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Convergence in distribution norms in the CLT for non identical distributed random variables

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Abstract

We study the convergence in distribution norms in the Central Limit Theorem for non identical distributed random variables that is

$$\varepsilon_n(f) := \mathbb{E}\Big(f\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i\Big)\Big) - \mathbb{E}\big(f(G)\big) \to 0$$

where Z_i, $i \in \mathbb{N}$, are centred independent random variables and G is a Gaussian random variable. We also consider local developments (Edgeworth expansion). This kind of results is well understood in the case of smooth test functions f. If one deals with measurable and bounded test functions (convergence in total variation distance), a well known theorem due to Prohorov shows that some regularity condition for the law of the random variables Z_i, $i \in \mathbb{N}$, on hand is needed. Essentially, one needs that the law of Z_i is locally lower bounded by the Lebesgue measure (Doeblin's condition). This topic is also widely discussed in the literature. Our main contribution is to discuss convergence in distribution norms, that is to replace the test function f by some derivative $\partial_\alpha f$ and to obtain upper bounds for $\varepsilon_n(\partial_\alpha f)$ in terms of the infinite norm of f. Some applications are also discussed: an invariance principle for the occupation time for random walks, small balls estimates and expected value of the number of roots of trigonometric polynomials with random coefficients.

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1 Introduction

The framework. We consider n independent (but not necessarily identically distributed) random variables Y_k, $k = 1, \ldots, n$, taking values in $\mathbb{R}^m$, which are centered and with identity covariance matrix. Moreover, we consider n matrices $C_{n,k} \in \mathrm{Mat}(d \times m)$ and we look to

$$S_n(Y) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} C_{n,k} Y_k. \tag{1.1}$$

Our aim is to obtain a Central Limit Theorem (CLT) as well as Edgeworth developments in this framework. The basic hypotheses are the following. We assume the normalization condition

$$\frac{1}{n} \sum_{k=1}^{n} C_{n,k} C_{n,k}^{*} = \mathrm{Id}_d, \tag{1.2}$$

where $*$ denotes transposition and $\mathrm{Id}_d \in \mathrm{Mat}(d \times d)$ is the identity matrix. Moreover we assume that for each $p \in \mathbb{N}$ there exists a constant $\mathrm{C}_p(Y) \geq 1$ such that

$$\max_{1 \leq k \leq n} \mathbb{E}(|C_{n,k} Y_k|^p) \leq \mathrm{C}_p(Y). \tag{1.3}$$

The case of smooth test functions. Let $\|f\|_{k,\infty}$ denote the norm in $W^{k,\infty}$, that is the uniform norm of f and of all its derivatives of order less or equal to k. First, we want to prove that

$$\left| \mathbb{E}(f(S_n(Y)) - \int_{\mathbb{R}^d} f(x) \gamma_d(x) dx \right| \leq \frac{\mathrm{C}_0}{n^{\frac{1}{2}}} \|f\|_{3,\infty} \tag{1.4}$$

where $\gamma_d(x) = (2\pi)^{-d/2} \exp(-\frac{1}{2} |x|^2)$ is the density of the standard normal law. This corresponds to the Central Limit Theorem (hereafter CLT). Moreover we look for some polynomials $\psi_{n,k} : \mathbb{R}^d \to \mathbb{R}$ such that for $N \in \mathbb{N}$ and for every $f \in C_b^{\widehat{N}}(\mathbb{R}^d)$, with $\widehat{N} = N(2\lfloor N/2 \rfloor + N + 5)$,

$$\left| \mathbb{E}(f(S_n(Z)) - \int_{\mathbb{R}^d} f(x) \Big(1 + \sum_{k=1}^{N} \frac{1}{n^{k/2}} \psi_{n,k}(x)\Big) \gamma_d(x) dx \right| \leq \frac{\mathrm{C}_N}{n^{\frac{1}{2}(N+1)}} \|f\|_{\widehat{N},\infty}. \tag{1.5}$$

This is Theorem 4.1, giving the Edgeworth development of order N. In the case of smooth test functions f (as it is the case in (1.5)), this topic has been widely discussed and well understood. One should mention the seminal paper by Essen [23] the books of Gnedenko and Kolmogorov [25], Petrov [32], Battacharaya and Rao [15] and Zolotarev [36]. Such development has been obtained by Sirazhdinov and Mamatov [35] in the case of identically distributed random variables and then by Götze and Hipp [26] in the non identically distributed case. A complete presentation of this topic may be found in the recent review paper by Bobkov [17]. The coefficients $\psi_{n,k}$ in the development (1.5) are linear combinations of Hermite polynomials. An explicit expression, in the one dimensional case, is given in [17]. Ourselves we give the explicit formula of these coefficients in the multi-dimensional case. This is important because, in the working paper [10], the development of order three, in the 2-dimensional case, is used in order to study invariance principles for the variance of trigonometric polynomials.

It is worth to mention that the classical approach is based on Fourier analysis. In our paper we use a different approach based on the Lindeberg method for Markov semigroups (this is inspired from works concerning the parametrix method for Markov semigroups in [13], see also Chatterjee [21]). This alternative approach is convenient for the proof of our main result concerning "distribution norms"(see below).

The case of general test functions. A second problem is to obtain the estimate (1.5) for test functions f which are not regular, in particular to replace $\|f\|_{\widehat{N},\infty}$ by $\|f\|_{\infty}$.

This amounts to estimate the error in total variation distance. In the case of identically distributed random variables, and for $N = 0$ (so at the level of the standard CLT), this problem has been widely studied. First of all, one may prove the convergence in Kolmogorov distance, that is for $f = 1_D$ where D is a rectangle. Many refinements of this type of result have been obtained by Battacharaya and Rao and they are presented in [15]. But it turns out that one may not prove such a result for a general measurable set D without assuming more regularity on the law of Y_k, $k \in \mathbb{N}$.

Indeed, consider the standard CLT, so take $m = d$, $C_{n,k} = \mathrm{Id}_d$ and Y_k, $k = 1, \ldots, n$, i.i.d. In his seminal paper [33] Prohorov proved that the convergence in total variation distance is equivalent to the fact that there exists r such that the law of $Y_1 + \cdots + Y_r$ has an absolutely continuous component. This is "essentially" equivalent to the Doeblin's condition that we present now (see Remark 2.1): we assume that there exists $r, \varepsilon > 0$ and there exists $y_k \in \mathbb{R}^m$ such that for every measurable set $A \subset B_r(y_k)$

$$\mathbb{P}(Y_k \in A) \geq \varepsilon \lambda(A) \tag{1.6}$$

where λ is the Lebesgue measure. Under (1.6) we are able to obtain (1.5) in total variation distance.

Let us finally mention another line of research which has been strongly developed in the last years: it consists in estimating the convergence in the CLT in entropy distance. This starts with the papers of Barron [14] and Johnson and Barron [28]. In these papers the case of identically distributed random variables is considered, but recently, Bobkov, Chistyakov and Götze [19] have obtained the estimate in entropy distance for the case of random variables which are no more identically distributed as well. We recall that the convergence in entropy distance implies the convergence in total variation distance, so such results are stronger. However, in order to work in entropy distance one has to assume that the law of $Z_{n,k} = C_{n,k} Y_k$ is absolutely continuous with respect to the Lebesgue measure and have finite entropy and this is more limiting than (1.6). So the hypotheses and the results are slightly different. Finally, other types of distances (W_p-transport distances) have been recently studied in [18, 20, 34].

Convergence in distribution norms. Consider first the particular case when $Z_{n,k} = C_{n,k} Y_k$ are identically distributed and have a density which is one time differentiable with derivative belonging to L^1. Then the law of $S_n(Y)$ is absolutely continuous with C^n density and then, in Proposition 2.12, we prove that for every $k \in \mathbb{N}$ and every multiindex α

$$\sup_x (1 + |x|^2)^k \left| \partial_\alpha p_{S_n}(x) - \partial_\alpha \gamma_d(x) \right| \leq \frac{C}{\sqrt{n}}$$

which is the standard convergence in distribution norms. Notice also that here we are at the level of the CLT and we are not able to deal with Edgeworth expansions.

Unfortunately we fail to obtain such a result in the general framework (which is the interesting case): this is moral because we do not assume that the laws of $C_{n,k} Y_k$, $k = 1, ..., n$ are absolutely continuous, and then the law of $S_n(Y)$ may have atoms. However we obtain a similar result, but we have to keep a "small error". Let us give a precise statement of our result. For a function $f \in C_p^q(\mathbb{R}^d)$ (q times differentiable with polynomial growth) we define $L_q(f)$ and $l_q(f)$ to be two constants such that

$$\sum_{0 \leq |\alpha| \leq q} |\partial_\alpha f(x)| \leq L_q(f)(1 + |x|)^{l_q(f)}. \tag{1.7}$$

Our main result is given in Theorem 2.3 and says the following: for a fixed $q \in \mathbb{N}$, there exist some constants $\mathrm{C}_N \geq 1 \geq \mathrm{c}_N > 0$ (depending on r, ε from (1.6) and on $\mathrm{C}_p(Y)$ from

(1.3)) such that for every multiindex γ with $|\gamma| = q$ and for every $f \in C_p^q(\mathbb{R}^d)$

$$\begin{aligned}&\Big|\mathbb{E}\Big(\partial_\gamma f(S_n(Z))\Big) - \int_{\mathbb{R}^d} \partial_\gamma f(x)\Big(1 + \sum_{k=1}^N \frac{1}{n^{k/2}}\psi_{n,k}(x)\Big)\gamma_d(x)dx\Big| \\ &\leq C_N\Big(L_q(f)e^{-c_N \times n} + \frac{1}{n^{\frac{1}{2}(N+1)}}L_0(f)\Big).\end{aligned} \tag{1.8}$$

However we fail to get convergence in distribution norms because $L_q(f)e^{-c_N\times n}$ appears in the upper bound of the error and $L_q(f)$ depends on the derivatives of f. But we are close to such a result: notice first that if $f_n = f * \phi_{\delta_n}$ is a regularization by convolution with $\delta_n = \exp(-\frac{c_N}{2q} \times n)$ then (1.8) gives

$$\Big|\mathbb{E}\Big(\partial_\gamma f_n(S_n(Z))\Big) - \int_{\mathbb{R}^d} \partial_\gamma f_n(x)\Big(1 + \sum_{k=1}^N \frac{1}{n^{k/2}}\psi_{n,k}(x)\Big)\gamma_d(x)dx\Big| \leq \frac{C_N}{n^{\frac{1}{2}(N+1)}}L_0(f). \tag{1.9}$$

We discuss now three applications.

Application 1: an invariance principle related to the local time. Let

$$S_n(k,Y) = \frac{1}{\sqrt{n}}\sum_{i=1}^k Y_i,$$

where $Y_1, \ldots, Y_n$ are independent and identically distributed random variables. We set $\varepsilon_n = n^{-\frac{1}{2}(1-\rho)}$ with $\rho \in (0,1)$ and in Theorem 3.1 we prove that, for every $\rho' < \rho$,

$$\Big|\frac{1}{n}\sum_{k=1}^n \mathbb{E}\Big(\frac{1}{2\varepsilon_n}1_{\{|S_n(k,Y)|\leq\varepsilon_n\}}\Big) - \mathbb{E}\Big(\int_0^1 \frac{1}{2\varepsilon_n}1_{\{|W_s|\leq\varepsilon_n\}}ds\Big)\Big| \leq \frac{C}{n^{\frac{1+\rho'}{2}}}.$$

with W_s a Brownian motion (we recall that $\int_0^1 \frac{1}{2\varepsilon_n}1_{\{|W_s|\leq\varepsilon_n\}}ds$ converges to the local time of W). Here the test function is $f_n(x) = \frac{1}{2\varepsilon_n}1_{|x|<\varepsilon_n}$ and this converges to the Dirac function. This example shows that (1.8) is an appropriate estimate in order to deal with some singular problems.

Application 2: small ball probabilities. We consider the case in which the matrices $C_{n,k}$ can depend on a parameter $u \in \mathbb{R}^\ell$, that is,

$$S_n(u,Y) = \frac{1}{\sqrt{n}}\sum_{k=1}^n C_{n,k}(u)Y_k, \quad u \in \mathbb{R}^\ell.$$

We assume that $u \mapsto C_{n,k}(u) \in \mathrm{Mat}(d\times d)$ is twice differentiable with bounded derivatives up to order two and that the covariance matrix field of $S_n(u,Y)$ is the identity matrix, that is, $\Sigma_n(u) = \frac{1}{n}\sum_{k=1}^n C_{n,k}(u)C_{n,k}^*(u) = \mathrm{Id}_d$. Then in Theorem 3.2 we prove the following estimate: if $d > \ell$, $a \geq 0$ and $\theta > \frac{a\ell}{d-\ell}$ then, for every $\varepsilon > 0$,

$$\mathbb{P}\Big(\inf_{|u|\leq n^a}|S_n(u,Y)| \leq \frac{1}{n^\theta}\Big) \leq \frac{C}{n^{\theta(d-\ell)-a\ell-\varepsilon}}. \tag{1.10}$$

This is done by applying (1.9) to the multiindex $\gamma = (1,\ldots,d)$ and the function $f = f_n$, with

$$f_n(x) = n^{-\theta d}\int_{-\infty}^{x_1} dx_2 \ldots \int_{-\infty}^{x_{d-1}} dx_d 1_{\{|x|<n^{-\theta}\}}(x).$$

Then (1.9) allows one to replace $S_n(u,Y)$ with a Gaussian random variable, and in this case we have a nice estimate of the error. We emphasize that, contrarily to the case of supremas of random processes, much less is known regarding infimas. As such, the last result can be seen as a preliminary step enabling one to switch to the Gaussian case for which more accurate tools are available.

Application 3: an invariance principle for the expected roots of trigonometric polynomials. Let $N_n(Y)$ be the number of roots in $(0,\pi)$ of the polynomial

$$P_n(t,Y)=\sum_{k=1}^n \big(Y_k^1\cos(kt)+Y_k^2\sin(kt)\big).$$

It is known, see e.g. [22], that if the Y_k's are replaced by independent and identically distributed standard normal random variables G_k's then

$$\lim_n \frac{1}{n}\mathbb{E}(N_n(G))=\frac{1}{\sqrt{3}}.$$

Note that the aforementioned asymptotic still holds when the Gaussian coefficients display some strong form of dependence [1]. In the recent paper [24], the above result has been proved for general independent and identically distributed random variables Y_k, $k\in\mathbb{N}$, which are centered and with variance one. In Theorem 3.4 we drop the assumption of being identically distributed: we prove that the same limit holds for $N_n(Y)$ when the Y_k's are independent and fulfill the Doeblin's condition. We stress that it is not completely clear whether the strategy used in [24] can be adapted to the setting of non-identically distributed coefficients since it is explicitly used at several moments in the proof that the characteristic function of each coefficients behaves *in a same way* near the origin, which is more restrictive that our normalization condition (1.2). Our main result enters in the following way: thanks to the Kac-Rice formula, we have

$$N_n(Y)=\lim_{\delta\to 0}\int_a^b |\partial_t P_n(t,Y)|\,1_{\{|P_n(t,Y)|\le\delta\}}\frac{dt}{2\delta},$$

so we apply (1.8) to the pair $(\partial_t P_n(t,Y),P_n(t,Y))$. Although the article only focuses on the expectation, we stress that this methods paves the way to an investigation of higher moments (and hence variance or CLT's) by using Kac-Rice formulas of higher order. This is actually the main content of the forthcoming article [10] which follows the series [27, 3, 2] of articles dedicated to this task in the Gaussian case.

2 Notation and main results

We fix $n\in\mathbb{N}$ and we consider n independent random variables $\{Y_k\}_{1\le k\le n}$, with $Y_k=(Y_k^1,\dots,Y_k^m)\in\mathbb{R}^m$, which are centered and whose covariance matrix is the identity. Let $\{C_{n,k}\}_{1\le k\le n}$ denote n matrices in $\mathrm{Mat}(d\times m)$ and set

$$\sigma_{n,k}=C_{n,k}C_{n,k}^*\in\mathrm{Mat}(d\times d),$$

$*$ denoting transposition, so $\sigma_{n,k}$ is the covariance matrix of the random variable $C_{n,k}Y_k$. We define

$$S_n(Y)=\frac{1}{\sqrt{n}}\sum_{k=1}^n C_{n,k}Y_k. \tag{2.1}$$

Sometimes, but not everywhere, we consider the normalizing condition

$$\frac{1}{n}\sum_{k=1}^n \sigma_{n,k}=\mathrm{Id}_d, \tag{2.2}$$

Id_d denoting the $d \times d$ identity matrix. Our aim is to compare the law of $S_n(Y)$ with the law of $S_n(G)$ where $G = (G_k)_{1\leq k\leq n}$ denote n standard independent Gaussian random variables. This is a CLT result (but we stress that it is not asymptotic) and we will obtain an Edgeworth development as well.

We assume that Y_k has finite moments of any order and more precisely,

$$\max_{1\leq k\leq n} \mathbb{E}(|C_{n,k}Y_k|^p) \leq \mathrm{C}_p(Y), \quad \forall\, p. \tag{2.3}$$

Notice that by (2.2) $|\sigma^{i,j}_{n,k}| \leq 1$ so we may assume without loss of generality that $\mathbb{E}(|C_{n,k}G_k|^p) \leq \mathrm{C}_p(Y)$ for the standard normal random variables as well.

2.1 Doeblin's condition and Nummelin's splitting

We say that the law of the random variable $Y \in \mathbb{R}^m$ is locally lower bounded by the Lebesgue measure if there exists $y_Y \in \mathbb{R}^d$ and $\varepsilon, r > 0$ such that for every non negative and measurable function $f : \mathbb{R}^d \to \mathbb{R}_+$

$$\mathbb{E}(f(Y)) \geq \varepsilon \textstyle\int f(y - y_Y) 1_{B(0,2r)}(y - y_Y)dy. \tag{2.4}$$

(2.4) is known as the Doeblin's condition. We denote by $\mathfrak{D}(r, \varepsilon)$ the class of the random variables which verify (2.4). Given $r > 0$ we consider the functions $a_r, \psi_r : \mathbb{R} \to \mathbb{R}_+$ defined by

$$a_r(t) = 1 - \frac{1}{1 - (\frac{t}{r} - 1)^2} \quad \text{and} \quad \psi_r(t) = 1_{\{|t|\leq r\}} + 1_{\{r<|t|\leq 2r\}} e^{a_r(|t|)}. \tag{2.5}$$

If $Y \in \mathfrak{D}(r, \varepsilon)$ then

$$\mathbb{E}(f(Y)) \geq \varepsilon \int f(y - y_Y)\psi_r(|y - y_Y|^2)dy.$$

The advantage of $\psi_r(|y - y_Y|^2)$ is that it is a smooth function (which replaces the indicator function of the ball) and (it is easy to check) that for each $l \in \mathbb{N}, p \geq 1$ there exists a universal constant $\mathrm{C}_{l,p} \geq 1$ such that

$$\psi_r(t)|a^{(l)}_r(|t|)|^p \leq \frac{\mathrm{C}_{l,p}}{r^{lp}} \tag{2.6}$$

where $a^{(l)}_r$ denotes the derivative of order l of a_r. Moreover one can check (see [8]) that if $Y \in \mathfrak{D}(r, \varepsilon)$ then it admits the following decomposition (the equality is understood as identity of laws):

$$Y = \chi V + (1 - \chi)U \tag{2.7}$$

where χ, V, U are independent random variables with the following laws:

$$\begin{aligned} \mathbb{P}(\chi = 1) = \varepsilon\mathfrak{m}_r \quad &\text{and} \quad \mathbb{P}(\chi = 0) = 1 - \varepsilon\mathfrak{m}_r, \\ \mathbb{P}(V \in dy) &= \frac{1}{\mathfrak{m}_r}\psi_r(|y - y_Y)|^2)dy \\ \mathbb{P}(U \in dy) = \frac{1}{1 - \varepsilon\mathfrak{m}_r}&(\mathbb{P}(Z \in dy) - \varepsilon\psi_r(|y - y_Y|^2)dy) \end{aligned} \tag{2.8}$$

with

$$\mathfrak{m}_r = \int \psi_r(|y - y_Y|^2)dy. \tag{2.9}$$

The decomposition (2.7) is also known as the Nummelin's splitting. We will see later on, specifically in next Section 5.1, that the noise coming from the Nummelin's decomposition allows one to set-up a Malliavin type calculus, which in turn will be our main tool in order to get our CLT result in distribution norms.

Remark 2.1. In his seminal paper [33] Prohorov considers a sequence of i.i.d. random variables X_n and proves that the convergence in the CLT holds in total variation distance if and only if the following hypothesis holds: there exists n_* such that the law of $X_1+...+X_{n_*}$ has an absolute continuous component, that is $X_1+...+X_{n_*} \sim \mu(dx)+p(x)dx$. Of course this is much weaker than Doeblin's condition, but, as long as we want to prove the CLT in total variation distance, we may proceed as follows: we denote $Y_k = X_{kn_*+1}+...+X_{(k+1)n_*}$ and take $Z_k = Y_{2k}+Y_{2k+1}$. Since the convolution of two functions from L^1 is a continuous function, $p*p$ is continuous and consequently locally lower bonded by the Lebesgue measure. So Z_k verifies Doeblin's condition. We prove the CLT in total variation for Z_k and then it easily follows for X_n (see Corollary 2.11 below). So, as long as one is concerned with the CLT the two conditions are (in the above sense) equivalent.

Remark 2.2. We stress that in [8] Proposition 2.4 there is fault: it is asserted that, if $X \sim \mu(dx)+p(x)dx$ then X satisfies the Doeblin's condition – and of course this is false if we do not ask p to be lower semicontinous. However, in Lemma $A.1$ from the appendix in the same paper, the lower continuity hypothesis is mentioned.

2.2 Main results

In order to give the expression of the terms which appear in the Edgeworth development we need to introduce some notation.

We say that α is a multiindex if $\alpha \in \{1,\ldots,d\}^k$ for some $k \geq 1$, and we set $|\alpha| = k$ its length. We allow the case $k=0$, giving the void multiindex $\alpha = \emptyset$.

Let α be a multiindex and set $k=|\alpha|$. For for $x \in \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$, we denote $x^\alpha = x_{\alpha_1}\cdots x_{\alpha_k}$ and $\partial_\alpha f(x) = \partial_{x_{\alpha_1}}\cdots\partial_{x_{\alpha_k}} f(x)$, the case $k=0$ giving $x^\emptyset = 1$ and $\partial_\emptyset f = f$. In the following, we denote with $C^k(\mathbb{R}^d)$ the set of the functions f such that $\partial_\alpha f$ exists and is continuous for any α with $|\alpha| \leq k$. The set $C^k_p(\mathbb{R}^d)$, resp. $C^k_b(\mathbb{R}^d)$, is the subset of $C^k(\mathbb{R}^d)$ such that $\partial_\alpha f$ has polynomial growth, resp. is bounded, for any α with $|\alpha| \leq k$. $C^\infty(\mathbb{R}^d)$, resp. $C^\infty_p(\mathbb{R}^d)$ and $C^\infty_b(\mathbb{R}^d)$, denotes the intersection of $C^k(\mathbb{R}^d)$, resp. of $C^k_p(\mathbb{R}^d)$ and of $C^k_b(\mathbb{R}^d)$, for every k. For $f \in C^k_p(\mathbb{R}^d)$ we define $L_k(f)$ and $l_k(f)$ to be some constants such that

$$\sum_{0\leq|\alpha|\leq k} |\partial_\alpha f(x)| \leq L_k(f)(1+|x|)^{l_k(f)}. \tag{2.10}$$

Notice that if $f \in C^\infty_b(\mathbb{R}^d)$ then $l_k(f)=0$ and $L_k(f) = \sum_{0\leq|\alpha|\leq k} \|\partial_\alpha f\|_\infty$.

Moreover, for a non negative definite matrix $\sigma \in \mathrm{Mat}(d\times d)$ we denote by L_σ the Laplace operator associated to σ, i.e.

$$L_\sigma = \sum_{i,j=1}^d \sigma^{i,j}\partial_{z_i}\partial_{z_j}. \tag{2.11}$$

For $r \geq 1$ and $l \geq 0$ we set

$$\Delta_{n,r}(\alpha) = \mathbb{E}((C_{n,r}Y_r)^\alpha) - \mathbb{E}((C_{n,r}G_r)^\alpha) \quad \text{and} \quad D^{(l)}_{n,r} = \sum_{|\alpha|=l} \Delta_{n,r}(\alpha)\partial_\alpha. \tag{2.12}$$

Notice that $D^{(l)}_{n,r} \equiv 0$ for $l=0,1,2$ and, by (2.3), for $l \geq 3$ and $|\alpha| = l$ then

$$|\Delta_{n,r}(\alpha)| \leq 2\mathrm{C}_l(Y), \quad r=1,\ldots,n. \tag{2.13}$$

We construct now the coefficients of our development. Let N be fixed: this is the order of the development that we will obtain. Given $1 \leq m \leq k \leq N$ we define

$$\begin{aligned}\Lambda_m &= \{((l_1,l_1'),...,(l_m,l_m')) : N+2 \geq l_i \geq 3, \lfloor N/2 \rfloor \geq l_i' \geq 0, i=1,...,m\},\\ \Lambda_{m,k} &= \{((l_1,l_1'),...,(l_m,l_m')) \in \Lambda_m : \sum_{i=1}^m l_i + 2\sum_{i=1}^m l_i' = k+2m\}.\end{aligned} \tag{2.14}$$

Then, for $1 \leq k \leq N$, we define the differential operator

$$\Gamma_{n,k} = \sum_{m=1}^k \sum_{((l_1,l_1'),...,(l_m,l_m'))\in\Lambda_{m,k}} \frac{1}{n^m} \sum_{1\leq r_1<...<r_m\leq n} \prod_{i=1}^m \frac{1}{l_i!} D_{n,r_i}^{(l_i)} \prod_{j=1}^m \frac{(-1)^{l_j'}}{2^{l_j'} l_j'!} L_{\sigma_{n,r_j}}^{l_j'}. \tag{2.15}$$

By using (2.3) and (2.13), one easily gets the following estimates:

$$|\Gamma_{n,k} f(x)| \leq C \times \mathrm{C}_{3k}(Y) L_{3k}(f)(1+|x|)^{l_{3k}(f)}, \quad f \in C_p^{3k}(\mathbb{R}^d), \tag{2.16}$$

where $L_{3k}(f)$ and $l_{3k}(f)$ are given in (2.10) and $C>0$ is a suitable constant which does not depend on n.

We introduce now the Hermite polynomials, we refer to Nualart [31] for definitions and properties. The Hermite polynomial H_m of order m on $\mathbb{R}$ is defined as

$$H_m(x) = (-1)^m e^{\frac{1}{2}x^2} \frac{d^m}{x^m} e^{-\frac{1}{2}x^2}. \tag{2.17}$$

For a multiindex $\alpha \in \{1,\dots,d\}^l$ we denote $\beta_i(\alpha) = \mathrm{card}\{j : \alpha_j = i\}$ and we define the Hermite polynomial on $\mathbb{R}^d$ corresponding to the multiindex α by

$$H_\alpha(x) = \prod_{i=1}^l H_{\beta_i(\alpha)}(x_i) \qquad \text{for} \qquad x=(x_1,...,x_d). \tag{2.18}$$

Equivalently, the Hermite polynomial H_α on $\mathbb{R}^d$ associated to the multiindex α is defined by

$$\mathbb{E}(\partial_\alpha f(W)) = \mathbb{E}(f(W)H_\alpha(W)) \quad \forall f \in C_p^\infty(\mathbb{R}^d) \tag{2.19}$$

where W is a standard normal random variable in $\mathbb{R}^d$. Moreover for a differential operator $\Gamma = \sum_{|\alpha|\leq k} a(\alpha)\partial_\alpha$, with $a(\alpha)\in\mathbb{R}$, we denote $H_\Gamma = \sum_{|\alpha|\leq k} a(\alpha)H_\alpha$ so that

$$\mathbb{E}(\Gamma f(W)) = \mathbb{E}(f(W)H_\Gamma(W)). \tag{2.20}$$

Finally we define

$$\Phi_{n,N}(x) = 1 + \sum_{k=1}^N \frac{1}{n^{k/2}} H_{\Gamma_{n,k}}(x) \text{ with } \Gamma_{n,k} \text{ defined in (2.15).} \tag{2.21}$$

The polynomial $\Phi_{n,N}$ gives the Edgeworth expansion of order N in the CLT, as stated in the following result, which represents the main result of this paper.

Theorem 2.3. *Assume that $Y_k \in \mathfrak{D}(r,\varepsilon), \forall k \in N$ for some $\varepsilon>0, r>0$. Let the normalizing condition (2.2) and the moment bounds condition (2.3) both hold. Let $N,q \in \mathbb{N}$ be fixed. We assume that n is sufficiently large in order to have*

$$n^{\frac{1}{2}(N+1)} e^{-\frac{\mathfrak{m}_r^2 n}{256}} \leq 1 \quad \textit{and} \quad n \geq 4(N+1)\mathrm{C}_2(Y).$$

There exists $C \geq 1$, depending on N and q only, such that for every multiindex γ with $|\gamma| = q$ and every $f \in C_p^q(\mathbb{R}^d)$

$$|\mathbb{E}(\partial_\gamma f(S_n(Y))) - \mathbb{E}(\partial_\gamma f(W)\Phi_{n,N}(W))| \leq C \times \mathrm{C}_*(Y)\Big(\frac{L_0(f)}{n^{\frac{1}{2}(N+1)}} + L_q(f) e^{-\frac{\mathfrak{m}_r^2}{32}\times n}\Big) \tag{2.22}$$

where $\mathrm{C}_*(Y)$ is a constant which depends on $q, l_q(f), N$ and $\mathrm{C}_p(Y)$ for $p = 2(N+3) \vee 2l_0(f)$..

Remark 2.4. The precise value of $\mathrm{C}_*(Y)$ is given by

$$\begin{aligned}\mathrm{C}_*(Y) = &\Big(1 \vee \frac{8}{\mathfrak{m}_r}\Big)^{2dp_2} 2^{(N+3)l_0(f)+l_q(f)}\, \hat{c}_{p_1,l_0(f)} c_{l_q(f)\vee(l_0(f)+p_1)} \\ &\times \frac{\mathrm{C}_{16dp_2}^{(4d+1)p_2}(Y)}{r^{p_2(p_2+1)}} \mathrm{C}_{2(N+3)}^{(\lfloor N/2\rfloor+2)(N+1)}(Y)\big(1+\mathrm{C}_{2l_0(f)}^{l_0(f)\vee(N+1)}(Y)\big)\end{aligned} \tag{2.23}$$

with $p_1 = q + (N+1)(N+3), \quad p_2 = q + p_1,$

$$\mathrm{c}_\rho = \int |\phi(z)|\,(1+|z|)^\rho dz, \quad \hat{c}_{p,l} = 1 \vee \max_{0\le|\alpha|\le p} \int (1+|x|)^l |\partial_\alpha \phi(x)| dx \tag{2.24}$$

in which ϕ denotes a super kernel (see next (5.15) and (5.16)).

Actually the coefficients $H_{\Gamma_{n,k}}(x)$ of the polynomial $\Phi_{n,N}(x)$ are cumbersome. The following corollary, whose proof is postponed in Section 5.3.2, gives a plain expansion of order three:

Corollary 2.5. *Let the set-up of Theorem 2.3 holds. For a multiindex α and $i,j \in \{1,\dots,d\}$, set*

$$c_n(\alpha) = \frac{1}{n}\sum_{r=1}^n \Delta_{n,r}(\alpha) \quad \text{and} \quad \bar{c}_n(\alpha,i,j) = \frac{1}{n}\sum_{r=1}^n \Delta_{n,r}(\alpha)\sigma_{n,r}^{ij}. \tag{2.25}$$

Then there exists $C \ge 1$, depending on N and q only, such that for every multiindex γ with $|\gamma| = q$ and every $f \in C_p^q(\mathbb{R}^d)$

$$\Big|\mathbb{E}\Big(\partial_\gamma f(S_n(Y))\Big) - \mathbb{E}\Big(\partial_\gamma f(W)\Big(1+\sum_{k=1}^3 \frac{1}{n^{k/2}}\mathcal{H}_{n,k}(W)\Big)\Big)\Big| \le C\mathrm{C}_*(Y)\Big(\frac{L_0(f)}{n^2} + L_q(f)e^{-\frac{\mathfrak{m}_r^2}{32}\times n}\Big) \tag{2.26}$$

where $\mathrm{C}_*(Y)$ is given in (2.24) and

$$\mathcal{H}_{n,1}(x) = \frac{1}{6}\sum_{|\alpha|=3} c_n(\alpha)H_\alpha(x), \tag{2.27}$$

$$\mathcal{H}_{n,2}(x) = \frac{1}{24}\sum_{|\alpha|=4} c_n(\alpha)\,H_\alpha(x) + \frac{1}{72}\sum_{|\alpha|=3}\sum_{|\beta|=3} c_n(\alpha)c_n(\beta)H_{(\alpha,\beta)}(x), \tag{2.28}$$

$$\begin{aligned}\mathcal{H}_{n,3}(x) = &-\frac{1}{12}\sum_{|\alpha|=3}\sum_{i,j=1}^2 \bar{c}_n(\alpha,i,j)H_{(\alpha,\beta)}(x) + \frac{1}{120}\sum_{|\alpha|=5} c_n(\alpha)H_\alpha(x) \\ &+\frac{1}{144}\sum_{|\alpha|=3}\sum_{|\beta|=4} c_n(\alpha)c_n(\beta)H_{(\alpha,\beta)} + \frac{1}{1296}\sum_{|\alpha|=3}\sum_{|\beta|=3}\sum_{|\gamma|=3} c_n(\alpha)c_n(\beta)c_n(\gamma)H_{(\alpha,\beta,\gamma)}(x).\end{aligned} \tag{2.29}$$

Remark 2.6. We stress that the coefficients of the Hermite polynomials appearing in $\mathcal{H}_{n,1}(x)$–$\mathcal{H}_{n,3}(x)$ depend on n (this is because we work with $C_{n,k}Y_k$, $k=1,\dots,n$, whose law depends on n) but in a bounded way. In fact, by the formula (2.25) and by (2.13), for $|\alpha| = l$ and $i,j \in \{1,\dots,d\}$,

$$|c_n(\alpha)| \le 2\mathrm{C}_l(Y) \text{ and } |\bar{c}_n(\alpha,i,j)| \le 4\mathrm{C}_l(Y)\mathrm{C}_2(Y), \quad \text{for every } n.$$

Remark 2.7. In the one dimensional case Bobkov obtained in [17] (see Proposition 14.1 therein) the following development using Hermite polynomials:

$$\Phi_{n,N}(x) = 1 + \sum \frac{1}{k_1!\dots k_N!}\Big(\frac{\gamma_{n,3}}{3!}\Big)^{k_1} \dots \Big(\frac{\gamma_{n,N+2}}{(N+2)!}\Big)^{k_N} \times H_k(x)$$

where $k = 3k_1 + \dots + (N+2)k_N$ and the summation is made over the non negative integers $k_1,\dots k_N$ such that $0 < k_1 + 2k_2 + \dots + Nk_N \le N$. And $\gamma_{n,p}$ is the p-cumulant of $S_n(Y)$. This is an alternative way to write the correctors which is ordered according to the powers of the Hermite polynomials (and of course, the two expressions are equivalent and one may pass from one to another).

The proof of Theorem 2.3 is done by using a Malliavin type calculus based on the random variables V_k's coming from the Nummelin's splitting associated to the Y_k's. This differential calculus is developed in next Section 5.1. The proof of Theorem 2.3 represents the main effort in this paper, so we postpone it to Section 5.3.1. As for Corollary 2.5, the proof consists in heavy but straightforward computations, so we postpone in Section 5.3.2.

We give now two slight variants of Theorem 2.3 which will be used in the following. First:

Proposition 2.8. *Let (2.2) and (2.3) hold. Assume that for some $n_* < n$ one has $Y_k \in \mathfrak{D}(r,\varepsilon)$ for $k \leq n-n_*$ and $\frac{1}{n}\sum_{k=1}^{n-n_*}\sigma_{n,k} \geq \frac{1}{2}\mathrm{Id}_d$. Then (2.22) holds true.*

The proof of Proposition 2.8 mimics the one of Theorem 2.3 so we postpone it as well, in next Section 5.3.3. This result will be used in the proof of Corollary 2.11 below.

Let us now show how to get the estimate in Theorem 2.3 without assuming the normalization condition (2.2). We assume that $\Sigma_n := \frac{1}{n}\sum_{k=1}^{n}\sigma_{n,k}$, is invertible and we denote $\overline{C}_{n,k} = \Sigma_n^{-1/2}C_{n,k}$. Then we construct $\Phi_{n,N}^{\Sigma_n}$ as in (2.15) by using $\overline{\Delta}_{n,k}(\alpha) = \mathbb{E}((\overline{C}_{n,k}Y_k)^\alpha) - \mathbb{E}((\overline{C}_{n,k}G_k)^\alpha)$.

Proposition 2.9. *Assume that $Y_k \in \mathfrak{D}(r,\varepsilon), \forall k \in \mathbb{N}$ for some $\varepsilon > 0, r > 0$ and $\Sigma_n = \frac{1}{n}\sum_{k=1}^{n}\sigma_{n,k}$ is invertible and condition (2.3) hold. Let $N, q \in \mathbb{N}$ be fixed. Then Theorem 2.3 holds as well and (2.22) reads: for a multiindex α with $|\alpha| = q$,*

$$\begin{aligned}&\left|\mathbb{E}(\partial_\alpha f(S_n(Y))) - \mathbb{E}(\partial_\alpha f(\Sigma_n^{1/2}W)\Phi_{n,N}^{\Sigma_n}(W))\right| \\ &\quad\leq C\underline{\lambda}_n^{-q} \times \mathrm{C}_*(Y)\Big(\frac{1}{n^{\frac{1}{2}(N+1)}}L_0(f) + L_q(f)e^{-\frac{\mathfrak{m}_{r_n}^2}{16}\times n}\Big)\end{aligned} \tag{2.30}$$

where W is a standard Gaussian random variable, $\mathrm{C}_*(Y)$ is given in (2.23) and $\underline{\lambda}_n$ is the lower eigenvalue of Σ_n.

Proof. For an invertible matrix $\sigma \in \mathrm{Mat}(d\times d)$ and for $f : \mathbb{R}^d \to \mathbb{R}$, let $f_\sigma(x) = f(\sigma x)$. A simple computation shows that

$$(\partial_\alpha f)(\sigma x) = \sum_{|\beta|=|\alpha|}(\sigma^{-1})^{\alpha,\beta}\partial_\beta f_\sigma(x),$$

where, for any two multiindexes α and β with $|\alpha| = q = |\beta|$,

$$(\sigma^{-1})^{\alpha,\beta} = \prod_{i=1}^{q}(\sigma^{-1})^{\alpha_i,\beta_i}.$$

We denote now $\overline{S}_n(Y) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\overline{C}_{n,k}Y_k = \Sigma_n^{-1/2}S_n(Y)$ verifies the normalization condition (2.2). So using (2.22) for $\overline{S}_n(Y)$ we obtain

$$\begin{aligned}\mathbb{E}(\partial_\alpha f(S_n(Y))) &= \mathbb{E}(\partial_\alpha f(\Sigma_n^{1/2}\overline{S}_n(Y))) = \sum_{|\beta|=q}(\Sigma_n^{-1/2})^{\alpha,\beta}\mathbb{E}(\partial_\beta f_{\Sigma_n^{1/2}}(\overline{S}_n(Y))) \\ &= \sum_{|\beta|=q}(\Sigma_n^{-1/2})^{\alpha,\beta}\big(\mathbb{E}(\partial_\beta f_{\Sigma_n^{1/2}}(W)\Phi_N^{\Sigma_n}(W)) + R_N^\beta(n)\big) \\ &= \mathbb{E}(\partial_\alpha f(\Sigma_n^{1/2}W)\Phi_N^{\Sigma_n}(W)) + \sum_{|\beta|=q}(\Sigma_n^{-1/2})^{\alpha,\beta}R_N^\beta(n).\end{aligned}$$

The estimate of $R_N(n)$ follows from $L_q(f_{\Sigma_n^{1/2}}) \leq \overline{\lambda}_n^q L_q(f)$ and $\sum_{|\beta|=q}(\Sigma_n^{-1/2})^{\alpha,\beta} \leq C\underline{\lambda}_n^{-q}\overline{\lambda}_n^{dq}$. ☐

Another immediate consequence of Theorem 2.3 is given by the following estimate for an “approximative density” of the law of $S_n(Y)$:

Proposition 2.10. *Assume that $Y_k \in \mathfrak{D}(r,\varepsilon)$ for some $\varepsilon > 0, r > 0$ and let (2.2) and (2.3) hold. Suppose that $n^{\frac{1}{2}(N+1)} e^{-\frac{\mathfrak{m}_r^2 n}{256}} \leq 1$ and $n \geq 4(N+1)\mathrm{C}_2(Y)$. Let δ_n be such that*

$$n^{(N+1)/2d} e^{-\frac{\mathfrak{m}_r^2}{32d}\times n} \leq \delta_n \leq \frac{1}{n^{\frac{1}{2}(N+1)}}.$$

Then

$$\left| \mathbb{E}\Big(\frac{1}{\delta_n^d} 1_{\{|S_n(Y)-a|\leq \delta_n\}}\Big) - \gamma_d(a)\Phi_{n,N}(a) \right| \leq \frac{C}{n^{\frac{1}{2}(N+1)}}, \tag{2.31}$$

where γ_d denotes the density of the standard normal law in $\mathbb{R}^d$.

Proof. Let $h(x) = \int_{-\infty}^{x_1} dx_1 ... \int_{-\infty}^{x_{d-1}} \frac{1}{\delta_n^d} 1_{\{|x-a|\leq\delta_n\}} dx_d$ so that $\frac{1}{\delta_n^d} 1_{\{|x-a|\leq\delta_n\}} = \partial_{x_1}...\partial_{x_d} h(x)$. Using Theorem 2.3

$$\begin{aligned} \mathbb{E}\Big(\frac{1}{\delta_n^d} 1_{\{|S_n(Y)-a|\leq\delta_n\}}\Big) &= \mathbb{E}(\partial_{x_1}...\partial_{x_d} h(S_n(Y))) = \mathbb{E}(\partial_{x_1}...\partial_{x_d} h(W)\Phi_{n,N}(W)) + R_N(n) \\ &= \mathbb{E}\Big(\frac{1}{\delta_n^d} 1_{\{|W-a|\leq\delta_n\}} \Phi_{n,N}(W)\Big) + R_N(n) \end{aligned}$$

with

$$|R_N(n)| \leq C\Big(\frac{1}{n^{\frac{1}{2}(N+1)}} + \frac{1}{\delta_n^d} e^{-\frac{\mathfrak{m}_r^2}{32}\times n}\Big) \leq \frac{C}{n^{\frac{1}{2}(N+1)}}$$

the last inequality being true by our choice of δ_n. Moreover

$$\begin{aligned} \mathbb{E}\Big(\frac{1}{\delta_n^d} 1_{\{|W-a|\leq\delta_n\}} \Phi_{n,N}(W)\Big) &= \int_{\mathbb{R}^d} \frac{1}{\delta_n^d} 1_{\{|y-a|\leq\delta_n\}})\Phi_{n,N}(y)\gamma_d(y)dy \\ &= \Phi_{n,N}(a)\gamma_d(a) + R'(n) \end{aligned}$$

with $|R'(n)| \leq \frac{C}{n^{\frac{1}{2}(N+1)}}$, as a further consequence of the choice of δ_n. □

We now prove a stronger version of Prohorov’s theorem. We consider a sequence of identical distributed, centered random variables $X_k \in \mathbb{R}^d$ which have finite moments of any order and we look to

$$S_n(X) = \frac{1}{\sqrt{n}} \sum_{k=1}^n X_k.$$

Following Prohorov we assume that there exist $n_* \in \mathbb{N}$ such that

$$\mathbb{P}(X_1 + \cdots + X_{n_*} \in dx) = \mu(dx) + \psi(x)dx \tag{2.32}$$

for some measurable non negative function ψ.

Corollary 2.11. *We assume that (2.32) holds. We fix $q, N \in \mathbb{N}$. There exist two constants $0 < \mathrm{c}_* \leq 1 \leq \mathrm{C}_*$, depending on N and q, such that the following holds: if*

$$n^{\frac{1}{2}(N+1)} e^{-\mathrm{c}_* n} \leq 1$$

then, for every multiindex γ with $|\gamma| \leq q$ and for every $f \in C_p^q(\mathbb{R}^d)$ one has

$$|\mathbb{E}(\partial_\gamma f(S_n(X))) - \mathbb{E}(\partial_\gamma f(W)\Phi_{n,N}(W))| \leq \mathrm{C}_*\Big(\frac{1}{n^{\frac{1}{2}(N+1)}} L_0(f) + L_q(f) e^{-\mathrm{c}_*\times n}\Big). \tag{2.33}$$

Proof. . We denote

$$Y_k = \sum_{i=2kn_*+1}^{2(k+1)n_*} X_i \quad \text{and} \quad Z_k = \frac{1}{\sqrt{n}} Y_k.$$

Notice that we may take ψ in (2.32) to be bounded with compact support. Then $\psi * \psi$ is continuous and so we may find some $r > 0, \varepsilon > 0$ and $y \in \mathbb{R}^d$ such that $\psi * \psi \geq \varepsilon 1_{B_{2r}(y)}$. It follows that $Y_k \in \mathfrak{D}(r, \varepsilon)$ and we may use Theorem 2.3 in order to obtain (2.33) for $n = 2n_* \times n'$ with $n' \in \mathbb{N}$. But this is not satisfactory because we claim that (2.33) holds for every $n \in \mathbb{N}$. This does not follow directly but needs to come back to the proof of Theorem 2.3 and to adapt it in the following way. Suppose that $2n_* n' \leq n < 2n_*(n'+1)$. Then

$$S_n(X) = S_{2n_*n'}(X) + \frac{1}{\sqrt{n}} \sum_{k=2n_*n'+1}^{n} X_k = \frac{1}{\sqrt{n}} \sum_{k=1}^{n'} Y_k + \frac{1}{\sqrt{n}} \sum_{k=2n_*n'+1}^{n} X_k.$$

Since X_k, $2n_*n' + 1 \leq k \leq n$, have no regularity property, we may not use them in the regularization arguments employed in the proof of Theorem 2.3. But Y_k, $1 \leq k \leq n'$ contain sufficient noise in order to achieve the proof (see the proof of Proposition 2.8 in next Section 5.3.3). □

2.3 Convergence in distribution norms

In this section we prove that, under some supplementary regularity assumptions on the laws of Y_k, $k \in \mathbb{N}$, Theorem 2.3 implies that the density of the law of $S_n(Y)$ converges in distribution norms to the Gaussian density. We consider the case $C_{n,k} \equiv C_k$, that is,

$$S_n(Y) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} C_k Y_k,$$

and we denote $\sigma_k = C_k C_k^*$. We assume that

$$0 < \underline{\sigma} \leq \inf_k \sigma_k \leq \sup_k \sigma_k \leq \overline{\sigma} < \infty \quad \text{and} \quad \sup_k \|Y_k\|_p^p < \infty. \tag{2.34}$$

In particular each σ_k is invertible. We denote $\gamma_k = \sigma_k^{-1}$. For a function $f \in C^1(\mathbb{R}^d)$ and for $k \in \mathbb{N}$ we denote

$$m_{1,k}(f) = \int_{\mathbb{R}^d} (1 + |x|)^k \, |\nabla f(x)| \, dx.$$

Proposition 2.12. *We fix $q \in \mathbb{N}$ and we also fix a polynomial P. Suppose that $Y_k \in \mathfrak{D}(r, \varepsilon)$, $k \in \mathbb{N}$, and (2.34) holds. Suppose moreover that*

$$\mathbb{P}(Y_k \in dy) = p_{Y_k}(y) dy \quad \textit{with} \quad p_{Y_k} \in C^1(\mathbb{R}^d) \quad \textit{for every for } i = 1, ..., q. \tag{2.35}$$

***A**. There exist some constants $\mathrm{c} \in (0,1)$ (depending on r and on ε) and $\mathrm{D}_q(P) \geq 1$ (depending on $q, \underline{\sigma}, \overline{\sigma}$ and on P) such that, if $n^{(q+1)/2} e^{-cn} \leq 1$, then for every $f \in C_p^q(\mathbb{R}^d)$ and every multiindex α with $|\alpha| \leq q$,*

$$|\mathbb{E}(P(S_n(Y)) \partial_\alpha f(S_n(Z)) - \mathbb{E}(P(S_n(G)) \partial_\alpha f(S_n(G))| \leq \frac{\mathrm{D}_q(P)}{\sqrt{n}} \prod_{i=1}^{q} m_{1, l_0(f) + l_0(P)}(p_{Y_i}) \times L_0(f). \tag{2.36}$$

***B**. Moreover, if p_{S_n} is the density of the law of $S_n(Y)$ then, if $n^{(d+q+1)/2} e^{-cn} \leq 1$, we have*

$$\sup_{x \in \mathbb{R}^d} |P(x)(\partial_\alpha p_{S_n}(x) - \partial_\alpha \gamma_d(x))| \leq \frac{\mathrm{D}_{q+d}(P)}{\sqrt{n}} \prod_{i=1}^{q+d} m_{1, l_0(f) + l_0(P)}(p_{Y_i}) \tag{2.37}$$

where γ_d is the density of the standard normal law in $\mathbb{R}^d$.

Proof. **A.** We proceed by recurrence on the degree i of the polynomial P. First we assume that $i=0$ (so that P is a constant) and we prove (2.36) for every $q\in\mathbb{N}$. We write

$$S_n(Y)=\frac{1}{\sqrt{n}}\sum_{k=1}^{n}C_kY_k=\frac{1}{\sqrt{n}}\sum_{k=1}^{q}C_kY_k+S_n^{(q)}(Y).$$

with

$$S_n^{(q)}(Z)=\frac{1}{\sqrt{n}}\sum_{k=q+1}^{n}C_kY_k.$$

Then we define

$$g(x)=\mathbb{E}\Big(f\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^{q}C_kY_k+x\Big)\Big)$$

and we have

$$\mathbb{E}(\partial_\alpha f(S_n(Y)))=\mathbb{E}(\partial_\alpha g(S_n^{(q)}(Y))).$$

Now using (2.30) with $N=0$ for $S_n^{(q)}(Y)$ we get

$$\mathbb{E}(\partial_\alpha g(S_n^{(q)}(Y)))=\mathbb{E}(\partial_\alpha g(S_n^{(q)}(G)))+R_n=\mathbb{E}\Big(\partial_\alpha f\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^{q}C_kY_k+S_n^{(q)}(G)\Big)\Big)+R_n \quad (2.38)$$

with

$$|R_n|\leq C\Big(\frac{1}{\sqrt{n}}L_0(g)+e^{-cn}L_q(g)\Big). \quad (2.39)$$

Let us estimate $L_q(g)$. We set $\gamma_k=\sigma_k^{-1}$. For $\alpha=(\alpha_1,...,\alpha_q)$ we have

$$(\partial_\alpha f)\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}C_ky_k+x\Big)=\sum_{\beta_1,\ldots,\beta_q=1}^{d}n^{q/2}\Big(\prod_{k=1}^{q}(\gamma_kC_k)^{\alpha_k,\beta_k}\Big)\times\partial_{y_1^{\beta_1}}....\partial_{y_q^{\beta_q}}\Big(f\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}C_ky_k+x\Big)\Big), \quad (2.40)$$

in which we have assumed that the Y_k's take values in $\mathbb{R}^m$. So

$$\begin{aligned}
\partial_\alpha g(x)&=\mathbb{E}\Big((\partial_\alpha f)\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^{q}C_kY_k+x\Big)\Big)\\
&=n^{q/2}\sum_{\beta_1,\ldots,\beta_q=1}^{m}\Big(\prod_{k=1}^{q}(\gamma_kC_k)^{\alpha_k,\beta_k}\Big)\int_{\mathbb{R}^{qm}}\partial_{y_1^{\beta_1}}....\partial_{y_q^{\beta_q}}\Big(f\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}C_ky_k+x\Big)\Big)\prod_{k=1}^{q}p_{Y_k}(y_k)dy_1...dy_q\\
&=(-1)^qn^{q/2}\sum_{\beta_1,\ldots,\beta_q=1}^{m}\Big(\prod_{k=1}^{q}(\gamma_kC_k)^{\alpha_k,\beta_k}\Big)\int_{\mathbb{R}^{qm}}f\Big(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}C_ky_k+x\Big)\prod_{k=1}^{q}\partial_{y_k^{\beta_k}}p_{Y_k}(y_k)dy_1...dy_q.
\end{aligned}$$

It follows that

$$\begin{aligned}
|\partial_\alpha g(x)|&\leq Cn^{q/2}L_0(f)\int_{\mathbb{R}^q}(1+|x|+\sum_{k=1}^{q}|y_k|)^{l_0(f)}\prod_{k=1}^{q}|\nabla p_{Y_k}(y_k)|\,dy_1...dy_q\\
&\leq Cn^{q/2}L_0(f)(1+|x|)^{l_0(f)}\prod_{k=1}^{q}m_{1,l_0(f)}(p_{Y_k}).
\end{aligned}$$

We conclude that $l_q(g)=l_0(f)$ and $L_q(g)\leq Cn^{q/2}L_0(f)\prod_{k=1}^{q}m_{1,l_0(f)}(p_{Y_k})$. The same is true for $q=0$ and so (2.39) gives

$$|R_n|\leq CL_0(f)\prod_{k=1}^{q}m_{1,l_0(f)}(p_{Y_k})\Big(\frac{1}{\sqrt{n}}+n^{q/2}e^{-cn}\Big)\leq CL_0(f)\prod_{k=1}^{q}m_{1,l_0(f)}(p_{Y_k})\times\frac{1}{\sqrt{n}}$$

the last inequality being true if $n^{q/2}e^{-cn}\leq n^{-1/2}$.

So (2.38) says that we succeed to replace $Y_k, q+1\leq k\leq n$ by $G_k, q+1\leq k\leq n$ and the price to be paid is $CL_0(f)\prod_{k=1}^{q}m_{1,l_0(f)}(p_{Y_k})\times\frac{1}{\sqrt{n}}$. Now we can do the same thing and replace $Y_k, 1 \le k \le q$ by $G_k, 1 \le k \le q$ and the price will be the same (here we use $C_k G_k, k = q+1, ..., 2q$ instead of $C_k Y_k, k = 1, ..., q$). So (2.36) is proved for polynomials P of degree $i = 0$.

We assume now that (2.36) holds for every polynomials of degree less or equal to $i-1$ and we prove it for a polynomial P of order i. We have

$$\partial_\alpha (P \times f) = \sum_{(\beta,\gamma)=\alpha} \partial_\beta P \times \partial_\gamma f$$

so that

$$P \times \partial_\alpha f = \partial_\alpha (P \times f) - \sum_{\substack{(\beta,\gamma)=\alpha \\ |\beta|\ge 1}} \partial_\beta P_i \times \partial_\gamma f.$$

Since $|\beta| \ge 1$ the polynomial $\partial_\beta P$ has degree at most $i-1$. Then the recurrence hypothesis ensures that (2.36) holds for $\partial_\beta P \times \partial_\gamma f$. Moreover, using again (2.36) for $g = P \times f$ we obtain (2.36) in which $L_0(g) \le L_0(P) L_0(f)$ and $l_0(g) \le l_0(P) + l_0(f)$ appear. So **A.** is proved.

Let us prove **B**. We denote $f_x(y) = \prod_{k=1}^d 1_{(x,\infty)}(y)$ and, for a multiindex $\alpha = (\alpha_1, ..., \alpha_q)$ we denote $\overline{\alpha} = (\alpha_1, ..., \alpha_q, 1, ..., d)$. Then, using a formal computation (which may de done rigorously by means of a regularization procedure) we obtain

$$\begin{aligned} P(x)\partial_\alpha p_{S_n}(x) &= \int \delta_0(y-x) P(y) \partial_\alpha p_{S_n}(y) dy \\ &= (-1)^q \sum_{(\beta,\gamma)=\alpha} \int \partial_\beta \delta_0(y-x) \partial_\gamma P(y) p_{S_n}(y) dy \\ &= (-1)^q \sum_{(\beta,\gamma)=\alpha} \int \partial_{\overline{\beta}} f_x(y) \partial_\gamma P(y) p_{S_n}(y) dy \\ &= (-1)^q \sum_{(\beta,\gamma)=\alpha} \mathbb{E}(\partial_{\overline{\beta}} f_x(S_n(Y)) \partial_\gamma P(S_n(Y))). \end{aligned}$$

A similar computation holds with $S_n(Y)$ replaced by $S_n(G)$. So we have

$$\begin{aligned} & |P(x)(\partial_\alpha p_{S_n}(x) - \partial_\alpha \gamma(x)| \\ \le & \sum_{(\beta,\gamma)=\alpha} \left| \mathbb{E}(\partial_{\overline{\beta}} f_x(S_n(Y)) \partial_\gamma P(S_n(Z))) - \mathbb{E}(\partial_{\overline{\beta}} f_x(S_n(G)) \partial_\gamma P(S_n(G))) \right| \\ \le & \frac{C_{q+d}(P)}{\sqrt{n}} \prod_{k=1}^{q+d} m_{1, l_0(f)+l_0(P)}(p_{Y_k}) \end{aligned}$$

the last inequality being a consequence of (2.36). □

Remark 2.13. We would like to obtain Edgeworth's expansions as well – but there is a difficulty: when we use the expansion for $S_n^{(q)}(Z)$ we are in the situation when the covariance matrix of $S_n^{(q)}(Z)$ is not the identity matrix. So the coefficients of the expansion are computed using a correction (see the definition of $\overline{\Delta}_k$ in Proposition 2.9). And this correction produces an error of order $n^{-1/2}$. This means that we are not able to go beyond this level (at least without supplementary technical effort).

3 Examples

3.1 An invariance principle related to the local time

In this section we consider a sequence of independent identically distributed, centered random variables Y_k, $k \in \mathbb{N}$, with finite moments of any order and we denote

$$S_n(k,Y)=\frac{1}{\sqrt{n}}\sum_{i=1}^{k}Y_i.$$

Our aim is to study the asymptotic behavior of the expectation of

$$L_n(Y)=\frac{1}{n}\sum_{k=1}^{n}\psi_{\varepsilon_n}(S_n(k,Y)) \quad \text{with} \quad \psi_{\varepsilon_n}(x)=\frac{1}{2\varepsilon_n}1_{\{|x|\leq\varepsilon_n\}}.$$

So $L_n(Y)$ appears as the occupation time of the random walk $S_n(k,Y), k=1,...,n$, and consequently, as $\varepsilon_n\to 0$, one expects that it has to be close to the local time in zero at time 1, denoted by l_1, of the Brownian motion. In fact, we prove now that $\mathbb{E}(L_n(Y))\to\mathbb{E}(l_1)$ as $n\to\infty$.

Theorem 3.1. *Let $\varepsilon_n=n^{-\frac{1}{2}(1-\rho)}$ with $\rho\in(0,1)$. We consider a centered random variable $Y\in\mathfrak{D}(r,\varepsilon)$ which has finite moments of any order and we take a sequence $Y_i, i\in\mathbb{N}$ of independent copies of Y. We define*

$$N(Y)=\max\{2k:\mathbb{E}(Y^{2k})=\mathbb{E}(G^{2k})\}-1\geq 1$$

and we denote $p_{N(Y)}=8(1+(N(Y)+1)(N(Y)+3))(4+(N(Y)+1)(N(Y)+3))$. For every $\eta<1$ there exists a constant C depending on r,ε,ρ,η and on $\|Y\|_{p_{N(Y)}}$ such that

$$\left|\mathbb{E}(L_n(Y))-\mathbb{E}(L_n(G))\right|\leq\frac{\mathrm{C}}{n^{\frac{1}{2}+\frac{\eta\rho N(Y)}{2}}}. \tag{3.1}$$

The above inequality holds for n which is sufficiently large in order to have

$$n^{\frac{1}{2}}\exp\Big(-\frac{\mathfrak{m}_r^2}{32}\times n^{\rho\eta}\Big)\leq\frac{1}{n^{\frac{1}{2}(N(Y)+1)\eta\rho}} \tag{3.2}$$

As a consequence, we have

$$\lim_{n\to\infty}\mathbb{E}(L_n(Y))=\mathbb{E}(l_1), \tag{3.3}$$

l_1 denoting the local time in the point 0 at time 1 of a Brownian motion.

Proof. All over this proof we denote by C a constant which depends on r,ε,ρ,η and on $\|Y\|_{p_{N(Y)}}$ (as in the statement of the lemma) and which may change from a line to another.

Step 1. We take $k_n=n^{\eta\rho}$. Suppose first that $k\leq k_n$. We write

$$\mathbb{E}(\psi_{\varepsilon_n}(S_n(k,Y)))=\frac{1}{\varepsilon_n}\Big(1-\mathbb{P}(|S_n(k,Y)|\geq\varepsilon_n)\Big)$$

so that

$$|\mathbb{E}(\psi_{\varepsilon_n}(S_n(k,Y)))-\mathbb{E}(\psi_{\varepsilon_n}(S_n(k,G)))|\leq\frac{1}{\varepsilon_n}\big(\mathbb{P}(|S_n(k,Y)|\geq\varepsilon_n)+\mathbb{P}(|S_n(k,G)|\geq\varepsilon_n)\big).$$

Using Chebyshev's inequality and Burkholder's inequality we obtain for every $p\geq 2$

$$\begin{aligned}\mathbb{P}(|S_n(k,Y)|\geq\varepsilon_n)&=\mathbb{P}\Big(\Big|\sum_{i=1}^{k}Y_i\Big|\geq\varepsilon_n\sqrt{n}\Big)\leq\frac{1}{(\varepsilon_n\sqrt{n})^p}\mathbb{E}\Big(\Big|\sum_{i=1}^{k}Y_i\Big|^p\Big)\\&\leq\frac{\mathrm{C}}{(\varepsilon_n\sqrt{n})^p}\Big(\sum_{i=1}^{k}\|Y_i\|_p^2\Big)^{p/2}\leq\frac{\mathrm{C}k^{p/2}}{(\varepsilon_n\sqrt{n})^p}=\frac{\mathrm{C}}{\varepsilon_n^p}\times\Big(\frac{k}{n}\Big)^{p/2}.\end{aligned}$$

And the same estimate holds with Y_i replaced by G_i. We conclude that

$$\Big|\mathbb{E}\Big(\frac{1}{n}\sum_{k=1}^{k_n}\psi_{\varepsilon_n}(S_n(k,Y))\Big)-\mathbb{E}\Big(\frac{1}{n}\sum_{k=1}^{k_n}\psi_{\varepsilon_n}(S_n(k,G))\Big)\Big|\leq\frac{\mathrm{C}}{\varepsilon_n^{p+1}}\times\frac{1}{n}\sum_{k=1}^{k_n}\Big(\frac{k}{n}\Big)^{p/2}$$
$$\leq\frac{\mathrm{C}}{\varepsilon_n^{p+1}}\times\int_0^{k_n/n}x^{p/2}dx=\frac{\mathrm{C}}{\varepsilon_n^{p+1}}\times\Big(\frac{k_n}{n}\Big)^{\frac{p}{2}+1}$$
$$=\frac{\mathrm{C}}{n^{\frac{p\rho}{2}(1-\eta)+\frac{1}{2}-(\eta-\frac{1}{2})\rho}}\leq\frac{\mathrm{C}}{n^{\frac{p\rho}{2}(1-\eta)}}.$$

We take $p=\frac{1+\rho\eta N(Y)}{\rho(1-\eta)}$ and we obtain

$$\Big|\mathbb{E}\Big(\frac{1}{n}\sum_{k=1}^{k_n}\psi_{\varepsilon_n}(S_n(k,Y))\Big)-\mathbb{E}\Big(\frac{1}{n}\sum_{k=1}^{k_n}\psi_{\varepsilon_n}(S_n(k,G))\Big)\Big|\leq\frac{\mathrm{C}}{n^{\frac{1}{2}+\frac{N(Y)}{2}\eta\rho}}.$$

Step 2. We fix now $k\geq k_n$ and we apply our Edgeworth development (2.22) to

$$\frac{1}{\sqrt{k}}\sum_{i=1}^{k}Y_i.$$

In particular the constants $\mathrm{C}_p(Y)$ defined in (2.3) are given by $\mathrm{C}_p(Y)=\|Y\|_p^p$. We denote

$$h_{\alpha,n}(x)=\int_{-\infty}^{\alpha x}\psi_{\varepsilon_n}(y)dy=h_{1,n}(\alpha x). \tag{3.4}$$

This gives $\psi_{\varepsilon_n}(x)=h'_{1,n}(x)$ and $h'_{\alpha,n}(x)=\alpha h'_{1,n}(\alpha x)$. Moreover, $\|h_{\alpha,n}\|_\infty\leq 1$ and $\|h'_{\alpha,n}\|_\infty\leq|\alpha|/\varepsilon_n$, so that

$$L_0(h_{\alpha,n})=1\quad\text{and}\quad L_1(h_{\alpha,n})=|\alpha|\times\frac{1}{\varepsilon_n}.$$

We now write

$$\begin{aligned}\mathbb{E}(\psi_{\varepsilon_n}(S_n(k,Y)))&=\mathbb{E}(h'_{1,n}(S_n(k,Y)))=\mathbb{E}\Big(h'_{1,n}\Big(\sqrt{\frac{k}{n}}\frac{1}{\sqrt{k}}\sum_{i=1}^{k}Y_i\Big)\Big)\\&=\sqrt{\frac{n}{k}}\mathbb{E}\Big(h'_{\sqrt{\frac{k}{n}},n}\Big(\frac{1}{\sqrt{k}}\sum_{i=1}^{k}Y_i\Big)\Big).\end{aligned}$$

We use now (2.22) with $f=h_{\sqrt{\frac{k}{n}},n}$ and here ∂_γ is the first order derivative. Then, by (2.22) with $N=N(Y)$

$$\mathbb{E}(\psi_{\varepsilon_n}(S_n(k,Y))=\sqrt{\frac{n}{k}}\Big(\mathbb{E}(h'_{\sqrt{\frac{k}{n}},n}(W_1)\Phi_{k,N(Y)}(W_1))+R_{N(Y)}(k)\Big)$$

where W denotes a Brownian motion and with

$$\begin{aligned}\big|R_{N(Y)}(k)\big|&\leq\frac{\mathrm{C}}{k^{(N(Y)+1)/2}}L_0(h_{\sqrt{\frac{k}{n}},n})+\mathrm{C}L_1(h'_{\sqrt{\frac{k}{n}},n})\exp\Big(-\frac{\mathfrak{m}_r^2}{32}\times k\Big)\\&\leq\frac{\mathrm{C}}{k^{(N(Y)+1)/2}}+\mathrm{C}\sqrt{\frac{k}{n}}\times\frac{1}{\varepsilon_n}\exp\Big(-\frac{\mathfrak{m}_r^2}{32}\times k\Big).\end{aligned}$$

Here C is the constant from (2.22) defined in (2.23). Notice that by (3.2), for $k\geq k_n=n^{\eta\rho}$ one has

$$\begin{aligned}\sqrt{\frac{k}{n}}\times\frac{1}{\varepsilon_n}\exp\Big(-\frac{\mathfrak{m}_r^2}{32}\times k\Big)&\leq n^{\frac{1}{2}}\exp\Big(-\frac{\mathfrak{m}_r^2}{32}\times n^{\rho\eta}\Big)\\&\leq\frac{1}{n^{\frac{1}{2}(N(Y)+1)\eta\rho}}=\frac{1}{k_n^{(N(Y)+1)/2}}\leq\frac{\mathrm{C}}{k^{(N(Y)+1)/2}},\end{aligned}$$

so that $\left|R_{N(Y)}(k)\right| \leq \mathrm{C}k^{-(N(Y)+1)/2}$. Then

$$\begin{aligned}\Big|\sum_{k=k_n}^{n}\sqrt{\frac{n}{k}}R_{N(Y)}(k)\frac{1}{n}\Big| &\leq \frac{\mathrm{C}}{n^{(N(Y)+1)/2}}\sum_{k=k_n}^{n}\frac{1}{(k/n)^{1+\frac{N(Y)}{2}}}\times\frac{1}{n}\\ &\leq \frac{\mathrm{C}}{n^{(N(Y)+1)/2}}\int_{k_n/n}^{1}\frac{ds}{s^{1+\frac{N(Y)}{2}}} = \frac{\mathrm{C}}{n^{(N(Y)+1)/2}}(n/k_n)^{\frac{N(Y)}{2}} = \frac{\mathrm{C}}{n^{\frac{1}{2}+\frac{N(Y)\rho\eta}{2}}}.\end{aligned}$$

We recall now that (see (2.21))

$$\Phi_{k,N(Y)}(x) = 1 + \sum_{l=1}^{N(Y)}\frac{1}{n^{l/2}}H_{\Gamma_{k,l}}(x)$$

with $H_{\Gamma_{k,l}}(x)$ linear combination of Hermite polynomials (see (2.15) and (2.20)). Notice that if l is odd then $\Gamma_{k,l}$ is a linear combination of differential operators of odd order (see the definition of $\Lambda_{m,l}$ in (2.14)). So $H_{\Gamma_{k,l}}$ is an odd function (as a linear combination of Hermite polynomials of odd order) so that $\psi_{\varepsilon_n}\times H_{\Gamma_{k,l}}$ is also an odd function. Since W_1 and $-W_1$ have the same law, it follows that

$$\begin{aligned}\mathbb{E}\Big(\psi_{\varepsilon_n}\Big(\sqrt{\frac{k}{n}}\times W_1\Big)H_{k,\Gamma_l}(W_1)\Big) &= \mathbb{E}\Big(\psi_{\varepsilon_n}\Big(\sqrt{\frac{k}{n}}\times(-W_1)\Big)H_{\Gamma_{k,l}}(-W_1)\Big)\\ &= -\mathbb{E}\Big(\psi_{\varepsilon_n}\Big(\sqrt{\frac{k}{n}}\times W_1\Big)H_{\Gamma_{k,l}}(W_1)\Big)\end{aligned}$$

and consequently

$$\sqrt{\frac{n}{k}}\times\mathbb{E}\Big(h'_{\sqrt{\frac{k}{n}},n}(W_1)H_{\Gamma_{k,l}}(W_1)\Big) = \mathbb{E}\Big(\psi_{\varepsilon_n}\Big(\sqrt{\frac{k}{n}}\times W_1\Big)H_{\Gamma_{k,l}}(W_1)\Big) = 0.$$

Moreover, by the definition of $N(Y)$, for $2l \leq N(Y)$ we have $\mathbb{E}(Y^{2l}) = \mathbb{E}(G^{2l})$ so that $H_{\Gamma_{k,2l}} = 0$. We conclude that

$$\begin{aligned}\sqrt{\frac{n}{k}}\mathbb{E}\Big(h'_{\sqrt{\frac{k}{n}},n}(W_1)\Phi_{k,N(Y)}(W_1)\Big) &= \sqrt{\frac{n}{k}}\mathbb{E}\Big(h'_{\sqrt{\frac{k}{n}},n}(W_1)\Big) = \mathbb{E}\Big(\psi_{\varepsilon_n}\Big(\sqrt{\frac{k}{n}}\times W_1\Big)\Big)\\ &= \mathbb{E}(\psi_{\varepsilon_n}(S_n(k,G))).\end{aligned}$$

We put now together the results from the first and the second step and we obtain (3.1).

Step 3. We prove (3.3). Recall first the representation formula

$$\mathbb{E}\Big(\int_0^1\psi_{\varepsilon_n}(W_s)ds\Big) = \mathbb{E}\Big(\int\psi_{\varepsilon_n}(a)l_1^a da\Big),$$

where l_1^a denotes the local time in $a\in\mathbb{R}$ at time 1, so that $l_1 = l_1^0$. Since $a\mapsto l_1^a$ is Hölder continuous of order $\frac{\rho'}{2}$ for every $\rho' < 1$, we obtain

$$\Big|\mathbb{E}\Big(\int_0^1\psi_{\varepsilon_n}(W_s)ds\Big) - \mathbb{E}(l_1^0)\Big| \leq \varepsilon_n^{\rho'/2} = \frac{1}{n^{\frac{\rho'(1-\rho)}{4}}}. \tag{3.5}$$

We prove now that, for every $\rho' < 1$ and n large enough,

$$\Big|\mathbb{E}\Big(\int_0^1\psi_{\varepsilon_n}(W_s)ds\Big) - \mathbb{E}(L_n(G))\Big| \leq \frac{C}{n^{\frac{1+\eta\rho}{2}}}. \tag{3.6}$$

To begin we notice that $S_n(k,G)$ has the same law as $W_{k/n}$, so that we write

$$\mathbb{E}\Big(\int_0^1\psi_{\varepsilon_n}(W_s)ds\Big) - \mathbb{E}(L_n(G)) = \mathbb{E}\Big(\sum_{k=1}^{n}\delta_k\Big),\ \text{with}\ \delta_k = \int_{k/n}^{(k+1)/n}(\psi_{\varepsilon_n}(W_s)-\psi_{\varepsilon_n}(W_{k/n}))ds.$$

As above, we take $k_n = n^{\rho\eta}$ and for $k \le k_n$, we have

$$\mathbb{E}(\delta_k) = -\frac{1}{2\varepsilon_n}\int_{k/n}^{(k+1)/n}\big(\mathbb{P}(|W_s| \ge \varepsilon_n) - \mathbb{P}(|W_{k/n}| \ge \varepsilon_n)\big)ds.$$

Since $\mathbb{P}(|W_s| \ge \varepsilon_n) \le C\exp(-\frac{\varepsilon^2}{2s})$, this immediately gives

$$|\mathbb{E}(\delta_k)| \le \frac{C}{n\varepsilon_n}\exp\Big(-\frac{1}{2}\varepsilon_n^2 \times \frac{n}{k+1}\Big) \le \frac{C}{n\varepsilon_n}\exp\Big(-\frac{1}{2}\varepsilon_n^2 \times \frac{n}{k_n+1}\Big) = \frac{C}{n\varepsilon_n}\exp\Big(-\frac{1}{2}n^{\rho(1-\eta)}\Big)$$

so that

$$\sum_{k=1}^{k_n}|\mathbb{E}(\delta_k)| \le \frac{C}{\varepsilon_n}\exp(-\frac{1}{8}n^{\rho(1-\eta)}) \le \frac{C}{n^{\frac{1+\eta\rho}{2}}},$$

for n large enough.

We consider now the case $k \ge k_n$. Using a formal computation, by applying the standard Gaussian integration by parts formula, we write

$$\begin{aligned}\mathbb{E}(\psi_{\varepsilon_n}(W_s) - \psi_{\varepsilon_n}(W_{k/n})) &= \frac{1}{2}\int_{k/n}^{s}\mathbb{E}(\psi''_{\varepsilon_n}(W_v))dv = \frac{1}{2}\int_{k/n}^{s}\mathbb{E}(\psi''_{\varepsilon_n}(\sqrt{v}W_1))dv \\ &= \int_{k/n}^{s}\mathbb{E}(h'''_{1,n}(\sqrt{v}W_1)H_3(W_1))dv = \int_{k/n}^{s}\frac{1}{2v^{3/2}}\mathbb{E}(h_{1,n}(\sqrt{v}W_1)H_3(W_1))dv,\end{aligned}$$

in which we have used (3.4) and where H_3 denotes the third Hermite polynomial. The above computation is formal because ψ_{ε_n} is not differentiable. But, since the first and the last term in the chain of equalities depends on ψ_{ε_n} only (and not on the derivatives) we may use regularization by convolution in order to do it rigorously. Notice also that the first equality is obtained using Ito's formula and the last one is obtained using integration by parts. It follows that

$$|\mathbb{E}(\delta_k)| \le \int_{k/n}^{(k+1)/n}ds\int_{k/n}^{s}\frac{1}{2v^{3/2}}\mathbb{E}(h_{1,\varepsilon_n}(\sqrt{v}W_1)\,|H_3(W_1)|)dv \le \frac{C}{n}\int_{k/n}^{(k+1)/n}\frac{1}{v^{3/2}}dv$$

and consequently

$$\sum_{k=k_n}^{n}|\mathbb{E}(\delta_k)| \le \frac{C}{n}\int_{k_n/n}^{1}\frac{1}{v^{3/2}}dv \le \frac{C}{n^{\frac{1+\eta\rho}{2}}}.$$

So (3.6) is proved, and this together with (3.5) and (3.1), give (3.3). □

3.2 Small ball estimates

We look to

$$S_n(u,Y) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}C_{n,k}(u)Y_k, \quad u \in \mathbb{R}^\ell, \tag{3.7}$$

where $Y_k \in \mathbb{R}^d$, $k \in \mathbb{N}$, and $C_{n,k}(u) \in \mathrm{Mat}(d \times d)$ (so, here $m = d$).

Theorem 3.2. *Suppose that* $\{Y_k\}_{k\in\mathbb{N}} \subset \mathfrak{D}(\varepsilon, r)$, *with* $\mathrm{M}_p(Y) = \sup_k \|Y_k\|_p < \infty$, *and that* $u \mapsto C^{i,j}_{n,k}(u)$ *is twice differentiable. We assume that for every* $n \in \mathbb{N}, k \le n$ *and* $u \in \mathbb{R}^\ell$

$$\|C_{n,k}\|_{2,\infty} := \sum_{i,j=1}^{d}\sum_{|\alpha|\le 2}\|\partial_u^\alpha C^{i,j}_{n,k}\|_{2,\infty} \le Q_{*,2} < \infty, \tag{3.8}$$

$$\frac{1}{n}\sum_{k=1}^{n}C_{n,k}(u)C^*_{n,k}(u) \ge \lambda_* > 0, \tag{3.9}$$

A. *There exist* $\mathrm{C} \ge 1$ *and* $\mathrm{c} > 0$ *such that for every* $\eta > 0$

$$\sup_{u\in\mathbb{R}^\ell}\mathbb{P}(|S_n(u,Y)| \le \eta) \le \mathrm{C}(\eta^d + e^{-\mathrm{c}n}). \tag{3.10}$$

B. *Suppose that $d > \ell$. Let $a \geq 0$ and $\theta > \frac{a\ell}{d-\ell}$. Then, for every $\varepsilon > 0$*

$$\mathbb{P}\Big(\inf_{|u|\leq n^a} |S_n(u,Y)| \leq \frac{1}{n^\theta}\Big) \leq \frac{\mathrm{C}}{n^{\theta(d-\ell)-a\ell-\varepsilon}} \tag{3.11}$$

The constant C depends on $\mathfrak{m}_r$ (from Doeblin's condition), on $Q_{*,2}, \lambda_*, \ell, d$ and on $\mathrm{M}_p(Y)$ for sufficiently large p.

We first prove the following lemma.

Lemma 3.3. *Under the hypotheses of Theorem 3.2, for every $q > \ell$, $i \in \{1, \dots, d\}$ and $R > 0$ one has*

$$\mathbb{E}\Big(\sup_{|u|\leq R} |\partial_i S_n(u,Y)|^q\Big) \leq \mathrm{C}R^\ell Q_{*,2}^q \mathrm{M}_q^q(Y). \tag{3.12}$$

where C is a constant which depends on q.

Proof. As an immediate consequence of Morrey's inequality one may find a universal constant C (independent of R) such that

$$\sup_{|u|\leq R} |\partial_i S_n(u,Y)| \leq \mathrm{C}\Big(\int_{|u|\leq R+1} |\partial_i S_n(u,Y)|^q + \sum_{j=1}^{\ell} |\partial_j\partial_i S_n(u,Y)|^q\, du\Big)^{1/q}$$

so that

$$\mathbb{E}(\sup_{|u|\leq R} |\partial_i S_n(u,Y)|^q) \leq \mathrm{C}\int_{|u|\leq R+1} \Big(\mathbb{E}\,|\partial_i S_n(u,Y)|^q + \sum_{j=1}^{\ell} \mathbb{E}\,|\partial_j\partial_i S_n(u,Y)|^q\Big) du.$$

Since

$$\partial_j\partial_i S_n(u,Y) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \partial_j\partial_i C_{n,k}(u) Y_k,$$

we can use the Burkholder's inequality for martingales and we obtain

$$\begin{aligned}
\mathbb{E}\,|\partial_j\partial_i S_n(u,Y)|^q &\leq \mathrm{C}\mathbb{E}\Big(\Big[\frac{1}{n}\sum_{k=1}^{n} \|\partial_j\partial_i C_{n,k}(u)(\partial_i\partial_\alpha C_{n,k}(u))^*\|^2 |Y_k|^2\Big]^{q/2}\Big) \\
&\leq \mathrm{C}Q_{*,2}^q \mathbb{E}\Big(\Big[\frac{1}{n}\sum_{k=1}^{n} |Y_k|^2\Big]^{q/2}\Big) \leq \mathrm{C}Q_{*,2}^q M_q^q(Y).
\end{aligned}$$

A similar estimate holds for $\mathbb{E}\,|\partial_i S_n(u,Y)|^q$, so that (3.12) is proved. □

Proof of Theorem 3.2. **A**. Let us prove (3.10). We take $\eta > 0$ and we consider the functions

$$\theta_{d,\eta}(x) = \frac{1}{(c_d\eta)^d} 1_{|x|\leq\eta}, \quad \Theta_{d,\eta}(x) = \int_{-\infty}^{x_1} dx_2 \dots \int_{-\infty}^{x_{d-1}} dx_d \theta_{d,\eta}(x) \tag{3.13}$$

with c_d such that $\int_{\mathbb{R}^d} \theta_{d,\eta}(x)dx = 1$. Then $\partial_1 \dots \partial_d \Theta_{d,\eta} = \theta_{d,\eta}$ so that

$$\mathbb{P}(|S_n(u,Y)| \leq \eta) = (c_d\eta)^d \mathbb{E}(\theta_{d,\eta}(S_n(u,Y)) = (c_d\eta)^d \mathbb{E}(\partial_1 \dots \partial_d \Theta_{d,\eta}(S_n(u,Y)).$$

We denote $S_n(t,G)$ the sum from (3.7) in which $Y_k, k \in \mathbb{N}$, are replaced by standard normal random variables and we use Theorem 2.3, specifically (2.22), in order to obtain

$$\mathbb{E}(\partial_1 \dots \partial_d \Theta_{d,\eta}(S_n(u,Y)) = \mathbb{E}(\partial_1 \dots \partial_d \Theta_{d,\eta}(S(u,G)) + \varepsilon_n(\eta)$$

where

$$|\varepsilon_n(\eta)| \leq \mathrm{C}\Big(\frac{1}{n^{1/2}} + \eta^{-d} e^{-\mathrm{c}n}\Big).$$

Here C is a constant which depends on $\mathfrak{m}_r$ (from Doeblin's condition) on $Q_{*,0}$ and on $\mathrm{M}_p(Y)$ for a sufficiently large p. We conclude that

$$\begin{aligned}\mathbb{P}(|S_n(u,Y)|\leq\eta) &= (c_d\eta)^d\mathbb{E}(\partial_1....\partial_d\Theta_{d,\eta}(S_n(u,Y)))\\ &\leq (c_d\eta)^d\big(\mathbb{E}(\partial_1....\partial_d\Theta_{d,\eta}(S(u,G)))+|\varepsilon_n(\eta)|\big)\\ &= \mathrm{C}\mathbb{P}(|S(u,G)|\leq\eta)+\mathrm{C}\eta^d\,|\varepsilon_n(\eta)|\,.\end{aligned}$$

Since $S_n(u,G)$ is a non degenerate Gaussian random variable we have $\mathbb{P}(|S_n(u,G)|\leq\eta)\leq\mathrm{C}\eta^d\lambda_*^{-d/2}$ and finally we get

$$\mathbb{P}(|S_n(t_\alpha,Y)|\leq\eta)\leq\mathrm{C}\eta^d(\lambda_*^{-d/2}+|\varepsilon_n(\eta)|)\leq\mathrm{C}(\eta^d+e^{-cn}).$$

B. We denote $R_n=n^a$, $\delta_n=n^{-\theta}$ and we take $h>0$ (to be chosen later on). For $\alpha\in\mathbb{Z}^\ell$ we denote $t_\alpha=(t_{\alpha_1},...,t_{\alpha_\ell})=(h\alpha_1,\ldots,h\alpha_\ell)$ and $I_\alpha=[t_{\alpha_1},t_{\alpha_1+1})\times\cdots\times[t_{\alpha_\ell},t_{\alpha_\ell+1})$, so, if $|u|\leq R_n$ then $u\in\cup_{|t_\alpha|\leq R_n}I_\alpha$. Moreover we denote

$$\omega_n=\inf_{|u|\leq R_n}|S_n(u,Y)|\,,\quad \omega_{n,\alpha}=\inf_{u\in I_\alpha}|S_n(u,Y)|$$

and we have

$$\omega_n\geq\min_{|t_\alpha|\leq R_n}\omega_{n,\alpha}.$$

If $\omega_{n,\alpha}<\delta_n$ then there is some $u_\alpha\in I_\alpha$ such that $|S_n(u_\alpha)|\leq\delta_n$. So, with $U_n=\sup_{|u|\leq R_n}|\nabla S_n(u,Y)|$, we have

$$|S_n(t_\alpha)|\leq|S_n(u_\alpha)|+U_nh\leq\delta_n+U_nh.$$

Now we take $\lambda>0$ (to be chosen later on) and we write, with $q>\ell$,

$$\begin{aligned}\mathbb{P}(\omega_n\leq\delta_n) &\leq \mathbb{P}(\omega_n\leq\delta_n,U_n\leq\lambda)+\mathbb{P}(U_n\geq\lambda)\\ &\leq \sum_{|t_\alpha|\leq R_n}\mathbb{P}(\omega_{n,\alpha}\leq\delta_n,U_n\leq\lambda)+\lambda^{-q}\mathbb{E}(U_n^q)\\ &\leq (R_n/h)^\ell\max_{|t_\alpha|\leq R_n}\mathbb{P}(|S_n(t_\alpha,Y)|\leq\delta_n+\lambda h)+\mathrm{C}\lambda^{-q}R_n^\ell Q_{*,2}^qM_q^q(Y)\\ &\leq \mathrm{C}(R_n/h)^\ell((\delta_n+\lambda h)^d+e^{-cn})+\mathrm{C}\lambda^{-q}R_n^\ell Q_{*,2}^qM_q^q(Y),\end{aligned}$$

in which we have used (3.12) and (3.10). We recall that $R_n=n^a$ and $\delta_n=n^{-\theta}$. We take $\lambda=n^\varepsilon$ for a sufficiently small $\varepsilon>0$ and $h=n^{-(\theta+\varepsilon)}$. Then, for large enough q, we get

$$\mathbb{P}(\omega_n\leq\delta_n)\leq Cn^{(a+\theta+\varepsilon)\ell}\times n^{-\theta d}+Cn^{-\varepsilon q}\times n^{\ell a}\leq Cn^{-(\theta d-(a+\theta+\varepsilon)\ell)}.\qquad\square$$

3.3 Expected number of roots for trigonometric polynomials: an invariance principle

In this section we look to trigonometric polynomials with random coefficients of the form

$$Q_n(t,Y)=\sum_{k=1}^n\big(Y_k^1\cos(kt)+Y_k^2\sin(kt)\big)$$

where $Y_k=(Y_k^1,Y_k^2),k\in\mathbb{N}$, are independent centered random variables such that $Y_k\in\mathfrak{D}(\varepsilon,r)$ for each k. Our aim is to estimate the asymptotic behavior, as $n\to\infty$, of the expected number of zeros in the interval $(0,\pi)$ of these polynomials. This clearly coincide with the number of zeros in $(0,n\pi)$ of the renormalized polynomials

$$P_n(t,Y)=\frac{1}{\sqrt{n}}\sum_{k=1}^n\Big(Y_k^1\cos\Big(\frac{kt}{n}\Big)+Y_k^2\sin\Big(\frac{kt}{n}\Big)\Big).\tag{3.14}$$

So we denote by $N_n(Y)$ the number of zeros of $P_n(t,Y)$ in $(0,n\pi)$. It is known that if we replace Y_k by G_k, independent standard normal random variables then (see [22, 24])

$$\lim_n \frac{1}{n}\mathbb{E}(N_n(G)) = \frac{1}{\sqrt{3}}.$$

Our aim is to prove that this remains true for any sequence $Y_k, k\in\mathbb{N}$ of independent but non necessarily identically distributed random variables. So we will prove:

Theorem 3.4. *Suppose that $Y=(Y_k)_{k\in\mathbb{N}}$ is a sequence of independent random variables in $\subset \mathfrak{D}(\varepsilon,r)$, having finite moments of any order. Then*

$$\left|\frac{1}{n}\mathbb{E}(N_n(Y)) - \frac{1}{n}\mathbb{E}(N_n(G))\right| \leq \frac{\mathrm{C}}{\sqrt{n}}. \tag{3.15}$$

Proof. . The first ingredient in the proof is Kac-Rice lemma that we recall now. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function and set

$$\omega_{a,b}(f) = \inf_{x\in[a,b]}(|f(x)|+|f'(x)|) \text{ and } \delta_{a,b}(f) = \min\{|f(a)|,|f(b)|,\omega_{a,b}(f)\}.$$

We denote by $N_{a,b}(f)$ the number of solutions of $f(t)=0$ for $t\in[a,b]$ and

$$I_{a,b}(f,\delta) = \int_a^b |f'(t)|\, 1_{\{|f(t)|\leq\delta\}}\frac{dt}{2\delta}, \quad \delta>0.$$

The Kac-Rice lemma says that if $\delta_{a,b}(f)>0$ then

$$N_{a,b}(f) = I_{a,b}(f,\delta) \quad \text{for} \quad \delta\leq\delta_{a,b}(f). \tag{3.16}$$

Notice that we also have, for every $\delta>0$,

$$I_{a,b}(\delta,f) \leq 1 + N_{a,b}(f'). \tag{3.17}$$

Indeed, we may assume that $N_{a,b}(f') = p < \infty$ and then we take $a=a_0\leq a_1 < .... < a_p \leq a_{p+1} = b$ to be the roots of f'. Since f is monotonic on each (a_i,a_{i+1}) one has $I_{a_i,a_{i+1}}(\delta,f)\leq 1$ so (3.17) holds.

We will use this result for $f(t) = P_n(t,Y)$ so we have $N_n(Y) = N_{0,n\pi}(P_n(t,Y))$. We denote $\delta_n(Y) = \delta_{0,n\pi}(P_n(t,Y)))$, we take $\theta=3$ and we write

$$\begin{aligned}\frac{1}{n}\mathbb{E}(N_n(Y)) =& \frac{1}{n}\mathbb{E}(N_n(Y)1_{\{\delta_n(Y)\leq n^{-\theta}\}}) - \frac{1}{n}\mathbb{E}(I_{0,n\pi}(\delta,P_n(\cdot,Y))1_{\{\delta_n(Y)\leq n^{-\theta}\}})\\ &+ \frac{1}{n}\mathbb{E}(I_{0,n\pi}(\delta,P_n(\cdot,Y)))\\ =:& A_n(Y) - A'_n(Y) + B_n(Y).\end{aligned}$$

A trigonometric polynomial of order n has at most $2n$ roots on $(0,\pi)$. So the number of roots of $P_n(t,Y)$ on $(0,\pi)$ is upper bonded by $2n$, so that consequently $N_n(Y)\leq 2n$. It follows that $A_n(Y)\leq 2\mathbb{P}(\delta_n(Y)\leq n^{-\theta})$. Since P'_n is also a trigonometric polynomial of order n, by (3.17) we also have $I_{0,n\pi}(\delta,P_n(.,Y))\leq 1+N_{0,n\pi}(P'_n(t,Y))\leq 2n+1$. It follows that $|A'_n(Y)|\leq 3\mathbb{P}(\delta_n(Y)\leq n^{-\theta})$.

We will use Theorem 3.2 and Theorem 2.3 for $S_n(t,Y) = (P_n(t,Y),P'_n(t,Y))$, so we have to check the hypotheses there. Notice that in this case we have $\ell=1, d=2$ and

$$C_{n,k}(t) = \begin{pmatrix} \cos(\frac{kt}{n}) & \sin(\frac{kt}{n}) \\ -\frac{k}{n}\sin(\frac{kt}{n}) & \frac{k}{n}\cos(\frac{kt}{n})\end{pmatrix}.$$

First, (3.8) trivially holds. Moreover, for every $\xi \in \mathbb{R}^2$ one has $|C_{n,k}(t)\xi|^2 = \frac{k^2}{n^2}$ so that

$$\frac{1}{n}\sum_{k=1}^{n}|C_{n,k}(t)\xi|^2 = \frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{n^2}|\xi|^2 \geq \int_0^1 x^2dx \times |\xi|^2 = \frac{1}{3}|\xi|^2.$$

This means that (3.9) holds with $\lambda_* = \frac{1}{3}$ and we are able to use (3.11) in order to get $A_n(Y) \leq \mathrm{C}/n$ and $|A_n'(Y)| \leq \mathrm{C}/n$. Moreover, by (3.16)

$$B_n(Y) = \frac{1}{n}\mathbb{E}\Big(\int_0^{n\pi}|P_n'(t,Y)|\,1_{\{|P_n(t,Y)|\leq\delta_n\}}\frac{dt}{\delta_n}\Big) \quad \text{with} \quad \delta_n = \frac{1}{n^\theta}.$$

We now use the Theorem 2.3 applied to $\Psi_{\delta_n}(x_1,x_2) = |x_2|\,\Theta_{1,\delta_n}(x_1)$ with Θ_{1,δ_n} defined in (3.13). Then

$$B_n(Y) = \frac{1}{n}\mathbb{E}\Big(\int_0^{n\pi}\partial_1\Psi_{\delta_n}(S_n(t,Y))dt\Big) \quad \text{with} \quad \delta_n = \frac{1}{n^\theta}.$$

We have $\|\Psi_{\delta_n}\|_{1,\infty} \leq \delta_n^{-1}$ so, using (2.22) we get

$$\big|\mathbb{E}\big(\partial_1\Psi_{\delta_n}(S_n(t,Y))\big) - \mathbb{E}\big(\partial_1\Psi_{\delta_n}(S_n(t,G))\big)\big| \leq C\Big(\frac{1}{\sqrt{n}} + n^3e^{-cn}\Big)$$

and this gives $|B_n(Y) - B_n(G)| \leq Cn^{-1/2}$. As above we have $A_n(G) \leq Cn^{-1}$ and $|A_n'(G)| \leq Cn^{-1}$ so we finally obtain (3.15). □

4 The case of smooth test functions

We first study a variant of our main Theorem 2.3, namely, we assume that $q = 1$ therein and we ask for a smooth function f. In this case, thanks to the regularity assumption for f, we do not need any Doeblin's condition. This will be used in a second step, where we will be able to relax the smoothness assumption for f by means of a regularization result from Malliavin calculus.

We come back to the notation introduced in Section 2. We just recall here the corrector polynomial $\Phi_{n,N}$ defined in (2.21):

$$\Phi_{n,N}(x) = 1 + \sum_{k=1}^{N}\frac{1}{n^{k/2}}H_{\Gamma_{n,k}}(x),$$

where $H_{\Gamma_{n,k}}$ is the Hermite polynomial associated with the differential operator $\Gamma_{n,k}$ defined in (2.15).

The result we prove in this section is the following:

Theorem 4.1. *Let $N \in \mathbb{N}$ be given. Suppose that the normalization property (2.2) and the moment bounds (2.3) both hold (the latter being sufficient for $p \leq N+3$). Then for every $f \in C_p^{2N(\lfloor N/2\rfloor+N+5)}(\mathbb{R}^d)$*

$$\begin{aligned}&|\mathbb{E}(f(S_n(Y)) - \mathbb{E}(f(W)\Phi_{n,N}(W))|\\ &\leq \mathcal{H}_N \mathrm{C}_{2(N+3)}^{2N(N+2\lfloor N/2\rfloor)}(Y)(1+\mathrm{C}_{2l_{\widehat{N}}(f)}(Y))^{2N+3}2^{(N+2)(l_{\widehat{N}}(f)+1)}L_{\widehat{N}}(f) \times \frac{1}{n^{\frac{N+1}{2}}}\end{aligned} \tag{4.1}$$

in which $\widehat{N} = N(2\lfloor N/2\rfloor + N + 5)$, $\mathcal{H}_N$ is a positive constant depending on N and W denotes a standard normal random variable in $\mathbb{R}^d$. As a consequence, taking $f(x) = x^\beta$ with $|\beta| = k$, one gets

$$|\mathbb{E}(S_n(Y)^\beta) - \mathbb{E}(W^\beta\Phi_N(W))| \leq \mathcal{H}_N \mathrm{C}_{2(N+3)}^{2N(N+2\lfloor N/2\rfloor)}(Y)(1+\mathrm{C}_{2k}(Y))^{2N+3}2^{(N+2)(k+1)} \times \frac{1}{n^{\frac{N+1}{2}}}. \tag{4.2}$$

In order to give the proof of Theorem 4.1, we introduce a decomposition allowing us to work with suitable semigroups. But first, in order to simplify the forthcoming notation, we set

$$Z_{n,k}=\frac{1}{\sqrt{n}}C_{n,k}Y_k \quad \text{and} \quad G_{n,k}=\frac{1}{\sqrt{n}}C_{n,k}G_k,$$

so that

$$S_n(Y)=\sum_{k=1}^n Z_{n,k}=:\overline{S}_n(Z) \quad \text{and} \quad S_n(G)=\sum_{k=1}^n G_{n,k}=:\overline{S}_n(G). \tag{4.3}$$

Notice that the covariance matrices of $Z_{n,k}$ and $G_{n,k}$ are both given by

$$\mathrm{Cov}(Z_{n,k})=\mathrm{Cov}(G_{n,k})=\overline{\sigma}_{n,k}=\frac{1}{n}\sigma_{n,k},$$

so the normalization condition (2.2) reads

$$\sum_{k=1}^n \overline{\sigma}_{n,k}=\mathrm{Id}_d.$$

Sketch of the proof. The proof of the above theorem is rather long and technical, so, in order to orient the reader, we give first a sketch of it. The strategy is based on the classical Lindeberg method but it turns out that it is convenient to do it in terms of semigroups (the so called Trotter's method). We define the Markov semigroup

$$P^{Z,n}_{k,p}f(x)=\mathbb{E}\Big(f\Big(x+\sum_{i=k}^{p-1}Z_{n,i}\Big)\Big) \tag{4.4}$$

with the convention $P^{Z,n}_{k,k}f=f$. Then Lindberg's decomposition gives

$$P^{Z,n}_{k,n+1}-P^{G,n}_{k,n+1}=\sum_{r=k}^n P^{Z,n}_{r+1,n+1}(P^{Z,n}_{r,r+1}-P^{G,n}_{r,r+1})P^{G,n}_{k,r}. \tag{4.5}$$

We use now Taylor expansion of order three. The terms of order one and two cancel (because the moments of order one and two of Y_r and G_r coincide) and we obtain

$$\begin{aligned}\delta_{n,r}f(x) &:= (P^{Z,n}_{r,r+1}-P^{G,n}_{r,r+1})f(x)=\mathbb{E}(f(x+Z_{n,r}))-\mathbb{E}(f(x+G_{n,r})) &(4.6)\\ &= \frac{1}{6}\sum_{|\alpha|=3}\int_0^1\mathbb{E}(\partial^\alpha f(\lambda Z_{n,k}+(1-\lambda)G_{n,k})(Z^\alpha_{n,k}-G^\alpha_{n,k}))d\lambda &(4.7)\end{aligned}$$

so one obtains $\|\delta_{n,r}f(x)\|_\infty\leq C\,\|f\|_{3,\infty}\,\frac{1}{n^{3/2}}$. We insert this in (4.5) and we obtain $P^{Z,n}_{k,n+1}-P^{G,n}_{k,n+1}\sim n\times\frac{1}{n^{3/2}}=\frac{1}{n^{1/2}}$. This is the proof of the classical CLT. Now, if we want to obtain Edgeworth development of order N, we have to go further. First we iterate (4.5) and we obtain

$$P^{Z,n}_{k,n+1}f=P^{G,n}_{k,n+1}f+\sum_{l=1}^N\sum_{1<r_1<\ldots<r_l<n}^n T_{r_1,\ldots,r_l}f+R^N_n f$$

with

$$\begin{aligned}T_{r_1,\ldots,r_l}f &= P^{G,n}_{r_l+1,n+1}\prod_{\hat{\imath}=1}^{l-1}\Big(\delta_{n,r_i}P^{G,n}_{r_{i-1},r_i}\Big)f \quad \textit{and}\\ R^N_n f &= \sum_{1<r_1<\ldots<r_{N+1}<n}^n P^{Z,n}_{r_{N+1}+1,n+1}\prod_{\hat{\imath}=1}^{N}\Big(\delta_{n,r_i}P^{G,n}_{r_{i-1},r_i}f\Big).\end{aligned}$$

Since each of $\delta_{n,r}f$ is of order $n^{-3/2}$ it follows that $R^N_n f$ is of order $n^{N+1}\times n^{-\frac{3}{2}(N+1)}=n^{-(N+1)/2}$.

We look now to $T_{r_1,\ldots,r_l}f$. We expect this term to be of order $n^{-\frac{3}{2}l}$, and indeed, it is. But we notice that δ_{n,r_i} contains information on the whole law of Y, and not only on its moments (see (4.6)). So, if we want to obtain the real coefficient of order $n^{-l/2}$ in the Edgeworth development, we have to replace $\delta_{n,r}$ by some $\widehat{\delta}_{n,r}$ which depends on the moments only - and this is done by using Taylor expansion as in (4.6). But, as we want to obtain a final error of order $n^{-(N+1)/2}$, the development of order three is no more sufficient and we need now a development of order $N+2$. This will involve differential operators of order less or equal to $N+2$ with coefficients computed by using the difference of moments given in (2.12). Here is that the Hermite polynomials come on, due to the following integration by parts formula: if G is a standard normal random variable then $\mathbb{E}(\partial^\alpha f(G)) = \mathbb{E}(f(G)H_\alpha(G))$ where ∂^α is the differential operator associated to the multi-index α and H_α is the Hermite polynomial corresponding to α. Collecting the terms of order $n^{-l/2}$ from all the $T'_{r_1,\ldots,r_l}$'s we get the corrector of order l in the Edgeworth expansion. These are the main ideas of the proof. However, the precise description of the coefficients of the Edgeworth expansion, turns out to be a very technical matter. We do all this through the following lemmas in this section.

We go no on and give the complete proofs. Let $N \in \{0,1,\ldots\}$. For $D_{n,r}$ given in (2.12), we define

$$T^0_{n,N,r}f(x) = \sum_{l=1}^{N+2} \frac{1}{n^{l/2}\,l!} D^{(l)}_{n,r}f(x). \tag{4.8}$$

Since $D^{(l)}_{n,r} \equiv 0$ for $l = 0,1,2$, the above sum actually begins with $l = 3$ and of course this is the basic fact. Then, with the convention $\sum_{l=3}^{2} = 0$, we have

$$T^0_{n,N,r}f(x) = \sum_{l=3}^{N+2} \frac{1}{n^{l/2}\,l!} D^{(l)}_{n,r}f(x).$$

We also define

$$\begin{aligned} T^{1,Z}_{n,N,r}f(x) &= \frac{1}{(N+2)!}\sum_{|\alpha|=N+3}\int_0^1 (1-\lambda)^{N+2}\mathbb{E}(\partial_\alpha f(x+\lambda Z_{n,r})Z^\alpha_{n,r})d\lambda \quad \text{and} \\ T^1_{n,N,r}f(x) &= T^{1,Z}_{n,N,r}f(x) - T^{1,G}_{n,N,r}f(x). \end{aligned} \tag{4.9}$$

For a matrix $\sigma \in \mathrm{Mat}(d\times d)$ we recall the Laplace operator L_σ associated to σ (see (2.11)) and we define

$$h^0_{N,\sigma}f(x) = f(x) + \sum_{l=1}^{\lfloor N/2\rfloor} \frac{(-1)^l}{n^l\,2^l l!} L^l_\sigma f(x), \tag{4.10}$$

$$h^1_{N,\sigma}f(x) = \frac{(-1)^{\lfloor N/2\rfloor+1}}{n^{\lfloor N/2\rfloor+1}\,2^{\lfloor N/2\rfloor+1}\lfloor N/2\rfloor!}\int_0^1 s^{\lfloor N/2\rfloor}\mathbb{E}(L^{\lfloor N/2\rfloor+1}_\sigma f(x+\sigma^{1/2}\sqrt{s}\,W))ds. \tag{4.11}$$

In (4.11), W stands for a standard Gaussian random variable. Then we define

$$U^0_{n,N,r}f(x) = \mathbb{E}(h^0_{N,\sigma_{n,r}}f(x+G_{n,r})) \quad \text{and} \quad U^1_{n,N,r}f(x) = h^1_{N,\sigma_{n,r}}f(x). \tag{4.12}$$

We now put our problem in a semigroup framework. For a sequence X_k, $k \geq 1$, of independent r.v.'s, for $1 \leq k \leq p$ we define

$$P^X_{k,k}f(x) = f \text{ and for } p > k \geq 1 \text{ then } P^X_{k,p}f(x) = \mathbb{E}\Big(f\Big(x + \textstyle\sum_{i=k}^{p-1} X_i\Big)\Big). \tag{4.13}$$

By using independence, we have the semigroup and the commutative property:

$$P^X_{k,p} = P^X_{r,p}P^X_{k,r} = P^X_{k,r}P^X_{r,p} \quad k \leq r \leq p. \tag{4.14}$$

We use $P^X_{k,p}$ with $X_k = Z_{n,k}$ and $X_k = G_{n,k}$, that we call $P^{Z,n}_{k,p}$ and $P^{G,n}_{k,p}$ because each local random variables depend on n.

Moreover, for $m = 1, \dots, N$ we denote

$$\begin{aligned} Q^{(m)}_{n,N,r_1,\dots,r_m} &= \sum_{\substack{\sum_{i=1}^m q_i + \sum_{i=1}^m q'_i > 0 \\ q_i, q'_i \in \{0,1\}}} \prod_{i=1}^m U^{q'_i}_{n,N,r_i} \prod_{j=1}^m T^{q_j}_{n,N,r_j} \quad \text{and} \\ R^{(m)}_{n,N,k} &= \sum_{k \le r_1 < \dots < r_m \le n} P^{G,n}_{r_m+1,n} P^{G,n}_{r_{m-1}+1,r_m} \cdots P^{G,n}_{r_1+1,r_2} P^{G,n}_{k,r_1} Q^{(m)}_{n,N,r_1,\dots,r_m}. \end{aligned} \tag{4.15}$$

Notice that in the first sum above the conditions $q_i, q'_i \in \{0,1\}$ and $q_1 + \dots + q_m + q'_1 + \dots + q'_m > 0$ say that at least one of $q_i, q'_i, i = 1, ..., m$ is equal to one. We notice that the operators T^1_{n,N,r_i} and $U^1_{n,N,\sigma_{r_i}}$ represent “remainders” and they are supposed to give small quantities of order $n^{-\frac{1}{2}(N+1)}$. So the fact that at least one q_i or q'_i is non null means that the product $(\prod_{i=1}^m U^{q'_i}_{n,N,r_i})(\prod_{i=1}^m T^{q_i}_{n,N,r_i})$ has at least one term which is a remainder (so is small), and consequently $R^{(m)}_{n,N,k}$ is a remainder also.

Finally we define

$$\begin{aligned} Q^{(N+1)}_{n,N,r_1,\dots,r_{N+1}} &= \prod_{i=1}^{N+1} (T^0_{N,r_i} + T^1_{N,r_i}) \quad \text{and} \\ R^{(N+1)}_{n,N,k} &= \sum_{k \le r_1 < \dots < r_{N+1} \le n} P^{Z,n}_{r_{N+1}+1,n} P^{G,n}_{r_N+1,r_{N+1}} \cdots P^{G,n}_{r_1+1,r_2} P^{G,n}_{k,r_1} Q^{(N+1)}_{n,N,r_1,\dots,r_{N+1}} \end{aligned} \tag{4.16}$$

As a preliminary result for Theorem 4.1, we study the following “backward Taylor formula”:

Lemma 4.2. *Let $\mathcal{N}_k$, $k \in \mathbb{N}$, denote independent centered Gaussian random variables in $\mathbb{R}^d$ with covariance matrix σ_k, $k \in \mathbb{N}$, and set $S_p = \sum_{k=1}^p \mathcal{N}_k$. For $\sigma \in \mathrm{Mat}(d \times d)$, we define*

$$\begin{aligned} H^0_{N,\sigma}\phi(x) &= \phi(x) + \sum_{l=1}^N \frac{(-1)^l}{2^l l!} L^l_\sigma \phi(x), \\ H^1_{N,\sigma}\phi(x) &= \frac{(-1)^{N+1}}{2^{N+1}N!} \int_0^1 s^N \mathbb{E}(L^{N+1}_\sigma \phi(x + \sigma^{1/2} W_s)) ds, \end{aligned}$$

where W is a $d-$dimensional Brownian motion independent of S_p. Then for every $\phi \in C^{2N+2}(\mathbb{R}^d)$ one has

$$\mathbb{E}(\phi(S_p)) = \mathbb{E}(H^0_{N,\sigma_{p+1}} \phi(S_{p+1})) + \mathbb{E}(H^1_{N,\sigma_{p+1}} \phi(S_p)) \tag{4.17}$$

Proof. We use the following property: for every $N \in \mathbb{N}$, $N \ge 0$, and $g \in C^{2N+2}_b(\mathbb{R}^d)$ one has

$$g(0) = \mathbb{E}(\sigma^{1/2} W_1)) + \sum_{l=1}^N \frac{(-1)^l}{2^\ell l!} \mathbb{E}(L^l_\sigma g(\sigma^{1/2} W_1)) + \frac{(-1)^{N+1}}{2^{N+1}N!} \int_0^1 s^N \mathbb{E}(L^{N+1}_\sigma g(\sigma^{1/2} W_s)) ds, \tag{4.18}$$

in which W denotes a standard Brownian motion in $\mathbb{R}^d$ and L_σ is given in (2.11). The decomposition (4.18) is proved in [8] (see Appendix C therein) in the case $\sigma = \mathrm{Id}$ and (4.18) represents a straightforward generalization to any covariance matrix σ.

We notice that $\mathcal{N}_{p+1}$ has the same law as $\sigma^{1/2}_{p+1} W_1$. We denote $\psi_\omega(x) = \phi(S_p(\omega) + x)$. Then, using the independence property and (4.18) we obtain

$$\begin{aligned}\mathbb{E}(\psi_\omega(0)) &= \mathbb{E}(\psi_\omega(\sigma_{p+1}^{1/2}W_1)) + \sum_{l=1}^{N}\frac{(-1)^l}{2^l l!}\mathbb{E}(L^l_{\sigma_{p+1}}\psi_\omega(\sigma_{p+1}^{1/2}W_1)) \\ &\quad + \frac{(-1)^{N+1}}{2^{N+1}N!}\int_0^1 s^N\mathbb{E}(L^{N+1}_{\sigma_{p+1}}\psi_\omega(\sigma_{p+1}^{1/2}W_s)))ds.\end{aligned}$$

Since $\mathbb{E}(L^l_{\sigma_{p+1}}\psi_\omega(\sigma_{p+1}^{1/2}W_1)) = \mathbb{E}(L^l_{\sigma_{p+1}}\phi(S_{p+1}))$ and $\mathbb{E}(L^N_{\sigma_{p+1}}\psi_\omega(\sigma_{p+1}^{1/2}W_1)) = \mathbb{E}(L^{N+1}_{\sigma_{p+1}}\phi(S_p + \sigma_{p+1}^{1/2}W_s))$ the above formula is exactly (4.17). □

We are now able to give our first result:

Proposition 4.3. *Let $N \geq 1$ and let $T^0_{n,N,r}$, h^0_{n,N,σ_r} and $R^{(m)}_{n,N,k}$, $m = 1,\ldots,N+1$, be given through (4.8), (4.10) and (4.15)–(4.16). Then for every $1 \leq k \leq n+1$ and $f \in C_p^{N(2\lfloor N/2\rfloor+N+5)}(\mathbb{R}^d)$ one has*

$$P^{Z,n}_{k,n+1}f = P^{G,n}_{k,n+1}f + \sum_{m=1}^{n}\sum_{k\leq r_1<\cdots<r_m\leq n}P^{G,n}_{k,n+1}\Big(\prod_{i=1}^m T^0_{n,N,r_i}\Big)\Big(\prod_{j=1}^m h^0_{N,\sigma_n,r_j}\Big)f + \sum_{m=1}^{N+1}R^{(m)}_{n,N,k}f. \tag{4.19}$$

Proof. **Step 1 (Lindeberg method).** We use the Lindeberg method in terms of semigroups: for $1 \leq k \leq n+1$

$$P^{Z,n}_{k,n+1} - P^{G,n}_{k,n+1} = \sum_{r=k}^{n}P^{Z,n}_{r+1,n+1}(P^{Z,n}_{r,r+1} - P^{G,n}_{r,r+1})P^{G,n}_{k,r}.$$

Then we define

$$A_{k,p} = 1_{1\leq k\leq p-1\leq n}\,(P^{Z,n}_{p-1,p} - P^{G,n}_{p-1,p})P^{G,n}_{k,p-1} \tag{4.20}$$

(here n is fixed so we do not stress the dependence of $A_{k,p}$ on n) and the above relation reads

$$P^{Z,n}_{k,n+1} = P^{G,n}_{k,n+1} + \sum_{r=k}^{n}P^{Z,n}_{r+1,n+1}A_{k,r+1}. \tag{4.21}$$

We will write (4.21) as a discrete time Volterra type equation (this is inspired from the approach to the parametrix method given in [13]: see equation (3.1) there). For a family of operators $F_{k,p}$, $k \leq p$ we define AF by

$$(AF)_{k,p} = \sum_{r=k}^{p-1}F_{r+1,p}A_{k,r+1}$$

and we write (4.21) in functional form:

$$P^{Z,n} = P^{G,n} + AP^{Z,n}. \tag{4.22}$$

By iteration,

$$P^{Z,n} = P^{G,n} + AP^{G,n} + \cdots + A^NP^{G,n} + A^{N+1}P^{Z,n}. \tag{4.23}$$

By the commutative property in (4.14), straightforward computations give

$$\begin{aligned}(A^mP^{G,n})_{k,p} &= 1_{k\leq p-m}\sum_{k\leq r_1<\cdots<r_m\leq p-2}P^{G,n}_{r_m+1,p-1}P^{G,n}_{r_{m-1}+1,r_m}\cdots P^{G,n}_{r_1+1,r_2}P^{G,n}_{k,r_1}(P^{Z,n}_{p-1,p} - P^{G,n}_{p-1,p})\times \\ &\quad\times(P^{Z,n}_{r_m,r_m+1} - P^{G,n}_{r_m,r_m+1})(P^{Z,n}_{r_{m-1},r_{m-1}+1} - P^{G,n}_{r_{m-1},r_{m-1}+1})\cdots(P^{Z,n}_{r_1,r_1+1} - P^{G,n}_{r_1,r_1+1}).\end{aligned} \tag{4.24}$$

Step 2 (Taylor formula). The drawback of (4.23) is that A depends on $P^{Z,n}$ also, see (4.20), so we use the Taylor's formula in order to eliminate this dependence We take into account (2.3) and we consider a Taylor approximation at the level of an error of order $n^{-\frac{N+2}{2}}$. We use the following expression for the Taylor's formula: for $f \in C_p^\infty(\mathbb{R}^d)$,

$$f(x+y) = f(x) + \sum_{p=1}^{N+2} \frac{1}{p!} \sum_{|\alpha|=p} \partial_\alpha f(x) y^\alpha + \frac{1}{(N+2)!} \sum_{|\alpha|=N+3} y^\alpha \int_0^1 (1-\lambda)^{N+2} \partial_\alpha f(x+\lambda y) d\lambda$$

Then we have, with $D_{n,r}^{(l)}$ defined in (2.12),

$$\begin{aligned}
&(P_{r,r+1}^{Z,n} - P_{r,r+1}^{G,n}) f(x) = \mathbb{E}(f(x+Z_{n,r})) - \mathbb{E}(f(x+G_{n,r})) = \sum_{l=1}^{N+2} \frac{1}{l!} D_r^{(l)} f(x) + \\
&\quad + \frac{1}{(N+2)!} \sum_{|\alpha|=N+3} \int_0^1 (1-\lambda)^{N+2} \big[\mathbb{E}(\partial_\alpha f(x+\lambda Z_{n,r}) Z_{n,r}^\alpha) \\
&\quad - \mathbb{E}(\partial_\alpha f(x+\lambda G_{n,r}) G_{n,r}^\alpha)\big] d\lambda \\
&= T_{n,N,r}^0 f(x) + T_{n,N,r}^1 f(x).
\end{aligned}$$

By using the independence property, one can apply commutativity and by using (4.24) we have

$$\begin{aligned}
&(A^m F)_{k,r+1} \\
&= 1_{k \le r+1-m} \sum_{k \le r_1 < \cdots < r_m \le r} F_{r_m+1,r+1} P_{r_{m-1}+1,r_m}^{G,n} \cdots P_{r_1+1,r_2}^{G,n} P_{k,r_1}^{G,n} \prod_{j=1}^m (T_{n,N,r_j}^0 + T_{n,N,r_j}^1).
\end{aligned} \tag{4.25}$$

Notice that the operator in (4.25) acts on $f \in C^{m(N+3)}$.

Step 3 (Backward Taylor formula). Since

$$P_{r_m+1,n+1}^{G,n} P_{r_{m-1}+1,r_m}^{G,n} \cdots P_{r_1+1,r_2}^{G,n} P_{k,r_1}^{G,n} f(x) = \mathbb{E}\Big(f\Big(x + \sum_{i=k}^n G_{n,k} - \sum_{j=1}^m G_{n,r_j}\Big)\Big),$$

the chain $P_{r_m+1,n}^{G,n} \cdots P_{r_1+1,r_2}^{G,n} P_{k,r_1}^{G,n}$ contains all the steps, except for the steps corresponding to $r_i, i = 1, ..., m$ (remark that for each i, $P_{r_i,r_i+1}^{G,n}$ is replaced with $T_{n,N,r_i}^0 + T_{n,N,r_i}^1$). In order to "insert" such steps we use the backward Taylor formula (4.17) up to order $\overline{N} = \lfloor N/2 \rfloor$. With $\overline{\sigma}_{n,k} = \frac{1}{n}\sigma_{n,r} = \mathrm{Cov}(G_{n,r})$, one has $H_{\overline{N},\overline{\sigma}_{n,r}}^0 = h_{N,\sigma_{n,r}}^0$ and $H_{\overline{N},\overline{\sigma}_{n,r}}^1 = h_{N,\sigma_{n,r}}^1$, $h_{N,\sigma_{n,r}}^0$ and $h_{N,\sigma_{n,r}}^1$ being given in (4.10) and (4.11) respectively. So, we have

$$\begin{aligned}
&P_{r_1+1,r_2}^{G,n} P_{k,r_1}^{G,n} f(x) = \mathbb{E}\Big(f\Big(x + \sum_{i=k}^{r_2-1} G_{n,i} - G_{n,r_1}\Big)\Big) \\
&\quad = \mathbb{E}\Big(h_{N,\sigma_{n,r_1}}^0 f\Big(x + \sum_{i=k}^{r_2-1} G_{n,i}\Big)\Big) + \mathbb{E}\Big(h_{N,\sigma_{n,r_1}}^1 f\Big(x + \sum_{i=k}^{r_2-1} G_{n,i} - G_{n,r_1}\Big)\Big) \\
&\quad = P_{r_1+1,r_2}^{G,n} P_{k,r_1}^{G,n} (P_{r_1,r_1+1}^{G,n} h_{N,\sigma_{n,r_1}}^0 + h_{N,\sigma_{n,r_1}}^1) f(x) \\
&\quad = P_{r_1+1,r_2}^{G,n} P_{k,r_1}^{G,n} (U_{n,N,r_1}^0 + U_{n,N,r_1}^1) f(x),
\end{aligned}$$

U_{n,N,r_1}^0 and U_{n,N,r_1}^1 being given in (4.12). For every $i = 1, 2, ..., m$, we use this formula in (4.25) evaluated in $r = n$ and we get

$$\begin{aligned}
(A^m P^{G,n})_{k,n+1} = &\sum_{k \le r_1 < \cdots < r_m \le n} P_{r_m+1,n+1}^{G,n} \cdots P_{r_1+1,r_2}^{G,n} P_{k,r_1}^{G,n} \\
&\times \Big(\prod_{i=1}^m (U_{n,N,r_i}^0 + U_{n,N,r_i}^1)\Big)\Big(\prod_{j=1}^m (T_{n,N,r_j}^0 + T_{n,N,r_j}^1)\Big).
\end{aligned} \tag{4.26}$$

Notice that the above operator acts on $C_p^{m(2\lfloor N/2\rfloor+N+5)}(\mathbb{R}^d)$. Our aim now is to isolate the principal term, that is the sum of the terms where only U^0_{n,N,r_i} and T^0_{n,N,r_i} appear. So, we use $Q^{(m)}_{n,N,r_1,\dots,r_m}$ defined in (4.15) and we write

$$\begin{aligned}(A^mP^{G,n})_{k,n+1} \quad &= \sum_{k\le r_1<\dots<r_m\le n} P^{G,n}_{r_m+1,n+1}\cdots P^{G,n}_{r_1+1,r_2}P^{G,n}_{k,r_1}\Big(\prod_{i=1}^m U^0_{n,N,r_i}\Big)\Big(\prod_{j=1}^m T^0_{n,N,r_j}\Big)\\ &\quad+\sum_{k\le r_1<\dots<r_m\le n} P^{G,n}_{r_m+1,n+1}\cdots P^{G,n}_{r_1+1,r_2}P^{G,n}_{k,r_1}Q^{(m)}_{n,N,r_1,\dots,r_m}.\end{aligned}$$

The second term is just $R^{(m)}_{n,N,k}$ in (4.15). In order to compute the first one we notice that for every $r'<r<r''$ we have

$$P^{G,n}_{r+1,r''}P^{G,n}_{r',r}P^{G,n}_{r,r+1}=P^{G,n}_{r',r''}$$

so that

$$P^{G,n}_{r_m+1,n+1}\cdots P^{G,n}_{r_1+1,r_2}P^{G,n}_{k,r_1}(\prod_{i=1}^m U^0_{n,N,r_i})=P^{G,n}_{k,n+1}(\prod_{i=1}^m h^0_{N,\sigma_{n,r_i}}).$$

Then, for $m=1,...,N$

$$(A^mP^{G,n})_{k,n+1}=\sum_{k\le r_1<\dots<r_m\le n}P^{G,n}_{k,n+1}\Big(\prod_{i=1}^m h^0_{N,\sigma_{n,r_i}}\Big)\Big(\prod_{i=1}^m T^0_{n,N,r_i}\Big)+R^{(m)}_{n,N,k}.$$

We treat now $A^{N+1}P^{Z,n}$. Using (4.25) we get

$$\begin{aligned}&(A^{N+1}P^{Z,n})_{k,n+1}\\ &=\sum_{k\le r_1<\dots<r_{N+1}\le n}P^{Z,n}_{r_{N+1}+1,n+1}P^{G,n}_{r_N+1,r_{N+1}}\cdots P^{G,n}_{r_1+1,r_2}P^{G,n}_{k,r_1}\prod_{i=1}^N(T^0_{n,N,r_i}+T^1_{n,N,r_i})=R^{(N+1)}_{n,N,k},\end{aligned}$$

which acts on $C_p^{N(N+3)}$. □

We give now some useful representations of the remainders.

Lemma 4.4. *Let $m\in\{1,...,N+1\}$ and $r_1<\dots<r_m\le n$ be fixed. Set $N_m:=m(2\lfloor N/2\rfloor+N+5)$ for $m\le N$ and $N_m=(N+1)(N+3)$ otherwise. Then, the operators $Q^{(m)}_{n,N,r_1,\dots,r_m}$ defined in (4.15) for $m=1,\dots,N$ and in (4.16) for $m=N+1$, can be written as*

$$Q^{(m)}_{n,N,r_1,\dots,r_m}f(x)=\frac{1}{n^{\frac{N+3m}{2}}}\sum_{3\le|\alpha|\le N_m}a_{n,r_1,\dots,r_m}(\alpha)\theta^\alpha_{n,r_1,\dots,r_m}\partial_\alpha f(x),\quad f\in C_p^{N_m}(\mathbb{R}^d),\tag{4.27}$$

where $a_{n,r_1,\dots,r_m}(\alpha)\in\mathbb{R}$ are suitable coefficients with the property

$$|a_{n,r_1,\dots,r_m}(\alpha)|\le(C\mathrm{C}_2^{\lfloor N/2\rfloor+1}(Y))^m,\tag{4.28}$$

and $\theta^\alpha_{n,r_1,\dots,r_m}:C_p^\infty(\mathbb{R}^d)\to C_p^\infty(\mathbb{R}^d)$ is an operator which verifies

$$|\theta^\alpha_{n,n,r_1,\dots,r_m}\partial_\alpha f(x)|\le\big(2^{l_{N_m}(f)+1}\mathrm{C}^{1/2}_{2(N+3)}(Z)(1+\mathrm{C}_{2l_{N_m}(f)}(Z))^2\big)^mL_{N_m}(f)(1+|x|)^{l_{N_m}(f)}\tag{4.29}$$

$C>0$ being a suitable constant. Moreover, $\theta^\alpha_{r_1,\dots,r_m}$ can be represented as

$$\theta^\alpha_{n,r_1,\dots,r_m}f(x)=\int_{(\mathbb{R}^d)^{2m}}f(x+y_1+\dots+y_{2m})\mu^\alpha_{n,r_1,\dots,r_m}(dy_1,\dots,dy_{2m})\tag{4.30}$$

where $\mu^\alpha_{n,r_1,\dots,r_m}$ is a finite signed measure such that $|\mu^\alpha_{n,r_1,\dots,r_m}|(\mathbb{R}^{2md})\le\mathrm{C}_*^m$, for a suitable constant C_* independent of n and depending just on N and on the moment bounds $\mathrm{C}_p(Y)$ for p large enough.

Proof. In a first step we construct the measures $\mu^{\alpha}_{n,r_1,\ldots,r_m}$ and the operators $\theta^{\alpha}_{n,r_1,\ldots,r_m}$ and in a second step we prove that the coefficients $a_{n,r_1,\ldots,r_m}(\alpha)$ verify (4.28). We start by representing $T^0_{n,N,r}$ defined in (4.8). Set

$$\nu^{0,\alpha}_{n,r}(dy) = \Delta_{n,r}(\alpha)\delta_0(dy), \quad |\alpha| = l \geq 3.$$

Notice that if $|\alpha| = l \geq 3$ then $|\Delta_{n,r}(\alpha)| \leq 2\mathrm{C}_l(Y)$. So, we have

$$\begin{aligned} &T^0_{n,N,r}f(x) = \sum_{l=3}^{N+2} \frac{1}{n^{\frac{l}{2}}} \sum_{|\alpha|=l} \frac{1}{l!} \int_{\mathbb{R}^d} \partial_\alpha f(x+y)\nu^{0,\alpha}_{n,r}(dy) \quad \text{with} \\ &\int_{\mathbb{R}^d} (1+|y|)^\gamma |\nu^{0,\alpha}_{n,r}|(dy) \leq 2\mathrm{C}_l(Y), \quad |\alpha| = l \leq N+2, \quad \gamma \geq 0. \end{aligned} \tag{4.31}$$

Concerning $T^1_{n,N,r}$ in (4.9), for $|\alpha| = N+3$ set $\nu^{1,\alpha}_{n,N,r}(dy) = n^{\frac{N+3}{2}} \int_0^1 (1-\lambda)^{N+2} (\frac{y}{\lambda})^\alpha \times \big[\mu_{\lambda Z_{n,r}}(dy) - \mu_{\lambda G_{n,r}}(dy)\big] d\lambda$, $\mu_{\lambda Z_{n,r}}$, resp. $\mu_{\lambda G_{n,r}}$, denoting the law of $\lambda Z_{n,r}$, resp. $\lambda G_{n,r}$. In other words,

$$\nu^{1,\alpha}_{n,N,r}(A) = n^{\frac{N+3}{2}} \int_0^1 (1-\lambda)^{N+2} \big[\mathbb{E}(Z^\alpha_{n,r} 1_{\lambda Z_{n,r}\in A}) - \mathbb{E}(G^\alpha_{n,r} 1_{\lambda G_{n,r}\in A})\big] d\lambda, \quad |\alpha| = N+3,$$

for every Borel set $A \subset \mathbb{R}^d$. Then we have

$$\begin{aligned} &T^1_{n,N,r}f(x) = \frac{1}{n^{\frac{1}{2}(N+3)}} \sum_{|\alpha|=N+3} \frac{1}{(N+2)!} \int_{\mathbb{R}^d} \partial_\alpha f(x+y)\nu^{1,\alpha}_{n,N,r}(dy) \quad \text{with} \\ &\int_{\mathbb{R}^d} (1+|y|)^\gamma |\nu^{1,\alpha}_{n,N,r}|(dy) \leq \frac{2^{\gamma+1}}{N+3} \mathrm{C}^{1/2}_{2(N+3)}(Y)(1+\mathrm{C}_{2\gamma}(Y))^{1/2}, \quad |\alpha| = N+3, \ \gamma \geq 0. \end{aligned} \tag{4.32}$$

We represent now the operator $U^0_{n,N,r}f(x) = \mathbb{E}(h^0_{N,\sigma_{n,r}} f(x+G_{n,r}))$ with $h^0_{N,\sigma_{n,r}}f$ defined in (4.10). Notice that

$$h^0_{N,\sigma_{n,r}} = \mathrm{Id} + \sum_{l=1}^{\lfloor N/2\rfloor} \frac{1}{n^l} \sum_{|\alpha|=2l} c_{\sigma_{n,r}}(\alpha)\partial_\alpha \quad \text{with} \quad c_\sigma(\alpha) = \frac{(-1)^l}{2^l l!} \prod_{k=1}^{l} \sigma^{\alpha_{2k-1},\alpha_{2k}}, \quad |\alpha| = 2l > 0.$$

So, by denoting $\rho^0_{\sigma_{n,r}}$ the law of $G_{n,r}$, we have

$$\begin{aligned} U^0_{n,N,r}f(x) &= \mathbb{E}(h^0_{N,\sigma_{n,r}} f(x+G_{n,r})) \\ &= \sum_{l=0}^{\lfloor N/2\rfloor} \frac{1}{n^l} \sum_{|\alpha|=2l} c_{\sigma_{n,r}}(\alpha) \int_{\mathbb{R}^d} \partial_\alpha f(x+y)\rho^0_{\sigma_{n,r}}(dy) \quad \text{with} \\ |c_{\sigma_{n,r}}(\alpha)| &\leq \frac{\mathrm{C}_2(Y)^l}{2^l l!} \quad \text{and} \quad \int_{\mathbb{R}^d} (1+|y|)^\gamma |\rho^0_{\sigma_{n,r}}|(dy) \leq 2^\gamma (1+\mathrm{C}_\gamma(Y)), \ \gamma \geq 0. \end{aligned} \tag{4.33}$$

We now obtain a similar representation for $h^1_{N,\sigma}f(x)$ defined in (4.11). Set

$$\rho^1_\sigma(dy) = \Big(\int_0^1 s^{\lfloor N/2\rfloor} \phi_{\sigma^{1/2}\sqrt{s}W}(y) ds\Big) dy,$$

in which $\phi_{\sigma^{1/2}\sqrt{s}W}$ denotes the density of a centered Gaussian r.v. with covariance matrix $s\sigma$. Then we write

$$\begin{aligned} &h^1_{N,\sigma}f(x) = \frac{1}{n^{\lfloor N/2\rfloor+1}} \sum_{|\alpha|=2(\lfloor N/2\rfloor+1)} b_{N,\sigma}(\alpha) \int_{\mathbb{R}^d} \partial_\alpha f(x+y)\rho^1_\sigma(dy) \text{ with} \\ &b_{N,\sigma}(\alpha) = \frac{(-1)^{\lfloor N/2\rfloor+1}}{2^{\lfloor N/2\rfloor+1}\lfloor N/2\rfloor!} \prod_{k=1}^{\lfloor N/2\rfloor+1} \sigma^{\alpha_{2k-1},\alpha_{2k}}. \end{aligned}$$

So, we have

$$
\begin{aligned}
U^1_{n,N,r}f(x) &= h^1_{N,\sigma_{n,r}}f(x)\\
&= \frac{1}{n^{\lfloor N/2\rfloor+1}}\sum_{|\alpha|=2(\lfloor N/2\rfloor+1)} b_{N,\sigma_{n,r}}(\alpha)\int \partial_\alpha f(x+y)\rho^1_{\sigma_{n,r}}(dy) \quad \text{with}\\
|b_{N,\sigma_{n,r}}(\alpha)| &\leq \frac{1}{2^{\lfloor N/2\rfloor+1}\lfloor N/2\rfloor!}\mathrm{C}_2(Y)^{\lfloor N/2\rfloor+1}\\
\text{and}\quad \int_{\mathbb{R}^d}(1+|y|)^\gamma|\rho^1_{\sigma_{n,r}}|(dy) &\leq \frac{2^\gamma}{\lfloor N/2\rfloor+1}(1+\mathrm{C}_\gamma(Y)),\ \gamma\geq 0.
\end{aligned}
\tag{4.34}
$$

Using (4.31), (4.32), (4.33) and (4.34) we obtain (4.27) with the measure $\mu^\alpha_{n,r_1,\ldots,r_m}$ from (4.30) constructed in the following way:

$$
\begin{aligned}
&\int_{\mathbb{R}^{d\times 2m}} f(y_1,\ldots,y_m,\bar y_1,\ldots,\bar y_m)\mu^\alpha_{n,r_1,\ldots,r_m}(dy_1,\ldots,dy_m,d\bar y_1,\ldots,d\bar y_m)\\
&= \int_{\mathbb{R}^{d\times 2m}} f(y_1,\ldots,y_m,\bar y_1,\ldots,\bar y_m)\eta_1(dy_1)\cdots\eta_m(dy_m)\bar\eta_1(d\bar y_1)\cdots\bar\eta_m(d\bar y_m)
\end{aligned}
$$

where η_i is one of the measures $\nu^{q,\beta}_{n,r_i}$, $q=0,1$, and $\bar\eta_i$ is one of the measures $\rho^q_{\sigma_{n,r_i}}$, $q=0,1$.

Let us check that the coefficients $a_{n,r_1,\ldots,r_m}(\alpha)$ which will appear in (4.27) verify the bounds in (4.28). Take first $m\in\{1,\ldots,N\}$. Then $Q^{(m)}_{n,r_1,\ldots,r_m}$ is the sum of $(\prod_{i=1}^m U^{q_i'}_{n,N,r_i})(\prod_{j=1}^m T^{q_j}_{n,N,r_j})$ where $q_i,q_i'\in\{0,1\}$ and at least one of them is equal to one. And $a_n^{r_1,\ldots,r_m}(\alpha)$ is the product of coefficients which appear in the representation of $U^{q_i'}_{n,N,r_i}$ and $T^{q_j}_{n,N,r_j}$. Recall that the coefficients of T^0_{n,N,r_j} are all bounded by $Cn^{-3/2}$ and the coefficients of T^1_{n,N,r_j} are bounded by $Cn^{-\frac12(N+3)}$. Moreover the coefficients of U^0_{n,N,r_i} are bounded by $C\mathrm{C}_2^{\lfloor N/2\rfloor}(Y)$ and the coefficients of U^1_{n,N,r_i} are bounded by $C\mathrm{C}_2^{\lfloor N/2\rfloor+1}(Y)n^{-(\lfloor N/2\rfloor+1)}$. Therefore, $(\prod_{i=1}^m U^{q_i'}_{n,N,r_i})(\prod_{j=1}^m T^{q_j}_{n,N,r_j})$ is upper bounded by

$$
\begin{aligned}
&\Big(\frac{C}{n^{\frac12(N+3)}}\Big)^{\sum_{i=1}^m q_i}\times\Big(\frac{C}{n^{3/2}}\Big)^{\sum_{i=1}^m(1-q_i)}\times\Big(\frac{C\mathrm{C}_2^{\lfloor N/2\rfloor+1}(Y)}{n^{\lfloor N/2\rfloor+1}}\Big)^{\sum_{i=1}^m q_i'}\times\big(C\mathrm{C}_2^{\lfloor N/2\rfloor}(Y)\big)^{\sum_{i=1}^m(1-q_i')}\\
&\leq\Big(\frac{1}{n^{\frac12 N}}\Big)^{\sum_{i=1}^m q_i}\times\frac{C^m}{n^{\frac{3m}{2}}}\times\Big(\frac{1}{n^{\lfloor N/2\rfloor+1}}\Big)^{\sum_{i=1}^m q_i'}\times(C\mathrm{C}_2^{\lfloor N/2\rfloor+1}(Y))^m\\
&\leq\frac{(C\mathrm{C}_2^{\lfloor N/2\rfloor+1}(Y))^m}{n^{\frac N2\sum_{i=1}^m q_i+(\lfloor N/2\rfloor+1)\sum_{i=1}^m q_i'+\frac{3m}{2}}}\leq\frac{(C\mathrm{C}_2^{\lfloor N/2\rfloor+1}(Y))^m}{n^{\frac N2(\sum_{i=1}^m q_i+\sum_{i=1}^m q_i')+\frac{3m}{2}}}\leq\frac{(C\mathrm{C}_2^{\lfloor N/2\rfloor+1}(Y))^m}{n^{\frac{N+3m}{2}}}.
\end{aligned}
$$

We finally prove (4.29). We have

$$
\begin{aligned}
&|\theta^\alpha_{n,r_1,\ldots,r_m}\partial_\alpha f(x)|\\
&\leq\int_{\mathbb{R}^{d\times 2m}}|\partial_\alpha f|\Big(x+\sum_{i=1}^m y_i+\sum_{j=1}^m\bar y_j\Big)|\eta|_1(dy_1)\cdots|\eta_m|(dy_m)|\bar\eta|_1(d\bar y_1)\cdots|\bar\eta_m|(d\bar y_m)\\
&\leq L_{N_m}(f)(1+|x|)^{l_{N_m}(f)}\Big(\prod_{i=1}^m\int_{\mathbb{R}^d}(1+|y|)^{l_{N_m}(f)}|\eta_i|(dy)\Big)\Big(\prod_{i=1}^m\int_{\mathbb{R}^d}(1+|y|)^{l_{N_m}(f)}|\bar\eta_i|(dy)\Big)\\
&\leq L_{N_m}(f)(1+|x|)^{l_{N_m}(f)}\Big((2\mathrm{C}_{N+2}(Y))\vee(2^{l_{N_m}(f)+1}\mathrm{C}^{1/2}_{2(N+3)}(Y)(1+\mathrm{C}_{2l_{N_m}(f)}(Y)\Big)^m\\
&\quad\times\Big(2^{l_{N_m}(f)}(1+\mathrm{C}_{l_{N_m}(f)}(Y))\Big)^m\\
&\leq\big(2^{l_{N_m}(f)+1}\mathrm{C}^{1/2}_{2(N+3)}(Y)(1+\mathrm{C}_{2l_{N_m}(f)}(Y))^2\big)^m L_{N_m}(f)(1+|x|)^{l_{N_m}(f)}
\end{aligned}
$$

because $\mathrm{C}_{N+2}(Y) \le \mathrm{C}_{2(N+3)}(Y)^{\frac{N+2}{2(N+3)}} \le \mathrm{C}_{2(N+3)}(Y)^{\frac{1}{2}}$. So the proof concerning $Q^{(m)}_{n,N,r_1,\dots,r_m}$, $m = 1,\dots,N$, is completed. The proof for $Q^{(N+1)}_{n,N,r_1,\dots,r_{N+1}}$ is clearly the same. □

We give now the representation of the "principal term":

Lemma 4.5. *Let the set-up of Proposition 4.3 hold. Then,*

$$\sum_{m=1}^{N}\sum_{1\le r_1<\ldots<r_m\le n}\Big(\prod_{i=1}^{m}T^0_{n,N,r_i}\Big)\Big(\prod_{j=1}^{m}h^0_{N,\sigma_{n,r_j}}\Big)=\sum_{k=1}^{N}\frac{1}{n^{k/2}}\Gamma_{n,k}+Q^0_{n,N} \tag{4.35}$$

with $\Gamma_{n,k}$ *defined in (2.15) and*

$$\begin{aligned} Q^0_{n,N} &= \frac{1}{n^{(N+1)/2}}\sum_{N+1\le|\alpha|\le N(N+2\lfloor N/2\rfloor)} c_{n,N}(\alpha)\partial_\alpha \\ \textit{with}\quad & |c_{n,N}(\alpha)|\le (C\mathrm{C}_{N+1}(Y)\mathrm{C}_2(Y))^{N(N+2\lfloor N/2\rfloor)} \end{aligned} \tag{4.36}$$

Proof. Let Λ_m and $\Lambda_{m,k}$ be the sets in (2.14). Notice that, for fixed m, the $\Lambda_{m,k}$'s are disjoint as k varies. Suppose that $m\in\{1,\dots,N\}$. Then $\Lambda_{m,k}=\emptyset$ if $k\notin\{m,\dots,N(N+2\lfloor N/2\rfloor)\}$ so that $\Lambda_m=\cup_{k=m}^{2N(N+2)}\Lambda_{m,k}$ and consequently

$$\cup_{m=1}^{N}\Lambda_m=\cup_{m=1}^{N}\cup_{k=m}^{N(N+2\lfloor N/2\rfloor)}\Lambda_{m,k}=\cup_{k=1}^{N(N+2\lfloor N/2\rfloor)}\cup_{m=1}^{k}\Lambda_{m,k}.$$

It follows that

$$\begin{aligned} &\sum_{m=1}^{N}\sum_{1\le r_1<\ldots<r_m\le n}\Big(\prod_{i=1}^{m}T^0_{n,N,r_i}\Big)\Big(\prod_{j=1}^{m}h^0_{N,\sigma_{n,r_j}}\Big)\\ =\;&\sum_{m=1}^{N}\sum_{l_1,..,l_m=3}^{N+2}\sum_{l_1',..,l_m'=0}^{\lfloor N/2\rfloor}\sum_{1\le r_1<\ldots<r_m\le n}\Big(\prod_{i=1}^{m}\frac{1}{n^{l_i/2}\,l_i!}D^{(l_i)}_{n,r_i}\Big)\Big(\prod_{j=1}^{m}\frac{(-1)^{l_j'}}{n^{l_j'}\,2^{l_j'}l_j'!}L^{l_j'}_{\sigma_{n,r_j}}\Big)\\ =\;&\sum_{k=1}^{N(N+2\lfloor N/2\rfloor)}\sum_{m=1}^{k}\sum_{(l_1,l_1'),\ldots,(l_m,l_m'))\in\Lambda_{m,k}}\sum_{1\le r_1<\ldots<r_m\le n}\frac{1}{n^{(k+2m)/2}}\Big(\prod_{i=1}^{m}\frac{1}{l_i!}D^{(l_i)}_{n,r_i}\Big)\\ &\times\Big(\prod_{j=1}^{m}\frac{(-1)^{l_j'}}{2^{l_j'}l_j'!}L^{l_j'}_{\sigma_{n,r_j}}\Big)\\ =\;&\sum_{k=1}^{N}\frac{1}{n^{k/2}}\Gamma_{n,k}+Q^0_{n,N} \end{aligned}$$

with

$$\begin{aligned} Q^0_{n,N}&=\frac{1}{n^{N+1/2}}\\ &\times\sum_{k=N+1}^{N(N+2\lfloor N/2\rfloor)}\sum_{m=1}^{k}\sum_{(l_1,l_1'),\ldots,(l_m,l_m'))\in\Lambda_{m,k}}\sum_{1\le r_1<\ldots<r_m\le n}\frac{n^{(N+1)/2}}{n^{(k+2m)/2}}\Big(\prod_{i=1}^{m}\frac{1}{l_i!}D^{l_i}_{n,r_i}\Big)\\ &\times\Big(\prod_{j=1}^{m}\frac{(-1)^{l_j'}}{2^{l_j'}l_j'!}L^{l_j'}_{\sigma_{n,r_j}}\Big), \end{aligned}$$

which is a differential operator of the form (4.36). Moreover,

$$\begin{aligned} |c_{n,N}(\alpha)|&\le n^m\times\frac{1}{n^m}\times\prod_{i=1}^{m}\Big(\frac{2\mathrm{C}_{N+1}(Y)}{l_i!}\Big)\times\prod_{i=1}^{m}\Big(\frac{\mathrm{C}_2^{l_i'}(Y)}{2^{l_i'}l_i'!}\Big)\le(C\mathrm{C}_{N+1}(Y)\mathrm{C}_2(Y))^m\\ &\le(C\mathrm{C}_{N+1}(Y)\mathrm{C}_2(Y))^m \end{aligned}$$

and the estimate in (4.36) follows. □

We are now ready for the

Proof of Theorem 4.1. We denote $P_n^{Z,n} = P_{1,n+1}^{Z,n}$ and $P_n^{G,n} = P_{1,n+1}^{G,n}$, so that, since $S_n(Y) = \overline{S}_n(Z)$,

$$\mathbb{E}(f(S_n(Y)) - \mathbb{E}(f(W)\Phi_N(W)) = P_n^{Z,n}f(0) - P_n^{G,n}\Big(\mathrm{Id} + \sum_{k=1}^N \frac{1}{n^{k/2}}\Gamma_{n,k}\Big)f(0).$$

Putting together (4.19) and (4.35), we can write

$$P_n^{Z,n}f(x) = P_n^{G,n}\Big(\mathrm{Id} + \sum_{k=1}^N \frac{1}{n^{k/2}}\Gamma_k\Big)f(x) + I_1f(x) + I_2f(x) + I_3f(x) \tag{4.37}$$

with

$$\begin{aligned} I_1f(x) &= \frac{1}{n^{(N+1)/2}}P_n^{G,n}Q_{n,N}^0 f(x),\\ I_2f(x) &= \sum_{1\le r_1<\ldots<r_{N+1}\le n} P_{r_{N+1}+1,n}^{Z,n}P_{r_N+1,r_{N+1}}^{G,n}\cdots P_{r_1+1,r_2}^{G,n}P_{k,r_1}^{G,n}Q_{n,N,r_1,\ldots,r_{N+1}}^{(N+1)}f(x)\\ I_3f(x) &= \sum_{m=1}^N\sum_{1\le r_1<\ldots<r_m\le n} P_{r_m+1,n}^{G,n}P_{r_{m-1}+1,r_m}^{G,n}\cdots P_{r_1+1,r_2}^{G,n}P_{k,r_1}^{G,n}Q_{n,N,r_1,\ldots,r_m}^{(m)}f(x), \end{aligned} \tag{4.38}$$

so it is sufficient to study the remaining terms I_1, I_2 and I_3 above. In such study, we will use the following easy consequence of Burkholder's inequality for discrete martingales: if $M_n = \sum_{k=1}^n \Delta_k$ with $\Delta_k, k = 1,\ldots,n$ independent centered random variables, then

$$\|M_n\|_p \le C\mathbb{E}\Big(\Big(\sum_{k=1}^n|\Delta_k|^2\Big)^{p/2}\Big)^{1/p} = C\Big\|\sum_{k=1}^n|\Delta_k|^2\Big\|_{p/2}^{1/2} \le C\Big(\sum_{k=1}^n\|\Delta_k\|_p^2\Big)^{1/2}. \tag{4.39}$$

We first estimate I_1f. Let us set $N_* = N(N+2\lfloor N/2\rfloor)$. So, (4.36) gives

$$\begin{aligned} |I_1f(x)| &\le \frac{1}{n^{(N+1)/2}}\sum_{N+1\le|\alpha|\le N_*}|c_n(\alpha)|\,|P_n^{G,n}\partial_\alpha f(x)|\\ &\le \frac{1}{n^{(N+1)/2}}\sum_{N+1\le|\alpha|\le N_*}|c_n(\alpha)|\,L_{N_*}(f)(1+|x|)^{l_{NN_*}(f)}\mathbb{E}\Big(\Big(1+\Big|\sum_{k=1}^n G_{n,k}\Big|\Big)^{l_{NN_*}(f)}\Big)\\ &\le \frac{1}{n^{\frac{N+1}{2}}}(C\mathrm{C}_{N+1}(Y)\mathrm{C}_2(Y))^{NN_*}L_{NN_*}(f)(1+|x|)^{l_{NN_*}(f)}\times 2^{l_{NN_*}(f)}(1+\mathrm{C}_{2l_{NN_*}(f)}(Y)), \end{aligned}$$

in which we have used the Burkholder inequality (4.39).

The study of I_2 and I_3 is similar, so we consider I_3. Take $m\in\{1,\ldots,N\}$. We use Lemma 4.4 (recall N_m given therein) and in particular (4.27):

$$\begin{aligned} &P_{r_m+1,n}^{G,n}P_{r_{m-1}+1,r_m}^{G,n}\cdots P_{r_1+1,r_2}^{G,n}P_{1,r_1}^{G,n}Q_{n,N,r_1,\ldots,r_m}^{(m)}f\\ &= \frac{1}{n^{\frac{N+3m}{2}}}\sum_{3\le|\alpha|\le N_m}a_{n,r_1,\ldots,r_m}(\alpha)P_{r_m+1,n}^{G,n}P_{r_{m-1}+1,r_m}^{G,n}\cdots P_{r_1+1,r_2}^{G,n}P_{1,r_1}^{G,n}\theta_{n,r_1,\ldots,r_m}^\alpha\partial_\alpha f. \end{aligned}$$

Notice that if $|g(x)|\le L(1+|x|)^l$ then

$$|P_{r_m+1,n}^{G,n}P_{r_{m-1}+1,r_m}^{G,n}\cdots P_{r_1+1,r_2}^{G,n}P_{1,r_1}^{G,n}g(x)| \le \mathbb{E}\Big(L\Big(1+\Big|x+\sum_{k=1}^n G_{n,k}1_{k\notin\{r_1,\ldots,r_m\}}\Big|\Big)^l\Big)$$

$$\le L(1+|x|)^l \mathbb{E}\Big(\Big(1+\Big|\sum_{k=1}^n G_{n,k}1_{k\notin\{r_1,\ldots,r_m\}}\Big|\Big)^l\Big)$$
$$\le L(1+|x|)^l\Big(1+\Big\|\sum_{k=1}^n G_{n,k}1_{k\notin\{r_1,\ldots,r_m\}}\Big\|_l\Big)^l.$$

Since the $G_{n,k}1_{k\notin\{r_1,\ldots,r_m\}}$'s are centered and independent, with $\|G_{n,k}1_{k\notin\{r_1,\ldots,r_m\}}\|_l \le \mathrm{C}_l(Y)/n^{l/2}$, we can use the Burkholder inequality (4.39), we get

$$\begin{aligned}|P^{G,n}_{r_m+1,n}P^{G,n}_{r_{m-1}+1,r_m}\cdots P^{G,n}_{r_1+1,r_2}P^{G,n}_{1,r_1}g(x)| &\le L(1+|x|)^l(1+\mathrm{C}_l^{1/l}(Y))^l\\ &\le 2^l(1+\mathrm{C}_l(Y))L(1+|x|)^l.\end{aligned} \tag{4.40}$$

We use now this inequality with $g=\theta^\alpha_{n,r_1,\ldots,r_m}\partial_\alpha f$: by applying (4.29) we get

$$|P^{G,n}_{r_m+1,n}P^{G,n}_{r_{m-1}+1,r_m}\cdots P^{G,n}_{r_1+1,r_2}P^{G,n}_{1,r_1}Q^{(m)}_{N,r_1,\ldots,r_m}f(x)| \le \mathcal{K}_{N,m}(f)(1+|x|)^{l_{N_m}(f)}$$

with

$$\mathcal{K}_{N,m}(f)=2^{l_{N_m}(f)}(1+\mathrm{C}_{l_{N_m}(f)}(Y))\big(2^{l_{N_m}(f)+1}\mathrm{C}^{1/2}_{2(N+3)}(Y)(1+\mathrm{C}_{2l_{N_m}(f)}(Y))^2\big)^m L_{N_m}(f).$$

Moreover, using (4.28)

$$\begin{aligned}&|P^{G,n}_{r_m+1,n}P^{G,n}_{r_{m-1}+1,r_m}\cdots P^{G,n}_{r_1+1,r_2}P^{G,n}_{1,r_1}Q^{(m)}_{n,N,r_1,\ldots,r_m}f(x)|\\ \le\ & \mathcal{K}_{N,m}(f)(1+|x|)^{l_{N_m}(f)}\frac{1}{n^{\frac12(N+3m)}}\sum_{0\le|\alpha|\le N+1}|a_{n,r_1,\ldots,r_m}(\alpha)|\\ \le\ & \mathcal{H}_N\mathcal{K}_{N,m}(f)(1+|x|)^{l_{N_m}(f)}\frac{1}{n^{\frac12(N+3m)}}(C\mathrm{C}_2^{\lfloor N/2\rfloor+1}(Y))^m,\end{aligned}$$

$\mathcal{H}_N$ denoting a constant depending on N only. Since the set $\{1\le r_1<\ldots<r_m\le n\}$ has less than n^m elements, we get

$$|I_3f(x)|\le N\times n^m\times\frac{1}{n^{\frac12(N+3m)}}\times\mathcal{H}_N\mathcal{K}_{N,m}(f)(1+|x|)^{l_{N_m}(f)}(C\mathrm{C}_2^{\lfloor N/2\rfloor+1}(Y))^m$$
$$\le N\mathcal{H}_N\mathcal{K}_{N,m}(f)(1+|x|)^{l_{N_m}(f)}(C\mathrm{C}_2^{\lfloor N/2\rfloor+1}(Y))^m\times\frac{1}{n^{\frac12(N+1)}}$$

The estimate for $I_2(f)$ is analogous. So, we get

$$\sum_{i=1}^3|I_if(x)|$$
$$\le\mathcal{H}_N\mathrm{C}^{2N(N+2\lfloor N/2\rfloor)}_{2(N+3)}(Y)(1+\mathrm{C}_{2l_{\widehat N}(f)}(Y))^{2N+3}2^{(N+2)(l_{\widehat N}(f)+1)}L_{\widehat N}(f)(1+|x|)^{l_{\widehat N}(f)}\times\frac{1}{n^{\frac{N+1}{2}}}$$

with $\widehat N=N(2\lfloor N/2\rfloor+N+5)$, and statement (4.1) follows. Concerning (4.2), it suffices to notice that for $f(x)=x^\beta$ with $|\beta|=k$ then $L_{\widehat N}(f)=1$ and $l_{\widehat N}(f)=k$. □

5 The case of general test functions

5.1 Differential calculus based on the Nummelin's splitting

In this section we use the variational calculus settled in [6, 5, 11, 12] in order to treat general test functions. Let us give the definitions and the notation.

We fix $r,\varepsilon>0$ and we consider a sequence of independent random variables $Y_k\in\mathfrak{D}(r,\varepsilon)$, $k\in\mathbb{N}$. Then, using the Nummelin's splitting (2.7) we write

$$Y_k=\chi_kV_k+(1-\chi_k)U_k, \tag{5.1}$$

the law of χ_k, V_k and U_k being given in (2.8). We assume that χ_k, V_k, U_k, $k \in \mathbb{N}$, are independent. We define $\mathcal{G} = \sigma(\chi_k, U_k, k \in \mathbb{N})$. A random variable $F = f(\omega, V_1, ..., V_n)$ is called a simple functional if f is $\mathcal{G} \times \mathcal{B}(\mathbb{R}^{d\times n})$ measurable and for each $\omega, f(\omega, \cdot) \in C_b^\infty(\mathbb{R}^{d\times n})$. We denote $\mathcal{S}$ the space of the simple functionals. Moreover we define the differential operator $D : \mathcal{S} \to l_2 := l_2(\mathbb{R}^d)$ by $D_{(k,i)}F = \chi_k \partial_{v_k^i} f(\omega, V_1, ..., V_n)$. Then the Malliavin covariance matrix of $F \in (F^1, ..., F^m) \in \mathcal{S}^m$ is defined as

$$\sigma_F^{i,j} = \left\langle DF^i, DF^j \right\rangle_{l_2} = \sum_{k=1}^\infty \sum_{p=1}^d D_{(k,p)}F^i \times D_{(k,p)}F^j, \quad i,j = 1, ..., m. \tag{5.2}$$

If σ_F is invertible we denote $\gamma_F = \sigma_F^{-1}$.

Moreover, we define the iterated derivatives $D^m : \mathcal{S} \to l_2^{\otimes m}$ by $D^{(m)}_{(k_1,i_1),...,(k_m,i_m)} = D_{(k_1,i_1)} \cdots D_{(k_m,i_m)}$ and on $\mathcal{S}$ we consider the norms

$$|F|_q^2 = |F|^2 + \sum_{m=1}^q |D^m F|^2_{l_2^{\otimes m}} = |F|^2 + \sum_{m=1}^q \sum_{k_1,...,k_m=1}^\infty \sum_{i_1,...,i_m=1}^d \left|D_{(k_1,i_1)}....D_{(k_m,i_m)}F\right|^2$$

and

$$\|F\|_{q,p} = (\mathbb{E}(|F|_q^p))^{1/p}. \tag{5.3}$$

We introduce now the Ornstein-Uhlenbeck operator L. We denote $\theta_{k,i} = \partial_i \ln p_{V_k}(V_k) = 2(V_k - y_Y)^i 1_{r<|V_k-y_Y|^2<2r} a_r'(|V_k - y_{Y_k}|^2)$, p_{V_k} being the density of V_k (see (2.8)) and a_r is given in (2.5). So, we define

$$LF = -\sum_{k=1}^\infty \sum_{i=1}^d \left(D_{(k,i)}D_{(k,i)}F + D_{(k,i)}F \times \theta_{k,i}\right). \tag{5.4}$$

Using elementary integration by parts on $\mathbb{R}^d$ one easily proves the following duality formula: for $F, G \in \mathcal{S}$

$$\mathbb{E}(\langle DF, DG\rangle_{l_2}) = \mathbb{E}(FLG) = \mathbb{E}(GLF). \tag{5.5}$$

Finally, for $q \geq 2$, we define

$$|||F|||_{q,p} = \|F\|_{q,p} + \|LF\|_{q-2,p}\,. \tag{5.6}$$

We recall now the basic computational rules and the integration by parts formulas. For $\phi \in C^1(\mathbb{R}^d)$ and $F = (F^1, ..., F^d) \in \mathcal{S}^d$ we have

$$D\phi(F) = \sum_{j=1}^d \partial_j \phi(F) DF^j, \tag{5.7}$$

and for $F, G \in \mathcal{S}$

$$L(FG) = FLG + GLF - 2\langle DF, DG\rangle\,. \tag{5.8}$$

The formula (5.7) is just the chain rule in the standard differential calculus and (5.8) is obtained using duality. Let $H \in \mathcal{S}$. We use the duality relation and (5.5) we obtain

$$\mathbb{E}(HFLG) = \mathbb{E}(\langle D(HF), DG\rangle_{l_2}) = \mathbb{E}(H\langle DF, DG\rangle_{l_2}) + \mathbb{E}(F\langle DH, DG\rangle_{l_2}).$$

A similar formula holds with GLF instead of FLG. We sum them and we obtain

$$\begin{aligned} \mathbb{E}(H(FLG + GLF)) &= 2\mathbb{E}(H\langle DF, DG\rangle_{l_2}) + \mathbb{E}(\langle DH, D(FG)\rangle_{l_2}) \\ &= 2\mathbb{E}(H\langle DF, DG\rangle_{l_2}) + \mathbb{E}(HL(FG)). \end{aligned}$$

We give now the integration by parts formula (this is a localized version of the standard integration by parts formula from Malliavin calculus).

Theorem 5.1. *Let $\eta > 0$ be fixed and let $\Psi_\eta \in C^\infty(\mathbb{R})$ be such that $1_{[\eta/2,\infty)} \leq \Psi_\eta \leq 1_{[\eta,\infty)}$ and for every $k \in \mathbb{N}$ one has $\|\Psi_\eta^{(k)}\|_\infty \leq C\eta^{-k}$. Let $F \in \mathcal{S}^d$ and $G \in \mathcal{S}$. For every $\phi \in C_p^\infty(\mathbb{R}^d)$, $\eta > 0$ and $i = 1, ..., d$*

$$\mathbb{E}(\partial_i \phi(F) G \Psi_\eta(\det \sigma_F)) = \mathbb{E}(\phi(F) H_i(F, G\Psi_\eta(\det \sigma_F))) \tag{5.9}$$

with

$$H_i(F, G\Psi_\eta(\det \sigma_F)) = \sum_{j=1}^d \big(G\Psi_\eta(\det \sigma_F)\big)\gamma_F^{i,j} LF^j + \big\langle D(G\Psi_\eta(\det \sigma_F)\gamma_F^{i,j}), DF^j \big\rangle_{l_2}. \tag{5.10}$$

Let $m \in \mathbb{N}, m \geq 2$ and $\alpha = (\alpha_1, ..., \alpha_m) \in \{1, ..., d\}^m$. Then

$$\mathbb{E}(\partial_\alpha \phi(F) G \Psi_\eta(\det \sigma_F)) = \mathbb{E}(\phi(F) H_\alpha(F, G\Psi_\eta(\det \sigma_F))) \tag{5.11}$$

with $H_\alpha(F, G\Psi_\eta(\det \sigma_F))$ defined by recurrence

$$H_{(\alpha_1,...,\alpha_m)}(F, G\Psi_\eta(\det \sigma_F)) := H_{\alpha_m}(F, H_{(\alpha_1,...,\alpha_{m-1})}(F, G\Psi_\eta(\det \sigma_F))).$$

The proof is standard, for details see e.g. [7, 11].

We give now useful estimates for the weights which appear in (5.11):

Lemma 5.2. *Let $q, m \in \mathbb{N}$ and $F \in \mathcal{S}^d$ and $G \in \mathcal{S}$. There exists a universal constant $\mathrm{C} \geq 1$ (depending on d, q, m only) such that for every multiindex α with $|\alpha| = q$ and every $\eta > 0$ one has*

$$\left|H_\alpha(F, \Psi_\eta(\det \sigma_F)G)\right|_m \leq \frac{\mathrm{C}}{\eta^{2q+m}} \times \mathcal{K}_{q,m}(F) \times |G|_{m+q}\,, \tag{5.12}$$

with

$$\mathcal{K}_{q,m}(F) = (|F|_{1,m+q+1} + |LF|_{m+q})^q (1 + |F|_{1,m+q+1})^{2d(2q+m)}. \tag{5.13}$$

In particular, taking $m = 0$ and $G = 1$ we have

$$\|H_\alpha(F, \Psi_\eta(\det \sigma_F))\|_p \leq \frac{C}{\eta^{2q}} \times \|\mathcal{K}_{q,0}(F)\|_p \tag{5.14}$$

The proof is straightforward but technical so we leave it for Appendix A.

We go now on and we give the regularization lemma. We recall that a super kernel $\phi : \mathbb{R}^d \to \mathbb{R}$ is a function which belongs to the Schwartz space $\mathbf{S}$ (infinitely differentiable functions which rapidly decrease at infinity), $\int \phi(x)dx = 1$, and such that for every multiindexes α and β, one has

$$\int y^\alpha \phi(y)dy = 0, \quad |\alpha| \geq 1, \tag{5.15}$$

$$\int |y|^m \left|\partial_\beta \phi(y)\right| dy < \infty. \tag{5.16}$$

As usual, for $|\alpha| = m$ then $y^\alpha = \prod_{i=1}^m y_{\alpha_i}$. Since super kernels play a crucial role in our approach we give here the construction of such an object (we follow [29] Section 3, Remark 1). We do it in dimension $d = 1$ and then we take tensor products. So, if $d = 1$ we take $\psi \in \mathbf{S}$ which is symmetric and equal to one in a neighborhood of zero and we define $\phi = \mathcal{F}^{-1}\psi$, the inverse of the Fourier transform of ψ. Since $\mathcal{F}^{-1}$ sends $\mathbf{S}$ into $\mathbf{S}$ the property (5.16) is verified. And we also have $0 = \psi^{(m)}(0) = i^{-m} \int x^m \phi(x)dx$ so (5.15) holds as well. We finally normalize in order to obtain $\int \phi = 1$.

We fix a super kernel ϕ. For $\delta \in (0, 1)$ and for a function f we define

$$\phi_\delta(y) = \frac{1}{\delta^d}\phi\Big(\frac{y}{\delta}\Big) \quad \text{and} \quad f_\delta = f * \phi_\delta,$$

the symbol $*$ denoting convolution. For $f \in C_p^k(\mathbb{R}^d)$, we recall the constants $L_k(f)$ and $l_k(f)$ in (2.10).

Lemma 5.3. *Let $F \in \mathcal{S}^d$ and $q, m \in \mathbb{N}$. There exists some constant $C \geq 1$, depending on d, m and q only, such that for every $f \in C^{q+m}_{\mathrm{pol}}(\mathbb{R}^d)$, every multiindex γ with $|\gamma| = m$ and every $\eta, \delta > 0$*

$$\begin{aligned} & |\mathbb{E}(\Psi_\eta(\det \sigma_F)\partial_\gamma f(x+F)) - \mathbb{E}(\Psi_\eta(\det \sigma_F)\partial_\gamma f_\delta(x+F))| \\ & \leq C\, 2^{l_0(f)} c_{l_0(f)+q} L_0(f) \left\| F \right\|_{2l_0(f)}^{l_0(f)} \mathcal{C}_{q+m}(F) \frac{\delta^q}{\eta^{2(q+m)}} (1+|x|)^{l_0(f)} \end{aligned} \tag{5.17}$$

with

$$c_p = \int |\phi(z)|\,(1+|z|)^p dz \quad \textit{and} \quad \mathcal{C}_p(F) = \left\| \mathcal{K}_{p,0}(F) \right\|_2, \tag{5.18}$$

$\mathcal{K}_{p,0}(F)$ being defined in (5.13). Moreover, for every $p > 1$

$$\begin{aligned} & |\mathbb{E}(\partial_\gamma f(x+F)) - \mathbb{E}(\partial_\gamma f_\delta(x+F))| \leq C(1+\left\| F \right\|_{pl_0(f)})^{l_0(f)} (1+|x|)^{l_m(f)} \\ & \times \Big(L_m(f) c_{l_m(f)} 2^{l_m(f)} \mathbb{P}^{(p-1)/p}(\det \sigma_F \leq \eta) + 2^{l_0(f)} c_{l_0(f)+q}\, L_0(f) \frac{\delta^q}{\eta^{2(q+m)}} \mathcal{C}_{q+m}(F) \Big). \end{aligned} \tag{5.19}$$

Proof. **A**. Using Taylor expansion of order q

$$\begin{aligned} \partial_\gamma f(x) - \partial_\gamma f_\delta(x) &= \int (\partial_\gamma f(x) - \partial_\gamma f(y))\phi_\delta(x-y)dy \\ &= \int I_{\gamma,q}(x,y)\phi_\delta(x-y)dy + \int R_{\gamma,q}(x,y)\phi_\delta(x-y)dy \end{aligned}$$

with

$$\begin{aligned} I_{\gamma,q}(x,y) &= \sum_{i=1}^{q-1} \frac{1}{i!} \sum_{|\alpha|=i} \partial_\alpha \partial_\gamma f(x)(x-y)^\alpha, \\ R_{\gamma,q}(x,y) &= \frac{1}{q!} \sum_{|\alpha|=q} \int_0^1 \partial_\alpha \partial_\gamma f(x+\lambda(y-x))(x-y)^\alpha (1-\lambda)^q d\lambda. \end{aligned}$$

Using (5.15) we obtain $\int I(x,y)\phi_\delta(x-y)dy = 0$ and by a change of variable we get

$$\int R_{\gamma,q}(x,y)\phi_\delta(x-y)dy = \frac{1}{q!} \sum_{|\alpha|=q} \int_0^1 \int dz \phi_\delta(z) \partial_\alpha \partial_\gamma f(x+\lambda z) z^\alpha (1-\lambda)^q d\lambda.$$

So, we have

$$\begin{aligned} & \mathbb{E}(\Psi_\eta(\det \sigma_F)\partial_\gamma f(x+F)) - \mathbb{E}(\Psi_\eta(\det \sigma_F)\partial_\gamma f_\delta(x+F)) \\ & = \mathbb{E}\Big(\int \Psi_\eta(\det \sigma_F) R_{\gamma,q}(x+F,y)\phi_\delta(x+F-y)dy \Big) \\ & = \frac{1}{q!} \sum_{|\alpha|=q} \int_0^1 \int dz \phi_\delta(z) \mathbb{E}\big(\Psi_\eta(\det \sigma_F)\partial_\alpha \partial_\gamma f(x+F+\lambda z)\big) z^\alpha (1-\lambda)^q d\lambda. \end{aligned}$$

Using integration by parts formula (5.11) (with $G = 1$)

$$\begin{aligned} & |\mathbb{E}(\Psi_\eta(\det \sigma_F)\partial_\alpha \partial_\gamma f(x+F+\lambda z))| \\ & = \left| \mathbb{E}(f(F+\lambda z) H_{(\gamma,\alpha)}(F, \Psi_\eta(\det \sigma_F)) \right| \\ & \leq L_0(f) \mathbb{E}((1+|x|+|z|+|F|)^{l_0(f)} \left| H_{(\gamma,\alpha)}(F, \Psi_\eta(\det \sigma_F)) \right|) \\ & \leq C(1+|x|)^{l_0(f)} (1+|z|)^{l_0(f)} L_0(f) \left\| F \right\|_{2l_0(f)}^{l_0(f)} \left\| H_{(\gamma,\alpha)}(F, \Psi_\eta(\det \sigma_F)) \right\|_2. \end{aligned}$$

The upper bound from (5.14) (with $p = 2$) gives

$$\left\| H_\alpha(F, \Psi_\eta(\det \sigma_F)) \right\|_2 \leq \frac{C}{\eta^{2(q+m)}} \times \left\| \mathcal{K}_{q+m,0}(F) \right\|_2$$

And since $\int dz\,|\phi_\delta(z)z^\alpha|\,(1+|z|)^{l_0(f)} \leq \delta^q \int |\phi(z)|\,(1+|z|)^{|\alpha|+l_0(f)}dz$ we conclude that

$$\begin{aligned}&|\mathbb{E}(\Psi_\eta(\det\sigma_F))\partial_\gamma f(x+F)) - \mathbb{E}(\Psi_\eta(\det\sigma_F))\partial_\gamma f_\delta(x+F))|\\ &\leq C(1+|x|)^{l_0(f)} c_{l_0(f)+q} L_0(f)\,\|F\|_{2l_0(f)}^{l_0(f)}\,\|\mathcal{K}_{q+m,0}(F)\|_2\,\frac{C\delta^q}{\eta^{2(q+m)}}\end{aligned}$$

and (5.17) holds. Concerning (5.19), we set $L_{\gamma,\delta} = L_0(\partial_\gamma f_\delta) \vee L_0(\partial_\gamma f)$ and $l_{\gamma,\delta} = l_0(\partial_\gamma f_\delta) \vee l_0(\partial_\gamma f)$. So, for every $p > 1$, we have

$$\begin{aligned}&\left|\mathbb{E}\big((1-\Psi_\eta(\det\sigma_F))\partial_\gamma f(x+F)\big) - \mathbb{E}\big((1-\Psi_\eta(\det\sigma_F))\partial_\gamma f_\delta(x+F)\big)\right|\\ \leq\;& 2L_{\gamma,\delta}\mathbb{E}\big((1-\Psi_\eta(\det\sigma_F))(1+|x|+|F|)^{l_{\gamma,\delta}}\big)\\ \leq\;& 2L_{\gamma,\delta}2^{l_{\gamma,\delta}}(1+|x|)^{l_{\gamma,\delta}}(1+\|F\|_{pl_0(f_\delta)\vee l_0(f)})^{l_0(f_\delta)\vee l_0(f)}\mathbb{P}^{(p-1)/p}(\det\sigma_F \leq \eta).\end{aligned}$$

So the proof of (5.19) will be completed as soon as we check that $l_0(\partial_\gamma f_\delta) \leq l_m(f)$ and $L_0(\partial_\gamma f_\delta) \leq L_m(f)\int(1+|y|)^{l_m(f)}\,|\phi(y)|\,dy = L_m(f)c_{l_m(f)}$:

$$\begin{aligned}|\partial_\gamma f_\delta(x)| &= \left|\int \partial_\gamma f(x-y)\phi_\delta(y)dy\right| \leq L_m(f)\int(1+|x-y|)^{l_m(f)}\,|\phi_\delta(y)|\,dy\\ &\leq L_m(f)(1+|x|)^{l_m(f)}\int(1+|y|)^{l_m(f)}\,|\phi(y)|\,dy.\end{aligned}$$ □

5.2 CLT and Edgeworth's development

In this section we take $F = S_n(Y) = \frac{1}{\sqrt{n}}\sum_{k=1}^n C_{n,k}Y_k$ defined in (2.1), and we recall that $\sigma_{n,k} = C_{n,k}C_{n,k}^* = \mathrm{Cov}(C_{n,k}Y_k)$. From now on, we assume that $Y_k \in \mathfrak{D}(r,\varepsilon)$ so we have the decomposition (5.1). Consequently

$$F = S_n(Y) = \frac{1}{\sqrt{n}}\sum_{k=1}^n C_{n,k}Y_k = \frac{1}{\sqrt{n}}\sum_{k=1}^n C_{n,k}(\chi_k V_k + (1-\chi_k)U_k).$$

We will use Lemma 5.3, so we estimate the quantities which appear in the right hand side of (5.17).

Lemma 5.4. *Let $Y_k \in \mathfrak{D}(2\varepsilon, r)$ and let the moment bounds condition (2.3) hold. For every $k \in \mathbb{N}$ and $p \geq 2$ there exists a constant C depending on k, p only, such that*

$$\sup_n \|S_n(Y)\|_{k,p} \leq 2\times \mathrm{C}_p(Y) \quad \textit{and} \quad \sup_n |||S_n(Y)|||_{k,p} \leq C\times\frac{\mathrm{C}_p(Y)}{r^k}. \tag{5.20}$$

Proof. Using the Burkholder inequality (4.39) and (2.3) we obtain $\|S_n(Y)\|_p \leq C\times\mathrm{C}_p(Y)$. We look now to the Sobolev norms. It is easy to see that, $S_n(Y)^i$ denoting the ith component of $S_n(Y)$,

$$D_{(k,j)}S_n(Y)^i = \frac{1}{\sqrt{n}}\chi_k C_{n,k}^{i,j} \quad \text{and} \quad D^{(l)}S_n(Y) = 0 \text{ for } l \geq 2.$$

Since $\frac{1}{n}\sum_{k=1}^n|\sigma_{n,k}| \leq \mathrm{C}_2(Y)$ it follows that

$$\|S_n(Y)\|_{k,p} \leq 2\mathrm{C}_p(Y) \quad \forall k \in \mathbb{N}, p \geq 2.$$

Moreover

$$LS_n(Y) = -\frac{1}{\sqrt{n}}\sum_{k=1}^n C_{n,k}LY_k = -\frac{1}{\sqrt{n}}\sum_{k=1}^n \chi_k C_{n,k}A_r(V_k),$$
$$\text{with} \quad A_r(V_k) = 1_{r<|V_k-y_Y|^2<2r}\times 2a_r'(|V_k-y_{Y_k}|^2)(V_k-y_{Y_k}).$$

We prove that

$$\|LS_n(Y)\|_{k,p} \le \frac{C}{r^k} \times \mathrm{C}_p(Y), \tag{5.21}$$

C depending on k, p but being independent of n.

Let $k=0$. The duality relation gives $\mathbb{E}(LY_k) = \mathbb{E}(\langle D1, DY_k\rangle_{l_2}) = 0$. Since the LY_k's are independent, we can apply (4.39) first and then (2.3), so that

$$\|LS_n(Y)\|_p \le C\Big(\frac{1}{n}\sum_{k=1}^n \|C_{n,k}A_r(V_k)\|_p^2\Big)^{1/2} \le C\Big(\mathrm{C}_2(Y)\frac{1}{n}\sum_{k=1}^n \|A_r(V_k)\|_p^2\Big)^{1/2}$$

By (2.6) $\mathbb{E}(|A_r(V_k)|^p) \le Cr^{-p}$ so $\|LS_n(Y)\|_p \le Cr^{-1} \times \mathrm{C}_2(Y)$ and (5.21) follows for $k=0$.

We take now $k=1$. We have

$$D_{(q,j)}LS_n(Y)^i = \frac{1}{\sqrt{n}}D_{(q,j)}\big(\chi_k C_{n,q}A_r(V_q)\big) = \frac{1}{\sqrt{n}}\chi_k C_{n,q}D_{(q,j)}A_r(V_q)$$

so that, using again (2.3),

$$|DLS_n(Y)|_{l_2}^2 = \frac{1}{n}\sum_{q=1}^n\sum_{j=1}^d \big|\chi_k C_{n,q}D_{(q,j)}A_r(V_q)\big|^2 \le C \times \frac{\mathrm{C}_2(Y)}{n}\sum_{q=1}^n\sum_{j=1}^d \big|D_{(q,j)}A_r(V_q)\big|^2.$$

We notice that $D_{(q,j)}A_r(V_q)$ is not null for $r < |V_q - y_{Y_q}|^2 < 2r$ and contains the derivatives of a_r up to order 2, possibly multiplied by polynomials in the components of $V_q - y_{Y_q}$ of order up to 2. Since $|V_q - y_{Y_q}|^2 \le 2r$, by using (2.6) one obtains $\mathbb{E}(|DLS_n(Y)|_{l_2}^p) \le Cr^{-2p} \times \mathrm{C}_2^{p/2}(Y)$, so (5.21) holds for $k=1$ also. And for higher order derivatives the proof is similar. □

Remark 5.5. For further use, we give here an upper estimate of the quantity $\|\mathcal{K}_{q,0}(F)\|_p$, with $\mathcal{K}_{q,0}(F)$ defined in (5.13), in the case $F = S_n(Y)$ (recall that $S_n(Y)$ takes values in $\mathbb{R}^d$). From (5.13), it follows that

$$\|\mathcal{K}_{q,0}(F)\|_p \le \||F|\|_{q,2qp}^q(1+\|F\|_{q+1,8dqp})^{4dq}.$$

So, for $F = S_n(Y)$ we use (5.20) and for a suitable constant C depending on d, q and p only, we obtain

$$\|\mathcal{K}_{q,0}(S_n(Y))\|_p \le C \times \frac{\mathrm{C}_{8dqp}^{(4d+1)q}(Y)}{r^{q(q+1)}}. \tag{5.22}$$

We give now estimates of the Malliavin covariance matrix. We have

$$\sigma_{S_n(Y)} = \frac{1}{n}\sum_{k=1}^n \chi_k\sigma_{n,k}.$$

We denote

$$\Sigma_n = \frac{1}{n}\sum_{i=1}^n \sigma_{n,i}, \quad \underline{\lambda}_n = \inf_{|\xi|=1}\langle\Sigma_n\xi,\xi\rangle, \quad \overline{\lambda}_n = \sup_{|\xi|=1}\langle\Sigma_n\xi,\xi\rangle. \tag{5.23}$$

For reasons which will be clear later on, we *do not* consider here the normalization condition $\Sigma_n = \mathrm{Id}_d$. We have the following result.

Lemma 5.6. *Let* $\eta = (\frac{\underline{\lambda}_n\mathfrak{m}_r}{2(1+2\overline{\lambda}_n)})^d$, $\overline{\lambda}_n$ *and* $\underline{\lambda}_n$ *being given in (5.23). Then*

$$\mathbb{P}(\det\sigma_{S_n(Y)} \le \eta) \le \frac{e^3\overline{c}_d}{9}\Big(\frac{2(1+2\overline{\lambda}_n)}{\underline{\lambda}_n\mathfrak{m}_r}\Big)^d \exp\Big(-\frac{\underline{\lambda}_n^2\mathfrak{m}_r^2}{16\overline{\lambda}_n}\times n\Big), \tag{5.24}$$

$\overline{c}_d$ *denoting a positive constant depending on the dimension* d *only.*

Proof. Since $\sigma_{n,k} = C_{n,k}C_{n,k}^*$ we have

$$\langle \sigma_{S_n(Y)}\xi,\xi\rangle = \frac{1}{n}\sum_{k=1}^n \chi_k \langle \sigma_{n,k}\xi,\xi\rangle = \frac{1}{n}\sum_{k=1}^n \chi_k \left|C_{n,k}\xi\right|^2.$$

Take $\xi_1,...,\xi_N \in S_{d-1} =: \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ such that the balls of centers ξ_i and radius $\eta^{1/d}$ cover S_{n-1}. One needs $N \leq \bar{c}_d\eta^{-1}$ points, where $\bar{c}_d$ is a constant depending on the dimension. It is easy to check that $\xi \mapsto \langle \sigma_{S_n(Z)}\xi,\xi\rangle$ is Lipschitz continuous with Lipschitz constant $2\overline{\lambda}_n$ so that $\inf_{|\xi|=1}\langle \sigma_{S_n(Z)}\xi,\xi\rangle \geq \inf_{i=1,\dots,N}\langle \sigma_{S_n(Z)}\xi_i,\xi_i\rangle - 2\overline{\lambda}_n\eta^{1/d}$. Consequently,

$$\begin{aligned}\mathbb{P}(\det \sigma_{S_n(Z)} \leq \eta) &\leq \mathbb{P}(\inf_{|\xi|=1} \left\langle \sigma_{S_n(Z)}\xi,\xi\right\rangle \leq \eta^{1/d}) \leq \sum_{i=1}^N \mathbb{P}(\left\langle \sigma_{S_n(Z)}\xi_i,\xi_i\right\rangle \leq \eta^{1/d} + 2\overline{\lambda}_n\eta^{1/d}) \\ &\leq \frac{\bar{c}_d}{\eta}\max_{i=1,\dots,N} \mathbb{P}\big(\langle \sigma_{S_n(Z)}\xi_i,\xi_i\rangle \leq \eta^{1/d}(1+2\overline{\lambda}_n)\big).\end{aligned}$$

So, it remains to prove that for every $\xi \in S_{d-1}$ and for the choice $\eta = (\frac{\underline{\lambda}_n \mathfrak{m}_r}{2(1+2\overline{\lambda}_n)})^d$,

$$\mathbb{P}\big(\langle \sigma_{S_n(Z)}\xi,\xi\rangle \leq (1+2\overline{\lambda}_n)\eta^{1/d}\big) \leq \frac{2e^3}{9}\exp\Big(-\frac{\underline{\lambda}_n^2\mathfrak{m}_r^2}{16\overline{\lambda}_n}\times n\Big).$$

We recall that $\mathbb{E}(\chi_k) = \mathfrak{m}_r$ and we write

$$\begin{aligned}&\mathbb{P}\big(\langle \sigma_{S_n(Z)}\xi,\xi\rangle \leq (1+2\overline{\lambda}_n)\eta^{1/d}\big) \\ &= \mathbb{P}\Big(\frac{1}{n}\sum_{k=1}^n(\chi_k - \mathfrak{m}_r)\left|C_{n,k}\xi\right|^2 \leq (1+2\overline{\lambda}_n)\eta^{1/d} - \mathfrak{m}_r\frac{1}{n}\sum_{k=1}^n \left|C_{n,k}\xi\right|^2\Big) \\ &\leq \mathbb{P}\Big(-\frac{1}{n}\sum_{k=1}^n(\chi_k - \mathfrak{m}_r)\left|C_{n,k}\xi\right|^2 \geq \underline{\lambda}_n\mathfrak{m}_r - (1+2\overline{\lambda}_n)\eta^{1/d}\Big)\end{aligned}$$

the last equality being true because, from (5.23),

$$\frac{1}{n}\sum_{k=1}^n \left|C_k\xi\right|^2 = \frac{1}{n}\sum_{k=1}^n \langle \sigma_{n,k}\xi,\xi\rangle = \langle \Sigma_n\xi,\xi\rangle \geq \underline{\lambda}_n \left|\xi\right|^2 = \underline{\lambda}_n.$$

So, we take $\eta = (\frac{\underline{\lambda}_n \mathfrak{m}_r}{2(1+2\overline{\lambda}_n)})^d$ and we get

$$\mathbb{P}\big(\langle \sigma_{S_n(Z)}\xi,\xi\rangle \leq (1+2\overline{\lambda}_n)\eta^{1/d}\big) \leq \mathbb{P}\Big(-\sum_{k=1}^n(\chi_k - \mathfrak{m}_r)\left|C_{n,k}\xi\right|^2 \geq \frac{\underline{\lambda}_n\mathfrak{m}_r}{2}\times n\Big)$$

We now use the following Hoeffding's inequality (in the slightly more general form given in [16] Corollary 1.4): if the differences X_k of a martingale M_n are such that $\mathbb{P}(|X_k| \leq b_k) = 1$ then $\mathbb{P}(M_n \geq x) \leq (2e^3/9)\exp(-|x|^2 \times n/(2(b_1^2 + \cdots + b_n^2)))$. Here, we choose $X_k = -(\chi_k - \mathfrak{m}_r)\left|C_{n,k}\xi\right|^2$. These are independent random variables and $|X_k| \leq 2\left|C_{n,k}\xi\right|^2$. Then

$$\begin{aligned}\mathbb{P}\Big(-\sum_{k=1}^n(\chi_k - \mathfrak{m}_r)\left|C_{n,k}\xi\right|^2 \geq \frac{\underline{\lambda}_n\mathfrak{m}_r}{2}\times n\Big) &\leq \frac{2e^3}{9}\exp\Big(-\frac{\underline{\lambda}_n^2\mathfrak{m}_r^2}{4}\times\frac{n}{4\sum_{k=1}^n\left|C_{n,k}\xi\right|^2}\Big) \\ &\leq \frac{2e^3}{9}\exp\Big(-\frac{\underline{\lambda}_n^2\mathfrak{m}_r^2}{16\overline{\lambda}_n}\times n\Big).\end{aligned}$$

□

We are now able to give the regularization lemma in our specific framework.

Lemma 5.7. *Let $h, q \in \mathbb{N}$. There exists a constant $C \geq 1$, depending just on h, q, such that for every $\delta > 0$, every multiindex γ with $|\gamma| = q$ and every $f \in C_p^q(\mathbb{R}^d)$ one has*

$$\begin{aligned}
&|\mathbb{E}(\partial_\gamma f(x+S_n(Y))) - \mathbb{E}(\partial_\gamma f_\delta(x+S_n(Y)))| \\
&\leq C\mathrm{C}_{2l_0(f)}^{1/2}(Y)\mathrm{Q}_{h,q}(Y)\Big(L_q(f)\exp\Big(-\frac{\underline{\lambda}_n^2\mathfrak{m}_r^2}{32\overline{\lambda}_n}\times n\Big) + L_0(f)\delta^h\Big)(1+|x|)^{l_q(f)}
\end{aligned} \tag{5.25}$$

with

$$\mathrm{Q}_{h,q}(Y) = 2^{l_q(f)} c_{l_q(f)\vee(l_0(f)+h)} \frac{\mathrm{C}_{16d(h+q)}^{(4d+1)(h+q)}(Y)}{r^{(h+q)(h+q+1)}}\Big(1 \vee \frac{2(1+2\overline{\lambda}_n)}{\underline{\lambda}_n\mathfrak{m}_r}\Big)^{2d(h+q)}, \tag{5.26}$$

c_p being given in (5.18).

Proof. We will use Lemma 5.3. Since $\mathcal{C}_{h+q}(S_n(Y)) = \|\mathcal{K}_{h,0}(S_n(Y))\|_2$, (5.22) gives

$$\mathcal{C}_{h+q}(S_n(Y)) \leq C \times \frac{\mathrm{C}_{16d(h+q)}^{(4d+1)(h+q)}(Y)}{r^{(h+q)(h+q+1)}},$$

C depending on d and $h+q$. And by using the Burkholder inequality (4.39), one has $\|S_n(Y)\|_{2l_0(f)}^{l_0(f)} \leq \mathrm{C}_{2l_0(f)}^{1/2}(Y)$. So (5.19) (with $p=2$) gives

$$\begin{aligned}
&|\mathbb{E}(\partial_\gamma f(x+S_n(Y))) - \mathbb{E}(\partial_\gamma f_\delta(x+S_n(Y)))| \\
&\leq C\mathrm{C}_{2l_0(f)}^{1/2}(Y)\, 2^{l_q(f)} c_{l_q(f)\vee(l_0(f)+h)} \frac{\mathrm{C}_{16d(h+q)}^{(4d+1)(h+q)}(Y)}{r^{(h+q)(h+q+1)}} \\
&\quad\times\Big(L_q(f)\mathbb{P}^{1/2}(\det\sigma_{S_n(Y)} \leq \eta) + L_0(f)\frac{\delta^h}{\eta^{2(h+q)}}\Big)(1+|x|)^{l_q(f)}.
\end{aligned}$$

We take now $\eta = (\frac{\underline{\lambda}_n\mathfrak{m}_r}{2(1+\overline{\lambda}_n)})^d$ and we use (5.24) in order to obtain

$$\begin{aligned}
&|\mathbb{E}(\partial_\gamma f(x+S_n(Y))) - \mathbb{E}(\partial_\gamma f_\delta(x+S_n(Y)))| \\
&\leq C\mathrm{C}_{2l_0(f)}^{1/2}(Y)\, 2^{l_q(f)} c_{l_q(f)\vee(l_0(f)+h)} \frac{\mathrm{C}_{16d(h+q)}^{(4d+1)(h+q)}(Y)}{r^{(h+q)(h+q+1)}} \\
&\times\Big(1 \vee \frac{2(1+2\overline{\lambda}_n)}{\underline{\lambda}_n\mathfrak{m}_r}\Big)^{2d(h+q)}\Big(L_q(f)\exp\Big(-\frac{\underline{\lambda}_n^2\mathfrak{m}_r^2}{32\overline{\lambda}_n}\times n\Big) + L_0(f)\delta^h\Big)(1+|x|)^{l_q(f)}. \quad \square
\end{aligned}$$

We are now able to characterize the regularity of the semigroup $P_n^{Z,n}$:

Proposition 5.8. *Let $f \in C_p^q(\mathbb{R}^d)$. If $|\gamma| = q$ then*

$$\begin{aligned}
\big|\mathbb{E}\big(\partial_\gamma f(x+S_n(Y))\big)\big| \leq\; & C \times 2^{l_q(f)}\mathrm{B}_q(Y)(1+\mathrm{C}_{2l_q(f)}^{l_q(f)}(Y))(1+|x|)^{l_q(f)}\times \\
&\times\Big[L_q(f)\exp\Big(-\frac{\underline{\lambda}_n^2\mathfrak{m}_r^2}{32\overline{\lambda}_n}\times n\Big) + L_0(f)\Big]
\end{aligned} \tag{5.27}$$

where

$$\mathrm{B}_q(Y) = \Big(1 \vee \frac{2(1+\overline{\lambda}_n)}{\underline{\lambda}_n\mathfrak{m}_r}\Big)^{2dq}\frac{\mathrm{C}_{16dq}^{(4d+1)q}(Y)}{r^{q(q+1)}} \tag{5.28}$$

and C is a constant depending on q only.

Proof. We take $\eta = (\frac{\underline{\lambda}_n\mathfrak{m}_r}{2(1+\overline{\lambda}_n)})^d$ and the truncation function Ψ_η and we write

$$\mathbb{E}(\partial_\gamma f(x+S_n(Y))) = I + J$$

with

$$I=\mathbb{E}(\partial_\gamma f(x+S_n(Y))(1-\Psi_\eta(\det\sigma_{S_n}))),\quad J=\mathbb{E}(\partial_\gamma f(x+S_n(Y))\Psi_\eta(\det\sigma_{S_n})).$$

We estimate first

$$\begin{aligned}
|I| &\leq L_q(f)\mathbb{E}((1+|x|+|S_n(Y)|)^{l_q(f)}(1-\Psi_\eta(\det\sigma_{S_n})))\\
&\leq L_q(f)\big(\mathbb{E}\big((1+|x|+|S_n(Y)|^{2l_q(f)})\big)\big)^{1/2}\mathbb{P}^{1/2}(\det\sigma_{S_n}\leq\eta)\\
&\leq CL_q(f)2^{l_q(f)}(1+|x|)^{l_q(f)}(1+\mathrm{C}_{2l_q(f)}^{l_q(f)}(Y))\Big(\frac{2(1+2\overline{\lambda}_n)}{\underline{\lambda}_n\mathfrak{m}_r}\Big)^{d/2}\exp\Big(-\frac{\underline{\lambda}_n^2\mathfrak{m}_r^2}{32\overline{\lambda}_n}\times n\Big),
\end{aligned}$$

in which we have used the Burkholder's inequality (4.39). In order to estimate J we use integration by parts and we obtain

$$\begin{aligned}
|J| &= \big|\mathbb{E}\big(f(x+S_n(Y))H_\gamma(S_n(Y),\Psi_\eta(\det\sigma_{S_n}))\big)\big|\\
&\leq L_0(f)\mathbb{E}\big((1+|x|+|S_n|)^{l_0(f)}\,|H_\gamma(S_n(Y),\Psi_\eta(\det\sigma_{S_n}))|\,\big)\\
&\leq CL_0(f)2^{l_0(f)}(1+|x|)^{l_0(f)}(1+\mathrm{C}_{2l_0(f)}^{l_0(f)}(Y))\big(\mathbb{E}(|H_\gamma(S_n(Y),\Psi_\eta(\det\sigma_{S_n}))|^2)\big)^{1/2}.
\end{aligned}$$

Then using (5.14) and (5.22)

$$\big\|H_\alpha(S_n(Y),\Psi_\eta(\det\sigma_{S_n(Y)}))\big\|_2\leq C\times\Big(\frac{2(1+\overline{\lambda}_n)}{\underline{\lambda}_n\mathfrak{m}_r}\Big)^{2dq}\times\frac{\mathrm{C}_{16dq}^{(4d+1)q}(Y)}{r^{q(q+1)}}(1+|x|)^{l_0(f)},$$

so that

$$|J|\leq C\times L_0(f)2^{l_0(f)}(1+|x|)^{l_0(f)}(1+\mathrm{C}_{2l_0(f)}^{l_0(f)}(Y))\times\Big(\frac{2(1+\overline{\lambda}_n)}{\underline{\lambda}_n\mathfrak{m}_r}\Big)^{2dq}\times\frac{\mathrm{C}_{16dq}^{(4d+1)q}(Y)}{r^{q(q+1)}}(1+|x|)^{l_0(f)}.$$

(5.27) now follows from the above estimates for I and J. □

5.3 Proofs of the results in Section 2

5.3.1 Proof of Theorem 2.3

Step 1. We assume first that $f\in C_p^{q+(N+1)(N+3)}(\mathbb{R}^d)$ and we prove that

$$\begin{aligned}
&|\mathbb{E}(\partial_\gamma f(S_n(Y)))-\mathbb{E}(\partial_\gamma f(W)\Phi_N(W))|\\
&\quad\leq\frac{C}{n^{\frac{N+1}{2}}}\times\widehat{\mathrm{C}}_{q+(N+1)+(N+3),N}(f,Y)\Big[L_{q+(N+1)(N+3)}(f)\,e^{-\frac{n}{128}}+L_0(f)\Big],
\end{aligned}\tag{5.29}$$

where C is a constant depending only on q and N and

$$\begin{aligned}
\widehat{\mathrm{C}}_{p,N}(f,Y)= &\ \mathrm{C}_2^{(\lfloor N/2\rfloor+1)(N+1)}(Y)2^{(N+3)l_p(f)}\mathrm{C}_{2(N+3)}^{(N+1)/2}(Y)\big(1+\mathrm{C}_{2l_p(f)}^{l_p(f)\vee(N+1)}(Y)\big)\times\\
&\times\Big(1\vee\frac{8}{\mathfrak{m}_r}\Big)^{2dp}\frac{\mathrm{C}_{16dp}^{(4d+1)p}(Y)}{r^{p(p+1)}}.
\end{aligned}\tag{5.30}$$

Notice that (5.29) is analogous to (2.22) but here $L_q(f)$ and $l_q(f)$ are replaced by $L_{q+(N+1)(N+3)}(f)$ and $l_{q+(N+1)(N+3)}(f)$. We will see in Step 2 how to drop the dependence on $q+(N+1)(N+3)$.

We recall (4.37) and (4.38): we have

$$\mathbb{E}\big(\partial_\gamma f(S_n(Y))\big)-\mathbb{E}\big(\partial_\gamma f(W)\Phi_{n,N}(W)\big)=P_n^{Z,n}(\partial_\gamma f)(0)-P_n^{G,n}\Big(\mathrm{Id}+\sum_{k=1}^N\frac{1}{n^{k/2}}\Gamma_{n,k}\Big)(\partial_\gamma f)(0)$$

$$=I_1(\partial_\gamma f)(0)+I_2(\partial_\gamma f)(0)+I_3(\partial_\gamma f)(0)$$

with

$$\begin{aligned} I_1 &= P_n^{G,n} Q^0_{n,N}, \\ I_2 &= \sum_{1\leq r_1<\cdots<r_{N+1}\leq n} P^{Z,n}_{r_{N+1}+1,n+1} P^{G,n}_{r_N+1,r_{N+1}} \cdots P^{G,n}_{r_1+1,r_2} P^{G,n}_{1,r_1} Q^{(N+1)}_{n,N,r_1,\ldots,r_{N+1}} \\ I_3 &= \sum_{m=1}^{N} \sum_{1\leq r_1<\cdots<r_m\leq n} P^{G,n}_{r_m+1,n+1} P^{G,n}_{r_{m-1}+1,r_m} \cdots P^{G,n}_{r_1+1,r_2} P^{G,n}_{1,r_1} Q^{(m)}_{n,N,r_1,\ldots,r_m} \end{aligned} \tag{5.31}$$

and (see (4.27) and (4.36))

$$\begin{aligned} Q^{(m)}_{N,r_1,\ldots,r_m} &= \frac{1}{n^{\frac{N+3m}{2}}} \sum_{3\leq|\alpha|\leq N_m} a_{n,r_1,\ldots,r_m}(\alpha)\theta^{\alpha}_{n,r_1,\ldots,r_m}\partial_\alpha, \\ Q^0_{N,n}f(x) &= \frac{1}{n^{(N+1)/2}} \sum_{N+1\leq|\alpha|\leq N(N+2\lfloor N/2\rfloor)} c_{n,N}(\alpha)\partial_\alpha, \end{aligned} \tag{5.32}$$

N_m being given in Lemma 4.4: $N_m = m(2\lfloor N/2\rfloor + N + 5)$ for $m \leq N$ and if $m = N+1$ then $N_m = (N+1)(N+3)$. By (4.28) and (4.36), the coefficient which appear above satisfy

$$|a_{n,r_1,\ldots,r_m}(\alpha)| \leq (C\mathrm{C}_2^{\lfloor N/2\rfloor+1}(Y))^m \quad \text{and} \quad |c_{n,N}(\alpha)| \leq (C\mathrm{C}_{N+1}(Y)\mathrm{C}_2(Y))^{N(N+2\lfloor N/2\rfloor)}. \tag{5.33}$$

We first estimate $I_2(\partial_\gamma f)$. Let us prove that for every $r_1 < \cdots < r_{N+1}$

$$\begin{aligned} &\left| P^{Z,n}_{r_{N+1}+1,n} P^{G,n}_{r_N+1,r_{N+1}} ... P^{G,n}_{r_1+1,r_2} P^{G,n}_{k,r_1} Q^{(N+1)}_{r_1,\ldots,r_{N+1}} \partial_\gamma f(x) \right| \leq \frac{C}{n^{\frac{4N+3}{2}}} \times \\ &\times \widehat{\mathrm{C}}_{q+(N+1)+(N+3),N}(f,Y)\left[L_{q+(N+1)(N+3)}(f)\, e^{-\frac{\mathrm{m}_r^2}{128}\,n} + L_0(f)\right](1+|x|)^{l_{q+(N+1)(N+3)}(f)} \end{aligned} \tag{5.34}$$

where C depends only on q and N and $\widehat{\mathrm{C}}_{p,N}(f,Y)$ is given by (5.30).

Recall that $\sigma_{n,r_i} \leq \frac{1}{n}\mathrm{C}_2(Y)$. We take $n \geq 4(N+1)\mathrm{C}_2(Y)$ so that

$$\frac{1}{n}\sum_{i=1}^{N+1} \sigma_{n,r_i} \leq \frac{1}{4}\mathrm{Id}_d. \tag{5.35}$$

Recall that $\frac{1}{n}\sum_{r=1}^{n} \sigma_{n,r} = \mathrm{Id}_d$. So we distinguish now two cases:

$$\textbf{Case 1:} \qquad \frac{1}{n}\sum_{r=r_{N+1}+1}^{n} \sigma_{n,r} \geq \frac{1}{2}\mathrm{Id}_d, \tag{5.36}$$

$$\textbf{Case 2:} \qquad \frac{1}{n}\sum_{r=1}^{r_{N+1}} \sigma_{n,r} \geq \frac{1}{2}\mathrm{Id}_d. \tag{5.37}$$

We treat Case 1. Notice that all the operators coming on in (5.31) commute so, using also (5.32) we obtain

$$\begin{aligned} &P^{Z,n}_{r_{N+1}+1,n+1} P^{G,n}_{r_N+1,r_{N+1}} ... P^{G,n}_{r_1+1,r_2} P^{G,n}_{k,r_1} Q^{(N+1)}_{N,r_1,\ldots,r_{N+1}} \partial_\gamma f(x) \\ &= \frac{1}{n^{(4N+3)/2}} \sum_{3\leq|\alpha|\leq(N+1)(N+3)} a_{n,r_1,\ldots,r_{N+1}}(\alpha)\theta^{\alpha}_{n,r_1,\ldots,r_{N+1}} \\ &\qquad \times P^{G,n}_{r_{N-1}+1,r_N} \cdots P^{G,n}_{r_1+1,r_2} P^{G,n}_{1,r_1} P^{Z,n}_{r_N+1,n} \partial_\gamma\partial_\alpha f(x). \end{aligned}$$

We use now (5.27) with $m = |\gamma|+|\alpha| \leq q+(N+1)(N+3)$ and $S_n(Y) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} C_{n,k}Y_k \equiv \sum_{k=1}^{n} Z_{n,k}$ replaced by $\sum_{k=r_{N+1}+1}^{n} Z_{n,k}$, whose covariance matrix is $\frac{1}{n}\sum_{k=r_{N+1}+1}^{n}\sigma_{n,r}$.

Under (5.36) we have $\frac{1}{2} \le \underline{\lambda}_n \le \overline{\lambda}_n \le 1$, so (5.27) gives

$$\begin{aligned}
&|P^{Z,n}_{r_{N+1}+1,n+1}\partial_\gamma\partial_\alpha f(x)|\\
&\quad \le C \times 2^{l_{q+(N+1)(N+3)}(f)}\widehat{\mathrm{B}}_{q+(N+1)(N+3)}(Y)(1+\mathrm{C}^{l_{q+(N+1)(N+3)}(f)}_{2l_{q+(N+1)(N+3)}(f)}(Y))\times\\
&\quad\quad \times \Big[L_{q+(N+1)(N+3)}(f)\, e^{-\frac{\mathfrak{m}_r^2}{128}n} + L_0(f)\Big](1+|x|)^{l_{q+(N+1)(N+3)}(f)}
\end{aligned}$$

where C is a constant depending only on q and N and $\widehat{\mathrm{B}}_p(Y)$ is the constant $\mathrm{B}_p(Y)$ given in (5.28) with $\underline{\lambda}_n = 1/2$ and $\overline{\lambda}_n = 1$, that is,

$$\widehat{\mathrm{B}}_p(Y) = \Big(1 \vee \frac{8}{\mathfrak{m}_r}\Big)^{2dp}\,\frac{\mathrm{C}^{(4d+1)p}_{16dp}(Y)}{r^{p(p+1)}}$$

Therefore, we can write

$$\begin{aligned}
l_0\big(P^{Z,n}_{r_{N+1}+1,n}\partial_\gamma\partial_\alpha f\big) =& l_{q+(N+1)(N+3)}(f),\\
L_0\big(P^{Z,n}_{r_{N+1}+1,n}\partial_\gamma\partial_\alpha f\big) =& C \times 2^{l_{q+(N+1)(N+3)}(f)}\widehat{\mathrm{B}}_{q+(N+1)(N+3)}(Y)(1+\mathrm{C}^{l_{q+(N+1)(N+3)}(f)}_{2l_{q+(N+1)(N+3)}(f)}(Y))\times\\
&\times \Big[L_{q+(N+1)(N+3)}(f)\, e^{-\frac{\mathfrak{m}_r^2}{128}n} + L_0(f)\Big].
\end{aligned}$$

Now, in the proof of Theorem 4.1 we have proven that (see (4.40)

$$|P^{G,n}_{r_{N-1}+1,r_N}\cdots P^{G,n}_{r_1+1,r_2}P^{G,n}_{1,r_1}g(x)| \le 2^{l_0(g)}(1+\mathrm{C}_{l_0(g)}(Y))L_0(g)(1+|x|)^{l_0(g)}$$

and following the proof of Lemma 4.4 we have

$$|\theta^\alpha_{n,r_1,\dots,r_m}g(x)| \le \big(2^{l_0(g)}\mathrm{C}^{1/2}_{2(N+3)}(Y)(1+\mathrm{C}_{2l_0(g)}(Y))^2\big)^m L_0(g)(1+|x|)^{l_0(g)}.$$

So, taking all estimates, we obtain

$$\begin{aligned}
&|\theta^\alpha_{n,r_1,\dots,r_{N+1}}P^{G,n}_{r_{N-1}+1,r_N}\cdots P^{G,n}_{r_1+1,r_2}P^{G,n}_{1,r_1}P^{Z,n}_{r_N+1,n+1}\partial_\gamma\partial_\alpha f(x)|\\
&\le C \times \mathrm{D}_{q+(N+1)(N+3)}(f,Y)\Big[L_{q+(N+1)(N+3)}(f)\, e^{-\frac{\mathfrak{m}_r^2}{128}n} + L_0(f)\Big](1+|x|)^{l_{q+(N+1)(N+3)}(f)}
\end{aligned}$$

where

$$\mathrm{D}_p(f,Y) = 2^{(N+3)l_p(f)}\mathrm{C}^{(N+1)/2}_{2(N+3)}(Y)\big(1+\mathrm{C}^{l_p(f)\vee(N+1)}_{2l_p(f)}(Y)\big)\Big(1 \vee \frac{8}{\mathfrak{m}_r}\Big)^{2dp}\,\frac{\mathrm{C}^{(4d+1)p}_{16dp}(Y)}{r^{p(p+1)}}$$

We use now formula (5.32) for $Q^{(N+1)}_{N,r_1,\dots,r_{N+1}}$ and the estimate (5.33) for the coefficients $a_{n,r_1,\dots,r_{N+1}}$ and we get

$$\begin{aligned}
&|P^{Z,n}_{r_{N+1}+1,n+1}P^{G,n}_{r_N+1,r_{N+1}}\dots P^{G,n}_{r_1+1,r_2}P^{G,n}_{1,r_1}Q^{(N+1)}_{r_1,\dots,r_{N+1}}\partial_\gamma f(x)|\\
\le& C\frac{1}{n^{\frac{4N+3}{2}}} \times \widehat{\mathrm{C}}_*(f,Y)\Big[L_{q+(N+1)(N+3)}(f)\, e^{-\frac{\mathfrak{m}_r^2}{128}n} + L_0(f)\Big](1+|x|)^{l_{q+(N+1)(N+3)}(f)}
\end{aligned}$$

where C depends only on q and N and $\widehat{\mathrm{C}}_*(f,Y)$ is given by

$$\widehat{\mathrm{C}}_*(f,Y) = \big(\mathrm{C}^{\lfloor N/2\rfloor+1}_2(Y)\big)^{N+1}\mathrm{D}_{q+(N+1)(N+3)}(f,Y).$$

Since $\widehat{\mathrm{C}}_*(f,Y) = \widehat{\mathrm{C}}_{q+(N+1)(N+3),N}(f,Y)$, (5.34) is proved in Case 1.

We deal now with Case 2, that is, we assume (5.37). We write

$$\begin{aligned}
&P^{Z,n}_{r_{N+1}+1,n+1}P^{G,n}_{r_N+1,r_{N+1}}...P^{G,n}_{r_1+1,r_2}P^{G,n}_{1,r_1}Q^{(N+1)}_{N,r_1,\dots,r_{N+1}}\partial_\gamma f(x)\\
&=\frac{1}{n^{\frac{4N+3}{2}}}\sum_{3\le|\alpha|\le(N+1)^2}a_{n,r_1,\dots,r_{N+1}}(\alpha)\theta^{\alpha}_{n,r_1,\dots,r_{N+1}}P^{Z,n}_{r_{N+1}+1,n+1}\\
&\quad\times P^{G,n}_{r_N+1,r_{N+1}}\cdots P^{G,n}_{r_1+1,r_2}P^{G,n}_{1,r_1}\partial_\gamma\partial_\alpha f(x).
\end{aligned}$$

Notice that

$$P^{G,n}_{r_N+1,r_{N+1}}\cdots P^{G,n}_{r_1+1,r_2}P^{G,n}_{1,r_1}\partial_\gamma\partial_\alpha f(x)=\mathbb{E}(\partial_\gamma\partial_\alpha f(x+G))$$

where G is a centered Gaussian random variable of variance $\frac{1}{n}\sum_{i=1}^{r_{N+1}}\sigma_{n,i}-\frac{1}{n}\sum_{i=1}^{N+1}\sigma_{n,r_i}\ge\frac{1}{4}\mathrm{Id}_d$, as it follows by using also (5.35). So standard integration by parts yields

$$|P^{G,n}_{r_N+1,r_{N+1}}...P^{G,n}_{r_1+1,r_2}P^{G,n}_{1,r_1}\partial_\gamma\partial_\alpha f(x)|\le CL_0(f)(1+|x|)^{l_0(f)}.$$

Now the proof follows as in the previous case. So (5.34) is proved in Case 2 as well.

Therefore, by summing over $r_1<r_2<\cdots<r_{N+1}\le n$ (giving a contribution of order n^{N+1}), inequality (5.34) gives

$$|I_2(\partial_\gamma f)(x)|\le\frac{C}{n^{\frac{N+1}{2}}}\times\widehat{\mathrm{C}}_*(f,Y)\Big[L_{q+(N+1)(N+3)}(f)\,e^{-\frac{\mathfrak{m}_r^2}{128}n}+L_0(f)\Big](1+|x|)^{l_{q+(N+1)(N+3)}(f)}$$

Exactly as in Case 2 presented above (using standard integration by parts with respect to the law of Gaussian random variables) we obtain

$$|I_1(\partial_\gamma f)(x)|+|I_3(\partial_\gamma f)(x)|\le\frac{C}{n^{\frac{N+1}{2}}}\times\widehat{\mathrm{C}}_*(f,Y)L_0(f)(1+|x|)^{l_0(f)}.$$

So, recalling that $\widehat{\mathrm{C}}_*(f,Y)=\widehat{\mathrm{C}}_{q+(N+1)(N+3),N}(f,Y)$, (5.29) is proved.

Step 2. We now come back and we replace $L_{q+(N+1)(N+3)}(f)$ by $L_q(f)$ in (5.29). We will use the regularization lemma. So we fix $\delta>0$ (to be chosen in a moment) and we write

$$|\mathbb{E}(\partial_\gamma f(S_n(Y)))-\mathbb{E}(\partial_\gamma f(W)\Phi_N(W))|\le A_\delta(f)+A'_\delta(f)+A''_\delta(f)$$

with

$$\begin{aligned}
A_\delta(f)&=\big|\mathbb{E}\big(\partial_\gamma f_\delta(S_n(Y))\big)-\mathbb{E}\big(\partial_\gamma f_\delta(W)\Phi_N(W)\big)\big|\\
A'_\delta(f)&=\big|\mathbb{E}\big(\partial_\gamma f(S_n(Y))\big)-\mathbb{E}\big(\partial_\gamma f_\delta(S_n(Y))\big)\big|,\\
A''_\delta(f)&=\big|\mathbb{E}\big(\partial_\gamma f(W)\Phi_N(W)\big)-\mathbb{E}\big(\partial_\gamma f_\delta(W)\Phi_N(W)\big)\big|.
\end{aligned}$$

We will use (5.29) for f_δ. Notice that $L_p(f_\delta)\le\hat{c}_{p,l_0(f)}L_0(f)\delta^{-p}$, with $\hat{c}_{p,l}=1\vee\max_{0\le|\alpha|\le p}\int(1+|x|)^l|\partial_\alpha\phi(x)|dx$, and $l_p(f_\delta)=l_0(f)$. So,

$$A_\delta(f)\le\frac{C}{n^{\frac{N+1}{2}}}\times\mathrm{H}_{q,N}(f,Y)L_0(f)\Big[\frac{1}{\delta^{q+(N+1)(N+3)}}\,e^{-\frac{\mathfrak{m}_r^2}{128}\times n}+1\Big],$$

where

$$\begin{aligned}
\mathrm{H}_{q,N}(f,Y)&=\mathrm{C}_2^{(\lfloor N/2\rfloor+1)(N+1)}(Y)2^{(N+3)l_0(f)}\mathrm{C}^{(N+1)/2}_{2(N+3)}(Y)\big(1+\mathrm{C}^{l_0(f)\vee(N+1)}_{2l_0(f)}(Y)\big)\times\\
&\times\Big(1\vee\frac{8}{\mathfrak{m}_r}\Big)^{2d(q+(N+1)(N+3))}\frac{\mathrm{C}^{(4d+1)(q+(N+1)(N+3))}_{16d(q+(N+1)(N+3))}(Y)}{r^{(q+(N+1)(N+3))(q+(N+1)(N+3)+1)}}\,\hat{c}_{q+(N+1)(N+3),l_0(f)}.
\end{aligned}$$

We use now (5.25) with $x=0$ and with some h to be chosen in a moment. We then obtain

$$A'_\delta(f)\le C\mathrm{C}^{1/2}_{2l_0(f)}(Y)\mathrm{Q}_{h,q}(Y)\Big(L_q(f)e^{-\frac{\mathfrak{m}_r^2}{32}\times n}+L_0(f)\delta^h\Big)$$

with $\mathrm{Q}_{h,q}(Y)$ given in (5.26). And we also have $A''_\delta(f) \leq CL_0(f)\delta^h$ (the proof is identical to the one of (5.19) but one employs usual integration by parts with respect to the Gaussian law). We put all this together and we obtain

$$|\mathbb{E}(\partial_\gamma f(S_n(Y))) - \mathbb{E}(\partial_\gamma f(W)\Phi_N(W))| \leq \frac{C}{n^{\frac{N+1}{2}}}\,\mathrm{H}_{q,N}(f,Y)L_0(f)\Big[\frac{1}{\delta^{q+(N+1)(N+3)}}\,e^{-\frac{\mathrm{m}_r^2 n}{128}}+1\Big]$$
$$+ C\mathrm{C}^{1/2}_{2l_0(f)}(Y)\mathrm{Q}_{h,q}(Y)L_q(f)e^{-\frac{\mathrm{m}_r^2 n}{32}} + CL_0(f)\delta^h$$

We take now δ such that

$$\delta^h = \frac{1}{\delta^{q+(N+1)(N+3)}}e^{-\frac{\mathrm{m}_r^2 n}{128}}$$

and $h = q + (N+1)(N+3)$, so that

$$\delta^h = e^{-\frac{\mathrm{m}_r^2 n}{128}\times\frac{h}{h+q+(N+1)(N+3)}} = e^{-\frac{\mathrm{m}_r^2 n}{256}}.$$

With this choice of h and δ we get

$$|\mathbb{E}(\partial_\gamma f(S_n(Y))) - \mathbb{E}(\partial_\gamma f(W)\Phi_N(W))| \leq C\,\mathrm{H}_{q,N}(f,Y)L_0(f)\Big(\frac{1}{n^{\frac{N+1}{2}}} + e^{-\frac{\mathrm{m}_r^2 n}{256}}\Big)$$
$$+ C\mathrm{C}^{1/2}_{2l_0(f)}(Y)\mathrm{Q}_{q+(N+1)(N+3),q}(Y)L_q(f)e^{-\frac{\mathrm{m}_r^2 n}{32}}$$

We take now n sufficiently large in order to have

$$n^{\frac{1}{2}(N+1)}e^{-\frac{\mathrm{m}_r^2 n}{256}} \leq 1.$$

The statement now follows by observing that, with $\mathrm{C}_*(Y)$ given in (2.23),

$$\mathrm{C}_*(Y) \geq \mathrm{H}_{q,N}(f,Y) \quad \text{and} \quad \mathrm{C}_*(Y) \geq \mathrm{C}^{1/2}_{2l_0(f)}(Y)\mathrm{Q}_{q+(N+1)(N+3),q}(Y). \qquad \square$$

5.3.2 Proof of Corollary 2.5

We first explicitly write the expression of the polynomials $H_{\Gamma_{n,k}}(x)$ for $k = 1, 2, 3$. Recall formula (2.15) for the kth operator $\Gamma_{n,k}$ and recall formula (2.14) for the set $\Lambda_{m,k}$ appearing in (2.15). Recall also formula (2.25) for $c_n(\alpha)$ and $d_n(\alpha,\beta)$.

Case $k = 1$. Then $m = 1$ and $l = 3, l' = 0$. So the first order operator is given by

$$\Gamma_{n,1} = \frac{1}{n}\sum_{r=1}^n \frac{1}{6}D^{(3)}_{n,r} = \frac{1}{6n}\sum_{r=1}^n\sum_{|\alpha|=3}\Delta_{n,r}(\alpha)\partial_\alpha,$$

so that, with $c_n(\alpha)$ given in (2.25),

$$H_{\Gamma_{n,1}}(x) = \frac{1}{6}\sum_{|\alpha|=3}c_n(\alpha)H_\alpha(x) = \mathcal{H}_{n,1}(x)$$

and formula (2.27) holds.

Case $k = 2$. Then $m = 1$ or $m = 2$, and we call $\Gamma'_{n,2}$ and $\Gamma''_{n,2}$ the corresponding operator. Suppose first that $m = 1$. Then we need that $l + 2l' = k + 2m = 4$. This means that we have $l = 4, l' = 0$. Then

$$\Gamma'_{n,2} = \frac{1}{n}\sum_{r=1}^n\frac{1}{24}D^{(4)}_{n,r} = \frac{1}{24\,n}\sum_{r=1}^n\sum_{|\alpha|=4}\Delta_{n,r}(\alpha)\partial_\alpha = \frac{1}{24}\sum_{|\alpha|=4}c_n(\alpha)\partial_\alpha.$$

Suppose now that $m=2$. Then we need that $l_1+l_2+2(l'_1+l'_2)=k+2m=6$. The only possibility is $l_1=l_2=3$, $l'_1=l'_2=0$ and the corresponding term is

$$\begin{aligned}\Gamma''_{n,2}&=\frac{1}{n^2}\sum_{0\leq r_1<r_2\leq n}\frac{1}{36}D^{(3)}_{n,r_1}D^{(3)}_{n,r_2}=\frac{1}{36\,n^2}\sum_{1\leq r_1<r_2\leq n}\sum_{|\alpha|=3}\sum_{|\beta|=3}\Delta_{n,r_1}(\alpha)\Delta_{n,r_2}(\beta)\partial_\alpha\partial_\beta\\&=\frac{1}{72\,n^2}\sum_{1\leq r_1\neq r_2\leq n}\sum_{|\alpha|=3}\sum_{|\beta|=3}\Delta_{n,r_1}(\alpha)\Delta_{n,r_2}(\beta)\partial_\alpha\partial_\beta.\end{aligned}$$

We notice that, for $|\alpha|=|\beta|=3$,

$$\frac{1}{n^2}\sum_{1\leq r_1\neq r_2\leq n}\Delta_{n,r_1}(\alpha)\Delta_{n,r_2}(\beta)=c_n(\alpha)c_n(\beta)-\frac{1}{n}d_n(\alpha,\beta)$$

with

$$\sup_n|d_n(\alpha,\beta)|\leq 4\mathrm{C}_3^2(Y),\quad |\alpha|=|\beta|=3. \tag{5.38}$$

So, by inserting,

$$\Gamma''_{n,2}=\frac{1}{72}\sum_{|\alpha|=3}\sum_{|\beta|=3}c_n(\alpha)c_n(\beta)\partial_\alpha\partial_\beta-\frac{1}{72\,n}\sum_{|\alpha|=3}\sum_{|\beta|=3}d_n(\alpha,\beta)\partial_\alpha\partial_\beta.$$

We conclude that

$$H_{\Gamma_{n,2}}(x)=H_{\Gamma'_{n,2}}(x)+H_{\Gamma''_{n,2}}(x)=\mathcal{H}_{n,2}(x)-\frac{1}{72\,n}\sum_{|\alpha|=3}\sum_{|\beta|=3}d_n(\alpha,\beta)H_{(\alpha,\beta)}(x),$$

$\mathcal{H}_{n,2}(x)$ being given in (2.28).

Case $k=3$. $m=1$. We need that $l+2l'=k+2m=5$. So $l=3,l'=1$ or $l=5,l'=0$. The operator term corresponding to $l=3,l'=1$ is

$$\Gamma^1_{n,3}=-\frac{1}{12\,n}\sum_{r=1}^{n}D^{(3)}_{n,r}L^1_{\sigma_{n,r}}=-\frac{1}{12}\sum_{|\alpha|=3}\sum_{i,j=1}^{d}\bar{c}_n(\alpha,i,j)\partial_\alpha\partial_i\partial_j,$$

$\bar{c}_n(\alpha,i,j)$ being given in (2.25). The term corresponding to $l=5,l'=0$ is

$$\Gamma^2_{n,3}=\frac{1}{n}\sum_{r=1}^{n}\frac{1}{5!}D^{(5)}_{n,r}=\frac{1}{5!\,n}\sum_{r=1}^{n}\sum_{|\alpha|=5}\Delta_{n,r}(\alpha)\partial_\alpha=\frac{1}{5!}\sum_{|\alpha|=5}c_n(\alpha)\partial_\alpha.$$

$m=2$. We need $l_1+l_2+2(l'_1+l'_2)=k+2m=7$. The only possibility is $l_1=3$, $l_2=4$, $l'_1=l'_2=0$ and $l_1=4$, $l_2=3$, $l'_1=l'_2=0$. The corresponding term is

$$\Gamma^3_{n,3}=\frac{2}{n^2}\sum_{1\leq r_1<r_2\leq n}\frac{1}{3!}D^{(3)}_{n,r_1}\frac{1}{4!}D^{(4)}_{n,r_2}=\frac{1}{3!4!}\sum_{|\alpha|=3}\sum_{|\beta|=4}\Big[c_n(\alpha)c_n(\beta)-\frac{1}{n}d_n(\alpha,\beta)\Big]\partial_\alpha\partial_\beta,$$

with

$$\sup_n|d_n(\alpha,\beta)|\leq C\,\mathrm{C}_3(Y)\mathrm{C}_4(Y)\leq C\,\mathrm{C}_4^2(Y),\quad |\alpha|=3,|\beta|=4. \tag{5.39}$$

$m=3$. We need $l_1+l_2+l_3+2(l'_1+l'_2+l'_3)=k+2m=3+6=9$. The only possibility is $l_1=l_2=l_3=3,l'_1=l'_2=l'_3=0$ and the corresponding term is

$$\begin{aligned}\Gamma^4_{n,3}&=\frac{1}{6^3\,n^3}\sum_{1\leq r_1<r_2<r_3\leq n}D^{(3)}_{n,r_1}D^{(3)}_{n,r_2}D^{(3)}_{n,r_3}\\&=\frac{1}{6^3\,n^3\,3!}\sum_{1\leq r_1\neq r_2\neq r_3\leq n}\sum_{|\alpha|=3}\sum_{|\beta|=3}\sum_{|\gamma|=3}\Delta_{n,r_1}(\alpha)\Delta_{n,r_2}(\beta)\Delta_{n,r_3}(\gamma)\partial_\alpha\partial_\beta\partial_\gamma,\end{aligned}$$

where the notation $r_1 \neq r_2 \neq r_3$ means that r_1, r_2, r_3 are all different. Now, straightforward computations give

$$\frac{1}{n^3} \sum_{1\leq r_1\neq r_2\neq r_3\leq n} \Delta_{n,r_1}(\alpha)\Delta_{n,r_2}(\beta)\Delta_{n,r_2}(\gamma) = c_n(\alpha)c_n(\beta)c_n(\gamma) - \frac{1}{n}e_n(\alpha,\beta,\gamma),$$

with

$$\sup_n |e_n(\alpha,\beta,\gamma)| \leq C \times \mathrm{C}_3^3(Y), \quad |\alpha| = |\beta| = |\gamma| = 3. \tag{5.40}$$

So, we obtain

$$\Gamma^4_{n,3} = \frac{1}{6^3\, n^3\, 3!} \sum_{|\alpha|=3}\sum_{|\beta|=3}\sum_{|\gamma|=3} c_n(\alpha)c_n(\beta)c_n(\gamma)\partial_\alpha\partial_\beta\partial_\gamma - \frac{1}{n}\sum_{|\alpha|=3}\sum_{|\beta|=3}\sum_{|\gamma|=3} e_n(\alpha,\beta,\gamma)\partial_\alpha\partial_\beta\partial_\gamma.$$

We conclude that

$$H_{\Gamma_{n,3}}(x) = \sum_{i=1}^4 H_{\Gamma^i_{n,3}}(x) = \mathcal{H}_{n,3}(x) + \frac{1}{n}\sum_{|\alpha|\leq 9} f_n(\alpha)H_\alpha(x),$$

with $\mathcal{H}_{n,3}(x)$ as in (2.29) and with

$$\sup_n |f_n(\alpha)| \leq C\mathrm{C}_4^3(Y). \tag{5.41}$$

By resuming, we get

$$\Phi_{n,N}(x) = 1 + \sum_{k=1}^3 \frac{1}{n^{k/2}}\mathcal{H}_{n,k}(x) + \frac{1}{n^2}\mathcal{P}_n(x),$$

where, taking into account (5.38), (5.39), (5.40) and (5.41),

$$\mathcal{P}_n(x) = \sum_{|\alpha|\leq 9} g_n(\alpha)H_\alpha(x) \quad \text{with} \quad \sup_n |g_n(\alpha)| \leq C \times \mathrm{C}_4^3(Y),$$

where C a universal constant. Therefore

$$\begin{aligned}
&\Big|\mathbb{E}\Big(\partial_\gamma f(S_n(Y))\Big) - \mathbb{E}\Big(\partial_\gamma f(W)\Big(1 + \sum_{k=1}^3 \frac{1}{n^{k/2}}\mathcal{H}_{n,k}(W)\Big)\Big)\Big| \\
&\leq \big|\mathbb{E}\big(\partial_\gamma f(S_n(Y))\big) - \mathbb{E}\big(\partial_\gamma f(W)\Phi_{n,N}(W)\big)\big| + \frac{1}{n^2}\big|\mathbb{E}\big(\partial_\gamma f(W)\mathcal{P}_n(W)\big)\big| \\
&\leq C \times \mathrm{C}_*(Y)\Big(\frac{L_0(f)}{n^{\frac{1}{2}(N+1)}} + L_q(f)e^{-\frac{\mathfrak{m}_r^2}{32}\times n}\Big) + \frac{1}{n^2}\big|\mathbb{E}\big(\partial_\gamma f(W)\mathcal{P}_n(W)\big)\big|
\end{aligned}$$

in which we have used (2.22). Now, by using the standard integration by parts for the Gaussian law, we have

$$\big|\mathbb{E}\big(\partial_\gamma f(W)\mathcal{P}_n(W)\big)\big| = \big|\mathbb{E}\big(f(W)G_\gamma(W,\mathcal{P}_n(W))\big)\big| \leq \|f(W)\|_2\|G_\gamma(W,\mathcal{P}_n(W))\|_2,$$

where $G_\gamma(W,\mathcal{P}_n(W))$ denote the weight from the integration by parts formula. Since $\mathcal{P}_n$ is a linear combination of Hermite polynomials with bounded coefficients, we have $\|G_\gamma(W,\mathcal{P}_n(W))\|_2 \leq C$, C depending on q. Moreover, $|f(W)| \leq L_0(f)(1+|W|^{l_0(f)})$, so

$$\big|\mathbb{E}\big(\partial_\gamma f(W)\mathcal{P}_n(W)\big)\big| \leq C \times L_0(f)\mathrm{C}^{l_0(f)}_{2l_0(f)}(Y).$$

The statement now follows. $\square$

5.3.3 Proof of Proposition 2.8

The idea is that, since $\sum_{k=1}^{n-m}\sigma_k \geq \frac{1}{2}I$, the random variables $Y_k, k \leq n-m$ contain sufficient noise in order to give the regularization effect.

We show the main changes in the estimate of $I_2(f)$ (for $I_1(f)$, $I_3(f)$ the proof is analogues). We split $P^{Z,n}_{r_{N+1}+1,n} = P^{Z,n}_{r_{N+1}+1,n-m}P^{Z,n}_{n-m,n}$ and we need to have sufficient noise in order that $P^{Z,n}_{r_{N+1}+1,n-m}$ gives the regularization effect. Then, the two cases described in (5.36) and (5.37) are replaced now by $\sum_{i=r_{N+1}+1}^{n-m}\sigma_{n,i} \geq \frac{1}{4}I$ and $\sum_{i=1}^{r_{N+1}+1}\sigma_{n,i} \geq \frac{1}{4}I$ respectively. And the condition (5.35) becomes $\sum_{i=1}^{N+1}\sigma_{n,r_i} \leq \frac{1}{8}I$. Then the proof follows exactly the same line. □

A Norms

The aim of this section is to prove Lemma 5.2. For $F=(F_1,\ldots,F_d)$ We work with the norms

$$\begin{aligned} |F|_{1,k} &= \sum_{j=1}^{d}\sum_{1=1}^{k}\left|D^iF_j\right|_{\mathcal{H}^{\otimes i}}, \qquad |F|_k = |F| + |F|_{1,k} \\ \|F\|_{1,k,p} &= \|\,|F|_{1,k}\,\|_p, \qquad \|F\|_{k,p} = \|F\|_p + \|F\|_{1,k,p}\,. \end{aligned}$$

To begin we give several easy computational rules:

$$|FG|_k \leq C\sum_{k_1+k_2=k}|F|_{k_1}|G|_{k_2}\,, \tag{A.1}$$

$$|\langle DF, DG\rangle|_k \leq C\sum_{k_1+k_2=k}|F|_{1,k_1+1}|G|_{1,k_2+1}\,, \tag{A.2}$$

$$\left|\frac{1}{G}\right|_k \leq \frac{C}{|G|}\sum_{l=0}^{k}\frac{|G|_k^l}{|G|^l}. \tag{A.3}$$

Now, for $F=(F_1,\ldots,F_d)$ we consider the Malliavin covariance matrix $\sigma_F^{i,j} = \left\langle DF^i, DF^j\right\rangle$ and, if $\det\sigma_F \neq 0$, we denote $\gamma_F = \sigma_F^{-1}$. We write

$$\gamma_F^{i,j} = \frac{\widehat{\sigma}_F^{i,j}}{\det\sigma_F}$$

where $\widehat{\sigma}_F^{i,j}$ is the algebraic complement . Then, using (A.1)

$$\left|\gamma_F^{i,j}\right|_k \leq C\sum_{k_1+k_2=k}\left|\widehat{\sigma}_F^{i,j}\right|_{k_1}\left|\frac{1}{\det\sigma_F}\right|_{k_2}.$$

By (A.1) and (A.2), $\left|\widehat{\sigma}_F^{i,j}\right|_{k_1} \leq C\,|F|_{1,k_1+1}^{2(d-1)}$ and $|\det\sigma_F|_{k_2} \leq C\,|F|_{1,k_2+1}^{2d}$. Then, using (A.3)

$$\left|\frac{1}{\det\sigma_F}\right|_{k_2} \leq \frac{C}{|\det\sigma_F|}\sum_{l=0}^{k_2}\frac{|\det\sigma_F|_{k_2}^l}{|\det\sigma_F|^l} \leq \frac{C}{|\det\sigma_F|}\sum_{l=0}^{k_2}\frac{|F|_{1,k_2+1}^{2ld}}{|\det\sigma_F|^l}$$

so that

$$\left|\gamma_F^{i,j}\right|_k \leq C\frac{|F|_{1,k+1}^{2(d-1)}}{|\det\sigma_F|}\sum_{l=0}^{k}\Big(\frac{|F|_{1,k+1}^{2d}}{|\det\sigma_F|}\Big)^l \leq C\frac{|F|_{1,k+1}^{2(d-1)}}{|\det\sigma_F|}\Big(1+\frac{|F|_{1,k+1}^{2d}}{|\det\sigma_F|}\Big)^k. \tag{A.4}$$

We denote

$$\alpha_k = \frac{|F|_{1,k+1}^{2(d-1)}\left(|F|_{1,k+1} + |LF|_k\right)}{|\det\sigma_F|}, \quad \beta_k = \frac{|F|_{1,k+1}^{2d}}{|\det\sigma_F|} \tag{A.5}$$

and

$$\mathcal{K}_{n,k}(F) = (|F|_{1,k+n+1} + |LF|_{k+n})^n (1 + |F|_{1,k+n+1})^{2d(2n+k)}. \tag{A.6}$$

We also recall that for $\eta > 0$, we consider a function $\Psi_\eta \in C^\infty(\mathbb{R})$ such that $1_{(0,\eta)} \leq \Psi_\eta \leq 1_{(0,2\eta)}$ and $\|\Psi_\eta^{(k)}\|_\infty \leq C_k \eta^{-k}, \forall k \in \mathbb{N}$. Then we take $\Phi_\eta = 1 - \Psi_\eta$.

Lemma A.1. ***A***. *For every $k, n \in \mathbb{N}$ there exists a universal constant C (depending on k and n) such that, for ω such that $\det \sigma_F(\omega) > 0$,*

$$\left| H_\rho^{(n)}(F, G) \right|_k \leq C \alpha_{k+n}^n \sum_{p_1+p_2=k+n} |G|_{p_2} (1 + \beta_{k+n})^{p_1}. \tag{A.7}$$

B. *For every $\eta > 0$*

$$\left| H_\rho^{(n)}(F, \Phi_\eta(\det \sigma_F) G) \right|_k \leq \frac{C}{\eta^{2n+k}} \times \mathcal{K}_{n,k}(F) \times |G|_{k+n}. \tag{A.8}$$

Proof. **A.** We first prove (A.7) for $n = 1$. We have

$$H_i^{(1)}(F, G) = -\sum_{j=1}^m G\gamma_F^{i,j} LF^j + G\langle D\gamma_F^{i,j}, DF^j\rangle + \gamma_F^{i,j}\langle DG, DF^j\rangle.$$

Using (A.1)

$$\begin{aligned}
&\left|H_i^{(1)}(F,G)\right|_k \\
&\leq C \sum_{k_1+k_2+k_3=k} \Big(|\gamma_F|_{k_1} |LF|_{k_2} |G|_{k_3} + |\gamma_F|_{k_1+1} |F|_{1,k_2+1} |G|_{k_3} + |\gamma_F|_{k_1} |F|_{1,k_2+1} |G|_{k_3+1} \Big) \\
&\leq C(|F|_{k+1} + |LF|_k) \sum_{p_1+p_2 \leq k} \Big(|\gamma_F|_{p_1+1} |G|_{p_2} + |\gamma_F|_{p_1} |G|_{p_2+1} \Big).
\end{aligned}$$

For $n > 1$, we use recurrence and we obtain

$$\left| H_\gamma^{(n)}(F, G) \right|_k \leq C(|F|_{k+n+1} + |LF|_{k+n})^n \sum_{p_1+\ldots+p_{n+1} \leq k+n-1} \prod_{i=1}^n |\gamma_F|_{p_i} \times |G|_{p_{n+1}}.$$

Then, using (A.1) first and (A.4) secondly, (A.7) follows.

B. Let $G_\eta = \Phi_\eta(\det \sigma_F) G)$. For every $p \in \mathbb{N}$ one has $|G_\eta|_p \leq C\eta^{-p} |G|_p |F|_{1,p+1}^d$. Moreover one has $H_\rho^{(n)}(F, G_\eta) = 1_{\{\det \sigma_\Phi > \eta/2\}} H_\rho^{(n)}(F, G_\eta)$. So (A.7) implies (A.8). ☐

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