean space. When D is a $C^{1,1}$ domain, we establish sharp two-sided estimates for the transition densities of a large class of subordinate Brownian motions in D whose scaling order is not necessarily strictly below 2. Our estimates are explicit and written in terms of the dimension, the Euclidean distance between two points, the distance to the boundary and the Laplace exponent of the corresponding subordinator only.

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1 Introduction

Transition densities of Lévy processes killed upon leaving an open set D are Dirichlet heat kernels of the generators of such processes on D. For example, the classical Dirichlet heat kernel, which is the fundamental solution of the heat equation in D with zero boundary values, is the transition density of Brownian motion killed upon leaving D. Since, except in some special cases, explicit forms of the Dirichlet heat kernels are impossible to obtain, obtaining sharp estimates of the Dirichlet heat kernels has been a fundamental problem both in probability theory and in analysis.

After the fundamental work in [12], sharp two-sided estimates for the Dirichlet heat kernel $p_D(t,x,y)$ of non-local operators in open sets have been studied a lot (see [2, 3, 6, 5, 7, 13, 11, 18, 17, 15, 14, 16, 19, 20, 25, 33, 34, 36]). In particular, very recently in [5, 19], sharp two-sided estimates of $p_D(t,x,y)$ were obtained for a large class of

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†Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Republic of Korea, E-mail: pkim@snu.ac.kr

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rotationally symmetric Lévy processes when the radial parts of their characteristic exponents satisfy weak scaling conditions whose upper scaling exponent is strictly less than 2. A still remaining open question in this direction is that, when the upper scaling exponent is not strictly less than 2, for how general discontinuous Lévy processes one can prove sharp two-sided estimates for their Dirichlet heat kernels. In this paper we investigate this question for subordinate Brownian motions, which form a very large class of Lévy processes.

A subordinate Brownian motion in $\mathbb{R}^d$ is a Lévy process which can be obtained by replacing the time of Brownian motion in $\mathbb{R}^d$ by an independent subordinator (i.e., an increasing Lévy process starting from 0). The subordinator used to define the subordinate Brownian motion X can be interpreted as "operational" time or "intrinsic" time. For this reason, subordinate Brownian motions have been used in applied fields a lot.

To obtain the sharp Dirichlet heat kernel estimates, it is necessary to know the sharp heat kernel estimates in $\mathbb{R}^d$. Recently heat kernel estimates for discontinuous Markov processes have been a very active research area and, for a large class of purely discontinuous Markov processes, the sharp heat kernel estimates were obtained in [4, 8, 10, 21, 22, 23, 32, 47, 48]. But except [43, 48], for the estimates of the heat kernel, a common assumption on the purely discontinuous Markov processes in $\mathbb{R}^d$ considered so far is that their weak scaling orders were always strictly between 0 and 2. Very recently in [43], the second-named author considered a large class of purely discontinuous subordinate Brownian motions whose weak scaling order is between 0 and 2 *including 2*, and succeeded in obtaining sharp heat kernel estimates of such processes. In this sense, the results in [43] extend earlier works in [4].

Motivated by [43], the main purpose of this paper is to establish sharp two-sided estimates of $p_D(t,x,y)$ for a large class of subordinate Brownian motions in $C^{1,1}$ open set whose weak scaling order is not necessarily strictly below 2. Our estimates are explicit and written in terms of the dimension d, the Euclidian distance $|x-y|$ for $x,y\in D$, the distance to the boundary of D for $x,y\in D$ and the Laplace exponent of the corresponding subordinator only. See Section 8 for examples, in particular, (8.2)–(8.3) for estimates of the Dirichlet heat kernels.

This paper is also motivated by [6, 7], and, several results and ideas in [7, 43] will be used here. It is shown in [6] that, when weak scaling orders of characteristic exponents of unimodal Lévy processes in $\mathbb{R}^d$ are strictly below 2, sharp estimates on the survival probabilities for the unimodal Lévy processes can be obtained without the information on sharp two-sided estimates for the Dirichlet heat kernels. Such estimates in [6] can not be used in the setting of this paper.

We will use the symbol ":=," which is read as "is defined to be." In this paper, for $a,b\in\mathbb{R}$ we denote $a\wedge b:=\min\{a,b\}$ and $a\vee b:=\max\{a,b\}$. By $B(x,r)=\{y\in\mathbb{R}^d: |x-y|<r\}$ we denote the open ball around $x\in\mathbb{R}^d$ with radius $r>0$. We also use convention $0^{-1}=+\infty$. For any open set V, we denote by $\delta_V(x)$ the distance of a point x to V^c. We sometimes write point $z=(z_1,\dots,z_d)\in\mathbb{R}^d$ as $(\widetilde{z},z_d)$ with $\widetilde{z}\in\mathbb{R}^{d-1}$.

Let $B=(B_t,\,t\geq 0)$ be a Brownian motion in $\mathbb{R}^d$ whose infinitesimal generator is Δ and let $S=(S_t,\,t\geq 0)$ be a subordinator which is independent of B. The process $X=(X_t: t\geq 0)$ defined by $X_t=B_{S_t}$ is a rotationally invariant (unimodal) Lévy process in $\mathbb{R}^d$ and is called a subordinate Brownian motion. Let ϕ be the Laplace exponent of S. That is,

$$\mathbb{E}[\exp\{-\lambda S_t\}]=\exp\{-t\phi(\lambda)\},\qquad \lambda>0.$$

Then the characteristic exponent of X is $\Psi(\xi)=\phi(|\xi|^2)$ and the infinitesimal generator X is $\phi(\Delta)=-\phi(-\Delta)$. It is known that the Laplace exponent ϕ is a Bernstein function with $\phi(0+)=0$, that is $(-1)^n\phi^{(n)}\le 0$, for all $n\ge 1$. Thus it has a representation

$$\phi(\lambda)=b\lambda+\int_0^\infty (1-e^{-\lambda t})\,\mu(dt), \tag{1.1}$$

where $b\ge 0$, and μ is a measure satisfying $\int_0^\infty (1\wedge t)\mu(dt)<\infty$, which is called the Lévy measure of S (or ϕ). In this paper, we will always assume that $b=0$ and $\mu(0,\infty)=\infty$. Note that $\phi'(\lambda)=\lambda\int_0^\infty e^{-\lambda t}\mu(dt)>0$. Due to the independence of B and S, the Lévy measure $\Pi(dx)$ of X has a density $j(|x|)$, given by

$$j(r)=\int_0^\infty (4\pi s)^{-d/2}e^{-\frac{|x|^2}{4s}}\mu(ds), \quad r>0. \tag{1.2}$$

It is well known that there exists $c_0=c_0(d)$ depending only on d such that

$$j(r)\le c_0\frac{\phi(r^{-2})}{r^d}, \quad r>0 \tag{1.3}$$

(see [3, (15)]). Moreover, since $\mu(0,\infty)=\infty$, X has transition density $p(t,x,y)=p(t,y-x)=p(t,|y-x|)$ and it is of the form

$$p(t,x)=\int_{(0,\infty)}(4\pi s)^{-d/2}e^{-\frac{|x|^2}{4s}}\,\mathbb{P}(S_t\in ds) \tag{1.4}$$

for $x\in\mathbb{R}^d$ and $t>0$.

We now introduce the following scaling conditions.

Definition 1.1. *Suppose f is a function from $(0,\infty)$ into $(0,\infty)$.*

(1) We say that f satisfies the lower scaling condition $L_a(\gamma,C_L)$ if there exist $a\ge 0$, $\gamma>0$ and $C_L\in(0,1]$ such that

$$\frac{f(\lambda t)}{f(\lambda)}\ge C_L t^\gamma \qquad \text{for all} \quad \lambda>a \text{ and } t\ge 1. \tag{1.5}$$

We say that f satisfies the lower scaling condition near infinity if the above constant a is strictly positive and we say f satisfies the lower scaling condition globally if $a=0$.

(2) We say f satisfies the upper scaling condition $U_a(\delta,C_U)$ if there exist $a\ge 0$, $\delta>0$ and $C_U\in[1,\infty)$ such that

$$\frac{f(\lambda t)}{f(\lambda)}\le C_U t^\delta \qquad \text{for all} \quad \lambda>a \text{ and } t\ge 1. \tag{1.6}$$

We say f satisfies the upper scaling condition near infinity if the above constant a is strictly positive and we say f satisfies the upper scaling condition globally if $a=0$.

For any open set $D\subset\mathbb{R}^d$, the *first exit time of* D by the process X is defined by the formula $\tau_D:=\inf\{t>0: X_t\notin D\}$ and we use X^D to denote the process obtained by killing the process X upon exiting D. By the strong Markov property, it can easily be verified that

$$p_D(t,x,y):=p(t,x,y)-\mathbb{E}_x[p(t-\tau_D,X_{\tau_D},y):\tau_D<t], \quad t>0, x,y\in D, \tag{1.7}$$

is the transition density of X^D. Note that from (1.4) we see that $\sup_{|x|\ge\beta,t>0}p(t,x)<\infty$ for all $\beta>0$. Using this estimate and the continuity of p, it is routine to show that $p_D(t,x,y)$ is symmetric and continuous (see [27]).

We say that $D \subset \mathbb{R}^d$ (when $d \ge 2$) is a $C^{1,1}$ open set with $C^{1,1}$ characteristics (R_0, Λ) if there exist a localization radius $R_0 > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$ there exist a $C^{1,1}$-function $\varphi = \varphi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi(0) = 0$, $\nabla\varphi(0) = (0, \dots, 0)$, $\|\nabla\varphi\|_\infty \le \Lambda$, $|\nabla\varphi(x) - \nabla\varphi(w)| \le \Lambda|x - w|$ and an orthonormal coordinate system CS_z of $z = (z_1, \cdots, z_{d-1}, z_d) := (\widetilde{z}, z_d)$ with origin at z such that $D \cap B(z, R_0) = \{y = (\tilde{y}, y_d) \in B(0, R_0) \text{ in } CS_z : y_d > \varphi(\tilde{y})\}$. The pair (R_0, Λ) will be called the $C^{1,1}$ characteristics of the open set D. Note that a $C^{1,1}$ open set D with characteristics (R_0, Λ) can be unbounded and disconnected, and the distance between two distinct components of D is at least R_0. By a $C^{1,1}$ open set in $\mathbb{R}$ with a characteristic $R_0 > 0$, we mean an open set that can be written as the union of disjoint intervals so that the infimum of the lengths of all these intervals is at least R_0 and the infimum of the distances between these intervals is at least R_0.

It is well-known that $C^{1,1}$ open set D with the characteristic (R_0, Λ) satisfies the interior and exterior ball conditions with the characteristic $R_1 > 0$, that is, there exists $R_1 > 0$ such that the following holds: for all $x \in D$ with $\delta_D(x) \le R_1$ there exist balls $B_1 \subset D$ and $B_2 \subset D^c$ whose radii are R_1 such that $x \in B_1$ and $\delta_{B_1}(x) = \delta_D(x) = \delta_{\overline{B_2}^c}(x)$. Without loss of generality whenever we consider a $C^{1,1}$ open set D with the characteristic (R_0, Λ), we will take R_0 as the characteristic of the interior and exterior ball conditions of D, that is, $R_1 = R_0$.

We say that *the path distance in a connected open set U is comparable to the Euclidean distance with characteristic λ_1* if for every x and y in U there is a rectifiable curve l in U which connects x to y such that the length of l is less than or equal to $\lambda_1|x - y|$. Clearly, such a property holds for all bounded $C^{1,1}$ domains (connected open sets), $C^{1,1}$ domains with compact complements, and a domain consisting of all the points above the graph of a bounded globally $C^{1,1}$ function.

In this paper, for the Laplace exponent ϕ of a subordinator, we define the function $H : (0, \infty) \to [0, \infty)$ by $H(\lambda) := \phi(\lambda) - \lambda\phi'(\lambda)$. The function H, which appeared earlier in the work of Jain and Pruitt [31], took a central role in [43] in obtaining the sharp heat kernel estimates of the transition density of the corresponding subordinate Brownian motion X in $\mathbb{R}^d$.

Obviously, this function H will also naturally appear in this paper in the estimates of the transition density of X in open subsets. Under the weak scaling assumptions on H we will obtain the sharp two-sided estimates of $p_D(t, x, y)$. Recall that $\delta_D(x)$ is the distance between x and the boundary of D.

In the main results of this paper, we will impose the following assumption: there exists a positive constant $c > 0$ such that

$$j(r) \le c j(r + 1), \quad r > 1. \tag{1.8}$$

Remark 1.2. A Bernstein function ϕ is called a complete Bernstein function if the Lévy measure μ has a completely monotone density $\mu(t)$, i.e., $(-1)^n D^n \mu \ge 0$ for every non-negative integer n. Note that, if ϕ is a complete Bernstein function then by [38, Lemma 2.1], there exists $c_1 > 1$ such that

$$\mu(r) \le c_1\, \mu(r + 1), \qquad \forall r > 1. \tag{1.9}$$

If H satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ then by [43, Lemma 2.6] and Remark 2.2, $c^{-2}H(r^{-1}) \le \mu(r, \infty) \le c_2 H(r^{-1})$ for $r < 2$. Using the monotonicity of μ and $U_a(\delta, C_U)$ of H, it is easy to see that $c^{-3}r^{-1}H(r^{-1}) \le \mu(r) \le c_3 r^{-1}H(r^{-1})$ for $r < 2$ (see the proof [37, Theorem 13.2.10]). Therefore, by [37, Proposition 13.3.5], we see that if ϕ is a complete Bernstein function and H satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$, then (1.8) holds.

We are now ready to state the main result of this paper.

Theorem 1.3. *Let $S=(S_t)_{t\ge 0}$ be a subordinator with zero drift whose Laplace exponent is ϕ and let $X=(X_t)_{t\ge 0}$ be the corresponding subordinate Brownian motion in $\mathbb{R}^d$. Assume that* (1.8) *holds and that H satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta<2$ and $\gamma>2^{-1}\mathbf{1}_{\delta\ge 1}$ for some $a>0$. Suppose that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0,Λ).*
(a) For every $T>0$, there exist constants c_1, C_0 and $a_U>0$ such that for every $(t,x,y)\in(0,T]\times D\times D$,

$$p_D(t,x,y)\le C_0\left(1\wedge\frac{1}{\sqrt{t\phi(1/\delta_D(x)^2)}}\right)\left(1\wedge\frac{1}{\sqrt{t\phi(1/\delta_D(y)^2)}}\right)p(t,x/3,y/3) \tag{1.10}$$

$$\le c_1\left(1\wedge\frac{1}{\sqrt{t\phi(1/\delta_D(x)^2)}}\right)\left(1\wedge\frac{1}{\sqrt{t\phi(1/\delta_D(y)^2)}}\right)$$
$$\times\left(\phi^{-1}(t^{-1})^{d/2}\wedge\left(\frac{tH(|x-y|^{-2})}{|x-y|^d}+\phi^{-1}(t^{-1})^{d/2}\exp[-a_U|x-y|^2\phi^{-1}(t^{-1})]\right)\right). \tag{1.11}$$

(b) When D is an unbounded, we further assume that H satisfies $L_0(\gamma_0, C_L)$ and $U_0(\delta, C_U)$ with $\delta<2$ and that the path distance in each connected component of D is comparable to the Euclidean distance with characteristic λ_1. Then for every $T>0$ there exist constants $c_2, a_L>0$ such that for every $(t,x,y)\in(0,T]\times D\times D$,

$$p_D(t,x,y)\ge c_2^{-1}\left(1\wedge\frac{1}{\sqrt{t\phi(1/\delta_D(x)^2)}}\right)\left(1\wedge\frac{1}{\sqrt{t\phi(1/\delta_D(y)^2)}}\right)$$
$$\times\left(\phi^{-1}(t^{-1})^{d/2}\wedge\left(\frac{tH(|x-y|^{-2})}{|x-y|^d}+\phi^{-1}(t^{-1})^{d/2}\exp[-a_L|x-y|^2\phi^{-1}(t^{-1})]\right)\right). \tag{1.12}$$

(c) If D is a bounded $C^{1,1}$ open set, then for each $T>0$ there exists $c_3\ge 1$ such that for every $(t,x,y)\in[T,\infty)\times D\times D$,

$$c_3^{-1}\frac{e^{-t\,\lambda^D}}{\sqrt{\phi(1/\delta_D(x)^2)\phi(1/\delta_D(y)^2)}}\le p_D(t,x,y)\le c_3\frac{e^{-t\,\lambda^D}}{\sqrt{\phi(1/\delta_D(x)^2)\phi(1/\delta_D(y)^2)}},$$

where $-\lambda^D<0$ is the largest eigenvalue of the generator of X^D.

We emphasize that we put the assumption $\gamma>2^{-1}\mathbf{1}_{\delta\ge 1}$ on lower scaling condition near infinity, not globally, i.e., we don't assume that $\gamma_0>2^{-1}\mathbf{1}_{\delta\ge 1}$ in Theorem 1.3(b).

When D is a *half space-like* domain, we have the global estimates for all $t>0$ on the Dirichlet heat kernel.

Theorem 1.4. *Let $S=(S_t)_{t\ge 0}$ be a subordinator with zero drift whose Laplace exponent is ϕ and let $X=(X_t)_{t\ge 0}$ be the corresponding subordinate Brownian motion in $\mathbb{R}^d$. Suppose that D is a domain consisting of all the points above the graph of a bounded globally $C^{1,1}$ function and H satisfies $L_0(\gamma, C_L)$ and $U_0(\delta, C_U)$ with $\delta<2$ and $\gamma>2^{-1}\mathbf{1}_{\delta\ge 1}$. Then there exist $c\ge 1$ and $a_L, a_U>0$ such that both* (1.11) *and* (1.12) *hold for all $(t,x,y)\in(0,\infty)\times D\times D$.*

The assumption that H satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta<2$ in Theorems 1.3 and 1.4 allows us to cover several interesting cases where the scaling order of the characteristic exponent $\Psi(\xi)=\phi(|\xi|^2)$ of X is 2.

The rest of the paper is organized as follows. In Section 2, we revisit [43] and improve one of the main results of [43] in Theorem 2.9. This result will be used in Sections 5–7 to show the sharp two-sided estimates of the Dirichlet heat kernel when ϕ satisfies the lower scaling condition near infinity or $H(\lambda)=\phi(\lambda)-\lambda\phi'(\lambda)$ satisfies the lower and upper scaling conditions near infinity. In Section 3 we first show that the scale-invariant parabolic Harnack inequality holds with explicit scaling in terms of Laplace exponent. Then using this we give some preliminary interior lower bound of the Dirichlet heat kernel. Using such lower bound of the Dirichlet heat kernel, Theorem 2.9, (4.1), and the estimates on exit probabilities in Section 4 we prove the estimates of the survival probabilities and the sharp two-sided estimates of the transition density $p_D(t,x,y)$ for the killed process X^D. This is done in Sections 5–6. As an application of Theorem 1.3, in Section 7 we establish the estimates on the Green functions in bounded $C^{1,1}$ domain. Section 8 contains some examples of subordinate Brownian motions and the sharp two-sided estimates of transition density and Green function of them.

In this paper, we use the following notations. For a Borel set W in $\mathbb{R}^d$, ∂W, $\overline{W}$ and $|W|$ denote the boundary, the closure and the Lebesgue measure of W in $\mathbb{R}^d$, respectively. For $s \in R$, $s_+ := s \vee 0$ Throughout the rest of this paper, the positive constants $a_0, a_1, T_1, M_0, M_1, \widetilde{R}, R_*, R_0, R_1, C, C_i$, $i = 0, 1, 2, \dots$, can be regarded as fixed, while the constants $c_i = c_i(a, b, c, \dots)$, $i = 0, 1, 2, \dots$, denote generic constants depending on $a, b, c, \dots$, whose exact values are unimportant. They start anew in each statement and each proof. The dependence of the constants on $\phi, \gamma, \delta, C_L, C_U$ and the dimension $d \geq 1$, may not be mentioned explicitly.

2 Preliminary heat kernel estimates in $\mathbb{R}^d$

Throughout this paper we assume that ϕ is the Laplace exponent of a subordinator S. Without loss of generality we assume that $\phi(1) = 1$. In this section we revisit [43] and improve the main result of [43] for the case that ϕ satisfies the lower scaling condition near infinity.

The Laplace exponent ϕ belongs to the class of Bernstein functions

$$\mathcal{BF} = \{f \in C^\infty(0,\infty)\colon f \geq 0,\, (-1)^{n-1} f^{(n)} \geq 0,\ n \in \mathbb{N}\}$$

with $\phi(0+) = 0$. Thus ϕ has a unique representation

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda y})\mu(dy), \tag{2.1}$$

where $b \geq 0$ and μ is a Lévy measure satisfying $\int_0^\infty (1 \wedge t)\mu(dt) < \infty$. Let Φ be denote the increasing function

$$\Phi(r) := \frac{1}{\phi(1/r^2)}, \qquad r > 0. \tag{2.2}$$

The next Proposition is a particular case of [43, Proposition 2.4]. Note that there is a typo in [43, Proposition 2.4]: $\alpha\phi^{-1}(\beta^{-1})$ in the display there should be $\alpha\phi^{-1}(\beta t^{-1})$.

Proposition 2.1 ([43, Proposition 2.4])**.** *There exist constants $\rho \in (0,1)$ and $\tau > 0$ such that for every subordinator S,*

$$\mathbb{P}\left(\frac{1}{2\phi^{-1}(t^{-1})} \leq S_t \leq \frac{1}{\phi^{-1}(\rho t^{-1})}\right) \geq \tau \quad \textit{for all } t > 0\,.$$

We recall the conditions $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ from Definition 1.1.

Remark 2.2. Suppose that f is non-decreasing.
(1) If f satisfies $L_b(\gamma, C_L)$ then f satisfies $L_a(\gamma, (a/b)^\gamma C_L)$ for all $a \in (0, b]$;

$$\frac{f(x\lambda)}{f(\lambda)} \geq C_L (a/b)^\gamma x^\gamma\,, \quad x \geq 1, \lambda \geq a\,. \tag{2.3}$$

In fact, suppose $a \le \lambda < b$ and $x \ge 1$. Then, $f(x\lambda) \ge C_L x^\gamma (\lambda/b)^\gamma f(b) \ge C_L x^\gamma (a/b)^\gamma f(\lambda)$ if $x\lambda > b$, and $f(x\lambda) \ge f(\lambda) \ge C_L x^\gamma (a/b)^\gamma f(\lambda)$ if $x\lambda \le b$.
(2) If f satisfies $U_b(\delta, C_U)$ then f satisfies $U_a(\delta, C_U f(b)/f(a))$ for all $a \in (0, b]$;

$$\frac{f(x\lambda)}{f(\lambda)} \le C_U \frac{f(b)}{f(a)} x^\delta , \quad x \ge 1, \lambda \ge a . \tag{2.4}$$

In fact, suppose $a \le \lambda < b$ and $x \ge 1$. Then, $f(x\lambda) \le C_U x^\delta (\lambda/b)^\delta f(b) \le C_U x^\delta f(b) \le C_U x^\delta f(b) f(\lambda)/f(a)$ if $x\lambda > b$, and $f(x\lambda) \le f(b) \le C_U x^\delta f(b) f(\lambda)/f(a)$ if $x\lambda \le b$.

Recall that $H(\lambda) = \phi(\lambda) - \lambda\phi'(\lambda)$. Note that, by the concavity of ϕ, $H(\lambda) = \phi(\lambda) - \lambda\phi'(\lambda) \ge 0$. Moreover, H is non-decreasing since $H'(\lambda) = -\lambda\phi''(\lambda) \ge 0$.

Using Remark 2.2, we have the following. c.f., [43, Lemma 2.1].

Lemma 2.3. *(a) For any $\lambda > 0$ and $x \ge 1$,*

$$\phi(\lambda x) \le x\phi(\lambda) \qquad \textit{and} \qquad H(\lambda x) \le x^2 H(\lambda) .$$

(b) Assume that the drift b of ϕ in the representation (2.1) is zero. If H satisfies $L_a(\gamma, C_L)$ (resp. $U_a(\delta, C_U)$), then ϕ satisfies $L_a(\gamma, C_L)$(resp. $U_a(\delta \wedge 1, C_U)$). Thus if either H or ϕ satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ then for every $M > 0$ there exist $c_1, c_2 > 0$ such that

$$c_1 \Big(\frac{R}{r}\Big)^{2\gamma} \ \le\ \frac{\Phi(R)}{\Phi(r)} \ \le\ c_2 \Big(\frac{R}{r}\Big)^{2(\delta\wedge 1)} \qquad \textit{for every } 0 < r < R < a^{-1}M. \tag{2.5}$$

By Remark 2.2 we also have

Lemma 2.4. *If ϕ satisfies $L_a(\gamma, C_L)$ for some $a > 0$, then for every $b \in (0, a]$,*

$$\frac{\phi^{-1}(\lambda x)}{\phi^{-1}(\lambda)} \le (a/b) C_L^{-1/\gamma} x^{1/\gamma} \qquad \textit{for all} \quad \lambda > \phi(b),\ x \ge 1 .$$

Throughout this paper, the process $X = (X_t : t \ge 0)$ is a subordinate Brownian motion whose characteristic exponent is $\phi(|x|^2)$. Recall that $x \to j(|x|)$ is the Lévy density of the subordinate Brownian motion X defined in (1.2), which gives rise to a Lévy system for X describing the jumps of X; For any $x \in \mathbb{R}^d$, stopping time τ (with respect to the filtration of X), and nonnegative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$ and $s \ge 0$ we have

$$\mathbb{E}_x \left[\sum_{s \le \tau} f(s, X_{s-}, X_s) \right] = \mathbb{E}_x \left[\int_0^\tau \left(\int_{\mathbb{R}^d} f(s, X_s, y) j(|X_s - y|) dy \right) ds \right] \tag{2.6}$$

(e.g., see [22, Appendix A]).

The next lemma holds for every symmetric Lévy process and it follows from [44, (3.2)] and [29, Corollary 1]. Recall that τ_D is the first exit time of D by the process X.

Lemma 2.5. *For any positive constants a, b, there exists $c = c(a, b, \phi) > 0$ such that for all $z \in \mathbb{R}^d$ and $t > 0$,*

$$\inf_{y \in B(z, a\Phi^{-1}(t)/2)} \mathbb{P}_y \left(\tau_{B(z, a\Phi^{-1}(t))} > bt \right) \ \ge\ c.$$

Recall that X has a transition density $p(t, x, y) = p(t, y - x) = p(t, |y - x|)$ of the form (1.4). We first consider the estimates of $p(t, x)$ under the assumption that ϕ satisfies $L_a(\gamma, C_L)$ for some $a > 0$. Note that $L_a(\gamma, C_L)$ implies $\lim_{\lambda\to\infty} \phi(\lambda) = \infty$.

By our Remark 2.2 and [43, Propositions 3.2 and 3.4], we have the following two upper bounds.

Proposition 2.6. *If ϕ satisfies $L_a(\gamma, C_L)$ for some $a > 0$, then for every $T > 0$ there exists $c = c(T) > 0$ such that for all $t \le T$ and $x \in \mathbb{R}^d$,*

$$p(t,x) \le c\,\phi^{-1}(t^{-1})^{d/2}.$$

Proposition 2.7. *If ϕ satisfies $L_a(\gamma, C_L)$ for some $a > 0$, then for every $T > 0$ there exist $c_1, c_2 > 0$ such that for all $t \le T$ and $x \in \mathbb{R}^d$ satisfying $t\phi(|x|^{-2}) \le 1$,*

$$p(t,x) \le c_1 \left(t|x|^{-d}H(|x|^{-2}) + \phi^{-1}(t^{-1})^{d/2}\exp[-c_2|x|^2\phi^{-1}(t^{-1})] \right).$$

Proposition 2.8. *If ϕ satisfies $L_a(\gamma, C_L)$ for some $a > 0$, then for every $T > 0$ there exists $c = c(T) > 0$ such that for all $t \le T$ and $x \in \mathbb{R}^d$,*

$$p(t,x) \ge c\,\phi^{-1}(t^{-1})^{\frac{d}{2}}\exp[-2^{-1}|x|^2\phi^{-1}(t^{-1})].$$

In particular, if additionally $t\phi(M|x|^{-2}) \ge 1$ holds for some $M > 0$, then we have

$$p(t,x) \ge c\,e^{-M/2}\phi^{-1}(t^{-1})^{\frac{d}{2}}. \tag{2.7}$$

Proof. We closely follow the proof of [43, Proposition 3.5]. Let $\rho \in (0,1)$ be the constant in Proposition 2.1 and, without loss of generality, we assume $T \ge \rho\phi^{-1}(a)$. Using (1.4) we get

$$\begin{aligned} p(t,x) &\ge (4\pi)^{-d/2}\int_{[2^{-1}\phi^{-1}(t^{-1})^{-1},\phi^{-1}(\rho t^{-1})^{-1}]} s^{-d/2}e^{-\frac{|x|^2}{4s}}\,\mathbb{P}(S_t \in ds) \\ &\ge (4\pi)^{-d/2}\phi^{-1}(\rho t^{-1})^{d/2}e^{-\frac{1}{2}|x|^2\phi^{-1}(t^{-1})}\mathbb{P}\left(2^{-1}\phi^{-1}(t^{-1})^{-1} \le S_t \le \phi^{-1}(\rho t^{-1})^{-1}\right). \end{aligned} \tag{2.8}$$

Let $b = \phi^{-1}(\rho/T)$. Note that, by Lemma 2.4, we have that for $0 < t < T = \rho\phi(b)^{-1}$,

$$\phi^{-1}(\rho t^{-1}) = \phi^{-1}(t^{-1})\frac{\phi^{-1}(\rho t^{-1})}{\phi^{-1}(t^{-1})} \ge (b/a)C_L^{1/\gamma}\rho^{1/\gamma}\,\phi^{-1}(t^{-1}). \tag{2.9}$$

Using (2.9), Proposition 2.1 and (2.8) we get

$$p(t,x) \ge c_2 e^{-\frac{1}{2}|x|^2\phi^{-1}(t^{-1})}\phi^{-1}(t^{-1})^{d/2}.$$

□

We now revisit [43].

Theorem 2.9. *Let $S = (S_t)_{t\ge 0}$ be a subordinator with zero drift whose Laplace exponent is ϕ and let $X = (X_t)_{t\ge 0}$ be the corresponding subordinate Brownian motion in $\mathbb{R}^d$ and $p(t,x,y) = p(t,y-x)$ be the transition density of X.*

If ϕ satisfies $L_a(\gamma, C_L)$ for some $a > 0$, then for every $T > 0$ there exist $c_1 = c_1(T,a) > 1$ and $c_2 = c_2(T,a) > 0$ such that for all $t \le T$ and $x \in \mathbb{R}^d$,

$$p(t,x) \le c_1\left(\phi^{-1}(t^{-1})^{d/2} \wedge \left(t|x|^{-d}H(|x|^{-2}) + \phi^{-1}(t^{-1})^{d/2}e^{-c_2|x|^2\phi^{-1}(t^{-1})}\right)\right), \tag{2.10}$$

$$j(|x|) \le c_1|x|^{-d}H(|x|^{-2}), \tag{2.11}$$

and

$$c_1^{-1}\phi^{-1}(t^{-1})^{d/2} \le p(t,x) \le c_1\phi^{-1}(t^{-1})^{d/2}, \quad \text{if } t\phi(|x|^{-2}) \ge 1. \tag{2.12}$$

Proof. (2.10) and (2.11) follow from Propositions 2.6 and 2.7. The estimates (2.12) follow from Remark 2.2, [43, Proposition 3.2] and Proposition 2.8. □

3 Parabolic Harnack inequality and preliminary lower bounds of $p_D(t,x,y)$

Throughout this section, we assume that ϕ *has no drift and satisfies* $L_a(\gamma, C_L)$ *for some* $a \ge 0$. Recall that $p_D(t,x,y)$ defined in (1.7) is the transition density for X^D, the subprocess of X killed upon leaving D.

Let $Z_s := (V_s, X_s)$ be the time-space process of X, where $V_s = V_0 - s$. The law of the time-space process $s \mapsto Z_s$ starting from (t,x) will be denoted as $\mathbb{P}^{(t,x)}$.

Definition 3.1. *A non-negative Borel measurable function $h(t,x)$ on $\mathbb{R} \times \mathbb{R}^d$ is said to be parabolic (or caloric) on $(a,b] \times B(x_0,r)$ if for every relatively compact open subset U of $(a,b] \times B(x_0,r)$, $h(t,x) = \mathbb{E}^{(t,x)}[h(Z_{\tau_U^Z})]$ for every $(t,x) \in U \cap ([0,\infty) \times \mathbb{R}^d)$, where $\tau_U^Z := \inf\{s > 0 : Z_s \notin U\}$.*

Recall that $\Phi(r) = \frac{1}{\phi(1/r^2)}$. In this section, we will first prove that X satisfies the scale-invariant parabolic Harnack inequality with explicit scaling in terms of Φ. That is,

Theorem 3.2. *Suppose that ϕ has no drift and satisfies $L_a(\gamma, C_L)$ for some $a \ge 0$. For every $M > 0$, there exist $c > 0$ and $c_1, c_2 \in (0,1)$ depending on d, γ and C_L (also depending on M and a if $a > 0$) such that for every $x_0 \in \mathbb{R}^d$, $t_0 \ge 0$, $R \in (0, a^{-1}M)$ and every non-negative function u on $[0,\infty) \times \mathbb{R}^d$ that is parabolic on $(t_0, t_0 + 4c_1\Phi(R)] \times B(x_0,R)$,*

$$\sup_{(t_1,y_1)\in Q_-} u(t_1,y_1) \le c \inf_{(t_2,y_2)\in Q_+} u(t_2,y_2),$$

where $Q_- = (t_0 + c_1\Phi(R), t_0 + 2c_1\Phi(R)] \times B(x_0, c_2R)$ and $Q_+ = [t_0 + 3c_1\Phi(R), t_0 + 4c_1\Phi(R)] \times B(x_0, c_2R)$.

Theorem 3.2 clearly implies the elliptic Harnack inequality. Thus this extends the main result of [29].

To prove Theorem 3.2, we first observe that for each $c_1, b > 0$ and every $r, t > 0$ satisfying $r\phi^{-1}(t^{-1})^{1/2} \ge c_1$ we have

$$\begin{aligned}
&\phi^{-1}(t^{-1})^{d/2}e^{-br^2\phi^{-1}(t^{-1})}/(tr^{-d}\phi(r^{-2})) = (\phi(r^{-2})t)^{-1}(r\phi^{-1}(t^{-1})^{1/2})^d e^{-br^2\phi^{-1}(t^{-1})}\\
&\le \sup_{a>0}[(\phi(a^2r^{-2})/\phi(r^{-2}))a^d e^{-ba^2}] \le \sup_{a>0} a^d(a\vee 1)^2 e^{-ba^2} =: c_2 < \infty.
\end{aligned}$$

Using this and the fact that $\phi \ge H$, we see that for each $b > 0$ there exists $c = c(b) > 0$ such that for all $t > 0, x \in \mathbb{R}^d$,

$$\begin{aligned}
&\phi^{-1}(t^{-1})^{d/2} \wedge \left(t|x|^{-d}H(|x|^{-2}) + \phi^{-1}(t^{-1})^{d/2}e^{-b|x|^2\phi^{-1}(t^{-1})}\right)\\
\le& c(\phi^{-1}(t^{-1})^{d/2} \wedge t|x|^{-d}\phi(|x|^{-2})).
\end{aligned} \tag{3.1}$$

Thus by [43] (for $a = 0$) and Proposition 2.8 and (2.10) (for $a > 0$) we have the following bounds: for $t \in (0,T]$ if $a > 0$ (for $t > 0$ if $a = 0$),

$$p(t,x) \le C\left((\Phi^{-1}(t))^{-d} \wedge \frac{t}{|x|^d\Phi(|x|)}\right), \quad x \in \mathbb{R}^d \tag{3.2}$$

and

$$p(t,x) \ge C^{-1}(\Phi^{-1}(t))^{-d}e^{-\frac{1}{2}|x|^2/(\Phi^{-1}(t))^2}, \tag{3.3}$$

where the above constant $C > 1$ depends on T if $a > 0$.

Now, using (3.2) and (3.3) we get the following lower bound.

Proposition 3.3. *Suppose that ϕ has no drift and satisfies $L_a(\gamma, C_L)$ for some $a \ge 0$. For every $M > 0$, there exist constants $c > 0$ and $\varepsilon \in (0,1/2)$ such that for every $x_0 \in \mathbb{R}^d$ and $r \in (0, a^{-1}M)$,*

$$p_{B(x_0,r)}(t,x,y) \ge c\,\frac{1}{(\Phi^{-1}(t))^d} \qquad \text{for } x,y \in B(x_0, \varepsilon\Phi^{-1}(t)) \text{ and } t \in (0, \Phi(\varepsilon r)]. \tag{3.4}$$

Proof. Since the proof for the case $a = 0$ is almost identical to the proof for the case $a > 0$, we will prove the proposition for the case $a > 0$ only. Fix $x_0 \in \mathbb{R}^d$ and let $B_r := B(x_0, r)$. The constant $\varepsilon \in (0, 1/2)$ will be chosen later. For $x, y \in B_{\varepsilon\Phi^{-1}(t)}$, we have $|x - y| \le 2\varepsilon\Phi^{-1}(t)$. So,

$$\frac{|x-y|^2}{2(\Phi^{-1}(t))^2} \le 2\varepsilon^2 \le 1/2. \tag{3.5}$$

Now combining (1.7), (3.2), (3.3) and (3.5) we have that for $x, y \in B_{\varepsilon\Phi^{-1}(t)}$ and $t \in (0, \Phi(\varepsilon r)]$,

$$\begin{aligned} p_{B_r}(t,x,y) \ge & C^{-1}\frac{e^{-2}}{(\Phi^{-1}(t))^d} \\ & - C\,\mathbb{E}_x\left[1_{\{\tau_{B_r}\le t\}}\left(\frac{1}{(\Phi^{-1}(t-\tau_{B_r}))^d} \wedge \frac{t-\tau_{B_r}}{|X_{\tau_{B_r}}-y|^d\Phi(|X_{\tau_{B_r}}-y|)}\right)\right]. \end{aligned} \tag{3.6}$$

Observe that

$$|X_{\tau_{B_r}} - y| \ge r - \varepsilon\Phi^{-1}(t) \ge (\varepsilon^{-1} - \varepsilon)\Phi^{-1}(t) \ge \Phi^{-1}(t), \quad \text{for all } t \in (0, \Phi(\varepsilon r)]$$

and so

$$\frac{t-\tau_{B_r}}{|X_{\tau_{B_r}}-y|^d\Phi(|X_{\tau_{B_r}}-y|)} \le \frac{t}{((\varepsilon^{-1}-\varepsilon)\Phi^{-1}(t))^d\Phi(\Phi^{-1}(t))} = \frac{(\varepsilon^{-1}-\varepsilon)^{-d}}{(\Phi^{-1}(t))^d}. \tag{3.7}$$

Consequently, we have from (3.6) and (3.7),

$$\begin{aligned} p_{B_r}(t,x,y) &\ge \frac{e^{-2}C^{-1}}{(\Phi^{-1}(t))^d} - C\frac{(\varepsilon^{-1}-\varepsilon)^{-d}}{(\Phi^{-1}(t))^d} \\ &\ge \left(e^{-2}C^{-1} - C(\varepsilon^{-1}-1)^{-d}\right)\frac{1}{(\Phi^{-1}(t))^d}. \end{aligned}$$

Choose $\varepsilon := ((2e^2C^2)^{1/d} + 1)^{-1} < 1/2$ so that $e^{-2}C^{-1} - C(\varepsilon^{-1} - \varepsilon)^{-d} \ge e^{-2}C^{-1}/2$. We now have $p_{B_r}(t, x, y) \ge 2^{-1}e^{-2}C^{-1}(\Phi^{-1}(t))^{-d}$ for $x, y \in B_{\varepsilon\Phi^{-1}(t)}$ and $t \in (0, \Phi(\varepsilon r)]$. □

Since for all $A > 0$

$$\mathbb{E}_x[\tau_{B(x,r)}] = \int_0^{A\Phi(r)} p_{B(x,r)}(t,x,y)dy + \sum_{k=0}^{\infty}\int_{A2^k\Phi(r)}^{A2^{k+1}\Phi(r)}\int_{B(x,r)} p_{B(x,r)}(t,x,y)dydt,$$

using (3.2) and (3.4) and the semigroup property, we can obtain that there exist constants $c_1, c_2 > 0$ such that

$$c_1\Phi(r) \le \mathbb{E}_x[\tau_{B(x,r)}] \le c_2\Phi(r), \quad x \in \mathbb{R}^d, r < 1. \tag{3.8}$$

We say (**UJS**) holds for J if there exists a positive constant c such that for every $y \in \mathbb{R}^d$,

$$J(y) \le \frac{c}{r^d}\int_{B(0,r)} J(y-z)dz \qquad \text{whenever } r \le |y|/2. \tag{\textbf{UJS}}$$

Proof of Theorem 3.2. Note that (**UJS**) always holds for our Lévy density $x \to j(|x|)$ since j is non-increasing. (see [9, page 1070]). Thus, using Proposition 3.3, (3.2) (for the case $a = 0$) and (**UJS**), we see that Theorem 3.2 for the case $a = 0$ is a special case of [24, Theorem 1.17 or Theorem 4.3 and (4.11)]. Moreover, using Proposition 3.3, (3.2) (for the case $a > 0$) and (**UJS**), the proof of Theorem 3.2 for the case $a > 0$ is almost identical to the proof for the case $a = 0$ in [24, Theorem 4.3]. We skip the details. □ .

For the remainder of this section, we use the convention that $\delta_D(\cdot) \equiv \infty$ when $D = \mathbb{R}^d$. For the next two results, D is an arbitrary nonempty open set.

Proposition 3.4. *Suppose that ϕ has no drift and satisfies $L_a(\gamma, C_L)$ for some $a \ge 0$. For every $T > 0$ and $b > 0$, there exists $c = c(a, T, b, \phi) > 0$ such that*

$$p_D(t,x,y) \ \ge \ c\,(\Phi^{-1}(t))^{-d}$$

for every $(t,x,y) \in (0, a^{-1}T) \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \ge b\Phi^{-1}(t) \ge 4|x-y|$.

Proof. Using Theorem 3.2, the proof for the case that ϕ satisfies $L_0(\gamma, C_L)$ is identical to that of [7, Proposition 3.4]. Even through the proof is similar, for reader's convenience we provide the proof for the case that ϕ satisfies $L_a(\gamma, C_L)$ for $a > 0$.

Without loss of generality we assume $a = 1$. We fix $b, T > 0$ and $(t,x,y) \in (0,T)\times D\times D$ satisfying $\delta_D(x) \wedge \delta_D(y) \ge b\Phi^{-1}(t) \ge 4|x-y|$. Since $|x-y| \le b\Phi^{-1}(t)/4$, we have

$$B(x, b\Phi^{-1}(t)/4) \subset B(y, b\Phi^{-1}(t)/2) \subset B(y, b\Phi^{-1}(t)) \subset D. \tag{3.9}$$

Thus, by the symmetry of p_D, Theorem 3.2 and Lemma 2.3(a), there exists $c_1 = c_1(b,T) > 0$ such that

$$p_{B(x,b\Phi^{-1}(t)/4)}(t/2,x,w) \ \le \ p_D(t/2,x,w) \ \le \ c_1 p_D(t,x,y) \quad \text{for every } w \in B(x, b\Phi^{-1}(t)/4).$$

This together with Lemma 2.5 yields that there exist $c_2, c_3 > 0$ such that

$$\begin{aligned} p_D(t,x,y) \ &\ge \ \frac{c_1^{-1}}{|B(x,b\Phi^{-1}(t)/4)|} \int_{B(x,b\Phi^{-1}(t)/4)} p_{B(x,b\Phi^{-1}(t)/4)}(t/2,x,w)dw \\ &= \ c_2(\Phi^{-1}(t))^{-d}\, \mathbb{P}_x\left(\tau_{B(x,b\Phi^{-1}(t)/4)} > t/2\right) \ \ge \ c_3\,(\Phi^{-1}(t))^{-d}. \end{aligned}$$

□

Proposition 3.5. *Suppose that ϕ has no drift and satisfies $L_a(\gamma, C_L)$ for some $a \ge 0$. For every $b, T > 0$, there exists a constant $c = c(a,b,T) > 0$ such that*

$$p_D(t,x,y) \ \ge \ c\,t\,j(|x-y|) \tag{3.10}$$

for every $(t,x,y) \in (0, a^{-1}T) \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \ge b\Phi^{-1}(t)$ and $b\Phi^{-1}(t) \le 4|x-y|$.

Proof. Again, using Proposition 3.4, the proof for the case that ϕ satisfies $L_0(\gamma, C_L)$ is the same as that of [7, Proposition 3.5], and for reader's convenience we provide the proof for the case that ϕ satisfies $L_a(\gamma, C_L)$ for $a > 0$.

Without loss of generality we assume $a = 1$. Throughout the proof we assume that $t \in (0,T)$. By Lemma 2.5, starting at $z \in B(y, (12)^{-1}b\Phi^{-1}(t))$, with probability at least $c_1 = c_1(b,T) > 0$ the process X does not move more than $(18)^{-1}b\Phi^{-1}(t)$ by time t. Thus, using the strong Markov property and the Lévy system in (2.6), we obtain

$$\begin{aligned} &\mathbb{P}_x\left(X_t^D \in B\big(y,\, 6^{-1}b\Phi^{-1}(t)\big)\right) \\ &\ge c_1 \mathbb{P}_x(X^D_{t\wedge\tau_{B(x,(18)^{-1}b\Phi^{-1}(t))}} \in B(y,\, (12)^{-1}b\Phi^{-1}(t)) \\ &\qquad\qquad\qquad \text{and } t\wedge\tau_{B(x,(18)^{-1}b\Phi^{-1}(t))} \text{ is a jumping time }) \\ &= c_1\mathbb{E}_x\left[\int_0^{t\wedge\tau_{B(x,(18)^{-1}b\Phi^{-1}(t))}} \int_{B(y,\,(12)^{-1}b\Phi^{-1}(t))} j(|X_s-u|)du ds\right]. \end{aligned} \tag{3.11}$$

Using the (**UJS**) property of j (see [9, page 1070]), we obtain

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{t\wedge\tau_{B(x,(18)^{-1}b\Phi^{-1}(t))}}\int_{B(y,\,(12)^{-1}b\Phi^{-1}(t))} j(|X_s-u|)duds\right]\\
=\;&\mathbb{E}_x\left[\int_0^t\int_{B(y,\,(12)^{-1}b\Phi^{-1}(t))} j(|X_s^{B(x,(18)^{-1}b\Phi^{-1}(t))}-u|)duds\right]\\
\geq\;& c_2\Phi^{-1}(t)^d\int_0^t \mathbb{E}_x\left[j(|X_s^{B(x,(18)^{-1}b\Phi^{-1}(t))}-y|)\right]ds\\
\geq\;& c_2\Phi^{-1}(t)^d\int_{t/2}^t\int_{B(x,(72)^{-1}b\Phi^{-1}(t/2))} j(|w-y|)p_{B(x,(18)^{-1}b\Phi^{-1}(t))}(s,x,w)dwds. \qquad (3.12)
\end{aligned}$$

Since, for $t/2<s<t$ and $w\in B(x,(72)^{-1}b\Phi^{-1}(t/2))$,

$$\delta_{B(x,(18)^{-1}b\Phi^{-1}(t))}(w)\geq(18)^{-1}b\Phi^{-1}(t)-(72)^{-1}b\Phi^{-1}(t/2)\geq 2^{-1}(18)^{-1}b\Phi^{-1}(s)$$

and

$$|x-w|<(72)^{-1}b\Phi^{-1}(t/2)\leq 4^{-1}(18)^{-1}b\Phi^{-1}(s),$$

we have by Proposition 3.4 that for $t/2<s<t$ and $w\in B(x,(72)^{-1}b\Phi^{-1}(t/2))$,

$$p_{B(x,(18)^{-1}b\Phi^{-1}(t))}(s,x,w)\geq c_3\,(\Phi^{-1}(s))^{-d}\geq c_3\,(\Phi^{-1}(t))^{-d}. \qquad (3.13)$$

Combining (3.11), (3.12) with (3.13) and applying (**UJS**) again, we get

$$\begin{aligned}
\mathbb{P}_x\left(X_t^D\in B\big(y,\,6^{-1}b\Phi^{-1}(t)\big)\right)\geq& c_4t\int_{B(x,(72)^{-1}b\Phi^{-1}(t/2))} j(|w-y|)dw\\
\geq& c_5t(\Phi^{-1}(t/2))^d j(|x-y|)\geq c_6t(\Phi^{-1}(t))^d j(|x-y|). \qquad (3.14)
\end{aligned}$$

In the last inequality we have used Lemma 2.3(a). Since by the semigroup property of p_D and Proposition 3.4,

$$\begin{aligned}
p_D(t,x,y)\;&=\;\int_D p_D(t/2,x,z)p_D(t/2,z,y)dz\\
&\geq\;\int_{B(y,\,b\Phi^{-1}(t/2)/6)} p_D(t/2,x,z)p_D(t/2,z,y)dz\\
&\geq\; c_7(\Phi^{-1}(t/2))^{-d}\,\mathbb{P}_x\left(X_{t/2}^D\in B(y,6^{-1}b\Phi^{-1}(t/2))\right),
\end{aligned}$$

the proposition now follows from this and (3.14). □

Recall that $B=(B_t:\,t\geq 0)$ is a Brownian motion in $\mathbb{R}^d$ and $S=(S_t:\,t\geq 0)$ a subordinator independent of B. Suppose that U is an open subset of $\mathbb{R}^d$. We denote by B^U the part process of B killed upon leaving U. The process $\{Z_t^U:t\geq 0\}$ defined by $Z_t^U=B_{S_t}^U$ is called a subordinate killed Brownian motion in U. Let $q_U(t,x,y)$ be the transition density of Z^U. Clearly, $Z_t^U=B_{S_t}$ for every $t\in[0,\zeta)$ where ζ is the lifetime of Z^U. Therefore we have

$$p_U(t,z,w)\geq q_U(t,z,w)\quad\text{for } (t,z,w)\in(0,\infty)\times U\times U. \qquad (3.15)$$

For a $C^{1,1}$ open set D in $\mathbb{R}^d$ with characteristics (R_0,Λ), consider a $z\in\partial D$ and a $C^{1,1}$-function $\varphi=\varphi_z:\mathbb{R}^{d-1}\to\mathbb{R}$ satisfying $\varphi(0)=0$, $\nabla\varphi(0)=(0,\dots,0)$, $\|\nabla\varphi\|_\infty\leq\Lambda$, $|\nabla\varphi(x)-\nabla\varphi(w)|\leq\Lambda|x-w|$ and an orthonormal coordinate system CS_z of $z=$

$(z_1, \cdots, z_{d-1}, z_d) := (\widetilde{z}, z_d)$ with origin at z such that $D \cap B(z, R_0) = \{y = (\tilde{y}, y_d) \in B(0, R_0)$ in $CS_z : y_d > \varphi(\widetilde{y})\}$. Define

$$\rho_z(x) := x_d - \varphi(\widetilde{x}) \text{ and } D_z(r_1, r_2) := \{y \in D : r_1 > \rho_z(y) > 0, |\widetilde{y}| < r_2\}, \quad r_1, r_2 > 0, \tag{3.16}$$

where $(\widetilde{x}, x_d)$ are the coordinates of x in CS_z. We also define

$$\kappa = \kappa(\Lambda) := (1 + (1 + \Lambda)^2)^{-1/2}. \tag{3.17}$$

It is easy to see that for every $z \in \partial D$ and $r \leq \kappa R_0$,

$$D_z(r, r) \subset D \cap B(z, r/\kappa). \tag{3.18}$$

It is well known (see, for instance [46, Lemma 2.2]) that there exists $L_0 = L_0$ $(R_0, \Lambda, d) > 0$ such that for every $z \in \partial D$ and $r \leq \kappa R_0$, one can find a $C^{1,1}$ domain $V_z(r)$ with characteristics $(rR_0/L_0, \Lambda L_0/r)$ such that $D_z(3r/2, r/2) \subset V_z(r) \subset D_z(2r, r)$. In this paper, given a $C^{1,1}$ open set D, $V_z(r)$ always refers to the $C^{1,1}$ domain above.

Proposition 3.6. *Suppose that ϕ has no drift and satisfies $L_a(\gamma, C_L)$ for some $a \geq 0$.*
(a) We assume that D is a connected $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ) such that the path distance of D is comparable to the Euclidean distance with characteristic λ_1. For any $T > 0$, there exist positive constants c_1 and c_2 depending on $R_0, \Lambda, \lambda_1, T, \phi, \gamma, C_L, a, b$ such that for every $(t, x, y) \in (0, T] \times D \times D$,

$$p_D(t, x, y) \geq c_1 \left(1 \wedge \frac{\delta_D(x)}{\Phi^{-1}(t)}\right) \left(1 \wedge \frac{\delta_D(y)}{\Phi^{-1}(t)}\right) \Phi^{-1}(t)^{-d} \exp\left(-\frac{c_2|x-y|^2}{\Phi^{-1}(t)^2}\right). \tag{3.19}$$

Moreover, there exist $c_3, c_4 > 0$ such that for all $z \in \partial D$, $r \leq \kappa R$ and $(t, x, y) \in (0, \Phi(r)] \times V_z(r) \times V_z(r)$,

$$p_{V_z(r)}(t, x, y) \geq c_3 \left(1 \wedge \frac{\delta_{V_z(r)}(x)}{\Phi^{-1}(t)}\right) \left(1 \wedge \frac{\delta_{V_z(r)}(y)}{\Phi^{-1}(t)}\right) \Phi^{-1}(t)^{-d} \exp\left(-\frac{c_4|x-y|^2}{\Phi^{-1}(t)^2}\right). \tag{3.20}$$

(b) Furthermore, if ϕ satisfies $L_0(\gamma, C_L)$ and D is a domain consisting of all the points above the graph of a bounded globally $C^{1,1}$ function, then (3.19) *holds for every $(t, x, y) \in (0, \infty) \times D \times D$.*

Proof. (a) Le $\rho \in (0,1)$ be the constant in Proposition 2.1. Without loss of generality we assume $T \geq \rho\phi(a)^{-1}$. Suppose that x and y are in D. Let $\widetilde{p}_D(t, z, w)$ be the transition density of B^D. By [26, Theorem 3.3] (see also [49, Theorem 1.2] where the comparability condition on the path distance in D with the Euclidean distance is used), there exist positive constants $c_1 = c_1(R_0, \Lambda, \lambda_0, T, \phi, \rho)$ and $c_2 = c_2(R_0, \Lambda, \lambda_0)$ such that for any $(s, z, w) \in (0, \phi^{-1}(\rho T^{-1})^{-1}] \times D \times D$,

$$\widetilde{p}_D(s, z, w) \geq c_1 \left(1 \wedge \frac{\delta_D(z)}{\sqrt{s}}\right) \left(1 \wedge \frac{\delta_D(w)}{\sqrt{s}}\right) s^{-d/2} e^{-c_2|z-w|^2/s}. \tag{3.21}$$

Recall that $q_D(t, x, y)$ is of the form

$$q_D(t, x, y) = \int_{(0,\infty)} \widetilde{p}_D(s, x, y) \mathbb{P}(S_t \in ds).$$

Using this and (3.21) we get

$$\begin{aligned}
&p_D(t,x,y) \ge q_D(t,x,y)\\
&\ge \int_{[2^{-1}\phi^{-1}(t^{-1})^{-1}, \phi^{-1}(\rho t^{-1})^{-1}]} \widetilde{p}_D(s,x,y)\mathbb{P}(S_t \in ds)\\
&\ge c_1 \int_{[2^{-1}\phi^{-1}(t^{-1})^{-1}, \phi^{-1}(\rho t^{-1})^{-1}]} \left(1 \wedge \frac{\delta_D(x)}{\sqrt{s}}\right)\left(1 \wedge \frac{\delta_D(y)}{\sqrt{s}}\right) s^{-d/2} e^{-c_2\frac{|x-y|^2}{s}} \mathbb{P}(S_t \in ds)\\
&\ge c_1 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{\phi^{-1}(\rho t^{-1})^{-1}}}\right)\left(1 \wedge \frac{\delta_D(y)}{\sqrt{\phi^{-1}(\rho t^{-1})^{-1}}}\right) \phi^{-1}(\rho t^{-1})^{d/2} e^{-2c_2|x-y|^2\phi^{-1}(t^{-1})}\\
&\quad \times \mathbb{P}\left(2^{-1}\phi^{-1}(t^{-1})^{-1} \le S_t \le \phi^{-1}(\rho t^{-1})^{-1}\right). \qquad (3.22)
\end{aligned}$$

Now, using (2.9) and Proposition 2.1, we conclude from (3.22) that

$$\begin{aligned}
&p_D(t,x,y)\\
\ge& c_3 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{\phi^{-1}(t^{-1})^{-1}}}\right)\left(1 \wedge \frac{\delta_D(y)}{\sqrt{\phi^{-1}(t^{-1})^{-1}}}\right) \phi^{-1}(t^{-1})^{d/2} e^{-2c_2|x-y|^2\phi^{-1}(t^{-1})}\\
=& c_3 \left(1 \wedge \frac{\delta_D(x)}{\Phi^{-1}(t)}\right)\left(1 \wedge \frac{\delta_D(y)}{\Phi^{-1}(t)}\right) \Phi^{-1}(t)^{-d} \exp\left(-2c_2 \frac{|x-y|^2}{\Phi^{-1}(t)^2}\right).
\end{aligned}$$

We have proved (3.19).

Using [40, (4.4)], we have that there exist $c_4, c_5 > 0$ such that for any $s \in (0, r^2]$ and any $z, w \in V_z(r)$,

$$\widetilde{p}_{V_z(r)}(s,z,w) \ge c_4 \left(1 \wedge \frac{\delta_{V_z(r)}(z)}{\sqrt{s}}\right)\left(1 \wedge \frac{\delta_{V_z(r)}(w)}{\sqrt{s}}\right) s^{-d/2} e^{-c_5|z-w|^2/s}. \qquad (3.23)$$

Since $t \le \Phi(r)$ if and only if $\phi^{-1}(t^{-1})^{-1} \le r^2$, we can repeat the proof of (3.19) and see that (3.20) holds true.

(b) Suppose that D is a domain consisting of all the points above the graph of a bounded globally $C^{1,1}$ function. Then by [46], (3.21) holds for all $(s,z,w) \in (0,\infty) \times D \times D$. Using this fact and the assumption $L_0(\gamma, C_L)$, one can follow the arguments in (a) line by line and prove (b). We skip the details. □

4 Key estimates

In this section we prove key estimates on exit distribution for X in $C^{1,1}$ open set with explicit decay rate.

Recall that $H(\lambda) = \phi(\lambda) - \lambda\phi'(\lambda)$, which is non-negative and non-decreasing on $(0,\infty)$. We remark here that H loses the information on the drift of ϕ.

Throughout this section we assume that H satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ for some $a > 0$ with $\delta < 2$ and the drift of the subordinator is zero.

Proposition 4.1. *For every $M > 0$ there exists $c = c(a, M) > 0$ such that for all $t > 0$ and $x \in B(0, M)$ satisfying $t\phi(|x|^{-2}) \le 1$ we have*

$$p(t,x) \ge ct|x|^{-d} H(|x|^{-2}).$$

Thus, for all $x \in B(0, M)$,

$$j(|x|) \ge c|x|^{-d} H(|x|^{-2}). \qquad (4.1)$$

Proof. The proof is just a combination of Proposition 3.5 and the proof of [43, Proposition 3.6]. We spell out the details for completeness. By [43, Proposition 2.8] there exist $L_1, L_2 > 1$ and $c_1 > 0$ such that for $|x| \le (aL_1)^{-1/2}$ and $t\phi(|x|^{-2}) \le 1$ it holds that

$$\mathbb{P}\left(|x|^2 \le S_t \le L_2|x|^2\right) \ge c_1 t H(|x|^{-2}) . \tag{4.2}$$

Without loss of generality, we assume that $M > (aL_1)^{-1/2}$ and consider the following two cases separately.
(1) $|x| \le (aL_1)^{-1/2}$ and $t\phi(|x|^{-2}) \le 1$: In this case, by (1.4) and (4.2) we obtain

$$\begin{aligned} p(t,x) &\ge (4\pi)^{-d/2} \int_{[|x|^2, L_2|x|^2]} s^{-d/2} e^{-\frac{|x|^2}{4s}} \mathbb{P}(S_t \in ds) \\ &\ge (4\pi)^{-d/2} L_2^{-d/2} |x|^{-d} e^{-1/4} \mathbb{P}(|x|^2 \le S_t \le L_2|x|^2) \\ &\ge c_1 (4\pi)^{-d/2} L_2^{-d/2} e^{-1/4} t|x|^{-d} H(|x|^{-2}) . \end{aligned}$$

(2) $(aL_1)^{-1/2} < |x| \le M$ and $t\phi(|x|^{-2}) \le 1$: In this case, $t \le \phi(|x|^{-2})^{-1} \le \phi(M^{-2})^{-1}$. Thus by Proposition 3.5 we obtain

$$p(t,x) \ge c_2 t j(|x|) \ge c_2 t j(M) \ge c_3 t|x|^{-d} H(|x|^{-2}) .$$

□

We now revisit [43] and improve the main result of [43] for the cases that H satisfies the lower and upper scaling conditions near infinity.

Theorem 4.2. *For every $T, M > 0$ there exists $c = c(a,T,M) > 0$ such that for all $t \le T$ and $x \in B(0,M)$,*

$$p(t,x) \ge c\left(\phi^{-1}(t^{-1})^{d/2} \wedge \left(t|x|^{-d} H(|x|^{-2}) + \phi^{-1}(t^{-1})^{d/2} e^{-2^{-1}|x|^2 \phi^{-1}(t^{-1})}\right)\right).$$

Proof. This theorem follows from Lemma 2.3(b), Propositions 2.8 and 4.1.

□

Let $T_A := \inf\{t > 0 : X_t \in A\}$, the first hitting time of X to A. Observe that for every Borel subset $A \subset U$ and $r > 0$, we have

$$\begin{aligned} &\mathbb{P}_x\left(T_A < \tau_U \wedge \Phi(r)\right) \ge \mathbb{P}_x\Big(\int_0^{\Phi(r)} \mathbf{1}_A(X_s^U) ds > 0\Big) \\ &\ge \frac{1}{\Phi(r)} \int_0^{\Phi(r)} \mathbb{P}_x\Big(\int_0^{\Phi(r)} \mathbf{1}_A(X_s^U) ds > u\Big) du \\ &= \frac{1}{\Phi(r)} \mathbb{E}_x \int_0^{\Phi(r)} \mathbf{1}_A(X_s^U) ds \ge \frac{1}{\Phi(r)} \int_0^{\Phi(r)} \int_A p_U(s,x,y) dy ds. \end{aligned} \tag{4.3}$$

Using Levy system, (2.11) and (3.8), we have that for $w \in \mathbb{R}^d$ and $0 < 4r \le R < 1$,

$$\begin{aligned} &\mathbb{P}_w\left(X_{\tau_{B(w,r)}} \in B(w,R)^c\right) \le \mathbb{E}_w[\tau_{B(w,r)}] \sup_{y \in B(w,r)} \int_{B(w,R)^c} j(|y-z|) dz \\ &\le \frac{c_1}{\phi(r^{-2})} \left(\int_R^1 H(s^{-2}) s^{-1} ds + c\right) \le c_2 \frac{H(R^{-2})}{\phi(r^{-2})} \le c_2 \frac{\phi(R^{-2})}{\phi(r^{-2})}. \end{aligned} \tag{4.4}$$

Now we prove the following estimate, which is inspired by the proof of [28, Lemma 5.3]. (See also [40, 41].) We recall that ρ_z, $D_z(r_1, r_2)$ and κ are defined in (3.16) and (3.17) respectively.

Proposition 4.3. *Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ open set with characteristics (R_0, Λ). Assume that H satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ and $\gamma > 2^{-1}\mathbf{1}_{\delta \geq 1}$ for some $a \geq 0$. Then there exists a constant $c = c(\phi, R_0, \Lambda) > 0$ such that for every $r \leq \kappa^{-1}(R_0 \wedge 1)/2$, $z \in D$ and $x \in D_z(2^{-3}r, 2^{-4}r)$,*

$$\mathbb{P}_x\big(X_{\tau_{D_z(r,r)}} \in D\big) \leq c\, \mathbb{P}_x\big(X_{\tau_{D_z(r,r)}} \in D_z(2r, r)\big).$$

Proof. Without loss of generality we assume $z = 0$. Let $E_2 := \{X_{\tau_{D_0(r,r)}} \in D\}$ and $E_1 := \{X_{\tau_{D_0(r,r)}} \in D_0(2r, r)\}$. We claim that $\mathbb{P}_x(E_2) \leq c_0 \mathbb{P}_x(E_1)$ for all $r \leq \kappa^{-1}(R_0 \wedge 1)/2$ and $x \in D_0(2^{-3}r, 2^{-4}r)$.

When $\delta < 1$, we use [30, Theorem 1.8] and get the claim immediately. Thus, throughout the proof we assume that $\delta \geq 1$.

Recall from the paragraph before Proposition 3.6 that, for $z \in \partial D$ and $r \leq \kappa R_0$, $V_0(r)$ is a $C^{1,1}$ domain with characteristics $(rR_0/L_0, \Lambda L_0/r)$ such that $D_0(3r/2, r/2) \subset V_0(r) \subset D_0(2r, r)$. Note that for $w \in D_0(2^{-3}r, 2^{-4}r)$, we have $\delta_{V_0(r)}(w) = \delta_D(w)$. Using this, (3.20) and (4.3), we have that for $w \in D_0(2^{-3}r, 2^{-4}r)$,

$$\begin{aligned}
&\mathbb{P}_w(E_1) \geq \mathbb{P}_w\big(\tau_{V_0(r)} > T_{D_0(5r/4, r/4) \setminus D_0(r, r/4))}\big) \\
&\geq \frac{1}{\Phi(r)} \int_{\Phi(r)/2}^{\Phi(r)} \int_{D_0(5r/4, r/4) \setminus D_0(r, r/4))} p_{V_0(r)}(s, w, y) dy ds \\
&\geq \frac{c_1}{\Phi(r)} \int_{\Phi(r)/2}^{\Phi(r)} \int_{D_0(5r/4, r/4) \setminus D_0(r, r/4))} \left(1 \wedge \frac{\delta_{V_0(r)}(w)}{\Phi^{-1}(s)}\right) \Phi^{-1}(s)^{-d} dy ds \\
&\geq c_2 \frac{\delta_D(w) r^d}{\Phi(r)} \int_{\Phi(r)/2}^{\Phi(r)} \frac{ds}{\Phi^{-1}(s)^{d+1}} \geq c_3 \frac{\delta_D(w)}{r}.
\end{aligned} \tag{4.5}$$

We define, for $i \geq 1$,

$$J_i = D_0(2^{-i-2}r, s_i) \setminus D_0(2^{-i-3}r, s_i), \quad s_i = \frac{1}{4}\left(\frac{1}{2} - \frac{1}{50}\sum_{j=1}^{i} \frac{1}{j^2}\right) r,$$

and $s_0 = s_1$. Note that $r/(10) < s_i < r/8$. For $i \geq 1$, set

$$d_i = d_i(r) = \sup_{z \in J_i} \mathbb{P}_z(E_2)/\mathbb{P}_z(E_1), \quad \widetilde{J}_i = D_0(2^{-i-2}r, s_{i-1}), \quad \tau_i = \tau_{\widetilde{J}_i}. \tag{4.6}$$

Repeating the argument leading to [40, (6.29)], we get that for $z \in J_i$ and $i \geq 2$,

$$\mathbb{P}_z(E_2) \leq \left(\sup_{1 \leq k \leq i-1} d_k\right) \mathbb{P}_z(E_1) + \mathbb{P}_z\left(X_{\tau_i} \in D \setminus \cup_{k=1}^{i-1} J_k\right). \tag{4.7}$$

For $i \geq 2$, define $\sigma_{i,0} = 0, \sigma_{i,1} = \inf\{t > 0 : |X_t - X_0| \geq 2^{-i-2}r\}$ and $\sigma_{i,m+1} = \sigma_{i,1} \circ \theta_{\sigma_{i,m}}$ for $m \geq 1$.

We first claim that for all $w \in \widetilde{J}_i$, $\mathbb{P}_w(X_{\sigma_{i,1}} \notin \widetilde{J}_i)$ is bounded below by a strictly positive constant. We prove the claim for $w \in \widetilde{J}_i \setminus D_0(2^{-i-3}r, s_{i-1}) = \{y \in D : 2^{-i-2}r > \rho_0(y) \geq 2^{-i-3}r, |\widetilde{y}| < s_{i-1}\}$. Since $\widetilde{J}_i = \{y = (\widetilde{y}, t) : 0 < t - \widehat{\varphi}(\widetilde{y}) < 2^{-i-2}r, \quad |\widetilde{y}| < s_{i-1}\}$ with $t := 2^{-i-2}r - y_d$ and $\widehat{\varphi}(\widetilde{y}) := -\varphi(\widetilde{y})$, the proof for the case $w \in D_0(2^{-i-3}r, s_{i-1})$ is same.

We choose $\varepsilon \in (0, 2^{-4}/\Lambda)$ small so that

$$(2\varepsilon^2 + 1)\left(\frac{3 + \varepsilon\Lambda}{1 - \Lambda\varepsilon}\right)^2 + 2\varepsilon^2 < 16. \tag{4.8}$$

Fix $w \in \widetilde{J}_i \setminus D_0(2^{-i-3}r, s_{i-1})$ and define $A := B((\widetilde{w}, w_d + 2^{-i-1}r), \varepsilon 2^{-i-4}r)$ and

$$V := B((\widetilde{w}, w_d - 2^{-i-4}r), 3 \cdot 2^{-i-2}r) \cap \{y_d > w_d - 2^{-i-4}r, |\widetilde{y} - \widetilde{w}| < \varepsilon(y_d - w_d + 2^{-i-4}r)\}.$$

For $y \in A$, we have $y_d - w_d \geq 2^{-i-1}r - |y_d - w_d - 2^{-i-1}r| > 2^{-i-1}r - \varepsilon 2^{-i-4}r > 2^{-i-2}r$. Thus, for $y \in A$ we have $y \notin B(w, 2^{-i-2}r)$, $|\widetilde{w} - \widetilde{y}| \leq \varepsilon 2^{-i-4}r < \varepsilon(y_d - w_d + 2^{-i-4}r)$ and

$$\rho_0(y) \geq y_d - w_d + \rho_0(w) - |\varphi(\widetilde{w}) - \varphi(\widetilde{y})| > (2^{-i-2} + (1-\varepsilon\Lambda)2^{-i-4})r > 2^{-i-2}r.$$

Therefore

$$A \subset V \setminus (\widetilde{J}_i \cup B(w, 2^{-i-2}r)). \tag{4.9}$$

If $y \in V \cap \widetilde{J}_i$ and $y_d < w_d$, then clearly $|y_d - w_d| = w_d - y_d \leq 2^{-i-4}r$. If $y \in V \cap \widetilde{J}_i$ and $y_d \geq w_d$, then $y_d - w_d = \rho_0(y) - \rho_0(w) + |\varphi(\widetilde{w}) - \varphi(\widetilde{y})| < 3 \cdot 2^{-i-4}r + \Lambda\varepsilon|y_d - w_d| + \Lambda\varepsilon 2^{-i-4}r$ so that $|y_d - w_d| < 2^{-i-4}r(3+\Lambda\varepsilon)/(1-\Lambda\varepsilon)$. Thus using (4.8), we have that for $y \in V \cap \widetilde{J}_i$,

$$\begin{aligned}
&|y-w|^2 \leq \varepsilon^2(|y_d - w_d| + 2^{-i-4}r)^2 + |y_d - w_d|^2 \leq (2\varepsilon^2+1)|y_d - w_d|^2 + 2\varepsilon^2(2^{-i-4}r)^2 \\
&\leq \left((2\varepsilon^2+1)\Big(\frac{3+\varepsilon\Lambda}{1-\Lambda\varepsilon}\Big)^2 + 2\varepsilon^2\right)(2^{-i-4}r)^2 < (2^{-i-2}r)^2,
\end{aligned}$$

which implies that

$$V \cap \widetilde{J}_i \subset B(w, 2^{-i-2}r) \tag{4.10}$$

On the other hand, for $y \in \frac{1}{2}A := B((\widetilde{w}, w_d + 2^{-i-1}r), \varepsilon 2^{-i-5}r)$, we have $\delta_V(w) \wedge \delta_V(y) \geq c_0 2^{-i-1}r$ and $|w - y| \leq 2^{-i}r$. Since we assume that $\gamma > 1/2$, we can find a large M so that

$$\frac{\Phi^{-1}(2s)}{\Phi^{-1}(s/M)} \leq c_4(2M)^{1/(2\gamma)} < \frac{Mc_0}{48} \text{ and } \frac{\Phi(s)}{\Phi(c_0 s)} \leq M \quad \text{for all } s \in (0,1).$$

Thus, when $\Phi(2^{-i-2}r)/2 \leq s \leq \Phi(2^{-i-2}r)$ and $|z_1 - z_2| \leq 3 \cdot 2^{-i}r/M$ with $\delta_V(z_i) \geq c_0 2^{-i-2}r$, we see that $|z_1 - z_2| \leq 12 \cdot 2^{-i-2}r/M \leq 12\Phi^{-1}(2s)/M \leq c_0\Phi^{-1}(s/M)/4$ and $\delta_V(z_i) \geq c_0 2^{-i-2}r \geq \Phi^{-1}(s/M)$ (because $M \geq \Phi(2^{-i-2}r)/\Phi(c_0 2^{-i-2}r) \geq s/\Phi(c_0 2^{-i-2}r)$). Thus, by Proposition 3.4, for such y, z and s, using this and a chaining argument through the semigroup property, we have

$$p_V(s, w, y) \geq c_6(2^{-i}r)^{-d}, \quad \text{for } \Phi(2^{-i-2}r)/2 \leq s \leq \Phi(2^{-i-2}r) \text{ and } y \in \frac{1}{2}A. \tag{4.11}$$

By (4.3) and (4.8)–(4.11), we have that for all $w \in \widetilde{J}_i \setminus D_0(2^{-i-3}r, s_{i-1})$,

$$\begin{aligned}
&\mathbb{P}_w\big(X_{\sigma_{i,1}} \notin \widetilde{J}_i\big) \geq \mathbb{P}_w(T_{\frac{1}{2}A} < \tau_V \wedge \Phi(2^{-i-2}r)) \\
&\geq \frac{1}{\Phi(2^{-i-2}r)} \int_{\Phi(2^{-i-2}r)/2}^{\Phi(2^{-i-2}r)} \int_{\frac{1}{2}A} p_V(s, w, y)\,dy\,ds \geq c_6 \frac{|\frac{1}{2}A|}{\Phi(2^{-i-2}r)} \int_{\Phi(2^{-i-2}r)/2}^{\Phi(2^{-i-2}r)} (2^{-i}r)^{-d}ds,
\end{aligned}$$

which is a positive constant independent of i. We have proved the claim.

Thus, we have that there exists $k_1 \in (0,1)$ such that

$$\mathbb{P}_w\big(X_{\sigma_{i,1}} \in \widetilde{J}_i\big) = 1 - \mathbb{P}_w\big(X_{\sigma_{i,1}} \notin \widetilde{J}_i\big) < k_1, \quad w \in \widetilde{J}_i. \tag{4.12}$$

For the purpose of further estimates, we now choose a positive integer $l \geq 1$ such that $k_1^l \leq 4^{-1}$. Next we choose $i_0 \geq 2$ large enough so that $2^{-i} < 1/(200li^3)$ for all $i \geq i_0$. Now we assume $i \geq i_0$. Using (4.12) and the strong Markov property we have that for $z \in J_i$,

$$\begin{aligned}
&\mathbb{P}_z(\tau_i > \sigma_{i,li}) \leq \mathbb{P}_z\big(X_{\sigma_{i,k}} \in \widetilde{J}_i, 1 \leq k \leq li\big) \\
&= \mathbb{E}_z\Big[\mathbb{P}_{X_{\sigma_{i,li-1}}}(X_{\sigma_{i,1}} \in \widetilde{J}_i) : X_{\sigma_{i,li-1}} \in \widetilde{J}_i, X_{\sigma_{i,k}} \in \widetilde{J}_i, 1 \leq k \leq li-2\Big] \\
&\leq \mathbb{P}_z\big(X_{\sigma_{i,k}} \in \widetilde{J}_i, 1 \leq k \leq li-1\big)k_1 \leq k_1^{li}.
\end{aligned} \tag{4.13}$$

Note that if $z \in J_i$ and $y \in D \setminus [\widetilde{J}_i \cup (\cup_{k=1}^{i-1} J_k)]$, then $|y-z| \geq (s_{i-1}-s_i) \wedge (2^{-3}-2^{-i-2})r = r/(200i^2)$. Furthermore, since $2^{-i-2}r < r/(200i^2)$, τ_i must be one of the $\sigma_{i,k}$'s, $k \leq li$. Hence, on $\{X_{\tau_i} \in D \setminus \cup_{k=1}^{i-1} J_k, \ \ \tau_i \leq \sigma_{i,li}\}$ with $X_0 = z \in J_i$, there exists k, $1 \leq k \leq li$, such that $|X_{\sigma_{i,k}} - X_0| = |X_{\tau_i} - X_0| > r/(200i^2)$. Thus for some $1 \leq k \leq li$,

$$\sum_{j=1}^{k} \left|X_{\sigma_{i,j}} - X_{\sigma_{i,j-1}}\right| > \frac{r}{200i^2}\,.$$

which implies for some $1 \leq k' \leq k \leq li$,

$$\left|X_{\sigma_{i,k'}} - X_{\sigma_{i,k'-1}}\right| \geq \frac{1}{k}\frac{r}{200i^2} \geq \frac{1}{li}\frac{r}{200i^2}.$$

Thus, using the strong Markov property and then using (4.4) (noting that $4 \cdot 2^{-i-2} < 1/(200li^3)$ for all $i \geq i_0$) we have

$$\begin{aligned}
&\mathbb{P}_z\left(X_{\tau_i} \in D \setminus \cup_{k=1}^{i-1} J_k, \ \ \tau_i \leq \sigma_{i,li}\right)\\
\leq &\sum_{k=1}^{li} \mathbb{P}_z\left(|X_{\sigma_{i,k}} - X_{\sigma_{i,k-1}}| \geq r/(200li^3), X_{\sigma_{i,k-1}} \in \widetilde{J}_i\right)\\
\leq &li \sup_{z\in \widetilde{J}_i} \mathbb{P}_z\left(|X_{\sigma_{i,1}} - z| \geq r/(200li^3)\right) \leq c_7 li \frac{\phi((200li^3)^2/r^2)}{\phi(2^{2(i+2)}r^{-2})}.
\end{aligned} \tag{4.14}$$

Since

$$\frac{\phi((200li^3)^2/r^2)}{\phi(2^{2(i+2)}r^{-2})} \geq c_8 \frac{(200li^3)^2}{(2^{2(i+2)})^2} \geq c_9 i^6 (4)^{-i},$$

by (4.5), (4.13), (4.14) and Lemma 2.3(b), for $z \in J_i$, $i \geq i_0$, we have

$$\begin{aligned}
&\frac{\mathbb{P}_z(X_{\tau_i} \in D \setminus \cup_{k=1}^{i-1} J_k)}{\mathbb{P}_z(E_1)} \leq \frac{1}{\mathbb{P}_z(E_1)}\left(k_1^{li} + c_{24} li \frac{\phi((200li^3)^2/r^2)}{\phi(2^{2(i+2)}r^{-2})}\right)\\
\leq &\frac{c_{10} i}{\mathbb{P}_z(E_1)} \frac{\phi((200li^3)^2/r^2)}{\phi(2^{2(i+2)}r^{-2})} \leq c_{11} i 2^i \frac{\phi((200li^3)^2/r^2)}{\phi(2^{2(i+2)}r^{-2})} \leq c_{12} i 2^i i^{6\gamma} (2^\gamma)^{-2i} \leq c_{13} i^{13} 2^{-(2\gamma-1)i}.
\end{aligned}$$

By this and (4.7), for $z \in J_i$ and $i \geq i_0$,

$$\frac{\mathbb{P}_z(E_2)}{\mathbb{P}_z(E_1)} \leq \sup_{1\leq k\leq i-1} d_k + \frac{\mathbb{P}_z(X_{\tau_i} \in D \setminus \cup_{k=1}^{i-1} J_k)}{\mathbb{P}_z(E_1)} \leq \sup_{1\leq k\leq i-1} d_k + c_{13} i^{13} 2^{-(2\gamma-1)i}.$$

This implies that

$$\sup_{r\leq \kappa^{-1}(R_0\wedge 1)/2} d_i(r) \leq \sup_{\substack{1\leq k\leq i_0-1\\ r\leq \kappa^{-1}(R_0\wedge 1)/2}} d_k(r) + c_{14} \sum_{k=i_0}^{\infty} k^{13} 2^{-(2\gamma-1)k} =: c_{15} < \infty.$$

Thus the claim above is valid, since $D_0(2^{-3}r, 2^{-4}r) \subset \cup_{k=1}^{\infty} J_k$. The proof is now complete. □

The next result should be well-known but we could not find any reference. We provide the full details.

Lemma 4.4. *For any non-negative locally integrable function* $t \to k(t)$ *on* $(0,\infty)$ *and every* $R > 0$, $s \in (0, R/2)$ *and* $\varepsilon \in (0, s/2)$,

$$\begin{aligned}
&\left(\int_{s+\varepsilon}^{R+s} + \int_{-R+s}^{s-\varepsilon}\right) ((t_+)^2 - s^2) k(|t-s|) dt\\
= &\int_{\varepsilon}^{R} (\mathbf{1}_{u<s} 2u^2 + \mathbf{1}_{u\geq s}(u^2 + s(2u-s))) k(|u|) du.
\end{aligned} \tag{4.15}$$

Thus,

$$P.V.\int_{-R+s}^{R+s}((t_+)^2-s^2)k(|t-s|)dt=\int_0^R(\mathbf{1}_{u<s}2u^2+\mathbf{1}_{u\geq s}(u^2+s(2u-s)))k(|u|)du.$$

Proof. Using the change of variables $u=t-s$ in the first integral and $u=s-t$ in the second integral, we get that for $\varepsilon\in(0,s/2)$,

$$\begin{aligned}&\left(\int_{s+\varepsilon}^{R+s}+\int_{-R+s}^{s-\varepsilon}\right)((t_+)^2-s^2)k(|t-s|)dt\\ =&\int_\varepsilon^R((s+u)^2-s^2)k(|u|)du+\int_\varepsilon^R([(s-u)_+]^2-s^2)k(|u|)du\\ =&\int_\varepsilon^R((s+u)^2+[(s-u)_+]^2-2s^2)k(|u|)du\\ =&\int_\varepsilon^s((s+u)^2+(s-u)^2-2s^2)k(|u|)du+\int_s^R((s+u)^2-2s^2)k(|u|)du\\ =&\int_\varepsilon^s 2u^2k(|u|)du+\int_s^R(u^2+s(2u-s))k(|u|)du.\end{aligned}$$

Letting $\varepsilon\to0$, we also have proved the second claim of the lemma. □

Lemma 4.5. *For every $R>0$ and $x=(\widetilde{0},x_d)\in\mathbb{R}^d$ with $x_d>0$,*

$$\begin{aligned}&\frac{1}{2d}\int_{B(0,R)}|z|^2j(|z|)dz\\ \leq& P.V.\int_{\{(\widetilde{w},w_d)\in\mathbb{R}^d:|\widetilde{w}|<R,|w_d-x_d|<R\}}([(w_d)_+]^2-x_d^2)j(|w-x|)dw \qquad (4.16)\\ \leq&\frac{1}{d}\int_{B(0,\sqrt{2}R)}|z|^2j(|z|)dz<\infty.\end{aligned}$$

Proof. By Lemma 4.4, for all small $\varepsilon\in(0,x_d/2)$,

$$\begin{aligned}&\int_{\{(\widetilde{w},w_d)\in\mathbb{R}^d:|\widetilde{w}|<R,|w_d-x_d|<R,|\widetilde{w}|^2+|w_d-x_d|^2>\varepsilon^2\}}([(w_d)_+]^2-x_d^2)j(|w-x|)dw\\ =&\int_{\{|\widetilde{w}|<R\}}\int_{\{\sqrt{(\varepsilon^2-|\widetilde{w}|^2)_+}<|w_d-x_d|<R\}}([(w_d)_+]^2-x_d^2)j((|w_d-x_d|^2+|\widetilde{w}|^2)^{1/2})dw_dd\widetilde{w}\\ =&\int_{\{|\widetilde{w}|<R\}}\int_{\sqrt{(\varepsilon^2-|\widetilde{w}|^2)_+}}^R(\mathbf{1}_{u<x_d}2u^2+\mathbf{1}_{u\geq x_d}(|u|^2+x_d(2u-x_d)))j((|u|^2+|\widetilde{w}|^2)^{1/2})dud\widetilde{w}.\end{aligned}$$

Thus by the monotone convergence theorem, (4.16) is equal to

$$\begin{aligned}&\frac12\int_{\{|\widetilde{w}|<R\}}\int_{-R}^R(\mathbf{1}_{|u|<x_d}2u^2+\mathbf{1}_{|u|\geq x_d}(u^2+x_d(2u-x_d)))j((|u|^2+|\widetilde{w}|^2)^{1/2})dud\widetilde{w}\\ &\geq\frac12\int_{B(0,R)}|u|^2j((|u|^2+|\widetilde{w}|^2)^{1/2})dud\widetilde{w}=\frac{1}{2d}\int_{B(0,R)}|z|^2j(|z|)dz.\end{aligned}$$

Since $x_d(2u-x_d)\leq u^2$, wee also have the upper bound as

$$\begin{aligned}&\frac12\int_{\{|\widetilde{w}|<R\}}\int_{-R}^R(\mathbf{1}_{|u|<x_d}2u^2+\mathbf{1}_{|u|\geq x_d}(u^2+x_d(2u-x_d)))j((|u|^2+|\widetilde{w}|^2)^{1/2})dud\widetilde{w}\\ &\leq\int_{B(0,\sqrt{2}R)}|u|^2j((|u|^2+|\widetilde{w}|^2)^{1/2})dud\widetilde{w}=\frac1d\int_{B(0,\sqrt{2}R)}|z|^2j(|z|)dz.\end{aligned}$$ □

Let $\psi(r)=1/H(r^{-2})$. We first note that $\Phi(r)\le\psi(r)$ and

$$c_1\Big(\frac{R}{r}\Big)^{2\gamma}\ \le\ \frac{\psi(R)}{\psi(r)}\ \le\ c_2\Big(\frac{R}{r}\Big)^{2\delta}\qquad \text{for every } 0<r<R<1. \tag{4.17}$$

Since

$$\int_0^r\frac{s}{\psi(s)}ds=\int_0^r sH(s^{-2})ds=\frac12\int_{r^{-2}}^\infty\frac{H(t)}{t^2}dt=-\frac12\int_{r^{-2}}^\infty(\frac{\phi(t)}{t})'dt=\frac{r^2}{2}\phi(r^{-2})=\frac{r^2}{2\Phi(r)},$$

Φ and ψ are also related as

$$\Phi(r)=\frac{r^2}{2\int_0^r\frac{s}{\psi(s)}ds}. \tag{4.18}$$

Using (4.1), (4.17) and (4.18), we get that for $R<1$,

$$\int_{B(0,R)}|z|^2j(|z|)dz\ge c_1^{-1}c_2(d)\int_0^R\frac{r}{\psi(r)}dr=\frac{c_2(d)}{2c_1}\frac{R^2}{\Phi(R)}, \tag{4.19}$$

and

$$\begin{aligned}&\int_{B(0,R)^c}j(|z|)dz\le c_2(d)(c_1\int_R^1\frac{dr}{r\psi(r)}+\int_1^\infty j(r)dr)=c_2(d)(\frac{c_1}{\psi(R)}\int_R^1\frac{\psi(R)}{r\psi(r)}dr+c_3)\\&\le\frac{c_2(d)c_1c_4(1-R^{2\delta})+c_3}{\psi(R)}\le\frac{c_2(d)c_1c_4+c_3}{\psi(R)}\le\frac{c_2(d)c_1c_4+c_3}{\Phi(R)}\end{aligned} \tag{4.20}$$

Choose

$$M_0:=4[c_1d(c_2(d)c_1c_4+c_3)/c_2(d)]^{1/2}>4. \tag{4.21}$$

By (4.19) and (4.20), if $r\le R/M_0$ then

$$r^2\int_{B(0,R)^c}j(|z|)dz\le R^2\frac{c_2(d)c_1c_4+c_3}{M_0^2\Phi(R)}\le\frac{c_2(d)}{8dc_1}\frac{R^2}{\Phi(R)}\le\frac{1}{4d}\int_{B(0,R)}|z|^2j(|z|)dz. \tag{4.22}$$

We use this constant M_0 in Lemma 4.6, Proposition 4.7 and Theorem 4.11 below.

For any function $f:\mathbb{R}^d\to\mathbb{R}$ and $x\in\mathbb{R}^d$, we define an operator as follows:

$$\begin{aligned}&\mathcal{L}f(x):=P.V.\int_{\mathbb{R}^d}(f(y)-f(x))j(|x-y|)dy,\\&\mathcal{D}(\mathcal{L}):=\left\{f\in C^2(\mathbb{R}^d):P.V.\int_{\mathbb{R}^d}(f(y)-f(x))j(|x-y|)dy\ \text{exists and is finite.}\right\}.\end{aligned}$$

Recall that $C_0^2(\mathbb{R}^d)$ is the collection of C^2 functions in $\mathbb{R}^d$ vanishing at infinity. It is well known that $C_0^2(\mathbb{R}^d)\subset\mathcal{D}(\mathcal{L})$ and that, by the rotational symmetry of X,

$$A|_{C_0^2(\mathbb{R}^d)}=\mathcal{L}|_{C_0^2(\mathbb{R}^d)} \tag{4.23}$$

where A is the infinitesimal generator of X. We also recall that $\delta_D(x)$ is the distance of the point x to D^c.

Lemma 4.6. *Suppose that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0,Λ). For any $z\in\partial D$ and $r\le(1\wedge R_0)/4$, we define*

$$f(y)=f_{r,z}(y):=(\delta_D(y))^2\mathbf{1}_{D\cap B(z,2r)}(y).$$

Then there exist $c=c(\phi,\Lambda,d)>1$ and $\widetilde{R}=\widetilde{R}(\phi,\Lambda,d)\in(0,(1\wedge R_0)/4)$ independent of z such that for all $r\le\widetilde{R}$, $\mathcal{L}f$ is well-defined in $D\cap B(z,r/M_0)$ and

$$c\frac{r^2}{\Phi(r)}\ge\mathcal{L}f(x)\ge c^{-1}\frac{r^2}{\Phi(r)}\qquad \textit{for all } x\in D\cap B(z,r/M_0). \tag{4.24}$$

Proof. Since the case of $d=1$ is easier, we give the proof only for $d\ge 2$. Without loss of generality we assume that $R<1$ and $\Lambda>1/R_0$. For $x\in D\cap B(z,r/M_0)$, choose $z_x\in\partial D$ be a point satisfying $\delta_D(x)=|x-z_x|$. Let φ be a $C^{1,1}$ function and $CS=CS_{z_x}$ be an orthonormal coordinate system with z_x chosen as the origin so that $\varphi(\widetilde{0})=0$, $\nabla\varphi(\widetilde{0})=(0,\dots,0)$, $\|\nabla\varphi\|_\infty\le\Lambda$, $|\nabla\varphi(\widetilde{y})-\nabla\varphi(\widetilde{z})|\le\Lambda|\widetilde{y}-\widetilde{z}|$, and $x=(\widetilde{0},x_d)$, $D\cap B(z_x,R_0)=\{y=(\widetilde{y},y_d)\in B(0,R_0)$ in $CS:y_d>\varphi(\widetilde{y})\}$. We fix the function φ and the coordinate system CS, and consider the truncated square function $[(y_d)_+]^2$ in CS. Let

$$\mathbb{B}_x=\mathbb{B}_x(r):=\{(\widetilde{w},w_d)\text{ in } CS:|\widetilde{w}|<r,|w_d-x_d|<r\}\subset B(z,2r),$$

and we define $\widehat{\varphi}:B(\widetilde{0},r)\to\mathbb{R}$ by $\widehat{\varphi}(\widetilde{y}):=2\Lambda|\widetilde{y}|^2$. Since $\nabla\varphi(\widetilde{0})=0$, by the mean value theorem we have $-2^{-1}\widehat{\varphi}(\widetilde{y})\le\varphi(\widetilde{y})\le 2^{-1}\widehat{\varphi}(\widetilde{y})$ for any $y\in D\cap B(x,r/2)$ and so that

$$\{z=(\widetilde{z},z_d)\in\mathbb{B}_x:z_d\ge\widehat{\varphi}(\widetilde{z})\}\subset D\cap\mathbb{B}_x\subset\{z=(\widetilde{z},z_d)\in\mathbb{B}_x:z_d\ge-\widehat{\varphi}(\widetilde{z})\}.$$

Let $A:=\{y\in\mathbb{B}_x:-\widehat{\varphi}(\widetilde{y})\le y_d\le\widehat{\varphi}(\widetilde{y})\}$ and $E:=\{y\in\mathbb{B}_x:y_d>\widehat{\varphi}(\widetilde{y})\}\subset D$ so that

$$\begin{aligned}&\int_{\mathbb{B}_x(r)}\left|[(y_d)_+]^2-(\delta_D(y))_+^2\right|j(|y-x|)dy\\ \le&\int_A(y_d^2+\delta_D(y)^2)j(|y-x|)dy+\int_E|y_d^2-\delta_D(y)^2|j(|y-x|)dy\\ \le&2^5\Lambda^2\int_A|\widetilde{y}|^4j(|\widetilde{y}|)dy+c_0r\int_E|y_d-\delta_D(y)|j(|y-x|)dy\end{aligned}\tag{4.25}$$

where we have used $y_d^2+\delta_D(y)^2\le 2(2\widehat{\varphi}(\widetilde{y}))^2=2(4\Lambda|\widetilde{y}|)^2$ for $y\in A$. We will show that the above is less than $c_1r^3/\Phi(r)$.

First, let $m_{d-1}(dy)$ be the Lebesgue measure on $\mathbb{R}^{d-1}$. Since $m_{d-1}(\{y:|\widetilde{y}|=s,-\widehat{\varphi}(\widetilde{y})\le y_d\le\widehat{\varphi}(\widetilde{y})\})\le c_2s^d$ for $0<s<r$, using polar coordinates for $|\widetilde{y}|=s$, by (4.18) and (2.11)

$$\int_A|\widetilde{y}|^4j(|\widetilde{y}|)dy\le r^3\int_A|\widetilde{y}|j(|\widetilde{y}|)dy\le c_3r^3\int_0^r\frac{s}{\psi(s)}ds=\frac{c_3r^5}{2\Phi(r)}.\tag{4.26}$$

Second, when $y\in E$, we have that $|y_d-\delta_D(y)|\le\Lambda|\widetilde{y}|$. Indeed, if $0<y_d\le\delta_D(y)$ and $y\in E$, $\delta_D(y)\le y_d+|\varphi(\widetilde{y})|\le y_d+\Lambda|\widetilde{y}|$. Since we assume that $\Lambda>1$, we have $|\widetilde{y}|^2+(R_0-y_d)^2<|\widetilde{y}|^2+(R_0-2\Lambda|\widetilde{y}|^2)^2<R^2$. Thus, if $y_d\ge\delta_D(y)$ and $y\in E$, using the interior ball condition, we have

$$\begin{aligned}&y_d-\delta_D(y)\le y_d-R_0+\sqrt{|\widetilde{y}|^2+(R_0-y_d)^2}\\ &=\frac{|\widetilde{y}|^2}{\sqrt{|\widetilde{y}|^2+(R_0-y_d)^2}+(R_0-y_d)}\ \le\ \frac{|\widetilde{y}|^2}{2(R_0-y_d)}\le\frac{|\widetilde{y}|^2}{R_0}\le\Lambda|\widetilde{y}|^2.\end{aligned}$$

Thus,

$$\int_E|y_d-\delta_D(y)|j(|y-x|)dy\le\Lambda\int_E|\widetilde{y}|^2j((|y_d-x_d|+|\widetilde{y}|)/2)dy_dd\widetilde{y}.\tag{4.27}$$

Since $E\subset\{(\widetilde{y},y_d):|\widetilde{y}|<r,\ \widehat{\varphi}(\widetilde{y})<y_d<\widehat{\varphi}(\widetilde{y})+2r\}$, using the polar coordinates for $|\widetilde{y}|=v$ and the change of the variable $s:=y_d-\widehat{\varphi}(v)$, we have by (2.11) and Lemma 2.3,

$$\int_E|\widetilde{y}|^2j((|y_d-x_d|+|\widetilde{y}|)/2)dy_dd\widetilde{y}\le c_4\int_0^r\int_0^{2r}\frac{dsdv}{\psi(v+|s+\widehat{\varphi}(v)-x_d|)}.\tag{4.28}$$

Using [39, Lemma 4.4] with non-increasing functions $f(s)\equiv 1$ and $g(s):=\psi(s)^{-1}$ and $x(r)=x_d-\widehat{\varphi}(r)$ and get

$$\int_0^r\int_0^{2r}\frac{dsdv}{\psi(v+|s+\widehat{\varphi}(v)-x_d|)}\le 2\int_0^{3r}(\int_0^u ds)\frac{du}{\psi(u)}\le\int_0^{3r}u\frac{du}{\psi(u)}.\tag{4.29}$$

Applying (4.26)–(4.29) to (4.25) and using (4.18), we have that

$$\int_{\mathbb{B}_x(r)} |[(y_d)_+]^2 - (\delta_D(y))_+^2| j(|y-x|)dy \le c_5 r \int_0^{3r} u \frac{du}{\psi(u)} \le c_6 r \frac{r^2}{\Phi(r)}. \tag{4.30}$$

On the other hand, since $x_d = \delta_D(x) \le r/M_0$, we see that $\mathcal{L}f(x)$ is well-defined and

$$\int_{\mathbb{B}_x(r)^c} (f(y) - x_d^2) j(|y-x|)dy \ge -x_d^2 \int_{\mathbb{B}_x(r)^c} j(|y-x|)dy \ge -(r/M_0)^2 \int_{B(0,r)^c} j(|z|)dz. \tag{4.31}$$

Thus, using our choice of the positive constant M_0, (4.19), (4.22) and Lemma 4.5, we have

$$\begin{aligned}\mathcal{L}f(x) &= P.V. \int_{\mathbb{B}_x(r)} (f(y) - x_d^2) j(|y-x|)dy + \int_{\mathbb{B}_x(r)^c} (f(y) - x_d^2) j(|y-x|)dy \\ &= P.V. \int_{\mathbb{B}_x(r)} ([(y_d)_+]^2 - x_d^2) j(|y-x|)dy + \int_{\mathbb{B}_x(r)^c} (f(y) - x_d^2) j(|y-x|)dy \\ &\quad + \int_{\mathbb{B}_x(r)} (f(y) - [(y_d)_+]^2) j(|y-x|)dy \qquad (4.32) \\ &\ge c_7 \frac{r^2}{\Phi(r)} - \int_{\mathbb{B}_x(r)} |[(y_d)_+]^2 - (\delta_D(y))_+^2| j(|y-x|)dy \ge (c_7 - rc_6) \frac{r^2}{\Phi(r)}. \qquad (4.33)\end{aligned}$$

Let $c_8 := (1 \wedge R_0 \wedge (c_7/c_6))/4$. Then, from (4.33) and (4.30) we conclude that for all $r \le c_8$, $z \in \partial D$ and $x \in D \cap B(z, r/M_0)$, $\mathcal{L}f(x) > 2^{-1} c_7 \frac{r^2}{\Phi(r)}$, and, by Lemma 4.5, (4.20), (4.30) and (4.32) we also have

$$\mathcal{L}f(x) \le c_9 \frac{r^2}{\Phi(r)} + r^2 \int_{\mathbb{B}_x(r)^c} j(|y-x|)dy + \int_{\mathbb{B}_x(r)} |f(y) - [(y_d)_+]^2| j(|y-x|)dy \le c_{10} \frac{r^2}{\Phi(r)}.$$

We have proved the lemma. □

Since (4.23) holds, we have Dynkin's formula for $\mathcal{L}$: for each $g \in C_c^2(\mathbb{R}^d)$ and any bounded open subset U of $\mathbb{R}^d$ we have

$$\mathbb{E}_x \int_0^{\tau_U} \mathcal{L}g(Z_t)dt = \mathbb{E}_x[g(Z_{\tau_U})] - g(x). \tag{4.34}$$

Note that, since H may not be comparable to ϕ, the next result can not be obtained using Lévy system and (4.1).

Proposition 4.7. *Suppose that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ). Let $\widetilde{R}$ be the constant in Lemma 4.6. There exists a constant $c > 0$ such that for any $z \in \partial D$, $r \le \widetilde{R}$, open set $U \subset D \cap B(z, r/M_0)$, and $x \in U$,*

$$\mathbb{P}_x(X_{\tau_U} \in B(z, 2r)) \ge c \frac{\mathbb{E}_x[\tau_U]}{\Phi(r)}.$$

Proof. Fix $z \in \partial D$, $r \le \widetilde{R}$ and an open set $U \subset D \cap B(z, r/M_0)$. Define $f(y) = (\delta_D(y))^2 \mathbf{1}_{D \cap B(z,2r)}(y)$. Then by Lemma 4.6, there exists $c_1 = c_1(\phi, \Lambda, d) \in (0,1)$ such that for all $r \le \widetilde{R}$ and $y \in D \cap B(z, r/M_0)$, $c_1^{-1} \frac{r^2}{\Phi(r)} \ge \mathcal{L}f(y) \ge c_1 \frac{r^2}{\Phi(r)}$. Let $v \ge 0$ be a smooth radial function such that $v(y) = 0$ for $|y| > 1$ and $\int_{\mathbb{R}^d} v(y)dy = 1$. For $k \ge 1$, define

$v_k(y) := 2^{kd}v(2^k y)$ and $f_r^{(k)} := v_k * f \in C_c^2(\mathbb{R}^d)$, and let $B_k := \{y \in U : \delta_U(y) \geq 2^{-k}\}$. We note that

$$\begin{aligned}&\int_{|w-y|>\varepsilon}(f_r^{(k)}(y) - f_r^{(k)}(w))j(|w-y|)dy\\ &= \int_{|u|<2^{-k}} v_k(u) \int_{|w-y|>\varepsilon} (f(y-u) - f(w-u))\, j(|w-y|)dydu.\end{aligned}$$

By letting $\varepsilon \downarrow 0$ and using the dominated convergence theorem, it follows that for $w \in B_k$ and all large k,

$$\mathcal{L}f_r^{(k)}(w) = \int_{|u|<2^{-k}} v_k(u)\mathcal{L}^u f(w)\, du \geq c_1 \frac{r^2}{\Phi(r)} \int_{|u|<2^{-k}} v_k(u)\, du = c_1 \frac{r^2}{\Phi(r)}.$$

Therefore, by the Dynkin's formula in (4.34) we have that for $x \in B_k$ and all large k,

$$c_1 r^2 \frac{\mathbb{E}_x[\tau_{B_k}]}{\Phi(r)} \leq \mathbb{E}_x \int_0^{\tau_{B_k}} \mathcal{L}f_r^{(k)}(X_s)ds \leq \mathbb{E}_x f_r^{(k)}(X_{\tau_{B_k}}).$$

By letting $k \to \infty$, for any $x \in U$, we conclude that

$$\mathbb{P}_x\left(X_{\tau_U} \in B(z, 2r)\right) \geq \frac{\mathbb{E}_x f(X_{\tau_U})}{\sup_{z \in \mathrm{supp}(f)\setminus U} f(z)} \geq c_1 \frac{\mathbb{E}_x[\tau_U]}{4\Phi(r)}.$$

□

Let X^d be the last coordinate of X and let L_t be the local time at 0 for $(\sup_{s\leq t} X_s^d) - X_t^d$. Using its right-continuous inverse L_s^{-1}, define the ascending ladder-height process as $H_s = X^d_{L_s^{-1}}$. We define V, the renewal function of the ascending ladder-height process H, as

$$V(x) = \int_0^\infty \mathbb{P}(H_s \leq x)ds, \quad x \in \mathbb{R}.$$

It is well-known that V is subadditive (see [1, p.74]). Note that, since the resolvent measure of X_t^d is absolutely continuous, by [45, Theorem 2], V is absolutely continuous and V and V' are harmonic for the process X_t^d on $(0,\infty)$. Thus, by the strong Markov property, $V((x_d)_+)$ and $V'((x_d)_+)$ are harmonic in the upper half space $\mathbb{R}_+^d := \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$ with respect to X. Furthermore, the function $V(r)$ is comparable to $\Phi(r)^{1/2}$ (see [4, Corollary 3]): there exists $c > 1$ such that

$$c^{-1}\Phi(r)^{1/2} \leq V(r) \leq c\Phi(r)^{1/2} \qquad \text{for any } \ r > 0. \tag{4.35}$$

Using [6, (2.23) and Lemma 3.5], we see that [30, Proposition 3.2] also holds in our setting. Moreover, if we assume (1.8), then we can use [42, Theorem 1] so that [30, Proposition 3.1] holds in our setting too. Therefore, by following the proof of [30, Proposition 3.3] line by line, we have the following.

Theorem 4.8. *Let $w(x) := V((x_d)_+)$. Suppose that* (1.8) *holds. Then, for any $x \in \mathbb{R}_+^d$, $\mathcal{L}w(x)$ is well-defined and $\mathcal{L}w(x) = 0$.*

We observe that, by a direct calculation using (4.18),

$$\left(\frac{s}{\Phi(s)^{1/2}}\right)' = \left(\left(\frac{s^2}{\Phi(s)}\right)^{1/2}\right)' = 2^{-1}\left(\frac{s^2}{\Phi(s)}\right)^{-1/2} 2\frac{s}{\psi(s)} = \frac{\Phi(s)^{1/2}}{\psi(s)}.$$

Thus, using this and the fact $\lim_{s\to 0} s\Phi(s)^{-1/2} = 0$ which also can be seen from (4.18), we have

$$\int_0^r \frac{\Phi(s)^{1/2}}{\psi(s)}ds = \int_0^r \left(\frac{s}{\Phi(s)^{1/2}}\right)' ds = \frac{r}{\Phi(r)^{1/2}}. \tag{4.36}$$

Lemma 4.9. *Assume that H satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ and $\gamma > 2^{-1}\mathbf{1}_{\delta\ge 1}$ for some $a \ge 0$. Let $\gamma_1 := \gamma \mathbf{1}_{\delta<1} + (2\gamma-1)\mathbf{1}_{\delta\ge 1} > 0$. There exist $c_1, c_2, c_3 > 0$ such that for all positive constants $R \le 1$ and $\lambda > 1$,*

$$\int_{R/\lambda}^{R} \frac{\Phi(t)^{1/2}}{\psi(t)t} dt \ge c_1(\lambda^{\gamma_1}-1)\frac{\Phi(R)^{1/2}}{\psi(R)} \ge c_2(\lambda^{\gamma_1}-1)R^{-\gamma_1}, \tag{4.37}$$

and

$$\int_{R}^{1} \frac{\Phi(t)^{1/2}}{\psi(t)t} dt \le c_3 \frac{\Phi(R)^{1/2}}{\psi(R)}. \tag{4.38}$$

Proof. If $\delta < 1$ then ψ and Φ are comparable near 0, thus, by (2.5) for $t \le R \le 1$,

$$\frac{\Phi(t)^{1/2}}{\Phi(R)^{1/2}} \frac{\psi(R)}{\psi(t)} \ge c_1 \frac{\Phi(R)^{1/2}}{\Phi(t)^{1/2}} \ge c_2 (t/R)^{-\gamma}.$$

By (4.17) and Lemma 2.3(a), if $\delta \ge 1$ then for $t \le R \le 1$,

$$\frac{\Phi(t)^{1/2}}{\Phi(R)^{1/2}} \frac{\psi(R)}{\psi(t)} \ge c_3 (t/R)(R/t)^{2\gamma} = c_3 (t/R)^{1-2\gamma}.$$

Thus, for $t \le R \le 1$,

$$\frac{\Phi(t)^{1/2}}{\Phi(R)^{1/2}} \frac{\psi(R)}{\psi(t)} \ge c_4 (t/R)^{-\gamma_1}. \tag{4.39}$$

Using (4.39) we have that for all $R \le 1$ and $\lambda > 1$,

$$\begin{aligned}
&\int_{R/\lambda}^{R} \frac{\Phi(t)^{1/2}}{\psi(t)t} dt = \frac{\Phi(R)^{1/2}}{\psi(R)} \int_{R/\lambda}^{R} \frac{\Phi(t)^{1/2}}{\Phi(R)^{1/2}} \frac{\psi(R)}{\psi(t)t} dt \ge c_4 \frac{\Phi(R)^{1/2}}{\psi(R)} R^{\gamma_1} \int_{R/\lambda}^{R} t^{-\gamma_1 - 1} dt \\
&= \frac{c_4}{\gamma_1} \frac{\Phi(R)^{1/2}}{\psi(R)} R^{\gamma_1} ((R/\lambda)^{-\gamma_1} - R^{-\gamma_1}) = \frac{c_4}{\gamma_1} (\lambda^{\gamma_1} - 1) \frac{\Phi(R)^{1/2}}{\psi(R)},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{R}^{1} \frac{\Phi(t)^{1/2}}{\psi(t)t} dt = \frac{\Phi(R)^{1/2}}{\psi(R)} \int_{R}^{1} \frac{\Phi(t)^{1/2}}{\Phi(R)^{1/2}} \frac{\psi(R)}{\psi(t)t} dt \le c_4^{-1} R^{\gamma_1} \frac{\Phi(R)^{1/2}}{\psi(R)} \int_{R}^{1} t^{-\gamma_1 - 1} dt \\
&= \frac{c_4^{-1}}{\gamma_1} R^{\gamma_1} (R^{-\gamma_1} - 1) \frac{\Phi(R)^{1/2}}{\psi(R)} \le \frac{c_4^{-1}}{\gamma_1} \frac{\Phi(R)^{1/2}}{\psi(R)}.
\end{aligned}$$

The second inequality in (4.37) also follows from (4.39) (with $R = 1$ and $t = R$). □

Proposition 4.10. *Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ open set with characteristics (R_0, Λ). Assume that (1.8) holds and that H satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $a > 0$, $\delta < 2$ and $\gamma > 2^{-1}\mathbf{1}_{\delta \ge 1}$. For any $z \in \partial D$ and $r \le 1 \wedge R_0$, we define*

$$h_r(y) = h_{r,z}(y) := V(\delta_D(y)) \mathbf{1}_{D \cap B(z,r)}(y).$$

Then, there exists $C_* = C_*(\phi, \Lambda, d) > 0$ independent of z such that $\mathcal{L}h_r$ is well-defined in $D \cap B(z, r/4)$ and

$$|\mathcal{L}h_r(x)| \le C_* \frac{\Phi(r)^{1/2}}{\psi(r)} \quad \text{for all } x \in D \cap B(z, r/4). \tag{4.40}$$

Proof. Since the case of $d = 1$ is easier, we give the proof only for $d \ge 2$. Without loss of generality we assume that $R_0 \le 1$ and $\Lambda > 1/R_0$.

For $x \in D \cap B(z, r/4)$, let $z_x \in \partial D$ be a point satisfying $\delta_D(x) = |x - z_x|$. Let φ be a $C^{1,1}$ function and $CS = CS_{z_x}$ be an orthonormal coordinate system with z_x as the origin so that $\varphi(\widetilde{0}) = 0$, $\nabla\varphi(\widetilde{0}) = (0, \ldots, 0)$, $\|\nabla\varphi\|_\infty \le \Lambda$, $|\nabla\varphi(\widetilde{y}) - \nabla\varphi(\widetilde{z})| \le \Lambda|\widetilde{y} - \widetilde{z}|$, and $x = (\widetilde{0}, x_d)$, $D \cap B(z_x, R_0) = \{y = (\widetilde{y}, y_d) \in B(0, R_0) \text{ in } CS : y_d > \varphi(\widetilde{y})\}$. We fix the function φ and the coordinate system CS, and define a function $g_x(y) = V(\delta_{\mathbb{R}^d_+}(y)) = V(y_d)$, where $\mathbb{R}^d_+ = \{y = (\widetilde{y}, y_d) \text{ in } CS : y_d > 0\}$ is the half space in CS.

Note that $h_r(x) = g_x(x)$, and that $\mathcal{L}(h_r - g_x) = \mathcal{L}h_r$ by Theorem 4.8. So, it suffices to show that $\mathcal{L}(h_r - g_x)$ is well defined and that there exists a constant $c_0 > 0$ independent of $x \in D \cap B(z, r/4)$ and $z \in \partial D$ such that

$$\int_{D \cup \mathbb{R}^d_+} |h_r(y) - g_x(y)| j(|x - y|) dy \le c_0 \frac{\Phi(r)^{1/2}}{\psi(r)}. \tag{4.41}$$

We define $\widehat{\varphi} : B(\widetilde{0}, r) \to \mathbb{R}$ by $\widehat{\varphi}(\widetilde{y}) := 2\Lambda|\widetilde{y}|^2$. Since $\nabla\varphi(\widetilde{0}) = 0$, by the mean value theorem we have $-2^{-1}\widehat{\varphi}(\widetilde{y}) \le \varphi(\widetilde{y}) \le 2^{-1}\widehat{\varphi}(\widetilde{y})$ for any $y \in D \cap B(x, r/2)$ and so that

$$\begin{aligned}\{z = (\widetilde{z}, z_d) \in B(x, r/2) : z_d \ge \widehat{\varphi}(\widetilde{z})\} &\subset D \cap B(x, r/2) \\ &\subset \{z = (\widetilde{z}, z_d) \in B(x, r/2) : z_d \ge -\widehat{\varphi}(\widetilde{z})\}.\end{aligned}$$

Let $A := \{y \in (D \cup \mathbb{R}^d_+) \cap B(x, r/4) : -\widehat{\varphi}(\widetilde{y}) \le y_d \le \widehat{\varphi}(\widetilde{y})\}$, $E := \{y \in B(x, r/4) : y_d > \widehat{\varphi}(\widetilde{y})\} \subset D$,

$$\begin{aligned}\mathrm{I} &:= \int_{B(x,r/4)^c} (h_r(y) + g_x(y)) j(|x - y|) dy = \int_{B(0,r/4)^c} (h_r(x+z) + g_x(x+z)) j(|z|) dz, \\ \mathrm{II} &:= \int_A (h_r(y) + g_x(y)) j(|x - y|) dy, \qquad \text{and} \qquad \mathrm{III} := \int_E |h_r(y) - g_x(y)| j(|x - y|) dy.\end{aligned}$$

First, since $h_r \le V(r)$ and $V(x_d + z_d) \le V(x_d) + V(|z|)$, we have

$$\begin{aligned}\mathrm{I} &\le V(r) \int_{B(0,r/4)^c} j(|z|) dz + \int_{B(0,r/4)^c} (V(x_d) + V(|z|)) j(|z|) dz \\ &\le c_1 V(r) \left(\int_{r/4}^1 j(s) s^{d-1} ds + 1 \right) + \left(\int_{r/4}^1 j(s) V(s) s^{d-1} ds + \int_1^\infty j(s) V(s) s^{d-1} ds \right) \\ &\le c_2 \left(\frac{\Phi(r)^{1/2}}{\psi(r)} + \int_{r/4}^1 \frac{\Phi(s)^{1/2} ds}{s\psi(s)} + 1 \right) \le c_3 \frac{\Phi(r)^{1/2}}{\psi(r)}\end{aligned} \tag{4.42}$$

In the second to last inequality above, we have used (2.11), (4.17), (4.35) and [6, Lemma 3.5]. In the last inequality above, we have used Lemma 4.9.

Second, let $m_{d-1}(dy)$ be the Lebesgue measure on $\mathbb{R}^{d-1}$. Since $m_{d-1}(\{y : |\widetilde{y}| = s, -\widehat{\varphi}(\widetilde{y}) \le y_d \le \widehat{\varphi}(\widetilde{y})\}) \le c_4 s^d$ for $0 < s < r/4$, and $h_r(y) + g_x(y) \le 2V(2\widehat{\varphi}(\widetilde{y})) \le 8(\Lambda + 1)V(|\widetilde{y}|)$, we get

$$\mathrm{II} \le 8(\Lambda + 1) \int_0^{r/4} \int_{|\widetilde{y}|=s} \mathbf{1}_A(y) V(|\widetilde{y}|) \nu(|\widetilde{y}|) m_{d-1}(dy) ds \le 8c_4(\Lambda + 1) \int_0^r V(s) j(s) s^d ds.$$

Thus, by (2.11), (4.35) and (4.36),

$$\mathrm{II} \le c_5 \int_0^r \frac{\Phi(s)^{1/2}}{\psi(s)} ds = c_5 \frac{r}{\Phi(r)^{1/2}} \le c_5 \frac{1}{\Phi(1)^{1/2}} = c_5. \tag{4.43}$$

Lastly, when $y \in E$, using $|y_d - \delta_D(y)| \le \Lambda|\widetilde{y}|$ (See the proof of Lemma 4.6.) we see that

$$\Lambda|\widetilde{y}| \le (y_d - \Lambda|\widetilde{y}|) \le y_d \wedge \delta_D(y) \quad \text{and} \quad y_d \vee \delta_D(y) - (y_d - \Lambda|\widetilde{y}|) \le 2\Lambda|\widetilde{y}|.$$

Thus we can the scale invariant Harnack inequality for X^d to V' (Theorem 3.2) and get

$$\begin{aligned} &|h_r(y) - g_x(y)| \le \Big(\sup_{u\in[y_d\wedge\delta_D(y),\, y_d\vee\delta_D(y)]} V'(u) \Big)|y_d - \delta_D(y)| \\ &\le \Big(\sup_{u\in[y_d-\Lambda|\widetilde{y}|,\, y_d\vee\delta_D(y)]} V'(u) \Big)|y_d - \delta_D(y)| \\ &\le c_6\Big(\inf_{u\in[y_d-\Lambda|\widetilde{y}|,\, y_d\vee\delta_D(y)]} V'(u) \Big)|y_d - \delta_D(y)| \le c_6 V'(y_d - \Lambda|\widetilde{y}|)\,|\widetilde{y}|^2. \end{aligned} \tag{4.44}$$

Since $V'(s) \le c_8 s^{-1}V(s) \le c_9 s^{-1}\Phi(s)^{1/2}$ by [42, Theorem 1] and (4.35), using (2.11) and the polar coordinates for $|\widetilde{y}| = v$ and the change of variable $s := y_d - \Lambda|v|$, we obtain

$$\begin{aligned} \mathrm{III} &\le c_6 \int_{\{(\widetilde{y},y_d):|\widetilde{y}|<r/4,\, \Lambda|\widetilde{y}|<y_d<2\Lambda|\widetilde{y}|+r/2\}} V'(y_d - \Lambda|\widetilde{y}|)|\widetilde{y}|^2 j(|x-y|)dy \\ &\le c_7 \int_0^{r/4}\int_0^{\Lambda r} \frac{V'(s)}{\psi((v^2+|s+\Lambda r - x_d|^2)^{1/2})}\frac{v^d}{(v^2+|s+\Lambda r - x_d|^2)^{d/2}}dy_d dv \\ &\le c_8 \int_0^{r/4}\int_0^{\Lambda r} \frac{s^{-1}\Phi(s)^{1/2}}{\psi((v^2+|s+\Lambda r - x_d|^2)^{1/2})}dy_d dv. \end{aligned}$$

Applying [39, Lemma 4.4] with non-increasing functions $s^{-1}\Phi(s)^{1/2}$ and $f(s) := \psi(s)^{-1}$ and $x(r) = x_d - \Lambda r$, we have that

$$\mathrm{III} \le c_9 \int_0^{2\Lambda r}\int_0^u \frac{\Phi(s)^{1/2}}{s} ds \frac{du}{\psi(u)} =: c_9\,\mathrm{IV}. \tag{4.45}$$

We claim that $\mathrm{IV} \le c_{10} < \infty$.

If $\delta < 1$ then $\psi(t)^{1/2}$, $\Phi(t)^{1/2}$ and $\int_0^u \frac{\Phi(s)^{1/2}}{s}ds$ are comparable near zero. Thus, by (2.5),

$$\mathrm{IV} \le c_{11}\int_0^{2\Lambda r}\Phi^{-1/2}(u)du \le c_{12}\int_0^{2\Lambda r} u^{-\delta}du \le c_{12}\int_0^{2\Lambda} u^{-\delta}du \le c_{13}.$$

If $\delta \ge 1$, using the assumption $\gamma > 2^{-1}$, we see from (4.17) that for $s < u < 2\Lambda r$,

$$\int_s^{2\Lambda r}\frac{\psi(s)}{\psi(u)}du \le c_{14}s^{2\gamma}\int_s^{2\Lambda r}u^{-2\gamma}du = \frac{c_{14}}{2\gamma-1}s^{2\gamma}(s^{1-2\gamma} - (2\Lambda r)^{1-2\gamma}) \le \frac{c_{14}}{2\gamma-1}s.$$

Thus, using (4.36) and the fact that $\frac{r}{\Phi(r)^{1/2}}$ is non-decreasing,

$$\begin{aligned} \mathrm{IV} &= \int_0^{2\Lambda r}\left(\int_s^{2\Lambda r}\frac{\psi(s)}{\psi(u)}du\right)\frac{\Phi(s)^{1/2}}{s\psi(s)}dsds \le \frac{c_{14}}{2\gamma-1}\int_0^{2\Lambda r}\frac{\Phi(u)^{1/2}}{\psi(u)}du \\ &= \frac{c_{14}}{2\gamma-1}\frac{2\Lambda r}{\Phi(2\Lambda r)^{1/2}} \le \frac{c_{14}}{2\gamma-1}. \end{aligned}$$

We have proved the claim $\mathrm{IV} \le c_{10} < \infty$. Combining (4.42)–(4.45) with this and using Lemma 4.9, we conclude that (4.41) holds. □

We are now ready to prove key estimates on exit probabilities.

Theorem 4.11. *Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ open set with characteristics (R_0, Λ). Assume that* (1.8) *holds and that H satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ and $\gamma > 2^{-1}\mathbf{1}_{\delta \geq 1}$ for some $a > 0$. Then there exist positive constants $R_* < (R_0 \wedge 1)/4$ and $c_1, c_2 > 1$ such that the following two estimates hold true.*
(a) For every $R \leq R_*$, $z \in \partial D$, open set $U \subset D \cap B(z, R)$ and $x \in U$,

$$\mathbb{E}_x [\tau_U] \geq c_1^{-1} \Phi(\delta_D(x))^{1/2} \Phi(R)^{1/2}. \tag{4.46}$$

(b) For every $R \leq R_*$, $z \in \partial D$ and $x \in D_z(2^{-3}R, 2^{-4}R)$,

$$\mathbb{E}_x \big[\tau_{D_z(R,R)}\big] \leq c_2 \Phi(R) \mathbb{P}_x \big(X_{\tau_{D_z(R,R)}} \in D_z(2R, R)\big) \leq c_1 c_2 \Phi\left(\delta_D(x)\right)^{1/2} \Phi(R)^{1/2}. \tag{4.47}$$

Proof. Fix $R \leq 1 \wedge R_0$ and without loss of generality, we assume $z = 0$. Define $h(y) = V(\delta_D(y)) \mathbf{1}_{D \cap B(0,R)}(y)$.

Using the same approximation argument in the proof of Proposition 4.7 and the Dynkin's formula, we have that, for every $\lambda \geq 4$, open set $U \subset D \cap B(0, \lambda^{-1}R)$ and $x \in U$,

$$\mathbb{E}_x \left[h_R \left(X_{\tau_U}\right)\right] + C_* \frac{\Phi(R)^{1/2}}{\psi(R)} \mathbb{E}_x \left[\tau_U\right] \geq V(\delta_D(x)) \geq \mathbb{E}_x \left[h_R \left(X_{\tau_U}\right)\right] - C_* \frac{\Phi(R)^{1/2}}{\psi(R)} \mathbb{E}_x \left[\tau_U\right],$$

where $C_* > 0$ is the constant in Proposition 4.10.

Since $j(|y-z|) \geq j(2|y|) \geq c_1 |y|^{-d} \psi(|y|)^{-1}$ for any $z \in D \cap B(0, \lambda^{-1}R)$ and $y \in D \cap (B(0,R) \setminus B(0, \lambda^{-1}R))$, by Lévy system we obtain

$$\begin{aligned} \mathbb{E}_x \left[h_R \left(X_{\tau_U}\right)\right] &\geq \mathbb{E}_x \int_{D \cap (B(0,R) \setminus B(0, \lambda^{-1}R))} \int_0^{\tau_U} j(|X_t - y|) dt h_R(y) dy \\ &\geq\ c_1\, \mathbb{E}_x \left[\tau_U\right] \int_{D \cap (B(0,R) \setminus B(0, \lambda^{-1}R))} |y|^{-d} \psi(|y|)^{-1} h_R(y) dy. \end{aligned}$$

Let $A := \{(\widetilde{y}, y_d) : 2\Lambda |\widetilde{y}| < y_d\}$. Since $y_d > 2\Lambda|\widetilde{y}| > 2\Lambda|\widetilde{y}|^2 > \varphi(\widetilde{y})$ for any $y \in A \cap B(0,R)$, we have $A \cap B(0,R) \subset D \cap B(0,R)$ and for any $y \in A \cap B(0,R)$,

$$\delta_D(y) \geq (2\Lambda)^{-1} \left(y_d - \varphi(\widetilde{y})\right) \geq (2\Lambda)^{-1} (y_d - \Lambda|\widetilde{y}|) > (4\Lambda)^{-1} y_d \geq (4\Lambda((2\Lambda)^{-2} + 1)^{1/2})^{-1} |y|.$$

By this and changing to polar coordinates with $|y| = t$ and (4.35), we obtain that

$$\begin{aligned} &\int_{D \cap (B(0,R) \setminus B(0, \lambda^{-1}R))} |y|^{-d} \psi(|y|)^{-1} h_R(y) dy \\ &\geq c_2 \int_{A \cap (B(0,R) \setminus B(0, \lambda^{-1}R))} |y|^{-d} \psi(|y|)^{-1} V(|y|) dy \geq c_3 \int_{\lambda^{-1}R}^{R} \frac{\Phi(t)^{1/2}}{\psi(t) t} dt. \end{aligned}$$

By Lemma 4.9, the above is great than $c_4(\lambda^{\gamma_1} - 1) \frac{\Phi(R)^{1/2}}{\psi(R)}$. Thus, we can use a λ_0 large (In fact, one can choose $\lambda_0 = (1 + c_1^{-1} c_4^{-1} 2C_*)^{-\gamma_1}$.) so that for all $\lambda \geq \lambda_0$, $R \in (0, 1 \wedge R_0)$ and for every open set $U \subset D \cap B(0, \lambda^{-1}R)$,

$$V(\delta_D(x)) \geq \mathbb{E}_x \left[h_R \left(X_{\tau_U}\right)\right] - C_* \frac{\Phi(R)^{1/2}}{\psi(R)} \mathbb{E}_x \left[\tau_U\right] \geq \frac{1}{2} \mathbb{E}_x \left[h_R \left(X_{\tau_U}\right)\right] \tag{4.48}$$

$$\text{and} \quad V(\delta_D(x)) \leq \mathbb{E}_x \left[h_R \left(X_{\tau_U}\right)\right] + C_* \frac{\Phi(R)^{1/2}}{\psi(R)} \mathbb{E}_x \left[\tau_U\right] \leq \frac{3}{2} \mathbb{E}_x \left[h_R \left(X_{\tau_U}\right)\right]. \tag{4.49}$$

By [7, Lemma 2.4], (4.49) and (4.35), we get

$$\begin{aligned} &\frac{2}{3} V(\delta_D(x)) \leq \mathbb{E}_x \left[h_R \left(X_{\tau_U}\right)\right] \leq V(R) \mathbb{P}_x \left(X_{\tau_U} \in D \cap B(0,R)\right) \\ &\leq c_5 V(R) \Phi(R)^{-1} \mathbb{E}_x [\tau_U] \leq c_6 \Phi(R)^{-1/2} \mathbb{E}_x [\tau_U]. \end{aligned} \tag{4.50}$$

Now, (4.46) follows from (4.35) and (4.50).

Let $\lambda_1 := \lambda_0 \vee M_0$ where M_0 is the constant in (4.21) and $U_1 := U_1(R) := D_0(\kappa\lambda_1^{-1}R, \kappa\lambda_1^{-1}R) \subset D \cap B(0, \lambda_1^{-1}R)$. Then, by (4.48) and Proposition 4.3 for all $x \in U_2 := D_0(2^{-3}\kappa\lambda_1^{-1}R, 2^{-4}\kappa\lambda_1^{-1}R)$

$$\begin{aligned}
&2V(\delta_D(x)) \geq \mathbb{E}_x\left[h_R\left(X_{\tau_{U_1}}\right)\mathbf{1}_{D_0(2\kappa\lambda_1^{-1}R,\kappa\lambda_1^{-1}R)}\right]\\
&\geq V(\kappa\lambda_1^{-1}R)\mathbb{P}_x\left(X_{\tau_{U_1}} \in D_0(2\kappa\lambda_1^{-1}R, \kappa\lambda_1^{-1}R)\right) \geq c_7\Phi(R)^{1/2}\mathbb{P}_x\left(X_{\tau_{U_1}} \in D\right).
\end{aligned} \tag{4.51}$$

Recall that $\widetilde{R}$ is the constant in Lemma 4.6 and Proposition 4.7. Applying Proposition 4.7 and (4.35) to (4.51), we conclude that for all $R \leq \widetilde{R}$ and all $x \in U_2$,

$$\begin{aligned}
\mathbb{E}_x[\tau_{U_1}] \leq c_8\Phi(R)\mathbb{P}_x\left(X_{\tau_{U_1}} \in D\right) &\leq c_9\Phi(R)\mathbb{P}_x\left(X_{\tau_{U_1}} \in D_0(2\kappa\lambda_1^{-1}R, \kappa\lambda_1^{-1}R)\right)\\
&\leq c_{10}\Phi(\delta_D(x))^{1/2}\Phi(R)^{-1/2}.
\end{aligned}$$

By taking $R_* = \widetilde{R}\lambda_1^{-1}\kappa$ we have proved (4.47). □

5 Upper bound estimates

In this section we discuss the upper bound of the Dirichlet heat kernels on $C^{1,1}$ open sets. *Throughout the remainder of this paper, we always assume that* (1.8) *holds, that* ϕ *has no drift and that* H *satisfies* $L_a(\gamma, C_L)$ *and* $U_a(\delta, C_U)$ *with* $\delta < 2$ *and* $\gamma > 2^{-1}\mathbf{1}_{\delta\geq 1}$ *for some* $a > 0$.

We first establish sharp estimates on the survival probability. Lemma 5.1 is proved in [6] when weak scaling order of characteristic exponent is strictly below 2. We emphasize here that results in [6] can not be used here.

Lemma 5.1. *Suppose* D *is a* $C^{1,1}$ *open set with the characteristic* (R_0, Λ). *Then for every* $T > 0$ *there exists* $C_1 = C_1(T, R_0, \Lambda) > 0$ *such that for* $t \in (0, T]$,

$$\mathbb{P}_x(\tau_D > t) \;\leq\; C_1\left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right), \quad \text{for all } x \in D, \tag{5.1}$$

and there exist $T_1 \in (0, \Phi(R_0)]$ *and* $C_2 > 0$ *such that for* $t \in (0, T_1]$,

$$\mathbb{P}_x(\tau_D > t) \geq C_2\left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right), \quad \text{for all } x \in D. \tag{5.2}$$

Proof. Recall thet $R_* > 0$ is the constant in Theorem 4.11. Let $b := \Phi(R_*/4)/T$ and $r_t := \Phi^{-1}(bt)$ for $t \leq T$ so that $r_t \leq R_*/4$. First note that, if $\delta_D(x) \geq 2^{-4}r_t$ then, by Lemma 2.5,

$$\mathbb{P}_x(\tau_D > t) \geq \mathbb{P}_x(\tau_{B(x,\delta_D(x))} > t) \geq \mathbb{P}_0(\tau_{B(0,2^{-4}r_t)} > t) = c_0 > 0. \tag{5.3}$$

We now assume that $\delta_D(x) < 2^{-4}r_t$. Let $z_x \in \partial D$ with $|x - z_x| = \delta_D(x)$. Then by [7, Lemma 2.4] and Theorem 4.11(b),

$$\begin{aligned}
&\mathbb{P}_x(\tau_D > t) = \mathbb{P}_x\left(\tau_{D\cap B(z_x,r_t)} = \tau_D > t\right) + \mathbb{P}_x\left(\tau_D > \tau_{D\cap B(z_x,r_t)} > t\right)\\
&\leq \mathbb{P}_x\left(\tau_{D\cap B(z_x,r_t)} > t\right) + \mathbb{P}_x\left(X_{\tau_{D\cap B(z_x,r_t)}} \in D\right)\\
&\leq t^{-1}\mathbb{E}_x\left[\tau_{D\cap B(z_x,r_t)}\right] + \mathbb{P}_x\left(X_{\tau_{D\cap B(z_x,r_t)}} \in D\right) \leq c_1\Phi(\delta_D(x))^{1/2}t^{-1/2}.
\end{aligned} \tag{5.4}$$

Recall that $D_z(r,r)$ is defined in (3.16). Let $U(x,t) := D_{z_x}(r_t,r_t)$. For the lower bound, we use the strong Markov property and Theorem 4.11(b) to get that for any $b \geq 1$ and $t \leq T/b$,

$$\begin{aligned}
&\mathbb{P}_x\left(\tau_D > bt\right) \\
\geq &\mathbb{P}_x\Big(\tau_{U(x,t)} < bt, X_{\tau_{U(x,t)}} \in D_{z_x}(2r_t,r_t), |X_{\tau_{U(x,t)}} - X_{\tau_{U(x,t)}+s}| \leq \frac{r_t}{4} \text{ for all } 0 < s < bt\Big) \\
\geq &\mathbb{P}_x\left(\tau_{U(x,t)} < bt, X_{\tau_{U(x,t)}} \in D_{z_x}(2r_t,r_t)\right) \mathbb{P}_0\left(\tau_{B_{r_t/4}} > bt\right) \\
\geq &\mathbb{P}_0\left(\tau_{B_{r_t/4}} > bt\right)\left(\mathbb{P}_x\big(X_{\tau_{U(x,t)}} \in D_{z_x}(2r_t,r_t)\big) - \mathbb{P}_x\big(\tau_{U(x,t)} \geq bt\big)\right) \\
\geq &\mathbb{P}_0\left(\tau_{B_{r_t/4}} > bt\right)\left(c_2 t^{-1}\mathbb{E}_x[\tau_{U(x,t)}] - b^{-1}t^{-1}\mathbb{E}_x\left[\tau_{U(x,t)}\right]\right). \qquad (5.5)
\end{aligned}$$

Take $b = \frac{2}{c_2} \vee 1$. Then, by Lemma 2.5 and Theorem 4.11(a) we have from (5.5) that for $t \leq T_0 := T/b$

$$\mathbb{P}_x\left(\tau_D > t\right) \geq \mathbb{P}_x\left(\tau_D > bt\right) \geq c_3 t^{-1}\mathbb{E}_x\left[\tau_{U(x,t)}\right] \geq \Phi(\delta_D(x))^{1/2} t^{-1/2}. \qquad (5.6)$$

Combining (5.3), (5.4) and (5.6), we have proved the lemma. □

Using [7, Lemmas 2.5 and 2.8], Lemma 5.1 and Theorem 4.11(b) we obtain the following upper bound of $p_D(t,x,y)$.

Lemma 5.2. *Suppose that D is a $C^{1,1}$ open set with characteristics (R_0,Λ). For each $T>0$, there exist constants $c = c(a,\phi,R_0,\Lambda,T) > 0$ and $a_0 = a_0(\phi,R_0,T) > 0$ such that for every $(t,x,y) \in (0,T] \times D \times D$ with $a_0\Phi^{-1}(t) \leq |x-y|$,*

$$\begin{aligned}
p_D(t,x,y) \leq &c\Big(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\Big)\Big(\sup_{(s,z):s\leq t, \frac{|x-y|}{2} \leq |z-y| \leq \frac{3|x-y|}{2}} p_D(s,z,y) \\
&+ \Big(\sqrt{t\Phi(\delta_D(y))} \wedge t\Big) j(|x-y|/3)\Big). \qquad (5.7)
\end{aligned}$$

Proof. Throughout the proof, we assume $t \in (0,T]$ and let $a_0 := 6R_*/\Phi^{-1}(T)$. Note that $a_0\Phi^{-1}(t)/6 \leq R_*$.

We first assume $\delta_D(x) \leq 2^{-7}a_0\Phi^{-1}(t)/3 \leq 2^{-7}|x-y|/3$ and let x_0 be a point on ∂D such that $\delta_D(x) = |x-x_0|$ and let $U_1 := B(x_0, a_0\Phi^{-1}(t)/(12)) \cap D$, $U_3 := \{z \in D : |z-x| > |x-y|/2\}$ and $U_2 := D \setminus (U_1 \cup U_3)$. Using Theorem 4.11(b) we have

$$\mathbb{E}_x\left[\tau_{U_1}\right] \leq \mathbb{E}_x\left[\tau_{D_{x_0}(a_0\Phi^{-1}(t)/(12), a_0\Phi^{-1}(t)/(12))}\right] \leq c_1\sqrt{t\Phi(\delta_D(x))}. \qquad (5.8)$$

Since $|z-x| > 2^{-1}|x-y| \geq a_0 2^{-1}\Phi^{-1}(t)$ for $z \in U_3$, we have for $u \in U_1$ and $z \in U_3$,

$$|u-z| \geq |z-x| - |x_0-x| - |x_0-u| \geq \frac{1}{2}|x-y| - \frac{1}{6}a_0\Phi^{-1}(t) \geq \frac{1}{3}|x-y|.$$

Thus, by the fact $U_1 \cap U_3 = \emptyset$ and the monotonicity of j,

$$\sup_{u\in U_1,\, z\in U_3} j(|u-z|) \leq \sup_{(u,z):|u-z|\geq \frac{1}{3}|x-y|} j(|u-z|) = j(|x-y|/3). \qquad (5.9)$$

On the other hand, for $z \in U_2$,

$$\frac{3}{2}|x-y| \geq |x-y| + |x-z| \geq |z-y| \geq |x-y| - |x-z| \geq \frac{|x-y|}{2} \geq a_0 2^{-1}\Phi^{-1}(t),$$

so

$$\sup_{s\le t,\, z\in U_2} p_D(s,z,y) \le \sup_{s\le t,\, \frac{|x-y|}{2}\le|z-y|\le\frac{3|x-y|}{2}} p_D(s,z,y). \tag{5.10}$$

Furtherover, by Lemma 5.1,

$$\begin{aligned}
&\int_0^t \mathbb{P}_x\left(\tau_{U_1}>s\right)\mathbb{P}_y\left(\tau_D>t-s\right)ds \le \int_0^t \mathbb{P}_x\left(\tau_D>s\right)\mathbb{P}_y\left(\tau_D>t-s\right)ds\\
&\le c_3\sqrt{\Phi(\delta_D(x))}\int_0^t s^{-1/2}\left(\sqrt{\frac{\Phi(\delta_D(y))}{t-s}}\wedge 1\right)ds\\
&\le c_4\sqrt{\Phi(\delta_D(x))}\left(\sqrt{\Phi(\delta_D(y))}\wedge\sqrt{t}\right).
\end{aligned}\tag{5.11}$$

Finally, applying [7, Lemma 2.5] and then (5.8), we have

$$\mathbb{P}_x\Big(X_{\tau_{U_1}}\in U_2\Big) \le \mathbb{P}_x\Big(X_{\tau_{U_1}}\in B(x_0, a\Phi^{-1}(t)/(12))^c\Big) \le \frac{c_5}{t}\,\mathbb{E}_x[\tau_{U_1}] \le c_6 t^{-1/2}\sqrt{\Phi(\delta_D(x))}.$$

Applying this and (5.8)–(5.11) to [7, Lemma 2.8] we conclude that

$$\begin{aligned}
p_D(t,x,y) &\le \left(\int_0^t \mathbb{P}_x\left(\tau_{U_1}>s\right)\mathbb{P}_y\left(\tau_D>t-s\right)ds\right)\sup_{u\in U_1,\, z\in U_3} j(|u-z|)\\
&\quad +\mathbb{P}_x\Big(X_{\tau_{U_1}}\in U_2\Big)\sup_{s\le t,\, z\in U_2} p_D(s,z,y)\\
&\le c_4\sqrt{\Phi(\delta_D(x))}\left(\sqrt{\Phi(\delta_D(y))}\wedge\sqrt{t}\right) j(|x-y|/3)\\
&\quad +c_6 t^{-1/2}\sqrt{\Phi(\delta_D(x))}\sup_{s\le t,\, \frac{|x-y|}{2}\le|z-y|\le\frac{3|x-y|}{2}} p_D(s,z,y).
\end{aligned}$$

If $\delta_D(x) > 2^{-7}a_0\Phi^{-1}(t)/3$, by Lemma 2.3(a),

$$\sqrt{\frac{\Phi(\delta_D(x))}{t}} \ge \sqrt{\frac{\Phi(a_0\Phi^{-1}(t)/(24))}{\Phi(\Phi^{-1}(t))}} \ge c_7 > 0.$$

Thus (5.7) is clear. Therefore we have proved (5.7). □

We now apply Lemma 5.2 to get the upper bound of the Dirichlet heat kernel.

Proof of Theorem 1.3(a): We will closely follow the argument in [7]. We fix $T>0$.

By [7, Lemma 2.7] and Proposition 2.6, for every $(t,x,y)\in(0,T]\times D\times D$,

$$p_D(t,x,y) \le c_1(\Phi^{-1}(t))^{-d}\left(\sqrt{\frac{\Phi(\delta_D(x))}{t}}\wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_D(y))}{t}}\wedge 1\right).$$

Recall that a_0 is the constant in Lemma 5.2. If $a_0\Phi^{-1}(t)\ge|x-y|$, by Proposition 3.4, $p(t,x-y)\ge c_2(\Phi^{-1}(t))^{-d}$. Thus for every $(t,x,y)\in(0,T]\times D\times D$ with $a_0\Phi^{-1}(t)\ge|x-y|$,

$$p_D(t,x,y) \le c_3\left(\sqrt{\frac{\Phi(\delta_D(x))}{t}}\wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_D(y))}{t}}\wedge 1\right)p(t,x-y). \tag{5.12}$$

We extend the definition of $p(t,w)$ by setting $p(t,w)=0$ for $t<0$ and $w\in\mathbb{R}^d$. For each fixed $x,y\in\mathbb{R}^d$ and $t>0$ with $|x-y|>8r$, one can easily check that $(s,w)\mapsto p(s,w-y)$ is a parabolic function in $(-\infty,\infty)\times B(x,2r)$. Suppose $\Phi^{-1}(t)\le|x-y|$ and let (s,z) with

$s \le t$ and $\frac{|x-y|}{2} \le |z-y| \le \frac{3|x-y|}{2}$. Then by Theorem 3.2, there is a constant $c_4 \ge 1$ so that for every $t \in (0, T]$,

$$\sup_{s \le t} p(s, z-y) \le c_4 p(t, z-y).$$

Using this and the monotonicity of $r \to p(t, r)$ we have

$$\sup_{s \le t, \frac{|x-y|}{2} \le |z-y| \le \frac{3|x-y|}{2}} p(s, z-y) \le c_4 \sup_{\frac{|x-y|}{2} \le |z-y| \le \frac{3|x-y|}{2}} p(t, z-y) = c_4 p(t, |x-y|/2). \quad (5.13)$$

Combining (5.13) and Lemma 5.2 and Proposition 3.5 and using the monotonicity of $r \to p(t, r)$, we have for every $(t, x, y) \in (0, T] \times D \times D$ with $a_0 \Phi^{-1}(t) \le |x-y|$,

$$\begin{aligned}
&p_D(t, x, y) \\
\le &c_5 \Big(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1 \Big) \Big(p(t, |x-y|/2) + \Big(\sqrt{t \Phi(\delta_D(y))} \wedge t \Big) j(|x-y|/3) \Big) \\
\le &c_6 \Big(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1 \Big) \left(p(t, |x-y|/2) + p(t, |x-y|/3) \right) \\
\le &2c_6 \Big(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1 \Big) p(t, |x-y|/3).
\end{aligned}$$

In view of (5.12), using the monotonicity of $r \to p(t, r)$ again, the last inequality in fact holds for all $(t, x, y) \in (0, T] \times D \times D$.

Thus by semigroup properties of p and p_D and the symmetry of $(x, y) \to p_D(t, x, y)$,

$$\begin{aligned}
p_D(t, x, y) = &\int_D p_D(t/2, x, z) p_D(t/2, y, z) dz \\
\le &c_7 \Big(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1 \Big) \Big(\sqrt{\frac{\Phi(\delta_D(y))}{t}} \wedge 1 \Big) \int_D p(t/2, |x-z|/3) p(t/2, |z-y|/3) dz \\
\le &c_8 \Big(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1 \Big) \Big(\sqrt{\frac{\Phi(\delta_D(y))}{t}} \wedge 1 \Big) \int_{\mathbb{R}^d} p(t/2, x/3, z) p(t/2, z, y/3) dz \\
= &c_8 \Big(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1 \Big) \Big(\sqrt{\frac{\Phi(\delta_D(y))}{t}} \wedge 1 \Big) p(t, |x-y|/3).
\end{aligned}$$

We have proved (1.10).

(1.11) follows from (1.10), Lemma 2.3 and Theorem 2.9 (applying to $p(t, |x-y|/3)$). □

6 Lower bound estimates

Recall that we always assume that (1.8) holds, that ϕ has no drift and that H satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ and $\gamma > 2^{-1} \mathbf{1}_{\delta \ge 1}$ for some $a > 0$.

Using Lemma 5.1 from Section 5, in this section we will prove Theorem 1.3(b). The main ideas in this section come from [7]. We first observe the following simple lemma.

Lemma 6.1. *The function* $\mathbb{H}(\lambda) := \sup_{t \in (0,1]} \mathbb{P}_0 \left(|X_t| > \lambda \Phi^{-1}(t) \right)$ *vanishes at* ∞*, that is,* $\lim_{\lambda \to \infty} \mathbb{H}(\lambda) = 0$.

Proof. By [7, Theorem 2.2] there exists a constant $c_1 = c_1(d) > 0$ such that

$$\mathbb{P}_0(|X_t| > r) \le c_1 \, t / \Phi(r) \quad \text{for } (t, r) \in (0, \infty) \times (0, \infty).$$

Noting $\phi^{-1}(t^{-1})^{1/2} = \Phi^{-1}(t)^{-1}$, the above inequality implies that

$$\sup_{t \in (0,1]} \mathbb{P}_0 \left(|X_t| > \lambda \Phi^{-1}(t) \right) \le c_1 \sup_{t \in (0,1]} \frac{t}{\Phi(\lambda \Phi^{-1}(t))} = c_1 \sup_{t \in (0,1]} t \phi(\lambda^{-2} \phi^{-1}(t^{-1})).$$

The condition $L_a(\gamma, C_L)$ and Remark 2.2 imply that for all $\lambda \geq 1$,

$$\sup_{t\in(0,1]} t\phi(\lambda^{-2}\phi^{-1}(t^{-1})) \leq \sup_{t\in(0,1]} \frac{\phi(\lambda^{-2}\phi^{-1}(t^{-1}))}{\phi(\phi^{-1}(t^{-1}))} \leq c_2\lambda^{-2\gamma},$$

which goes to zero as $\lambda \to \infty$. □

We now discuss some lower bound estimates of $p_D(t,x,y)$. We first note that by Lemma 5.1, there exist $C_3 \geq 1$ and $T_1 \in (0, 1 \wedge \Phi(R_0)]$ such that for all $x \in D$ and $t \in (0, T_1]$,

$$C_3^{-1}\left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right) \leq \mathbb{P}_x(\tau_D > t) \leq C_3 \left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right). \tag{6.1}$$

For $x \in D$ we use z_x to denote a point on ∂D such that $|z_x - x| = \delta_D(x)$ and $\mathbf{n}(z_x) := (x - z_x)/|z_x - x|$. By a simple geometric argument, one can easily see that

$$x + r\mathbf{n}(z_x) \in D \quad \text{for all } x \in D \text{ and } r \in [0, R_0/2]. \tag{6.2}$$

Lemma 6.2. *There exist $a_1 > 0$ and $M_1 > 1 \vee 4a_1$ such that for all $a \in (0, a_1]$, $x \in D$ and $t \in (0, T_1]$, we have that*

$$\mathbb{P}_x\left(X_t \in D \cap B(\xi_x^a(t), M_1\Phi^{-1}(t)) \text{ and } \Phi(\delta_D(X_t)) > at\right) \geq (2C_3)^{-1}\left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right)$$

where $\xi_x^a(t) := x + a\Phi^{-1}(t)\mathbf{n}(z_x)$ and C_3 and T_1 are the constants in (6.1).

Proof. By (1.10) and a change of variable, for every $a > 0$, $t \in (0, T_1]$ and $x \in D$,

$$\begin{aligned}
&\int_{\{u\in D:\Phi(\delta_D(u))\leq at\}} p_D(t,x,u)du \\
\leq &C_0\left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right)\int_{\{u\in D:\Phi(\delta_D(u))\leq at\}}\left(\sqrt{\frac{\Phi(\delta_D(u))}{t}} \wedge 1\right)p(t,|x-u|/3)du \\
\leq &C_0\sqrt{a}\left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right)\int_{\{u\in D:\Phi(\delta_D(u))\leq at\}} p(t,|x-u|/3)du \\
\leq &C_0\sqrt{a}\left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right)\int_{\mathbb{R}^d} p(t,|x-u|/3)du \\
= &C_0 3^d\sqrt{a}\left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right)\int_{\mathbb{R}^d} p(t,w)dw = C_0 3^d\sqrt{a}\left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right).
\end{aligned} \tag{6.3}$$

Choose $a_1 > 0$ small so that $C_0 3^d\sqrt{a_1} \leq (4C_3)^{-1}$ where C_3 is the constant in (6.1).

For the rest of the proof, we assume that $x \in D$, $a \in (0, a_1]$ and $t \in (0, T_1]$. Since $\xi_x^a(t) = x + a\Phi^{-1}(t)\mathbf{n}(z_x)$, for every $\lambda \geq 2a_1$ and $u \in D \cap B(\xi_x^a(t), \lambda\Phi^{-1}(t))^c$, we have

$$|x-u| \geq |\xi_x^a(t)-u| - |x-\xi_x^a(t)| \geq |\xi_x^a(t)-u| - a_1\Phi^{-1}(t) \geq (1-\frac{a_1}{\lambda})|\xi_x^a(t)-u| \geq \frac{1}{2}|\xi_x^a(t)-u|.$$

Thus using this, (1.10) and the monotonicity of $r \to p(t,r)$, we have that for every $\lambda \geq 2a_1$,

$$\begin{aligned}
&\int_{D\cap B(\xi_x^a(t),\lambda\Phi^{-1}(t))^c} p_D(t,x,u)du \\
\leq & C_0 \left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right) \int_{D\cap B(\xi_x^a(t),\lambda\Phi^{-1}(t))^c} p(t,|x-u|/3)du \\
\leq & C_0 \left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right) \int_{D\cap B(\xi_x^a(t),\lambda\Phi^{-1}(t))^c} p(t,|\xi_x^a(t)-u|/6)du \\
\leq & C_0 \left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right) \int_{B(0,\lambda\Phi^{-1}(t))^c} p(t,6^{-1}y)dy \\
\leq & C_0 6^d \mathbb{H}(6^{-1}\lambda) \left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right).
\end{aligned} \tag{6.4}$$

By Lemma 6.1, we can choose $M_1 > 1 \vee 4a_1$ large so that $C_0 6^d \mathbb{H}(6^{-1}M_1) < (4C_3)^{-1}$. Then by (6.1)–(6.4) and our choice of a_1 and M_1, we conclude that

$$\begin{aligned}
&\int_{\{u\in D\cap B(\xi_x^a(t),M_1\Phi^{-1}(t)):\Phi(\delta_D(u))>at\}} p_D(t,x,u)du \\
=&\int_D p_D(t,x,u)du - \int_{D\cap B(\xi_x^a(t),M_1\Phi^{-1}(t))^c} p_D(t,x,u)du - \int_{\{u\in D:\Phi(\delta_D(u))\leq at\}} p_D(t,x,u)du \\
\geq & (2C_3)^{-1}\left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1\right).
\end{aligned}$$

□

The next result is easy to check (see the proof of [20, Lemma 2.5] for a similar computation). We skip the proof.

Lemma 6.3. *For any given positive constants c_1, r_1, T and $r_2 > r_1$, there is a positive constant $c_2 = c_2(r_1, r_2, T, c_1, \phi)$ so that*

$$\phi^{-1}(t^{-1})^{d/2} e^{-c_1|x-y|^2\phi^{-1}(t^{-1})} \leq c_2 t r^{-d} H(r^{-2}) \qquad \text{for every } r_1 \leq r < r_2(a\wedge 1)^{-1}, t \in (0,T].$$

Proof of Theorem 1.3(b): It is clear that any bounded $C^{1,1}$ open set has the property that the path distance in any connected component of D is comparable to the Euclidean distance.

By (4.1) and [43, Proposition 3.6], we have

$$j(|x-y|) \geq c_0 |x-y|^{-d} H(|x-y|^{-2}), \quad \text{for all } x, y \in D \tag{6.5}$$

Recall that $a_1 > 0$ and $M_1 > 1 \vee 4a_1$ are the constants in Lemma 6.2 and C_3 and T_1 are the constants in (6.1). We also recall that for $x \in D$, $z_x \in \partial D$ such that $|z_x - x| = \delta_D(x)$ and $\mathbf{n}(z_x) = (x - z_x)/|z_x - x|$. Without loss of the generality we assume that $T > 3T_1$.

Let $a_2 := a_1 \wedge (2^{-1}R_0/\Phi^{-1}(T))$. For $x \in D$ and $t \in (0,T]$, let $\xi_x(t) := x + a_2\Phi^{-1}(t)\mathbf{n}(z_x)$. Note that $\xi_x(t) \in D$ by (6.2). Define

$$\mathcal{B}(x,t) := \left\{z \in D \cap B(\xi_x(t), M_1\Phi^{-1}(t)) : \delta_D(z) > a_2\Phi^{-1}(t)\right\}. \tag{6.6}$$

Observe that, we have

$$\delta_D(u) \wedge \delta_D(v) \geq a_2\Phi^{-1}(t), \quad \text{for every } (u,v) \in \mathcal{B}(x,t) \times \mathcal{B}(y,t), \tag{6.7}$$

and

$$|x-y| - 2a_2\Phi^{-1}(t) \leq |\xi_x(t) - \xi_y(t)| \leq |x-y| + 2a_2\Phi^{-1}(t), \tag{6.8}$$

Using (6.8) we also have that for every $(u,v) \in \mathcal{B}(x,t) \times \mathcal{B}(y,t)$,

$$\begin{aligned}
&|x-y| - \frac{5}{2} M_1 \Phi^{-1}(t) \le |x-y| - 2(M_1 + a_2)\Phi^{-1}(t) \le |u-v| \\
&\le |x-y| + |u - \xi_x(t)| + |v - \xi_y(t)| + 2a_2 \Phi^{-1}(t) \\
&\le |x-y| + 2(M_1 + a_2)\Phi^{-1}(t) \le |x-y| + 3M_1 \Phi^{-1}(t). \qquad (6.9)
\end{aligned}$$

Step1: Suppose $t \in (0, 3T_1]$ and x and y are in the same connected component. By the semigroup property of p_D,

$$\begin{aligned}
p_D(t,x,y) \ &\ge \int_{\mathcal{B}(y,t)} \int_{\mathcal{B}(x,t)} p_D(t/3,x,u) p_D(t/3,u,v) p_D(t/3,v,y)\, du dv \\
&\ge \left(\inf_{(u,v) \in \mathcal{B}(x,t) \times \mathcal{B}(y,t)} p_D(t/3,u,v) \right) \int_{\mathcal{B}(y,t)} p_D(t/3,x,u) du \int_{\mathcal{B}(x,t)} p_D(t/3,v,y) dv. \qquad (6.10)
\end{aligned}$$

When $|x-y| \le 3M_1\Phi^{-1}(t)$, by (6.7) and (6.9) $|u-v| \le 6M_1\Phi^{-1}(t)$ and $\delta_D(u) \wedge \delta_D(v) \ge a_2\Phi^{-1}(t)$ for $(u,v) \in \mathcal{B}(x,t) \times \mathcal{B}(y,t)$. Thus using Theorem 3.2 and Lemma 2.3(a) and Proposition 3.4, we get

$$p_D(t/3,u,v) \ge c_0 p_D(c_1 t, u, u) \ge c_2 \Phi^{-1}(t)^{-d} \quad \text{for every } (u,v) \in \mathcal{B}(x,t) \times \mathcal{B}(y,t). \qquad (6.11)$$

When $|x-y| > 3M_1\Phi^{-1}(t)$, we have by (6.9) that for $(u,v) \in \mathcal{B}(x,t) \times \mathcal{B}(y,t)$,

$$\frac{a_2}{4}\Phi^{-1}(t/3) \le \frac{1}{2} M_1 \Phi^{-1}(t) \le |u-v| \le (2|x-y|) \wedge (|x-y| + 3M_1\Phi^{-1}(T)).$$

Thus, by Lemma 2.3(a), Propositions 3.5 and 3.6(a) we have that for $|x-y| > 3M_1\Phi^{-1}(t)$ and $t \le 3T_1$,

$$\begin{aligned}
&\inf_{(u,v) \in \mathcal{B}(x,t) \times \mathcal{B}(y,t)} p_D(t/3,u,v) \\
\ge &\inf_{\substack{(u,v): 2^{-2} a_2 \Phi^{-1}(t/3) \le |u-v| \le (2|x-y|) \wedge (|x-y| + 3M_1\Phi^{-1}(T)) \\ \delta_D(u) \wedge \delta_D(v) > a_2 \Phi^{-1}(t/3)}} p_D(t/3,u,v) \\
\ge &c_3 \inf_{\substack{(u,v): \\ |u-v| \le (2|x-y|) \wedge (|x-y| + 3M_1\Phi^{-1}(T))}} \left(tj(|u-v|) + \phi^{-1}((t/3)^{-1})^{d/2} e^{-c_4 |u-v|^2 \phi^{-1}((t/3)^{-1})} \right) \\
\ge &c_5 \left(tj((2|x-y|) \wedge (|x-y| + 3M_1 \Phi^{-1}(T))) + \phi^{-1}(t^{-1})^{d/2} e^{-c_6 |x-y|^2 \phi^{-1}(t^{-1})} \right). \qquad (6.12)
\end{aligned}$$

We now apply Lemma 6.2, (6.12) and (6.11) to (6.10) and use (6.5) to obtain (1.12) for $t \le 3T_1$ and x and y in the same connected component.

Step2: Suppose $t \in (3T_1, T]$ and x and y are in the same connected component. By semigroup property of p_D and Lemma 6.2,

$$\begin{aligned}
p_D(t,x,y) \ &\ge \int_{\mathcal{B}(y,T_1)} \int_{\mathcal{B}(x,T_1)} p_D(T_1,x,u) p_D(t - 2T_1, u, v) p_D(T_1, v, y)\, du dv \\
&\ge \left(\inf_{(u,v) \in \mathcal{B}(x,T_1) \times \mathcal{B}(y,T_1)} p_D(t - 2T_1, u, v) \right) \int_{\mathcal{B}(y,T_1)} \int_{\mathcal{B}(x,T_1)} p_D(T_1,x,u) p_D(T_1,v,y)\, du dv \\
&\ge (2C_3)^{-2} \left(\inf_{(u,v) \in \mathcal{B}(x,T_1) \times \mathcal{B}(y,T_1)} p_D(t - 2T_1, u, v) \right) \left(\sqrt{\frac{\Phi(\delta_D(x))}{T_1}} \wedge 1 \right) \left(\sqrt{\frac{\Phi(\delta_D(y))}{T_1}} \wedge 1 \right) \\
&\ge (2C_3)^{-2} \left(\inf_{(u,v) \in \mathcal{B}(x,T_1) \times \mathcal{B}(y,T_1)} p_D(t - 2T_1, u, v) \right) \left(\sqrt{\frac{\Phi(\delta_D(x))}{t}} \wedge 1 \right) \left(\sqrt{\frac{\Phi(\delta_D(y))}{t}} \wedge 1 \right)
\end{aligned}$$
(6.13)

When $|x-y| \le 3M_1\Phi^{-1}(t)$, by (6.7) and (6.9) $|u-v| \le c_7\Phi^{-1}(T_1)$ and $\delta_D(u)\wedge\delta_D(v) \ge a_2\Phi^{-1}(T_1)$ for $(u,v)\in\mathcal{B}(x,T_1)\times\mathcal{B}(y,T_1)$. Thus using Theorem 3.2 and Lemma 2.3(a) and Proposition 3.4, we get that for every $(u,v)\in\mathcal{B}(x,T_1)\times\mathcal{B}(y,T_1)$,

$$p_D(t-2T_1,u,v) \ge c_8 p_D(c_9T_1,u,u) \ge c_{10}\Phi^{-1}(c_9T_1)^{-d} \ge c_{11}\Phi^{-1}(t)^{-d}. \tag{6.14}$$

When $|x-y| > 3M_1\Phi^{-1}(t)$, we have by (6.9) that for $(u,v)\in\mathcal{B}(x,T_1)\times\mathcal{B}(y,T_1)$,

$$\frac{a_2}{4}\Phi^{-1}(t-2T_1) \le \frac{1}{2}M_1\Phi^{-1}(t) \le |u-v| \le (2|x-y|)\wedge(|x-y|+3M_1\Phi^{-1}(T)).$$

Thus, by Lemma 2.3(a), Propositions 3.5 and 3.6(1) we have that for $|x-y|>3M_1\Phi^{-1}(t)$ and $3T_1<t\le T$,

$$\begin{aligned}
&\inf_{(u,v)\in\mathcal{B}(x,T_1)\times\mathcal{B}(y,T_1)} p_D(t-2T_1,u,v)\\
&\ge \inf_{\substack{(u,v):2^{-2}a_2\Phi^{-1}(t-2T_1)\le|u-v|\le(2|x-y|)\wedge(|x-y|+3M_1\Phi^{-1}(T_1))\\ \delta_D(u)\wedge\delta_D(v)>a_2\Phi^{-1}(T_1)}} p_D(t-2T_1,u,v)\\
&\ge \inf_{\substack{(u,v):2^{-2}a_2\Phi^{-1}(t-2T_1)\le|u-v|\le(2|x-y|)\wedge(|x-y|+3M_1\Phi^{-1}(T))\\ \delta_D(u)\wedge\delta_D(v)>a_2\Phi^{-1}((a_2T_1/T)(t-2T_1))}} p_D(t-2T_1,u,v)\\
&\ge c_{12} \inf_{\substack{(u,v):\\ |u-v|\le(2|x-y|)\wedge(|x-y|+3M_1\Phi^{-1}(T))}} \Big(tj(|u-v|)+\phi^{-1}((t/3)^{-1})^{d/2}e^{-c_{13}|u-v|^2\phi^{-1}((t/3)^{-1})}\Big)\\
&\ge c_{14}\Big(tj((2|x-y|)\wedge(|x-y|+3M_1\Phi^{-1}(T)))+\phi^{-1}(t^{-1})^{d/2}e^{-c_{15}|x-y|^2\phi^{-1}(t^{-1})}\Big).
\end{aligned} \tag{6.15}$$

Combining (6.13) and (6.15) and using (6.5) we obtain (1.12) for $t\in(3T_1,T]$ and x and y are in the same connected component.
Step3: Suppose $t\in(0,T]$ and x and y are in different connected components. We use Proposition 6.4. Then, thanks to (6.5) and Lemma 6.3, we see that (1.12) still holds. □

Note that, in the proof of Theorem 1.3(b), the assumptions that D is connected and the path distance in D is comparable to the Euclidean distance, are only used to apply Proposition 3.6. Thus, following the proof of Theorem 1.3(b) without applying Proposition 3.6, we have the following.

Proposition 6.4. *For every $C^{1,1}$ open set D and $T>0$, there exist constants $c>0, M_1>1$ such that for every $(t,x,y)\in(0,T]\times D\times D$,*

$$\begin{aligned}
&p_D(t,x,y)\\
&\ge c\left(\sqrt{\frac{\Phi(\delta_D(x))}{t}}\wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_D(y))}{t}}\wedge 1\right)\begin{cases} t\dfrac{H(|x-y|^{-2})}{|x-y|^d} & \text{if } |x-y|>3M_1\Phi^{-1}(t),\\ \Phi^{-1}(t)^{-d} & \text{if } |x-y|\le 3M_1\Phi^{-1}(t).\end{cases}
\end{aligned}$$

Proof of Theorem 1.3(c): Since D is bounded and j is non-increasing, Theorem 1.3(a) and Proposition 6.4 imply that for every $(x,y)\in D\times D$,

$$c^{-1}\Phi(\delta_D(x))^{1/2}\Phi(\delta_D(y))^{1/2} \le p_D(1,x,y) \le c\Phi(\delta_D(x))^{1/2}\Phi(\delta_D(y))^{1/2}.$$

Using this, the proof of Theorem 1.3(c) is almost identical to that of [19, Theorems 1.3(iii)], so we omit the proofs. □

Proof of Theorem 1.4. Either by the proof of Theorem 1.3 or by applying the main result in [43] and our Propositions 3.5 and 3.6(1) to [7, Theorem 4.1 and 4.5], the theorem holds true when D is an upper half space $\{x=(\widetilde{x},x_d)\in\mathbb{R}^d : x_d>0\}$. Then using the "push inward" method of [25] (see also [5, Theorem 5.8]) and our short time heat kernel estimates in Theorem 1.3, one can obtain global sharp two-sided Dirichlet heat kernel estimates when D is a domain consisting of all the points above the graph of a bounded globally $C^{1,1}$ function. We skip the proof since it would be almost identical to the one of [5, Theorem 5.8]. □

7 Green function estimates

In next two sections we use the notation $f(x) \asymp g(x), x \in I$, which means that there exist constants $c_1, c_2 > 0$ such that $c_1 f(x) \le g(x) \le c_2 g(x)$ for $x \in I$.

Recall that $\Phi(r) = (\phi(1/r^2))^{-1}$ where ϕ is the Laplace exponent ϕ of the subordinator S. When ϕ satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ for some $a > 0$, Green function estimates for the corresponding subordinate Brownian motion were already discussed in [19]. In this section we discuss Green function estimates when ϕ has no drift and that H satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ and $\gamma > 2^{-1}\mathbf{1}_{\delta \ge 1}$ for some $a > 0$.

By the exactly same proof as the one of [19, Lemma 7.1], we have the following.

Lemma 7.1. *For every $r \in (0, 1]$ and every open subset U of $\mathbb{R}^d$,*

$$\frac{1}{2}\left(1 \wedge \frac{r^2 \Phi(\delta_U(x))^{1/2}\Phi(\delta_U(y))^{1/2}}{\Phi(|x-y|)}\right) \le \left(1 \wedge \frac{r\Phi(\delta_U(x))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)\left(1 \wedge \frac{r\Phi(\delta_U(y))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)$$
$$\le 1 \wedge \frac{r^2 \Phi(\delta_U(x))^{1/2}\Phi(\delta_U(y))^{1/2}}{\Phi(|x-y|)}. \tag{7.1}$$

Since ϕ has no drift and satisfies $L_a(\gamma, C_L)$, by [35, Lemma 1.3] for every $M > 0$, we have

$$r\Phi'(r) \asymp \Phi(r) \quad \text{for } r \in (0, M]. \tag{7.2}$$

Note that, by Lemma 2.4, for every $T > 0$, there exists $C_T > 1$ such that

$$\frac{\Phi^{-1}(r)}{\Phi^{-1}(R)} \ge C_T^{-1}\left(\frac{r}{R}\right)^{1/(2\gamma)} \quad \text{for } 0 < r \le R \le T. \tag{7.3}$$

Moreover, by Lemma 2.3,

$$\frac{\Phi^{-1}(r)}{\Phi^{-1}(R)} \le \left(\frac{r}{R}\right)^{1/2} \quad \text{for } 0 < r \le R < \infty. \tag{7.4}$$

Recall $x_+ = x \vee 0$.

Lemma 7.2. *For $T, b, r > 0$ and $d = 1, 2$, set*

$$h_{T,d}(b,r) = b + \Phi(r)\int_{\Phi(r)/T}^{1}\left(1 \wedge \frac{ub}{\Phi(r)}\right)\frac{1}{u^2(\Phi^{-1}(u^{-1}\Phi(r)))^d}\,du + \frac{\Phi(r)}{r^d}\left(1 \wedge \frac{b}{\Phi(r)}\right). \tag{7.5}$$

Then, for $0 < r \le \Phi^{-1}(T/2)$ and $0 < b \le T/2$,

$$h_{T,d}(b,r) \asymp \frac{b}{r^d} \wedge \left(\frac{b}{\Phi^{-1}(b)^d} + \left(\int_r^{\Phi^{-1}(b)} \frac{\Phi(s)}{s^{d+1}}ds\right)_+\right).$$

Proof. (a) The lemma for $d = 1$ is given in [19, Lemma 7.2] under the assumption that ϕ satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with for some $a > 0$ $\delta < 2$. Using (7.4) instead of the assumption $U_a(\delta, C_U)$ with $\delta < 2$, the proof of (a) is the same as the that of [19, Lemma 7.2].

(b) We now assume that $d=2$. Using (7.2)–(7.4), the proof is a simple modification of the one of [19, Lemma 7.2]. We provide the proof in details for the readers' convenience.

For (b,r) with $0<b<\Phi(r)\le T/2$,

$$\begin{aligned} h_{T,2}(b,r) &\asymp b + b\int_{\Phi(r)/T}^1 \frac{du}{u(\Phi^{-1}(u^{-1}\Phi(r)))^2} + \frac{b}{r^2} \\ &= b + \frac{b}{r^2}\int_{\Phi(r)/T}^1 \left(\frac{\Phi^{-1}(\Phi(r))}{\Phi^{-1}(u^{-1}\Phi(r))}\right)^2 u^{-1}du + \frac{b}{r^2}. \end{aligned}$$

Since $\Phi(r)\le T/2$, by (7.3)–(7.4) we have

$$0<c_2=c_1^{-1}\int_{1/2}^1 u^{\frac{1}{\gamma}-1}du \le \int_{\Phi(r)/T}^1 \left(\frac{\Phi^{-1}(\Phi(r))}{\Phi^{-1}(u^{-1}\Phi(r))}\right)^2 u^{-1}du \le c_1\int_0^1 du = c_1<\infty.$$

Thus, for $0<b<\Phi(r)\le T/2$, we have

$$h_{T,2}(b,r)\asymp \frac{b}{r^2}. \tag{7.6}$$

On the other hand, using the change of variable $u=\Phi(r)/\Phi(s)$ and integration by parts, we have that for (b,r) with $\Phi(r)\le b\le T/2$,

$$\begin{aligned} &h_{T,2}(b,r) \\ =&b+\Phi(r)\int_{\Phi(r)/b}^1 \frac{du}{u^2(\Phi^{-1}(u^{-1}\Phi(r)))^2} + b\int_{\Phi(r)/T}^{\Phi(r)/b} \frac{du}{u(\Phi^{-1}(u^{-1}\Phi(r)))^2} + \frac{\Phi(r)}{r^2} \\ =&b+\int_r^{\Phi^{-1}(b)} \frac{\Phi'(s)}{s^2}ds + b\int_{\Phi^{-1}(b)}^{\Phi^{-1}(T)} \frac{\Phi'(s)}{s^2\Phi(s)}ds + \frac{\Phi(r)}{r^2} \\ =&b+\left(\frac{b}{\Phi^{-1}(b)^2}-\frac{\Phi(r)}{r^2}\right) + 2\int_r^{\Phi^{-1}(b)} \frac{\Phi(s)}{s^3}ds + b\int_{\Phi^{-1}(b)}^{\Phi^{-1}(T)} \frac{\Phi'(s)}{s^2\Phi(s)}ds + \frac{\Phi(r)}{r^2} \\ =&b+\frac{b}{\Phi^{-1}(b)^2} + 2\int_r^{\Phi^{-1}(b)} \frac{\Phi(s)}{s^3}ds + b\int_{\Phi^{-1}(b)}^{\Phi^{-1}(T)} \frac{\Phi'(s)}{s^2\Phi(s)}ds. \end{aligned} \tag{7.7}$$

Since $b\le T/2$, by (7.4) and the fact that Φ^{-1} is increasing,

$$\frac{1}{\Phi^{-1}(b)^2}-\frac{1}{\Phi^{-1}(T)^2}\asymp \frac{1}{\Phi^{-1}(b)^2}\ge c_4 \tag{7.8}$$

for some $c_4>0$. Using (7.2) and (7.8) in the second integral in (7.7), we get that for (b,r) with $\Phi(r)\le b\le T/2$,

$$\begin{aligned} h_{T,2}(b,r) &\asymp& b+\frac{b}{\Phi^{-1}(b)^2}+\int_r^{\Phi^{-1}(b)}\frac{\Phi(s)}{s^3}ds + b\int_{\Phi^{-1}(b)}^{\Phi^{-1}(T)}\frac{1}{s^3}ds \\ &=& b+\frac{b}{\Phi^{-1}(b)^2}+\int_r^{\Phi^{-1}(b)}\frac{\Phi(s)}{s^3}ds + \frac{b}{2}\left(\frac{1}{\Phi^{-1}(b)^2}-\frac{1}{\Phi^{-1}(T)^2}\right) \\ &\asymp& \frac{b}{\Phi^{-1}(b)^2}+2\int_r^{\Phi^{-1}(b)}\frac{\Phi(s)}{s^3}ds. \end{aligned} \tag{7.9}$$

Since $\Phi(s)$ is an increasing function, when $0<\Phi(r)\le b$, we have

$$\frac{b}{\Phi^{-1}(b)^2}+2\int_r^{\Phi^{-1}(b)}\frac{\Phi(s)}{s^3}ds \le \frac{b}{\Phi^{-1}(b)^2}+2b\int_r^{\Phi^{-1}(b)}\frac{1}{s^3}ds = \frac{b}{r^2},$$

while when $\Phi(r) \geq b > 0$,

$$\frac{b}{\Phi^{-1}(b)^2} + \left(\int_r^{\Phi^{-1}(b)} \frac{\Phi(s)}{s^3} ds\right)_+ = \frac{b}{\Phi^{-1}(b)^2} \geq \frac{b}{r^2}.$$

Thus combining this with (7.6) and (7.9) we establishes the lemma. □

Recall that the Green function $G_D(x,y)$ of X on D is defined as

$$G_D(x,y) = \int_0^\infty p_D(t,x,y)dt.$$

As an application of Theorems 1.3 and 1.4, we derive the sharp two sided estimates on the Green functions of X on bounded $C^{1,1}$ open sets. For notational convenience, let

$$a(x,y) := \sqrt{\Phi(\delta_D(x))}\sqrt{\Phi(\delta_D(y))} \tag{7.10}$$

and

$$g(x,y) := \begin{cases} \dfrac{\Phi(|x-y|)}{|x-y|^d}\left(1 \wedge \dfrac{\Phi(\delta_D(x))}{\Phi(|x-y|)}\right)^{1/2}\left(1 \wedge \dfrac{\Phi(\delta_D(y))}{\Phi(|x-y|)}\right)^{1/2}, & \text{when } d > 2 \\ \dfrac{a(x,y)}{|x-y|^d} \wedge \left(\dfrac{a(x,y)}{\Phi^{-1}(a(x,y))^d} + \left(\displaystyle\int_{|x-y|}^{\Phi^{-1}(a(x,y))} \dfrac{\Phi(s)}{s^{d+1}} ds\right)^+\right), & \text{when } d \leq 2. \end{cases} \tag{7.11}$$

Theorem 7.3. *Assume that* (1.8) *holds, that ϕ has no drift and that H satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $\delta < 2$ and $\gamma > 2^{-1}\mathbf{1}_{\delta \geq 1}$ for some $a > 0$. Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$, $d \geq 1$, with characteristics (R_0, Λ).*

(i) *There exists $c_1 > 0$ depending only on diam$(D), R_0, \Lambda, d$ and ϕ such that*

$$G_D(x,y) \geq c_1 \frac{\Phi(|x-y|)}{|x-y|^d}\left(1 \wedge \frac{\Phi(\delta_D(x))}{\Phi(|x-y|)}\right)^{1/2}\left(1 \wedge \frac{\Phi(\delta_D(y))}{\Phi(|x-y|)}\right)^{1/2}, \quad x,y \in D.$$

(ii) *There exists $c_2 > 0$ depending only on diam$(D), R_0, \Lambda, d$ and ϕ such that*

$$G_D(x,y) \leq c_2 \frac{a(x,y)}{|x-y|^d}, \quad x,y \in D.$$

(iii) *If D is connected, then*

$$G_D(x,y) \asymp g(x,y), \quad x,y \in D.$$

Proof. Put $T = 2\Phi(\mathrm{diam}(D))$.
(i) Let $M_1 > 0$ be the constant in Proposition 6.4 with our T. By Proposition 6.4 for every $(t,x,y) \in (0,T] \times D \times D$ with $|x-y| \leq 3M_1\Phi^{-1}(t)$,

$$p_D(t,x,y) \geq c_1 \left(1 \wedge \frac{\Phi(\delta_D(x))}{t}\right)^{1/2}\left(1 \wedge \frac{\Phi(\delta_D(y))}{t}\right)^{1/2}(\Phi^{-1}(t))^{-d}.$$

Thus, noting that $2\Phi(|x-y|) \leq T$, we have

$$\begin{aligned} G_D(x,y) &\geq c_1 \int_{\Phi(|x-y|/(3M_1))}^{2\Phi(|x-y|)} \left(1 \wedge \frac{\Phi(\delta_D(x))}{t}\right)^{1/2}\left(1 \wedge \frac{\Phi(\delta_D(y))}{t}\right)^{1/2}(\Phi^{-1}(t))^{-d} dt \\ &\geq \frac{c_1 2^{-1}}{\Phi^{-1}(2\Phi(|x-y|))^d}\left(1 \wedge \frac{\Phi(\delta_D(x))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)\left(1 \wedge \frac{\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)\int_{\Phi(|x-y|/(3M_1))}^{2\Phi(|x-y|)} dt \\ &\geq c_2 \frac{\Phi(|x-y|)}{|x-y|^d}\left(1 \wedge \frac{\Phi(\delta_D(x))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)\left(1 \wedge \frac{\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)^{1/2}}\right). \end{aligned}$$

We have proved part (i) of the theorem.

(ii) It follows from Theorem 1.3(c) that

$$\int_T^\infty p_D(t,x,y)dt \asymp \Phi(\delta_D(x))^{1/2}\Phi(\delta_D(y))^{1/2}, \quad x,y\in D. \tag{7.12}$$

By Theorem 1.3(a) and (3.1), there exists $c_3>0$ such that for $(t,x,y)\in(0,T]\times D\times D$,

$$\begin{aligned} & p_D(t,x,y) \\ \le & c_3\left(1\wedge\frac{\Phi(\delta_D(x))}{t}\right)^{1/2}\left(1\wedge\frac{\Phi(\delta_D(y))}{t}\right)^{1/2}\left((\Phi^{-1}(t))^{-d}\wedge\frac{t}{|x-y|^d\Phi(|x-y|)}\right). \end{aligned} \tag{7.13}$$

By the change of variable $u=\frac{\Phi(|x-y|)}{t}$ and the fact that $t\to\Phi^{-1}(t)$ is increasing, we have

$$\begin{aligned} & \int_0^T\left(1\wedge\frac{\Phi(\delta_D(x))}{t}\right)^{1/2}\left(1\wedge\frac{\Phi(\delta_D(y))}{t}\right)^{1/2}\left((\Phi^{-1}(t))^{-d}\wedge\frac{t}{|x-y|^d\Phi(|x-y|)}\right)dt \\ = \quad & \frac{\Phi(|x-y|)}{|x-y|^d}\left(\int_{\Phi(|x-y|)/T}^1+\int_1^\infty\right)u^{-2}\left(\left(\frac{\Phi^{-1}(ut)}{\Phi^{-1}(t)}\right)^d\wedge u^{-1}\right) \\ & \qquad\qquad \times\left(1\wedge\frac{\sqrt{u}\Phi(\delta_D(x))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)\left(1\wedge\frac{\sqrt{u}\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)du \\ \asymp \quad & \frac{\Phi(|x-y|)}{|x-y|^d}\int_{\Phi(|x-y|)/T}^1 u^{-2}\left(\frac{|x-y|}{\Phi^{-1}(u^{-1}\Phi(|x-y|))}\right)^d\left(1\wedge\frac{ua(x,y)}{\Phi(|x-y|)}\right)du \\ & +\frac{\Phi(|x-y|)}{|x-y|^d}\int_1^\infty u^{-3}\left(1\wedge\frac{\sqrt{u}\Phi(\delta_D(x))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)\left(1\wedge\frac{\sqrt{u}\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)du \\ =: \quad & I+II. \end{aligned} \tag{7.14}$$

In the fourth line of the display above, we used Lemma 7.1.

Since $\Phi(|x-y|)/a(x,y)\ge\Phi(|x-y|)/\Phi(\mathrm{diam}(D))\ge 2\Phi(|x-y|)/T$, by (7.4),

$$\begin{aligned} I \le & \frac{a(x,y)}{|x-y|^d}\int_{\Phi(|x-y|)/T}^1\frac{|x-y|^d}{\Phi^{-1}(u^{-1}\Phi(|x-y|))^d}\,u^{-1}du \\ = & \frac{a(x,y)}{|x-y|^d}\int_{\Phi(|x-y|)/T}^1\left(\frac{\Phi^{-1}(\Phi(|x-y|))}{\Phi^{-1}(u^{-1}\Phi(|x-y|))}\right)^d u^{-1}du \\ \le & c_4\frac{a(x,y)}{|x-y|^d}\int_0^1 u^{\frac{d}{2}-1}du \;=\; 2c_4d^{-1}\frac{a(x,y)}{|x-y|^d}. \end{aligned} \tag{7.15}$$

On the other hand, by Lemma 7.1

$$\begin{aligned} II \;\le\; & \frac{\Phi(|x-y|)}{|x-y|^d}\int_1^\infty u^{-2}\left(u^{-1/2}\wedge\frac{\Phi(\delta_D(x))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)\left(u^{-1/2}\wedge\frac{\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)du \\ \le & \frac{\Phi(|x-y|)}{|x-y|^d}\int_1^\infty u^{-2}\left(1\wedge\frac{\Phi(\delta_D(x))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)\left(1\wedge\frac{\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)du \\ = & \frac{\Phi(|x-y|)}{|x-y|^d}\left(1\wedge\frac{\Phi(\delta_D(x))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)\left(1\wedge\frac{\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)\le\frac{a(x,y)}{|x-y|^d}. \end{aligned} \tag{7.16}$$

Part (ii) of the theorem now follows from (7.12), (7.13), (7.15) and (7.16).

(iii) For the remainder of the proof we assume either that D is connected or that H satisfies $L_a(\gamma,C_L)$ and $U_a(\delta,C_U)$ with $\delta<2$ for some $a>0$. Then by Theorems 1.3(b) and 1.3

$$p_D(t,x,y)\ge c_5\left(1\wedge\frac{\Phi(\delta_D(x))}{t}\right)^{1/2}\left(1\wedge\frac{\Phi(\delta_D(y))}{t}\right)^{1/2}(\Phi^{-1}(t))^{-d}e^{-c_0\frac{|x-y|^2}{\Phi^{-1}(t)^2}}. \tag{7.17}$$

Since by (7.3)

$$\frac{|x-y|}{\Phi^{-1}(\Phi(|x-y|)/u)} = \frac{\Phi^{-1}(\Phi(|x-y|))}{\Phi^{-1}(\Phi(|x-y|)/u)} \le c_6 u^{1/(2\gamma)} \quad \text{if } u>1,$$

using this, by the change of variable $u = \frac{\Phi(|x-y|)}{t}$ and the fact that $t \to \Phi^{-1}(t)$ is increasing, we have

$$\begin{aligned}
&\int_0^T p_D(t,x,y)dt \\
\ge\ & c_5 \int_0^T \left(1 \wedge \frac{\Phi(\delta_D(x))}{t}\right)^{1/2} \left(1 \wedge \frac{\Phi(\delta_D(y))}{t}\right)^{1/2} (\Phi^{-1}(t))^{-d} e^{-c_0 \frac{|x-y|^2}{\Phi^{-1}(t)^2}} dt \\
=\ & c_5 \frac{\Phi(|x-y|)}{|x-y|^d} \left(\int_{\Phi(|x-y|)/T}^1 + \int_1^\infty\right) u^{-2} \left(\frac{\Phi^{-1}(ut)}{\Phi^{-1}(t)}\right)^d e^{-c_0 \frac{|x-y|^2}{\Phi^{-1}(\Phi(|x-y|)/u)^2}} \\
&\qquad \times \left(1 \wedge \frac{\sqrt{u}\Phi(\delta_D(x))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)\left(1 \wedge \frac{\sqrt{u}\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)^{1/2}}\right) du \\
\ge\ & c_5 e^{-c_0} \frac{\Phi(|x-y|)}{|x-y|^d} \int_{\Phi(|x-y|)/T}^1 u^{-2} \left(\frac{|x-y|}{\Phi^{-1}(u^{-1}\Phi(|x-y|))}\right)^d \left(1 \wedge \frac{ua(x,y)}{\Phi(|x-y|)}\right) du \\
&+ c_5 \frac{\Phi(|x-y|)}{|x-y|^d} \int_1^\infty u^{-2} e^{-c_0 c_6^2 u^{1/\gamma}} \left(1 \wedge \frac{\sqrt{u}\Phi(\delta_D(x))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)\left(1 \wedge \frac{\sqrt{u}\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)^{1/2}}\right) du \\
=:\ & c_5(I + III). \qquad (7.18)
\end{aligned}$$

Clearly, we have

$$III \ge \frac{\Phi(|x-y|)}{|x-y|^d} \left(1 \wedge \frac{\Phi(\delta_D(x))^{1/2}}{\Phi(|x-y|)^{1/2}}\right)\left(1 \wedge \frac{\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)^{1/2}}\right) \int_1^\infty u^{-2} e^{-c_0 c_6^2 u^{1/\gamma}} du. \quad (7.19)$$

Suppose that $d \le 2$. Let $h_{T,d}(a,r)$ be defined as in (7.5). Since $a(x,y) \le \Phi(\mathrm{diam}(D)) = T/2$, we have by (7.12)–(7.14), (7.16), (7.18), (7.19) and Lemma 7.1 that $G_D(x,y) \asymp h_T(a(x,y), |x-y|)$. Now, part (iii) of the theorem for $d \le 2$ follows from Lemmas 7.2.

Suppose $2 < d$, then we have that

$$\begin{aligned}
&\frac{\Phi(|x-y|)}{|x-y|^d} \int_{\Phi(|x-y|)/T}^1 u^{-2} \left(\frac{|x-y|}{\Phi^{-1}(u^{-1}\Phi(|x-y|))}\right)^d \left(1 \wedge \frac{u\Phi(\delta_D(x))^{1/2}\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)}\right) du \\
=&\frac{\Phi(|x-y|)}{|x-y|^d} \int_{\Phi(|x-y|)/T}^1 u^{-2} \left(\frac{|x-y|}{\Phi^{-1}(u^{-1}\Phi(|x-y|))}\right)^d \left(1 \wedge \frac{u\Phi(\delta_D(x))^{1/2}\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)}\right) du \\
\le& c_7 \frac{\Phi(|x-y|)}{|x-y|^d} \left(1 \wedge \frac{\Phi(\delta_D(x))^{1/2}\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)}\right) \int_0^1 u^{d/2-2}\, du \\
=& \frac{2c_7}{d-2} \frac{\Phi(|x-y|)}{|x-y|^d} \left(1 \wedge \frac{\Phi(\delta_D(x))^{1/2}\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)}\right). \qquad (7.20)
\end{aligned}$$

The case $d > 2$ of (iii) now follows from part (i) of the theorem, Lemma 7.1, (7.12), (7.13), (7.16) and (7.20). □

We now consider the Green function estimates for half space-like domains. Here we will give a sketch of the proofs only.

The proof of the next lemma is very similar (and simpler) to the one of Lemma 7.2 so we skip the proof.

Lemma 7.4. *Suppose that* (1.8) *holds, that ϕ has no drift and that H satisfies $L_0(\gamma, C_L)$ and $U_0(\delta, C_U)$ with $\delta < 2$ and $\gamma > 2^{-1}\mathbf{1}_{\delta\geq 1}$. For $b, r > 0$ and $d = 1, 2$, set*

$$h_d(b,r) = \Phi(r)\int_0^1 \left(1 \wedge \frac{ub}{\Phi(r)}\right) \frac{1}{u^2(\Phi^{-1}(u^{-1}\Phi(r)))^d}\, du + \frac{\Phi(r)}{r^d}\left(1 \wedge \frac{b}{\Phi(r)}\right).$$

Then, for $r, b > 0$,

$$h_d(b,r) \asymp \frac{b}{r^d} \wedge \left(\frac{b}{\Phi^{-1}(b)^d} + \left(\int_r^{\Phi^{-1}(b)} \frac{\Phi(s)}{s^{d+1}} ds\right)_+\right).$$

Recall that $g(x, y)$ is defined in (7.11).

Theorem 7.5. *Let $S = (S_t)_{t\geq 0}$ be a subordinator with zero drift whose Laplace exponent is ϕ and let $X = (X_t)_{t\geq 0}$ be the corresponding subordinate Brownian motion in $\mathbb{R}^d$. Suppose that D is a domain consisting of all the points above the graph of a bounded globally $C^{1,1}$ function and H satisfies $L_0(\gamma, C_L)$ and $U_0(\delta, C_U)$ with $\delta < 2$. Then*

$$G_D(x,y) \asymp g(x,y), \quad \text{for } x, y \in D.$$

Proof. By Theorem 1.4 and (3.1),

$$\begin{aligned} &G_D(x,y) \\ \leq& c_1 \int_0^\infty \left(1 \wedge \frac{\Phi(\delta_D(x))}{t}\right)^{1/2} \left(1 \wedge \frac{\Phi(\delta_D(y))}{t}\right)^{1/2} \left((\Phi^{-1}(t))^{-d} \wedge \frac{t}{|x-y|^d\Phi(|x-y|)}\right) dt \end{aligned}$$

and

$$G_D(x,y) \geq c_2 \int_0^\infty \left(1 \wedge \frac{\Phi(\delta_D(x))}{t}\right)^{1/2} \left(1 \wedge \frac{\Phi(\delta_D(y))}{t}\right)^{1/2} (\Phi^{-1}(t))^{-d} e^{-c_0 \frac{|x-y|^2}{\Phi^{-1}(t)^2}} dt.$$

Thus by following the argument in Theorem 7.3 one can easily see that for $d > 2$,

$$G_D(x,y) \asymp \frac{\Phi(|x-y|)}{|x-y|^d} \left(1 \wedge \frac{\Phi(\delta_D(x))}{\Phi(|x-y|)}\right)^{1/2} \left(1 \wedge \frac{\Phi(\delta_D(y))}{\Phi(|x-y|)}\right)^{1/2}.$$

and, for $d \leq 2$, $G_D(x, y) \asymp h_d(a(x, y), |x - y|)$. Thus the theorem follows by this and Lemmas 2.3(b) and 7.4. □

8 Examples

Suppose that D is a bounded $C^{1,1}$ open set with $\text{diam}(D) < 1/2$ and ϕ is either

$$\text{(i) } \phi(\lambda) = \frac{\lambda}{\log(1+\lambda^{\beta/2})}, \quad \text{where } \beta \in (0,2), \quad \text{or} \quad \text{(ii) } \phi(\lambda) = \frac{\lambda}{\log(1+\lambda)} - 1.$$

Then $\phi(\lambda) - \lambda\phi'(\lambda)$ satisfies $L_a(\gamma, C_L)$ and $U_a(\delta, C_U)$ with $2^{-1} < \gamma < \delta < 2$ where $a = 0$ for the case (i) and $a > 0$ for the case (ii). It is easy to check that we have

$$\phi^{-1}(\lambda) \asymp \lambda \log \lambda \quad \text{and} \quad H(\lambda) \asymp \frac{\lambda}{(\log \lambda)^2}, \quad \lambda \geq 2.$$

Moreover,

$$\Phi(r) = 1/\phi(1/r^2) \asymp r^2 \log(1/r) \quad \text{for } 0 < r \leq 1/2, \tag{8.1}$$

and

$$\frac{1}{\sqrt{\phi(1/\lambda^2)}} \asymp \lambda\sqrt{\log(\lambda^{-1})}, \qquad \lambda \le 1/2\,.$$

Thus by Theorem 1.3, for $0 < t < 1/2$,

$$\begin{aligned} p_D(t,x,y) \ge c_1 &\left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\sqrt{\log(1/\delta_D(x))}\right)\left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\sqrt{\log(1/\delta_D(y))}\right) \\ \times &\left[\left(t^{-d/2}\Big(\log\frac{1}{t}\Big)^{d/2}\right) \wedge \left(\frac{t(\log\frac{1}{|x-y|})^{-2}}{|x-y|^{d+2}} + t^{-d/2}\Big(\log\frac{1}{t}\Big)^{-d/2} e^{-a_L\frac{|x-y|^2}{t}\log\frac{1}{t}}\right)\right], \end{aligned} \tag{8.2}$$

and

$$\begin{aligned} p_D(t,x,y) \le c_2 &\left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\sqrt{\log(1/\delta_D(x))}\right)\left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\sqrt{\log(1/\delta_D(y))}\right) \\ \times &\left[\left(t^{-d/2}\Big(\log\frac{1}{t}\Big)^{d/2}\right) \wedge \left(\frac{t(\log\frac{1}{|x-y|})^{-2}}{|x-y|^{d+2}} + t^{-d/2}\Big(\log\frac{1}{t}\Big)^{-d/2} e^{-a_U\frac{|x-y|^2}{t}\log\frac{1}{t}}\right)\right]. \end{aligned} \tag{8.3}$$

We now assume that $d = 2$ and D is a bounded $C^{1,1}$ open set in $\mathbb{R}^2$ with sufficiently small diameter. We will give the sharp estimates of the Green function on D.

There is a constant $c_0 \in (0,1)$ so that

$$c_0\left(\frac{s}{\log(1/s)}\right)^{1/2} \le \Phi^{-1}(s) \le c_0^{-1}\left(\frac{s}{\log(1/s)}\right)^{1/2} \qquad \text{for } s \in (0, \Phi(1/2)]. \tag{8.4}$$

Suppose $0 < r \le \Phi^{-1}(b) \le 1/2$. Then

$$\begin{aligned} &\int_r^{\Phi^{-1}(b)} \frac{\Phi(s)}{s^3}ds \asymp \int_r^{\Phi^{-1}(b)} \frac{\log(1/s)}{s}ds \\ = \;& \frac{1}{2}\left((\log(1/r))^2 - (\log(1/\Phi^{-1}(b)))^2\right) = \frac{1}{2}\log^+(\Phi^{-1}(b)/r)\log^+(1/(r\Phi^{-1}(b))). \end{aligned} \tag{8.5}$$

Let

$$\begin{aligned} b(x,y) &:= \delta_D(x)\delta_D(y)\sqrt{\log(1/\delta_D(x))\log(1/\delta_D(y))} \\ &\asymp \Phi(\delta_D(x))^{1/2}\Phi(\delta_D(y))^{1/2} = a(x,y). \end{aligned} \tag{8.6}$$

Note that by (8.4)

$$\Phi^{-1}(a(x,y)) \asymp \left(\frac{b(x,y)}{\log(1/b(x,y))}\right)^{1/2} \tag{8.7}$$

and, so

$$\frac{a(x,y)}{\Phi^{-1}(a(x,y))^2} \asymp \log\left[b(x,y)^{-1}\right]. \tag{8.8}$$

Applying expressions $\Phi(|x-y|) \asymp |x-y|^2\log(1/|x-y|)$ and (8.5)–(8.8) to Theorem 7.3(iii), we have the following explicit estimates:

$$\begin{aligned} &G_D(x,y) \\ \asymp &\frac{b(x,y)}{|x-y|^2} \wedge \left(\log^+\left[\frac{b(x,y)}{|x-y|^2\log(1/b(x,y))}\right]\log^+\left[\frac{\log(1/b(x,y))}{|x-y|^2 b(x,y)}\right] + \log\left[b(x,y)^{-1}\right]\right). \end{aligned}$$

References

[1] Jean Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996. MR-1406564

[2] Krzysztof Bogdan and Tomasz Grzywny, *Heat kernel of fractional Laplacian in cones*, Colloq. Math. **118** (2010), no. 2, 365–377. MR-2602155

[3] Krzysztof Bogdan, Tomasz Grzywny, and Michał Ryznar, *Heat kernel estimates for the fractional Laplacian with Dirichlet conditions*, Ann. Probab. **38** (2010), no. 5, 1901–1923. MR-2722789

[4] Krzysztof Bogdan, Tomasz Grzywny, and Michał Ryznar, *Density and tails of unimodal convolution semigroups*, J. Funct. Anal. **266** (2014), no. 6, 3543–3571. MR-3165234

[5] Krzysztof Bogdan, Tomasz Grzywny, and Michał Ryznar, *Dirichlet heat kernel for unimodal Lévy processes*, Stochastic Process. Appl. **124** (2014), no. 11, 3612–3650. MR-3249349

[6] Krzysztof Bogdan, Tomasz Grzywny, and Michał Ryznar, *Barriers, exit time and survival probability for unimodal Lévy processes*, Probab. Theory Related Fields **162** (2015), no. 1–2, 155–198. MR-3350043

[7] Zhen-Qing Chen and Panki Kim, *Global Dirichlet heat kernel estimates for symmetric Lévy processes in half-space*, Acta Appl. Math. **146** (2016), 113–143. MR-3569587

[8] Zhen-Qing Chen, Panki Kim, and Takashi Kumagai, *Weighted Poincaré inequality and heat kernel estimates for finite range jump processes*, Math. Ann. **342** (2008), no. 4, 833–883. MR-2443765

[9] Zhen-Qing Chen, Panki Kim, and Takashi Kumagai, *On heat kernel estimates and parabolic Harnack inequality for jump processes on metric measure spaces*, Acta Math. Sin. (Engl. Ser.) **25** (2009), no. 7, 1067–1086. MR-2524930

[10] Zhen-Qing Chen, Panki Kim, and Takashi Kumagai, *Global heat kernel estimates for symmetric jump processes*, Trans. Amer. Math. Soc. **363** (2011), no. 9, 5021–5055. MR-2806700

[11] Zhen-Qing Chen, Panki Kim, and Renming Song, *Dirichlet heat kernel estimates for* $\Delta^{\alpha/2} + \Delta^{\beta/2}$, Illinois J. Math. **54** (2010), no. 4, 1357–1392. MR-2981852

[12] Zhen-Qing Chen, Panki Kim, and Renming Song, *Heat kernel estimates for the Dirichlet fractional Laplacian*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 5, 1307–1329. MR-2677618

[13] Zhen-Qing Chen, Panki Kim, and Renming Song, *Two-sided heat kernel estimates for censored stable-like processes*, Probab. Theory Related Fields **146** (2010), no. 3–4, 361–399. MR-2574732

[14] Zhen-Qing Chen, Panki Kim, and Renming Song, *Heat kernel estimates for* $\Delta + \Delta^{\alpha/2}$ *in* $C^{1,1}$ *open sets*, J. Lond. Math. Soc. (2) **84** (2011), no. 1, 58–80. MR-2819690

[15] Zhen-Qing Chen, Panki Kim, and Renming Song, *Dirichlet heat kernel estimates for fractional Laplacian with gradient perturbation*, Ann. Probab. **40** (2012), no. 6, 2483–2538. MR-3050510

[16] Zhen-Qing Chen, Panki Kim, and Renming Song, *Global heat kernel estimate for relativistic stable processes in exterior open sets*, J. Funct. Anal. **263** (2012), no. 2, 448–475. MR-2923420

[17] Zhen-Qing Chen, Panki Kim, and Renming Song, *Global heat kernel estimates for relativistic stable processes in half-space-like open sets*, Potential Anal. **36** (2012), no. 2, 235–261. MR-2892325

[18] Zhen-Qing Chen, Panki Kim, and Renming Song, *Sharp heat kernel estimates for relativistic stable processes in open sets*, Ann. Probab. **40** (2012), no. 1, 213–244. MR-2917772

[19] Zhen-Qing Chen, Panki Kim, and Renming Song, *Dirichlet heat kernel estimates for rotationally symmetric Lévy processes*, Proc. Lond. Math. Soc. (3) **109** (2014), no. 1, 90–120. MR-3237737

[20] Zhen-Qing Chen, Panki Kim, and Renming Song, *Dirichlet heat kernel estimates for subordinate Brownian motions with Gaussian components*, J. Reine Angew. Math. **711** (2016), 111–138. MR-3456760

[21] Zhen-Qing Chen and Takashi Kumagai, *Heat kernel estimates for stable-like processes on* d*-sets*, Stochastic Process. Appl. **108** (2003), no. 1, 27–62. MR-2008600

[22] Zhen-Qing Chen and Takashi Kumagai, *Heat kernel estimates for jump processes of mixed types on metric measure spaces*, Probab. Theory Related Fields **140** (2008), no. 1-2, 277–317. MR-2357678

[23] Zhen-Qing Chen, Takashi Kumagai, and Jian Wang, *Stability of heat kernel estimates for symmetric jump processes on metric measure spaces*. To appear in Memoirs Amer. Math. Soc. arXiv:1604.04035v3 [math.PR] MR-3378838

[24] Zhen-Qing Chen, Takashi Kumagai, and Jian Wang, *Stability of parabolic Harnack inequalities for symmetric non-local Dirichlet forms*. To appear in J. Eur. Math. Soc. (JEMS) arXiv:1609.07594v3 [math.PR]

[25] Zhen-Qing Chen and Joshua Tokle, *Global heat kernel estimates for fractional Laplacians in unbounded open sets*, Probab. Theory Related Fields **149** (2011), no. 3–4, 373–395. MR-2776619

[26] Sungwon Cho, *Two-sided global estimates of the Green's function of parabolic equations*, Potential Anal. **25** (2006), no. 4, 387–398. MR-2255354

[27] Kai Lai Chung and Zhong Xin Zhao, *From Brownian motion to Schrödinger's equation*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 312, Springer-Verlag, Berlin, 1995. MR-1329992

[28] Qingyang Guan, *Boundary Harnack inequality for regional fractional Laplacian*. arXiv:0705.1614v3

[29] Tomasz Grzywny, *On Harnack inequality and Hölder regularity for isotropic unimodal Lévy processes*, Potential Anal. **41** (2014), no. 1, 1–29. MR-3225805

[30] Tomasz Grzywny, Kyung-Youn Kim, and Panki Kim, *Estimates of Dirichlet heat kernel for symmetric Markov processes*. arXiv:1512.02717 [math.PR]

[31] Naresh C. Jain and William E. Pruitt, *Lower tail probability estimates for subordinators and nondecreasing random walks*, Ann. Probab. **15** (1987), no. 1, 75–101. MR-0877591

[32] Kamil Kaleta and PawełSztonyk, *Estimates of transition densities and their derivatives for jump Lévy processes*, J. Math. Anal. Appl. **431** (2015), no. 1, 260–282. MR-3357585

[33] Kyung-Youn Kim, *Global heat kernel estimates for symmetric Markov processes dominated by stable-like processes in exterior $C^{1,\eta}$ open sets*, Potential Anal. **43** (2015), no. 2, 127–148. MR-3374107

[34] Kyung-Youn Kim and Panki Kim, *Two-sided estimates for the transition densities of symmetric Markov processes dominated by stable-like processes in $C^{1,\eta}$ open sets*, Stochastic Process. Appl. **124** (2014), no. 9, 3055–3083. MR-3217433

[35] Panki Kim and Ante Mimica, *Green function estimates for subordinate Brownian motions: stable and beyond*, Trans. Amer. Math. Soc. **366** (2014), no. 8, 4383–4422. MR-3206464

[36] Panki Kim and Renming Song, *Dirichlet heat kernel estimates for stable processes with singular drift in unbounded $C^{1,1}$ open sets*, Potential Anal. **41** (2014), no. 2, 555–581. MR-3232039

[37] Panki Kim, Renming Song, and Zoran Vondraček, *Potential theory of subordinate Brownian motions revisited*, Stochastic analysis and applications to finance, Interdiscip. Math. Sci., vol. 13, World Sci. Publ., Hackensack, NJ, 2012, pp. 243–290. MR-2986850

[38] Panki Kim, Renming Song, and Zoran Vondraček, *Two-sided Green function estimates for killed subordinate Brownian motions*, Proc. Lond. Math. Soc. (3) **104** (2012), no. 5, 927–958. MR-2928332

[39] Panki Kim, Renming Song, and Zoran Vondraček, *Global uniform boundary Harnack principle with explicit decay rate and its application*, Stochastic Process. Appl. **124** (2014), no. 1, 235–267. MR-3131293

[40] Panki Kim, Renming Song, and Zoran Vondraček, *Potential theory of subordinate killed Brownian motion*. To appear in *Trans. Amer. Math. Soc.*. arXiv:1610.00872v5 [math.PR]

[41] Panki Kim, Renming Song, and Zoran Vondraček, *On the boundary theory of subordinate killed Lévy processes*. arXiv:1705.02595v4 [math.PR]

[42] Tadeusz Kulczycki and Michał Ryznar, *Gradient estimates of harmonic functions and transition densities for Lévy processes*, Trans. Amer. Math. Soc. **368** (2016), no. 1, 281–318. MR-3413864

[43] Ante Mimica, *Heat kernel estimates for subordinate Brownian motions*, Proc. Lond. Math. Soc. (3) **113** (2016), no. 5, 627–648. MR-3570240

[44] William E. Pruitt, *The growth of random walks and Lévy processes*, Ann. Probab. **9** (1981), no. 6, 948–956. MR-0632968

[45] Martin L. Silverstein, *Classification of coharmonic and coinvariant functions for a Lévy process*, Ann. Probab. **8** (1980), no. 3, 539–575. MR-0573292

[46] Renming Song, *Estimates on the Dirichlet heat kernel of domains above the graphs of bounded $C^{1,1}$ functions*, Glas. Mat. Ser. III **39(59)** (2004), no. 2, 273–286. MR-2109269

[47] Paweł Sztonyk, *Transition density estimates for jump Lévy processes*, Stochastic Process. Appl. **121** (2011), no. 6, 1245–1265. MR-2794975

[48] Paweł Sztonyk, *Estimates of densities for Lévy processes with lower intensity of large jumps*, Math. Nachr. **290** (2017), no. 1, 120–141. MR-3604626

[49] Qi S. Zhang, *The boundary behavior of heat kernels of Dirichlet Laplacians*, J. Differential Equations **182** (2002), no. 2, 416–430. MR-1900329

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