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A heat flow approach to the Godbillon-Vey class*

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Abstract

We give a heat flow derivation for the Godbillon Vey class. In particular we prove that if (M, g) is a compact Riemannian manifold with a codimension 1 foliation $\mathcal{F}$, defined by an integrable 1-form ω such that $||\omega|| = 1$, then the Godbillon-Vey class can be written as $[-\mathcal{A}\omega \wedge d\omega]_{dR}$ for an operator $\mathcal{A} : \Omega^*(M) \to \Omega^*(M)$ induced by the heat flow.

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1 Introduction

Let (M, g) be a compact Riemannian manifold with a codimension 1 foliation $\mathcal{F}$ defined by an integrable 1-form ω on M, this is $\ker(\omega) = T\mathcal{F}$. The integrability of ω guarantees the existence of a 1-form η such that $d\omega = \eta \wedge \omega$. In [4] Godbillon and Vey proved that the 3-form $\eta \wedge d\eta$ defines a cohomology class $gv(\mathcal{F}) \in H^3_{dR}(M)$ that depends only on $\mathcal{F}$. Since then, many studies and aproaches had been given in order to interpret this class (see, for example, the work of S. Hurder [5] and the references therein for a good account of it).

The main purpouse of this work is to give a heat flow expression for the Godbillon Vey class. The idea is the following: Consider a drifted Brownian motion as a solution of a Stochastic Differential Equation and the associated flow ϕ_t (see, for example, [1], [6]). Denote by $\phi_t^*\omega$ to the action of this flow on the 1-form ω, and let ω_t be the 1-form defined by (see [3], [7])

$$\omega_t(v) = \mathbb{E}[\phi_t^*\omega(v)] \qquad v \in \mathcal{X}(M).$$

Then ω_t is a heat flow perturbation of ω, since $\omega_0 = \omega$. Our main result is the following

Theorem *The Godbillon Vey class of $\mathcal{F}$, denoted by $gv(\mathcal{F})$, is given by*

$$gv(\mathcal{F}) = -\left[\left.\frac{d}{dt}\right|_{t=0} (\omega_t \wedge d\omega_t) \right].$$

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2 Godbillon-Vey class

Let M be a compact differentiable manifold and denote by $\Omega^*(M)$ the space of differential forms over M. Recall that the exterior differential $d : \Omega^*(M) \to \Omega^*(M)$ and the inner product $\mathbf{i}_X : \Omega^*(M) \to \Omega^*(M)$, with respect to a vector field X, satisfy the following basic formulae: if α is a $k-$form and β is another differential form, then

$$d(\omega \wedge \beta) = d\omega \wedge \beta + (-1)^k \alpha \wedge d\beta,$$

and

$$\mathbf{i}_X(\alpha \wedge \beta) = \mathbf{i}_X \alpha \wedge \beta + (-1)^k \alpha \wedge \beta.$$

Let ω be an integrable 1-form M, this is ω satisfy $d\omega \wedge \omega = 0$, and consider a Riemannian metric g on M such that $||\omega|| = 1$. Denote by $\mathcal{F}$ to the the codimension 1 foliation defined by the integrable subbundle $E = \ker(\omega)$ of TM. The integrability of ω guarantees the existence of a 1-form η in M such that $d\omega = \eta \wedge \omega$. Since

$$(\eta - \eta(\omega^\sharp)\omega) \wedge \omega = \eta \wedge \omega = d\omega,$$

we can choose η, such that

$$\eta(\omega^\sharp) = 0,$$

without loosing generality.

The Godbillon Vey class of $\mathcal{F}$ is defined by de Rham cohomology class

$$gv(\mathcal{F}) = [\eta \wedge d\eta].$$

Let g_E the metric on E induced by g. By the Nash theorem we can do an isometric immersion of M into a $\mathbb{R}^N$ for N large enough. The gradients of the height functions associated to the immersion defines vector fields $\{\tilde{X}_1, \dots, \tilde{X}_N\}$. Consider their projections $\{X_1, \dots, X_N\}$ to E, then, by the usual argument of isometric immersion, the Laplace operators Δ_E on the leaves $L \in \mathcal{F}$ can be written as

$$\Delta_E = \sum_{i=1}^N X_i^2.$$

Lemma 2.1. *The Laplace operator on M can be decomposed as follows*

$$\Delta_M = (\omega^\sharp)^2 + \Delta_E - \nabla_{\omega^\sharp}\omega^\sharp.$$

Proof. For a smooth function f we obtain

$$\begin{aligned} div(\nabla f) &= \mathrm{Tr}\{(u,v) \to g(u, \nabla_v \nabla f)\} \\ &= g(\omega^\sharp, \nabla_{\omega^\sharp} \nabla f) + \sum_{i=1}^N g(X_i, \nabla_{X_i} \nabla f) \\ &= (\omega^\sharp)^2 f - (\nabla_{\omega^\sharp}\omega^\sharp) f + \Delta_E f. \end{aligned}$$

□

Fix a filtered probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Let $B = (B_0, \dots, B_N)$ be a Brownian motion on $\mathbb{R}^{N+1}$ and let V be a vector field on M. Denote by $X_0 = \omega^\sharp$ and $Z = -\frac{1}{2}\nabla_{\omega^\sharp}\omega^\sharp$. The solution of the Stratonovitch stochastic differential equation

$$\begin{aligned} dx_t &= (V + Z)(x_t)\, dt + \sum_{i=0}^N X_i(x_t) \circ dB_t^n, \\ x_0 &= x, \end{aligned}$$

is a diffusion process with infinitesimal generator given by $V + \frac{1}{2}\Delta_M$, which is a drifted Brownian motion on M.

Since M is compact we can guarantee the existence of a solution flow $\phi : \mathbb{R}_+ \times \Omega \times M \to M$ for this equation (see, for example, [1] or [6]). Associated to the flow ϕ there is the heat semigroup $\{P_t : \Omega^k(M) \to \Omega^k(M)\}$ acting on the space of differential forms by

$$(P_t\alpha)(v_1, \ldots, v_k) = \mathbb{E}[\alpha(\phi_{t*}v_1, \ldots, \phi_{t*}v_k)].$$

It is well known that $P_t\omega$ solves the evolution equation (see, for example, Kunita [7] or Elworthy et. al. [3])

$$\frac{d}{dt}(P_t\alpha) = \left(L_{V+Z} + \frac{1}{2}\sum_{i=0}^{N} L^2_{X_i}\right)(P_t\alpha), \tag{2.1}$$

$$P_0\alpha = \alpha. \tag{2.2}$$

Also $P_t \circ d = d \circ P_t$.

We observe that, in general,

$$P_t(\alpha \wedge \beta) \neq P_t\alpha \wedge P_t\beta.$$

Remark 2.2. In [3], Elworthy, Le-Jan and Li, study the divergent operator $\hat{\delta} = \Omega^*(M) \to \Omega^*(M)$ defined by

$$\hat{\delta} = \sum_{i=0}^{N} i_{X_i} L_{X_i}.$$

Following them, if $\mathcal{A} = \sum_{i=0}^{n} L^2_{X_i}$, we can show that

$$\mathcal{A} = d\hat{\delta} + \hat{\delta}d.$$

Therefore, the operator $\mathcal{A}$ is a kind of Hodge Laplacian.

We can see that:

Lemma 2.3. *Let $\alpha,\ \beta$ be differential forms. Then*

$$d\frac{d}{dt}\left(P_t\alpha \wedge P_t\beta\right) = \frac{d}{dt}d\left(P_t\alpha \wedge P_t\beta\right).$$

Proof. It follows from (2.1) and that $L_X \circ d = d \circ L_X$. □

Now, we apply the above formalism to the integrable 1-form ω that defines the foliation. Denote $\omega_t = P_t\omega$.

Theorem 2.4. *With the notation above,*

$$gv(\mathcal{F}) = -\left[\left.\frac{d}{dt}\right|_{t=0}(\omega_t \wedge d\omega_t)\right].$$

In order to prove this result we need some lemmata.

Lemma 2.5. *Let ω be a 1-form such that $d\omega = \eta \wedge \omega$ and X a vector field on M. Then*

$$L_X\omega \wedge d\omega = d(\boldsymbol{i}_X\omega \wedge \omega),$$

and

$$L_X\omega \wedge dL_X\omega = d\beta + (\boldsymbol{i}_X\omega \wedge d\boldsymbol{i}_X\eta - \boldsymbol{i}_X\eta \wedge d\boldsymbol{i}_X\omega) \wedge d\omega + (\boldsymbol{i}_X\omega)^2\eta \wedge d\eta,$$

for a $2-$form β.

Proof. Since $d\omega = \eta \wedge \omega$, then

$$\eta \wedge d\omega = 0, \qquad \omega \wedge d\omega = 0 \quad \text{and} \quad d\eta \wedge \omega = 0.$$

To show the first expression follows we calculate

$$\begin{aligned} (L_X\omega) \wedge d\omega &= (d\mathbf{i}_X\omega + \mathbf{i}_X d\omega) \wedge d\omega \\ &= d(\mathbf{i}_X\omega \wedge d\omega) + (\mathbf{i}_X\eta) \wedge \omega \wedge d\omega - (\mathbf{i}_X\omega) \wedge \eta \wedge d\omega \\ &= d(\mathbf{i}_X\omega \wedge d\omega). \end{aligned}$$

To prove the second expression we observe that

$$\begin{aligned} \mathbf{i}_X d\omega \wedge d\mathbf{i}_X d\omega &= (\mathbf{i}_X\eta \wedge \omega - \mathbf{i}_X\omega \wedge \eta) \wedge d(\mathbf{i}_X\eta \wedge \omega - \mathbf{i}_X\omega \wedge \eta) \\ &= -\mathbf{i}_X\eta \wedge \omega \wedge d\mathbf{i}_X\omega \wedge \eta - \mathbf{i}_X\omega \wedge \eta \wedge d\mathbf{i}_X\eta \wedge \omega \\ &\quad +(\mathbf{i}_X\omega)^2\eta \wedge d\eta \\ &= (\mathbf{i}_X\omega \wedge d\mathbf{i}_X\eta - \mathbf{i}_X\eta \wedge d\mathbf{i}_X\omega) \wedge d\omega + (\mathbf{i}_X\omega)^2\eta \wedge d\eta, \end{aligned}$$

therefore

$$\begin{aligned} L_X\omega \wedge L_X d\omega &= (d\mathbf{i}_X\omega + \mathbf{i}_X d\omega) \wedge dL_X\omega \\ &= d(\mathbf{i}_X\omega \wedge dL_X\omega) + \mathbf{i}_X d\omega \wedge dL_X\omega \\ &= d(\mathbf{i}_X\omega \wedge dL_X\omega) + (\mathbf{i}_X\omega \wedge d\mathbf{i}_X\eta - \mathbf{i}_X\eta \wedge d\mathbf{i}_X\omega) \wedge d\omega \\ &\quad + (\mathbf{i}_X\omega)^2\eta \wedge d\eta. \end{aligned}$$

□

Lemma 2.6. *Let $\{X_i\}_{i=0}^N$ be the vector fields over M defined as above, $\mathcal{A} = \sum_{i=0}^N L_{X_i}^2$ and ω a 1-form such that $||\omega|| = 1$ and $d\omega = \eta \wedge \omega$, then*

$$[\mathcal{A}\omega \wedge d\omega]_{dR} = -[\eta \wedge d\eta]_{dR}.$$

Proof. By usual computations and Lemma 2.5 we have,

$$\begin{aligned} (L_X^2\omega) \wedge d\omega &= L_X(L_X\omega \wedge d\omega) - L_X\omega \wedge dL_X\omega \\ &= d(\mathbf{i}_X\omega \wedge \omega) - (\mathbf{i}_X\omega)^2\eta \wedge d\eta + \\ &\quad -(\mathbf{i}_X\omega \wedge d\mathbf{i}_X\eta - \mathbf{i}_X\eta \wedge d\mathbf{i}_X\omega) \wedge d\omega + d\beta, \end{aligned}$$

for an arbitrary vector field X. Specializing on X_i and doing the sum we observe that

$$\sum_{i=0}^N (\mathbf{i}_{X_i}\omega)^2 = (\mathbf{i}_{X_0}\omega)^2 = ||\omega|| = 1,$$

and

$$\sum_{i=0}^N (\mathbf{i}_{X_i}\omega d\mathbf{i}_{X_i}\eta - \mathbf{i}_{X_i}\eta d\mathbf{i}_{X_i}\omega) \quad = \quad \mathbf{i}_{X_0}\omega d\mathbf{i}_{X_0}\eta - \mathbf{i}_{X_0}\eta d\mathbf{i}_{X_0}\omega = 0.$$

Therefore,

$$\sum_{i=0}^n (L_{X_i}^2\omega) \wedge d\omega = -\eta \wedge d\eta + d\gamma,$$

for a $2-$form $\gamma = \alpha + \beta$. □

Now we have the ingredients to prove our main result.

Proof of theorem 2.4. We calculate

$$\begin{aligned}
\frac{d}{dt}(\omega_t \wedge d\omega_t) &= \left(L_{(V+Z)}\omega_t \wedge d\omega_t + \omega_t \wedge dL_{(V+Z)}\omega_t\right) \\
&\quad + \frac{1}{2}\sum_{i=0}^{N}\left(L^2_{X_i}\omega_t \wedge d\omega_t + \omega_t \wedge dL^2_{X_i}\omega_t\right) \\
&= 2(L_{(V+Z)}\omega_t \wedge d\omega_t) + d(L_{(V+Z)}\omega_t \wedge \omega_t) \\
&\quad + \frac{1}{2}\sum_{i=0}^{N}\left(2L^2_{X_i}\omega_t \wedge d\omega_t + d(\omega_t \wedge L^2_{X_i}\omega_t)\right) \\
&= 2(L_{(V+Z)}\omega_t \wedge d\omega_t) + \mathcal{A}\omega_t \wedge d\omega_t \\
&\quad + \frac{1}{2}d\left(\omega_t \wedge \mathcal{A}\omega_t\right) + d(L_{(V+Z)}\omega_t \wedge \omega_t).
\end{aligned}$$

Then, at $t = 0$

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \omega_t \wedge d\omega_t &= 2(L_{(V+Z)}\omega \wedge d\omega) + \mathcal{A}\omega \wedge d\omega \\
&\quad + \frac{1}{2}d\left(\omega \wedge \mathcal{A}\omega\right) + L_{(V+Z)}\omega \wedge \omega.
\end{aligned}$$

By the first statement of Lemma 2.5,

$$L_{V+Z}\omega \wedge d\omega = d(\mathbf{i}_{(V+Z)}\omega \wedge d\omega),$$

and by Lemma 2.6

$$\mathcal{A}\omega \wedge d\omega = -\eta \wedge d\eta + d\alpha,$$

for a 2-form α. Thus

$$\left.\frac{d}{dt}\right|_{t=0} \omega_t \wedge d\omega_t = -\eta \wedge d\eta + d\gamma,$$

for a 2-form γ. □

Corollary 2.7. *With the notation of Remark 2.2, if $\hat{\delta} d\omega = 0$ then $gv(\mathcal{F}) = 0$.*

Corollary 2.8. *For all $k \geq 1$, the differential forms*

$$\gamma_k = \omega \wedge (d\mathcal{A}\omega)^k,$$

are exact.

Proof. When $k = 1$ then

$$\omega \wedge d\mathcal{A}\omega = \eta \wedge d\eta + d\beta,$$

which is closed. For $k > 1$ we have that

$$\begin{aligned}
\gamma_k &= \omega \wedge d\mathcal{A}\omega \wedge (d\mathcal{A}\omega)^{k-1} \\
&= \eta \wedge d\eta \wedge (d\mathcal{A}\omega)^{k-1} \\
&= \eta \wedge d\eta \wedge (d\mathcal{A}\omega) \wedge (d\mathcal{A}\omega)^{k-2} \\
&= \pm d(\eta \wedge d\eta \wedge \mathcal{A}\omega \wedge (d\mathcal{A}\omega)^{k-2}) \pm (d\eta)^2 \wedge \mathcal{A}\omega \wedge (d\mathcal{A}\omega)^{k-2}) \\
&= \pm d(\eta \wedge d\eta \wedge \mathcal{A}\omega \wedge (d\mathcal{A}\omega)^{k-2}).
\end{aligned}$$

□

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