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About the constants in the Fuk-Nagaev inequalities

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Abstract

In this paper we give efficient constants in the Fuk-Nagaev inequalities. Next we derive new upper bounds on the weak norms of martingales from our Fuk-Nagaev type inequality.

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1 Introduction and previous results

In this paper, we are interested in the deviation on the right of sums of unbounded independent random variables with finite variances. So, let $X_1, X_2, \ldots, X_n$ be a finite sequence of independent random variables with finite variances. Set

$$S_n = X_1 + X_2 + \cdots + X_n \text{ and } \sigma^2 = \mathbb{E}(X_1^2) + \mathbb{E}(X_2^2) + \cdots + \mathbb{E}(X_n^2). \tag{1.1}$$

Assume that $\mathbb{E}(S_n) = 0$. Then the so-called Tchebichef-Cantelli inequality states that

$$\mathbb{P}(S_n \geq x) \leq \sigma^2/(x^2 + \sigma^2) \text{ for any } x > 0. \tag{1.2}$$

Setting $z = 1 + (x/\sigma)^2$, this inequality is equivalent to

$$\mathbb{P}\big(S_n \geq \sigma\sqrt{z-1}\,\big) \leq 1/z \text{ for any } z > 1. \tag{1.3}$$

Assume now that the random variables $X_1, X_2, \ldots, X_n$ have a finite Laplace transform on $[0, +\infty[$ and satisfy the subGaussian condition below:

$$\log \mathbb{E}(e^{tX_1}) + \log \mathbb{E}(e^{tX_2}) + \cdots + \log \mathbb{E}(e^{tX_n}) \leq \tfrac{1}{2}\sigma^2 t^2 \text{ for any } t > 0. \tag{1.4}$$

Then the usual Chernoff calculation yields

$$\mathbb{P}(S_n \geq \sigma\sqrt{2\log z}\,) \leq 1/z \text{ for any } z > 1. \tag{1.5}$$

For large values of z this inequality is clearly much sharper than the Cantelli inequality. However (1.4) is too restrictive. A less restrictive condition is the existence of moments of order $q > 2$ for the positive parts of the random variables $X_1, X_2, \ldots, X_n$. Set

$$X_{i+} = \max(0, X_i) \text{ and } C_q(X) = \big(\mathbb{E}(X_{1+}^q) + \mathbb{E}(X_{2+}^q) + \cdots + \mathbb{E}(X_{n+}^q)\big)^{1/q} \tag{1.6}$$

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for any $q \geq 1$. If $\mathbb{E}(X_i) = 0$ for any i in $[1, n]$, then an adequate version of the Fuk-Nagaev inequalities (see Fuk (1973) or Nagaev (1979), Corollary 1.8) yields

$$\mathbb{P}(S_n \geq x\,) \leq \left(\frac{(q+2)C_q(X)}{qx} \right)^q + \exp\left(-\frac{2x^2}{(q+2)^2 e^q \sigma^2} \right) \text{ for any } x > 0 \tag{1.7}$$

and any $q > 2$ such that $C_q(X) < \infty$. Therefrom

$$\mathbb{P}(S_n \geq a_q \sigma \sqrt{2 \log z} + b_q C_q(X) z^{1/q}\,) \leq 2/z \text{ for any } z > 1, \tag{1.8}$$

with $a_q = (1 + q/2)e^{q/2}$ and $b_q = 1 + (2/q)$. Fan, Grama and Liu (2017) obtain an extension of (1.7) to the case of martingales (*see* their Corollary 2.5), with the same constants a_q and b_q. In Section 3 of this paper, we will prove the following maximal version of (1.8) with the optimal constant a_q. Let $S_n^* = \max(0, S_1, \ldots, S_n)$: under the assumptions of (1.7),

$$\mathbb{P}\big(S_n^* > \sigma\sqrt{2\log z} + \big(1 + (2/q) + (q/3)\mathbf{1}_{q>3}\big) C_q(X) z^{1/q}\,\big) \leq 1/z \text{ for any } z > 1. \tag{1.9}$$

In the iid case, one can derive immediately the bounded law of the iterated logarithm (with the exact constant) from (1.9), which shows (in spirit) that the constant $a_q = 1$ appearing here cannot be further improved. In Section 4 we give similar inequalities under weak moments conditions. In Section 5 we apply the results of Sections 3 and 4 to get constants in the weak Rosenthal inequalities of Carothers and Dilworth (1988). The results of Sections 3, 4 and 5 are given in the more general setting of martingale differences sequences. Section 2 deals with preliminary results, which are the starting point of this paper.

2 Preliminary results

In this section, we introduce some definitions and technical tools which will be used all along the paper. We start with the definition of the tail function, the quantile function and the integrated quantile function.

Definition 2.1. *Let X be a real-valued random variable. Then the tail function H_X of X is defined by $H_X(t) = \mathbb{P}(X > t)$. The quantile function Q_X of X is the cadlag inverse of H_X (note that Q_X is nonincreasing).*

The basic property of Q_X is: $x < Q_X(u)$ if and only if $H_X(x) > u$. This property ensures that $Q_X(U)$ has the same distribution as X for any random variable U with the uniform distribution over $[0, 1]$.

Definition 2.2. *The integrated quantile function $\tilde{Q}_X$ of the real-valued and integrable random variable X is defined by $\tilde{Q}_X(u) = u^{-1} \int_0^u Q_X(s)ds$ (since Q_X is nonincreasing, $\tilde{Q}_X$ is a nonincreasing function).*

We start by a byproduct of Doob's inequality, which is a reformulation of Lemma 1 in Dubins and Gilat (1978).

Lemma 2.3. *Let $(M_0, M_1, \ldots, M_n)$ be a submartingale in L^1 such that $M_0 \geq 0$ almost surely. Set $M_n^* = \max(M_0, M_1, \ldots, M_n)$. Then $Q_{M_n^*}(u) \leq \tilde{Q}_{M_n}(u)$ for any u in $]0, 1]$.*

Proof. The assumption $M_0 \geq 0$, which seems to be necessary (this assumption ensures that $Q_{M_n^*} \geq 0\,$), is omitted in Dubins and Gilat (1978). Therefore I give a proof below. From the Doob inequality and Lemma 2.1(a) in Rio (2000), for any $x \geq 0$,

$$x\,\mathbb{P}(M_n^* \geq x) \leq \mathbb{E}(M_n \mathbf{1}_{M_n^* \geq x}) \leq \int_0^{\mathbb{P}(M_n^* \geq x)} Q_{M_n}(s)ds.$$

For u in $]0,1]$, let $x = Q_{M_n^*}(u)$. Then $\mathbb{P}(M_n^* \geq x) = \mathbb{P}\big(Q_{M_n^*}(U) \geq Q_{M_n^*}(u)\big) \geq u > 0$. Hence $Q_{M_n^*}(u) \leq \tilde{Q}_{M_n}\big(\mathbb{P}(M_n^* \geq x)\big) \leq \tilde{Q}_{M_n}(u)$, since $\tilde{Q}_{M_n}$ is nonincreasing. $\square$

We now recall some elementary properties of the quantile function and the integrated quantile function. These properties are given and proved in Pinelis (2014).

Proposition 2.4. *Let X and Y be real-valued and integrable random variables. Then, for any u in $]0,1]$,*

(i) $Q_X(u) \leq \tilde{Q}_X(u)$, (ii) $\mathbb{P}(X > Q_X(u)) \leq u$, (iii) $\tilde{Q}_{X+Y}(u) \leq \tilde{Q}_X(u) + \tilde{Q}_Y(u)$.

Let us also recall the variational expression of $\tilde{Q}_X$, which can be found in Rockafellar and Uryasev (2000) or Pinelis (2014).

$$\tilde{Q}_X(u) = \inf\big\{t + u^{-1}\mathbb{E}\big((X-t)_+\big) : t \in \mathbb{R}\,\big\} \text{ for any } u \in]0,1]. \tag{2.1}$$

Consider now a real-valued random variable X with a finite Laplace transform on a right neighborhood of 0. Define ℓ_X by

$$\ell_X(t) = \log \mathbb{E}\big(\exp(tX)\big) \text{ for any } t \geq 0. \tag{2.2}$$

Define the transformation $\mathcal{T}$ on the class Ψ of convex functions $\psi : [0,\infty[\mapsto [0,\infty]$ such that $\psi(0) = 0$ by

$$\mathcal{T}\psi(x) = \inf\big\{t^{-1}(\psi(t) + x) : t \in]0,\infty[\big\} \text{ for any } x \geq 0 \tag{2.3}$$

and the function Q_X^* by

$$Q_X^*(u) = \mathcal{T}\ell_X(\log(1/u)). \text{ for any } u \in]0,1]. \tag{2.4}$$

As noted by Rio (2000, p. 159), $\mathcal{T}\ell_X$ is the inverse function of the Legendre transform of ℓ_X. Furthermore the following properties are valid.

Proposition 2.5. (i) *For any real-valued and integrable random variable X with a finite Laplace transform on a right neighborhood of 0, $\tilde{Q}_X \leq Q_X^*$.* (ii) *$\mathcal{T}$ is subadditive on Ψ.*

Proof. We refer to Pinelis (2014) for a proof of (i). We now prove (ii). Let ψ_0 and ψ_1 be elements of Ψ. It is enough to prove that, for $s > 0$ and $t > 0$, there exists $z > 0$ such that

$$t^{-1}(\psi_0(t) + x) + s^{-1}(\psi_1(s) + x) \geq z^{-1}(\psi_0(z) + \psi_1(z) + x). \tag{2.5}$$

Let $z = st/(s+t)$. From the convexity of the above functions and the facts that $\psi_0(0) = 0$ and $\psi_1(0) = 0$, $\psi_0(z) \leq s\psi_0(t)/(s+t)$ and $\psi_1(z) \leq t\psi_1(s)/(s+t)$, which ensures that (2.5) holds true for this choice of z. Hence $\mathcal{T}$ is subadditive. $\square$

3 Fuk-Nagaev inequalities under strong moments assumptions

Throughout this section, $(M_j)_{0\leq j\leq n}$ is a martingale in L^2 with respect to a nondecreasing filtration $(\mathcal{F}_j)_j$, such that $M_0 = 0$. We set $X_j = M_j - M_{j-1}$ for any positive j. We assume that, for some constant $q > 2$,

$$\|\mathbb{E}(X_j^2 \mid \mathcal{F}_{j-1})\|_\infty < \infty \text{ and } \|\mathbb{E}(X_{j+}^q \mid \mathcal{F}_{j-1})\|_\infty < \infty \text{ for any integer } j \in [1,n]. \tag{3.1}$$

We set

$$\sigma = \Big\|\sum_{j=1}^n \mathbb{E}(X_j^2 \mid \mathcal{F}_{j-1})\Big\|_\infty^{1/2} \text{ and } C_q(M) = \Big\|\sum_{j=1}^n \mathbb{E}(X_{j+}^q \mid \mathcal{F}_{j-1})\Big\|_\infty^{1/q}. \tag{3.2}$$

The increments $X_1, X_2, \ldots, X_n$ are said to be conditionally symmetric if, for any j in $[1,n]$ the conditional law of X_j given $\mathcal{F}_{j-1}$ is symmetric. The main result of this section is Theorem 3.1 below.

Theorem 3.1. *Let $(M_j)_{0\le j\le n}$ be a martingale in L^2 satisfying (3.1), such that $M_0 = 0$. Then*

$$\tilde{Q}_{M_n}(1/z) \le \sigma\sqrt{2\log z} + C_q(M)(\beta_q z^{1/q} + \mathbf{1}_{q>3}(e/3)\log z) \qquad (a)$$

for any $z > 1$, where $\beta_q = 1 + \min(1/5, 1/q) + (1/q)$ and $(C_q(M), \sigma)$ is defined by (3.2). Furthermore, if the increments $X_1, X_2, \ldots, X_n$ are conditionally symmetric, then

$$\tilde{Q}_{M_n}(1/z) \le \sigma\sqrt{2\log z} + C_q(M)(1 + (1/4) + 1/q)z^{1/q} \qquad (b)$$

for any q in $]3, 4]$ and any $z > 1$.

From Theorem 3.1 and Lemma 2.3, we immediately get the corollary below.

Corollary 3.2. *Under the assumptions of Theorem 3.1(a), for any $z > 1$,*

$$\mathbb{P}\big(\max(M_0, M_1, \ldots, M_n) > \sigma\sqrt{2\log z} + C_q(M)\big(\beta_q z^{1/q} + \mathbf{1}_{q>3}(e/3)\log z\,\big)\,\big) \le 1/z.$$

Remark 3.3. Note that $\beta_q \le 1 + (2/q)$. Hence Corollary 3.2 improves (1.8) for any value of z in the case $q \le 3$. If $q > 3$, the elementary inequality $e\log z \le qz^{1/q}$ can be used to replace $(e/3)\log z$ by $(q/3)z^{1/q}$ in the above inequalities, which proves that Corollary 3.2 implies (1.9).

Proof of Theorem 3.1(a). We prove Theorem 3.1(a) in the case $C_q(M) = 1$. The general case follows by dividing the random variables by $C_q(M)$. Let $y = z^{1/q}$. Set

$$\bar{X}_j = \min(X_j, y) \text{ and } \bar{M}_n = \bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_n. \qquad (3.3)$$

From Proposition 2.4(iii) and Proposition 2.5(i),

$$\tilde{Q}_{M_n}(1/z) \le Q^*_{\bar{M}_n}(1/z) + \tilde{Q}_{M_n - \bar{M}_n}(1/z). \qquad (3.4)$$

Next, from (2.1) and the fact that $M_n - \bar{M}_n \ge 0$,

$$\tilde{Q}_{M_n - \bar{M}_n}(1/z) \le z\mathbb{E}(M_n - \bar{M}_n). \qquad (3.5)$$

Let us now bound up $\mathbb{E}(M_n - \bar{M}_n)$. Let $\eta_j = X_j - \bar{X}_j$. Then

$$\mathbb{E}(M_n - \bar{M}_n) = \textstyle\sum_{j=1}^n \mathbb{E}\big(\mathbb{E}(\eta_j \mid \mathcal{F}_{j-1})\big). \qquad (3.6)$$

Now

$$\mathbb{E}(\eta_j \mid \mathcal{F}_{j-1}) = \int_y^\infty \mathbb{P}(X_j > s \mid \mathcal{F}_{j-1})ds \le \frac{1}{qy^{q-1}} \int_y^\infty qs^{q-1}\mathbb{P}(X_j > s \mid \mathcal{F}_{j-1})ds,$$

which ensures that

$$\mathbb{E}(\eta_j \mid \mathcal{F}_{j-1}) \le q^{-1}y^{1-q}\mathbb{E}(X_{j+}^q \mid \mathcal{F}_{j-1}). \qquad (3.7)$$

Hence

$$\mathbb{E}(M_n - \bar{M}_n) \le q^{-1}y^{1-q}\mathbb{E}\big(\textstyle\sum_{j=1}^n \mathbb{E}(X_{j+}^q \mid \mathcal{F}_{j-1})\,\big) \le q^{-1}y^{1-q}, \qquad (3.8)$$

since $\sum_{j=1}^n \mathbb{E}(X_{j+}^q \mid \mathcal{F}_{j-1}) \le 1$ almost surely. Combining (3.5) and (3.8), we then get that

$$\tilde{Q}_{M_n - \bar{M}_n}(1/z) \le q^{-1}y^{1-q}z = q^{-1}z^{1/q}. \qquad (3.9)$$

In view of (3.4) and (3.9), it remains to prove that

$$Q^*_{\bar{M}_n}(1/z) \le \sigma\sqrt{2\log z} + (\min(1/5, 1/q) + 1)z^{1/q} + \mathbf{1}_{q>3}(e/3)\log z \text{ for any } z > 1. \qquad (3.10)$$

In order to prove (3.10), we will bound the Laplace transform of $\bar{M}_n$ via the lemma below.

Lemma 3.4. *Let $Z_1, Z_2, \dots, Z_n$ be a finite sequence of random variables with finite variances, adapted to a nondecreasing filtration $(\mathcal{F}_j)_j$. Suppose furthermore that, for any j in $[1,n]$, $\mathbb{E}(Z_j \mid \mathcal{F}_{j-1}) \le 0$ almost surely. Let $T_n = Z_1 + Z_2 + \cdots + Z_n$. Then, for any positive t, $\log \mathbb{E}\big(e^{tT_n}\big) \le \ell(t)$, where*

$$\ell(t) = \Big\| \sum_{j=1}^n \mathbb{E}(Z_j^2 \mid \mathcal{F}_{j-1}) \Big\|_\infty \frac{t^2}{2} + \sum_{k=3}^\infty \Big\| \sum_{j=1}^n \mathbb{E}(Z_{j+}^k \mid \mathcal{F}_{j-1}) \Big\|_\infty \frac{t^k}{k!}.$$

Proof of Lemma 3.4. From the elementary inequality $e^x \le 1 + x + (x^2/2) + \sum_{k\ge 3}(x_+^k/k!)$, we infer that, for any positive t,

$$\begin{aligned}\mathbb{E}\big(e^{tZ_j} \mid \mathcal{F}_{j-1}\big) &\le 1 + \mathbb{E}(Z_j \mid \mathcal{F}_{j-1})\, t + \mathbb{E}(Z_j^2 \mid \mathcal{F}_{j-1}) \frac{t^2}{2} + \sum_{k=3}^\infty \mathbb{E}(Z_{j+}^k \mid \mathcal{F}_{j-1}) \frac{t^k}{k!} \\ &\le \exp\bigg(\mathbb{E}(Z_j^2 \mid \mathcal{F}_{j-1}) \frac{t^2}{2} + \sum_{k=3}^\infty \mathbb{E}(Z_{j+}^k \mid \mathcal{F}_{j-1}) \frac{t^k}{k!} \bigg) \text{ a.s.} \qquad (3.11)\end{aligned}$$

Define now the random variables $W_j(t)$ by $W_0(t) = 1$ and

$$W_j(t) = W_{j-1}(t) \exp\bigg(tZ_j - \mathbb{E}(Z_j^2 \mid \mathcal{F}_{j-1}) \frac{t^2}{2} - \sum_{k=3}^\infty \mathbb{E}(Z_{j+}^k \mid \mathcal{F}_{j-1}) \frac{t^k}{k!} \bigg) \text{ for } j \in [1,n].$$

Then, from (3.11) $(W_j(t))_{0\le j\le n}$ is a positive supermartingale adapted to $(\mathcal{F}_j)_{0\le j\le n}$, which ensures that $\mathbb{E}(W_n(t)) \le \mathbb{E}(W_0(t)) = 1$. Since $W_n(t)\exp(\ell(t)) \ge \exp(tT_n)$ almost surely, it implies Lemma 3.4. ☐

We now apply Lemma 3.4 to the random variables $\bar X_1, \bar X_2, \dots, \bar X_n$. Noticing that $\bar X_j \le X_j$, which ensures that $\mathbb{E}(\bar X_j \mid \mathcal{F}_{j-1}) \le 0$ and that $\bar X_j^2 \le X_j^2$, which implies that $\big\| \sum_{j=1}^n \mathbb{E}(\bar X_j^2 \mid \mathcal{F}_{j-1}) \big\|_\infty \le \sigma^2$, we thus get that

$$\log \mathbb{E}\big(e^{t\bar M_n}\big) \le \sigma^2 t^2/2 + \textstyle\sum_{k=3}^\infty \gamma_k\, t^k/k!, \quad \text{where} \quad \gamma_k = \big\| \sum_{j=1}^n \mathbb{E}(\bar X_{j+}^k \mid \mathcal{F}_{j-1}) \big\|_\infty. \qquad (3.12)$$

Usually the coefficients γ_k are bounded up by $\sigma^2 y^{k-2}$. However this upper bound does not take into accounts the assumption on the moments of order q. Here we need the more precise upper bound below.

Proposition 3.5. *Let $Z_1, Z_2, \dots, Z_n$ be a finite sequence of random variables, adapted to a nondecreasing filtration $(\mathcal{F}_j)_j$. Suppose furthermore that $\max(Z_1, Z_2, \dots, Z_n) \le c$ a.s. for some positive c and that*

$$\Big\| \sum_{j=1}^n \mathbb{E}(Z_{j+}^2 \mid \mathcal{F}_{j-1}) \Big\|_\infty \le V \ \text{ and } \ \Big\| \sum_{j=1}^n \mathbb{E}(Z_{j+}^q \mid \mathcal{F}_{j-1}) \Big\|_\infty \le 1.$$

Then, first $\sum_{j=1}^n \mathbb{E}(Z_{j+}^k \mid \mathcal{F}_{j-1}) \le V^{(q-k)/(q-2)}$ almost surely, for any real $k \in [2,q]$ and second $\sum_{j=1}^n \mathbb{E}(Z_{j+}^k \mid \mathcal{F}_{j-1}) \le c^{k-q}$ almost surely, for any real $k \ge q$.

Proof of Proposition 3.5. Noting that $Z_{j+}^k \le Z_{j+}^q c^{k-q}$ for any $k \ge q$, one immediately gets the second assertion. We now prove the first assertion. From the convexity of the exponential function, $(q-2)(Z_{j+}/a)^{k-2} \le (k-2)(Z_{j+}/a)^{q-2} + (q-k)$ for any k in $[2,q]$ and any positive a. Multiplying this inequality by $a^{k-2} Z_{j+}^2$, we get that

$$(q-2) Z_{j+}^k \le (k-2) a^{k-q} Z_{j+}^q + (q-k) a^{k-2} Z_{j+}^2.$$

Taking the conditional expectation with respect to $\mathcal{F}_{j-1}$ and summing then on j, we infer that

$$(q-2)\sum_{j=1}^{n}\mathbb{E}(Z_{j+}^{k}\mid\mathcal{F}_{j-1})\leq(k-2)a^{k-q}\sum_{j=1}^{n}\mathbb{E}(Z_{j+}^{q}\mid\mathcal{F}_{j-1})+(q-k)a^{k-2}\sum_{j=1}^{n}\mathbb{E}(Z_{j+}^{2}\mid\mathcal{F}_{j-1})$$
$$\leq(k-2)a^{k-q}+(q-k)a^{k-2}V\ \text{ a.s.}$$

Choosing $a=V^{-1/(q-2)}$ in this inequality, we then get the first part of Proposition 3.5. □

Let $\psi_q(s)=\sum_{k\geq q}(s^k/\,k!\,)$. Define

$$v=\sigma^2,\ \ell_0(t)=v\,\frac{t^2}{2},\ \ell_1(t)=\sum_{2<k<q}v^{(q-k)/(q-2)}\,\frac{t^k}{k!}\ \text{ and }\ell_2(t)=y^{-q}\,\psi_q(yt). \tag{3.13}$$

From (3.12) and Proposition 3.5 applied to $Z_j=\bar{X}_j$ with $c=y$ and $V=\sigma^2$,

$$\log\mathbb{E}\big(e^{t\bar{M}_n}\big)\leq\ell_0(t)+\ell_1(t)+\ell_2(t)\ \text{ for any } t>0. \tag{3.14}$$

Consequently, from Proposition 2.5(ii) and (2.4),

$$Q^*_{\bar{M}_n}(1/z)\leq\min\big(\mathcal{T}\ell_0(x)+\mathcal{T}(\ell_1+\ell_2)(x),\mathcal{T}(\ell_0+\ell_1)(x)+\mathcal{T}\ell_2(x)\big),\ \text{ where } x=\log z. \tag{3.15}$$

Let us bound up $\mathcal{T}\ell_2(x)$. Choosing $t=(x/y)$ in (2.3) yields

$$\mathcal{T}\ell_2(x)\leq y+y^{1-q}x^{-1}\psi_q(x)=y+ye^{-x}x^{-1}\psi_q(x), \tag{3.16}$$

since $y^q=z=e^x$. Now the function $x\mapsto e^{-x}x^{-1}\psi_q(x)$ is uniformly bounded on $]0,\infty[$, as shown by the lemma below.

Lemma 3.6. *For any $q>2$ and any positive x, $e^{-x}x^{-1}\psi_q(x)\leq\min(1/q,1/5)$.*

Proof of Lemma 3.6. $\psi_q(x)=\sum_{k\geq q}(x^k/k!)\leq(x/q)\sum_{j\geq q-1}(x^j/j!)\leq xe^x/q$, which gives the first bound. Now $e^{-x}x^{-1}\psi_q(x)\leq x^{-1}e^{-x}(e^x-1-x-x^2/2):=g(x)$ for any $q>2$. Let us bound the maximum of g: the function g is increasing on $[0,x_0]$ and decreasing on $[x_0,\infty[$, where x_0 is the unique positive solution of the equation $e^x-1-x-(x^2/2)=(x^3/2)$. Consequently $\sup_{x>0}g(x)=x_0^2e^{-x_0}/2$. Now one can prove that $x_0\geq x_1=3.35$. Since $x\mapsto x^2e^{-x}$ is decreasing on $]2,\infty[$, it implies that $x_0^2e^{-x_0}\leq x_1^2e^{-x_1}=0.394$. Hence $\sup_{x>0}g(x)\leq 1/5$, which completes the proof of Lemma 3.6. □

From (3.16) and Lemma 3.6,

$$\mathcal{T}\ell_2(x)\leq y+(y/x)\ell_2(x/y)\leq\alpha_q y,\ \text{ where } \alpha_q=1+\min(1/q,1/5). \tag{3.17}$$

Proof of (3.10) for $q\leq 3$. Then $\ell_1=0$. Furthermore, an elementary calculation gives

$$\mathcal{T}\ell_0(x)=\sigma\sqrt{2x}=\sigma\sqrt{2\log z}. \tag{3.18}$$

Then (3.10) follows from (3.15), (3.17) and (3.18).
Proof of (3.10) for $q>3$. Applying (2.3) to $\ell_1+\ell_2$ with $t=x/y$, we get that

$$\mathcal{T}(\ell_1+\ell_2)(x)\leq y+(y/x)\ell_2(x/y)+(y/x)\ell_1(x/y)\leq\alpha_q y+(y/x)\ell_1(x/y).$$

Now recall that $y=z^{1/q}=e^{x/q}$. Hence $(x/qy)\leq\sup_{s>0}se^{-s}=(1/e)$ or, equivalently,

$$(x/y)\leq(q/e). \tag{3.19}$$

Now $t \mapsto t^{-2}\ell_1(t)$ is increasing. Thus, using (3.19), $(y/x)\ell_1(x/y) \leq (x/y)(e/q)^2\ell_1(q/e)$. Since $y \geq 1$ and $q \geq 3$, it follows from the above inequalities that

$$\mathcal{T}(\ell_1 + \ell_2)(x) \leq \alpha_q y + (ex/3)\ell_1(q/e). \tag{3.20}$$

Next

$$\ell_0(t) + \ell_1(t) \leq vt^2 \sum_{k \geq 2} v^{(2-k)/(q-2)} \frac{t^{k-2}}{k!} \leq \frac{vt^2}{2(1 - v^{-1/(q-2)}t/3)}\,.$$

From the above bound and Inequality (2.17), page 30 in Bercu, Delyon and Rio (2015),

$$\mathcal{T}(\ell_0 + \ell_1)(x) \leq \sigma\sqrt{2x} + v^{-1/(q-2)}(x/3). \tag{3.21}$$

Putting together the inequalities (3.15), (3.17), (3.18) and (3.20), we then get that

$$Q^*_{\bar{M}_n}(1/z) \leq \sigma\sqrt{2x} + \alpha_q y + (x/3)\min\big(e\ell_1(q/e), v^{-1/(q-2)}\,\big), \tag{3.22}$$

where $x = \log z,\ y = z^{1/q}$. It remains to prove that

$$\min\big(e\ell_1(q/e), v^{-1/(q-2)}\,\big) \leq e. \tag{3.23}$$

If $v^{-1/(q-2)} \leq e$, (3.23) is trivial. Otherwise $v^{1/(q-2)} < (1/e)$, which ensures that $v^{(q-k)/(q-2)} \leq e^{k-q}$ for any k in $]2, q[$. Then $\ell_1(q/e) \leq e^{-q}\sum_{2<k<q}(q^k/k!) \leq 1$, which implies (3.23). Hence (3.10) holds true, which completes the proof of Theorem 3.1(a). □

Proof of Theorem 3.1(b). It is enough to prove Theorem 3.1(b) in the case $C_q(M) = 1$. Set $y = z^{1/q}$ and define the random variables $\bar{X}_j$ by $\bar{X}_j = \max(-y, \min(X_j, y))$ for j in $[1, n]$. Set $\bar{M}_n = \bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_n$. Then

$$M_n - \bar{M}_n = \textstyle\sum_{j=1}^n (X_j - \bar{X}_j) \leq \sum_{j=1}^n (X_{j+} - y)_+.$$

Now (3.4) is still valid, and applying (3.5)–(3.8) to the term on right hand (instead of $M_n - \bar{M}_n$), we find that (3.9) is also still valid. Consequently it only remains to prove that

$$Q^*_{\bar{M}_n}(1/z) \leq \sigma\sqrt{2\log z} + (5/4)z^{1/q} \ \text{ for any } z > 1. \tag{3.24}$$

Since the increments $\bar{X}_j$ are conditionally symmetric, the conditional moments of order $2k+1$ vanish. Hence, similarly to (3.11), we have

$$\mathbb{E}\big(e^{t\bar{X}_j} \mid \mathcal{F}_{j-1}\big) \leq \exp\bigg(\mathbb{E}(X_j^2 \mid \mathcal{F}_{j-1})\,\frac{t^2}{2} \,+\, 2\sum_{k=2}^{\infty} \mathbb{E}(\bar{X}_{j+}^{2k} \mid \mathcal{F}_{j-1})\,\frac{t^{2k}}{(2k)!}\bigg) \ \text{ a.s.} \tag{3.25}$$

Now, proceeding as in the proof of Lemma 3.4, we get that, for any $t > 0$,

$$\log\mathbb{E}\big(e^{t\bar{M}_n}\big) \leq \sigma^2\frac{t^2}{2} \,+ 2\sum_{k=2}^{\infty}\delta_k\,\frac{t^{2k}}{(2k)!}\,, \ \text{ where } \ \delta_k = \Big\|\sum_{j=1}^n \mathbb{E}(\bar{X}_{j+}^{2k} \mid \mathcal{F}_{j-1})\Big\|_\infty. \tag{3.26}$$

Applying then Proposition 3.5 to the random variables $\bar{X}_j$,

$$\log\mathbb{E}\big(e^{t\bar{M}_n}\big) \leq \ell_0(t) + \ell_2(t) \ \text{ where } \ell_0(t) = \sigma^2\frac{t^2}{2} \ \text{ and } \ \ell_2(t) = \frac{2}{z}\sum_{k=2}^{\infty}\frac{(ty)^{2k}}{(2k)!}. \tag{3.27}$$

Recall that $\mathcal{T}\ell_0(x) = \sigma\sqrt{2x}$. Hence, from (2.4), (3.27) and the subadditivity of $\mathcal{T}$,

$$Q^*_{\bar{M}_n}(1/z) \leq \sigma\sqrt{2x} + \mathcal{T}\ell_2(x), \ \text{ where } x = \log z. \tag{3.28}$$

Next, applying (3.3) with $t = (x/y)$ and noticing that $z = e^x$,

$$\mathcal{T}\ell_2(x) \leq y + (y/x)\ell_2(x/y) = y + 2(y/x)e^{-x}(\cosh(x) - 1 - x^2/2).$$

Now $x^{-1}(\cosh(x) - 1 - x^2/2) = \sum_{k=2}^{\infty} x^{2k-1}/(2k)! \leq \sinh(x)/4$, which ensures that

$$2(y/x)e^{-x}(\cosh(x) - 1 - x^2/2) \leq ye^{-x}\sinh(x)/2 \leq (y/4).$$

Hence $\mathcal{T}\ell_2(x) \leq (5y/4)$, which, together with (3.28), implies (3.24). □

4 Fuk-Nagaev inequalities under weak moments assumptions

Throughout this section, $(M_j)_{0\le j\le n}$ is a martingale in L^2 with respect to a nondecreasing filtration $(\mathcal{F}_j)_j$, such that $M_0 = 0$. We set $X_j = M_j - M_{j-1}$ for any positive j. We assume that, for some constant $r > 2$,

$$\|\mathbb{E}(X_j^2 \mid \mathcal{F}_{j-1})\|_\infty < \infty \ \text{ and } \ \|\sup_{t>0}\big(t^r\, \mathbb{P}(X_{j+} > t \mid \mathcal{F}_{j-1})\big)\|_\infty < \infty \tag{4.1}$$

for any j in $[1, n]$. We set

$$\sigma = \Big\| \sum_{j=1}^n \mathbb{E}(X_j^2 \mid \mathcal{F}_{j-1})\Big\|_\infty^{1/2} \ \text{ and } \ C_r^w(M) = \Big\| \sup_{t>0}\Big(t^r \sum_{j=1}^n \mathbb{P}(X_{j+} > t \mid \mathcal{F}_{j-1})\Big)\Big\|_\infty^{1/r} \tag{4.2}$$

(the letter w in $C_r^w(M)$ means weak). Let us now state our main result.

Theorem 4.1. *Let $(M_j)_{0\le j\le n}$ be a martingale in L^2 satisfying (4.1), such that $M_0 = 0$. Then, for any $z > 1$,*

$$\tilde{Q}_{M_n}(1/z) \le \sigma\sqrt{2\log z} + C_r^w(M)\mu_r z^{1/r},$$

where $\mu_r = 2 + \max(4/3, r/3)$ and $(C_r^w(M), \sigma)$ is defined by (4.2).

From Theorem 4.1 and Lemma 2.3, we immediately get the corollary below.

Corollary 4.2. *Under the assumptions of Theorem 4.1, for any $z > 1$,*

$$\mathbb{P}\big(\max(M_0, M_1, \ldots, M_n) > \sigma\sqrt{2\log z} + C_r^w(M)\mu_r z^{1/r}\big) \le 1/z.$$

Remark 4.3. From the Markov inequality $C_r^w(M) \le C_r(M)$. The constant μ_r appearing here can be improved. Nevertheless $\mu_r \le 10/3$ for any r in $]2, 4]$, which shows that Corollary 4.2 is suitable for numerical applications.

Proof of Theorem 4.1. As in Section 3, it is enough to prove Theorem 3.1 in the case $C_r^w(M) = 1$. Let then $y = z^{1/r}$. Set

$$\bar{X}_j = \min(X_j, y) \ \text{ and } \ \bar{M}_n = \bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_n. \tag{4.3}$$

The upper bounds (3.4) and (3.5) are still valid. Let us now bound up $\mathbb{E}(M_n - \bar{M}_n)$. Let $\eta_j = X_j - \bar{X}_j$. Then

$$\mathbb{E}(M_n - \bar{M}_n) = \mathbb{E}\big(\textstyle\sum_{j=1}^n \mathbb{E}(\eta_j \mid \mathcal{F}_{j-1})\big). \tag{4.4}$$

Let

$$A_r = \sup_{t>0}\big(t^r \textstyle\sum_{j=1}^n \mathbb{P}(X_{j+} > t \mid \mathcal{F}_{j-1})\big). \tag{4.5}$$

If $C_r^w(M) = 1$, then $A_r \le 1$ a.s., whence

$$\sum_{j=1}^n \mathbb{E}(\eta_j \mid \mathcal{F}_{j-1}) = \int_y^\infty \Big(\sum_{j=1}^n \mathbb{P}(X_j > s \mid \mathcal{F}_{j-1})\Big)ds \le A_r \int_y^\infty s^{-r} ds \le \frac{y^{1-r}}{r-1} \quad \text{a.s.}$$

It follows that $\mathbb{E}(M_n - \bar{M}_n) \le (r-1)^{-1} y^{1-r}$. Applying now (3.5), we get that

$$\tilde{Q}_{M_n - \bar{M}_n}(1/z) \le (r-1)^{-1} y^{1-r} z = (r-1)^{-1} z^{1/r}. \tag{4.6}$$

In view of (3.4) and (4.6), it remains to prove that

$$Q^*_{\bar{M}_n}(1/z) \le \sigma\sqrt{2\log z} + \big(2 + \max(4/3, r/3) - 1/(r-1)\big) z^{1/r} \ \text{ for any } z > 1. \tag{4.7}$$

We now prove (4.7). As in Dedecker, Gouëzel and Merlevède (2016), we will apply inequalities involving strong moments of order $q > r$ to the variables $\bar{X}_n$. We will choose q in $\mathbb{N}$. Let k be any integer such that $k > r$. We start by bounding up $C_k(\bar{M})$:

$$\sum_{j=1}^{n} \mathbb{E}(\bar{X}_{j+}^k \mid \mathcal{F}_{j-1}) = \int_0^y ks^{k-1}\Big(\sum_{j=1}^{n} \mathbb{P}(X_{j+} > s \mid \mathcal{F}_{j-1})\Big)ds \le A_r \int_0^y ks^{k-1-r}ds,$$

where A_r is defined by (4.5). Since $A_r \le 1$ a.s., it follows that

$$\Big\| \sum_{j=1}^{n} \mathbb{E}(\bar{X}_{j+}^k \mid \mathcal{F}_{j-1}) \Big\|_\infty \le \frac{k\, y^{k-r}}{k-r} \quad \text{for any } k > q. \tag{4.8}$$

From the above upper bound, for any integer $q > r$,

$$\sum_{k \ge q+1} \Big\| \sum_{j=1}^{n} \mathbb{E}(\bar{X}_{j+}^k \mid \mathcal{F}_{j-1}) \Big\|_\infty \frac{t^k}{k!} \le \ell_2(t), \ \text{ where } \ell_2(t) = \frac{y^{1-r}t}{q+1-r} \sum_{m \ge q} \frac{(ty)^m}{m!}\,. \tag{4.9}$$

Define now the positive real C_q and $Z_1, Z_2, \ldots, Z_n$ by

$$C_q = \big(qy^{q-r}/(q-r)\big)^{1/q} \ \text{ and } Z_j = C_q^{-1}\bar{X}_j \ \text{ for any } j \in [1, n]. \tag{4.10}$$

Then the random variables $Z_1, Z_2, \ldots, Z_n$ fulfill the conditions of Proposition 3.5 with $V = (\sigma/C_q)^2$ and $c = (y/C_q)$. Applying Proposition 3.5 to $Z_1, Z_2, \ldots, Z_n$ for k in $[2, q]$ and multiplying by C_q^k, we then get that

$$\Big\| \sum_{j=1}^{n} \mathbb{E}(\bar{X}_{j+}^k \mid \mathcal{F}_{j-1}) \Big\|_\infty \le \sigma^2 \big(C_q^q/\sigma^2\big)^{(k-2)/(q-2)} \text{for any integer } k \in [2, q]. \tag{4.11}$$

From the above upper bound, for any integer $q > r$,

$$\sum_{k=2}^{q} \Big\| \sum_{j=1}^{n} \mathbb{E}(\bar{X}_{j+}^k \mid \mathcal{F}_{j-1}) \Big\|_\infty \frac{t^k}{k!} \le \ell_{01}(t), \ \text{ where } \ell_{01}(t) = \sigma^2 \sum_{k=2}^{q} \Big(\frac{C_q^q}{\sigma^2}\Big)^{\frac{k-2}{q-2}} \frac{t^k}{k!}\,. \tag{4.12}$$

Applying now Lemma 3.4, we get that

$$\log \mathbb{E}\big(e^{t\bar{M}_n}\big) \le \ell(t) := \ell_{01}(t) + \ell_2(t) \ \text{ for any } t > 0. \tag{4.13}$$

Hence, by Proposition 2.5(ii),

$$Q^*_{\bar{M}_n}(1/z) \le \mathcal{T}\ell(x) \le \mathcal{T}\ell_{01}(x) + \mathcal{T}\ell_2(x), \ \text{ where } x = \log z. \tag{4.14}$$

Now, on the one hand, proceeding exactly as in Section 3 (*see* in particular (3.21), page 7),

$$\mathcal{T}\ell_{01}(x) \le \sigma\sqrt{2x} + (C_q^q/\sigma^2)^{1/(q-2)}(x/3) \tag{4.15}$$

and on the other hand, choosing $t = (x/y)$ in (2.3),

$$\mathcal{T}\ell_2(x) \le y + (y/x)\ell_2(x/y) = y + \frac{ye^{-x}}{(q+1-r)} \sum_{m \ge q} \frac{x^m}{m!} \le y + y/(q+1-r). \tag{4.16}$$

If $(C_q^q/\sigma^2)^{1/(q-2)} \le (q-1)y/x$, from (4.14) and the two above ineqalities,

$$Q^*_{\bar{M}_n}(1/z) \le \sigma\sqrt{2x} + y(1/(q+1-r) + (q+2)/3) \ \text{ where } x = \log z \text{ and } y = z^{1/r}. \tag{4.17}$$

Otherwise $(\sigma^2/C_q^q)^{1/(q-2)} < x/(qy-y)$, whence

$$\sigma^2(C_q^q/\sigma^2)^{(k-2)/(q-2)} = C_q^q(\sigma^2/C_q^q)^{(q-k)/(q-2)} \leq C_q^q\big(\,(q-1)y/x\,\big)^{k-q}.$$

Then, using the definition (4.10) of C_q,

$$(y/x)\ell_{01}(x/y) \leq y\,\frac{C_q^q y^{-q}x^{q-1}}{(q-1)^q}\,\sum_{k=2}^{q}\frac{(q-1)^k}{k!} \leq y\left(\frac{qx^{q-1}y^{-r}}{q-r}\right)\frac{e^{q-1}}{(q-1)^q}. \tag{4.18}$$

Now $x^{q-1}y^{-r} = x^{q-1}e^{-x} \leq (q-1)^{q-1}e^{1-q}$ and, consequently,

$$(y/x)\ell_{01}(x/y) \leq yq/(q-1)(q-r). \tag{4.19}$$

Applying now (2.3) with $t = (x/y)$ to ℓ and using (4.16) and (4.19), we get that

$$\mathcal{T}\ell(x) \leq y(1 + 1/(q+1-r) + q/(q-1)(q-r)). \tag{4.20}$$

Finally, from (4.14), (4.17) and (4.20),

$$Q^*_{\tilde M_n}(1/z) \leq \sigma\sqrt{2\log z} + z^{1/r}\delta_r, \tag{4.21}$$

where $\delta_r = \big(1 + 1/(q+1-r) + \max\big((q-1)/3, q/(q-1)(q-r)\big)\,\big)$.

In view of (4.21), it remains to prove that

$$\delta_r + 1/(r-1) \leq 2 + \max(4/3, r/3). \tag{4.22}$$

In order to prove this inequality, we separate three cases. For r in $]2, 8/3]$, set $q = 4$. Then $(q-1)/3 = 1 \geq q/(q-1)(q-r)$ and $\delta_r + 1/(r-1) = 2 + 1/(5-r) + 1/(r-1) \leq 10/3$ for any r in $]2, 8/3]$, which implies (4.7).

For r in $]8/3, 4]$, set $q = 5$. Then $(q-1)/3 = 4/3 \geq q/(q-1)(q-r)$. Consequently

$$\delta_r + 1/(r-1) \leq (7/3) + 1/(6-r) + 1/(r-1) \leq 10/3$$

for any r in $]8/3, 4]$, which implies (4.22).

If $r \geq 4$, we choose the integer $q = q_r$ such that $q_r - 1 < r + 1 \leq q_r$. Noticing that $(q-1)/3 \geq q/(q-1)(q-r)$ for any $r \geq 4$, we get that

$$\delta_r + 1/(r-1) \leq 1 + 1/(q+1-r) + (q-1)/3 + 1/(r-1) \leq 1 + 1/(q+1-r) + q/3$$

for $r \geq 4$. Set $s = q - r$. Then $1/(q+1-r) + q/3 = 1/(s+1) + s/3 + r/3 \leq 1 + r/3$, since s lies in $[1, 2]$. Hence $\delta_r + 1/(r-1) \leq 2 + (r/3)$ for any $r \geq 4$, which completes the proof of (4.22). Finally (4.7) holds true, whence Theorem 4.1. □

5 Upper bounds for weak norms of martingales

In this section, we apply the results of Sections 3 and 4 to weak norms of martingales. Let X be a real-valued and integrable random variable. For $r \geq 1$, let

$$\Lambda_r^+(X) = \sup_{t>0} t\,(\mathbb{P}(X > t))^{1/r} \text{ and } \Lambda_r(X) = \max(\Lambda_r^+(X), \Lambda_r^+(-X)\,). \tag{5.1}$$

Then Λ_r is a quasi-norm on the space weak-L^r of real-valued random variables X such that $\Lambda_r(|X|) < \infty$. From the properties of Q_X given in Section 2, one can easily get the well-known equalities

$$\Lambda_r^+(X) = \sup_{u\in]0,1]} u^{1/r}Q_X(u) \text{ and } \Lambda_r(X) = \max(\Lambda_r^+(X), \Lambda_r^+(-X)). \tag{5.2}$$

Define now, for $r > 1$,

$$\tilde{\Lambda}_r^+(X) = \sup_{u \in]0,1]} u^{(1/r)-1} \int_0^u Q_X(s)ds \text{ and } \tilde{\Lambda}_r(X) = \max(\tilde{\Lambda}_r^+(X), \tilde{\Lambda}_r^+(-X)). \tag{5.3}$$

Elementary arguments show that

$$\Lambda_r^+(X) \le \tilde{\Lambda}_r^+(X) \le \left(\tfrac{r}{r-1}\right)\Lambda_r^+(X) \text{ and } \Lambda_r(X) \le \tilde{\Lambda}_r(X) \le \left(\tfrac{r}{r-1}\right)\Lambda_r(X). \tag{5.4}$$

Furthermore, from Proposition 2.4(iii), $\tilde{\Lambda}_r^+$ and $\tilde{\Lambda}_r$ satisfy the triangle inequality. It follows that $\tilde{\Lambda}_r$ is a norm on the space weak-L^r.

Let us now recall the extension of Rosenthal's inequalities to weak-L^r spaces. Let $X_1, X_2, \dots, X_n$ be a finite sequence of independent centered random variables. Let $r > 2$. Suppose that the random variables belong to the space weak-L^r. Set $M_0 = 0$ and $M_n = X_1 + X_2 + \cdots + X_n$. Then, according to Theorem 2.2 in Carothers and Dilworth (1988), there exist positive constants a_r and b_r such that

$$\Lambda_r(|M_n|) \le a_r\sigma + b_r C_r^w \text{ where } \sigma^2 = \operatorname{Var} M_n \text{ and } C_r^w = \sup_{t>0} t\Big(\sum_{k=1}^n \mathbb{P}(|X_k| > t)\Big)^{1/r}. \tag{5.5}$$

From Theorem 4.1, we get the following constants in the maximal version of this inequality.

Theorem 5.1. *Let $(M_j)_{0 \le j \le n}$ be a martingale in L^2 satisfying (4.1), such that $M_0 = 0$. Set $M_n^* = \max(M_0, M_1, \dots, M_n)$. Then, for any real $r > 2$,*

$$\Lambda_r(M_n^*) \le \tilde{\Lambda}_r^+(M_n) \le \sigma\sqrt{(r/e)} + C_r^w(M)\mu_r \tag{a}$$

and

$$\tilde{\Lambda}_r(M_n) \le \sigma\sqrt{(r/e)} + \max(C_r^w(M), C_r^w(-M))\,\mu_r, \tag{b}$$

where $\mu_r = 2 + \max(4/3, r/3)$ and $(C_r^w(M), \sigma)$ is defined by (4.2).

Proof. (b) follows immediately from (a) applied to M_n and $-M_n$. Let us prove (a). By Theorem 4.1,

$$\tilde{\Lambda}_r^+(M_n) \le \sigma \sup_{z \ge 1}\big(z^{-1/r}\sqrt{2\log z}\,\big) + C_r^w(M)\mu_r. \tag{5.6}$$

Let $s = (2/r)\log z$. Then $z^{-1/r}\sqrt{2\log z} = \sqrt{rs\exp(-s)} \le \sqrt{(r/e)}$, which implies the right hand side of (a). Now $M_n^* \ge 0$. Therefrom $\Lambda_r(M_n^*) = \Lambda_r(|M_n^*|) = \Lambda_r^+(M_n^*)$. Now, using Lemma 2.3, we get that $\Lambda_r^+(M_n^*) \le \tilde{\Lambda}_r^+(M_n)$, which implies the left hand side of (a). □

To conclude this paper, we now compare the weak norms estimates that can be derived from Theorem 3.1 with weak norms estimates derived from Rosenthal's inequalities. Assume that the increments $X_1, X_2, \dots, X_n$ are independent and symmetric. Then, starting from Theorem 3.1 and proceeding exactly as in the above proof, one obtains that, for q in $]2, 4]$,

$$\Lambda_q(M_n^*) \le \sigma\sqrt{(q/e)} + \zeta_q C_q(M), \tag{5.7}$$

where $\zeta_q = (6/5) + (1/q)$ for q in $]2, 3]$ and $\zeta_q = (5/4) + (1/q)$ for q in $]3, 4]$. Now let Y be a random variable with Gaussian law $\mathcal{N}(0, 1)$. For q in $]2, 4]$, by Theorems 6.1 and 7.1 in Figiel et al. (1997),

$$\mathbb{E}(|M_n|^q) \le \sigma^q\mathbb{E}(|Y|^q) + \textstyle\sum_{j=1}^n \mathbb{E}\big(|X_j|^q\big). \tag{5.8}$$

From the Lévy symmetrization inequality, $\mathbb{P}(M_n^* > x) \le \mathbb{P}(|M_n| > x)$. Hence

$$\|M_n^*\|_q^q \le \mathbb{E}\big(|M_n|^q\big) \le \sigma^q\mathbb{E}(|Y|^q) + 2\textstyle\sum_{j=1}^n \mathbb{E}\big(X_{j+}^q\big). \tag{5.9}$$

Now, by the Markov inequality $\Lambda_q(M_n^*) \leq \|M_n^*\|_q$. Consequently

$$\Lambda_q(M_n^*) \leq \sigma \|Y\|_q + 2^{1/q} C_q(M). \tag{5.10}$$

Using the Stirling formula, one can prove that, for any $q > 2$,

$$\|Y\|_q^q = \pi^{-1/2} 2^{q/2} \Gamma((q+1)/2) \geq (q/e)^{q/2} (1 - 1/q)^{q/2} \sqrt{2e} \geq (q/e)^{q/2} \sqrt{e/2}.$$

Hence $\|Y\|_q > \sqrt{(q/e)}$. It follows that, for independent and identically distributed symmetric random variables in L^q, (5.7) is more efficient for large values of n. Note however that $2^{1/q} < \zeta_q$, so that one cannot compare (5.7) and (5.10) in the general case.

Remark 5.2. For q in $]2,3]$, from Theorem 3.1 in Section 3, (5.7) holds without the symmetry condition. For q in $]2,3]$, (5.8) also holds true without the symmetry condition, thanks to Theorem 5.1 in Pinelis (2015). However one cannot derive (5.10) from (5.8) in the nonsymmetric case.

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