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Homogenization via sprinkling

Itai Benjamini[a] and Vincent Tassion[b]

[a]The Weizmann Institute, Rehovot 76100, Israel. E-mail: itai.benjamini@weizmann.ac.il
[b]Département de Mathématiques Université de Genève, 2-4 rue du lievre Case postale 64, 1211 Genève 4, Switzerland.
E-mail: vincent.tassion@unige.ch

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Abstract. We show that a superposition of an ε-Bernoulli bond percolation and any everywhere percolating subgraph of $\mathbb{Z}^d$, $d \geq 2$, results in a connected subgraph, which after a renormalization dominates supercritical Bernoulli percolation. This result, which confirms a conjecture from (*J. Math. Phys.* **41** (2000) 1294–1297), is mainly motivated by obtaining finite volume characterizations of uniqueness for general percolation processes.

Résumé. On considère un sous-graphe de $\mathbb{Z}^d$, $d \geq 2$, dont toutes les composantes connexes sont infinies. On montre que la superposition d'un tel sous-graphe avec une ε-percolation forme un graphe connexe qui, convenablement renormalisé, domine une percolation de Bernoulli surcritique. Ce résultat confirme une conjecture énoncée dans (*J. Math. Phys.* **41** (2000) 1294–1297), et sa motivation principale est d'obtenir des caractérisations en volume fini de l'unicité de l'amas infini pour des processus de percolation généraux.

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1. Introduction

Consider a deterministic subset X of the edges of the standard d-dimensional lattice $\mathbb{Z}^d$, $d \geq 2$. Assume that X is percolating everywhere, meaning that every vertex of $\mathbb{Z}^d$ is in an infinite connected component of the graph $(\mathbb{Z}^d, X)$. Example of such graphs are foliation by lines or spanning forests. Consider then the random set of edges $Y = X \cup \omega$, obtained by adding to X the open edges ω of a Bernoulli percolation with density $\varepsilon > 0$. We prove that for every choice of X and every $\varepsilon > 0$, the graph Y is almost surely connected and has large scale geometry similar to that of supercritical Bernoulli percolation. In [2], this result was already proved in dimension $d = 2$, and conjectured for higher dimensions. The proof of [2] for $d = 2$ relies strongly on planar duality, and cannot extend to higher dimensions. In this paper, we develop new robust methods that allow us to extend the result of [2] to any dimension $d \geq 2$. This is the content of Theorem 1 below. The main step in our proof is of independent interest (see Lemma 1.1). We obtain a finite-volume characterization for the uniqueness of the infinite connected component in Y. More precisely, we show that with high probability, all the points in the ball of radius n are connected by a path of Y which lies inside the ball of radius $2n$. This finite-size criterion approach to uniqueness is in the same spirit of the original proof of uniqueness for Bernoulli percolation (see [1] and the recent work of [6]).

As a consequence of the finite-size criterion mentioned above, we show that a renormalized version of Y dominates highly supercritical percolation. In particular, Y percolates in sufficiently thick slabs and in half-spaces. This result is analogous to the Grimmett–Marstrand theorem [8], and we expect most of the properties of supercritical percolation to hold for Y (see [7], Chapters 7 and 8). The Grimmett–Marstrand theorem is a fundamental and powerful tool in supercritical Bernoulli percolation, but its proof does not provide directly quantitative estimates and relies on the specific symmetries of $\mathbb{Z}^d$ (see [12] for an extension to some graphs with less symmetries than the canonical $\mathbb{Z}^d$). We hope that the method in the present paper could be useful to obtain quantitative and robust proofs of the Grimmett–Marstrand theorem.

We do not require any symmetry hypothesis on the set X. Thus, the percolation process Y is not necessarily invariant under the symmetries of $\mathbb{Z}^d$. Therefore, the uniqueness of the infinite cluster cannot be derived from the Burton–Keane theorem [5]. In this sense, our result can be seen as a generalization of the Burton–Keane theorem. Recently, Teixeira [14] considered general percolation processes with high marginals on graphs with polynomial volume growth. Under some additional assumptions but without requiring symmetries or invariance, he obtains uniqueness of the infinite cluster. Since Y does not necessarily have high marginals, our uniqueness result is not implied by Teixeira's work.

We also obtain that the critical value for Bernoulli percolation on Y satisfies $p_c(Y) < 1$. This is related to the initial motivation in [2] for introducing the random set Y. Proving that $p_c(Y) < 1$ can be seen as an intermediate question in order to understand under which conditions a random infinite subgraph G of $\mathbb{Z}^d$ has $p_c(G) < 1$. Finding such conditions is very challenging, and related to famous open problems in percolation, e.g. the absence of infinite cluster at criticality for Bernoulli percolation (see [2] for more details). We give related questions in Section 1.2.

1.1. *Main results*

We prove the following theorem, which confirms a conjecture of Benjamini, Häggström and Schramm [2]. (If needed, see Section 1.4 for notation and definitions.)

Theorem 1. *Let X be a fixed everywhere percolating subgraph of $\mathbb{Z}^d$, and let $Y = X \cup \omega$ be obtained from X by adding an ε-percolation ω. For any $\varepsilon > 0$, the following hold.*

(i) *The subgraph Y is connected a.s.*
(ii) *The critical parameter for Bernoulli percolation on Y satisfies $p_c(Y) < 1$ a.s.*
(iii) *The subgraph Y percolates in the upper half-space a.s.*
(iv) *There exists $L = L(\varepsilon, d)$ such that Y percolates in the slab $\mathbb{Z}^2 \times \{0, \dots, L\}$ a.s.*
(v) *For every fixed $p < 1$, a renormalized version of Y stochastically dominates a p-Bernoulli percolation.*

The precise meaning of Item (v) is the following. For $n \ge 0$, set $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$. Define the percolation process $Y^{(n)}$ on $\mathbb{Z}^d$ by declaring an edge $e = \{x, y\} \in \mathbb{E}^d$ open if the vertex $2nx$ is Y-connected to $2ny$ inside $n(x + y) + \Lambda_{2n}$. Item (v) occurs if for every $p < 1$, there exists $n \ge 1$ such that the process $Y^{(n)}$ dominates stochastically a p-Bernoulli percolation (see [10] for more details on stochastic domination).

Remarks.

(a) As in [2], a straightforward extension of our proof shows that an analogue of Theorem 1 holds if we only assume that X is densely percolating. A subgraph X is said to be densely percolating if there exists R such that every box $2R.z + \Lambda_R$, $z \in \mathbb{Z}^2$, intersects an infinite component of X. In this framework, Item (i) needs to be replaced by uniqueness of the infinite cluster, and the definition of renormalization in Item (v) has to be slightly adapted.

(b) Once we know that a rescaled version of Y dominates supercritical Bernoulli percolation, one could use the known results for Bernoulli percolation to obtain other properties of Y (see [7], Chapters 7 and 8).

1.2. *Questions*

Let us begin by rewriting the question of [2] that motivates the problem studied here.

Question 1. Is there an invariant, finite energy percolation X on $\mathbb{Z}^d$, which a.s. percolates and satisfies $p_c(X) = 1$?

An invariant percolation is a probability measure on the percolation configuration that is invariant under the symmetries of $\mathbb{Z}^d$. The finite energy property was considered in [13]. In the sense of Lyons and Schramm [11], it corresponds to insertion and deletion tolerance: given an edge e, the conditional probability that e is present (resp. absent) given the status of all the other edges is positive. Benjamini, Häggström and Schramm [2] showed that Question 1 has a positive answer if we replace the finite energy condition by the insertion tolerance. They construct an insertion tolerant invariant process X (obtained by adding and ε-percolation to a well-chosen invariant percolation), that percolates but satisfies $p_c(X) = 1$.

Adding an ε-percolation to a percolation process is an easy way to build insertion tolerant processes. Of course, one may add a more general process instead. Our proof of Theorem 1 uses strongly that our process was constructed by adding an ε-percolation. With Question 1 in mind, it would be interesting to understand the effect of adding a more general process. We suggest the following question.

Question 2. Let X be a fixed everywhere percolating subgraph of $\mathbb{Z}^d$. Let η be a percolation process such that $\eta \neq \varnothing$ almost surely. Assume that η is ergodic with respect to the translations and invariant with respect to the whole automorphisms of the grid. Which properties among (i), (ii), (iii), (iv) and (v) are satisfied by $Y := X \cup \eta$?

Let us give a particular case. In $\mathbb{Z}^3$, consider a superposition of two independent ergodic invariant spanning forests. It is a.s. connected?

Another natural generalization of the problem treated in this paper is to consider graphs other than the hypercubic lattice. In [3] it is shown that non-amenable graphs, admit a spanning forest that stays disconnected after adding the edges of an ε-percolation, for ε sufficiently small. A positive answer to the following question would show that, in the context of transitive graphs with $p_c < 1$, the existence of an everywhere percolating disconnected subgraph that remains disconnected after adding an ε-sprinkling is equivalent to non-amenability.

Question 3. Let G be a transitive amenable graph with critical value for Bernoulli percolation satisfying $p_c(G) < 1$. Let X be an everywhere percolating subgraph of G, and let $Y = X \cup \omega$ be obtained from X by adding an ε-percolation ω. Is Y connected almost surely?

As an intermediate step toward Question 3, one can first consider the question for transitive graphs of polynomial volume growth (a framework in which our methods are more likely to be adapted).

Another perspective is to study the simple random walk on a superposition of an everywhere percolating subgraph of the lattice and an independent sprinkling (for example, verify diffusivity in this framework). Adding the sprinkling can be viewed as homogenisation, this suggests to study spaces of harmonic functions on this environment (in the spirit of [4]).

1.3. *Organization of the paper*

For the rest of the paper, *we fix the values of $d \geq 2$ and $\varepsilon > 0$*. In the proof, we will introduce constants, denoted by $C_0, C_1, \ldots$ By convention, the constants are elements of $(0, \infty)$, they may depend on d and ε, but never depend on any other parameter of the model. In particular, they never depend on the chosen everywhere percolating subgraph X, or the size of the box n.

In Section 2, we study the effect of an ε-percolation on a finite highly connected multigraph. More precisely we consider a multigraph G with $O(N^{2d})$ vertices, possibly with multi-edges, that is N-connected (G is connected and remains connected if we erase any set of N edges). We show that an ε-percolation on such a multigraph is connected with high probability.

The main new ideas are presented in Section 3, where we prove the following lemma.

Lemma 1.1. *Let X be a fixed everywhere percolating subset of edges of $\mathbb{Z}^d$. Let $Y = X \cup \omega$ be obtained from X by adding an ε-percolation ω. For every $n \geq 1$, we have*

$$\mathbb{P}[\text{For all } x, y \in \Lambda_n, x \text{ is } Y\text{-connected to } y \text{ inside } \Lambda_{2n}] \geq 1 - C_3 e^{-C_1\sqrt{n}}. \tag{1.1}$$

Let us present the strategy used to prove Lemma 1.1. The general idea is to interpret the process Y inside the box Λ_{2n} as an ε-percolation on a highly connected finite multigraph G, and then apply the result of Section 2.

We give the construction of G in a particular case. Consider an everywhere percolating subgraph X of $\mathbb{Z}^d$. Its restriction to the finite box Λ_{2n} gives a partition $\Lambda_{2n} = K_1 \sqcup \cdots \sqcup K_\ell$ of the box Λ_{2n} into finite clusters. Assume that each cluster K_i, $i = 1, \dots, \ell$, intersects the box $\Lambda_{2n-\sqrt{n}}$: roughly speaking, each cluster K_i must "cross" the annulus $\Lambda_{2n} \setminus \Lambda_{2n-\sqrt{n}}$. This assumption is very strong but illustrates well the general idea we are using. In this special case, contracting each cluster $K_1, \dots, K_\ell$ into one point results in a $\sqrt{n}$-connected multigraph G (we keep the multiple edges during the contraction procedure). Indeed, consider a set $S = \bigcup_{i \in I} K_i$, where I is a nonempty strict subset of $\{1, \dots, \ell\}$ (the edge-boundary of S separates the multigraph into at least two connected pieces). This set S must "cross" the annulus $\Lambda_{2n} \setminus \Lambda_{2n-\sqrt{n}}$, which implies that its edge-boundary must satisfy $|\Delta S| \geq \sqrt{n}$. The result of Section 2 implies that an ε-percolation on G results in a connected graph with high probability, which directly gives (1.1).

In the general case, we use the same strategy except that we cannot guarantee that all the clusters get connected after one ε-sprinkling: we first show that one sprinkling in Λ_{2n} shrinks the number of clusters intersecting Λ_n by a large factor with high probability. Then we use a finite number of sprinklings in Λ_{2n} to reduce the number of clusters intersecting Λ_n to one, with high probability.

In Section 4, we deduce Theorem 1 from Lemma 1.1. That section uses standard renormalization and stochastic domination tools, which were already used by Benjamini, Häggström and Schramm [2] to treat the case $d = 2$.

1.4. *Notation, definitions*

Let $(\mathbb{Z}^d, \mathbb{E}^d)$, $d \geq 2$, be the standard d-dimensional hypercubic lattice. For $r, R > 0$ (not necessarily integer), we write $\Lambda_r := [-r, r]^d \cap \mathbb{Z}^d$ and $A_{r,R} := \Lambda_R \setminus \Lambda_r$. For $z \in \mathbb{Z}^d$, we denote by $\Lambda_r(z) := z + \Lambda_r$ the translate of Λ_r by vector z.

In this paper, we call subgraph of $\mathbb{Z}^d$ a set of edges $X \subset \mathbb{E}^d$, and identify it to the graph $(\mathbb{Z}^d, X)$. (Notice that the vertex set of every subgraph of $\mathbb{Z}^d$ considered here will always be the entirety of $\mathbb{Z}^d$.) We say that two vertices $x, y \in \mathbb{Z}^d$ are X-connected, if there exists a sequence of disjoint vertices $x_1, \dots, x_\ell \in \mathbb{Z}^d$, such that $x_1 = x$, $x_\ell = y$, and for every $1 \leq i < \ell$ the edge $\{x_i, x_{i+1}\}$ belongs to X. We say that x and y are X-connected inside $S \subset \mathbb{Z}^d$ if, in addition, all the x_i's belong to S.

A subgraph X is said to be everywhere percolating if all its connected components are infinite. In other words, a subgraph is everywhere percolating if for every vertex x in $\mathbb{Z}^d$, x is X-connected to infinity. We say that X percolates in $S \subset \mathbb{Z}^d$ if the graph induced by X on S contains an infinite connected component.

We call p-Bernoulli percolation (or simply p-percolation) the random subgraph ω of $\mathbb{Z}^d$, constructed as follows: each edge of $\mathbb{E}^d$ is examined independently of the other and is declared to be an element of ω with probability p.

2. Percolation on a highly connected finite graph

In this section, we show that an ε-percolation on any N-connected finite multigraph with $O(N^{2d})$ vertices connects all the vertices with large probability. Let us begin with the definitions needed to state the result. A finite multigraph is given by a pair $G = (V, E)$: V is a finite set of vertices, and E is a multiset of pairs of unordered vertices (multiple edges and loops are allowed). We work with multigraphs rather than standard graphs because we will consider multigraphs obtained from other graphs by *contraction*, and we want to keep track of the multiple edges created by the contraction procedure. A multigraph $G = (V, E)$ is said to be N-connected[1] if it is connected, and removing any set of N edges does not disconnect it. In other words, for any subset $F \subset E$ such that $|F| \leq N$, the multigraph $(V, E \setminus F)$ is connected. An ε-percolation on G is defined equivalently as on $\mathbb{Z}^d$: it is a random set of edges $\omega \subset E$ such that the events $\{e \in \omega\}$ are independent, each of them having probability equal to ε.

Proposition 2. *Let $N \in \mathbb{N}$. Let $G = (V, E)$ be an N-connected multigraph with $|V| \leq N^{2d}$. Let $\omega \subset E$ be an ε-percolation on G. Then*

$$\mathbb{P}\big[\textit{The graph } (V, \omega) \textit{ is connected}\big] \geq 1 - C_2 e^{-C_1 N}.$$

[1] In graph theory, the terminology "N-edge-connected" is used in this case to distinguish with vertex-connectivity.

We begin with the following lemma, which says that adding an $(\varepsilon/4d)$-percolation on a subgraph of G either connects all the vertices or shrinks the number of connected components by a factor smaller than $1/\sqrt{N}$ (with large probability). We will then prove the proposition by applying this lemma $4d$ times. Given $X \subset E$, we write $K(X)$ for the number of connected components in the graph (V, X).

Lemma 2.1. *Let $N \in \mathbb{N}$. Let $G = (V, E)$ be a N-connected multigraph with $|V| \le N^{2d}$. Let ω_1 be an $(\varepsilon/4d)$-percolation on G. For every $X \subset E$, we have*

$$\mathbb{P}\left[K(X \cup \omega_1) > 1 \vee \frac{K(X)}{\sqrt{N}}\right] \le C_0 e^{-C_1 N}. \tag{2.1}$$

Proof. We can assume that $K(X) > 1$ (if $K(X) = 1$, then Equation (2.1) is trivially true). We say that a set of vertices $S \subset V$ is generated by X, if it can be exactly written

$$S = S_1 \cup \cdots \cup S_m, \tag{2.2}$$

where $S_1, \dots, S_m$ are disjoint connected components of X. For such a set S, we define $\mathsf{m}(S) := m$ as the number of connected components in the decomposition (2.2). Notice that every non-empty set generated by X satisfies $1 \le \mathsf{m}(S) \le K(X)$. The case $\mathsf{m}(S) = 1$ corresponds to S being a single connected component of (V, X), and $\mathsf{m}(S) = K(X)$ corresponds to $S = V$.

If $K(X \cup \omega_1) > 1 \vee K(X)/\sqrt{N}$, then there must exist a connected component $S \subsetneq V$ of $X \cup \omega_1$ that satisfies $1 \le \mathsf{m}(S) < \sqrt{N}$. By the union bound, we obtain

$$\mathbb{P}\left[K(X \cup \omega_1) > 1 \vee \frac{K(X)}{\sqrt{N}}\right] \le \sum_{\substack{S \subsetneq V \\ 1 \le \mathsf{m}(S) < \sqrt{N}}} \mathbb{P}[S \text{ is a connected component of } X \cup \omega_1]. \tag{2.3}$$

Now, if a non-empty set $S \subsetneq V$ is a connected component of $X \cup \omega_1$, then all the edges connecting a vertex of S to a vertex in $V \setminus S$ must be ω_1-closed. Since the multigraph G is N-connected there are at least N such edges, and we obtain

$$\mathbb{P}[S \text{ is a connected component of } X \cup \omega_1] \le (1 - \varepsilon/4d)^N. \tag{2.4}$$

Plugging Equation (2.4) in (2.3), we find

$$\begin{aligned}
\mathbb{P}\left[K(X \cup \omega_1) > 1 \vee \frac{K(X)}{\sqrt{N}}\right] &\le \left|\left\{S : 1 \le \mathsf{m}(S) < \sqrt{N}\right\}\right| (1 - \varepsilon/4d)^N \\
&\le \sum_{1 \le k < \sqrt{N}} \binom{K(X)}{k} (1 - \varepsilon/4d)^N \\
&\le \sqrt{N} N^{2d\sqrt{N}} (1 - \varepsilon/4d)^N \\
&\le C_0 e^{-C_1 N}.
\end{aligned}$$

In the third line we use that $K(X) \le |V| \le N^{2d}$ to bound the binomial coefficient by $N^{2d\sqrt{N}}$. □

Proof of Proposition 2. Let $\omega_1, \dots, \omega_{4d}$ be $4d$ independent $(\varepsilon/4d)$-percolations on G. Set $\eta_0 = \varnothing$ and $\eta_k := \omega_1 \cup \cdots \cup \omega_k$. By Lemma 2.1 we have, for all $1 \le k \le 4d$,

$$\mathbb{P}\left[K(\eta_k) > 1 \vee \left(K(\eta_{k-1})/\sqrt{N}\right)\right] \le C_0 e^{-C_1 N}.$$

Since $K(\eta_0) = |V| \le N^{2d}$, we find by induction

$$\mathbb{P}\left[K(\eta_k) > N^{2d - k/2}\right] \le k C_0 e^{-C_1 N}.$$

Setting $k = 4d$ in the equation above, we obtain that η_{4d} is connected with probability larger than $1 - C_2 e^{-C_1 N}$. This concludes the proof, since η_{4d} is stochastically dominated by an ε-percolation. $\square$

3. Proof of Lemma 1.1

Let X be an everywhere percolating subgraph of $\mathbb{Z}^d$, let $n \geq 1$. We define $\mathcal{C}_{8dn}(X)$ as the set of connected components for the graph induced by X on the box Λ_{8dn} (two vertices are in the same connected component if they are X-connected inside Λ_{8dn}). Fix n_0 such that, for every $n \geq n_0$, the size the boundary of Λ_{8dn} is smaller than n^d. (We call boundary of Λ_n the set $\Lambda_n \setminus \Lambda_{n-1}$.) Notice that n_0 depends only on the dimension d. Since any element of $\mathcal{C}_{8dn}(X)$ contains at least a vertex at the boundary of Λ_{8dn}, we also have, for every $n \geq n_0$,

$$\left|\mathcal{C}_{8dn}(X)\right| \leq n^d. \tag{3.1}$$

For $0 < a \leq b \leq 8dn$, define $U_{a,b}(X)$ as the number of sets $C \in \mathcal{C}_{8dn}(X)$ such that $C \cap \Lambda_a = \varnothing$ and $C \cap \Lambda_b \neq \varnothing$. In other words,

$$U_{a,b}(X) = \mathrm{Card}\left\{C \in \Lambda_{8dn} : a < \mathrm{d}(0, C) \leq b\right\},$$

where $\mathrm{d}(0, C)$ denotes the L^∞-distance between the origin and the set C. Define $U_{0,b}$ as the number of sets $C \in \mathcal{C}_{8dn}(X)$ such that $C \cap \Lambda_b \neq \varnothing$. That is

$$U_{0,b}(X) = \mathrm{Card}\left\{C \in \Lambda_{8dn} : \mathrm{d}(0, C) \leq b\right\}.$$

Notice that $U_{a,b}(X)$ depends on n, but we keep this dependence implicit to lighten the notation.

Lemma 3.1. *Fix an everywhere percolating subgraph X of $\mathbb{Z}^d$. Let $n_0 \leq n \leq m \leq m + \sqrt{n} \leq 8dn$. Let ω be an ε-percolation restricted to $A_{m,m+2\sqrt{n}}$, then*

$$\mathbb{P}\left[U_{0,m}(X \cup \omega) > 1 \vee U_{m,m+2\sqrt{n}}(X)\right] \leq C_2 e^{-C_1\sqrt{n}}.$$

(In the statement of the lemma, the ε-percolation restricted to $A_{m,m+2\sqrt{n}}$ is defined by $\omega = \eta \cap A_{m,m+2\sqrt{n}}$, where η is an ε-percolation in $\mathbb{Z}^d$.)

Proof. Let us first assume $U_{m,m+\sqrt{n}}(X) = 0$. In this case, the proof is easier and we directly show that

$$\mathbb{P}\left[U_{0,m}\left(X \vee \omega'\right) > 1\right] \leq C_2 e^{-C_1\sqrt{n}}, \tag{3.2}$$

where $\omega' = \omega \cap A_{m,m+\sqrt{n}}$ is the restriction of ω to the annulus $A_{m,m+\sqrt{n}}$.

Let G be the multigraph obtained from $\Lambda_{m+\sqrt{n}}$ by the following contraction procedure.

1. Start with $\Lambda_{m+\sqrt{n}}$, with the standard graph structure induced by $\mathbb{Z}^d$.
2. Examine the elements of $\mathcal{C}_{8dn}(X)$ one after the other. For every $C \in \mathcal{C}_{8dn}(X)$, contract all the points $x \in C \cap \Lambda_{m+\sqrt{n}}$ into one vertex.

Since $U_{m,m+\sqrt{n}}(X) = 0$, the multigraph G has exactly $U_{0,m}(X)$ vertices, and $U_{0,m}(X \vee \omega')$ corresponds the number of connected components resulting from an ε-percolation on G. Therefore, Equation (3.2) follows from Proposition 2, applied to G with $N = \sqrt{n}$. Indeed, the multigraph G has at most $(\sqrt{n})^{2d}$ vertices by (3.1), and is $\sqrt{n}$-connected. To see that G is $\sqrt{n}$-connected, observe that any non-trivial union of elements of $\mathcal{C}_{8dn}(X)$ that intersect $\Lambda_{m+\sqrt{n}}$ must "cross" the annulus $A_{m,m+\sqrt{n}}$ (this follows from the hypothesis $U_{m,m+\sqrt{n}}(X) = 0$).

We now turn to the case $U_{m,m+\sqrt{n}}(X) > 0$, in which we show

$$\mathbb{P}\left[U_{0,m}(X \cup \omega) \leq U_{m,m+2\sqrt{n}}(X)\right] \geq 1 - C_2 e^{C_1\sqrt{n}}. \tag{3.3}$$

The strategy is very similar to the one used in the case $U_{m,m+\sqrt{n}}(X)=0$, except that here we need to consider carefully the clusters that intersect the annulus $A_{m,m+2\sqrt{n}}$ but not the box Λ_m. More precisely, we partition the elements of $\mathcal{C}_{8dn}(X)$ *intersecting the box* $\Lambda_{m+2\sqrt{n}}$, into the following two types:

- the bulk-clusters, defined as the elements $C\in\mathcal{C}_{8dn}(X)$ such that $C\cap\Lambda_m\neq\varnothing$;
- the boundary-clusters, defined as the elements $C\in\mathcal{C}_{8dn}(X)$ such that $C\cap\Lambda_m=\varnothing$.

Notice that

$$U_{0,m+2\sqrt{n}}(X)=U_{0,m}(X)+U_{m,m+2\sqrt{n}}(X),$$

where $U_{0,m}(X)$ counts the bulk-clusters, and $U_{m,m+2\sqrt{n}}(X)$ counts the boundary-clusters.

Define $\tilde{C}$ as the union of all the boundary-clusters. The hypothesis $U_{m,m+\sqrt{n}}(X)>0$ implies that at least one boundary-cluster intersects the annulus $A_{m,m+\sqrt{n}}$. Thus, we have

$$\tilde{C}\cap\Lambda_{m+\sqrt{n}}\neq\varnothing.$$

We now construct a $\sqrt{n}$-connected multigraph G by the following contraction procedure.

1. Start with $\Lambda_{m+2\sqrt{n}}$, with the graph structure induced by $\mathbb{Z}^d$.
2. For every bulk-cluster $C\in\mathcal{C}_{8dn}(X)$, contract all the points $x\in C\cap\Lambda_{m+2\sqrt{n}}$ into one vertex.
3. Contract all the vertices x that belong to $\tilde{C}\cap\Lambda_{m+2\sqrt{n}}$ into one vertex.

To see that the multigraph is $\sqrt{n}$-connected, observe that all the bulk-clusters and $\tilde{C}$ cross the annulus $A_{m+\sqrt{n},m+2\sqrt{n}}$. Proposition 2 applied with $N=\sqrt{n}$, ensures than an ε-percolation on G connects all its vertices with probability larger than $1-C_2e^{-C_1\sqrt{n}}$. This proves Equation (3.3), because the ε-percolation ω can be interpreted as an ε-percolation on G, except that the $U_{m,m+2\sqrt{n}}(X)$ boundary-clusters were "artificially" merged into one point in the construction of G. □

Lemma 3.2. *Fix an everywhere percolating subgraph X of $\mathbb{Z}^d$. Let $n_0\le n\le m\le m+2n\le 8dn$. Let ω be an ε-percolation restricted to the annulus $A_{m,m+2n}$, then*

$$\mathbb{P}\left[U_{0,m}(X\cup\omega)>1\vee\frac{U_{0,m+2n}(X)}{\sqrt{n}}\right]\le C_2e^{-C_1\sqrt{n}}. \tag{3.4}$$

Proof. First observe that

$$U_{0,m}(X)+\sum_{i=0}^{\lfloor\sqrt{n}\rfloor-1}U_{m+2i\sqrt{n},m+2(i+1)\sqrt{n}}(X)\le U_{0,m+2n}(X).$$

Among the $1+\lfloor\sqrt{n}\rfloor$ terms summed on the left hand side, at least one of them must be smaller than or equal to $U_{0,m+2n}(X)/\sqrt{n}$. If $U_{0,m}(X)\le U_{0,m+2n}(X)/\sqrt{n}$, then Equation (3.4) is trivially true. Otherwise, one can fix i such that $U_{m+2i\sqrt{n},m+2(i+1)\sqrt{n}}(X)\le U_{0,m+2n}(X)/\sqrt{n}$. By Lemma 3.1, we have

$$\mathbb{P}\left[U_{0,m+2i\sqrt{n}}(X\cup\omega)>1\vee\frac{U_{0,m+2n}(X)}{\sqrt{n}}\right]\le C_2e^{-C_1\sqrt{n}}.$$

Then, use the inequality $U_{0,m}(X\cup\omega)\le U_{0,m+2i\sqrt{n}}(X\cup\omega)$ to conclude the proof. □

Proof of Lemma 1.1. Since n_0 depends only on d, it is sufficient to prove Equation (1.1) in Lemma 1.1 for $n\ge n_0$ (recall that n_0 was defined at the beginning of Section 3). We wish to apply Lemma 3.2 recursively in the $2d$ disjoint annuli

$$A_{(8d-2)n,8dn},A_{(8d-4)n,(8d-2)n},\dots,A_{4dn,(4d+2)n}.$$

For $i = 1, \dots, 2d$, set $m(i) = (8d - 2i)n$, and $\omega_i = \omega \cap \Lambda_{m(i),8dn}$. By Lemma 3.2, we have for all $i < 2d$

$$\mathbb{P}\left[U_{0,m(i+1)}(X \cup \omega_{i+1}) > 1 \vee \frac{U_{0,m(i)}(X \cup \omega_i)}{\sqrt{n}}\right] \le C_2 e^{-C_1\sqrt{n}}. \tag{3.5}$$

By Equation (3.1), we have $U_{0,8dn}(X) \le n^d$ for all $n \ge n_0$. This implies that

$$\begin{aligned}\mathbb{P}\big[U_{0,4dn}(X \cup \omega) = 1\big] &\ge \mathbb{P}\left[\text{For all } i,\ U_{0,m(i+1)}(X \cup \omega_{i+1}) \le 1 \vee \frac{U_{0,m(i)}(X \cup \omega_i)}{\sqrt{n}}\right] \\ &\ge 1 - 2dC_2 e^{-C_1\sqrt{n}},\end{aligned}$$

where the last line follows from Equation (3.5). This proves that with probability larger than $1 - C_3 e^{-C_1\sqrt{n}}$, all the vertices of Λ_{4n} are $(X \cup \omega)$-connected inside Λ_{8n}. Lemma 1.1 follows straightforwardly. □

4. Proof of Theorem 1

Let X be a fixed everywhere percolating subgraph of $\mathbb{Z}^d$, and let $Y = X \cup \omega$ be obtained from X by adding an ε-percolation ω. In this section, we will show how to derive Theorem 1 from the following estimate, stated in Lemma 1.1.

$$\mathbb{P}[\text{For all } x, y \in \Lambda_n,\ x \text{ is } Y\text{-connected to } y \text{ inside } \Lambda_{2n}] \ge 1 - C_3 e^{-C_1\sqrt{n}}.$$

Recall that the constants C_1, C_3 do not depend on the underlying everywhere percolating graph X. Thus, by considering translates of X, we show that for every $z \in \mathbb{Z}^d$,

$$\mathbb{P}\big[\text{For all } x, y \in \Lambda_n(z),\ x \text{ is } Y\text{-connected to } y \text{ inside } \Lambda_{2n}(z)\big] \ge 1 - C_3 e^{-C_1\sqrt{n}}. \tag{4.1}$$

4.1. *Proof of Item* (v)

Let $p < 1$. By Corollary 1.4 in [10], one can pick $p' < 1$ such that any 3-dependent[2] percolation process Z on $\mathbb{Z}^d$, satisfying for every edge $e \in \mathbb{E}^d$

$$\mathbb{P}[e \in Z] > p'$$

dominates stochastically a p-Bernoulli percolation.

Recall that the process $Y^{(n)}$ is defined by setting $\{x, y\} \in Y^{(n)}$ if $2nx$ is Y-connected to $2ny$ inside $\Lambda_{4n}(nx + ny)$. One can easily verify that for every $n \ge 1$, the process $Y^{(n)}$ is 3-dependent. Thus, in order to show that for some n, $Y^{(n)}$ dominates a p-Bernoulli percolation, one only need to prove that for every edge $\{x, y\} \in \mathbb{E}^d$,

$$\mathbb{P}\big[2nx \text{ is } Y\text{-connected to } 2ny \text{ inside } \Lambda_{2n}(nx + ny)\big] > p'. \tag{4.2}$$

Since both $2nx$ and $2ny$ belong to $\Lambda_n(nx + ny)$, we get that Equation (4.2) holds for n large enough, by applying Equation (4.1) with $z = nx + ny$.

4.2. *Proof of Items* (iii) *and* (iv)

Let $p > p_c(\mathbb{Z}^2)$. By Item (v), one can pick n such that $Y^{(n)}$ dominates a p-Bernoulli percolation on $\mathbb{Z}^d$. It is known that a p-Bernoulli percolation percolates in the half-plane $\{2, 3, \dots\} \times \mathbb{Z} \times \{0_{\mathbb{Z}^{d-2}}\}$ (see e.g. [9]). By stochastic domination, the same holds for $Y^{(n)}$. This implies that Y percolates in the half space $\mathbb{N} \times \mathbb{Z}^{d-1}$. When $d \ge 3$, it also implies that Y percolates in the slab $\mathbb{Z}^2 \times \{-2n, \dots, 2n\}^{d-2}$.

[2] A percolation process Z is said to be 3-dependent if, given two sets of edges A and B such that any edge of A is at L^∞ distance at least 3 from any edge of B, the processes $Z \cap A$ and $Z \cap B$ are independent.

4.3. *Proof of Item (ii)*

We begin as in the proof of Item (v). By stochastic domination arguments, one can fix $p' < 1$ such that any 3-dependent percolation process Z with $\mathbb{P}[e \in Z] > p'$ percolates in $\mathbb{Z}^d$. Let $p'' \in (p', 1)$. By Equation (4.1), one can fix n such that for every edge $\{x, y\} \in \mathbb{E}^d$,

$$\mathbb{P}\big[2nx \text{ is } Y\text{-connected to } 2ny \text{ inside } n(x+y) + \Lambda_{2n}\big] > p''.$$

Let η_q be a q-Bernoulli percolation process in $\mathbb{Z}^d$, independent of ω, and set $Y_q = Y \cap \eta_q$. This way, Y_q corresponds exactly to a q-percolation on Y. Choose $q < 1$ such that all the edges of Λ_{2n} belong to η_q with probability larger than $1 - (p'' - p')$. This way, for fixed z, the processes Y_q and Y differs in the box $z + \Lambda_{2n}$ with probability smaller than $(p'' - p')$. Thus, for every edge $\{x, y\} \in \mathbb{E}^d$, we have

$$\begin{aligned}
&\mathbb{P}\big[2nx \text{ is } Y_q\text{-connected to } 2ny \text{ inside } n(x+y) + \Lambda_{2n}\big] \\
&\quad \geq \mathbb{P}\big[2nx \text{ is } Y\text{-connected to } 2ny \text{ inside } n(x+y) + \Lambda_{2n}\big] - \big(p'' - p'\big) \\
&\quad > p'.
\end{aligned}$$

By the same stochastic argument that we already used several times, this is enough to guarantee that the process Y_q percolates in $\mathbb{Z}^d$. This proves that $p_c(Y) \leq q < 1$ almost surely.

4.4. *Proof of Item (i)*

Let $\mathcal{U}_n$ be the event that all the pairs of points in Λ_n are Y-connected in $\mathbb{Z}^d$. By Lemma 1.1, we have

$$\sum \mathbb{P}\big[\mathcal{U}_n^c\big] < \infty.$$

Thus, by the Borel–Cantelli Lemma, we have $\mathbb{P}[\bigcup_n \bigcap_{m \geq n} \mathcal{U}_m] = 1$, which implies that Y is almost surely connected.

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