

PERFECT HEDGING IN ROUGH HESTON MODELS

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Rough volatility models are known to reproduce the behavior of historical volatility data while at the same time fitting the volatility surface remarkably well, with very few parameters. However, managing the risks of derivatives under rough volatility can be intricate since the dynamics involve fractional Brownian motion. We show in this paper that surprisingly enough, explicit hedging strategies can be obtained in the case of rough Heston models. The replicating portfolios contain the underlying asset and the forward variance curve, and lead to perfect hedging (at least theoretically). From a probabilistic point of view, our study enables us to disentangle the infinite-dimensional Markovian structure associated to rough volatility models.

1. Introduction. It has been recently shown in [12] that rough fractional processes enable us to reproduce very accurately the behavior of historical volatility time series. More precisely, the dynamic of their logarithm is quite similar to that of a fractional Brownian motion with Hurst parameter of order 0.1. Recall that a fractional Brownian motion W^H with Hurst parameter $H \in (0, 1)$ can be built from a classical two-sided Brownian motion W through the Mandelbrot–van Ness representation:

$$W_t^H = \frac{1}{\Gamma(H + 1/2)} \int_{-\infty}^0 ((t - s)^{H - \frac{1}{2}} - (-s)^{H - \frac{1}{2}}) dW_s \\ + \frac{1}{\Gamma(H + 1/2)} \int_0^t (t - s)^{H - \frac{1}{2}} dW_s.$$

The fractional Brownian motion has Hölder regularity $H - \varepsilon$ for any $\varepsilon > 0$. Hence fractional volatility models with small Hurst parameter are referred to as rough volatility models.

Beyond historical data modeling, rough volatility models provide excellent fits and dynamics for the whole volatility surface,¹ in particular for the at-the-money skew, with very few scalar parameters (typically three); see [4, 11, 12]. One of the only potential drawbacks of such models in practice is the difficulty to price

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¹In fact, this has been clearly established in [4, 12] for the so-called rough Bergomi model, where the log-volatility follows a fractional Brownian motion. However, one can show that this remains true for other rough volatility models such as the rough Heston model introduced in [10].

and hedge derivatives with them. Indeed, although some promising approaches have been recently introduced (see [5]), due to the non-Markovian nature of the fractional Brownian motion, running efficient Monte-Carlo methods remains an intricate task in the rough volatility context; see [19].

However, it is shown in [10] that in the specific case of the so-called rough Heston model, fast pricing of derivatives can be obtained. The rough Heston model of [10] is a natural extension² to the rough framework of the classical Heston model of [15]. Indeed, the dynamic of the price S on a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is defined as follows:

$$\begin{aligned}
 (1) \quad dS_t &= S_t \sqrt{V_t} dW_t, \\
 V_t &= V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \lambda (\theta - V_u) du \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} v \sqrt{V_u} dB_u.
 \end{aligned}$$

Here, the parameters $\lambda, \theta, V_0, S_0$ and v are positive, $\alpha \in (1/2, 1)$ and $W = \rho B + \sqrt{1 - \rho^2} B^\perp$ with (B, B^\perp) a two-dimensional \mathbb{F} -Brownian motion and $\rho \in [-1, 1]$. From [10], the fractional stochastic differential equation (1) admits a unique weak solution and this solution has sample paths with Hölder regularity $\alpha - 1/2 - \varepsilon$ almost surely, for any $\varepsilon > 0$. Note also that in the case $\alpha = 1$, we retrieve the classical Heston model. Surprisingly enough, it is proved in [10] that a semi-closed formula *à la Heston* also holds for the characteristic function of the log-price in the rough Heston model. This formula is very similar to that obtained in the classical Heston case, except that the classical time-derivative in the Riccati equation has to be replaced by a fractional derivative. Indeed, we have

$$\mathbb{E}[\exp(ia \log(S_T/S_0))] = \exp(g_1(a, t) + V_0 g_2(a, t)),$$

where

$$g_1(a, t) = \theta \lambda \int_0^t h(a, s) ds, \quad g_2(a, t) = I^{1-\alpha} h(a, t),$$

and h is the unique continuous solution of the following fractional Riccati equation:

$$\begin{aligned}
 D^\alpha h(a, s) &= \frac{1}{2}(-a^2 - ia) + (ia\rho v - \lambda)h(a, s) + \frac{v^2}{2}h^2(a, s), \\
 I^{1-\alpha} h(a, 0) &= 0,
 \end{aligned}$$

with $I^{1-\alpha}$ and D^α the fractional integral and derivative operators defined in Appendix A. When $\alpha = 1$, this result does coincide with the classical Heston's result.

²Actually there is no really standard definition for the rough Heston model and other versions can be considered; see [13].

Furthermore, efficient numerical pricing procedures for vanilla options can be easily designed from it; see [10].

Thus, the relevance of the rough Heston model is twofold: it enjoys the nice modeling properties of rough volatility models and, as the classical Heston model, it offers a pricing methodology which does not rely on Monte Carlo simulations. However, the interest of having a pricing procedure is of course limited if it does not go along with a hedging strategy. Being able to build a hedging portfolio essentially means computing conditional expectations of the form $C_t = \mathbb{E}[f(S_T)|\mathcal{F}_t]$, where f is a deterministic payoff function. In the classical Heston case, the Markovian structure of the model is very helpful to do it. In the rough case, this task is much more intricate since the variance process is neither a Markov process nor a semimartingale.

To tackle this issue, we first study the conditional laws in rough Heston models. We actually prove a very nice stability property. Indeed, we show that conditional on \mathcal{F}_t , the law of the rough Heston model is still that of a rough Heston model, provided that the mean-reversion level θ is replaced by a time-dependent one. Hence we generalize our definition of the rough Heston model, allowing for the mean-reversion level to depend on time. Then using Hawkes processes as in [10], we are able to compute the extended characteristic function of the log-price in generalized rough Heston models, that is,

$$(2) \quad \mathbb{E}[\exp(z \log(S_t/S_0))]$$

for $z = a + ib$, with $b \in \mathbb{R}$ and a in some subset of \mathbb{R} to be defined later. From an explicit expression of (2), we can deduce a semiclosed formula for C_t following, for example, the approach in [7].

Our most important result is the fact that we are able to identify the relevant state variables in rough Heston models, namely the underlying and the so-called forward variance curve: $(\mathbb{E}[V_{s+t}|\mathcal{F}_t])_{0 \leq s \leq T-t}$. Indeed, we show that C_t can be written

$$C_t = C(T - t, S_t, (\mathbb{E}[V_{s+t}|\mathcal{F}_t])_{s \geq 0}),$$

with $C()$ an explicit deterministic function. The above formula shows rigorously that the hedging instruments needed with rough models are the spot price and the forward variance curve, an idea already emphasized in [4]. Such result is also in the spirit of the approach developed in [6]. More precisely, we show that the dynamic of the option price satisfies

$$dC_t = \partial_S C(T - t, S_t, (\mathbb{E}[V_{s+t}|\mathcal{F}_t])_{s \geq 0}) dS_t + \partial_V C(T - t, S_t, (\mathbb{E}[V_{s+t}|\mathcal{F}_t])_{s \geq 0}) \cdot (d\mathbb{E}[V_{s+t}|\mathcal{F}_t])_{s \geq 0},$$

where $\partial_S C$ is the derivative of C with respect to the underlying (the so-called delta) and $\partial_V C$ is the Fréchet derivative of C according to the forward variance curve and $\partial_V C \cdot \xi$ denotes $\partial_V C$ applied on the curve ξ , as $\partial_V C$ is seen as a linear application.

From this expression, we readily obtain hedging strategies in terms of underlying and forward variance curve. Of course, in practice, one cannot really trade the whole forward variance curve. However, approximations can be built using liquid variance swaps or vanilla options.

Note also that using generalized rough Heston models enables us to perfectly fit the initial forward variance curve through the time varying mean-reversion parameter. Thus, one reproduces the rough behavior observed in historical volatility time-series, the whole implied volatility surface, including the at-the-money skew and the forward variance curve, and has access to fast pricing and hedging methods.

The paper is organized as follows. In Section 2, we investigate conditional laws of rough Heston models and introduce generalized rough Heston models with time-dependent mean-reversion level. Using Hawkes processes, we derive in Section 3 the characteristic function of the log-price in generalized rough Heston models, emphasizing the role of the forward variance curve. We also discuss useful sufficient conditions for finite moments of the underlying price. Finally, we design our hedging strategies in Section 4. Some proofs are relegated to Section 5 and some technical results are given in Appendices A–C.

2. Conditional laws of rough Heston models. The goal of this paper is to understand how to price and hedge vanilla options with maturity $T > 0$ and payoff $f(S_T)$ in the rough Heston framework (1). Thus, a first step is to characterize the law of the process $(S_t^{t_0}, V_t^{t_0})_{t \geq 0} = (S_{t+t_0}, V_{t+t_0})_{t \geq 0}$ conditional on \mathcal{F}_{t_0} , for a fixed $t_0 > 0$. Indeed, in order to derive the option price dynamic and to build hedging portfolios, one needs to be able to compute $\mathbb{E}[f(S_T)|\mathcal{F}_t]$, $0 \leq t \leq T$.

To state our result on conditional laws of rough Heston models, it is convenient to introduce a generalized version of Model (1), allowing for time-varying mean-reversion level.

DEFINITION 2.1 (Generalized rough Heston model). On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, we define a generalized rough Heston model by

$$\begin{aligned}
 dS_t &= S_t \sqrt{V_t} dW_t, \\
 V_t &= V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \lambda (\theta^0(u) - V_u) du \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \nu \sqrt{V_u} dB_u.
 \end{aligned}
 \tag{3}$$

Here, the parameters λ , S_0 and ν are positive, $\alpha \in (1/2, 1)$ and $W = \rho B + \sqrt{1 - \rho^2} B^\perp$ with (B, B^\perp) a two-dimensional \mathbb{F} -Brownian motion and $\rho \in [-1, 1]$. Moreover, θ^0 is a deterministic function, continuous on \mathbb{R}_+^* satisfying

$$\forall u > 0; \quad \theta^0(u) \geq -\frac{V_0}{\lambda \Gamma(1 - \alpha)} u^{-\alpha},
 \tag{4}$$

and

$$(5) \quad \forall \varepsilon > 0 \exists K_\varepsilon > 0; \forall u \in (0, 1]; \quad |\theta^0(u)| \leq K_\varepsilon u^{-\frac{1}{2}-\varepsilon}.$$

Condition (5) is a regularity condition on θ^0 at zero. It is a technical one already used in [10]. Condition (4) enables us to ensure the nonnegativity of V . Remark that Condition (4) and Condition (5) obviously imply that V_0 is nonnegative. Indeed,

$$V_0 \geq -\lambda\Gamma(1 - \alpha)t^\alpha\theta^0(t),$$

which goes to zero as t vanishes since $\alpha > 1/2$. Note also that under these two conditions, the fractional stochastic differential equation (3) admits a unique weak solution; see Theorem 3.1 and associated references.

We now give our result for the conditional laws of generalized rough Heston models [which Model (1) is a particular case of]. Let $(S_t, V_t)_{t \geq 0}$ be defined by (3). We have the following theorem, proved in Section 5.1.

THEOREM 2.1. *The law of the process $(S_t^{t_0}, V_t^{t_0})_{t \geq 0} = (S_{t+t_0}, V_{t+t_0})_{t \geq 0}$ is that of a generalized rough Heston model with the following dynamic:*

$$\begin{aligned} dS_t^{t_0} &= S_t^{t_0} \sqrt{V_t^{t_0}} dW_t^{t_0}, \quad S_0^{t_0} = S_{t_0}, \\ V_t^{t_0} &= V_{t_0} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \lambda(\theta^{t_0}(u) - V_u^{t_0}) du \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} v \sqrt{V_u^{t_0}} dB_u^{t_0}, \end{aligned}$$

with $(W_t^{t_0}, B_t^{t_0})_{t \geq 0} = (W_{t_0+t} - W_{t_0}, B_{t_0+t} - B_{t_0})_{t \geq 0}$ a two-dimensional Brownian motion with correlation ρ , independent of \mathcal{F}_{t_0} and

$$\begin{aligned} \theta^{t_0}(u) &= \theta^0(t_0 + u) + \frac{\alpha}{\lambda\Gamma(1 - \alpha)} \int_0^{t_0} (t_0 - v + u)^{-1-\alpha} (V_v - V_{t_0}) dv \\ &\quad + \frac{(u + t_0)^{-\alpha}}{\lambda\Gamma(1 - \alpha)} (V_0 - V_{t_0}), \end{aligned}$$

which is an \mathcal{F}_{t_0} -measurable function continuous on \mathbb{R}_+^* such that Conditions (4) and (5) (where the index 0 should be replaced by t_0) are satisfied.

Hence the class of generalized rough Heston models is stable with respect to conditioning. The conditional law of a rough Heston model is still that of a rough Heston model. The only difference is a modification in the mean-reversion level function. In particular, when considering the usual rough Heston model (1), the constant parameter θ becomes an \mathcal{F}_{t_0} measurable function when taking conditional law at time t_0 . This result will be crucial to derive hedging strategies in the rough Heston framework, and more generally to understand the state variables associated to rough Heston-type dynamics.

3. Characteristic function of generalized rough Heston models. The goal of this section is to derive the extended characteristic functions for the log-price in the rough Heston model (3). This together with Theorem 2.1 will enable us to derive conditional characteristic functions, leading to hedging strategies. The first step to achieve this goal is to build a suitable sequence of processes converging to the generalized rough Heston model of Definition 2.1. Then we will be able to do computations on these processes (notably deriving characteristic functions), and pass them to the limit to obtain results for generalized rough Heston models.

3.1. *Generalized rough Heston models as limit of nearly unstable Hawkes processes.* In [10], a microscopic price model, based on two-dimensional Hawkes processes, is built so that it converges in the long run after suitable rescaling to a rough Heston log-price (with constant mean-reversion). Then, characteristic functions are obtained from this result. Such method could easily be extended to obtain a generalized rough Heston model in the limit. However, it would only enable us to compute (2) with $a = 0$. This is not enough so that classical Fourier inversion methods such as that in [7] can be rigorously applied to compute prices and hedging portfolios.

Thus, we use another approach in this section, quite similar to that of [16]. We consider a sequence of one-dimensional Hawkes processes $(N_t^T)_{t \geq 0}$, indexed by $T > 0$ going to infinity, with intensity given by

$$\lambda_t^T = \mu_T + \int_0^t a_T \varphi(t - s) dN_s^T,$$

where μ_T and a_T are positive constants with $a_T < 1$ and $\varphi : \mathbb{R}_+ - \{0\} \mapsto \mathbb{R}_+$ is integrable such that $\int_0^\infty \varphi = 1$. In [16], it is shown that provided

$$(6) \quad x^\alpha \int_x^\infty \varphi(s) ds \xrightarrow{x \rightarrow \infty} \frac{1}{\Gamma(1 - \alpha)}, \quad \alpha \in (1/2, 1),$$

and

$$T^\alpha (1 - a_T) \xrightarrow{T \rightarrow \infty} \lambda, \quad T^{1-\alpha} \mu_T \xrightarrow{T \rightarrow \infty} \lambda/\nu^2,$$

for some positive constants λ and ν , a suitably rescaled version of the intensity process λ_t^T asymptotically behaves as the variance process of a rough Heston model with constant mean-reversion parameter such as (1) and with initial variance equal to zero. To obtain a time-dependent mean-reversion level and a nonzero starting value in the limit, we are inspired by an idea in [10], where it is shown that a time-dependent μ_T is a way to modify some parameters in the limit. More precisely, we consider the following assumption, where $f^{\alpha,1}$ denotes the Mittag-Leffler density function defined in Appendix A.1.

ASSUMPTION 3.1. There exist $\lambda, \nu > 0, \alpha \in (1/2, 1)$ and $V_0 \geq 0$ such that for $T > 1/\lambda^{-1/\alpha}$ and $t \geq 0$,

$$\lambda_t^T = \mu_T \zeta^T(t) + \int_0^t \varphi^T(t-s) dN_s^T,$$

where

$$a_T = 1 - \lambda T^{-\alpha}, \quad \mu_T = (\lambda/\nu^2)T^{\alpha-1}, \quad \varphi^T = a_T \varphi,$$

with $\varphi = f^{\alpha,1}$ and

$$\begin{aligned} \zeta^T(t) &= V_0 \left(\frac{1}{1-a_T} \left(1 - \int_0^t \varphi^T(t-s) ds \right) - \int_0^t \varphi^T(t-u) du \right) \\ &\quad + \int_0^t \varphi^T(t-u) \theta^0(u/T) du, \end{aligned}$$

where $\theta^0(\cdot)$ satisfies the assumptions of Definition 2.1.

Note that we are working in the so-called nearly unstable case for Hawkes processes since the L^1 norm of the kernel φ^T converges to one. Furthermore, remark that (6) is satisfied; see Appendix A.1.

REMARK 3.1. Remark that ζ^T can also be written as follows:

$$\begin{aligned} \zeta^T(t) &= \int_0^t \varphi^T(t-u) \theta^0(u/T) du \\ &\quad + V_0 \left(\frac{T^\alpha}{\lambda} \int_t^\infty \varphi(s) ds + \lambda T^{-\alpha} \int_0^t \varphi(s) ds \right). \end{aligned}$$

Therefore, using that $I^{1-\alpha} \varphi^T(t) = \int_t^\infty \varphi^T$ (see Appendix A.1), together with Condition (4) we get

$$\begin{aligned} \zeta^T(t) &\geq -\frac{V_0}{\lambda \Gamma(1-\alpha)} T^\alpha \int_0^t \varphi^T(t-u) u^{-\alpha} du \\ &\quad + V_0 \left(\frac{T^\alpha}{\lambda} \int_t^\infty \varphi(s) ds + \lambda T^{-\alpha} \int_0^t \varphi(s) ds \right) \\ &= -\frac{V_0}{\lambda} T^\alpha \int_t^\infty \varphi^T(s) ds + V_0 \left(\frac{T^\alpha}{\lambda} \int_t^\infty \varphi(s) ds + \lambda T^{-\alpha} \int_0^t \varphi(s) ds \right) \\ &= V_0 \left(\int_t^\infty \varphi(s) ds + \lambda T^{-\alpha} \int_0^t \varphi(s) ds \right). \end{aligned}$$

This shows that ζ^T is a nonnegative function, and thus that the intensity process λ_t^T is well defined.

We define $M_t^T = N_t^T - \int_0^t \lambda_s^T ds$ and

$$X_t^T = v^2 \frac{1 - a_T}{T^{\alpha\lambda}} N_{iT}^T, \quad \Lambda_t^T = v^2 \frac{1 - a_T}{T^{\alpha\lambda}} \int_0^{tT} \lambda_s^T ds,$$

$$Z_t^T = v \sqrt{\frac{1 - a_T}{T^{\alpha\lambda}}} M_{iT}^T.$$

Conditions (4) and (5) on the function θ^0 allow us to adapt the proofs in [10, 16] in a straightforward way to obtain the following result.

THEOREM 3.1. *Let $t_0 > 0$. As $T \rightarrow \infty$, under Assumption 3.1, the process $(\Lambda_t^T, X_t^T, Z_t^T)_{t \in [0, t_0]}$ converges in law for the Skorokhod topology to (Λ, X, Z) , where:*

- $\Lambda_t = X_t = \int_0^t V_s ds$.
- $Z_t = \int_0^t \sqrt{V_s} dB_s$ which is a continuous martingale.
- V is the unique weak solution of the rough stochastic differential equation

$$V_t = V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \lambda(\theta^0(s) - V_s) ds$$

$$+ \frac{v}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \sqrt{V_s} dB_s,$$

where B is a Brownian motion. Furthermore, the process V is nonnegative and has Hölder regularity $\alpha - 1/2 - \varepsilon$ for any $\varepsilon > 0$.

Theorem 3.1 will be one of the key results to obtain the extended characteristic function of the log-price in generalized rough Heston models.

3.2. Conditions for finite moments in generalized rough Heston models. Recall that we aim at computing (2) with $\Re(z) \neq 0$. A preliminary step toward this is to derive sufficient conditions for the finiteness of the moments of S_t and $\exp(\int_0^t V_s ds)$ in generalized rough Heston models. To obtain such result, we use Theorem 3.1. Let $a \in \mathbb{R}$. First, note that

$$(S_t)^a = (S_0)^a \exp\left(a\rho \int_0^t \sqrt{V_s} dB_s - \frac{a}{2} \int_0^t V_s ds + a\sqrt{1 - \rho^2} \int_0^t \sqrt{V_s} dB_s^\perp\right).$$

Consequently, we have

$$\mathbb{E}[(S_t)^a] = (S_0)^a \mathbb{E}\left[\exp\left(a\rho \int_0^t \sqrt{V_s} dB_s + \frac{1}{2}(-a + a^2(1 - \rho^2)) \int_0^t V_s ds\right)\right].$$

Now define

$$M_t = \exp\left(a\rho \int_0^t \sqrt{V_s} dB_s - \frac{a^2\rho^2}{2} \int_0^t V_s ds\right).$$

The process M_t is a positive local martingale and actually, by Proposition B.1 in Appendix B, a true martingale. Define the corresponding probability measure \mathbb{Q} :

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = M_t.$$

By the Girsanov theorem, under \mathbb{Q} ,

$$B_t^{\mathbb{Q}} = B_t - a\rho \int_0^t \sqrt{V_s} ds$$

is a \mathbb{F} -Brownian motion. Consequently, under \mathbb{Q} , V defined in (3) is still the variance process of a generalized rough Heston model, but with different parameters:

$$(7) \quad \begin{aligned} V_t &= V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \tilde{\lambda} (\tilde{\theta}^0(u) - V_u) du \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} v \sqrt{V_u} dB_u^{\mathbb{Q}}, \end{aligned}$$

where

$$\tilde{\lambda} = \lambda - \rho v a, \quad \tilde{\theta}^0(t) = \frac{\lambda \theta^0(t)}{\lambda - \rho v a},$$

provided that $\lambda - \rho v a > 0$. Hence we obtain

$$(8) \quad \mathbb{E}[(S_t)^a] = (S_0)^a \mathbb{E}_{\mathbb{Q}} \left[\exp \left(\frac{1}{2} (-a + a^2) \int_0^t V_s ds \right) \right].$$

Therefore, a sufficient condition on a for

$$(9) \quad \mathbb{E}_{\mathbb{Q}} \left[\exp \left(\frac{1}{2} (-a + a^2) \int_0^t V_s ds \right) \right] < \infty$$

will readily imply a sufficient condition for the finiteness of $\mathbb{E}[(S_t)^a]$.

We now explain how to derive such condition. Recall that from Theorem 3.1, $v^2 T^{-2\alpha} N_{iT}^T$ converges in law to $\int_0^t V_s ds$. Thus, we look first for a condition on $a \in \mathbb{R}$ for which

$$(10) \quad \mathbb{E}[\exp(av^2 T^{-2\alpha} N_{iT}^T)] < \infty,$$

for large enough $T > 0$ and fixed $t > 0$. This is done using a population interpretation of Hawkes processes; see Appendix C.1. It leads us to a sufficient condition on $a \in \mathbb{R}$ for (9). Furthermore, we are able to compute explicitly the expectation in (10); see Appendix C.1. Thus, we can pass to the limit as T goes to infinity and then obtain an explicit expression for the expectation in (9). More precisely, we have the following result whose proof is given in Section 5.2, where $a_0(t)$ is defined for $t > 0$ by

$$a_0(t) = \frac{1}{2v^2} \left(\lambda + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \right)^2.$$

THEOREM 3.2. *Let V be the variance process of the generalized rough Heston model (3). For any $t > 0$ and $a < a_0(t)$,*

$$\mathbb{E}\left[\exp\left(a \int_0^t V_s ds\right)\right] < \infty$$

and

$$\mathbb{E}\left[\exp\left(a \int_0^t V_s ds\right)\right] = \exp\left(\int_0^t g(a, t-s)\left(\lambda\theta^0(s) + \frac{V_0s^{-\alpha}}{\Gamma(1-\alpha)}\right) ds\right),$$

where $g(a, \cdot)$ is the unique continuous solution of the following fractional Riccati equation:

$$D^\alpha g(a, s) = a - \lambda g(a, s) + \frac{v^2}{2} g(a, s)^2, \quad s \leq t, \quad I^{1-\alpha} g(a, 0) = 0.$$

For any $0 \leq s \leq t$, this function satisfies

$$g(a, s) \leq \frac{c}{v^2} \left(\lambda + \frac{s^{-\alpha}}{\Gamma(1-\alpha)} + v\sqrt{a_0(s) - a} \right)$$

for some constant $c > 0$. Furthermore, for fixed $0 \leq s \leq t$, $a \mapsto g(a, s)$ is nondecreasing and $s \mapsto g(a, s)$ is nonincreasing on $[0, t]$ if $a < 0$ and nondecreasing if $a > 0$.

Let S_t denote the price in the generalized rough Heston model of Definition 2.1. Using (8), we obtain the following corollary on the moments of S_t .

COROLLARY 3.1. *Let $t > 0$. Assume*

$$\lambda - \rho va > 0, \quad a_-(t) < a < a_+(t),$$

where

$$a_-(t) = \frac{v^2 - 2\rho v X(t) + \sqrt{\Delta(t)}}{2v^2(1 - \rho^2)}, \quad a_+(t) = \frac{v^2 - 2\rho v X(t) - \sqrt{\Delta(t)}}{2v^2(1 - \rho^2)},$$

with

$$X(t) = \lambda + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad \Delta(t) = 4v^2 X(t)^2 + v^4 - 4\rho v^3 X(t).$$

Then we have

$$\mathbb{E}[(S_t)^a] < \infty.$$

Furthermore,

$$\mathbb{E}[(S_t)^a] = (S_0)^a \exp\left(\int_0^t h(a, t-s)\left(\lambda\theta^0(s) + \frac{V_0s^{-\alpha}}{\Gamma(1-\alpha)}\right) ds\right),$$

where $h(a, \cdot)$ is the unique continuous solution of the following fractional Riccati equation:

$$D^\alpha h(a, s) = \frac{a^2 - a}{2} - (\lambda - \rho va)h(a, s) + \frac{v^2}{2}h(a, s)^2, \quad s \leq t,$$

$$I^{1-\alpha}h(a, 0) = 0.$$

REMARK 3.1. Note that if we formally take $\alpha = 1$ in Corollary 3.1, our model coincides with the classical Heston model. In that case, $X(t) = \lambda$ and, therefore, a_- and a_+ do not depend on t . Moreover, the set of $a \in \mathbb{R}$ such that

$$\lambda - \rho va > 0, \quad a_- \leq a \leq a_+,$$

exactly corresponds to that of $a \in \mathbb{R}$ for which

$$\forall t \geq 0, \quad \mathbb{E}[(S_t)^a] < \infty;$$

see [2] for further details on moment explosions for the classical Heston model.

PROOF OF COROLLARY 3.1. Recall that from (8),

$$\mathbb{E}[(S_t)^a] = (S_0)^a \mathbb{E}_{\mathbb{Q}} \left[\exp \left(\frac{1}{2}(-a + a^2) \int_0^t V_s ds \right) \right].$$

From Theorem 3.2 and the fact that under \mathbb{Q} , V follows (7), this quantity is finite if $\lambda - \rho va > 0$ and

$$\frac{1}{2}(-a + a^2) < \tilde{a}_0(t) = \frac{1}{2v^2} \left(\tilde{\lambda} + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \right)^2 = \frac{1}{2v^2} \left(\lambda - \rho va + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \right)^2.$$

This is equivalent to

$$a^2 v^2 (1 - \rho^2) + a(-v^2 + 2X(t)\rho v) - X(t)^2 < 0.$$

The conditions on $a \in \mathbb{R}$ stated in Corollary 3.1 follow. Finally, the expression of $\mathbb{E}[(S_t)^a]$ is easily obtained using (8) together with Theorem 3.2. \square

3.3. *Characteristic functions of generalized rough Heston models.* We are now ready to derive the characteristic functions of generalized rough Heston models. Let $t > 0$. We want to compute

$$R(z, t) = \mathbb{E}[\exp(z \log(S_t/S_0))],$$

where $z \in \mathbb{C}$ satisfies

$$(11) \quad z = a + ib, \quad a, b \in \mathbb{R}, \quad \lambda - \rho va > 0, \quad a_-(t) < a < a_+(t),$$

where $a_-(t)$ and $a_+(t)$ are defined in Corollary 3.1. Recall that from Corollary 3.1, (11) implies that $\exp(z \log(S_t/S_0))$ is integrable and, therefore, $R(z, t)$ is well defined.

Using the same computations as in the preceding sections, we get

$$(12) \quad R(z, t) = \mathbb{E}_{\mathbb{Q}} \left[\exp \left(ib\rho \int_0^t \sqrt{V_s} dB_s^{\mathbb{Q}} + \frac{1}{2}(\rho^2 b^2 + z^2 - z) \int_0^t V_s ds \right) \right].$$

As already seen, under \mathbb{Q} , V still follows the variance process of a generalized rough Heston model driven by the Brownian motion $B^{\mathbb{Q}}$; see (7). Thus, we need to study

$$G(z, x, t) = \mathbb{E} \left[\exp \left(ix \int_0^t \sqrt{V_s} dB_s + z \int_0^t V_s ds \right) \right],$$

with $x \in \mathbb{R}$, $z \in \mathbb{C}$ such that $\Re(z) < a_0(t)$, [$a_0(t)$ is defined in Theorem 3.2], and V is the variance process of a generalized rough Heston model. To do so, we use again Theorem 3.1. Indeed, $(v^2 T^{-2\alpha} N_{tT}^T, v T^{-\alpha} M_{tT}^T)$ converges in law as T goes to infinity to $(\int_0^t V_s ds, \int_0^t \sqrt{V_s} dB_s)$. Computing

$$\mathbb{E}[\exp(ixvT^{-\alpha} M_{tT}^T + zv^2 T^{-2\alpha} N_{tT}^T)]$$

and passing to the limit, we obtain the following result whose proof is given in Section 5.3.

THEOREM 3.3. *Let V be the variance process of the generalized rough Heston model (3). For any $t > 0$, $b \in \mathbb{R}$ and $z \in \mathbb{C}$ such that $\Re(z) < a_0(t)$,*

$$G(z, x, t) = \exp \left(\int_0^t \xi(z, x, t-s) \left(\lambda \theta^0(s) + \frac{V_0 s^{-\alpha}}{\Gamma(1-\alpha)} \right) ds \right),$$

where $\xi(z, x, \cdot)$ is the unique continuous solution of the following fractional Riccati equation:

$$D^\alpha \xi(z, x, s) = z - \frac{x^2}{2} + (ixv - \lambda)\xi(z, x, s) + \frac{v^2}{2}\xi(z, x, s)^2, \quad s \leq t,$$

$$I^{1-\alpha} \xi(z, x, 0) = 0.$$

The following corollary is readily obtained from Theorem 3.3 together with (12).

COROLLARY 3.2. *Let $t > 0$ and $z \in \mathbb{C}$ satisfying (11). We have*

$$R(z, t) = \exp \left(\int_0^t h(z, t-s) \left(\lambda \theta^0(s) + \frac{V_0 s^{-\alpha}}{\Gamma(1-\alpha)} \right) ds \right),$$

where $h(z, \cdot)$ is the unique continuous solution of the following fractional Riccati equation:

$$D^\alpha h(z, s) = \frac{1}{2}(z^2 - z) + (z\rho v - \lambda)h(z, s) + \frac{v^2}{2}h(z, s)^2, \quad s \leq t,$$

$$I^{1-\alpha} h(z, 0) = 0.$$

3.4. *Connection with the forward variance curve.* We now show how the characteristic function given in Corollary 3.2 can be written as a functional of the forward variance curve $(\mathbb{E}[V_t])_{t \geq 0}$. This property will be crucial in the next section when computing hedging portfolios. We first remark that the time-dependent parameter θ^0 can be directly linked to the forward variance curve through the following result.

PROPOSITION 3.1. *Let V be the variance process of the generalized rough Heston model (3). For any $t \geq 0$, we have*

$$(13) \quad \mathbb{E}[V_t] = V_0(1 - F^{\alpha,\lambda}(t)) + \int_0^t f^{\alpha,\lambda}(t-s)\theta^0(s) ds,$$

where $F^{\alpha,\lambda}$ and $f^{\alpha,\lambda}$ are defined in the Appendix A.1. Furthermore, θ^0 can be written as a functional of the forward variance curve as follows:

$$(14) \quad \lambda\theta^0(t) + V_0 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = D^\alpha \mathbb{E}[V_t] + \lambda \mathbb{E}[V_t], \quad t > 0.$$

PROOF. In the same way as in [16], we can show that for any $t \geq 0$,

$$\mathbb{E} \left[\int_0^t V_s ds \right] < \infty.$$

So we have that $t \mapsto \mathbb{E}[V_t]$ is locally integrable. Moreover, $f^{\alpha,\lambda}$ is square-integrable; see Appendix A.1. Thus, we obtain that for any $t \geq 0$,

$$\int_0^t f^{\alpha,\lambda}(t-s)^2 \mathbb{E}[V_s] ds < \infty.$$

Therefore,

$$\mathbb{E} \left[\int_0^t f^{\alpha,\lambda}(t-s) \sqrt{V_s} dB_s \right] = 0.$$

Writing the dynamic of V under the following form as in [16]:

$$(15) \quad V_t = V_0(1 - F^{\alpha,\lambda}(t)) + \int_0^t f^{\alpha,\lambda}(t-s)\theta^0(s) ds + \frac{\nu}{\lambda} \int_0^t f^{\alpha,\lambda}(t-s) \sqrt{V_s} dB_s,$$

we deduce (13). Now using the Fubini theorem and noting that $I^{1-\alpha} f^{\alpha,\lambda} = \lambda(1 - F^{\alpha,\lambda})$ (see Appendix A.1), we get that for any $t \geq 0$,

$$I^{1-\alpha} \mathbb{E}[V_t] = V_0 \frac{t^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} + \int_0^t \lambda(1 - F^{\alpha,\lambda}(t-s))(\theta^0(s) - V_0) ds.$$

Using Fubini again, this can be rewritten

$$\begin{aligned} I^{1-\alpha} \mathbb{E}[V_t] &= V_0 \frac{t^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} + \int_0^t \lambda(\theta^0(s) - V_0) ds \\ &\quad - \lambda \int_0^t \int_0^s f^{\alpha,\lambda}(s-u)(\theta^0(u) - V_0) du ds. \end{aligned}$$

Then from (13) we derive

$$I^{1-\alpha}\mathbb{E}[V_t] = V_0 \frac{t^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} + \int_0^t \lambda(\theta^0(s) - V_0) ds - \lambda \int_0^t (\mathbb{E}[V_s] - V_0) ds.$$

We finally obtain (14) by differentiating this last equality. \square

REMARK 3.2. Assume that the forward variance curve $t \mapsto \mathbb{E}[V_t]$ is observed on the market through the implied volatility surface or liquid variance swaps, and that this curve admits a fractional derivative of order α . Then the mean-reversion function θ^0 can be chosen so that the model is consistent with this market forward variance curve by taking

$$\lambda\theta^0(t) = D^\alpha(\mathbb{E}[V_\cdot] - V_0)(t) + \lambda\mathbb{E}[V_t].$$

From Corollary 3.2 together with Proposition 3.1, we can eventually write the characteristic function of the log-price as a functional of the forward variance curve. Thus, it indicates that the forward variance curve is a relevant state variable in generalized rough Heston models. Such type of phenomena also appears in the class of models developed in [6]. More precisely, we have the following corollary.

COROLLARY 3.3. Let $t > 0$ and $z \in \mathbb{C}$ satisfying (11). We have

$$R(z, t) = \exp\left(\int_0^t \chi(z, t-s)\mathbb{E}[V_s] ds\right),$$

where

$$\chi(z, t) = \frac{1}{2}(z^2 - z) + z\rho\nu h(z, t) + \frac{\nu^2}{2}h(z, t)^2,$$

with $h(z, \cdot)$ the unique continuous solution of the fractional Riccati equation given in Corollary 3.2.

Thus, characteristic functions and, therefore, conditional characteristic functions of the log-price can be written in term of the forward variance curve. This shows that this object plays the role of state variable in this infinite-dimensional fractional setting. Actually, this result could probably be understood in a more general framework of affine processes; see [1, 8].

PROOF OF COROLLARY 3.3. By Lemma A.2 in Appendix A, for any $0 \leq s \leq t$,

$$(16) \quad h(z, s) = \int_0^s \frac{1}{\lambda} f^{\alpha, \lambda}(s-u)\chi(z, u) du.$$

Moreover, from (13) together with the fact that $I^{1-\alpha} f^{\alpha,\lambda} = \lambda(1 - F^{\alpha,\lambda})$ (see Appendix A.1), we have

$$\mathbb{E}[V_s] = \int_0^s \frac{1}{\lambda} f^{\alpha,\lambda}(s - u) \left(\lambda \theta^0(u) + V_0 \frac{u^{-\alpha}}{\Gamma(1 - \alpha)} \right) du.$$

Then, using the Fubini theorem, we obtain

$$\begin{aligned} \int_0^t \chi(z, t - s) \mathbb{E}[V_s] ds &= \int_0^t \left(\int_0^{t-s} \frac{1}{\lambda} f^{\alpha,\lambda}(t - s - u) \chi(z, u) du \right) \\ &\quad \times \left(\lambda \theta^0(s) + V_0 \frac{s^{-\alpha}}{\Gamma(1 - \alpha)} \right) ds \end{aligned}$$

and, therefore,

$$\int_0^t \chi(z, t - s) \mathbb{E}[V_s] ds = \int_0^t h(z, t - s) \left(\lambda \theta^0(s) + V_0 \frac{s^{-\alpha}}{\Gamma(1 - \alpha)} \right) ds.$$

The result follows from Corollary 3.2. \square

4. Hedging under generalized rough Heston models. We consider a generalized rough Heston model with the additional assumption that $\rho \leq 0$. We show in this section how to compute explicitly hedging portfolios for vanilla options in such model. We treat here the case of a European call option with maturity $T > 0$ and strike $K > 0$. Nevertheless, the approach can be easily extended to other vanilla payoffs.

It is easy to see that we can find $a > 1$ such that the conditions of Corollary 3.1 are satisfied for any $t \geq 0$. Therefore, for any $t \geq 0$,

$$\mathbb{E}[(S_t)^a] < \infty.$$

We define the call option price process

$$C_t = \mathbb{E}[(S_T - K)_+ | \mathcal{F}_t], \quad 0 \leq t \leq T.$$

We write

$$X_t = \log(S_t), \quad t \geq 0$$

and

$$g(x) = e^{-ax} (e^x - K)_+, \quad x \in \mathbb{R}.$$

We have $g \in \mathbb{L}^1(\mathbb{R}) \cap \mathbb{L}^2(\mathbb{R})$ and, therefore,

$$g(x) = \frac{1}{2\pi} \int_{b \in \mathbb{R}} \hat{g}(-b) e^{ibx} db,$$

where $\hat{g} \in \mathbb{L}^1(\mathbb{R}) \cap \mathbb{L}^2(\mathbb{R})$ is the Fourier transform of g . Note that we are able to compute explicitly \hat{g} :

$$\hat{g}(b) = \frac{e^{(1-a+ib)\log(K)}}{(ib-a)(ib-a+1)}, \quad b \in \mathbb{R}.$$

We then deduce by Fubini theorem that

$$(17) \quad C_t = \mathbb{E}[g(X_T)e^{aX_T} | \mathcal{F}_t] = \frac{1}{2\pi} \int_{b \in \mathbb{R}} \hat{g}(-b) P_t^T(a+ib) db,$$

where

$$P_t^T(a+ib) = \mathbb{E}[\exp((a+ib)X_T) | \mathcal{F}_t].$$

Using the fact that conditional on \mathcal{F}_t , S still follows a generalized rough Heston dynamic together with Corollary 3.3, we obtain

$$\begin{aligned} & \mathbb{E}[\exp((a+ib)\log(S_T/S_t) | \mathcal{F}_t)] \\ &= \exp\left(\int_0^{T-t} \chi(a+ib, T-t-s) \mathbb{E}[V_{s+t} | \mathcal{F}_t] ds\right), \end{aligned}$$

where χ is defined in Corollary 3.3. Thus,

$$(18) \quad \begin{aligned} P_t^T(a+ib) &= \exp\left((a+ib)\log(S_t)\right) \\ &+ \int_0^{T-t} \chi(a+ib, T-t-s) \mathbb{E}[V_{s+t} | \mathcal{F}_t] ds. \end{aligned}$$

Hence, from (18), we deduce that $P_t^T(a+ib)$ is a deterministic functional of the underlying spot price S_t and the forward variance curve until maturity T : $\mathbb{E}[V_{t+u} | \mathcal{F}_t]$, $0 \leq u \leq T-t$.

Let

$$\begin{aligned} \mathcal{V}_{\alpha,\lambda} &= \left\{ \xi : \mathbb{R}_+ \mapsto \mathbb{R}_+, \xi(t) = \int_0^t \frac{s^{-\alpha}}{\lambda \Gamma(1-\alpha)} f^{\alpha,\lambda}(t-s) \theta_\xi(s) ds, \right. \\ & \left. \theta_\xi \text{ is continuous on } \mathbb{R}_+ \right\}. \end{aligned}$$

The space $\mathcal{V}_{\alpha,\lambda}$ is a metric space containing

$$\begin{aligned} \mathcal{V}_{\alpha,\lambda}^+ &= \{ \xi \in \mathcal{V}_{\alpha,\lambda}, \theta_\xi > 0 \text{ and for any } t > 0, \theta_\xi(t) = \xi(0) + t^\alpha \lambda \Gamma(1-\alpha) \theta_\xi^0(t), \\ & \theta_\xi^0 \text{ satisfies (5)} \}, \end{aligned}$$

which is the set of all possible forward variance curves produced by generalized rough Heston models. Note that from the same computations as for Proposition 3.1, we get the uniqueness of the function θ_ξ for each $\xi \in \mathcal{V}_{\alpha,\lambda}$ since we have

$$\theta_\xi(t) = (D^\alpha \xi(t) + \lambda \xi(t)) \Gamma(1-\alpha) t^\alpha, \quad t > 0.$$

We equip $\mathcal{V}_{\alpha,\lambda}$ with the following complete metric:

$$d_{\alpha,\lambda}(\xi, \zeta) = \|\theta_\xi - \theta_\zeta\| \wedge 1\|_\infty.$$

From (17) and (18), we get that the spot price and the forward variance curve are the relevant state variables for the call price process. Indeed, there exists a deterministic functional $C : \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathcal{V}_{\alpha,\lambda} \mapsto \mathbb{R}$ such that

$$C_t = C(T - t, S_t, (\mathbb{E}[V_{s+t}|\mathcal{F}_t])_{s \geq 0}), \quad t \in [0, T],$$

where for any $t \geq 0, S \in \mathbb{R}_+$ and $\xi \in \mathcal{V}_{\alpha,\lambda}$

$$(19) \quad C(t, S, \xi) = \frac{1}{2\pi} \int_{b \in \mathbb{R}} \hat{g}(-b)L(a + ib, t, S, \xi) db,$$

with

$$L(a + ib, t, S, \xi) = \exp\left((a + ib) \log(S) + \int_0^t \chi(a + ib, t - s)\xi(s) ds \right).$$

In the following proposition, proved in Section 5.4, we give some useful regularity properties of the functional C .

PROPOSITION 4.1. *Let $\xi \in \mathcal{V}_{\alpha,\lambda}^+, S > 0, t > 0$ and assume $|\rho| < 1$. The function $C(t, \cdot, \xi)$ defined in (19) is differentiable in S and its derivative is given by*

$$\partial_S C(t, S, \xi) = \frac{1}{2\pi} \int_{b \in \mathbb{R}} \frac{a + ib}{S} \hat{g}(-b)L(a + ib, t, S, \xi) db.$$

Moreover, the function $C(t, S, \cdot)$ is differentiable in the sense of Fréchet in ξ , with derivative such that for any $\zeta \in \mathcal{V}_{\alpha,\lambda}$,

$$\begin{aligned} & \partial_V C(t, S, \xi) \cdot \zeta \\ &= \int_0^t \left(\frac{1}{2\pi} \int_{b \in \mathbb{R}} \hat{g}(-b)L(a + ib, t, S, \xi) \chi(a + ib, t - s) db \right) \zeta(s) ds. \end{aligned}$$

We end this section by stating our result showing how one can build a hedging portfolio by trading the underlying and the forward variance curve.

THEOREM 4.1. *For any time $t \in [0, T]$, we have*

$$\begin{aligned} C_t &= C_0 + \int_0^t \partial_S C(T - u, S_u, \mathbb{E}[V_{\cdot+u}|\mathcal{F}_u]) dS_u \\ &+ \int_0^t \partial_V C(T - u, S_u, \mathbb{E}[V_{\cdot+u}|\mathcal{F}_u]) \cdot (d\mathbb{E}[V_{\cdot+u}|\mathcal{F}_u]), \end{aligned}$$

where

$$\partial_V C(T - u, S_u, \mathbb{E}[V_{\cdot+u}|\mathcal{F}_u]) \cdot (d\mathbb{E}[V_{\cdot+u}|\mathcal{F}_u])$$

denotes

$$\int_0^{T-u} \left(\frac{1}{2\pi} \int_{b \in \mathbb{R}} \hat{g}(-b)L(a + ib, T - u, S_u, \mathbb{E}[V_{\cdot+u}|\mathcal{F}_u]) \times \chi(a + ib, T - u - s) db \right) d\mathbb{E}[V_{s+u}|\mathcal{F}_u] ds,$$

with $d\mathbb{E}[V_x|\mathcal{F}_u]$ the Itô differential at time u of the martingale $M_u = \mathbb{E}[V_x|\mathcal{F}_u]$, $u \leq x$.

REMARK 4.1. Using (15), we show that

$$d\mathbb{E}[V_{s+u}|\mathcal{F}_u] = \frac{1}{\lambda} f^{\alpha,\lambda}(s) v \sqrt{V_u} dB_u.$$

The proof of Theorem 4.1 is given in Section 5.5. This result shows that in an idealistic setting where the underlying asset and the forward variance curve can be traded (in continuous time), perfect replication can be obtained in generalized rough Heston models. Of course, in practice, this strategy will be discretized and one will use liquid variance swaps or European options instead of the forward variance curve.

REMARK 4.2. It is interesting to remark that the price function $C(t, S, \xi)$ is solution of a Feynman–Kac-type partial differential equation. Let us define the following derivative according to time $t > 0$:

$$\partial_t C(t, S, \xi) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} (C(t - \varepsilon, S, \xi_{\varepsilon+}) - C(t, S, \xi)).$$

We easily have that $L(a + ib, t, S, \xi)$ is solution of the following infinite-dimensional PDE:

$$0 = \partial_t L + \frac{1}{2} (S\sqrt{\xi_0})^2 \partial_S^2 L + \frac{1}{2} (v\sqrt{\xi_0}) \partial_V^2 L \cdot \left(\frac{1}{\lambda} f^{\alpha,\lambda}, \frac{1}{\lambda} f^{\alpha,\lambda} \right) + \rho(S\sqrt{\xi_0})(v\sqrt{\xi_0}) \partial_{S,V}^2 L \cdot \left(\frac{1}{\lambda} f^{\alpha,\lambda} \right),$$

with the initial condition $L(a + ib, 0, S, \xi) = S^{a+ib}$.

As in Proposition 4.1, we can show that C is twice differentiable in S and in V (in the sense of Fréchet for V), and that $\partial_t C$ is well defined. So we can deduce that C satisfies the same PDE:

$$0 = \partial_t C + \frac{1}{2} (S\sqrt{\xi_0})^2 \partial_S^2 C + \frac{1}{2} (v\sqrt{\xi_0}) \partial_V^2 C \cdot \left(\frac{1}{\lambda} f^{\alpha,\lambda}, \frac{1}{\lambda} f^{\alpha,\lambda} \right) + \rho(S\sqrt{\xi_0})(v\sqrt{\xi_0}) \partial_{S,V}^2 C \cdot \left(\frac{1}{\lambda} f^{\alpha,\lambda} \right),$$

with the initial condition $C(0, S, \xi) = (S - K)_+$.

Note that $\partial_V^2 C \cdot (\frac{1}{\lambda} f^{\alpha,\lambda}, \frac{1}{\lambda} f^{\alpha,\lambda})$ [resp. $\partial_{S,V}^2 C \cdot (\frac{1}{\lambda} f^{\alpha,\lambda})$] is the second Fréchet derivative of C (resp., the first Fréchet derivative of $\partial_S C$) applied on $(\frac{1}{\lambda} f^{\alpha,\lambda}, \frac{1}{\lambda} f^{\alpha,\lambda})$ (resp., $\frac{1}{\lambda} f^{\alpha,\lambda}$) which is well defined even though $\frac{1}{\lambda} f^{\alpha,\lambda}$ does not belong to the metric space $\mathcal{V}_{\alpha,\lambda}$.

5. Proofs. The notion of fractional integrals and derivatives are heavily used in the proofs. Notation, definitions and useful results related to them are given in Appendix A.

5.1. Proof of Theorem 2.1.

Finding the dynamic of $V_t^{t_0}$ conditional on \mathcal{F}_{t_0} . Using the stochastic Fubini theorem, we can show that $I^{1-\alpha} V$ is a semimartingale and for $t > 0$,

$$(I^{1-\alpha} V)_t = V_0 \int_0^t \frac{s^{-\alpha}}{\Gamma(1-\alpha)} ds + \int_0^t \lambda(\theta^0(s) - V_s) ds + \int_0^t v\sqrt{V_s} dB_s.$$

Therefore,

$$\frac{1}{\Gamma(1-\alpha)} \int_0^{t+t_0} (t+t_0-u)^{-\alpha} V_u du$$

is equal to

$$\begin{aligned} &\frac{1}{\Gamma(1-\alpha)} \int_0^{t_0} (t_0-u)^{-\alpha} V_u du + V_0 \int_{t_0}^{t+t_0} \frac{1}{\Gamma(1-\alpha)} u^{-\alpha} du \\ &+ \int_{t_0}^{t+t_0} \lambda(\theta^0(u) - V_u) du + \int_{t_0}^{t+t_0} v\sqrt{V_u} dB_u. \end{aligned}$$

Using a change of variable, this can be written

$$\begin{aligned} &\frac{1}{\Gamma(1-\alpha)} \int_0^{t_0} (t_0-u)^{-\alpha} V_u du + V_0 \int_0^t \frac{1}{\Gamma(1-\alpha)} (t_0+u)^{-\alpha} du \\ &+ \int_0^t \lambda(\theta^0(u+t_0) - V_u^{t_0}) du + \int_0^t v\sqrt{V_u^{t_0}} dB_u^{t_0}, \end{aligned}$$

where $(B_t^{t_0})_{t \geq 0} = (B_{t+t_0} - B_{t_0})_{t \geq 0}$ is a Brownian motion independent of \mathcal{F}_{t_0} . Moreover, remarking that

$$I^{1-\alpha} V_t^{t_0} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-u)^{-\alpha} V_u^{t_0} du = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t+t_0} (t+t_0-u)^{-\alpha} V_u du$$

is equal to

$$\frac{1}{\Gamma(1-\alpha)} \int_0^{t+t_0} (t+t_0-u)^{-\alpha} V_u du - \frac{1}{\Gamma(1-\alpha)} \int_0^{t_0} (t+t_0-u)^{-\alpha} V_u du$$

and that

$$\frac{1}{\Gamma(1-\alpha)}((t_0-u)^{-\alpha} - (t+t_0-u)^{-\alpha}) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t (t_0-u+v)^{-1-\alpha} dv,$$

we derive

$$\begin{aligned} I^{1-\alpha} V_t^{t_0} &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{t_0} \int_0^t (t_0-u+v)^{-1-\alpha} dv V_u du \\ &\quad + \int_0^t \frac{1}{\Gamma(1-\alpha)} (t_0+u)^{-\alpha} du V_0 \\ &\quad + \int_0^t \lambda(\theta^0(u+t_0) - V_u^{t_0}) du + \int_0^t v \sqrt{V_u^{t_0}} dB_u^{t_0}. \end{aligned}$$

This can be written as follows:

$$(20) \quad V_{t_0} \frac{t^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} + \int_0^t \lambda(\theta^{t_0}(u) - V_u^{t_0}) du + \int_0^t v \sqrt{V_u^{t_0}} dB_u^{t_0},$$

with $(\theta^{t_0}(u))_{u \geq 0}$ a function which is \mathcal{F}_{t_0} measurable and defined by

$$\begin{aligned} \theta^{t_0}(u) &= \theta^0(t_0+u) + \frac{\alpha}{\lambda\Gamma(1-\alpha)} \int_0^{t_0} (t_0-v+u)^{-1-\alpha} (V_v - V_{t_0}) dv \\ &\quad + \frac{(u+t_0)^{-\alpha}}{\lambda\Gamma(1-\alpha)} (V_0 - V_{t_0}). \end{aligned}$$

Properties of θ^{t_0} . It is clear that θ^{t_0} is continuous on \mathbb{R}_+^* . Moreover it is easy to see that for any $u > 0$:

$$\begin{aligned} \theta^{t_0}(u) &= \theta^0(t_0+u) + \frac{\alpha}{\lambda\Gamma(1-\alpha)} \int_0^{t_0} (t_0-v+u)^{-1-\alpha} V_v dv \\ &\quad + \frac{1}{\lambda\Gamma(1-\alpha)} (V_0(u+t_0)^{-\alpha} - V_{t_0}u^{-\alpha}). \end{aligned}$$

Since V is a nonnegative process and θ^0 satisfies (4), we obtain that θ^{t_0} also satisfies (4). Finally, for fixed $\varepsilon > 0$, V being $\alpha - 1/2 - \varepsilon$ Hölder continuous, there exists for almost each $\omega \in \Omega$ a positive constant $c_\varepsilon(\omega)$ such that for any $x, y \in [0, t_0]$:

$$|V_x - V_y| \leq c_\varepsilon(\omega) |x - y|^{\alpha-1/2-\varepsilon}.$$

Thus, by integration by parts, we obtain for any $u \in (0, t_0]$

$$\begin{aligned} &\left| \int_0^{t_0} (t_0-v+u)^{-1-\alpha} (V_v - V_{t_0}) dv \right| \\ &\leq c_\varepsilon(\omega) \int_0^{t_0} (t_0-v+u)^{-1-\alpha} (t_0-v)^{\alpha-1/2-\varepsilon} dv \end{aligned}$$

$$\begin{aligned}
 &= c_\varepsilon(\omega)u^{-1/2-\varepsilon} \int_0^{t_0/u} (x+1)^{-1-\alpha} x^{\alpha-1/2-\varepsilon} dx \\
 &\leq c_\varepsilon(\omega)u^{-1/2-\varepsilon} \int_0^\infty (x+1)^{-1-\alpha} x^{\alpha-1/2-\varepsilon} dx.
 \end{aligned}$$

Thus, θ^{t_0} satisfies Condition (5) almost surely.

End of the proof. We complete the proof noting that from (20) and the stochastic Fubini theorem we have that

$$\int_0^t V_s^{t_0} ds = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} I^{1-\alpha} V_s^{t_0} ds$$

is equal to

$$\begin{aligned}
 &V_{t_0}t + \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^s (s-u)^{\alpha-1} \lambda(\theta^{t_0}(u) - V_u^{t_0}) du ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^s (s-u)^{\alpha-1} \nu \sqrt{V_u^{t_0}} dB_u^{t_0} ds.
 \end{aligned}$$

Hence by differentiating the previous equality, we conclude that the dynamic of (S^{t_0}, V^{t_0}) is given by

$$\begin{aligned}
 S_t^{t_0} &= S_{t_0} \exp\left(\int_0^t \sqrt{V_u^{t_0}} dW_u^{t_0} - \frac{1}{2} \int_0^t V_u^{t_0} du\right), \\
 V_t^{t_0} &= V_{t_0} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \lambda(\theta^{t_0}(u) - V_u^{t_0}) du \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \nu \sqrt{V_u^{t_0}} dB_u^{t_0},
 \end{aligned}$$

where $(W_t^{t_0})_{t \geq 0} = (W_{t+t_0} - W_{t_0})_{t \geq 0}$ is a Brownian motion independent of \mathcal{F}_{t_0} and with correlation ρ with B^{t_0} .

5.2. *Proof of Theorem 3.2.* We work here with the sequence of Hawkes processes N^T defined in Section 3.1 such that the intensities satisfy Assumption 3.1. Recall that for $t \geq 0$, from Theorem 3.1, $\nu^2 T^{-2\alpha} N_{tT}^T$, converges in law as T goes to infinity to

$$\int_0^t V_s ds,$$

where V is solution of the fractional stochastic differential equation (3). A key step for the proof of Theorem 3.2 is to show that for suitable $a \in \mathbb{R}$,

$$(21) \quad \mathbb{E}[\exp(a\nu^2 T^{-2\alpha} N_{tT}^T)] \xrightarrow{T \rightarrow \infty} \mathbb{E}\left[\exp\left(a \int_0^t V_s ds\right)\right].$$

Applying (31) in Appendix C.1 on the Hawkes process N^T , we write

$$\mathbb{E}[\exp(av^2T^{-2\alpha}N_{iT}^T)] = \exp\left(\int_0^t \lambda \zeta^T(T(t-s))g^T(a,s) ds\right),$$

with

$$g^T(a,t) = v^{-2}T^\alpha(\exp(av^2T^{-2\alpha})\mathbb{E}[\exp(av^2T^{-2\alpha}N_{iT}^{f,T})] - 1),$$

where $N^{f,T}$ is the Hawkes process of children cluster (with migrant rate φ^T and kernel φ^T); see Appendix C.1 for details. Moreover, from Lemma A.5, $\lambda \zeta^T(Ts)$ converges pointwise as T goes to infinity to

$$\lambda \theta^0(s) + \frac{V_0 s^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0 < s \leq t.$$

Therefore, it is left to study the convergence of the function g^T .

Uniform boundedness of g^T . From now on, c denotes a positive constant that may vary from line to line.

From (30) in the Appendix C, for each $t > 0$,

$$(22) \quad g^T(a,t) < \infty$$

provided

$$av^2T^{-2\alpha} \leq \int_0^{iT} \varphi^T - 1 - \log\left(\int_0^{iT} \varphi^T\right).$$

Moreover note that

$$T^{2\alpha} \left(\int_0^{iT} \varphi^T - 1 - \log\left(\int_0^{iT} \varphi^T\right)\right) \underset{T \rightarrow \infty}{\sim} \frac{T^{2\alpha}}{2} \left(1 - \int_0^{iT} \varphi^T\right)^2.$$

Using $\varphi^T = (1 - \lambda T^{-\alpha})f^{\alpha,1}$ and $\int_0^\infty f^{\alpha,1} = 1$, we obtain

$$T^{2\alpha} \left(\int_0^{iT} \varphi^T - 1 - \log\left(\int_0^{iT} \varphi^T\right)\right) \underset{T \rightarrow \infty}{\sim} \frac{T^{2\alpha}}{2} \left(\int_{iT}^\infty f^{\alpha,1} + \lambda T^{-\alpha} \int_0^{iT} f^{\alpha,1}\right)^2,$$

which converges to

$$\frac{1}{2} \left(\lambda + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right)^2,$$

thanks to Appendix A.1. Thus, Property (22) is satisfied for large enough $T > T_0(a,t,\lambda,\nu)$ and $a < a_0(t)$ with

$$a_0(t) = \frac{1}{2v^2} \left(\lambda + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right)^2.$$

Furthermore, as $N^{f,T} \leq N^{\infty,T}$ (which is the Galton–Watson process defined in the Appendix C.2), using (34) in Appendix C.2, we obtain

$$\mathbb{E}[\exp(av^2 T^{-2\alpha} N_{tT}^{f,T})] \leq \sum_{n \geq 0} \frac{v_T(t)^n e^{-v_T(t)(n+1)}}{n!} (n+1)^{n-1} e^{av^2 T^{-2\alpha} n},$$

with $v_T(t) = \int_0^t \varphi^T$. It is also easy to see from (34) [by taking $a = 0$ and $v = 1$ in (34)] that

$$1 = \sum_{n \geq 0} \frac{e^{-(n+1)}}{n!} (n+1)^{n-1}.$$

Consequently, we obtain

$$\begin{aligned} g^T(a, t) &\leq v^{-2} T^\alpha \sum_{n \geq 0} \frac{e^{-(n+1)}}{n!} (n+1)^{n-1} \\ &\quad \times (v_T(t)^n e^{(1-v_T(t))(n+1)} e^{av^2 T^{-2\alpha}(n+1)} - 1) \\ &= \frac{1}{v^2 v_T(t)} T^\alpha \sum_{n \geq 0} \frac{e^{-(n+1)}}{n!} (n+1)^{n-1} (e^{x_T(t)(n+1)} - v_T(t)), \end{aligned}$$

where

$$x_T(t) = 1 - v_T(t) + \log(v_T(t)) + av^2 T^{-2\alpha},$$

which is nonpositive for $T > T_0(a, t, \lambda, v)$. Therefore,

$$g^T(a, t) \leq \frac{1}{v^2 v_T(t)} T^\alpha (1 - v_T(t)).$$

Assume now $a \leq 0$, we use again $N^{f,T} \leq N^{\infty,T}$ and (34) to get

$$v^{-2} T^\alpha \sum_{n \geq 0} \frac{e^{-(n+1)}}{n!} (n+1)^{n-1} (e^{x_T(t)(n+1)} - 1) \leq g^T(a, t) \leq 0.$$

By the Stirling formula,

$$\frac{e^{-(n+1)}}{n!} (n+1)^{n-1} \underset{n \rightarrow \infty}{\sim} \frac{1}{\sqrt{2\pi(n+1)^3}}.$$

Thus,

$$-c \frac{1}{v^2 v_T(t)} T^\alpha \sum_{n \geq 0} \frac{1}{(n+1)^{3/2}} (1 - e^{x_T(t)(n+1)}) \leq g^T(a, t) \leq 0.$$

We deduce from Lemma A.7 that

$$-\frac{c}{v^2 v_T(t)} \sqrt{-T^{2\alpha} x_T(t)} \leq g^T(a, t) \leq 0.$$

Therefore, for any $a < a_0(t)$,

$$|g^T(a, t)| \leq c \frac{1}{v^2 v_T(t)} (T^\alpha (1 - v_T(t)) + \sqrt{-T^{2\alpha} x_T(t)}).$$

Finally, note that

$$T^\alpha (1 - v_T(t)) \rightarrow \lambda + \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}$$

and

$$T^{2\alpha} x_T(t) \rightarrow v^2 (a_0(t) - a),$$

as T goes to infinity. Eventually,

$$(23) \quad \limsup_{T \rightarrow \infty} |g^T(a, t)| \leq c v^{-2} \left(\lambda + \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} + v \sqrt{(a_0(t) - a)} \right).$$

Uniform convergence of g^T . We now fix $t_0 > 0$ and $a < a_0(t_0)$. The function $t \mapsto g^T(a, t)$ being monotone and such that $g^T(a, 0) = 0$, we have that for any $0 \leq t \leq t_0$,

$$|g^T(a, t)| \leq |g^T(a, t_0)|.$$

Moreover, from the previous section, there exists $T_0(t_0, a, \lambda, v) > 0$ such that

$$\sup_{T \geq T_0} |g^T(a, t_0)| < \infty.$$

Hence, $g^T(a, \cdot)$ is uniformly bounded in $0 \leq t \leq t_0$ and $T \geq T_0$. We now assume that $T \geq T_0$.

Applying (31) in Appendix C.1 on the Hawkes process $N^{f,T}$, we obtain that for any $t \in [0, t_0]$,

$$v^2 T^{-\alpha} g^T(a, t) + 1 = \exp \left(v^2 T^{-2\alpha} a + v^2 T^{1-\alpha} \int_0^t \varphi^T(Ts) g^T(a, t-s) ds \right).$$

By taking the logarithm of the previous expression, we write

$$\begin{aligned} & v^2 T^{-2\alpha} a + v^2 T^{1-\alpha} \int_0^t \varphi^T(Ts) g^T(a, t-s) ds \\ &= v^2 T^{-\alpha} g^T(a, t) - \frac{v^4}{2} T^{-2\alpha} g^T(a, t)^2 - \varepsilon_1^T(t), \end{aligned}$$

where $|T^{3\alpha} \varepsilon_1^T|$ is uniformly bounded in $t \in [0, t_0]$ and $T \geq T_0$. Hence

$$\begin{aligned} g^T(a, t) &= T \int_0^t \varphi^T(Ts) g^T(a, t-s) ds + a T^{-\alpha} \\ &\quad + \frac{v^2}{2} T^{-\alpha} g^T(a, t)^2 + \frac{T^\alpha}{v^2} \varepsilon_1^T(t). \end{aligned}$$

Thanks to Lemma A.1,

$$g^T(a, t) = aT^{1-\alpha} \int_0^t \psi^T(Ts) ds + \frac{v^2}{2} T^{1-\alpha} \int_0^t \psi^T(Ts) g^T(a, t-s)^2 ds + \varepsilon_2^T(t),$$

where

$$\varepsilon_2^T(t) = aT^{-\alpha} + \frac{v^2}{2} T^{-\alpha} g^T(a, t)^2 + \frac{T^\alpha}{v^2} \varepsilon_1^T(t) + \frac{T^{\alpha+1}}{v^2} \int_0^t \psi^T(Ts) \varepsilon_1^T(t-s) ds$$

and $\psi^T = \sum_{k \geq 1} (\varphi^T)^{*k}$.³ Note that $T^\alpha \varepsilon_2^T$ is uniformly bounded in $t \in [0, t_0]$ and $T \geq T_0$. Recall also that using Laplace transform computations as in [16], we get

$$(24) \quad \lambda T^{1-\alpha} \psi^T(Tt) = a_T f^{\alpha, \lambda}(t).$$

Thus,

$$g^T(a, t) = \int_0^t \frac{1}{\lambda} f^{\alpha, \lambda}(t-s) \left(a + \frac{v^2}{2} g^T(a, s)^2 \right) ds + \varepsilon^T(t),$$

with $\varepsilon^T(t) = \varepsilon_2^T(t) - T^{-\alpha} \int_0^t f^{\alpha, \lambda}(t-s) \left(a + \frac{v^2}{2} g^T(a, s)^2 \right) ds$. As done in the proof of Proposition 6.5 in [10], using that $T^\alpha \varepsilon^T$ and $g^T(a, \cdot)$ are uniformly bounded in $t \in [0, t_0]$ and $T \geq T_0$, together with Lemma A.3, we deduce that $g^T(a, \cdot)$ is a Cauchy sequence on $C([0, t_0], \mathbb{R})$. Therefore, it converges to a continuous function $g(a, \cdot)$ solution of the following equation:

$$g(a, t) = \int_0^t \frac{1}{\lambda} f^{\alpha, \lambda}(t-s) \left(a + \frac{v^2}{2} g(a, s)^2 \right) ds.$$

By Lemma A.2, it is equivalent to the fractional Riccati equation

$$D^\alpha g(a, t) = a - \lambda g(a, t) + \frac{v^2}{2} g(a, t)^2, \quad I^{1-\alpha} g(a, 0) = 0,$$

which admits a unique continuous solution (the uniqueness being an obvious corollary of Lemma A.3). Finally remark that from (23)

$$|g(a, t)| \leq \frac{c}{v^2} \left(\lambda + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + v \sqrt{a_0(t) - a} \right).$$

³Recall that $(\varphi^T)^{*1} = \varphi$ and $(\varphi^T)^{*k}(t) = \int_0^t \varphi(t-s) \cdot (\varphi^T)^{*k-1}(s) ds$.

REMARK 5.1. Note that for $a \geq 0$, $t \mapsto g(-a, t)$ is nonincreasing and since $g(-a, 0) = 0$, we obtain the following inequality:

$$\begin{aligned} g(-a, t) &= \int_0^t \frac{1}{\lambda} f^{\alpha, \lambda}(t-s) \left(a + \frac{v^2}{2} g(-a, s)^2 \right) ds \\ &\leq \frac{1}{\lambda} F^{\alpha, \lambda}(t) \left(-a + \frac{v^2}{2} g(-a, t)^2 \right). \end{aligned}$$

From this inequality, we get for $t > 0$

$$g(-a, t) \leq \frac{1 - \sqrt{1 + \frac{2v^2 a}{\lambda^2} F^{\alpha, \lambda}(t)^2}}{\frac{v^2}{\lambda} F^{\alpha, \lambda}(t)}.$$

End of the proof. We know that for any $t \in [0, t_0]$ and for fixed $a < a_0(t_0)$,

$$\mathbb{E}[\exp(av^2 T^{-2\alpha} N_{iT}^T)] = \exp\left(\int_0^t \lambda \zeta^T(T(t-s)) g^T(a, s) ds\right).$$

Then, from the uniform convergence of $g^T(a, \cdot)$ to $g(a, \cdot)$ together with Lemma A.4, Lemma A.5 and the dominated convergence theorem, we obtain

$$\mathbb{E}[\exp(av^2 T^{-2\alpha} N_{iT}^T)] \rightarrow \exp\left(\int_0^t g(a, t-s) \left(\lambda \theta^0(s) + \frac{V_0 s^{-\alpha}}{\Gamma(1-\alpha)} \right) ds\right)$$

as T goes to infinity. By the Fatou lemma, we deduce

$$\mathbb{E}\left[\exp\left(a \int_0^t V_s ds\right)\right] < \infty.$$

We complete the proof by showing (21). The case $a \leq 0$ being obvious, we assume that $0 < a < a_0(t_0)$. Let $\varepsilon > 0$ such that $a(1 + \varepsilon) < a_0(t_0)$. From the computations above, there exists $T_0(t, a, \lambda, v, \varepsilon)$ such that

$$\sup_{T \geq T_0} \mathbb{E}[\exp(a(1 + \varepsilon)v^2 T^{-2\alpha} N_{iT}^T)] < \infty.$$

Therefore, $(\exp(av^2 T^{-2\alpha} N_{iT}^T))_{T \geq T_0}$ is uniformly integrable and we conclude that

$$\mathbb{E}[\exp(av^2 T^{-2\alpha} N_{iT}^T)] \rightarrow \mathbb{E}\left[\exp\left(a \int_0^t V_s ds\right)\right].$$

This completes the proof of Theorem 3.2.

5.3. *Proof of Theorem 3.3.* In this section, we place ourselves in the framework of generalized rough Heston models (3) and compute for $0 \leq t \leq t_0$,

$$G(z, x, t) = \mathbb{E}\left[\exp\left(z \int_0^t V_s ds + ix \int_0^t \sqrt{V_s} dB_s\right)\right],$$

with $x \in \mathbb{R}$ and $z \in \mathbb{C}$ such that $\Re(z) < a_0(t)$ [$a_0(t)$ is defined in Theorem 3.2]. It has been shown in the proof of Theorem 3.2 that there exists $T_0 > 0$ such that

$$\exp(z\nu^2 T^{-2\alpha} N_{iT}^T + ix\nu T^{-\alpha} M_{iT}^T)$$

is uniformly integrable for fixed t and $T \geq T_0$. We have that

$$(25) \quad \mathbb{E}[\exp(z\nu^2 T^{-2\alpha} N_{iT}^T + ix\nu T^{-\alpha} M_{iT}^T)]$$

is equal to

$$\mathbb{E}\left[\exp\left((z\nu^2 T^{-2\alpha} + ix\nu T^{-\alpha})N_{iT}^T - ix\nu T^{-\alpha} \int_0^{iT} \int_0^s \varphi^T(s-u) dN_u^T ds - ix\nu T^{-\alpha} \mu_T \int_0^{iT} \zeta^T(s) ds\right)\right].$$

Let

$$f^T(t) = z\nu^2 T^{-2\alpha} + ix\nu T^{-\alpha} - ix\nu T^{-\alpha} \int_0^t \varphi^T(s) ds.$$

Using the Fubini theorem, we get that (25) is also equal to

$$\mathbb{E}\left[\exp\left(\int_0^{iT} f^T(tT-s) dN_s^T - ix\frac{\lambda}{\nu} \int_0^t \zeta^T(sT) ds\right)\right].$$

Hence we deduce from Lemma A.4 and Lemma A.5 that

$$\begin{aligned} G(z, x, t) &= \lim_{T \rightarrow \infty} \mathbb{E}[\exp(z\nu^2 T^{-2\alpha} N_{iT}^T + ix\nu T^{-\alpha} M_{iT}^T)] \\ &= \exp\left(-\frac{ix}{\nu} \int_0^t \lambda \theta^0(s) + \frac{V_0 s^{-\alpha}}{\Gamma(1-\alpha)} ds\right) \\ &\quad \times \lim_{T \rightarrow \infty} \mathbb{E}\left[\exp\left(\int_0^{iT} f^T(tT-s) dN_s^T\right)\right]. \end{aligned}$$

Passing to the limit. Applying (35) in Appendix C.3 on the Hawkes process N^T with the function f^T , we have that for large enough T ,

$$\exp\left(\int_0^{iT} f^T(tT-s) dN_s^T\right)$$

is integrable and

$$\mathbb{E}\left[\exp\left(\int_0^{iT} f^T(tT-s) dN_s^T\right)\right] = \exp\left(\int_0^t \lambda \zeta^T(T(t-s)) k^T(z, x, s) ds\right),$$

where

$$k^T(z, x, t) = \frac{1}{\nu^2} T^\alpha (e^{f^T(tT)} \mathbb{E}[e^{\int_0^{iT} f^T(tT-u) dN_u^{f^T}}] - 1).$$

Furthermore, from Lemma A.5, $\lambda\zeta^T(Ts)$ converges pointwise as T tends to infinity to

$$\lambda\theta^0(s) + \frac{V_0s^{-\alpha}}{\Gamma(1-\alpha)}, \quad s \leq t.$$

As in Section 5.2, we show the uniform boundedness of k^T and then its uniform convergence.

Uniform boundedness of k^T . We start by noting that for $t \in [0, t_0]$,

$$\begin{aligned} |k^T(z, x, t)| \leq & \frac{1}{v^2} T^\alpha (|e^{f^T(tT)} \mathbb{E}[e^{\int_0^{tT} f^T(tT-u) dN_u^{f,T}}] \\ & - e^{i\Im[f^T(tT)]} \mathbb{E}[e^{\int_0^{tT} i\Im[f^T(tT-u)] dN_u^{f,T}}] | \\ & + |e^{i\Re[f^T(tT)} \mathbb{E}[e^{\int_0^{tT} i\Re[f^T(tT-u)] dN_u^{f,T}}] - 1|). \end{aligned}$$

Using that $\Re[f^T] = \Re(z)v^2T^{-2\alpha}$ together with the following inequality,

$$\begin{aligned} & |e^{f^T(tT)} \mathbb{E}[e^{\int_0^{tT} f^T(tT-u) dN_u^{f,T}}] - e^{i\Re[f^T(tT)]} \mathbb{E}[e^{\int_0^{tT} i\Re[f^T(tT-u)] dN_u^{f,T}}] | \\ & \leq |e^{\Re[f^T(tT)} \mathbb{E}[e^{\int_0^{tT} \Re[f^T(tT-u)] dN_u^{f,T}}] - 1|, \end{aligned}$$

we derive

$$|k^T(z, x, t)| \leq |k^T(\Re(z), 0, t)| + |k^T(i\Im(z), x, t)|.$$

In Section 5.2, we have already shown that $k^T(\Re(z), 0, t)$ is uniformly bounded in $t \in [0, t_0]$, for large enough T . It is now left to show the uniform boundedness of $k^T(i\Im(z), x, t)$. So now we take $z = ia$ where $a \in \mathbb{R}$. First, remark that f^T becomes

$$\begin{aligned} f^T(t) &= iav^2T^{-2\alpha} + ixvT^{-\alpha} \left(1 - \int_0^t \varphi^T(s) ds \right) \\ &= i(av^2 + xv\lambda)T^{-2\alpha} + ixvT^{-\alpha} \int_t^\infty \varphi^T(s) ds. \end{aligned}$$

We write

$$\tilde{X}_t^T = f^T(tT) + \int_0^{tT} f^T(tT-s) dN_s^{f,T}.$$

It is easy to see that

$$|f^T(tT)| \leq |a|v^2T^{-2\alpha} + |x|vT^{-\alpha}.$$

Furthermore,

$$\mathbb{E} \left[\int_0^{tT} f^T(tT-s) dN_s^{f,T} \right]$$

is equal to

$$T^{-2\alpha}i(av^2 + \lambda xv) \int_0^{tT} \mathbb{E}[\lambda_s^{f,T}] ds + ixvT^{-\alpha} \int_0^{tT} \left(\int_{tT-s}^\infty \varphi^T(u) du \right) \mathbb{E}[\lambda_s^{f,T}] ds,$$

where $\lambda^{f,T}$ is the intensity of the cluster of children Hawkes process $N^{f,T}$; see Appendix C.1. We recall its definition:

$$\lambda_u^{f,T} = \varphi^T(u) + \int_0^u \varphi^T(u-s) dN_s^{f,T}.$$

Using Lemma A.1, we know that

$$\lambda_u^{f,T} = \psi^T(u) + \int_0^u \psi^T(u-s) dM_s^{f,T},$$

where $M^{f,T} = N^{f,T} - \int_0^\cdot \lambda_s^{f,T} ds$ is the martingale associated to $N^{f,T}$. Thanks to (24), we obtain

$$\mathbb{E}[\lambda_{tT}^{f,T}] = \frac{a_T f^{\alpha,\lambda}(t)}{\lambda} T^{\alpha-1}.$$

Therefore,

$$\int_0^{tT} \mathbb{E}[\lambda_s^{f,T}] ds \leq \frac{F^{\alpha,\lambda}(t)}{\lambda} T^\alpha \leq \frac{F^{\alpha,\lambda}(t_0)}{\lambda} T^\alpha \leq cT^\alpha.$$

Moreover, using that $y \in \mathbb{R}_+ \rightarrow y^\alpha \int_y^\infty \varphi^T$ is uniformly bounded in y and T and $I^{1-\alpha} f^{\alpha,\lambda} = \lambda(1 - F^{\alpha,\lambda})$ (see Appendix A.1), we obtain

$$T \int_0^t \int_{T(t-s)}^\infty \varphi^T(u) du \mathbb{E}[\lambda_{sT}^{f,T}] ds \leq \frac{c}{\lambda} \int_0^t (t-s)^{-\alpha} f^{\alpha,\lambda}(s) ds \leq c.$$

We deduce then that

$$|\mathbb{E}[\tilde{X}_t^T]| \leq cT^{-\alpha}(|a|v^2 + |x|v).$$

Using that there exists $c > 0$ such that for any $y \in \mathbb{R}$,

$$|e^{iy} - 1 - iy| \leq cy^2,$$

we get

$$|\mathbb{E}[e^{\tilde{X}_t^T} - 1]| \leq c(|\mathbb{E}[\tilde{X}_t^T]| + \mathbb{E}[|\tilde{X}_t^T|^2]).$$

We have

$$\mathbb{E}[|\tilde{X}_t^T|^2] \leq 2|f^T(t)|^2 + 2\mathbb{E}\left[\left|\int_0^{tT} f^T(tT-s) dN_s^{f,T}\right|^2\right],$$

and

$$\begin{aligned} & \mathbb{E}\left[\left|\int_0^{tT} f^T(tT - s) dN_s^{f,T}\right|^2\right] \\ & \leq 2\left(\mathbb{E}\left[\left|\int_0^{tT} f^T(tT - s) dM_s^{f,T}\right|^2\right] + \mathbb{E}\left[\left|\int_0^{tT} f^T(tT - s) \lambda_s^{f,T} ds\right|^2\right]\right). \end{aligned}$$

Since $\langle M^{f,T}, M^{f,T} \rangle = \int_0^\cdot \lambda^{f,T}(s) ds$, we obtain

$$\begin{aligned} \mathbb{E}\left[\left|\int_0^{tT} f^T(tT - s) dM_s^{f,T}\right|^2\right] &= \mathbb{E}\left[\int_0^{tT} |f^T(tT - s)|^2 \lambda_s^{f,T} ds\right] \\ &\leq cT^{-\alpha}(|a|v^2 + |x|v)^2. \end{aligned}$$

Using the Fubini theorem, we derive

$$\begin{aligned} \int_0^{tT} f^T(tT - s) \lambda_s^{f,T} ds &= \int_0^{tT} f^T(tT - s) \psi^T(s) ds \\ &\quad + \int_0^{tT} \int_0^{tT-s} f^T(tT - s - u) \psi^T(u) du dM_s^{f,T}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \mathbb{E}\left[\left|\int_0^{tT} f^T(tT - s) \lambda_s^{f,T} ds\right|^2\right] \\ & \leq 2\left|\int_0^{tT} f^T(tT - s) \psi^T(s) ds\right|^2 \\ & \quad + 2\int_0^{tT} \left|\int_0^{tT-s} f^T(tT - s - u) \psi^T(u) du\right|^2 \mathbb{E}[\lambda_s^{f,T}] ds \\ & \leq c(|a|v^2 + |x|v)^2 T^{-2\alpha} \left(1 + \int_0^{tT} \mathbb{E}[\lambda_s^{f,T}] ds\right) \\ & \leq c(|a|v^2 + |x|v)^2 T^{-\alpha}. \end{aligned}$$

We eventually deduce

$$|k^T(ia, x, t)| \leq \frac{c}{v^2}(c(a, x) + c(a, x)^2), \quad c(a, x) = v^2|a| + v|x|.$$

End of the proof. Using the same computations as in Section 5.2, we show that for fixed $z \in \mathbb{C}$ and $x \in \mathbb{R}$ such that $\Re(z) < a_0(t_0)$, $k^T(z, x, \cdot)$ is a Cauchy sequence in $C([0, t_0], \mathbb{C})$ and therefore converges uniformly to $k(z, x, \cdot)$ solution of

$$k(z, x, t) = ix \frac{1}{v} + \int_0^t \frac{1}{\lambda} f^{\alpha, \lambda}(t - s) \left(z + \frac{v^2}{2} k(z, x, t)^2\right) ds.$$

Therefore, we deduce

$$G(z, x, t) = \exp\left(\int_0^t \xi(z, x, t-s) \left(\lambda\theta^0(s) + \frac{V_0s^{-\alpha}}{\Gamma(1-\alpha)}\right) ds\right),$$

where $\xi(z, x, t) = k(z, x, t) - ix/v$, which is solution of the following equation:

$$\xi(z, x, t) = \int_0^t \frac{1}{\lambda} f^{\alpha,\lambda}(t-s) \left(z - \frac{x^2}{2} + ibv\xi(z, x, s) + \frac{v^2}{2}\xi(z, x, s)^2\right) ds.$$

By Lemma A.2, this is equivalent to the following fractional Riccati equation:

$$D^\alpha \xi(z, x, t) = z - \frac{x^2}{2} + (ixv - \lambda)\xi(z, x, t) + \frac{v^2}{2}\xi(z, x, t)^2,$$

$$I^{1-\alpha} \xi(z, x, 0) = 0.$$

We end this section with the following remarks which will be useful in the proof of Proposition 4.1.

REMARK 5.2. From the definition of k^T ,

$$\Re(k^T(z, x, t)) \leq k^T(\Re(z), 0, t).$$

Passing to the limit as T goes to infinity, we get

$$\Re(\xi(z, x, t)) \leq \xi(\Re(z), 0, t) = g(\Re(z), t),$$

with g defined in Theorem 3.2.

REMARK 5.3. From the proof of uniform boundedness of k^T and using the inequality for g in Theorem 3.2, we get that for any $t \in [0, t_0]$,

$$|\xi(z, x, t)| \leq c(1 + \sqrt{|\Re(z)|} + \Im(z)^2 + x^2),$$

where c is a positive constant, $x \in \mathbb{R}$ and $z \in \mathbb{C}$ such that $\Re(z) < a_0(t_0)$.

5.4. *Proof of Proposition 4.1.* We fix $S > 0$, $t > 0$, $\xi \in \mathcal{V}_{\alpha,\lambda}^+$ and $a > 1$ such that $\mathbb{E}[(S_t)^a] < \infty$. Using the same computations as in the proof of Corollary 3.3, we get

$$L(a + ib, t, S, \xi) = \exp\left((a + ib) \log(S) + \int_0^t \frac{s^{-\alpha}}{\Gamma(1-\alpha)} h(a + ib, t-s) \theta_\xi(s) ds\right),$$

where h is the unique continuous solution of the fractional Riccati equation in Corollary 3.2. Moreover, thanks to Remark 5.2, we have that for any $t > 0$ and $b \in \mathbb{R}$,

$$\Re(h(a + ib, s)) \leq q(b, s), \quad s \leq t,$$

where $q(b, \cdot)$ is the unique continuous solution of the following fractional Riccati equation:

$$D^\alpha q(b, s) = \frac{a^2 - a}{2} - (1 - \rho^2) \frac{b^2}{2} - (\lambda - \rho\nu a)q(b, s) + \frac{\nu^2}{2}q(b, s)^2, \quad s \leq t,$$

$$I^{1-\alpha}q(b, 0) = 0.$$

Note also that for large $|b|$, $\frac{a^2 - a}{2} - (1 - \rho^2) \frac{b^2}{2}$ is negative and, therefore, using Remark 5.1,

$$q(b, s) \leq M(b, s) = \frac{1 - \sqrt{1 + \nu^2 \frac{(1-\rho^2)b^2 - (a^2 - a)}{(\lambda - \rho\nu a)^2} F^{\alpha, \lambda - \rho\nu a}(s)^2}}{\frac{\nu^2}{\lambda - \rho\nu a} F^{\alpha, \lambda - \rho\nu a}(s)}, \quad s \leq t.$$

By the dominated convergence theorem, we have

$$\int_0^t \frac{s^{-\alpha}}{\Gamma(1 - \alpha)} M(b, t - s) \theta_\xi(s) ds \underset{b \rightarrow \infty}{\sim} -|b| \frac{\sqrt{1 - \rho^2}}{\nu} \int_0^t \frac{s^{-\alpha}}{\Gamma(1 - \alpha)} \theta_\xi(s) ds.$$

Consequently, there exists $c(t, \xi) > 0$ such that for any $b \in \mathbb{R}$,

$$|L(a + ib, t, S, \xi)| \leq S^a \exp(-c(t, \xi)(-1 + |b|)).$$

Moreover, it is easy to see that for any $b \in \mathbb{R}$, $L(a + ib, t, \cdot, \xi)$ is differentiable in S and that

$$\partial_S L(a + ib, t, S, \xi) = \frac{a + ib}{S} L(a + ib, t, S, \xi).$$

Using (17) together with the dominated convergence theorem, we conclude that C is differentiable in the first variable S and that

$$\partial_S C(t, S, \xi) = \frac{1}{2\pi} \int_{b \in \mathbb{R}} \hat{g}(-b) \frac{a + ib}{S} L(a + ib, t, S, \xi) db.$$

Now let $\zeta \in \mathcal{V}_{\alpha, \lambda}$ and $\varepsilon_0 > 0$ such that $\theta_\xi(s) - \varepsilon_0 |\theta_\zeta(s)| > 0$ for any $s \in [0, t]$. We have that for any $\varepsilon \neq 0$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$,

$$(26) \quad \frac{1}{\varepsilon} |L(a + ib, t, S, \xi + \varepsilon \zeta) - L(a + ib, t, S, \xi)|$$

is equal to

$$\begin{aligned} & S^a \exp\left(\int_0^t \frac{s^{-\alpha}}{\Gamma(1 - \alpha)} \Re(h(a + ib, t - s))(\theta_\xi(s) - \varepsilon |\theta_\zeta(s)|) ds\right) \\ & \times \frac{1}{\varepsilon} \left| \exp\left(\int_0^t \frac{s^{-\alpha}}{\Gamma(1 - \alpha)} \varepsilon h(a + ib, t - s) \theta_\zeta(s) ds\right) \right. \\ & \left. - \exp\left(\int_0^t \frac{s^{-\alpha}}{\Gamma(1 - \alpha)} \varepsilon h(a + ib, t - s) |\theta_\zeta(s)| ds\right) \right|. \end{aligned}$$

Recall that for large $|b|$, $\Re(h(a + ib, s))$ is nonpositive for any $s \leq t$. Since there exists $c > 0$ such that for any $z, z' \in \mathbb{C}$ such that $\Re(z) \leq 0$ and $\Re(z') \leq 0$,

$$|\exp(z) - \exp(z')| \leq c|z - z'|,$$

we conclude that (26) is dominated by

$$cS^\alpha \left(\int_0^t \frac{s^{-\alpha}}{\Gamma(1 - \alpha)} |h(a + ib, t - s)| |\theta_\zeta(s)| ds \right) \\ \times \exp \left(\int_0^t \frac{s^{-\alpha}}{\Gamma(1 - \alpha)} \Re(h(a + ib, t - s)) (\theta_\xi(s) - \varepsilon_0 |\theta_\zeta(s)|) ds \right).$$

Using the same arguments as previously, we get that there exists $c(t, \xi, \zeta, \varepsilon_0) > 0$ such that

$$\exp \left(\int_0^t \frac{s^{-\alpha}}{\Gamma(1 - \alpha)} \Re(h(a + ib, t - s)) (\theta_\xi(s) - \varepsilon_0 |\theta_\zeta(s)|) ds \right) \\ \leq \exp(-c(t, \xi, \zeta, \varepsilon_0)(-1 + |b|)).$$

From Remark 5.3, we know that there exists $c(t) > 0$ such that for any $s \in [0, t]$ and $b \in \mathbb{R}$,

$$|h(a + ib, s)| \leq c(t)(1 + b^2).$$

Moreover, note that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} L(a + ib, t, S, \xi + \varepsilon \zeta) - L(a + ib, t, S, \xi)$$

is equal to

$$L(a + ib, t, S, \xi) \int_0^t \chi(a + ib, t - s) \zeta_s ds.$$

Consequently, by the dominated convergence theorem, $C(t, S, \cdot)$ is differentiable in ξ in the direction of ζ in the Fréchet sense and

$$\partial_V C(t, S, \xi) \cdot \zeta \\ = \frac{1}{2\pi} \int_{b \in \mathbb{R}} \hat{g}(-b) L(a + ib, t, S, \xi) \left(\int_0^t \chi(a + ib, t - s) \zeta_s ds \right) db.$$

5.5. Proof of Theorem 4.1. We first show that

$$\int_0^{T-t} \chi(a + ib, s) \mathbb{E}[V_{T-s} | \mathcal{F}_t] ds$$

is equal to

$$(27) \quad \int_0^T \chi(a + ib, s) \mathbb{E}[V_{T-s}] ds - \int_0^t \chi(a + ib, T - s) V_s ds \\ + \int_0^t h(a + ib, T - s) v \sqrt{V_s} dB_s.$$

Recall that from equation (15) we get

$$V_s = \mathbb{E}[V_s] + \int_0^s \frac{1}{\lambda} f^{\alpha,\lambda}(s-u) \nu \sqrt{V_u} dB_u.$$

This together with the Fubini theorem give

$$\begin{aligned} & \int_0^t \chi(a+ib, T-s) V_s ds \\ &= \int_0^t \chi(a+ib, T-s) \mathbb{E}[V_s] ds \\ & \quad + \int_0^t \left(\int_0^{t-u} \frac{1}{\lambda} f^{\alpha,\lambda}(s) \chi(a+ib, T-u-s) ds \right) \nu \sqrt{V_u} dB_u. \end{aligned}$$

We also have that for $s \in [0, T-t]$,

$$(28) \quad \mathbb{E}[V_{T-s} | \mathcal{F}_t] = \mathbb{E}[V_{T-s}] + \int_0^t \frac{1}{\lambda} f^{\alpha,\lambda}(T-s-u) \nu \sqrt{V_u} dB_u.$$

Then similarly,

$$\int_0^{T-t} \chi(a+ib, s) \mathbb{E}[V_{T-s} | \mathcal{F}_t] ds$$

is equal to

$$\begin{aligned} & \int_0^{T-t} \chi(a+ib, s) \mathbb{E}[V_{T-s}] ds \\ & \quad + \int_0^t \left(\int_0^{T-t} \frac{1}{\lambda} f^{\alpha,\lambda}(T-s-u) \chi(a+ib, s) ds \right) \nu \sqrt{V_u} dB_u. \end{aligned}$$

This can also be written

$$\begin{aligned} & \int_t^T \chi(a+ib, T-s) \mathbb{E}[V_s] ds \\ & \quad + \int_0^t \left(\int_{t-u}^{T-u} \frac{1}{\lambda} f^{\alpha,\lambda}(s) \chi(a+ib, T-u-s) ds \right) \nu \sqrt{V_u} dB_u. \end{aligned}$$

Finally, we obtain that

$$\int_0^t \chi(a+ib, T-s) V_s ds + \int_0^{T-t} \chi(a+ib, s) \mathbb{E}[V_{T-s} | \mathcal{F}_t] ds$$

is equal to

$$\begin{aligned} & \int_0^T \chi(a+ib, T-s) \mathbb{E}[V_s] ds \\ & \quad + \int_0^t \left(\int_0^{T-u} \frac{1}{\lambda} f^{\alpha,\lambda}(s) \chi(a+ib, T-u-s) ds \right) \nu \sqrt{V_u} dB_u. \end{aligned}$$

Thus, (27) is directly deduced from the last relation and (16). Now using (27) together with the Itô formula, we derive

$$P_t^T(a + ib) = P_0^T(a + ib) + \int_0^t (a + ib) P_s^T(a + ib) \sqrt{V_s} dW_s + \int_0^t P_s^T(a + ib) h(a + ib, T - s) v \sqrt{V_s} dB_s.$$

Then by (17) together with the stochastic Fubini theorem and Proposition 4.1, we get

$$C_t = C_0 + \int_0^t \partial_S C(T - u, S_u, \mathbb{E}[V_{\cdot+u} | \mathcal{F}_u]) dS_u + \frac{1}{2\pi} \int_0^t \left(\int_{b \in \mathbb{R}} \hat{g}(-b) P_u^T(a + ib) h(a + ib, T - u) db \right) v \sqrt{V_u} dB_u.$$

Furthermore, using again (16) together with Fubini theorem, we obtain that

$$\frac{1}{2\pi} \int_0^t \left(\int_{b \in \mathbb{R}} \hat{g}(-b) P_u^T(a + ib) h(a + ib, T - u) db \right) v \sqrt{V_u} dB_u$$

is equal to

$$\int_0^t \left(\frac{1}{2\pi} \int_0^{T-u} \int_{b \in \mathbb{R}} \hat{g}(-b) P_u^T(a + ib) \chi(a + ib, T - u - s) db ds \right) \times \frac{1}{\lambda} f^{\alpha, \lambda}(s) v \sqrt{V_u} dB_u.$$

This last quantity can be expressed in term of the forward variance curve thanks to (28).

APPENDIX A: FRACTIONAL CALCULUS

We define the fractional integral of order $r \in (0, 1]$ of a function f as

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_0^t (t - s)^{r-1} f(s) ds,$$

whenever the integral exists, and its the fractional derivative of order $r \in [0, 1)$ as

$$D^r f(t) = \frac{1}{\Gamma(1 - r)} \frac{d}{dt} \int_0^t (t - s)^{-r} f(s) ds,$$

whenever it exists.

We gather in this section some useful technical results related to fractional calculus.

A.1. Mittag-Leffler functions. Let $(\alpha, \beta) \in (\mathbb{R}_+^*)^2$. The Mittag-Leffler function $E_{\alpha,\beta}$ is defined and for $z \in \mathbb{C}$ by

$$E_{\alpha,\beta}(z) = \sum_{n \geq 0} \frac{z^n}{\Gamma(\alpha n + \beta)}.$$

For $(\alpha, \lambda) \in (0, 1) \times \mathbb{R}_+$ we also define

$$f^{\alpha,\lambda}(t) = \lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0,$$

$$F^{\alpha,\lambda}(t) = \int_0^t f^{\alpha,\lambda}(s) ds, \quad t \geq 0.$$

The function $f^{\alpha,\lambda}$ is a density function on \mathbb{R}_+ called the Mittag-Leffler density function. The following properties of $f^{\alpha,\lambda}$ and $F^{\alpha,\lambda}$ can be found in [14, 17, 18]. We have

$$f^{\alpha,\lambda}(t) \underset{t \rightarrow 0^+}{\sim} \frac{\lambda}{\Gamma(\alpha)} t^{\alpha-1}, \quad f^{\alpha,\lambda}(t) \underset{t \rightarrow \infty}{\sim} \frac{\alpha}{\lambda \Gamma(1-\alpha)} t^{-(\alpha+1)}$$

and

$$F^{\alpha,\lambda}(t) = 1 - E_{\alpha,1}(-\lambda t^\alpha), \quad F^{\alpha,\lambda}(t) \underset{t \rightarrow 0^+}{\sim} \frac{\lambda}{\Gamma(\alpha+1)} t^\alpha,$$

$$1 - F^{\alpha,\lambda}(t) \underset{t \rightarrow \infty}{\sim} \frac{1}{\lambda \Gamma(1-\alpha)} t^{-\alpha}.$$

Note also that from obvious computations, we get $I^{1-\alpha} f^{\alpha,\lambda} = \lambda(1 - F^{\alpha,\lambda})$. Finally, for $\alpha \in (1/2, 1)$, $f^{\alpha,\lambda}$ is square-integrable and its Laplace transform is given for $z \geq 0$ by

$$\hat{f}^{\alpha,\lambda}(z) = \int_0^\infty f^{\alpha,\lambda}(s) e^{-zs} ds = \frac{\lambda}{\lambda + z^\alpha}.$$

A.2. Wiener–Hopf equations. The following result enables us to solve Wiener–Hopf-type equations; see, for example, [3] for details.

LEMMA A.1. *Let g be a measurable locally bounded function from \mathbb{R} to \mathbb{R}^d and $\phi : \mathbb{R}_+ \mapsto \mathcal{M}^d(\mathbb{R})$ be a matrix-valued function with integrable components such that the spectral radius of $\int_0^\infty \phi(s) ds$ is strictly smaller than 1. Then there exists a unique locally bounded function f from \mathbb{R} to \mathbb{R}^d solution of*

$$f(t) = g(t) + \int_0^t \phi(t-s) \cdot f(s) ds, \quad t \geq 0$$

given by

$$f(t) = g(t) + \int_0^t \psi(t-s) \cdot g(s) ds, \quad t \geq 0,$$

where $\psi = \sum_{k \geq 1} \phi^{*k}$.⁴

⁴Recall that $\phi^{*1} = \phi$ and $\phi^{*k}(t) = \int_0^t \phi(t-s) \cdot \phi^{*(k-1)}(s) ds$.

A.3. Fractional differential equations. We now give some useful results about fractional differential equations. The next lemma can be found in [20].

LEMMA A.2. *Let h be a continuous function from $[0, 1]$ to \mathbb{R} , $\alpha \in (0, 1]$ and $\lambda \in \mathbb{R}$. There is a unique solution to the equation*

$$D^\alpha y(t) = \lambda y(t) + h(t), \quad y(0) = 0$$

given by

$$y(t) = \int_0^t (t - s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t - s)^\alpha) h(s) ds.$$

We also have the following result whose proof can be found in [10].

LEMMA A.3. *Let h be a nonnegative continuous function from $[0, 1]$ to \mathbb{R} such that for any $t \in [0, 1]$,*

$$h(t) \leq \varepsilon + C \int_0^t f^{\alpha,\lambda}(t - s) h(s) ds,$$

for some $\varepsilon \geq 0$ and $C \geq 0$. Then for any $t \in [0, 1]$,

$$h(t) \leq C' \varepsilon,$$

with

$$C' = 1 + C\lambda \int_0^1 s^{\alpha-1} E_{\alpha,\alpha}(\lambda(C - 1)s^\alpha) ds > 0.$$

In particular, if $\varepsilon = 0$ then $h = 0$.

A.4. Further results.

LEMMA A.4. *There exists a positive constant c such that for any $T > 1/\lambda^{-1/\alpha}$ and $t \in (0, 1)$:*

$$\zeta^T(tT) \leq c \left(1 + \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} \right).$$

PROOF. Note that by Remark 3.1, we have

$$\begin{aligned} \zeta^T(tT) &= \int_0^{tT} \varphi^T(tT - u) \theta^0(u/T) du \\ &\quad + V_0 \left(\frac{T^\alpha}{\lambda} \int_{tT}^\infty \varphi(s) ds + \lambda T^{-\alpha} \int_0^{tT} \varphi(s) ds \right). \end{aligned}$$

Thanks to Appendix A.1, we have that for each $t \in (0, 1]$:

$$T^\alpha \int_{tT}^\infty \varphi \leq ct^{-\alpha}.$$

Moreover, by using condition (5) and the fact that $\alpha > 1/2$, we write that for each $t \in (0, 1]$

$$\theta^0(t) \leq ct^{-\alpha}.$$

Thus,

$$\zeta^T(tT) \leq c \int_0^{tT} \varphi(Tt - u)(u/T)^{-\alpha} du + c(1 + t^{-\alpha}).$$

Using Appendix A.1, we obtain

$$\int_0^{tT} \varphi(Tt - u)(u/T)^{-\alpha} du = \Gamma(1 - \alpha)T^\alpha \int_{tT}^\infty \varphi \leq ct^{-\alpha},$$

which completes the proof. \square

LEMMA A.5. For each $t \in (0, 1]$, as T tends to infinity, $\zeta^T(tT)$ defined by Assumption 3.1 converges to

$$V_0 \frac{t^{-\alpha}}{\lambda \Gamma(1 - \alpha)} + \theta^0(t).$$

PROOF. Let $t > 0$. We have

$$\begin{aligned} \zeta^T(tT) &= a_T \int_0^t T \varphi(T(t - s)) \theta^0(s) ds \\ &\quad + V_0 \left(\frac{T^\alpha}{\lambda} \int_{tT}^\infty \varphi(s) ds + \lambda T^{-\alpha} \int_0^{tT} \varphi(s) ds \right). \end{aligned}$$

Moreover, from Appendix A.1,

$$V_0 \left(\frac{T^\alpha}{\lambda} \int_{tT}^\infty \varphi(s) ds + \lambda T^{-\alpha} \int_0^{tT} \varphi(s) ds \right)$$

converges to

$$V_0 \frac{t^{-\alpha}}{\lambda \Gamma(1 - \alpha)}.$$

Moreover, since θ^0 is continuous in t , for any $\varepsilon > 0$ there exists $\eta > 0$ such that for any $s \in [t - \eta, t]$,

$$|\theta^0(s) - \theta^0(t)| \leq \varepsilon.$$

Hence from Appendix A.1 together with the fact that φ is nonincreasing, we obtain

$$\begin{aligned} &\left| \int_0^t T \varphi(T(t - s)) (\theta^0(s) - \theta^0(t)) ds \right| \\ &\leq \varepsilon \int_0^{T\eta} \varphi + \int_0^{t-\eta} T \varphi(T(t - s)) (|\theta^0(t)| + |\theta^0(s)|) ds \\ &\leq \varepsilon + T \varphi(T\eta) \int_0^t (|\theta^0(t)| + |\theta^0(s)|) ds \leq 2\varepsilon \end{aligned}$$

for large enough T . Thus, $\int_0^t T \varphi(T(t - s)) \theta^0(s) ds$ converges to $\theta^0(t)$. \square

LEMMA A.6. *If $\theta^0 : (0, 1] \mapsto \mathbb{R}$ satisfies Condition (5), then for any $0 < \varepsilon < \alpha - 1/2$,*

$$t \mapsto \int_0^t f^{\alpha,\lambda}(t-s)\theta^0(s) ds$$

has Hölder smoothness $\alpha - 1/2 - \varepsilon$ on $[0, 1]$.

PROOF. Using Proposition A.2 in [16], we obtain that for any $\eta \in (0, \alpha)$,

$$\int_0^t f^{\alpha,\lambda}(t-s)\theta^0(s) ds = \int_0^t D^\eta f^{\alpha,\lambda}(t-s)I^\eta\theta^0(s).$$

Taking $\eta = 1/2 + \varepsilon$, we have that $I^\eta\theta^0$ is a bounded function. Then, using Proposition A.3 in [16], we obtain that our function has Hölder regularity equal to $\alpha - \eta = \alpha - 1/2 - \varepsilon$. \square

Let $x \geq 0$. We define

$$S(x) = \sum_{n \geq 0} \frac{1}{(n+1)^{3/2}}(1 - e^{-x(n+1)}).$$

We have the following lemma.

LEMMA A.7. *There exists $c > 0$ such that for any $x \geq 0$*

$$S(x) \leq c\sqrt{x}.$$

PROOF. We have

$$S(x) = \sum_{n \geq 0} \frac{1}{(n+1)^{3/2}}(1 - e^{-x}) \sum_{0 \leq k \leq n} e^{-kx}.$$

This can be rewritten

$$S(x) = (1 - e^{-x}) \sum_{k \geq 0} \xi_k e^{-kx},$$

with $\xi_k = \sum_{n \geq k} \frac{1}{(n+1)^{3/2}}$, which is equivalent to $2/\sqrt{k+1}$ as k tends to infinity. Thus, there exists $c > 0$ such that, for any $x \geq 0$,

$$S(x) \leq c(1 - e^{-x}) \sum_{k \geq 0} \frac{1}{\sqrt{k+1}} e^{-(k+1)x}.$$

We conclude using that

$$\sum_{k \geq 0} \frac{1}{\sqrt{k+1}} e^{-(k+1)x} \leq \sum_{k \geq 0} \int_k^{k+1} \frac{1}{\sqrt{y}} e^{-yx} dy = \frac{\Gamma(1/2)}{\sqrt{x}}$$

together with the fact that

$$1 - e^{-x} \leq cx. \quad \square$$

APPENDIX B: MARTINGALE PROPERTY OF THE PRICE IN THE GENERALIZED ROUGH HESTON MODEL

PROPOSITION B.1. *The process S defined by the generalized rough Heston model in Definition 2.1 is a \mathbb{F} -martingale.*

PROOF. Let $t_0 > 0$ such that $1/2 < a_0(t_0)$. Thanks to Theorem 3.2, Novikov’s criterion holds:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^{t_0} V_s ds\right)\right] < \infty.$$

Therefore, $(S_u)_{0 \leq u \leq t_0}$ is a martingale and $\mathbb{E}[S_{t_0}] = S_0$.

Now, assume that for a given $n \in \mathbb{N}$, $\mathbb{E}[S_{nt_0}] = S_0$. Recall that conditional on \mathcal{F}_{nt_0} , the law of $(S_t^{nt_0}, V_t^{nt_0})_{t \geq 0} = (S_{t+nt_0}, V_{t+nt_0})_{t \geq 0}$ is still that of a rough Heston model with the following dynamic:

$$\begin{aligned} dS_t^{nt_0} &= S_t^{nt_0} \sqrt{V_t^{nt_0}} dW_t^{nt_0}, \\ V_t^{nt_0} &= V_{nt_0} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \lambda(\theta^{nt_0}(u) - V_u^{nt_0}) du \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} v \sqrt{V_u^{nt_0}} dB_u^{nt_0}, \end{aligned}$$

where θ^{nt_0} is a \mathcal{F}_{nt_0} -measurable function satisfying almost surely Conditions (4) and (5), and $(W^{nt_0}, B^{nt_0}) = (W_{\cdot+nt_0} - W_{nt_0}, B_{\cdot+nt_0} - B_{nt_0})$ is a Brownian motion independent of \mathcal{F}_{nt_0} . Since $1/2 < a_0(t_0)$, we have again Novikov’s criterion

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^{t_0} V_s^{nt_0} ds\right) \middle| \mathcal{F}_{nt_0}\right] < \infty.$$

Therefore, $\mathbb{E}[S_{t_0}^{nt_0} | \mathcal{F}_{nt_0}] = S_{nt_0}$ and so

$$\mathbb{E}[S_{(n+1)t_0}] = \mathbb{E}[S_{nt_0}] = S_0.$$

Consequently, for any $n \in \mathbb{N}$,

$$\mathbb{E}[S_{nt_0}] = S_0,$$

which completes the proof. \square

APPENDIX C: MOMENTS PROPERTIES FOR HAWKES PROCESSES

Here we consider a one-dimensional Hawkes process N with intensity

$$\lambda_t = \mu(t) + \int_0^t \varphi(t-s) dN_s,$$

such that $\mu, \varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ are locally integrable and $\int_0^\infty \varphi < 1$. We are interested in a sufficient condition on $a > 0$ so that

$$(29) \quad \mathbb{E}[e^{aN_t}] < \infty.$$

We will show that (29) holds provided

$$(30) \quad a \leq \int_0^t \varphi - 1 - \log\left(\int_0^t \varphi\right).$$

To do so, we recall the branching structure of Hawkes processes.

C.1. Branching structure of Hawkes processes. We recall that the Hawkes process N can be viewed as a population process in which migrants arrive according to a nonhomogenous Poisson process N^0 with intensity μ . Each migrant gives birth to children according to a nonhomogenous Poisson process with intensity φ and each child also gives birth to children according to a nonhomogenous Poisson process with the same intensity and so on.

Therefore, it is easy to see that the cluster of children created by a migrant has the law of a Hawkes process N^f with the same kernel function φ but with migrant rate φ . So, the intensity of N^f is given by

$$\lambda_t^f = \varphi(t) + \int_0^t \varphi(t-s) dN_s^f.$$

Using the branching structure of the Hawkes process, we can see that we easily derive the following equality in law:

$$N_t = N_t^0 + \sum_{1 \leq k \leq N_t^0} N_{t-T_k}^{f,k},$$

where the $(T_k)_{k \geq 1}$ are the arrival times of the migrants and $(N^{f,k})_{k \geq 1}$ are independent copies of N^f , independent of N^0 . Then we can show that, for $a \geq 0$,

$$(31) \quad \mathbb{E}[e^{aN_t}] = \exp\left(\int_0^t \mu(t-s)(e^a \mathbb{E}[e^{aN_s^f}] - 1) ds\right);$$

see [10]. This is smaller than

$$\exp\left(\int_0^t \mu(s) ds (e^a \mathbb{E}[e^{aN_t^f}] - 1)\right).$$

Consequently, a sufficient condition to obtain (29) is

$$(32) \quad \mathbb{E}[e^{aN_t^f}] < \infty.$$

C.2. Galton–Watson structure and exponential moments. Let us consider now the Hawkes process N^f . Using the population interpretation given in the previous section on this process, N_t^f is the number of migrants and children arrived up to time t . Let $t > 0$. We define the process N^∞ from N^f as follows:

- We consider $N_t^{(0)}$ the number of migrants arrived up to time t , which is a Poisson variable with parameter $\nu = \int_0^t \varphi$.
- For each migrant arrived at time $T_k < t$, we consider the number of children of first generation made by the migrant during a period of time t , which is also a Poisson variable with parameter ν , independent of $N_t^{(0)}$. We denote by X_t^1 the set of all those children and $N_t^{(1)} = \#(X_t^1)$ their total number.
- For each child of n th generation of the set X_t^n , we consider the number of its children that are made during a period of time t , which is also a Poisson variable with parameter ν , independent of the previous generations. We denote X_t^{n+1} the set of all those children and $N_t^{(n+1)} = \#(X_t^{n+1})$ their total number.

It is clear that $X_t = \bigcup_{n \geq 0} X_t^n$ contains all the individuals of the Hawkes process N^f arrived up to time t . So,

$$N_t^\infty = \#(X_t) = \sum_{n \geq 0} N_t^{(n)} \geq N_t^f.$$

Thus, a sufficient condition to obtain (32) is

$$(33) \quad \mathbb{E}[e^{aN_t^\infty}] < \infty.$$

Now remark that $(N_t^{(n)})_{n \geq 0}$ is a Galton–Watson process. Indeed,

$$N_t^{(n+1)} = \sum_{1 \leq k \leq N_t^{(n)}} \xi_{k,n+1}; \quad n \geq 0,$$

where $(\xi_{k,n})_{k,n \geq 1}$ are i.i.d. Poisson random variables with parameter ν , independent of the $N_t^{(k)}$.

We classically have (see, e.g., [9])

$$\mathbb{P}[N_t^\infty = n] = \frac{\nu^n e^{-\nu(n+1)}}{n!} (n+1)^{n-1}.$$

Consequently,

$$(34) \quad \mathbb{E}[e^{aN_t^\infty}] = \sum_{n \geq 0} \frac{\nu^n e^{-\nu(n+1)}}{n!} (n+1)^{n-1} e^{an}.$$

Using the Stirling formula, we get

$$\frac{\nu^n e^{-\nu(n+1)}}{n!} (n+1)^{n-1} e^{an} \underset{n \rightarrow \infty}{\sim} \frac{(\nu e^{1-\nu+a})^n}{\sqrt{2\pi n^3}} e^{1-\nu}.$$

Hence (33) holds if and only if

$$\nu e^{1-\nu+a} \leq 1,$$

which is equivalent to

$$a \leq \int_0^t \varphi - 1 - \log\left(\int_0^t \varphi\right).$$

C.3. A useful equality. Let us consider $g : t \mapsto R_+ \rightarrow \mathbb{R}$ continuous and $a \in \mathbb{R}$ satisfying (30). We know that $\exp(\int_0^t f(t-s) dN_s)$ is integrable, where $f = a + ig$. Using the branching structure of Hawkes processes presented in Appendix C.1, we deduce the following equality in law:

$$\int_0^t f(t-s) dN_s = \int_0^t f(t-s) dN_s^0 + \sum_{1 \leq k \leq N_t^0} \int_0^{t-T_k} f(t-T_k-s) dN_s^{f,k}.$$

Therefore, we can show that

$$\begin{aligned} (35) \quad & \mathbb{E}\left[\exp\left(\int_0^t f(t-s) dN_s\right)\right] \\ &= \exp\left(\int_0^t \mu(t-s)(e^{f(s)} \mathbb{E}[e^{\int_0^s f(s-u) dN_u^f}] - 1) ds\right). \end{aligned}$$

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