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CENTRAL LIMIT THEOREM FOR AN ADAPTIVE RANDOMLY REINFORCED URN MODEL

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The generalized Pólya urn (GPU) models and their variants have been investigated in several disciplines. However, typical assumptions made with respect to the GPU do not include urn models with a diagonal replacement matrix, which arise in several applications, specifically in clinical trials. To facilitate mathematical analyses of models in these applications, we introduce an adaptive randomly reinforced urn model that uses accruing statistical information to adaptively skew the urn proportion toward specific targets. We study several probabilistic aspects that are important in implementing the urn model in practice. Specifically, we establish the law of large numbers and a central limit theorem for the number of sampled balls. To establish these results, we develop new techniques involving last exit times and crossing time analyses of the proportion of balls in the urn. To obtain precise estimates in these techniques, we establish results on the harmonic moments of the total number of balls in the urn. Finally, we describe our main results in the context of an application to response-adaptive randomization in clinical trials. Our simulation experiments in this context demonstrate the ease and scope of our model.

1. Introduction. A generalized Pólya urn (GPU) model [4] is characterized by a pair $(Y_{1,n}, Y_{2,n})$ of random variables representing the number of balls of two colors, red and white, for instance. The process evolves as follows: at time $n = 0$, the process starts with $(y_{1,0}, y_{2,0})$ balls. A ball is drawn at random. If the color is red, the ball is returned to the urn along with the random numbers $(D_{11,1}, D_{12,1})$ of red and white balls, respectively; otherwise, the ball is returned to the urn along with the random numbers $(D_{21,1}, D_{22,1})$ of red and white balls. Let $Y_{1,1} = y_{1,0} + D_{11,1}$ and $Y_{2,1} = y_{2,0} + D_{12,1}$ denote the urn composition when the sampled ball is red; similarly, let $Y_{1,1} = y_{1,0} + D_{21,1}$ and $Y_{2,1} = y_{2,0} + D_{22,1}$ denote the urn composition when the sampled ball is white. The process is repeated yielding the collection $\{(Y_{1,n}, Y_{2,n}); n \geq 1\}$. The quantities $R_1 = \{(D_{11,n}, D_{12,n}); n \geq 1\}$ and

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$R_2 = \{(D_{21,n}, D_{22,n}); n \geq 1\}$ are collections of independent and identically distributed (i.i.d.) nonnegative integer valued random variables, and R_1 is assumed to be independent of R_2. We refer to

$$D_n = \begin{bmatrix} D_{11,n} & D_{12,n} \\ D_{21,n} & D_{22,n} \end{bmatrix}$$

as a replacement matrix.

In this paper, we focus on an extension of the randomly reinforced urn (RRU) model, a variant of the randomized Pólya urn (RPU) models, whose replacement matrix is given by

$$D_n = \begin{bmatrix} D_{11,n} & D_{12,n} \\ D_{21,n} & D_{22,n} \end{bmatrix} \equiv \begin{bmatrix} D_{1,n} & 0 \\ 0 & D_{2,n} \end{bmatrix},$$

where the random variables $D_{1,n}$ and $D_{2,n}$ are supported on $[0, \infty)$, rather than on the set of nonnegative integers. Let $m_1 := \boldsymbol{E}[D_{1,n}]$ and $m_2 := \boldsymbol{E}[D_{2,n}]$. For the RRU model, the law of large numbers (LLN) was established in [18], that is,

$$(1.1) \qquad Z_n = \frac{Y_{1,n}}{Y_{1,n} + Y_{2,n}} \overset{\text{a.s.}}{\to} \begin{cases} 1 \cdot \mathbf{1}_{\{m_1 > m_2\}} + 0 \cdot \mathbf{1}_{\{m_1 < m_2\}}, & \text{if } m_1 \neq m_2, \\ Z_\infty, & \text{if } m_1 = m_2, \end{cases}$$

where $\overset{\text{a.s.}}{\to}$ stands for almost sure convergence and Z_∞ is a random variable supported on $(0, 1)$. The properties of the distribution of Z_∞ were studied in [2, 3]. Denoting $\{(N_{1,n}, N_{2,n}); n \geq 1\}$, the number of balls of red and white colors sampled from the urn, one can deduce from the above LLN that $N_{1,n}/n$ converges to the same limit as Z_n.

From (1.1), the limit of Z_n is either 1 or 0 when $m_1 \neq m_2$. However, in applications it is common to target a specific value $\rho \in (0, 1)$ for the limit of Z_n. This was achieved in [1], where the modified randomly reinforced urn (MRRU) model was introduced. The MRRU model is an RRU model with two fixed thresholds $0 < \rho_2 \leq \rho_1 < 1$, such that if $Z_n < \rho_2$, no white balls are replaced in the urn, while if $Z_n > \rho_1$, no red balls are replaced in the urn. These changes occur at random times that, in general, depend on m_1 and m_2. Thus, even if the sequences $\{D_{1,n}; n \geq 1\}$ and $\{D_{2,n}; n \geq 1\}$ are i.i.d., the replacement matrices of the MRRU model are not i.i.d., since they have the following representation:

$$D_n = \begin{bmatrix} D_{1,n} \cdot \mathbf{1}_{\{Z_{n-1} \leq \rho_1\}} & 0 \\ 0 & D_{2,n} \cdot \mathbf{1}_{\{Z_{n-1} \geq \rho_2\}} \end{bmatrix}.$$

The LLN for the MRRU when $m_1 \neq m_2$ is established in [13] and has the following form:

$$Z_n \overset{\text{a.s.}}{\to} \rho_1 \cdot \mathbf{1}_{\{m_1 > m_2\}} + \rho_2 \cdot \mathbf{1}_{\{m_1 < m_2\}}.$$

A second-order result for Z_n, namely the asymptotic distribution of Z_n after appropriate centering and scaling, was derived in [13]. *However, the validity of the central limit theorem (CLT) for $N_{1,n}/n$ in the MRRU model is not known.*

A critical issue in the implementation of the MRRU model is that ρ_1 and ρ_2 are typically unknown. In this paper, we use the accruing information concerning the balls in the urn to provide random thresholds which converge a.s. to specified targets. Specifically, our replacement matrix becomes adaptive in the sense that it reduces to

$$D_n = \begin{bmatrix} D_{1,n} \cdot \mathbf{1}_{\{Z_n \le \hat{\rho}_{1,n}\}} & 0 \\ 0 & D_{2,n} \cdot \mathbf{1}_{\{Z_n \ge \hat{\rho}_{2,n}\}} \end{bmatrix}, \tag{1.2}$$

where $\hat{\rho}_{1,n}$ and $\hat{\rho}_{2,n}$ represent the random thresholds which depend on the accruing information. We call this urn model an *adaptive randomly reinforced urn* (ARRU), to distinguish it from the RRU and the MRRU. In this paper, we investigate the asymptotic properties of the ARRU model when $m_1 \neq m_2$. Specifically, we establish the LLN for Z_n and $N_{1,n}/n$, and the CLT for $N_{1,n}/n$. Before concluding this section, we describe recent literature which is similar in spirit to the present work but is different from the above proposed model.

Let $H_n := \boldsymbol{E}[D_n|\mathcal{F}_{n-1}]$, where $\mathcal{F}_{n-1}$ is the "information" up to time $(n-1)$. This is referred to as the generating matrix. Asymptotic properties of the urn composition for homogeneous GPU, that is, $H_n = H$ for all $n \geq 1$, have been studied in [4] under the assumption that H is irreducible. In [22], the extended Pólya urn (EPU) is defined as a GPU such that all the rows of H sum to the same positive constant, that is,

$$H\mathbf{1} = c\mathbf{1}. \tag{1.3}$$

Under the additional assumption that H has simple eigenvalues, second-order asymptotic properties on the proportion of sampled color extracted from the urn are obtained. In [16], the limiting distribution of the proportion of sampled balls for homogeneous urn models are derived. In [5], weak consistency and asymptotic normality of the urn composition for the nonhomogeneous GPU are established. However, in [5], the sequence $\{H_n; n \geq 1\}$ is deterministic and converges to a matrix H satisfying (1.3). Bai et al. [7] and Bai and Hu [6] extended [5] to random generating matrices and established the almost sure convergence of the proportion of sampled balls. They also investigate the second-order properties. A key assumption in [6, 7] is (1.3). In [25], the sequence of generating matrices is defined as a function of adaptive estimators, which guarantees the convergence of H_n to a limiting matrix H satisfying (1.3). For "immigrated" urn models, theoretical results have been obtained in [26] under the assumptions (1.3), or $H\mathbf{1} < 0$. These extensions do not include the RRU model, where H_n is diagonal, nonnegative and (1.3) is not satisfied. For distributional results concerning large Pólya urns, see [8]. We now describe an application to the clinical trial literature (see [11]). For applications to computer science, we refer the reader to [17].

1.1. *Applications to clinical trials.* Urn models have a long history of applications in clinical trials, by providing randomization procedures that target certain objectives (for a review, see [19]). In this context, patients are sequentially allocated to treatments according to the sampled colors and the associated responses are used to update the urn. This is referred to as *response-adaptive*, since the probability of assignment depends on information about the treatment performance. For a literature review on response-adaptive designs in clinical trials, see [15, 20]. In an RRU model, responses to treatments are transformed by a utility function to obtain the reinforcement values, in such a way that the better treatment has higher reinforcement values. Hence, in this context, as can be seen from (1.1) treatment allocation to patients using the RRU yields a more ethically appealing allocation. However, in some situations, especially when the superiority of a certain treatment is not absolutely clear, response-adaptive designs can be used to target a certain proportion $\rho \in (0, 1)$ of patients to be allocated to a better performing treatment and, at the same time, obtain improved inferential properties at the end of the experiment. The inferential properties depend on the optimality criteria chosen and several of these are described in [21]. For this reason, in [1] the RRU was modified (yielding MRRU) to asymptotically attain any target allocation proportion, $\rho \in (0, 1)$. This guarantees that the MRRU design has an asymptotic allocation within (0, 1) thereby incorporating ethical constraints (namely, assigning more subjects to the superior treatment) and improving inferential properties as shown in [14]. The main issue is that, in the MRRU, ρ_1 and ρ_2 are functions of unknown parameters (see [21]). The ARRU model presented in this paper allows ρ_1 and ρ_2 to be functions of such unknown parameters, and adaptively updates by substituting sequential estimates of the parameters. The limiting results in this paper demonstrate that such procedures target the unknown optimal allocation and provide an appropriate randomization procedure for use in practice. We demonstrate, using simulations, that the limit properties hold even for moderate sample sizes.

1.2. *Structure of the paper.* The paper is organized as follows. In Section 2, we present the notation and assumptions concerning the ARRU model and related main results. Specifically, in Section 2.1, we present the LLN; in Section 2.2, we present the CLT under the assumption that the thresholds are updated at exponentially changing times. Section 2.3 is devoted to the implications of the main results in the context of clinical trials.

In Section 3, we describe several fundamental results concerning the ARRU model that are needed in the proof of the CLT. Specifically, we prove that the harmonic moments of the total number of balls in the ARRU are uniformly bounded. Then we use this to obtain a uniform L_1-bound for the distance between the urn proportion at successive update times and the adaptive thresholds. In Section 4, the proofs of the main results are provided, while Section 5 contains results of a simulation study. Section 6 contains extensions to multi-color urn models.

Finally, some remarks concerning proofs are in order. The LLN and CLT for $N_{1,n}/n$ are deduced using the asymptotic properties of Z_n. For this reason, in several results of this paper, we will provide a detailed probabilistic description of the sequence $\{Z_n; n \geq 1\}$.

2. Model assumptions, notation and main results. We begin by describing our model precisely. Let $\boldsymbol{\xi_1} = \{\xi_{1,n}; n \geq 1\}$ and $\boldsymbol{\xi_2} = \{\xi_{2,n}; n \geq 1\}$ be two sequences of i.i.d. random variables, with probability distributions μ_1 and μ_2, respectively. Without loss of generality, we assume that the support S of $\xi_{1,n}$ and $\xi_{2,n}$ to be the same. Consider an urn containing $y_{1,0} > 0$ red balls and $y_{2,0} > 0$ white balls, and define $y_0 = y_{1,0} + y_{2,0}$. At time $n = 1$, a ball is drawn at random from the urn and its color is observed. Let the random variable X_1 be such that

$$X_1 = \begin{cases} 1, & \text{if the extracted ball is red,} \\ 0, & \text{if the extracted ball is white.} \end{cases}$$

We assume X_1 to be independent of the sequences $\boldsymbol{\xi_1}$ and $\boldsymbol{\xi_2}$. Note that X_1 is a Bernoulli random variable with parameter $z_0 = y_{1,0}/y_0$.

Let $\hat{\rho}_{1,0}$ and $\hat{\rho}_{2,0}$ be two random variables such that $\hat{\rho}_{1,0}, \hat{\rho}_{2,0} \in (0, 1)$ and $\hat{\rho}_{1,0} \geq \hat{\rho}_{2,0}$ a.s. Let $u : S \to [a, b]$, $0 < a \leq b < \infty$. If $X_1 = 1$ and $z_0 \leq \hat{\rho}_{1,0}$, we return the extracted ball to the urn together with $D_{1,1} = u(\xi_{1,1})$ new red balls. While, if $X_1 = 0$ and $z_0 \geq \hat{\rho}_{2,0}$, we return it to the urn together with $D_{2,1} = u(\xi_{2,1})$ new white balls. If $X_1 = 1$ and $z_0 > \hat{\rho}_{1,0}$, or if $X_1 = 0$ and $z_0 < \hat{\rho}_{2,0}$, the urn composition is not modified. To ease notation, let us denote $W_{1,0} = \mathbf{1}_{\{z_0 \leq \hat{\rho}_{1,0}\}}$ and $W_{2,0} = \mathbf{1}_{\{z_0 \geq \hat{\rho}_{2,0}\}}$. Formally, the extracted ball is always replaced in the urn together with

$$X_1 D_{1,1} W_{1,0} + (1 - X_1) D_{2,1} W_{2,0}$$

new balls of the same color; now, the urn composition becomes

$$\begin{cases} Y_{1,1} = y_{1,0} + X_1 D_{1,1} W_{1,0}, \\ Y_{2,1} = y_{2,0} + (1 - X_1) D_{2,1} W_{2,0}. \end{cases}$$

Set $Y_1 = Y_{1,1} + Y_{2,1}$ and $Z_1 = Y_{1,1}/Y_1$. Now, by iterating the above procedure, we define $\hat{\rho}_{1,1}$ and $\hat{\rho}_{2,1}$ to be two random variables, measurable with respect to the σ-algebra $\mathcal{F}_1 = \sigma(X_1, X_1\xi_{1,1} + (1 - X_1)\xi_{2,1})$, with $\hat{\rho}_{1,1}, \hat{\rho}_{2,1} \in (0, 1)$ and $\hat{\rho}_{1,1} \geq \hat{\rho}_{2,1}$ a.s.

Let $\hat{\rho}_{1,n}$ and $\hat{\rho}_{2,n}$ be two random variables, measurable with respect to the σ-algebra:

$$\mathcal{F}_n = \sigma\big(X_1, X_1\xi_{1,1} + (1 - X_1)\xi_{2,1}, \ldots, X_n, X_1\xi_{1,n} + (1 - X_n)\xi_{2,n}\big),$$

with $\hat{\rho}_{1,n}, \hat{\rho}_{2,n} \in (0, 1)$ and $\hat{\rho}_{1,n} \geq \hat{\rho}_{2,n}$ a.s. We will refer to $\hat{\rho}_{j,n}$ $j = 1, 2$ as threshold parameters. At time $(n + 1)$, a ball is extracted and let $X_{n+1} = 1$ if the ball is red and $X_{n+1} = 0$ otherwise. Then the ball is returned to the urn together with

$$X_{n+1} D_{1,n+1} W_{1,n} + (1 - X_{n+1}) D_{2,n+1} W_{2,n}$$

balls of the same color, where $D_{1,n+1} = u(\xi_{1,n+1})$, $D_{2,n+1} = u(\xi_{2,n+1})$, $W_{1,n} = \mathbf{1}_{\{Z_n \leq \hat{\rho}_{1,n}\}}$, $W_{2,n} = \mathbf{1}_{\{Z_n \geq \hat{\rho}_{2,n}\}}$, and $Z_n = Y_{1,n}/(Y_{1,n} + Y_{2,n})$. Formally,

$$\begin{cases} Y_{1,n+1} = y_{1,0} + \sum_{i=1}^{n+1} X_i D_{1,i} W_{1,i-1}, \\ Y_{2,n+1} = y_{2,0} + \sum_{i=1}^{n+1} (1 - X_i) D_{2,i} W_{2,i-1}, \end{cases}$$

and to simplify the notation let $Y_{n+1} = Y_{1,n+1} + Y_{2,n+1}$. If $X_{n+1} = 1$ and $Z_n > \hat{\rho}_{1,n}$, that is, $W_{1,n} = 0$, or if $X_{n+1} = 0$ and $Z_n < \hat{\rho}_{2,n}$, that is, $W_{2,n} = 0$, the urn composition does not change at time $(n + 1)$. Note that condition $\hat{\rho}_{1,n} \geq \hat{\rho}_{2,n}$ a.s., which implies $W_{1,n} + W_{2,n} \geq 1$, ensures that the urn composition can change with positive probability for any $n \geq 1$, since the replacement matrix (1.2) is never a zero matrix. Since, conditionally on the σ-algebra $\mathcal{F}_n$, X_{n+1} is assumed to be independent of $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$, X_{n+1} is conditionally Bernoulli distributed with parameter Z_n.

We will denote by $N_{1,n}$ and $N_{2,n}$ the number of red and white sampled balls, respectively, after the first n draws, that is $N_{1,n} = \sum_{i=1}^{n} X_i$ and $N_{2,n} = \sum_{i=1}^{n}(1 - X_i)$. Let ρ_1 and ρ_2 be two constants such that $0 < \rho_2 \leq \rho_1 < 1$. We will adopt the following notation:

$$\hat{\rho}_n \equiv \hat{\rho}_{1,n} \mathbf{1}_{\{m_1 > m_2\}} + \hat{\rho}_{2,n} \mathbf{1}_{\{m_1 < m_2\}}, \qquad \rho \equiv \rho_1 \mathbf{1}_{\{m_1 > m_2\}} + \rho_2 \mathbf{1}_{\{m_1 < m_2\}}.$$

Let $m_1 = \int u(y)\mu_1(dy)$ and $m_2 = \int u(y)\mu_2(dy)$ be the means of $\{D_{1,n}; n \geq 1\}$ and $\{D_{2,n}; n \geq 1\}$, respectively. The urn process is then repeated for all $n \geq 1$. We assume throughout the paper that the following condition holds:

$$m_1 \neq m_2. \tag{2.1}$$

2.1. *Law of large numbers*. Our first result is concerned with the LLN.

THEOREM 2.1. *Under the assumptions* (2.1) *and* $\hat{\rho}_n \xrightarrow{\text{a.s.}} \rho$, *we have that*

$$\lim_{n \to \infty} Z_n = \rho \qquad \text{a.s.} \tag{2.2}$$

From Theorem 2.1, we can obtain the convergence of sampled balls, namely $N_{1,n}/n$.

COROLLARY 2.1. *Under the assumptions* (2.1) *and* $\hat{\rho}_n \xrightarrow{\text{a.s.}} \rho$, *we have that*

$$\lim_{n \to \infty} \frac{N_{1,n}}{n} = \rho \qquad \text{a.s.} \tag{2.3}$$

2.2. *Central limit theorem.* We next study the limit distribution of proportion of sampled balls $\frac{N_{1,n}}{n}$. By the description of the model, $\frac{N_{1,n}}{n}$ depends on the sequence $\hat{\rho}_{j,n},\ j=1,2$. However, frequent changes to $\hat{\rho}_{j,n}$ may lead to an erratic behavior of the sequence $\frac{N_{1,n}}{n}$. To stabilize the behavior of $\{\frac{N_{1,n}}{n}; n\geq 1\}$, we fix a constant $q>1$ and introduce the sequence $\{\tilde{\rho}_{j,n}; n\geq 0\}$, where

$$\tilde{\rho}_{j,n} := \hat{\rho}_{j,[q^i]} \qquad \text{where } [q^i] \leq n < [q^{i+1}],\ j=1,2,\ i\in\mathbb{N}; \tag{2.4}$$

that is, we update the threshold parameters only at exponential times $\{[q^i], i=1,2,\dots\}$. An alternative definition of $\tilde{\rho}_{j,n}$ $j=1,2$, which is used in some proofs, is the following: for $n\geq 1$,

$$(\tilde{\rho}_{1,n}, \tilde{\rho}_{2,n}) \equiv (\hat{\rho}_{1,[q^{k_n}]}, \hat{\rho}_{2,[q^{k_n}]}), \qquad k_n := [\log_q(n)]. \tag{2.5}$$

We will denote by $\tilde{\rho}_n = \tilde{\rho}_{1,n}\mathbf{1}_{\{m_1>m_2\}} + \tilde{\rho}_{2,n}\mathbf{1}_{\{m_1<m_2\}}$. We now turn to the statement of the CLT. In the following, $\xrightarrow{d}$ represents the convergence in distribution.

THEOREM 2.2. *Let $\tilde{\rho}_{1,n}$ and $\tilde{\rho}_{2,n}$ be as in* (2.4). *Assume that for any $\varepsilon>0$ and $j=1,2$, there exists $0<c_1<\infty$ such that for large n*

$$\boldsymbol{P}\big(|\hat{\rho}_{j,n}-\rho_j|>\varepsilon\big) \leq c_1\exp\big(-n\varepsilon^2\big). \tag{2.6}$$

Then, *under the assumption* (2.1), *we have that*

$$\sqrt{n}\left(\frac{N_{1,n}}{n} - \bar{\rho}_n\right) \xrightarrow{d} \mathcal{N}\big(0, \rho(1-\rho)\big), \tag{2.7}$$

where $\bar{\rho}_n = \frac{\sum_{i=1}^n \tilde{\rho}_{i-1}}{n}$.

REMARK 2.1. The result of Theorem 2.2 continues to hold if (2.6) is not satisfied, but $\hat{\rho}_n \xrightarrow{\text{a.s.}} \rho$ and the following conditions hold:

(c1) $\limsup_{n\to\infty} \sqrt{n}\boldsymbol{E}[|\hat{\rho}_n-\rho|]<\infty$.

(c2) There exists $\varepsilon\in(0,1/2)$ such that $\hat{\rho}_{j,n}\in[\varepsilon,1-\varepsilon]$ a.s. for any $n\geq 1$, $j=1,2$.

REMARK 2.2. The asymptotic distribution established in (2.7) does not depend on the value of $q>1$, and its main role is to reduce the frequency of updates in $\hat{\rho}_{j,n}$, $j=1,2$. This has practical significance since, in real time implementation of the model, updates to the database (which contains the accruing information) are performed a limited number of times to reduce cost. In a clinical trial, a data and safety monitoring board meets periodically to examine the updated database and make decisions about the future course of the trial. Interim decisions can be made using available responses collectively, rather than one-by-one. For additional technical remarks, see Remark 4.1 below.

Theorem 2.2 introduces an asymptotic bias for $N_{1,n}/n$ given by $(\bar{\rho}_n - \rho)$. We show that this bias is exactly of order $O(n^{-1/2})$; our next proposition makes this observation precise.

PROPOSITION 2.1. *Let $\tilde{\rho}_{1,n}$ and $\tilde{\rho}_{2,n}$ be as in (2.4). Assume also that (2.1) holds. Then, if either (2.6) or $\limsup_{n\to\infty} n\boldsymbol{E}[|\hat{\rho}_n - \rho|^2] < \infty$ holds, then*

$$\limsup_{n\to\infty} n \cdot \boldsymbol{E}\big[|\bar{\rho}_n - \rho|^2\big] < \infty. \tag{2.8}$$

In the case when $\hat{\rho}_{1,n} = \rho_1$ and $\hat{\rho}_{2,n} = \rho_2$ for any $n \geq 0$, Theorem 2.2 provides a CLT for the allocation proportion of MRRU model. This is summarized in the following corollary.

COROLLARY 2.2. *In a MRRU, under the assumption (2.1), we have that*

$$\sqrt{n}\left(\frac{N_{1,n}}{n} - \rho\right) \xrightarrow{d} \mathcal{N}\big(0, \rho(1-\rho)\big).$$

2.3. *Application to clinical trials (revisited).* Consider two competing treatments $\mathcal{T}_1$ and $\mathcal{T}_2$. The random variables $\xi_{1,n}$ and $\xi_{2,n}$ are interpreted as the potential responses to treatments $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, given by subjects that sequentially enter the trial. At all times $n \geq 1$, a subject is allocated to a treatment according to the color of the sampled ball and a new response is collected. Note that only one response is observable from every subject, that is, $X_n\xi_{1,n} + (1 - X_n)\xi_{2,n}$. The function u maps the responses into reinforcement values $D_{1,n}$ and $D_{2,n}$ that update the urn. Typically, u is chosen such that $\mathcal{T}_1$ (or $\mathcal{T}_2$) is considered the superior treatment when $m_1 > m_2$ ($m_1 < m_2$). We assume there exists a unique superior treatment, which is formally stated in assumption (2.1).

We now describe the role of the sequences $\{\hat{\rho}_{1,n}; n \geq 1\}$ and $\{\hat{\rho}_{2,n}; n \geq 1\}$ in clinical trails. Assume the distributions μ_1 and μ_2 are parametric, depending on the vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, respectively, with $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \in \Theta \subset R^d$, with $d \geq 1$. Let $\hat{\boldsymbol{\theta}}_n = (\hat{\boldsymbol{\theta}}_{1,n}, \hat{\boldsymbol{\theta}}_{2,n})$ be an estimator of $\boldsymbol{\theta}$ after the first n allocations, so that $\hat{\boldsymbol{\theta}}_n$ is measurable with respect to the σ-algebra $\mathcal{F}_n$. We assume that the distributions μ_1 and μ_2 are parametrically independent, in the sense that μ_1 does not depend on $\boldsymbol{\theta}_2$ and μ_2 does not depend on $\boldsymbol{\theta}_1$. Hence, $\hat{\boldsymbol{\theta}}_{1,n}$ is computed with the $N_{1,n}$ observations $\{\xi_{1,i} : X_i = 1, i \leq n\}$, while $\hat{\boldsymbol{\theta}}_{2,n}$ is computed with the $N_{2,n}$ observations $\{\xi_{2,i} : X_i = 0, i \leq n\}$. Thus, $\{\hat{\rho}_{1,n}; n \geq 1\}$ and $\{\hat{\rho}_{2,n}; n \geq 1\}$ are defined as follows:

$$\hat{\rho}_{1,n} := f_1(\hat{\boldsymbol{\theta}}_n) \quad \text{and} \quad \hat{\rho}_{2,n} := f_2(\hat{\boldsymbol{\theta}}_n), \qquad \forall n \geq 1, \tag{2.9}$$

where $f_1 : \Theta \to (0, 1)$ and $f_2 : \Theta \to (0, 1)$ are two continuous functions such that

$$f_1(\boldsymbol{x}) \geq f_2(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in \Theta;$$

this implies $\hat{\rho}_{1,n} \geq \hat{\rho}_{2,n}$ a.s. for every $n \geq 1$. Moreover, set

$$\rho_1 := f_1(\boldsymbol{\theta}) \quad \text{and} \quad \rho_2 := f_2(\boldsymbol{\theta}).$$

The LLN presented in Theorem 2.1 suggests a direct interpretation for the functions f_1 and f_2 in a clinical trial context: $f_1(\boldsymbol{\theta})$ and $f_2(\boldsymbol{\theta})$ represent the desired limiting allocations for the sequence $N_{1,n}/n$, in case the superior treatment is $\mathcal{T}_1$ ($m_1 > m_2$) or $\mathcal{T}_2$ ($m_1 < m_2$), respectively. This is a great improvement, since the design can target an arbitrary known function of all the parameters of the response distributions.

Ideally, f_1 and f_2 are chosen to obtain good statistical properties from the design. Typically, in clinical trials, a design is constructed to satisfy certain optimality criteria related to its statistical performances (e.g., power; see [21]). Letting $\eta(\boldsymbol{\theta})$ denote the limit proportion of subjects to be allocated to treatment $\mathcal{T}_1$, this design can be obtained by the urn model described in Section 2 by choosing $f_1(\boldsymbol{\theta}) = f_2(\boldsymbol{\theta}) = \eta(\boldsymbol{\theta})$. However, in some experiments, ethical aspects are important and the main goal may be to assign fewer subjects to the inferior treatment; in this case we choose $f_1(\boldsymbol{\theta}) \simeq 1$ and $f_2(\boldsymbol{\theta}) \simeq 0$. Designs requiring both ethical and statistical goals can also be obtained from our design, by setting $f_1(\boldsymbol{\theta}) \geq \eta(\boldsymbol{\theta}) \geq f_2(\boldsymbol{\theta})$. For instance, we may take

$$\begin{aligned} f_1(\boldsymbol{\theta}) &= p \cdot \eta(\boldsymbol{\theta}) + (1-p) \cdot 1, \\ f_2(\boldsymbol{\theta}) &= p \cdot \eta(\boldsymbol{\theta}) + (1-p) \cdot 0, \qquad p \in (0,1], \end{aligned} \tag{2.10}$$

where p is a biasing term, which introduces a trade-off between the ethics and statistical properties.

Finally, it is worth emphasizing that conditions $\hat{\rho}_n \stackrel{\text{a.s.}}{\to} \rho$ and (2.6) required in the LLN of Theorem 2.1 and in the CLT of Theorem 2.2, respectively, are straightforwardly satisfied when we take $\hat{\boldsymbol{\theta}}_n$ to be maximum likelihood estimators (MLEs) for $\boldsymbol{\theta}$.

Moreover, condition (c2) in Remark 2.1 is equivalent of the assumption that the ranges of f_1 and f_2 are subsets of $[\varepsilon, 1-\varepsilon]$, for some $\varepsilon \in (0, 1/2)$.

3. Harmonic moments and related asymptotics.

3.1. *Harmonic moments*. In this subsection, we show that the harmonic moments of the total number of balls in the urn are uniformly bounded. This is a key result which is needed in several probabilistic estimates, and in particular in the proof of the CLT. More specifically, as explained previously the results concerning the asymptotic behavior of $N_{1,n}/n$, depend critically on the behavior of $(Z_n - \hat{\rho}_n)$. In Section 3.2, we provide bounds for $Y_n(Z_n - \hat{\rho}_n)$, by using comparison arguments with the MRRU model. Now, to replace the random scaling Y_n by the deterministic scaling n, one needs to investigate the behavior of n/Y_n. Our next theorem provides a precise estimates of the jth moment of n/Y_n for any $j \geq 0$.

THEOREM 3.1. *Under the assumptions* (2.1) *and* (2.6), *for any* $j > 0$, *we have that*

$$\sup_{n\to\infty} \boldsymbol{E}\left[\left(\frac{n}{Y_n}\right)^j\right] < \infty.$$

In the proof of Theorem 3.1, we need the following lemma that provides an upper bound on the increments of the urn process Z_n, by imposing a condition on the total number of balls in the urn Y_n. Hence, the proof of Theorem 3.1 is reported after the following result.

LEMMA 3.1. *For any* $\varepsilon \in (0, 1)$, *we have that*

$$\left\{Y_n > b\left(\frac{1-\varepsilon}{\varepsilon}\right)\right\} \subseteq \{|Z_{n+1} - Z_n| < \varepsilon\}. \tag{3.1}$$

PROOF. The difference $(Z_{n+1} - Z_n)$ can be expresses as follows:

$$\frac{Y_{1,n} + X_{n+1}W_{1,n}D_{1,n+1}}{Y_n + X_{n+1}W_{1,n}D_{1,n+1} + (1 - X_{n+1})W_{2,n}D_{2,n+1}} - \frac{Y_{1,n}}{Y_n}.$$

Consider $\{Z_{n+1} > Z_n\}$, since the case $\{Z_{n+1} < Z_n\}$ is analogous. Note that $\{Z_{n+1} > Z_n\}$ implies $\{X_{n+1} = 1\}$ and $\{W_{1,n} = 1\}$. Then, since $D_{1,n+1} < b$ a.s., on the set $\{Z_{n+1} > Z_n\}$ we have

$$\begin{aligned} Z_{n+1} - Z_n &\le \frac{Y_{1,n} + D_{1,n+1}}{Y_n + D_{1,n+1}} - \frac{Y_{1,n}}{Y_n} \\ &= \frac{D_{1,n+1}}{D_{1,n+1} + Y_n}(1 - Z_n) \le \frac{b}{b + Y_n} < \varepsilon, \end{aligned}$$

where the last inequality follows from $\{Y_n > b(1 - \varepsilon)/\varepsilon\}$ in (3.1). $\square$

PROOF OF THEOREM 3.1. In this proof, when we have a set of integers $\{[a_1], \dots, [b_1]\}$ with $a_1, b_1 \notin \mathbb{N}$, to ease notation we will just write $\{a_1, \dots, b_1\}$, omitting the symbol $[\cdot]$. First, note that, since $D_{1,i}, D_{2,i} \ge a$ a.s. for any $i \ge 1$ and $Y_0 > 0$, we have that

$$\begin{aligned} Y_n &= Y_0 + \sum_{i=1}^{n} \big(D_{1,i}X_iW_{1,i-1} + D_{2,i}(1 - X_i)W_{2,i-1}\big) \\ &\ge Y_0 + a \cdot \sum_{i=1}^{n} \big(X_iW_{1,i-1} + (1 - X_i)W_{2,i-1}\big) \qquad (3.2) \\ &\ge Y_0 + a \cdot \sum_{i=n\beta}^{n} \big(X_iW_{1,i-1} + (1 - X_i)W_{2,i-1}\big), \end{aligned}$$

for any $\beta \in (0,1)$. To keep calculation transparent, we choose $\beta = 1/2$. We recall that, by construction, we have that $W_{1,i-1}, W_{2,i-1} \in \{0;1\}$ and $W_{1,i-1} + W_{2,i-1} \geq 1$ for any $i \geq 1$; hence, the random variables $X_i W_{1,i-1} + (1-X_i)W_{2,i-1}$ are, conditionally on the σ-algebra $\mathcal{F}_{i-1}$, Bernoulli distributed with parameter greater than or equal to $\min\{Z_{i-1}; 1-Z_{i-1}\}$. Hence, the behavior of Y_n is intrinsically related to the behavior of Z_n.

Thus, let us introduce the sets $A_{d,n}$ (down), $A_{c,n}$ (center) and $A_{u,n}$ (up) as follows:

$$A_{d,n} := \left\{ \bigcup_{n/2 \leq i \leq n} \{Z_i < c\} \right\},$$

$$A_{c,n} := \left\{ \bigcap_{n/2 \leq i \leq n} \{Z_i \in [c, 1-c]\} \right\},$$

$$A_{u,n} := \left\{ \bigcup_{n/2 \leq i \leq n} \{Z_i > 1-c\} \right\},$$

where $c \in (0,1)$ will be appropriately fixed more ahead in the proof. Then we perform the following decomposition on the behavior of $\{Z_i; n/2 \leq i \leq n\}$:

$$\boldsymbol{E}\left[\left(\frac{n}{Y_n}\right)^j\right] \leq \left(\frac{n}{Y_0}\right)^j \cdot \boldsymbol{P}(A_{d,n}) + \boldsymbol{E}\left[\left(\frac{n}{Y_n}\right)^j \mathbf{1}_{A_{c,n}}\right] + \left(\frac{n}{Y_0}\right)^j \cdot \boldsymbol{P}(A_{u,n}).$$

On the set $A_{c,n}$ the process $\{Z_i; n/2 \leq i \leq n\}$ is bounded away from the extreme values $\{0;1\}$. Hence, we can use comparison arguments with a sequence of i.i.d. Bernoulli random variables with parameter c to get the boundedness of $\boldsymbol{E}[(n/Y_n)^j \mathbf{1}_{A_{c,n}}]$. After that, we will focus on proving that $\boldsymbol{P}(A_{d,n})$ and $\boldsymbol{P}(A_{u,n})$ converge to zero with a sub-exponential rate.

First, note that on the set $A_{c,n}$ the random variables

$$X_i W_{1,i-1} + (1-X_i) W_{2,i-1}$$

are, conditionally on the σ-algebra $\mathcal{F}_{i-1}$, Bernoulli with parameter greater than or equal to c for any $i = n/2, \ldots, n$. Hence, if we introduce $\{B_i; i \geq 1\}$ a sequence of i.i.d. Bernoulli random variables with parameter c, from (3.2) we have that

$$\boldsymbol{E}\left[\left(\frac{n}{Y_n}\right)^j \mathbf{1}_{A_{c,n}}\right] \leq \frac{1}{a^j} \boldsymbol{E}\left[\left(\frac{n}{Y_0/a + \sum_{i=n/2}^n B_i}\right)^j\right].$$

We now show that

$$\limsup_{n\to\infty} \boldsymbol{E}\left[\left(\frac{n}{K_0 + \sum_{i=1}^n B_i}\right)^j\right] < \infty,$$

with $K_0 = Y_0/a$. To this end, we apply Theorem 2.1 of [12], with $n_0 = 1$, $p = j$, $Z_{i,n} = B_i + Y_0/n$ for $i \leq n$. All the assumptions of the theorem are satisfied in our case. In fact, at first we have $\boldsymbol{E}[\bar{Z}_{n_0}^{-p}] < \infty$ because $\boldsymbol{E}[(Y_0 + B_1)^{-j}] \leq K_0^{-j} < \infty$.

Next, note that $Z_{i,n}$ are identically distributed for all $i \leq n$, since B_i are i.i.d. Bernoulli with parameter c. Finally, $\bar{Z}_n$ converges in distribution, since $\bar{Z}_n = \sum_{i=1}^n B_i/n + K_0 \xrightarrow{\text{a.s.}} c + K_0$. Hence, by Theorem 2.1 of [12], it follows that $\boldsymbol{E}[\bar{Z}_n^{-p}]$ is uniformly integrable. As a consequence,

$$\limsup_{n\to\infty} \boldsymbol{E}\left[\left(\frac{n}{K_0 + \sum_{i=1}^n B_i}\right)^j\right] = \limsup_{n\to\infty} \boldsymbol{E}[\bar{Z}_n^{-p}] < \infty.$$

Now, we will prove that $\boldsymbol{P}(A_{d,n})$ and $\boldsymbol{P}(A_{u,n})$ converge to zero with a sub-exponential rate. To this end, we will show that $\hat{\rho}_{1,n}$ and $\hat{\rho}_{2,n}$ are bounded away from 0 and 1 with a probability that converges to 1 with a sub-exponential rate. Formally, fix $\varepsilon > 0$, such that $\rho_1 + \varepsilon < 1$ and $\rho_2 - \varepsilon > 0$, let $\alpha \in (0, 1)$ and for any $n \geq 1$ define the following sets:

$$A_{1,n} := \Big\{ \sup_{i \geq \alpha\sqrt{n}} \{\hat{\rho}_{1,i}\} > \rho_1 + \varepsilon \Big\},$$

$$A_{2,n} := \Big\{ \inf_{i \geq \alpha\sqrt{n}} \{\hat{\rho}_{2,i}\} < \rho_2 - \varepsilon \Big\},$$

$$A_{3,n} := \Big\{ \inf_{i \geq \alpha\sqrt{n}} \big\{\min\{1 - \hat{\rho}_{1,i}; \hat{\rho}_{2,i}\}\big\} \geq \min\{1 - \rho_1; \rho_2\} - \varepsilon \Big\},$$

where we recall that $\hat{\rho}_{1,i}$ and $\hat{\rho}_{2,i}$ are the adaptive thresholds. Note that $A_{1,n} \cup A_{2,n} \cup A_{3,n} = \Omega$. We have that

$$\begin{aligned} \boldsymbol{P}(A_{d,n}) &\leq \boldsymbol{P}(A_{1,n}) + \boldsymbol{P}(A_{2,n}) + \boldsymbol{P}(A_{3,n} \cap A_{d,n}), \\ \boldsymbol{P}(A_{u,n}) &\leq \boldsymbol{P}(A_{1,n}) + \boldsymbol{P}(A_{2,n}) + \boldsymbol{P}(A_{3,n} \cap A_{u,n}). \end{aligned}$$

First, we prove that $\boldsymbol{P}(A_{1,n})$ and $\boldsymbol{P}(A_{2,n})$ converge to zero with a sub-exponential rate. Consider the term $\boldsymbol{P}(A_{1,n})$. From the definition of $A_{1,n}$, we obtain

$$\boldsymbol{P}(A_{1,n}) = \boldsymbol{P}\bigg(\bigcup_{i \geq \alpha\sqrt{n}} \{\hat{\rho}_{1,i} > \rho_1 + \varepsilon\}\bigg) \leq \sum_{i \geq \alpha\sqrt{n}} \boldsymbol{P}(\hat{\rho}_{1,i} > \rho_1 + \varepsilon).$$

From (2.6), for large i we have that

$$\boldsymbol{P}(\hat{\rho}_{1,i} > \rho_1 + \varepsilon) \leq c_1 \exp(-i\varepsilon^2),$$

with $0 < c_1 < \infty$. Hence, using the fact that Y_n is increasing, we have that

$$\begin{aligned} \boldsymbol{P}(A_{1,n}) &\leq \sum_{i \geq \alpha\sqrt{n}} \boldsymbol{P}(\hat{\rho}_{1,i} > \rho_1 + \varepsilon) \\ &\leq c_1 \sum_{i \geq \alpha\sqrt{n}} \exp(-i\varepsilon^2) \\ &= c_1 \exp(-\alpha\sqrt{n}\varepsilon^2). \end{aligned}$$

Similar arguments can be applied to prove $\boldsymbol{P}(A_{2,n}) \to 0$ with a sub-exponential rate. Finally, we show that $\boldsymbol{P}(A_{3,n} \cap A_{d,n})$ and $\boldsymbol{P}(A_{3,n} \cap A_{u,n})$ converge to zero with a sub-exponential rate. Consider $\boldsymbol{P}(A_{3,n} \cap A_{d,n})$, since the proof for $\boldsymbol{P}(A_{3,n} \cap A_{u,n})$ is analogous. First, let us introduce $\phi := \min\{\rho_2; 1 - \rho_1\}$, and rewrite $A_{3,n}$ as follows:

$$A_{3,n} = \Big\{ \inf_{i \geq \alpha\sqrt{n}} \{\hat{\rho}_{2,n}; 1 - \hat{\rho}_{1,n}\} \geq \phi - \varepsilon \Big\}.$$

Define the set $\tilde{A}_{d,n}$ as follows:

$$\tilde{A}_{d,n} := \Big\{ \bigcap_{\alpha\sqrt{n} \leq i \leq n/2} \{Z_i < c\} \Big\}.$$

We now set an appropriate value of c such that

$$\{A_{3,n} \cap A_{d,n}\} \subset \{A_{3,n} \cap \tilde{A}_{d,n}\}, \tag{3.3}$$

for any $n \geq 1$. To this end, we need to find c such that $\{Z_i \geq c\} \subset \{Z_{i+1} \geq c\}$ for any $i \geq \alpha\sqrt{n}$. First, note that on the set $A_{3,n}$, $\{\hat{\rho}_{2,i} \geq (\phi - \varepsilon)\}$ for any $i \geq \alpha\sqrt{n}$. Hence, for any $c < (\phi - \varepsilon)$, if $\{c \leq Z_i \leq (\phi - \varepsilon)\}$ we have $W_{2,i} = 0$, that implies $Z_{i+1} \geq Z_i$ and so $Z_{i+1} \geq c$. Alternatively, if $\{Z_i \geq (\phi - \varepsilon) > c\}$, the set $\{Z_{i+1} \leq Z_i\}$ is possible, and hence we have to bound the increments of Z_n to guarantee that $Z_{i+1} \geq c$, that is, find c such that

$$|Z_{i+1} - Z_i| < (\phi - \varepsilon) - c, \qquad \forall i \geq 0.$$

Using (3.1), we obtain

$$c \leq p_0 := \frac{Y_0}{Y_0 + b} \cdot (\phi - \varepsilon). \tag{3.4}$$

This guarantees that (3.3) holds for any $n \geq 1$.

Next, we show that $\boldsymbol{P}(A_{3,n} \cap \tilde{A}_{d,n})$ converges to zero with a sub-exponential rate. To this end, first note that on the set $A_{3,n}$, we have $\hat{\rho}_{2,i} > \rho_2 - \varepsilon$ for any $i = \alpha\sqrt{n}, \dots, n/2$; moreover, on the set $\tilde{A}_{d,n}$, we have $Z_i < p_0$ for any $i = \alpha\sqrt{n}, \dots, n/2$. These considerations imply that $W_{2,i} = 0$ and $W_{1,i} = 1$ for any $i = \alpha\sqrt{n}, \dots, n/2$, on the set $A_{3,n} \cap \tilde{A}_{d,n}$. Hence, we can write

$$Z_{n/2} = \frac{Y_{1,\alpha\sqrt{n}} + \sum_{i=\alpha\sqrt{n}}^{n/2} X_i D_{1,i}}{Y_{\alpha\sqrt{n}} + \sum_{i=\alpha\sqrt{n}}^{n/2} X_i D_{1,i}} \geq \frac{y_{1,0} + a\sum_{i=\alpha\sqrt{n}}^{n/2} X_i}{(y_0 + \alpha\sqrt{n}b) + a\sum_{i=\alpha\sqrt{n}}^{n/2} X_i}, \tag{3.5}$$

where the inequality is because $Y_{1,\alpha\sqrt{n}} \geq y_{1,0}$, $Y_{\alpha\sqrt{n}} \leq y_0 + \alpha\sqrt{n}b$ and $D_{1,i} \geq a$ a.s. for any $i \geq 1$. Now, define for any $n \geq 1$ the set $A_{4,n}$ as follows:

$$A_{4,n} := \Big\{ \sum_{i=\alpha\sqrt{n}}^{n/2} X_i > \frac{p_0}{a(1 - p_0)}(y_0 + \alpha\sqrt{n}b) \Big\},$$

and consider the set $A_{3,n} \cap \tilde{A}_{d,n} \cap A_{4,n}$. On the set $A_{3,n} \cap \tilde{A}_{d,n}$ we can use the definition of $A_{4,n}$ in (3.5), obtaining

$$\{A_{3,n} \cap \tilde{A}_{d,n} \cap A_{4,n}\} \subset \{\{Z_{n/2} > p_0\} \cap \tilde{A}_{d,n}\}.$$

However, $\{Z_{n/2} > p_0\} \cap \tilde{A}_{d,n} = \varnothing$. Hence, $\boldsymbol{P}(A_{3,n} \cap \tilde{A}_{d,n} \cap A_{4,n}) = 0$ and it is sufficient to show that $\boldsymbol{P}(A_{3,n} \cap \tilde{A}_{d,n} \cap A_{4,n}^C)$ converges to zero with a sub-exponential rate.

Toward this, note that on the set $A_{3,n} \cap \tilde{A}_{d,n}$ we have $Z_{i+1} \geq Z_i$ for any $i = \alpha\sqrt{n}, \dots, n/2$, since we previously showed that $W_{2,i} = 0$ and $W_{1,i} = 1$. Hence, on the set $A_{3,n} \cap \tilde{A}_{d,n}$, $\{X_i, i = \alpha\sqrt{n}, \dots, n/2\}$ are conditionally Bernoulli random variables with parameter $p_i \geq Z_{\alpha\sqrt{n}}$ a.s. Now, let us denote by $\{\varrho_{i,n}; i = 1, \dots, n/2 - \alpha\sqrt{n}\}$ a sequence of i.i.d. Bernoulli random variables with parameter $z_{0,n}$, defined as

$$z_{0,n} := \frac{y_{1,0}}{y_0 + \alpha\sqrt{n}b} \leq Z_{\alpha\sqrt{n}} \qquad \text{a.s.};$$

it follows that $\boldsymbol{P}(A_{3,n} \cap \tilde{A}_{d,n} \cap A_{4,n}^C)$ is less than or equal than

$$\boldsymbol{P}\left(\sum_{i=1}^{n/2-\alpha\sqrt{n}} \varrho_{i,n} \leq \frac{p_0}{a(1-p_0)}(y_0 + \alpha\sqrt{n}b)\right). \tag{3.6}$$

Finally, we use the following Chernoff's upper bound for i.i.d. random variables in [0, 1] (see [10]):

$$\boldsymbol{P}\big(S_n \leq c_0 \cdot \boldsymbol{E}[S_n]\big) \leq \exp\left(-\frac{(1-c_0)^2}{2} \cdot \boldsymbol{E}[S_n]\right), \tag{3.7}$$

with $c_0 \in (0, 1)$ and $S_n = \sum_i^n X_i$. In our case, we have that (3.6) can be written as $\boldsymbol{P}(S_n \leq c_n \cdot \boldsymbol{E}[S_n])$, where $S_n = \sum_{i=1}^{n/2-\alpha\sqrt{n}} \varrho_{i,n}$ and

$$\boldsymbol{E}[S_n] = \left(\frac{n}{2} - \alpha\sqrt{n}\right)\frac{y_{1,0}}{(y_0 + \alpha\sqrt{n}b)} \quad \text{and} \quad c_n = \frac{p_0}{a(1-p_0)} \frac{(y_0 + \alpha\sqrt{n}b)^2}{y_{1,0}(n/2 - \alpha\sqrt{n})};$$

since $\alpha > 0$ can be chosen arbitrary small, we can define an integer n_0 such that $c_n < c_0$ for any $n \geq n_0$, so that

$$\boldsymbol{P}\big(S_n \leq c_n \cdot \boldsymbol{E}[S_n]\big) \leq \boldsymbol{P}\big(S_n \leq c_0 \cdot \boldsymbol{E}[S_n]\big).$$

Hence, by using (3.7), for any $n \geq n_0$ we have that

$$\boldsymbol{P}\big(A_{3,n} \cap A_{4,n}^C\big) \leq \exp\left(-\frac{(1-c_0)^2}{2} \cdot \boldsymbol{E}[S_n]\right),$$

which converges to zero with a sub-exponential rate since

$$\boldsymbol{E}[S_n] = \frac{y_{1,0}(n/2 - \alpha\sqrt{n})}{y_0 + \alpha\sqrt{n}b} \sim \frac{n}{\sqrt{n}} = \sqrt{n}.$$

This completes the proof. $\square$

REMARK 3.1. The result of Theorem 3.1 can also be obtained by relaxing assumption (2.6). In that case, we need conditions $\hat{\rho}_n \xrightarrow{\text{a.s.}} \rho$ and (c2) to be satisfied. Then the proof is similar by taking $A_{1,n} = \{\sup_{i \geq \alpha\sqrt{n}}\{\hat{\rho}_{1,i}\} > 1 - \varepsilon\}$, $A_{2,n} = \{\inf_{i \geq \alpha\sqrt{n}}\{\hat{\rho}_{2,i}\} < \varepsilon\}$, and $A_{3,n} = \{\inf_{i \geq \alpha\sqrt{n}}\{\min\{1 - \hat{\rho}_{1,i}; \hat{\rho}_{2,i}\}\} \geq \varepsilon\}$. Then, $\boldsymbol{P}(A_{1,n}) = \boldsymbol{P}(A_{2,n}) = 0$ for any $n \geq 1$.

3.2. *A uniform bound.* In this subsection, we provide a uniform bound for the scaled difference between Z_t and $\tilde{\rho}_t$. To make precise statements, we start by introducing additional notation. Set $\Delta_{j,k} := \operatorname{sign}(m_1 - m_2)(\tilde{\rho}_{q^j+k} - Z_{q^j+k})$ and $\tilde{T}_{j,k} := Y_{q^j+k}\Delta_{j,k}$, for any $j \geq 1$ and any $k = 1, \dots, d_j$, where $d_j := q^{j+1} - q^j$. Note that, using (2.4) for any $k \in \{1, \dots, d_j\}$ $\Delta_{j,k} = \operatorname{sign}(m_1 - m_2)(\hat{\rho}_{q^j} - Z_{q^j+k})$. Let $\{\tau_j; j \geq 1\}$ be a sequence of stopping times defined as follows:

$$\tau_j := \begin{cases} \inf\{k \geq 1 : \tilde{T}_{j,k} \in [-b, 0]\}, & \text{if } \{k \geq 1 : \tilde{T}_{j,k} \in [-b, 0]\} \neq \varnothing; \\ \infty, & \text{otherwise.} \end{cases} \tag{3.8}$$

In Theorem 3.2, we provide a L_1-uniform bound for the scaled distance between the urn proportion Z_{q^j+k} and the threshold $\tilde{\rho}_{q^j+k}$, on the set $\{\tau_j \leq k\}$.

THEOREM 3.2. *Let $\tilde{\rho}_{1,n}$ and $\tilde{\rho}_{2,n}$ be as in* (2.4). *Then, under the assumptions* (2.1) *and* (2.6), *there exists a constant $C > 0$ such that*

$$\sup_{j \geq 1} \sup_{1 \leq k \leq d_j} \boldsymbol{E}\big[q^j \cdot |\Delta_{j,k}| \mathbf{1}_{\{\tau_j \leq k\}}\big] < C, \tag{3.9}$$

where $d_j = q^{j+1} - q^j$.

The proof uses comparison arguments with the MRRU model and related asymptotic results. Hence, we first present the results concerning the MRRU model in Section 3.2.1. The proof of Theorem 3.2 is then reported in Section 3.2.2.

3.2.1. *Estimates for the MRRU model.* In this subsection, we present some probabilistic estimates concerning the MRRU model which are needed in the proof of Theorem 3.2. We recall that for the MRRU the threshold are fixed, that is, $\hat{\rho}_{j,n} = \rho_j$ for any $n \geq 1$, $j = 1, 2$. Hence, in this subsection we consider $W_{1,n} = \mathbf{1}_{\{Z_n \leq \rho_1\}}$ and $W_{2,n} = \mathbf{1}_{\{Z_n \geq \rho_2\}}$. We start by introducing some quantities related to the MRRU model. Let $\{T_n; n \geq 0\}$ be the process defined as

$$T_n := \operatorname{sign}(m_1 - m_2) \cdot Y_n(\rho - Z_n), \tag{3.10}$$

which is sometimes useful to represent it as follows:

$$T_n = \operatorname{sign}(m_1 - m_2) \cdot \big(\rho Y_{2,n} - (1 - \rho) Y_{1,n}\big).$$

Then, let t_0 be the following stopping time:

$$t_0 := \inf\{k \geq 0 : T_k \in [-b, 0]\}. \tag{3.11}$$

Let

$$L_n := \{0 \le k \le n : T_{n-k} \in [-b, 0]\},$$

and let $\{s_n; n \ge 1\}$ be a sequence of random times defined as follows:

$$s_n = \begin{cases} \inf\{L_n\}, & \text{if } L_n \neq \varnothing; \\ \infty, & \text{otherwise,} \end{cases} \tag{3.12}$$

where we recall that b is the maximum value of the urn reinforcements, that is, $D_{1,n}, D_{2,n} \le b$ a.s. for any $n \ge 1$. Note that by definition $\{s_n = \infty\} = \{t_0 > n\}$. In Theorem 3.3, we provide the L_2-uniform bound for $Y_n(Z_n - \rho)$, on the set $\{t_0 \le n\}$.

THEOREM 3.3. *For an MRRU, under the assumption* (2.1), *there exists a constant $C > 0$ such that*

$$\sup_{n \ge 1} \boldsymbol{E}\big[\big(Y_n|\rho - Z_n|\big)^2 | t_0 \le n\big] \le C. \tag{3.13}$$

The proof of the above theorem uses the boundedness of the moments of the excursion times s_n, which is provided in Theorem 3.4 below.

THEOREM 3.4. *For an MRRU, under the assumption* (2.1), *there exists a constant $C > 0$ such that*

$$\sup_{n \ge 1}\big\{\boldsymbol{E}\big[s_n^2 | t_0 \le n\big]\big\} \le C.$$

In the proof of Theorem 3.4, we need to couple the MRRU model with a particular urn model $\{\tilde{Z}_n; n \ge 1\}$. The processes are coupled, in the sense that: (i) the potential reinforcements are the same, that is, $\tilde{D}_{1,n} = D_{1,n}$ and $\tilde{D}_{2,n} = D_{2,n}$ a.s.; (ii) the drawing process is defined on the same probability space, that is, $\tilde{U}_n = U_n$ a.s. where $\{U_n; n \ge 1\}$ and $\{\tilde{U}_n; n \ge 1\}$ are i.i.d. uniform random variables such that $X_{n+1} := \mathbf{1}_{\{U_{n+1} < Z_n\}}$ and $\tilde{X}_{n+1} := \mathbf{1}_{\{\tilde{U}_{n+1} < \tilde{Z}_n\}}$ for any $n \ge 1$, respectively.

We now describe the urn model $\{\tilde{Z}_n; n \ge 1\}$. Fix a constant $\tilde{y}_0 \in (0, Y_0]$ and $z_0 = \rho_1$. The process $\{\tilde{Z}_n; n \ge 1\}$ evolves as follows: if $s_{n-1} = 0$, that is, $Z_{n-1} \ge \rho_1$, then $\tilde{X}_n = \mathbf{1}_{\{\tilde{U}_n < \rho_1\}}$ and

$$\begin{cases} \tilde{Y}_{1,n} = \rho_1 \cdot \tilde{y}_0 + \tilde{X}_n \tilde{D}_{1,n}, \\ \tilde{Y}_{2,n} = (1 - \rho_1) \cdot \tilde{y}_0 + (1 - \tilde{X}_n)\tilde{D}_{2,n}; \end{cases} \tag{3.14}$$

if $s_{n-1} = k \ge 1$, that is, $Z_{n-1} < \rho_1$, then $\tilde{X}_n = \mathbf{1}_{\{\tilde{U}_n < \tilde{Z}_{n-1}\}}$ and

$$\begin{cases} \tilde{Y}_{1,n} = \tilde{Y}_{1,n-1} + \tilde{X}_n \tilde{D}_{1,n}, \\ \tilde{Y}_{2,n} = \tilde{Y}_{2,n-1} + (1 - \tilde{X}_n)\tilde{D}_{2,n}; \end{cases} \tag{3.15}$$

where $\tilde{Y}_n := \tilde{Y}_{1,n} + \tilde{Y}_{2,n}$ and $\tilde{Z}_n := \tilde{Y}_{1,n}/\tilde{Y}_n$. The urn model is well-defined since s_{n-1} is $\mathcal{F}_{n-1}$-measurable. It is worth noticing that $\tilde{Z}_n$ represents a generalized Pólya urn evaluated after exactly $(s_{n-1}+1)$ steps, with initial composition $\rho_1\tilde{y}_0$ red and $\rho_1(1-\tilde{y}_0)$ white balls. In the next lemma, we state an important relation among the MRRU model and the process $\{\tilde{Z}_n; n \geq 1\}$, needed in the proof of Theorem 3.4.

LEMMA 3.2. *Consider the urn model $\{\tilde{Z}_n; n \geq 1\}$ defined in (3.14) and (3.15) coupled with the MRRU process $\{Z_n; n \geq 1\}$. Let $\tilde{T}_n := \operatorname{sign}(m_1 - m_2) \cdot \tilde{Y}_n(\rho - \tilde{Z}_n)$ for any $n \geq 1$. Then, on the set $\{\exists j < n : T_j \leq 0\}$, we have that*

$$\{T_n > 0\} \subset \{\tilde{T}_n \geq T_n\}.$$

PROOF. Without loss of generality, assume $m_1 > m_2$, which implies $\rho = \rho_1$ and $T_n = Y_n(\rho_1 - Z_n)$. Noting $\tilde{T}_n = \rho_1\tilde{Y}_{2,n} - (1-\rho_1)\tilde{Y}_{1,n}$, we now use induction to complete the proof of the lemma. Note that, on the set {there exists $j < n$: $T_j \leq 0$}, s_n is almost surely finite. On the set $\{s_n = 0\}$, that is, $\{T_n \leq 0\}$, we can immediately see that $\{T_{n+1} > 0\}$ implies $\{\tilde{T}_{n+1} \geq T_{n+1}\}$ and $\{\tilde{Z}_{n+1} \leq Z_{n+1}\}$. In fact, from $\{T_n \leq 0\}$ and $\{T_{n+1} > 0\}$ we have $X_{n+1} = 0$ and $W_{2,n} = 1$, so that

$$\begin{aligned} T_{n+1} &= T_n + \rho_1 D_{2,n+1} \leq \rho_1 D_{2,n+1} = \rho_1 \tilde{D}_{2,n+1} = \tilde{T}_{n+1}, \\ Z_{n+1} &= \frac{Z_n Y_n}{Y_n + D_{2,n+1}} \geq \frac{\rho_1 \tilde{y}_0}{\tilde{y}_0 + \tilde{D}_{2,n+1}} = \tilde{Z}_{n+1}. \end{aligned}$$

Now, consider the set $\{s_n \geq 1\}$ and assume by induction hypothesis that

$$\{\tilde{T}_i \geq T_i > 0, \tilde{Z}_i \leq Z_i < \rho_1, \forall i = n - s_n + 1, \dots, n\} \cap \{T_{n+1} > 0\}. \tag{3.16}$$

Then we will show that $\tilde{T}_{n+1} \geq T_{n+1}$ and $\tilde{Z}_{n+1} \leq Z_{n+1}$. Since $T_n = \rho_1 Y_{2,n} - (1 - \rho_1)Y_{1,n}$, we note that

$$T_{n+1} = T_{n-s_n} + \sum_{i=n-s_n+1}^{n+1} \left[\rho_1(1-X_i)D_{2,i}W_{2,i-1} - (1-\rho_1)X_i D_{1,i} W_{1,i-1}\right],$$

where we recall that for the MRRU model $W_{1,i} = \mathbf{1}_{\{Z_i \leq \rho_1\}}$ and $W_{2,i} = \mathbf{1}_{\{Z_i \geq \rho_2\}}$. Since $T_{n-s_n} \leq 0$, $W_{2,i-1} \leq 1$, by (3.16) $W_{1,i} = 1$ for any $i = n - s_n + 1, \dots, n$, $X_{n-s_n+1} = 0$, and by construction $\tilde{D}_{1,i} = D_{1,i}$ and $\tilde{D}_{2,i} = D_{2,i}$, we have

$$T_{n+1} \leq \sum_{i=n-s_n+1}^{n+1} \left[\rho_1(1-X_i)\tilde{D}_{2,i} - (1-\rho_1)X_i\tilde{D}_{1,i}\right].$$

Moreover, by (3.16) we have $X_{i+1} = \mathbf{1}_{\{U_{i+1} < Z_i\}} \geq \mathbf{1}_{\{\tilde{U}_{i+1} < \tilde{Z}_i\}} = \tilde{X}_{i+1}$ for any $i = n - s_n + 1, \dots, n$. Hence, we can write

$$T_{n+1} \leq \rho_1 \sum_{i=n-s_n+1}^{n+1} (1-\tilde{X}_i)\tilde{D}_{2,i} - (1-\rho_1) \sum_{i=n-s_n+1}^{n+1} \tilde{X}_i\tilde{D}_{1,i} = \tilde{T}_{n+1}.$$

Similarly, we can prove that $\tilde{Z}_{n+1} \leq Z_{n+1}$. Note that

$$Z_{n+1} = \frac{Z_{n-s_n} Y_{n-s_n} + \sum_{i=n-s_n+1}^{n+1} X_i D_{1,i} W_{1,i-1}}{Y_{n-s_n} + \sum_{i=n-s_n+1}^{n+1} X_i D_{1,i} W_{1,i-1} + \sum_{i=n-s_n+1}^{n+1} (1 - X_i) D_{2,i} W_{2,i-1}}.$$

Now, since $Z_{n-s_n} \geq \rho_1$, $Y_{n-s_n} \geq \tilde{y}_0$ and $X_{i+1} \geq \tilde{X}_{i+1}$ for any $i = n - s_n + 1, \ldots, n$, it follows that

$$Z_{n+1} \geq \frac{\rho_1 Y_0 + \sum_{i=n-s_n+1}^{n+1} \tilde{X}_i \tilde{D}_{1,i}}{\tilde{y}_0 + \sum_{i=n-s_n+1}^{n+1} \tilde{X}_i \tilde{D}_{1,i} + \sum_{i=n-s_n+1}^{n+1} (1 - \tilde{X}_i) \tilde{D}_{2,i}} = \tilde{Z}_{n+1},$$

which completes our proof by induction. □

PROOF OF THEOREM 3.4. Without loss of generality, assume $m_1 > m_2$, which implies $\rho = \rho_1$ and $T_n = Y_n(\rho_1 - Z_n)$. The structure of the proof is the following. The aim is to show that $\boldsymbol{P}(s_n = k | t_0 \leq n)$ converges to zero fast enough such that $\boldsymbol{E}[s_n^2 | t_0 \leq n]$ is bounded. To this end, we consider the urn model $\{\tilde{Z}_n; n \geq 1\}$ defined in (3.14) and (3.15) coupled with the MRRU model, such that $\boldsymbol{P}(s_n = k | t_0 \leq n)$ can be expressed in terms of $\{\tilde{Z}_n; n \geq 1\}$. After some calculations, this is provided by Lemma 3.2. Moreover, we compare $\{\tilde{Z}_n; n \geq 1\}$ with a generalized Pólya urn model, whose moments are uniformly bounded. First, for any $n \geq 1$ note that

$$\boldsymbol{E}[s_n^2 | t_0 \leq n] = \sum_{k=1}^{n} k^2 \boldsymbol{P}(s_n = k | t_0 \leq n),$$

since $\boldsymbol{P}(s_n = \infty | t_0 \leq n) = \boldsymbol{P}(t_0 > n | t_0 \leq n) = 0$. In fact, by definition $t_0 \leq n - s_n$ a.s.

Before considering the urn model $\{\tilde{Z}_n; n \geq 1\}$, we express $\boldsymbol{P}(s_n = k | t_0 \leq n)$ in terms of $\{T_n; n \geq 1\}$. Note that in the MRRU, if $T_j \geq -b$ for some $j \geq 0$, then $\boldsymbol{P}(T_n < -b) = 0$ for any $n \geq j$. In fact, when $T_n \geq 0$ ($Z_n \leq \rho_1$) we have $T_{n+1} \geq -b$, because the reinforcements are bounded by b and so $|T_{n+1} - T_n| < b$ a.s.; while when $-b \leq T_n < 0$ ($Z_n > \rho_1$) we have $T_{n+1} \geq T_n \geq -b$, because $Z_n > \rho_1$ implies $W_{1,n} = 0$ and so the urn is not reinforced by red balls, that is, $T_{n+1} \geq T_n$. As a consequence, since $T_{t_0} \geq -b$ by definition, on the set $\{n \geq t_0\}$, we have $\{T_n \notin [-b, 0]\} \subset \{T_n > 0\}$. Hence, since $t_0 \leq n - s_n$, we have for all $1 \leq k \leq n$

$$\begin{aligned}
\boldsymbol{P}(s_n = k | t_0 \leq n) &= \boldsymbol{P}\left(\bigcap_{i=0}^{k-1} \{T_{n-i} > 0\} \cap \{T_{n-k} \leq 0\} \,\middle|\, t_0 \leq n\right) \\
&\leq \boldsymbol{P}\left(\bigcap_{i=0}^{k-1} \{T_{n-i} > 0\} \,\middle|\, \{T_{n-k} \leq 0\} \cap \{t_0 \leq n\}\right) \qquad (3.17) \\
&\leq \boldsymbol{P}\left(\bigcap_{i=0}^{k-1} \{T_{n-i} > 0\} \,\middle|\, T_{n-k} \leq 0\right),
\end{aligned}$$

where the last inequality follows using $\{T_{n-k} \le 0\} \subseteq \{t_0 \le n\}$. To deal with (3.17), we consider the urn model $\{\tilde{Z}_n; n \ge 1\}$ defined in (3.14) and (3.15). From Lemma 3.2, we have that, on the set {there exists $j < n : T_j \le 0$}, the event $\{T_n > 0\}$ implies $\{\tilde{T}_n \ge T_n\}$. Hence, we have that

$$\begin{aligned}
\boldsymbol{P}\left(\bigcap_{i=0}^{k-1}\{T_{n-i} > 0\}\middle| T_{n-k} \le 0\right) &\le \boldsymbol{P}\left(\bigcap_{i=0}^{k-1}\{\tilde{T}_{n-i} > 0\}\middle| T_{n-k} \le 0\right) \\
&= \boldsymbol{P}\left(\bigcap_{i=1}^{k}\{Z_i^G < \rho_1\}\right),
\end{aligned} \tag{3.18}$$

by construction, where $\{Z_i^G; i \ge 1\}$ is the proportion of red balls of a generalized Pólya urn, starting with a proportion of $Z_0^G = \rho_1$ and an initial number of balls $Y_0^G = \tilde{y}_0$, and the same reinforcements distributions as $D_{1,n}$ and $D_{2,n}$.

Now, let s^G be the first time the process Z_i^G is above ρ_1, that is,

$$s^G := \begin{cases} \inf\{i \ge 1 : Z_i^G \ge \rho_1\}, & \text{if } \{i \ge 1 : Z_i^G \ge \rho_1\} \ne \varnothing; \\ \infty, & \text{otherwise.} \end{cases}$$

It can be shown using standard arguments that there exists $k_0 \in \mathbb{N}$ such that for any $k \ge k_0$, there exist $0 < c_1, c_2 < \infty$,

$$\boldsymbol{P}(s^G = k) \le c_1 \exp(-c_2 k),$$

which implies that $\boldsymbol{E}[\exp(\gamma s^G)] < \infty$ for some $\gamma > 0$.

Now, returning to (3.18), we have that

$$\boldsymbol{P}\left(\bigcap_{i=1}^{k}\{Z_i^G < \rho_1\}\right) = \boldsymbol{P}(s^G > k) \le \frac{\boldsymbol{E}[(s^G)^4]}{k^4} = \frac{C_4}{k^4}.$$

Thus, we have for any $k \ge 1$,

$$\boldsymbol{P}(s_n = k | t_0 \le n) \le \frac{C_4}{k^4},$$

and hence

$$\begin{aligned}
\boldsymbol{E}[s_n^2 | t_0 \le n] &= \sum_{k=1}^{n} k^2 \boldsymbol{P}(s_n = k | t_0 \le n) \\
&\le C_4 \cdot \sum_{k=1}^{n} \frac{1}{k^2} < C < \infty.
\end{aligned}$$

This completes the proof. □

PROOF OF THEOREM 3.3. Without loss of generality, assume $m_1 > m_2$, which implies $\rho = \rho_1$. Since $T_n = Y_n(\rho_1 - Z_n)$, we want to prove

$$\sup_{n\ge1} \boldsymbol{E}[T_n^2 | t_0 \le n] < \infty.$$

Let s_n be the random time defined in (3.12). Then, since $|T_{i+1} - T_i| \le b$ a.s. for any $i \ge 1$ and from (3.12) $T_{n-s_n} \in [-b, 0]$, we have

$$\begin{aligned}
\boldsymbol{E}\big[T_n^2 | t_0 \le n\big] &= \sum_{l=0}^{n} \boldsymbol{E}\big[T_n^2 | \{s_n = l\} \cap \{t_0 \le n\}\big] \boldsymbol{P}(s_n = l | t_0 \le n) \\
&= b^2 + \sum_{l=1}^{n} \boldsymbol{E}\left[\left(\sum_{i=n-l}^{n-1} (T_{i+1} - T_i) + T_{n-l}\right)^2 \Big| \{s_n = l\}\right] \\
&\quad \times \boldsymbol{P}(s_n = l | t_0 \le n) \\
&\le \sum_{l=0}^{n} (l+1)^2 b^2 \boldsymbol{P}(s_n = l | t_0 \le n).
\end{aligned}$$

Now, using $(l+1)^2 \le 4l^2$, we have that

$$\sum_{l=0}^{n} (l+1)^2 b^2 \boldsymbol{P}(s_n = l | t_0 \le n) \le 4b^2 \cdot \boldsymbol{E}\big[s_n^2 | t_0 \le n\big].$$

Finally, using Theorem 3.4 we have that the last quantity is uniformly bounded by a constant C independent of n, so the proof is concluded. $\square$

REMARK 3.2. From the proof of Theorem 3.4, we have that the constant C is independent of the initial proportion Z_0. Moreover, C provides a uniform bound for any other MRRU with initial number of balls $\ge Y_0$.

3.2.2. *Proof of Theorem* 3.2.

PROOF. Without loss of generality, assume $m_1 > m_2$, which implies $\rho = \rho_1$. First, fix $j \in \mathbb{N}$ and apply Cauchy–Schwarz, so obtaining

$$\big(\boldsymbol{E}[q^j \cdot |\Delta_{j,k}| \mathbf{1}_{\{\tau_j \le k\}}]\big)^2 \le \boldsymbol{E}[(\tilde{T}_{j,k})^2 \mathbf{1}_{\{\tau_j \le k\}}] \boldsymbol{E}\left[\left(\frac{q^j}{Y_{q^j}}\right)^2\right].$$

Since $\boldsymbol{E}[(q^j / Y_{q^j})^2]$ is uniformly bounded by Theorem 3.1, it remains to prove that

$$\boldsymbol{E}\big[(\tilde{T}_{j,k})^2 \mathbf{1}_{\{\tau_j \le k\}}\big] < C,$$

for any $j \ge 1$ and any $k = 1, \dots, d_j$. To this end, fix $j \in \mathbb{N}$ and note that since $\tilde{\rho}_{1,q^j+k} = \hat{\rho}_{1,q^j}$ for any $k \in \{1, \dots, d_j\}$, the process $\{Z_{q^j+k}; k = 1, \dots, d_j\}$ can be considered as the urn proportion of the MRRU model, with initial composition (Y_{1,q^j}, Y_{2,q^j}) and fixed threshold parameters $\hat{\rho}_{1,q^j}$ and $\hat{\rho}_{2,q^j}$. Then, for each $j \in \mathbb{N}$ we can apply Theorem 3.3, with t_0 defined in (3.11) equal to τ_j, so obtaining

$$\boldsymbol{E}\big[(\tilde{T}_{j,k})^2 \mathbf{1}_{\{\tau_j \le k\}}\big] \le C_j, \tag{3.19}$$

where C_j is a constant depending on the initial composition (Y_{1,q^j},Y_{2,q^j}). However, from Remark 3.2 we have that there exists a uniform bound $C > 0$ such that $C_j \leq C$ for any $j \geq 1$, since all the processes $\{Z_{q^j+k}, k = 1, \dots, d_j\}$ $j \geq 1$ can be considered as an MRRU with initial number of balls $\geq Y_0$. This completes the proof. $\square$

4. Proofs of the main results. Here, we present the proofs of the results described in Section 2. Section 4.1 is dedicated to the proof of Theorem 2.1 (LLN) and the related preliminary results. Then, in Section 4.2 we report the proof of Theorem 2.2 (CLT) together with Theorem 4.1, a new result needed to complete that proof. In the last subsections, the proofs of the remaining results of Section 2 are gathered.

4.1. *Proof of the LLN.* We start by reporting some preliminary results needed in the proof of the LLN. Initially, we show that the number of balls sampled from the urn $N_{1,n}$, $N_{2,n}$ and the total number of balls in the urn Y_n, increase to infinity almost surely. To this end, we first need to show a lower bound for the increments of the process Y_n, which is given by the following.

LEMMA 4.1. *For any $i \geq 1$, we have that*

$$\boldsymbol{E}[Y_i - Y_{i-1}|\mathcal{F}_{i-1}] \geq a \cdot \left(\frac{\min\{y_{1,0}; y_{2,0}\}}{y_0 + (i-1)b}\right).$$

PROOF. First, note that

$$Y_i - Y_{i-1} = X_i D_{1,i} W_{1,i-1} + (1 - X_i) D_{2,i} W_{2,i-1}.$$

Since X_i and $D_{1,i}$ are conditionally independent with respect to $\mathcal{F}_{i-1}$, and $W_{1,i-1}$ is $\mathcal{F}_{i-1}$-measurable, we have that

$$\begin{aligned} \boldsymbol{E}[Y_i - Y_{i-1}|\mathcal{F}_{i-1}] &= \big(m_1 Z_{i-1} W_{1,i-1} + m_2(1 - Z_{i-1}) W_{2,i-1}\big) \\ &\geq a \cdot \big(Z_{i-1} W_{1,i-1} + (1 - Z_{i-1}) W_{2,i-1}\big), \end{aligned}$$

where the last inequality is because $m_1, m_2 \geq a$. We recall that the variables $W_{1,i-1}$ and $W_{2,i-1}$ can only take the values 0 and 1, and by construction we have that $W_{1,i-1} + W_{2,i-1} \geq 1$ for any $i \geq 1$; then we can give a further lower bound

$$\boldsymbol{E}[Y_i - Y_{i-1}|\mathcal{F}_{i-1}] \geq a \cdot \big(\min\{Z_{i-1}; 1 - Z_{i-1}\}\big). \tag{4.1}$$

Finally, the result follows by noting that

$$\min\{Z_{i-1}; 1 - Z_{i-1}\} = \frac{\min\{Y_{1,i-1}; Y_{2,i-1}\}}{Y_{i-1}} \geq \frac{\min\{y_{1,0}; y_{2,0}\}}{y_0 + (i-1)b}. \qquad \square$$

Here, we present the lemma on the divergence of the sequences Y_n, $N_{1,n}$ and $N_{2,n}$. This result is obtained by using the conditional Borel–Cantelli lemma.

LEMMA 4.2. *Consider the urn model presented in Section 2. Then*:

(a) $Y_n \stackrel{\text{a.s.}}{\to} \infty$;
(b) $\min\{N_{1,n}; N_{2,n}\} \stackrel{\text{a.s.}}{\to} \infty$.

PROOF. We begin with the proof of part (a). First, notice that $Y_n = \sum_{i=1}^n (Y_i - Y_{i-1}) + y_0$. Then, by Theorem 1 in [9], it is sufficient to show that

$$\left\{\omega \in \Omega : \sum_{i=1}^{\infty} [Y_i - Y_{i-1} | \mathcal{F}_{i-1}] = \infty\right\}$$

occurs with probability one. To this end, we will now use the lower bound of Lemma 4.1, so obtaining

$$\sum_{i=1}^{n} \boldsymbol{E}[Y_i - Y_{i-1} | \mathcal{F}_{i-1}] \geq a\left(\sum_{i=1}^{n} \frac{\min\{y_{1,0}; y_{2,0}\}}{y_0 + (i-1)b}\right) \stackrel{\text{a.s.}}{\to} \infty.$$

Hence, we have that $Y_n \stackrel{\text{a.s.}}{\to} \infty$. We now turn to the proof of part (b).

We will show that $N_{1,n} \stackrel{\text{a.s.}}{\to} \infty$, since the proof for $N_{2,n}$ is analogous. Since $N_{1,n} = \sum_{i=1}^n X_i$, by Theorem 1 in [9], it is sufficient to show that

$$\left\{\omega \in \Omega : \sum_{i=1}^{\infty} \boldsymbol{P}(X_i | \mathcal{F}_{i-1}) = \infty\right\}$$

occurs with probability one. Then we obtain

$$\sum_{i=1}^{n} \boldsymbol{P}(X_i | \mathcal{F}_{i-1}) = \sum_{i=1}^{n} Z_i \geq \sum_{i=1}^{n} \frac{y_{1,0}}{y_0 + (i-1)b} \stackrel{\text{a.s.}}{\to} \infty.$$

Hence, we have that $N_{1,n} \stackrel{\text{a.s.}}{\to} \infty$. □

The following lemma corresponds to Theorem 2.1 of [1], and it is needed in the proof of Theorem 2.1. This result provides multiple equivalent ways to show the almost sure convergence of a real-valued process. We consider a general real-valued process $\{Z_n; n \geq 0\}$ and two real numbers d (down) and u (up), with $d < u$. The result requires two sequences of times $t_j(d,u)$ and $\tau_j(d,u)$ defined as follows: for each $j \geq 0$, $t_j(d,u)$ represents the time of the first up-cross of u after $\tau_{j-1}(d,u)$, and $\tau_j(d,u)$ represents the time of the first down-cross of d after t_j. Note that $t_j(d,u)$ and $\tau_j(d,u)$ are stopping times, since the events $\{t_j(d,u) = k\}$ and $\{\tau_j(d,u) = k\}$ depend on $\{Z_n; n \leq k\}$, which are measurable with respect to $\mathcal{F}_k$. We omit the proof since it is reported in Theorem 2.1 of [1], using the same notation.

LEMMA 4.3. *Let* $\{Z_n; n \geq 0\}$ *be a real-valued process in* $[0,1]$. *Let* $\tau_{-1}(d,u) = -1$ *and define for every* $j \geq 0$ *two stopping times*:

$$
\begin{aligned}
&t_j(d,u) = \begin{cases} \inf\{n > \tau_{j-1}(d,u) : Z_n > u\}, & \textit{if } \{n > \tau_j(d,u) : Z_n > u\} \neq \varnothing; \\ +\infty, & \textit{otherwise}, \end{cases} \\
&\tau_j(d,u) = \begin{cases} \inf\{n > t_j(d,u) : Z_n < d\}, & \textit{if } \{n > t_{j-1}(d,u) : Z_n < d\} \neq \varnothing; \\ +\infty, & \textit{otherwise}. \end{cases}
\end{aligned}
\tag{4.2}
$$

Then the following are equivalent:

(a) Z_n *converges a.s.*;

(b) *for any* $0 < d < u < 1$,

$$\lim_{j\to\infty} \boldsymbol{P}\big(t_j(d,u) < \infty\big) = 0;$$

(c) *for any* $0 < d < u < 1$,

$$\sum_{j\geq 1} \boldsymbol{P}\big(t_{j+1}(d,u) = \infty | t_j(d,u) < \infty\big) = \infty;$$

using the convention that $\boldsymbol{P}(t_{j+1}(d,u) = \infty | t_j(d,u) < \infty) = 1$ *when* $\boldsymbol{P}(t_j(d,u) = \infty) = 1$.

The following lemma provides lower bounds for the total number of balls in the urn at the times of up-crossings Y_{t_j}. The lemma gets used in the proof of Theorem 2.1, where conditioning on a fixed number of up-crossing ensures to have at least a number of balls Y_n determined by the lower bounds of this lemma. This result has been taken by Lemma 2.1 of [1]. We omit the proof since adaptive thresholds does not play any role during up-crossings and the proof reported in Lemma 2.1 of [1] carries over to our model, with D_n replaced by Y_n.

LEMMA 4.4. *For any* $0 < d < u < 1$, *we have that*

$$Y_{t_j(d,u)} \geq \left(\frac{u(1-d)}{d(1-u)}\right) Y_{t_{j-1}(d,u)} \geq \cdots \geq \left(\frac{u(1-d)}{d(1-u)}\right)^j Y_{t_0(d,u)}.$$

The following lemma provides a uniform bound for the generalized Pólya urn with same reinforcement means, which is needed in the proof of Theorem 2.1. This result has been taken from Lemma 3.2 of [1]. The proof is omitted since it is reported in [1].

LEMMA 4.5. *Consider a generalized Pólya urn with* $m_1 = m_2$. *If* $Y_0 \geq 2b$, *for every* $h > 0$ *we have that*

$$\boldsymbol{P}\Big(\sup_{n\geq 1} |Z_n - Z_0| \geq h\Big) \leq \frac{b}{Y_0}\left(\frac{4}{h^2} + \frac{2}{h}\right).$$

PROOF OF THEOREM 2.1. Without loss of gnerality, assume $m_1 > m_2$, which implies $\hat{\rho}_n = \hat{\rho}_{1,n}$ and $\rho = \rho_1$. We divide the proof in two steps:

(a) $\boldsymbol{P}(\limsup_{n\to\infty} Z_n = \rho_1) = 1$,
(b) $\boldsymbol{P}(\lim_{n\to\infty} Z_n \textit{ exists}) = 1$.

Proof of part (a):
We begin by proving that $\boldsymbol{P}(\limsup_{n\to\infty} Z_n \le \rho_1) = 1$. To this end, we show that there cannot exist $\varepsilon > 0$ and $\rho' > \rho_1$ such that

$$\boldsymbol{P}\Bigl(\limsup_{n\to\infty} Z_n > \rho_1'\Bigr) \ge \varepsilon > 0. \tag{4.3}$$

We prove this by contradiction using a comparison argument with an RRU model. The proof involves last exit time arguments. Now, suppose (4.3) holds and let $A_1 := \{\limsup_{n\to\infty} Z_n > \rho_1'\}$. Let

$$R_1 := \left\{k \ge 0 : \hat{\rho}_{1,k} \ge \frac{\rho_1' + \rho_1}{2}\right\},$$

and denote the last time the process $\{\hat{\rho}_{1,n}; n \ge 1\}$ is above $(\rho_1' + \rho_1)/2$ by

$$t_{\frac{\rho_1'+\rho_1}{2}} = \begin{cases} \sup\{R_1\}, & \text{if } R_1 \ne \varnothing; \\ 0, & \text{otherwise}. \end{cases}$$

Since $\hat{\rho}_{1,n} \overset{\text{a.s.}}{\to} \rho_1$, then we have that $\boldsymbol{P}(t_{\frac{\rho_1'+\rho_1}{2}} < \infty) = 1$. Hence, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\boldsymbol{P}(t_{\frac{\rho_1'+\rho_1}{2}} > n_\varepsilon) \le \frac{\varepsilon}{2}. \tag{4.4}$$

Setting $B_1 := \{t_{\frac{\rho_1'+\rho_1}{2}} > n_\varepsilon\}$ and using (4.4), it follows that

$$\varepsilon \le \boldsymbol{P}(A_1) \le \varepsilon/2 + \boldsymbol{P}\bigl(A_1 \cap B_1^c\bigr).$$

Now, we show that $\boldsymbol{P}(A_1 \cap B_1^c) = 0$. Setting

$$C_1 = \left\{\omega \in \Omega : \liminf_{n\to\infty} Z_n < \frac{\rho_1' + \rho_1}{2}\right\},$$

we decompose $\boldsymbol{P}(A_1 \cap B_1^c)$ as follows:

$$\boldsymbol{P}\bigl(A_1 \cap B_1^c\bigr) \le \boldsymbol{P}(E_1) + \boldsymbol{P}(E_2),$$

where $E_1 = A_1 \cap B_1^c \cap C_1$ and $E_2 = A_1 \cap B_1^c \cap C_1^c$.

Consider the term $\boldsymbol{P}(E_2)$. Note that on the set C_1^c, we have $\{\liminf_{n\to\infty} Z_n \ge \frac{\rho_1'+\rho_1}{2}\}$ and on the set B_1^c we have $\{\hat{\rho}_{1,n} \le \frac{\rho_1'+\rho_1}{2}\}$ for any $n \ge n_\varepsilon$. Hence, since $B_1^c \cap C_1^c \supset E_2$, on the set E_2 we have that $W_{1,n} = \mathbf{1}_{\{Z_n \le \hat{\rho}_{1,n}\}} \overset{\text{a.s.}}{\to} 0$. Then, letting

$\tau_W := \sup\{k \geq 1 : W_{1,k} = 1\}$ we have $\boldsymbol{P}(E_2 \cap \{\tau_W < \infty\}) = \boldsymbol{P}(E_2)$ and, on the set E_2, for any $n \geq \tau_W$ the ARRU model can be written as follows:

$$\begin{cases} Y_{1,n+1} = Y_{1,\tau_W}, \\ Y_{2,n+1} = Y_{2,\tau_W} + \sum_{i=\tau_W}^{n+1} (1 - X_i) D_{2,i}, \end{cases}$$

where $W_{1,i-1} = 0$ for any $i \geq \tau_W$, and $W_{2,i-1} = 1$ because $W_{2,i-1} + W_{2,i-1} \geq 1$ by construction. Now, consider an RRU model $\{Z_i^R; i \geq 1\}$ with initial composition $(Y_{1,0}^R, Y_{2,0}^R) = (Y_{1,\tau_W}, Y_{2,\tau_W})$ a.s.; the reinforcements are defined as $D_{1,i}^R = 0$ and $D_{2,i}^R = D_{2,\tau_W+i}$ for any $i \geq 1$ a.s.; the drawing process is modeled by $X_{i+1}^R := \mathbf{1}_{\{U_i^R < Z_i^R\}}$ and $U_i^R = U_{\tau_W+i}$ a.s., where $\{U_n; n \geq 1\}$ is the sequence such that $X_{n+1} = \mathbf{1}_{\{U_n < Z_n\}}$ for any $n \geq 1$. Formally, this RRU model can be described for any $n \geq 1$ as follows:

$$\begin{cases} Y_{1,n+1}^R = Y_{1,0}^R = Y_{1,\tau_W}, \\ Y_{2,n+1}^R = Y_{2,0}^R + \sum_{i=0}^{n+1} (1 - X_i^R) D_{2,i}^R = Y_{2,\tau_W} + \sum_{i=\tau_W}^{n+\tau_W+1} (1 - X_i) D_{2,i}. \end{cases}$$

Hence, on the set E_2 we have that

$$(Y_{1,n}, Y_{2,n}) = (Y_{1,n-\tau_W}^R, Y_{2,n-\tau_W}^R) \qquad \text{a.s.},$$

for any $n \geq \tau_W$. Since from [18] $\boldsymbol{P}(\limsup_{n\to\infty} Z_n^R = 0) = 1$, on the set E_2 we have that $\{\limsup_{n\to\infty} Z_n = 0\}$. This is incompatible with the set A_1 which includes E_2. Hence, $\boldsymbol{P}(E_2) = 0$.

We now turn to the proof that $\boldsymbol{P}(E_1) = 0$. To this end, let

$$\tau_\varepsilon := \inf\left\{k \geq n_\varepsilon : \left\{Z_k < \frac{\rho_1' + \rho_1}{2}\right\} \cap \left\{Y_k > \frac{b}{(\rho_1' - \rho_1)/2}\right\}\right\}$$

and note that, since by Lemma 4.2 $Y_n \stackrel{\text{a.s.}}{\to} \infty$, $\boldsymbol{P}(C_1 \cap \{\tau_\varepsilon < \infty\}) = \boldsymbol{P}(C_1)$. Moreover, on the set B_1^c we have that $\{\hat{\rho}_{1,n} \leq \frac{\rho_1'+\rho_1}{2}\}$ for any $n \geq n_\varepsilon$. We now show by induction that on the set $B_1^c \cap C_1$ we have $\{Z_n < \rho_1' \ \forall n \geq \tau_\varepsilon\}$. By definition, we have $Z_{\tau_\varepsilon} < \frac{\rho_1'+\rho_1}{2}$, and by Lemma 3.1 this implies $Z_{\tau_\varepsilon+1} < \rho_1'$; now, consider an arbitrary $n > \tau_\varepsilon$; if $Z_n < \frac{\rho_1'+\rho_1}{2}$, then by Lemma 3.1 we have $Z_{n+1} < \rho_1'$; if $\frac{\rho_1'+\rho_1}{2} < Z_n < \rho_1'$ we have $W_{1,n} = 0$ and so $Z_{n+1} \leq Z_n < \rho_1'$. Hence, since $B_1^c \cap C_1 \subset E_1$, on the set E_1 we have $\{Z_n < \rho_1' \ \forall n \geq \tau_\varepsilon\}$. This is incompatible with the set A_1 which also includes E_1. Hence, $\boldsymbol{P}(E_1) = 0$.

Combining all together we have $\varepsilon \leq \varepsilon/2 + \boldsymbol{P}(E_1) + \boldsymbol{P}(E_2) = \varepsilon/2$, which is impossible. Thus, we conclude that $\boldsymbol{P}(A_1^c) = \boldsymbol{P}(\limsup_{n\to\infty} Z_n \leq \rho_1) = 1$.

We now prove that $\boldsymbol{P}(\limsup_{n\to\infty} Z_n \geq \rho_1) = 1$. To this end, we show that there cannot exist $\varepsilon > 0$ and $\rho' < \rho_1$ such that

$$\boldsymbol{P}\Big(\limsup_{n\to\infty} Z_n < \rho_1'\Big) \geq \varepsilon > 0. \tag{4.5}$$

We prove this by contradiction, using a comparison argument with an RRU model. Now suppose (4.5) holds and let $A_2 := \{\limsup_{n\to\infty} Z_n < \rho_1'\}$. Let

$$R_2 := \left\{k \geq 0 : \hat{\rho}_{1,k} < \frac{\rho_1' + \rho_1}{2}\right\},$$

and define the last time the process $\{\hat{\rho}_{1,n}; n \geq 1\}$ is less than $(\rho_1' + \rho_1)/2$ by

$$\tau_{\frac{\rho_1'+\rho_1}{2}} = \begin{cases} \sup\{R_2\}, & \text{if } R_2 \neq \varnothing; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\hat{\rho}_{1,n} \overset{\text{a.s.}}{\to} \rho_1$, then we have that $\boldsymbol{P}(\tau_{\frac{\rho_1'+\rho_1}{2}} < \infty) = 1$. Hence, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\boldsymbol{P}(\tau_{\frac{\rho_1'+\rho_1}{2}} > n_\varepsilon) \leq \frac{\varepsilon}{2}. \tag{4.6}$$

Setting $B_2 := \{\tau_{\frac{\rho_1'+\rho_1}{2}} > n_\varepsilon\}$ and using (4.6), it follows that

$$\varepsilon \leq \boldsymbol{P}(A_2) \leq \varepsilon/2 + \boldsymbol{P}\big(A_2 \cap B_2^c\big).$$

Let $E_3 := A_2 \cap B_2^c$. We now show that $\boldsymbol{P}(E_3) = 0$. On the set A_2, we have $\{\liminf_{n\to\infty} Z_n \leq \rho_1'\}$ and on the set B_2^c, we have $\{\hat{\rho}_{1,n} \geq \frac{\rho_1'+\rho_1}{2}\}$ for any $n \geq n_\varepsilon$. Hence, on the set E_3 we have that $W_{1,n} = \mathbf{1}_{\{Z_n \leq \hat{\rho}_{1,n}\}} \overset{\text{a.s.}}{\to} 1$. Then, letting $\tau_W := \sup\{k \geq 1 : W_{1,n} = 0\}$ we have $\boldsymbol{P}(E_3 \cap \{\tau_W < \infty\}) = \boldsymbol{P}(E_3)$. Now, analogously to the proof of $\boldsymbol{P}(E_2) = 0$, we can use comparison arguments with an RRU model to show that on the set E_3 we have $\{\limsup_{n\to\infty} Z_n = 1\}$. This is incompatible with the set A_2, which also includes E_3. Hence, $\boldsymbol{P}(E_3) = 0$.

Combining all together we have $\varepsilon \leq \varepsilon/2 + \boldsymbol{P}(E_3) = \varepsilon/2$, which is impossible. Thus, we conclude that the event $A_2^c = \{\limsup_{n\to\infty} Z_n \geq \rho_1\}$ occurs with probability one.

Proof of part (b):
In part (a), we have shown that $\boldsymbol{P}(\limsup_{n\to\infty} Z_n = \rho_1) = 1$. Therefore, if the process $\{Z_n; n \geq 1\}$ converges almost surely, then its limit has to be equal to ρ_1. First, let d, u, γ and ρ_1' $(d < u < \gamma < \rho_1' < \rho_1)$ be four constants in $(0, 1)$. Let $\{\tau_j(d, u); j \geq 1\}$ and $\{t_j(d, u); j \geq 1\}$ be the sequences of random variables defined in (4.2). Since d and u are fixed in this proof, we sometimes denote $\tau_j(d, u)$ by τ_j and $t_j(d, u)$ by t_j. It is easy to see that τ_n and t_n are stopping times with respect to $\{\mathcal{F}_n; n \geq 1\}$.

Recall that, by Lemma 4.3, we have that for every $0 < d < u < 1$

$$Z_n \text{ converges a.s.} \quad \Leftrightarrow \quad \boldsymbol{P}\big(t_n(d,u) < \infty\big) \to 0,$$
$$\Leftrightarrow \quad \sum_{n=1}^{\infty} \boldsymbol{P}\big(t_{n+1}(d,u) = \infty | t_n(d,u) < \infty\big) = \infty.$$

Now, to prove that Z_n converges a.s., it is sufficient to show that

$$\boldsymbol{P}\big(t_n(d,u) < \infty\big) \to 0,$$

for all $0 < d < u < 1$. Suppose Z_n does not converge almost surely. Since $\boldsymbol{P}(t_n < \infty)$ is a nonincreasing sequence, $\boldsymbol{P}(t_n < \infty)$ converges to $\phi_1 > 0$. We will show that for large j there exists a constant $\phi < 1$ dependent on ϕ_1, such that

$$\boldsymbol{P}(t_{j+1} < \infty | t_j < \infty) \leq \phi. \tag{4.7}$$

This result implies that $\sum_n \boldsymbol{P}(t_{n+1} = \infty | t_n < \infty) = \infty$, establishing by Lemma 4.3 that $\boldsymbol{P}(t_n < \infty) \to 0$, which is a contradiction.

Consider the term $\boldsymbol{P}(t_{i+1} < \infty | t_i < \infty)$. First, let us denote by $\tau_{\rho_1'}$ the last time the process $\hat{\rho}_{1,n}$ is below ρ_1', that is,

$$\tau_{\rho_1'} = \begin{cases} \sup\{n \geq 1 : \hat{\rho}_{1,n} \leq \rho_1'\}, & \text{if } \{n \geq 1 : \hat{\rho}_{1,n} \leq \rho_1'\} \neq \varnothing; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\hat{\rho}_{1,n} \overset{\text{a.s.}}{\to} \rho_1$, we have that $\boldsymbol{P}(\tau_{\rho_1'} < \infty) = 1$. Hence, for any $\varepsilon \in (0, \frac{1}{2})$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\frac{1}{\phi_1} \boldsymbol{P}(\tau_{\rho_1'} > n_\varepsilon) \leq \varepsilon. \tag{4.8}$$

By denoting $\boldsymbol{P_i}(\cdot) = \boldsymbol{P}(\cdot | t_i < \infty)$ and using $t_i \leq \tau_i \leq t_{i+1}$ we obtain

$$\boldsymbol{P}(t_{i+1} < \infty | t_i < \infty) \leq \boldsymbol{P_i}(\tau_i < \infty).$$

Hence,

$$\boldsymbol{P_i}(\tau_i < \infty) \leq \boldsymbol{P_i}\big(\{\tau_i < \infty\} \cap \{\tau_{\rho_1'} \leq n_\varepsilon\}\big) + \boldsymbol{P_i}(\tau_{\rho_1'} > n_\varepsilon). \tag{4.9}$$

We start with the second term in (4.9). Note that

$$\boldsymbol{P_i}(\tau_{\rho_1'} > n_\varepsilon) \leq \frac{\boldsymbol{P}(\tau_{\rho_1'} > n_\varepsilon)}{\boldsymbol{P}(t_i < \infty)} \leq \frac{\boldsymbol{P}(\tau_{\rho_1'} > n_\varepsilon)}{\phi_1} \leq \varepsilon,$$

where the last inequality follows from (4.8).

Now, consider the first term in (4.9). Since the probability is conditioned on the set $\{t_i < \infty\}$, in what follows we will consider the urn process at times n after the stopping time t_i. Since we want to show (4.7) for large i, we can choose an integer $i \geq n_\varepsilon$ and

$$i > \log_{\frac{u(1-d)}{d(1-u)}} \left(\frac{b}{Y_0(\gamma - u)} \right),$$

so that:

(i) $t_i \geq i \geq n_\varepsilon$ a.s.;
(ii) from Lemma 4.4, we have that $Y_{\tau_i} > b/(\gamma - u)$ a.s.

These two properties imply that, on the set $\{n \geq t_i\}$:

(i) $\hat{\rho}_{1,n} \geq \rho_1'$, since from $\{\tau_{\rho_1'} \leq n_\varepsilon\}$ we have that $n \geq \tau_{\rho_1'}$ a.s.;
(ii) $Z_{t_i} \in (u, \gamma)$, since $Z_{t_i - 1} \leq u$ and $Z_{t_i} > u$ and from Lemma 3.1 we have that $|Z_n - Z_{n-1}| < (\gamma - u)$.

Now, let us define two sequences of stopping times $\{t_n^*; n \geq 1\}$ and $\{\tau_n^*; n \geq 1\}$, where t_n^* represents the first time after τ_{n-1}^* the process Z_{t_i+n} up-crosses ρ_1', while τ_n^* represents the first time after t_n^* the process Z_{t_i+n} down-crosses γ. Formally, let $\tau_0^* = 0$ and define for every $j \geq 1$ the following stopping times:

$$
\begin{aligned}
t_j^* &= \begin{cases} \inf\{n > \tau_{j-1}^* : Z_{\tau_i+n} > \rho_1'\}, & \text{if } \{n > \tau_j^* : Z_{\tau_i+n} > \rho_1'\} \neq \varnothing; \\ +\infty, & \text{otherwise;} \end{cases} \\
\tau_j^* &= \begin{cases} \inf\{n > t_j^* : Z_{\tau_i+n} \leq \gamma\}, & \text{if } \{n > t_{j-1}^* : Z_{\tau_i+n} \leq \gamma\} \neq \varnothing; \\ +\infty, & \text{otherwise.} \end{cases}
\end{aligned}
\tag{4.10}
$$

Note that, since $Z_{t_i+\tau_j^*-1} \geq \gamma$ and $Z_{t_i+\tau_j^*} < \gamma$, from (ii) we have that $Z_{t_i+\tau_j^*} \in (u, \gamma)$.

For any $j \geq 0$, let $\{\tilde{Z}_n^j; n \geq 1\}$ be an RRU model defined as follows:

(1) $(\tilde{Y}_{1,0}^j, \tilde{Y}_{2,0}^j) = (Y_{1,t_i+\tau_j^*}, Y_{1,t_i+\tau_j^*}\frac{u+d}{2-u-d})$ a.s., which implies that $\tilde{Z}_0^j = \frac{u+d}{2}$;

(2) the drawing process is modeled by $\tilde{X}_{n+1}^j = \mathbf{1}_{\{\tilde{U}_{n+1}^j < \tilde{Z}_n^j\}}$, where $\tilde{U}_{n+1}^j = U_{t_i+\tau_j^*+n+1}$ a.s. and U_n is such that $X_n = \mathbf{1}_{\{U_n < Z_{n-1}\}}$;

(3) the reinforcements are defined as $\tilde{D}_{2,n+1}^j = D_{2,t_i+\tau_j^*+n+1} + (m_1 - m_2)$, $\tilde{D}_{1,n+1}^j = D_{1,t_i+\tau_j^*+n+1}$ a.s.; this means $\boldsymbol{E}[\tilde{D}_{1,n}^j] = \boldsymbol{E}[\tilde{D}_{2,n}^j]$ for any $n \geq 1$;

(4) the urn process evolves as an RRU model, that is, for any $n \geq 0$,

$$
\begin{cases}
\tilde{Y}_{1,n+1}^j = \tilde{Y}_{1,n}^j + \tilde{X}_{n+1}^j \tilde{D}_{1,n+1}^j, \\
\tilde{Y}_{2,n+1}^j = \tilde{Y}_{2,n}^j + (1 - \tilde{X}_{n+1}^j)\tilde{D}_{2,n+1}^j, \\
\tilde{Y}_{n+1}^j = \tilde{Y}_{1,n+1}^j + \tilde{Y}_{2,n+1}^j, \\
\tilde{Z}_{n+1}^j = \dfrac{\tilde{Y}_{1,n+1}^j}{\tilde{Y}_{n+1}^j}.
\end{cases}
$$

We will compare the process $\{Z_{t_i+n}; n \geq 1\}$ with the ARRU process $\{Z_{t_i+n}; n \geq 1\}$. Note that at time n, we have defined only the processes $\tilde{Z}^j$ such that $\tau_j^* < n$.

We will prove, by induction, that on the set $\{\tau_{\rho_1'} \leq n_\varepsilon\}$, for any $j \in \mathbb{N}$ and for any $n \leq t_{j+1}^* - \tau_j^*$:

$$\tilde{Z}_n^j < Z_{t_i+\tau_j^*+n}, \qquad \tilde{Y}_{2,n}^j \geq Y_{2,t_i+\tau_j^*+n}, \qquad \tilde{Y}_{1,n}^j < Y_{1,t_i+\tau_j^*+n}. \tag{4.11}$$

In other words, we will show, provided that $t_i > \tau_{\rho_1'}$, that for each $j \geq 1$ the process $\tilde{Z}_n^j$ is always dominated by the original process $Z_{t_i+\tau_j^*+n}$, as long as $Z_{t_i+\tau_j^*+n}$ is dominated by ρ_1' (i.e., for $n \leq t_{j+1}^* - \tau_j^*$). By construction, we have that

$$\tilde{Z}_0^j = \frac{d+u}{2} < u < Z_{t_i+\tau_j^*}, \qquad \tilde{Y}_{1,0}^j = Y_{1,t_i+\tau_j^*},$$

which immediately implies $\tilde{Y}_{2,0}^j > Y_{2,t_i+\tau_j^*}$. To this end, we assume (4.11) by induction hypothesis. First, we will show that $\tilde{Y}_{2,n+1}^j > Y_{2,t_i+\tau_j^*+n+1}$. Since from (4.11) $\tilde{Z}_n^j < Z_{t_i+\tau_j^*+n}$ for $n \leq t_{j+1}^* - \tau_j^*$, by construction we obtain that

$$\tilde{X}_{n+1}^j = \mathbf{1}_{\{\tilde{U}_n^j < \tilde{Z}_n^j\}} \leq \mathbf{1}_{\{U_{t_i+\tau_j^*+n} < Z_{t_i+\tau_j^*+n}\}} = X_{t_i+\tau_j^*+n+1}.$$

As a consequence, since $W_n \leq 1$ for any $n \geq 1$, we have that

$$\begin{aligned}
(Y_{2,t_i+\tau_j^*+n+1} - Y_{2,t_i+\tau_j^*+n}) &= (1 - X_{t_i+\tau_j^*+n+1}) D_{2,t_i+\tau_j^*+n+1} W_{2,t_i+\tau_j^*+n} \\
&\leq (1 - \tilde{X}_{n+1}^j) \tilde{D}_{2,n+1}^j \\
&= (\tilde{Y}_{2,n+1}^j - \tilde{Y}_{2,n}^j),
\end{aligned}$$

which using hypothesis (4.11) implies $\tilde{Y}_{2,n+1}^j > Y_{2,t_i+\tau_j^*+n+1}$. Similarly, we now show that $\tilde{Y}_{1,n+1}^j \leq Y_{1,t_i+\tau_j^*+n+1}$. We have

$$(Y_{1,t_i+\tau_j^*+n+1} - Y_{1,t_i+\tau_j^*+n}) = X_{t_i+\tau_j^*+n+1} D_{1,t_i+\tau_j^*+n+1} W_{1,t_i+\tau_j^*+n}.$$

From (i), we have that, as long as Z remains below ρ_1', Z is also above the process $\hat{\rho}_{1,n}$. Since we consider the behavior of $Z_{t_i+\tau_j^*+n}$ when it is below ρ_1', that is, $n \leq \tau_{j+1}^* - t_j^*$, we have that $W_{1,t_i+\tau_j^*+n} = 1$. Thus,

$$(Y_{1,t_i+\tau_j^*+n+1} - Y_{1,t_i+\tau_j^*+n}) \geq \tilde{X}_{n+1}^j \tilde{D}_{1,n+1}^j = (\tilde{Y}_{1,n+1}^j - \tilde{Y}_{1,n}^j),$$

which using hypothesis (4.11) implies $\tilde{Y}_{1,n+1}^j \leq Y_{1,t_i+\tau_j^*+n+1}$. Thus, we have shown that, on the set $\{\tau_{\rho_1'} \leq n_\varepsilon\}$, for any $n \leq t_{j+1}^* - \tau_j^*$, $\tilde{Z}_{n+1}^j < Z_{t_i+\tau_j^*+n+1}$, $\tilde{Y}_{1,n+1}^j \leq Y_{1,t_i+\tau_j^*+n+1}$ and $\tilde{Y}_{2,n+1}^j > Y_{2,t_i+\tau_j^*+n+1}$ hold.

Now, for any $j \geq 1$, let T_j be the stopping time for $\tilde{Z}_n^j$ to exit from (d, u), that is,

$$T_j = \begin{cases} \inf\{R_3\}, & \text{if } R_3 \neq \varnothing; \\ +\infty, & \text{otherwise,} \end{cases}$$

where $R_3 := \{n \ge 1 : \tilde{Z}_n^j \le d \text{ or } \tilde{Z}_n^j \ge u\}$. Note that, on the set $\{\tau_{\rho_1'} \le n_\varepsilon\}$,

$$\{\tau_i < \infty\} = \Big\{\inf_{n\ge 1}\{Z_{t_i+n}\} < d\Big\} \subset \Big\{\bigcup_{j:\tau_j^*\le n}\Big\{\inf_{n\ge 1}\{\tilde{Z}^j_{n-\tau_j^*}\} < d\Big\}\Big\}$$

$$\subset \Big\{\bigcup_{j=0}^{\infty}\{T_j < \infty\}\Big\}.$$

Hence,

$$\boldsymbol{P_i}\big(\{\tau_i < \infty\} \cap \{\tau_{\rho_1'} \le n_\varepsilon\}\big) \le \boldsymbol{P_i}\Big(\Big\{\bigcup_{j=0}^{\infty}\{T_j < \infty\}\Big\} \cap \{\tau_{\rho_1'} \le n_\varepsilon\}\Big)$$

$$\le \sum_{j=0}^{\infty}\boldsymbol{P_i}\big(\{T_j < \infty\} \cap \{\tau_{\rho_1'} \le n_\varepsilon\}\big).$$

Consider a single term of the series; by setting $h = \frac{u-d}{2}$ we get

$$\boldsymbol{P_i}\big(\{T_j < \infty\} \cap \{\tau_{\rho_1'} \le n_\varepsilon\}\big) \le \boldsymbol{P_i}\Big(\Big\{\sup_{n\ge 1}|\tilde{Z}_n^j - \tilde{Z}_0^j| \ge h\Big\} \cap \{\tau_{\rho_1'} \le n_\varepsilon\}\Big)$$

$$\le \boldsymbol{P_i}\Big(\sup_{n\ge 1}|\tilde{Z}_n^j - \tilde{Z}_0^j| \ge h\Big).$$

Note that $\{\tilde{Z}_n^j; n \ge 1\}$ is the proportion of red balls in an RRU model with same reinforcement means. Then, using Lemma 4.5, we obtain

$$\boldsymbol{P_i}\Big(\sup_{n\ge 1}|\tilde{Z}_n^j - \tilde{Z}_0^j| \ge h\Big) = \boldsymbol{E_i}\Big[\boldsymbol{P}\Big(\Big\{\sup_{n\ge 1}|\tilde{Z}_n^j - \tilde{Z}_0^j| \ge h\Big\}|\mathcal{F}_{\tau_i+t_j^*}\Big)\Big]$$

$$\le \boldsymbol{E_i}\Big[\frac{b}{Y_{t_j^*}}\Big]\Big(\frac{4}{h^2} + \frac{2}{h}\Big),$$

where $\boldsymbol{E_i}[\cdot] = \boldsymbol{E}[\cdot|t_i < \infty]$. Moreover, using Lemma 4.4, the right-hand side can be expressed as

$$\boldsymbol{E_i}\Big[\frac{b}{Y_{t_i}}\Big]\Big(\frac{\rho_1'(1-\gamma)}{\gamma(1-\rho_1')}\Big)^j\Big(\frac{4}{h^2} + \frac{2}{h}\Big).$$

Since from Lemma 4.2 Y_n converges a.s. to infinity, and since $\tau_i \to \infty$ a.s. because $\tau_i \ge i$, we have that $\boldsymbol{E_i}[Y_{t_i}^{-1}]$ tends to zero as i increases. As a consequence, we can choose an integer i large enough such that

$$\boldsymbol{E_i}\Big[\frac{b}{Y_{t_i}}\Big]\Big(\frac{4}{h^2} + \frac{2}{h}\Big)\Big(\frac{1-\rho_1'}{1-\rho_1'/\gamma}\Big) < \frac{1}{2},$$

which setting $\phi = 1/2 + \varepsilon$ implies (4.7), that is,

$$\boldsymbol{P}(t_{i+1} < \infty | t_i < \infty) \le \phi < 1.$$

This completes the proof. $\square$

PROOF OF COROLLARY 2.1. This corollary has been proved in Proposition 2.1 of [13] for the MRRU. That proof is only based on the fact that the urn proportion Z_n converges a.s. to a value within the interval $(0,1)$, while the reinforcement rules do not play any role. Hence, the proof used in [13] can be applied to the ARRU, since $Z_n \stackrel{\text{a.s.}}{\to} \rho \in (0,1)$ for ARRU using Theorem 2.1. $\square$

4.2. *Proof of the central limit theorem.* Before the proof of Theorem 2.2, we recall that $\{\tau_j; j \geq 1\}$ is the sequence defined in (3.8) as follows:

$$\tau_j := \begin{cases} \inf\{k \geq 1 : \tilde{T}_{j,k} \in [-b, 0]\}, & \text{if } \{k \geq 1 : \tilde{T}_{j,k} \in [-b, 0]\} \neq \varnothing; \\ \infty, & \text{otherwise.} \end{cases}$$

Fix $\nu \in (0, 1/2)$ and, for any $j \geq 1$, let $r_j := q^{j\frac{1+\nu}{2}}$ and $\mathcal{R}_j := \{\tau_j > r_j\}$. The following theorem is critical to the proof of Theorem 2.2.

THEOREM 4.1. *Let $\tilde{\rho}_{1,n}$ and $\tilde{\rho}_{2,n}$ be as in* (2.4). *Then, under the assumptions* (2.1) *and* (2.6), *we have that*

$$\boldsymbol{P}(\mathcal{R}_j, i.o.) = 0. \tag{4.12}$$

We delay the proof of this theorem to Section 4.2.1.

PROOF OF THEOREM 2.2. Without loss of generality, assume $m_1 > m_2$, which implies $\rho = \rho_1$. To prove the main result, we establish

(a) $\sqrt{n}(\frac{N_{1,n}}{n} - \frac{\sum_{i=1}^n Z_{i-1}}{n}) \stackrel{d}{\to} \mathcal{N}(0, \rho_1(1-\rho_1))$, and
(b) $\sqrt{n}(\frac{\sum_{i=1}^n Z_{i-1}}{n} - \frac{\sum_{i=1}^n \tilde{\rho}_{1,i-1}}{n}) \stackrel{\text{a.s.}}{\to} 0$.

Finally, result (2.7) is obtained by using Slutsky's theorem to combine (a) and (b) together.

Proof of part (a): Let us define a random variable $J_{ni} := \frac{1}{\sqrt{n}}(X_i - \boldsymbol{E}[X_i|\mathcal{F}_{i-1}])$, for any $n, i \in \mathbb{N}$ with $i \leq n$. Then, for each $n \in \mathbb{N}$, the sequence $\{S_{nj} = \sum_{i=1}^j J_{ni}; 1 \leq j \leq n\}$ is a martingale. Now we apply the Martingale CLT (MCLT). First, note that $J_{ni}^2 \leq 1/n$ for any $n \in \mathbb{N}$ and $|J_{ni}| < \varepsilon$ for any $n \geq \varepsilon^{-2}$; thus,

$$\sum_{i=1}^n \boldsymbol{E}[J_{ni}^2 \mathbf{1}_{\{|J_{ni}|>\varepsilon\}}|\mathcal{F}_{i-1}] \leq \sum_{i=1}^{[\varepsilon^{-2}]+1} 1/n = \frac{[\varepsilon^{-2}]+1}{n} \to 0.$$

Also,

$$\begin{aligned} \boldsymbol{E}[J_{ni}^2|\mathcal{F}_{i-1}] &= \frac{1}{n} \cdot \boldsymbol{E}[(X_{ni} - \boldsymbol{E}[X_{ni}|\mathcal{F}_{i-1}])^2|\mathcal{F}_{i-1}] \\ &= \frac{Z_{i-1}(1-Z_{i-1})}{n}; \end{aligned}$$

since $\hat{\rho}_{1,n} \xrightarrow{a.s.} \rho_1$, from Theorem 2.1 we get $Z_n \xrightarrow{a.s.} \rho_1$, which implies

$$\sum_{i=1}^{n} \boldsymbol{E}[J_{ni}^2|\mathcal{F}_{i-1}] = \frac{\sum_{i=1}^{n} Z_{i-1}(1-Z_{i-1})}{n} \xrightarrow{a.s.} \rho_1(1-\rho_1).$$

From MCLT [15], it follows that

$$\begin{aligned}\frac{1}{\sqrt{n}} \cdot \sum_{i=1}^{n}\big(X_i - \boldsymbol{E}[X_i|\mathcal{F}_{i-1}]\big) &= \sqrt{n}\biggl(\frac{\sum_{i=1}^{n} X_i}{n} - \frac{\sum_{i=1}^{n} Z_{i-1}}{n}\biggr)\\ &\xrightarrow{d} \mathcal{N}\big(0, \rho_1(1-\rho_1)\big).\end{aligned}$$

We now turn to the proof of part (b). We first express

$$\begin{aligned}\sqrt{n}\biggl(\frac{\sum_{i=1}^{n} Z_{i-1}}{n} - \frac{\sum_{i=1}^{n} \tilde{\rho}_{1,i-1}}{n}\biggr) &= \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}(Z_i - \tilde{\rho}_{1,i})\\ &= B_{1n} + B_{2n},\end{aligned}$$

where

$$B_{1n} := \frac{1}{\sqrt{n}}\sum_{i=0}^{[q^{k_n}]}(Z_i - \tilde{\rho}_{1,i}), \qquad B_{2n} := \frac{1}{\sqrt{n}}\sum_{i=[q^{k_n}]+1}^{n-1}(Z_i - \tilde{\rho}_{1,i}),$$

and we recall k_n is defined in (2.5) as $k_n := [\log_q(n)]$, with $q > 1$. We begin with B_{1n}. Note that

$$\sum_{i=0}^{[q^{k_n}]}(Z_i - \tilde{\rho}_{1,i}) = \sum_{j=1}^{k_n-1}\sum_{i=1}^{d_j}(Z_{q^j+i} - \hat{\rho}_{1,q^j}) = \sum_{j=1}^{k_n-1}\sum_{i=1}^{d_j}(-\Delta_{j,i}),$$

where we recall that $d_j = q^{j+1} - q^j$ and $\Delta_{j,i} = \hat{\rho}_{1,q^j} - Z_{q^j+i}$ for any $j \geq 1$ and $1 \leq i \leq d_j$. Hence,

$$|B_{1n}| = \frac{1}{\sqrt{n}} \cdot \left|\sum_{j=1}^{k_n-1}\sum_{i=1}^{d_j}(-\Delta_{j,i})\right| \leq \frac{1}{\sqrt{n}} \cdot \sum_{j=1}^{k_n-1}\biggl(\frac{\sum_{i=1}^{d_j}|\Delta_{j,i}|}{\sqrt{d_j}}\biggr)\sqrt{d_j};$$

similarly,

$$|B_{2n}| \leq \frac{1}{\sqrt{n}} \cdot \biggl(\frac{\sum_{i=1}^{d_{k_n}}|\Delta_{k_n,i}|}{\sqrt{d_{k_n}}}\biggr)\sqrt{d_{k_n}}.$$

Now, defining for any $j \geq 1$

$$b_j := \frac{\sum_{i=1}^{d_j}|\Delta_{k_n,i}|}{\sqrt{d_j}}, \tag{4.13}$$

it follows that

$$|B_{1n}| + |B_{2n}| \leq \frac{1}{\sqrt{n}} \cdot \sum_{j=1}^{k_n} b_j \sqrt{d_j}.$$

Now, we have

$$\begin{aligned}|B_{1n}| + |B_{2n}| &\leq \frac{1}{\sqrt{n}} \cdot \sum_{j=1}^{k_n/2-1} b_j \sqrt{d_j} + \frac{1}{\sqrt{n}} \cdot \sum_{j=k_n/2}^{k_n} b_j \sqrt{d_j} \\ &\leq \left(\frac{\sup_{i\geq 1}\{b_i\}}{\sqrt[4]{n}}\right) \cdot H_{1n} + \Big(\sup_{i\geq k_n/2}\{b_i\}\Big) \cdot H_{2n},\end{aligned}$$

where

$$H_{1n} := \frac{1}{\sqrt[4]{n}} \sum_{j=1}^{k_n/2-1} \sqrt{d_j}, \qquad H_{2n} := \frac{1}{\sqrt{n}} \sum_{j=1}^{k_n} \sqrt{d_j}.$$

Using $d_j = (q-1)q^j$, we express

$$H_{1n} = \frac{\sqrt{q-1}}{\sqrt[4]{n}} \cdot \sum_{j=1}^{k_n/2-1} (\sqrt{q})^j = \left(\frac{\sqrt{q}^{k_n/2} - 1}{\sqrt[4]{n}}\right) \cdot \left(\frac{\sqrt{q-1}}{\sqrt{q}-1}\right),$$

$$H_{2n} = \frac{\sqrt{q-1}}{\sqrt{n}} \cdot \sum_{j=1}^{k_n} (\sqrt{q})^j = \left(\frac{\sqrt{q}^{k_n+1} - \sqrt{q}^{k_n/2}}{\sqrt{n}}\right) \cdot \left(\frac{\sqrt{q-1}}{\sqrt{q}-1}\right).$$

Since $n \geq q^{k_n}$, it follows that $H_{1n} \leq C$ and $H_{2n} \leq \sqrt{q}C$, where $C = (\frac{\sqrt{q-1}}{\sqrt{q}-1})$. Thus,

$$|B_{1n}| + |B_{2n}| \leq \left(\frac{\sup_{i\geq 1}\{b_i\}}{\sqrt[4]{n}}\right) \cdot C + \Big(\sup_{i\geq k_n/2}\{b_i\}\Big) \cdot \sqrt{q}C.$$

To conclude the proof, we will show that $b_j \stackrel{\text{a.s.}}{\to} 0$.

First, fix an arbitrary constant $\nu \in (0, 1/2)$ and let $r_j := q^{j\frac{1+\nu}{2}}$ for any $j \geq 1$; then write

$$\begin{aligned}b_j &= \frac{1}{\sqrt{d_j}} \sum_{i=1}^{d_j} |\Delta_{j,i}| = \left(\frac{1}{\sqrt{d_j}} \sum_{i=1}^{r_j} |\Delta_{j,i}|\right) + \left(\frac{1}{\sqrt{d_j}} \sum_{i=r_j+1}^{d_j} |\Delta_{j,i}|\right) \\ &= F_{1j} + F_{2j}.\end{aligned}$$

Let us consider term F_{1j}, we have that

$$F_{1j} = \frac{r_j}{\sqrt{d_j}} \cdot \left(\frac{1}{r_j} \sum_{i=1}^{r_j} |\Delta_{j,i}|\right) = \frac{[q^{j\frac{\nu}{2}}]}{\sqrt{q-1}} \cdot \left(\frac{1}{r_j} \sum_{i=1}^{r_j} |\Delta_{j,i}|\right),$$

since $d_j = (q-1)q^j$ and $r_j/\sqrt{q^j} = q^{j\frac{\nu}{2}}$. Now, for any $i = 1, \dots, r_j$ we note that

$$|\Delta_{j,i}| \le |Z_{q^j+i} - Z_{q^j}| + |\Delta_{j-1,d_{j-1}}| + |\hat{\rho}_{1,q^{j-1}} - \hat{\rho}_{1,q^j}|;$$

hence, we have $F_{1j} \le E_{1j} + E_{2j} + E_{3j}$, where

$$E_{1j} := \frac{[q^{j\frac{\nu}{2}}]}{\sqrt{q-1}} \cdot \left(\frac{1}{r_j}\sum_{i=1}^{r_j} |Z_{q^j+i} - Z_{q^j}|\right),$$

$$E_{2j} := \frac{[q^{j\frac{\nu}{2}}]}{\sqrt{q-1}} \cdot |\Delta_{j-1,d_{j-1}}|,$$

$$E_{3j} := \frac{[q^{j\frac{\nu}{2}}]}{\sqrt{q-1}} \cdot |\hat{\rho}_{1,q^{j-1}} - \hat{\rho}_{1,q^j}|.$$

Let us consider the term E_{1j}. By Lemma 3.1 we have $|Z_k - Z_{k-1}| \le b/Y_{k-1}$, and hence

$$E_{1j} \le \frac{[q^{j\frac{\nu}{2}}]}{\sqrt{q-1}} \cdot \frac{br_j}{Y_{q^j}} = \left(\frac{b}{\sqrt{q-1}}\right) \cdot \left(\frac{q^{j(\frac{1}{2}+\nu)}}{Y_{q^j}}\right).$$

Then by using Markov's inequality we obtain, since $q > 1$,

$$\sum_{j=1}^{\infty} \boldsymbol{P}\left(\frac{q^{j(\frac{1}{2}+\nu)}}{Y_{q^j}} > \varepsilon\right) \le \frac{1}{\varepsilon}\sum_{j=1}^{\infty} \boldsymbol{E}\left[\frac{q^j}{Y_{q^j}}\right] q^{-j(\frac{1}{2}-\nu)} \le \frac{C}{\varepsilon}\sum_{j=1}^{\infty} q^{-j(\frac{1}{2}-\nu)} < \infty,$$

where $C = \sup_{k\ge1}\{\boldsymbol{E}[k/Y_k]\}$ is finite from Theorem 3.1. Thus, from the Borel–Cantelli lemma it follows that $E_{1j} \xrightarrow{\text{a.s.}} 0$.

Now, consider the term E_{2j}. We have

$$\boldsymbol{P}\left(\lim_{k\to\infty}\bigcup_{j\ge k}\{E_{2j} > \varepsilon\}\right) \le \boldsymbol{P}\left(\lim_{k\to\infty}\bigcup_{j\ge k}\mathcal{R}_j\right) + \boldsymbol{P}\left(\lim_{k\to\infty}\bigcup_{j\ge k}\left\{\frac{[q^{(j+1)\frac{\nu}{2}}]}{\sqrt{q-1}} \cdot |\Delta_{j,d_j}| > \varepsilon\right\} \cap \mathcal{R}_j^c\right),$$

where the term $\boldsymbol{P}(\lim_{k\to\infty}\bigcup_{j\ge k}\mathcal{R}_j) = 0$ from Theorem 4.1. Then, by using Markov's inequality we obtain

$$\sum_{j=1}^{\infty} \boldsymbol{P}\left(\left\{\frac{[q^{(j+1)\frac{\nu}{2}}]}{\sqrt{q-1}} \cdot |\Delta_{j,d_j}| > \varepsilon\right\} \cap \mathcal{R}_j^c\right) \le M,$$

where

$$M := \frac{1}{\varepsilon}\sum_{j=1}^{\infty} \boldsymbol{E}\left[\frac{[q^{(j+1)\frac{\nu}{2}}]}{\sqrt{q-1}} \cdot |\Delta_{j,d_j}| \mathbf{1}_{\mathcal{R}_j^C}\right].$$

Now, for any $j \geq 1$ let us introduce the set $\mathcal{Q}_j := \{\tau_j > d_j\}$. Using $\mathcal{R}_j^C \subseteq \mathcal{Q}_j^C$ from $r_j \leq d_j$, and by multiplying and dividing by q^{j+1}, we have that

$$\begin{aligned} M &= \frac{1}{\varepsilon\sqrt{q-1}} \sum_{j=1}^{\infty} \boldsymbol{E}\big[q^{j+1}|\Delta_{j,d_j}|\mathbf{1}_{\mathcal{R}_j^C}\big] \cdot q^{-(j+1)(1-\frac{\nu}{2})} \\ &\leq \frac{1}{\varepsilon\sqrt{q-1}} \sum_{j=1}^{\infty} \boldsymbol{E}\big[q^{j+1}|\Delta_{j,d_j}|\mathbf{1}_{\mathcal{Q}_j^C}\big] \cdot q^{-(j+1)(1-\frac{\nu}{2})} \\ &\leq \frac{1}{\varepsilon\sqrt{q-1}} \Big(\sup_{k\geq 1}\{\boldsymbol{E}\big[q^{k+1}|\Delta_{k,d_k}|\mathbf{1}_{\mathcal{Q}_k^C}\big]\}\Big) \sum_{j=1}^{\infty} q^{-(j+1)(1-\frac{\nu}{2})} < \infty, \end{aligned}$$

where the finiteness follows from Theorem 3.2 and the result follows from the Borel–Cantelli lemma since $q > 1$.

Let us consider the term E_{3j}. For any $\varepsilon > 0$, by using Markov's inequality we have

$$\boldsymbol{P}(E_{3j} > \varepsilon) \leq \frac{1}{\varepsilon\sqrt{q-1}} \boldsymbol{E}\big[q^{j\frac{\nu}{2}} \cdot |\hat{\rho}_{1,q^j} - \hat{\rho}_{1,q^{j-1}}|\big].$$

The right-hand side of the above expression can be rewritten as follows:

$$\frac{q^{-j(\frac{1-\nu}{2})}}{\varepsilon\sqrt{q-1}} \boldsymbol{E}\big[q^{\frac{j}{2}} \cdot |\hat{\rho}_{1,q^j} - \hat{\rho}_{1,q^{j-1}}|\big].$$

Now, by decomposing the last expectation into

$$\boldsymbol{E}\big[q^{\frac{j}{2}} \cdot |\hat{\rho}_{1,q^j} - \hat{\rho}_{1,q^{j-1}}|\big] = \boldsymbol{E}\big[q^{\frac{j}{2}} \cdot |\rho_1 - \hat{\rho}_{1,q^{j-1}}|\big] + \boldsymbol{E}\big[q^{\frac{j}{2}} \cdot |\rho_1 - \hat{\rho}_{1,q^j}|\big],$$

we can see that

$$\sum_{j=1}^{\infty} \boldsymbol{P}(E_{3j} > \varepsilon) \leq \left(\frac{2\sup_{k\geq 1}\{\boldsymbol{E}[q^{\frac{k}{2}} \cdot |\rho_1 - \hat{\rho}_{1,q^k}|]\}}{\varepsilon\sqrt{q-1}}\right) \sum_{j=1}^{\infty} q^{-j(\frac{1-\nu}{2})},$$

which is finite because of (2.6). Hence, by another application the Borel–Cantelli lemma, $E_{3j} \xrightarrow{\text{a.s.}} 0$; then we have $F_{1j} \xrightarrow{\text{a.s.}} 0$.

Finally, let us consider term F_{2j}. First, we multiply and divide by $(d_j - r_j)q^{-\frac{j}{2}}$ to obtain $F_{2j} = c_j F_{3j}$, where

$$c_j = \frac{d_j - r_j}{q^{\frac{j}{2}}\sqrt{d_j}}, \qquad F_{3j} = \frac{1}{d_j - r_j} \sum_{i=r_j+1}^{d_j} q^{\frac{j}{2}}|\Delta_{j,i}|.$$

Since $c_j \to \sqrt{q-1}$, let us focus on F_{3j}. Since $\boldsymbol{P}(\mathcal{R}_j, \text{i.o.}) = 0$ (Theorem 4.1), it is sufficient to show that $F_{3j}\mathbf{1}_{R_j^c} \xrightarrow{\text{a.s.}} 0$. For any $\varepsilon > 0$, by Markov's inequality it follows that

$$\boldsymbol{P}(\{F_{3j} > \varepsilon\} \cap \mathcal{R}_j^C) \leq \frac{1}{\varepsilon}\left(\frac{1}{d_j - r_j} \sum_{i=r_j+1}^{d_j} \boldsymbol{E}[q^{\frac{j}{2}} |\Delta_{j,i}| \mathbf{1}_{\mathcal{R}_j^C}]\right).$$

Now, since

$$\sum_{i=r_j+1}^{d_j} \boldsymbol{E}[q^{\frac{j}{2}} |\Delta_{j,i}| \mathbf{1}_{\mathcal{R}_j^C}] \leq (d_j - r_j)\Big(\max_{i=r_j+1,\dots,d_j} \{\boldsymbol{E}[q^{\frac{j}{2}} |\Delta_{j,i}| \mathbf{1}_{\mathcal{R}_j^C}]\}\Big),$$

we have that

$$\begin{aligned}
\boldsymbol{P}(\{F_{3j} > \varepsilon\} \cap \mathcal{R}_j^C) &\leq \frac{1}{\varepsilon}\Big(\max_{i=r_j+1,\dots,d_j} \{\boldsymbol{E}[q^{\frac{j}{2}} |\Delta_{j,i}| \mathbf{1}_{\mathcal{R}_j^C}]\}\Big) \\
&= \frac{1}{\varepsilon}\Big(\max_{i=r_j+1,\dots,d_j} \{\boldsymbol{E}[q^{j} |\Delta_{j,i}| \mathbf{1}_{\mathcal{R}_j^C}]\}\Big) q^{-\frac{j}{2}} \\
&\leq \frac{1}{\varepsilon}\Big(\sup_{k\geq 1}\Big\{\max_{i=[r_k]+1,\dots,d_k} \{\boldsymbol{E}[q^{k} |\Delta_{k,i}| \mathbf{1}_{\mathcal{R}_k^C}]\}\Big\}\Big) q^{-\frac{j}{2}} \\
&\leq C q^{-\frac{j}{2}},
\end{aligned}$$

where the last inequality follows from Theorem 3.2. Now, summing over j we have that

$$\sum_{j=1}^{n} \boldsymbol{P}(\{F_{3j} > \varepsilon\} \cap \mathcal{R}_j^C) \leq C \sum_{j=1}^{n} q^{-\frac{j}{2}} < \infty.$$

Finally, using the Borel–Cantelli lemma we get that $F_{2j} \xrightarrow{\text{a.s.}} 0$, which concludes the proof. $\square$

REMARK 4.1. As a follow up to Remark 2.2, the condition $q > 1$ allows us to establish $\sqrt{n}(\frac{\sum_{i=1}^n Z_{i-1}}{n} - \bar{\rho}_n) \xrightarrow{\text{a.s.}} 0$. As q decreases to 1, the behavior is not clear and requires further analysis. Evidently, the remaining term $\sqrt{n}(\frac{N_{1,n}}{n} - \frac{\sum_{i=1}^n Z_{i-1}}{n})$ converges to a normal random variable which does not depend on the value of q, and hence the asymptotic distribution in (2.7) is the same for any $q > 1$.

4.2.1. *Proof of Theorem* 4.1.

PROOF. Without loss of generality, assume $m_1 > m_2$, which implies $\hat{\rho}_n = \hat{\rho}_{1,n}$ and $\rho = \rho_1$. To prove (4.12), we need to study the sequence of sets $\{\mathcal{R}_j; j \geq 1\}$. On the set $\mathcal{R}_j$, the urn proportions do not cross the thresholds at times $q^j, \dots, q^j + r_j$, where, as before, $q > 1$. Hence, $\mathcal{R}_j$ will be included in $\mathcal{A}_j \cup \mathcal{B}_j$, where $\mathcal{A}_j$ and $\mathcal{B}_j$ represent the events in which the urn proportion is always above and below, respectively, the thresholds at times $q^j, \dots, q^j + r_j$. To show that $\mathcal{A}_j$ and $\mathcal{B}_j$ cannot occur i.o., we need to appropriately express them by using the following scaling processes:

(a) $\tilde{T}_{j,k} = Y_{q^j+k}\Delta_{j,k} = Y_{q^j+k}(\hat{\rho}_{1,q^j} - Z_{q^j+k})$, defined for any $j \geq 1$ and any $k = 1, \dots, d_j$. This process describes the closeness between the urn proportions and the adaptive thresholds.

(b) $T_n = Y_n(\rho_1 - Z_n)$, defined for any $n \geq 1$. This process describes the closeness between the urn proportions and the limit of the threshold's sequence.

(c) $T^{(\rho_1)}_{j,k} := Y_{q^j+k}(\rho_1 - \hat{\rho}_{1,q^j})$, defined for any $j \geq 1$ and $k = 1, \dots, d_j$. This process describes the closeness between the adaptive thresholds and their limit.

Let us now define formally the sets $\mathcal{A}_j$ and $\mathcal{B}_j$. First, note that if the urn proportion crosses the threshold at time $(q^j + k)$, then $\tilde{T}_{q^j+k} \cdot \tilde{T}_{q^j+k-1} < 0$, since only one among $\tilde{T}_{q^j+k}$ and $\tilde{T}_{q^j+k-1}$ is within the interval $[-b, 0]$. Thus, from the definition of τ_j in (3.8), we have that

$$\{\Delta_{j,k-1} \cdot \Delta_{j,k} < 0\} \subseteq \{\tau_j \leq k\}.$$

This implies that

$$\begin{aligned}\mathcal{R}_j &\subset \left\{\bigcap_{k=1}^{r_j}\{\Delta_{j,k-1} \cdot \Delta_{j,k} > 0\}\right\} \\ &= \left\{\bigcap_{k=1}^{r_j}\{\Delta_{j,k} < 0\}\right\} \cup \left\{\bigcap_{k=1}^{r_j}\{\Delta_{j,k} > 0\}\right\}.\end{aligned}$$

Since $Y_{q^j+k}\Delta_{j,k} = T_{q^j+k} - T^{(\rho_1)}_{j,k}$, we can write $\mathcal{R}_j \subseteq \mathcal{A}_j \cup \mathcal{B}_j$, where

$$\mathcal{A}_j := \bigcap_{k=1}^{r_j} \mathcal{D}_{j,k}, \qquad \mathcal{B}_j := \bigcap_{k=1}^{r_j} \mathcal{D}^C_{j,k},$$

$$\mathcal{D}_{j,k} := \{T_{q^j+k} < T^{(\rho_1)}_{j,k}\}, \qquad k = 1, \dots, r_j.$$

The idea is to prove that these events cannot occur infinitely often; to this end, consider $\mathcal{A}_j$ (for instance) and rewrite the set $\mathcal{D}_{j,r_j}$ as

$$\mathcal{D}_{j,r_j} = \{T_{q^j+r_j} < T^{(\rho_1)}_{j,r_j}\} = \left\{\sum_{i=1}^{r_j}(T_{q^j+i} - T_{q^j+i-1}) < T^{(\rho_1)}_{j,r_j} - T_{q^j}\right\}, \tag{4.14}$$

where the last inequality follows using telescopic series. In the set $\mathcal{D}_{j,r_j}$ we have a sum of bounded random variables, that is, $(T_{q^j+i} - T_{q^j+i-1})$, whose means are strictly positive on $\mathcal{A}_j$, because $\mathcal{A}_j$ in included in $\bigcap_{k=1}^{r_j-1} \mathcal{D}_{j,k}$; hence, provided that the difference $(T^{(\rho_1)}_{j,r_j} - T_{q^j})$ increases with j slower than r_j, we could prove that the set cannot occur infinitely often. Roughly speaking, it means that, if the adaptive threshold $\hat\rho_{1,q^j}$ is not far enough from the urn proportion Z_{q^j}, then the average increments of the urn proportion make very likely that Z_{q^j+k} crosses $\hat\rho_{1,q^j}$ before $q^j + r_j$. Similar arguments apply for $\mathcal{B}_j$. More formally, fix $\varepsilon > 0$ and define the set $\mathcal{C}_j$ as follows:

$$\mathcal{C}_j := \{|T^{(\rho_1)}_{j,r_j} - T_{q^j}| > \varepsilon j^2 q^{\frac{j}{2}}\},$$

so that $\mathcal{C}_j^C$ is the set where the difference $|T^{(\rho_1)}_{j,r_j} - T_{q^j}|$ increases with j slower than r_j. Hence, it follows that

$$\mathcal{R}_j \subseteq \{\mathcal{A}_j - \mathcal{C}_j\} \cup \{\mathcal{B}_j - \mathcal{C}_j\} \cup \mathcal{C}_j,$$

and the result (4.12) is obtained by showing that

$$\boldsymbol{P}(\mathcal{A}_j - \mathcal{C}_j, \text{i.o.}) = \boldsymbol{P}(\mathcal{B}_j - \mathcal{C}_j, \text{i.o.}) = \boldsymbol{P}(\mathcal{C}_j, \text{i.o.}) = 0.$$

We will now prove that $\boldsymbol{P}(\mathcal{A}_j - \mathcal{C}_j, \text{i.o.}) = \boldsymbol{P}(\mathcal{B}_j - \mathcal{C}_j, \text{i.o.}) = 0$. From (4.14) we note that, on the set $\mathcal{C}_j^C$,

$$\mathcal{D}_{j,r_j} \subseteq \left\{\sum_{i=1}^{r_j}(T_{q^j+i} - T_{q^j+i-1}) < \varepsilon j^2 q^{\frac{j}{2}}\right\} = \mathcal{E}_j,$$

$$\mathcal{D}^C_{j,r_j} \subseteq \left\{\sum_{i=1}^{r_j}(T_{q^j+i} - T_{q^j+i-1}) > -\varepsilon j^2 q^{\frac{j}{2}}\right\} = \mathcal{F}_j.$$

As a consequence, we have

$$\mathcal{A}_j - \mathcal{C}_j \subseteq \left\{\bigcap_{k=1}^{r_j-1} \mathcal{D}_{j,k} \cap \mathcal{E}_j\right\},$$

$$\mathcal{B}_j - \mathcal{C}_j \subseteq \left\{\bigcap_{k=1}^{r_j-1} \mathcal{D}^C_{j,k} \cap \mathcal{F}_j\right\}.$$

Now, consider the increments $(T_{q^j+i} - T_{q^j+i-1})$ for $i = 1, \dots, r_j$ contained in the sets $\mathcal{E}_j$ and $\mathcal{F}_j$ above; recall that

$$\begin{aligned}(T_{q^j+i} - T_{q^j+i-1}) = \rho_1(1 - X_{q^j+i})D_{2,q^j+i}W_{2,q^j+i-1} \\ - (1-\rho_1)X_{q^j+i}D_{1,q^j+i}W_{1,q^j+i-1}.\end{aligned}$$

Fix an arbitrarily small $\varepsilon_1 > 0$ and introduce two collections of i.i.d. random variables $(A_1, \dots, A_{r_j})$ and $(B_1, \dots, B_{r_j})$ defined as follows:

$$A_i := \rho_1(1 - \mathbf{1}_{\{U_{q^j+i} < \rho_1+\varepsilon_1\}})D_{2,q^j+i},$$

$$B_i := \rho_1(1 - \mathbf{1}_{\{U_{q^j+i} < \rho_1-\varepsilon_1\}})D_{2,q^j+i} - (1-\rho_1)\mathbf{1}_{\{U_{q^j+i} < \rho_1-\varepsilon\}}D_{1,q^j+i},$$

where $(U_{q^j+1}, \dots, U_{q^{j+1}})$ are the i.i.d. $(0, 1)$ uniform random variables such that $X_{q^j+i} := \mathbf{1}_{\{U_{q^j+i} < Z_{q^j+i-1}\}}$.

First note that, by construction, on the set $\mathcal{A}_j$ we have $\bigcap_{k=1}^{r_j} \{Z_{q^j+k} > \tilde{\rho}_{1,q^j+k}\}$, and hence $\mathcal{A}_j \subset \bigcap_{k=1}^{r_j} \{W_{1,q^j+k} = 0\}$. Thus, since using (2.6) we have $Z_n \overset{\text{a.s.}}{\to} \rho_1$ by Theorem 2.1, on the set $\mathcal{A}_j$ we have that

$$\{(T_{q^j+i} - T_{q^j+i-1}) \geq A_i\}, \qquad i \in 1, \dots, r_j,$$

occurs with probability 1 as $n \to \infty$. Similarly, by construction, on the set $\mathcal{B}_j$ we have $\bigcap_{k=1}^{r_j} \{Z_{q^j+k} < \tilde{\rho}_{1,q^j+k}\}$, and hence $\mathcal{B}_j \subset \bigcap_{k=1}^{r_j} \{W_{1,q^j+k} = 1\}$. Thus, using $Z_n \overset{\text{a.s.}}{\to} \rho_1$, on the set $\mathcal{A}_j$ we have that the event

$$\{(T_{q^j+i} - T_{q^j+i-1}) \leq B_i\}, \qquad i \in 1, \dots, r_j,$$

occurs with probability 1 as $n \to \infty$. As a consequence, for large j, we have that

$$\boldsymbol{P}(\mathcal{A}_j - \mathcal{C}_j, \text{i.o.}) \leq \boldsymbol{P}\left(\sum_{i=1}^{r_j} A_i < \varepsilon j^2 q^{\frac{j}{2}}, \text{i.o.}\right) \quad \text{and}$$

$$\boldsymbol{P}(\mathcal{B}_j - \mathcal{C}_j, \text{i.o.}) \leq \boldsymbol{P}\left(\sum_{i=1}^{r_j} B_i > -\varepsilon j^2 q^{\frac{j}{2}}, \text{i.o.}\right).$$

Set

$$P_{Aj} := \boldsymbol{P}\left(\sum_{i=1}^{r_j} A_i < \varepsilon j^2 q^{\frac{j}{2}}\right) \quad \text{and} \quad P_{Bj} := \boldsymbol{P}\left(\sum_{i=1}^{r_j} B_i > -\varepsilon j^2 q^{\frac{j}{2}}\right).$$

We will now use Chernoff's upper bounds for i.i.d. bounded random variables A_i and B_i [see (3.7)]. First, notice that:

(1) $\boldsymbol{E}[A_i] = \rho_1(1 - \rho_1 - \varepsilon)m_2 > 0$,
(2) $\boldsymbol{E}[B_i] = \rho_1(1 - \rho_1 + \varepsilon)m_2 - (1 - \rho_1)(\rho_1 - \varepsilon)m_1 < 0$,
(3) $|A_i|, |B_i| < b$ a.s. for any $i \geq 1$.

Note that P_{Aj} can be written as $\boldsymbol{P}(S_j \leq c_j \cdot \boldsymbol{E}[S_j])$, where $S_j = \sum_{i=1}^{r_j}(A_i/b)$ and

$$c_j = \frac{\varepsilon j^2 q^{\frac{j}{2}}}{r_j \boldsymbol{E}[A_1]/b};$$

since $c_j \to 0$, we can define an integer j_0 such that $c_j < c_0$ for any $j \geq j_0$, so that

$$\boldsymbol{P}(S_j \leq c_j \cdot \boldsymbol{E}[S_j]) \leq \boldsymbol{P}(S_j \leq c_0 \cdot \boldsymbol{E}[S_j]).$$

Hence, by using (3.7), for any $j \geq j_0$ we have that

$$P_{Aj} \leq \exp\left(-\frac{(1 - c_0)^2}{2} \cdot \boldsymbol{E}[S_j]\right),$$

which converges to zero exponentially fast since

$$\boldsymbol{E}[S_j] = r_j \frac{\boldsymbol{E}[A_1]}{b} \sim q^{j\frac{1+\nu}{2}}.$$

We can repeat the same arguments for P_{Bj}, with the i.i.d. random variables $(-B_i + b)/2b \in (0, 1)$ for $i = 1, \dots, r_j$; in this case, c_j tends to a constant $c < 1$, so that the proof follows with $c_0 \in (c, 1)$. Thus,

$$\sum_{j=1}^{\infty} (P_{Aj} + P_{Bj}) < \infty,$$

yielding

$$\boldsymbol{P}(\mathcal{A}_j - \mathcal{C}_j, \text{i.o.}) = \boldsymbol{P}(\mathcal{B}_j - \mathcal{C}_j, \text{i.o.}) = 0.$$

We will now show that $\boldsymbol{P}(\mathcal{C}_j, \text{i.o.}) = 0$. Note that since $|T_{j,r_j}^{(\rho_1)}| \leq |T_{j,d_j}^{(\rho_1)}|$ and

$$T_{q^j} = Y_{q^j}(\rho_1 - \hat{\rho}_{1,q^{j-1}}) + Y_{q^j}(\hat{\rho}_{1,q^{j-1}} - Z_{q^j}) = T_{j-1,d_{j-1}}^{(\rho_1)} + \tilde{T}_{j-1,d_{j-1}},$$

it follows that

$$\bigl|T_{j,r_j}^{(\rho_1)} - T_{q^j}\bigr| \leq \bigl|T_{j,d_j}^{(\rho_1)}\bigr| + \bigl|T_{j-1,d_{j-1}}^{(\rho_1)}\bigr| + |\tilde{T}_{j-1,d_{j-1}}|,$$

which implies that

$$\{\mathcal{C}_j, \text{i.o.}\} \subset \left\{\bigl|T_{j,d_j}^{(\rho_1)}\bigr| > \frac{\varepsilon}{3} j^2 q^{\frac{j}{2}}, \text{i.o.}\right\} \cup \left\{|\tilde{T}_{j,d_j}| > \frac{\varepsilon}{3} j^2 q^{\frac{j}{2}}, \text{i.o.}\right\}.$$

Now, since $Y_n \leq Y_0 + bn$, it follows that

$$\{\mathcal{C}_j, \text{i.o.}\} \subset \{\mathcal{G}_{1j}, \text{i.o.}\} \cup \{\mathcal{G}_{2j}, \text{i.o.}\},$$

where

$$\mathcal{G}_{1j} := \left\{\left(\frac{Y_0}{bq^{j+1}} + 1\right) q^{\frac{j}{2}} |\rho_1 - \hat{\rho}_{1,q^j}| > j^2 \frac{\varepsilon}{3qb}\right\},$$

$$\mathcal{G}_{2j} := \left\{\left(\frac{Y_0}{bq^{j+1}} + 1\right) q^{\frac{j}{2}} |Z_{q^{j+1}} - \hat{\rho}_{1,q^j}| > j^2 \frac{\varepsilon}{3bq}\right\}.$$

We will now show that $\boldsymbol{P}(\mathcal{G}_{1j}, \text{i.o.}) = 0$. By using the Markov's inequality we have

$$\begin{aligned}
\sum_{j=1}^{\infty} \boldsymbol{P}(\mathcal{G}_{1j}) &\leq \frac{3qb}{\varepsilon} \sum_{j=1}^{\infty} \left(\frac{Y_0}{bq^{j+1}} + 1\right) \frac{\boldsymbol{E}[q^{\frac{j}{2}} |\rho_1 - \hat{\rho}_{1,q^j}|]}{j^2} \\
&= \frac{3qb}{\varepsilon} \left(\frac{Y_0}{bq} + 1\right) C \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty,
\end{aligned}$$

where

$$C := \sup_{k\geq 1} \boldsymbol{E}\{[q^{\frac{k}{2}}|\rho_1 - \hat{\rho}_{1,q^k}|]\} < \infty$$

from (2.6). Hence, using the Borel–Cantelli lemma, it follows that $\boldsymbol{P}(\mathcal{G}_{1j}, \text{i.o.}) = 0$.

Now, consider $\mathcal{G}_{2j}$. Let $\mathcal{H}_j := \{j^{-2}q^{\frac{j}{2}} \cdot |\Delta_{j,d_j}| > \varepsilon\}$ and since

$$\boldsymbol{P}(\mathcal{G}_{2j}, \text{i.o.}) = \boldsymbol{P}(\mathcal{H}_j, \text{i.o.})$$

we now focus on $\mathcal{H}_j$. First, for each $j \geq 1$, we recall that $\mathcal{Q}_j = \{\tau_j > d_j\}$ and we decompose $\mathcal{H}_j$ as follows:

$$\mathcal{H}_j \subseteq \mathcal{Q}_j \cup \{\mathcal{H}_j \cap \mathcal{Q}_j^C\},$$

which leads to

$$\boldsymbol{P}(\mathcal{H}_j, \text{i.o.}) \leq \boldsymbol{P}(\mathcal{Q}_j, \text{i.o.}) + \boldsymbol{P}(\mathcal{H}_j \cap \mathcal{Q}_j^C, \text{i.o.}).$$

First, consider $\boldsymbol{P}(\mathcal{H}_j \cap \mathcal{Q}_j^C, \text{i.o.})$. By using Markov's inequality, we have

$$\begin{aligned}\sum_{j=1}^{\infty} \boldsymbol{P}(\mathcal{H}_j \cap \mathcal{Q}_j^C) &\leq \sum_{j=1}^{\infty} \boldsymbol{E}[q^j \cdot |\Delta_{j,d_j}|\mathbf{1}_{\mathcal{Q}_j^C}]\frac{q^{-\frac{j}{2}}}{\varepsilon j^2} \\ &\leq \frac{(\sup_{k\geq 1}\{\boldsymbol{E}[q^k \cdot |\Delta_{k,d_k}|\mathbf{1}_{\mathcal{Q}_k^C}]\})}{\varepsilon} \sum_{j=1}^{\infty} \frac{q^{-\frac{j}{2}}}{j^2},\end{aligned}$$

which is finite from Theorem 3.2. Hence, again from the Borel–Cantelli lemma we have that

$$\boldsymbol{P}(\mathcal{H}_j \cap \mathcal{Q}_j^C, \text{i.o.}) = 0.$$

We will now show that $\boldsymbol{P}(\mathcal{Q}_j, \text{i.o.}) = 0$. To this end, we can follow the same arguments used in the first part of this proof, except that here we define

$$\mathcal{C}_j := \{|T_{j,d_j}^{(\rho_1)} - T_{q^j}| > \varepsilon q^j\}.$$

In this case, to show $\boldsymbol{P}(\mathcal{C}_j, \text{i.o.}) = 0$ we have to prove that the following two events cannot occur infinitely often:

(i) $\mathcal{G}_{3j} := \{(\frac{Y_0}{bq^{j+1}} + 1)|\rho_1 - \hat{\rho}_{1,q^j}| > \frac{\varepsilon}{2qb}\}$,

(ii) $\mathcal{G}_{4j} := \{(\frac{Y_0}{bq^j} + 1)|\rho_1 - Z_{q^j}| > \frac{\varepsilon}{2b}\}$.

Result (i) is implied by (2.6), while (ii) follows from Theorem 2.1. Hence, we have that

$$\boldsymbol{P}(\mathcal{C}_j, \text{i.o.}) = 0.$$

Then, similar to the first part of the proof, we deal with the sets $\mathcal{A}_j - \mathcal{C}_j$ and $\mathcal{B}_j - \mathcal{C}_j$ by applying Chernoff's upper bound to the probabilities

$$P_{Aj} = \boldsymbol{P}\left(\sum_{i=1}^{d_j} A_i < \varepsilon q^j\right) \quad \text{and} \quad P_{Bj} = \boldsymbol{P}\left(\sum_{i=1}^{d_j} B_i > -\varepsilon q^j\right),$$

which implies $\sum_{j=1}^{\infty} P_{Aj} < \infty$ and $\sum_{j=1}^{\infty} P_{Bj} < \infty$. Hence, from the Borel–Cantelli lemma we get

$$\boldsymbol{P}(\mathcal{A}_j - \mathcal{C}_j, \text{i.o.}) = \boldsymbol{P}(\mathcal{B}_j - \mathcal{C}_j) = 0,$$

which implies $\boldsymbol{P}(\mathcal{Q}_j, \text{i.o.}) = 0$. This completes the proof. $\square$

REMARK 4.2. The result of Theorem 4.1 continues to hold if (2.6) is not satisfied, but $\hat{\rho}_n \xrightarrow{\text{a.s.}} \rho$ and condition (c1) hold. Moreover, since in the proof we use Theorem 3.1, if (2.6) does not hold condition (c2) must be assumed (see Remark 3.1).

4.3. *Proof of Proposition* 2.1.

PROOF. Without loss of generality, assume $m_1 > m_2$, which implies $\hat{\rho}_n = \hat{\rho}_{1,n}$ and $\rho = \rho_1$. First, we have

$$\boldsymbol{E}\left[n|\bar{\rho}_{1,n} - \rho_1|^2\right] = \frac{1}{n}\boldsymbol{E}\left[\left|\sum_{i=0}^{n-1}(\tilde{\rho}_{1,i} - \rho_1)\right|^2\right], \tag{4.15}$$

and note that

$$\begin{aligned}\sum_{i=0}^{n-1}(\tilde{\rho}_{1,i} - \rho_1) &= \sum_{j=0}^{k_n}\sum_{i=0}^{d_j}(\tilde{\rho}_{1,q^j+i} - \rho_1)\mathbf{1}_{\{q^{k_n}+i \le n\}} \\ &= \sum_{j=0}^{k_n-1} d_j(\hat{\rho}_{1,q^j} - \rho_1) + \left(n - q^{k_n}\right)(\hat{\rho}_{1,q^{k_n}} - \rho_1),\end{aligned}$$

where we recall k_n is defined in (2.5) as $k_n := [\log_q(n)]$. Since $d_j = (q-1)q^j$, the LHS of (4.15) is equal to

$$\frac{(q-1)^2}{n}\boldsymbol{E}\left[\left|\sum_{j=0}^{k_n-1}(\sqrt{q})^j \cdot \left(\sqrt{q}^j(\hat{\rho}_{1,q^j} - \rho_1)\right) + \left(\frac{n - q^{k_n}}{q-1}\right)(\hat{\rho}_{1,q^{k_n}} - \rho_1)\right|^2\right],$$

and, defining $c_j := \sqrt{q}^j|\hat{\rho}_{1,q^j} - \rho_1|$, we can rewrite the last expression as follows:

$$\frac{(q-1)^2}{n}\boldsymbol{E}\left[\left(\sum_{j=0}^{k_n-1}(\sqrt{q})^j \cdot c_j + \left[\frac{n - q^{k_n}}{\sqrt{q}^{k_n}(q-1)}\right]c_{k_n}\right)^2\right].$$

Now, using the Cauchy–Schwarz inequality and using $(\frac{n-q^{k_n}}{\sqrt{q}^{k_n}(q-1)}) \leq \sqrt{q}^{k_n}$, the above expectation is less than or equal to

$$K_n := \sum_{j_1=0}^{k_n} \sum_{j_2=0}^{k_n} (\sqrt{q})^{j_1} (\sqrt{q})^{j_2} \cdot \sqrt{\boldsymbol{E}[c_{j_1}^2]\boldsymbol{E}[c_{j_2}^2]}.$$

Now, by the symmetry in K_n, we can use the following decomposition:

$$\begin{aligned}\sum_{j_1=0}^{k_n} \sum_{j_2=0}^{k_n} (\cdot) &= \sum_{j_1=0}^{\sqrt{k_n}} \sum_{j_2=0}^{\sqrt{k_n}} (\cdot) + 2 \sum_{j_1=0}^{\sqrt{k_n}} \sum_{j_2=\sqrt{k_n}}^{k_n} (\cdot) + \sum_{j_1=\sqrt{k_n}}^{k_n} \sum_{j_2=\sqrt{k_n}}^{k_n} (\cdot) \\ &\leq 2 \sum_{j_1=0}^{k_n} \sum_{j_2=0}^{\sqrt{k_n}} (\cdot) + \sum_{j_1=\sqrt{k_n}}^{k_n} \sum_{j_2=\sqrt{k_n}}^{k_n} (\cdot),\end{aligned}$$

we obtain

$$\begin{aligned}K_n &\leq \sup_{j\geq 1}\{\boldsymbol{E}[c_j^2]\} \cdot 2 \sum_{j_1=0}^{k_n} \sum_{j_2=0}^{\sqrt{k_n}} (\sqrt{q})^{j_1} (\sqrt{q})^{j_2} \\ &\quad + \max_{\sqrt{k_n}\leq j\leq k_n}\{\boldsymbol{E}[c_j^2]\} \cdot \sum_{j_1=\sqrt{k_n}}^{k_n} \sum_{j_2=\sqrt{k_n}}^{k_n} (\sqrt{q})^{j_1} (\sqrt{q})^{j_2} \\ &= K_{1n} + K_{2n}.\end{aligned}$$

Now, consider K_{1n}; we have that

$$K_{1n} \leq \sup_{j\geq 1}\{\boldsymbol{E}[c_j^2]\} \cdot 2\left(\frac{(\sqrt{q})^{\sqrt{k_n}+1} - 1}{\sqrt{q} - 1}\right)\left(\frac{(\sqrt{q})^{k_n+1} - 1}{\sqrt{q} - 1}\right),$$

and by multiplying for $(q-1)^2/n$ we obtain

$$2\left(\frac{q-1}{\sqrt{q}-1}\right)^2 \sup_{j\geq 1}\{\boldsymbol{E}[c_j^2]\} \cdot \left(\frac{(\sqrt{q})^{\sqrt{k_n}+1} - 1}{\sqrt{n}}\right)\left(\frac{(\sqrt{q})^{k_n+1} - 1}{\sqrt{n}}\right).$$

Using (2.6), we have that $\sup_{j\geq 1}\{\boldsymbol{E}[c_j^2]\}$ is finite. Moreover, since $n \leq q^{k_n+1}$ by definition of k_n, we have that

$$\left(\frac{(\sqrt{q})^{k_n+1} - 1}{\sqrt{n}}\right) \leq \sqrt{q}, \qquad \left(\frac{(\sqrt{q})^{\sqrt{k_n}+1} - 1}{\sqrt{n}}\right) \to 0.$$

Similarly, we can consider K_{2n} and write

$$K_{2n} \leq \max_{\sqrt{k_n}\leq j\leq k_n}\{\boldsymbol{E}[c_j^2]\} \cdot \left(\frac{(\sqrt{q})^{k_n+1} - 1}{\sqrt{q} - 1}\right)^2.$$

Then, by multiplying for $(q-1)^2/n$ we obtain

$$\left(\frac{q-1}{\sqrt{q}-1}\right)^2 \max_{\sqrt{k_n}\le j\le k_n}\{\boldsymbol{E}[c_j^2]\}\cdot\left(\frac{(\sqrt{q})^{k_n+1}-1}{\sqrt{n}}\right)^2,$$

and from (2.6) and $n \le q^{k_n+1}$ we have $\max_{\sqrt{k_n}\le j\le k_n}\{\boldsymbol{E}[c_j^2]\}$ is finite and

$$\left(\frac{(\sqrt{q})^{k_n+1}-1}{\sqrt{n}}\right)^2 \le q.$$

Then, combining all together, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\boldsymbol{E}\big[n|\bar{\rho}_{1,n}-\rho_1|^2\big] &\le \limsup_{n\to\infty}\frac{(q-1)^2}{n}K_{2n}\\ &\le q(1+\sqrt{q})^2\cdot\limsup_{n\to\infty}\boldsymbol{E}\big[n|\hat{\rho}_{1,n}-\rho_1|^2\big],\end{aligned}$$

which is finite because of condition (2.6). □

4.4. *Proof of Corollary* 2.2. To prove this result, we apply Theorem 2.2 to the urn model with fixed thresholds, that is, $\tilde{\rho}_{1,n}=\rho_1$ and $\tilde{\rho}_{2,n}=\rho_2$ for all $n\ge 0$, since in this case $\bar{\rho}_n=\rho$ for all $n\ge 0$.

5. Simulation studies. In this section we describe some simulation studies that illustrate the theoretical results presented in Section 2 in the context of clinical trials. We recall from Section 2.3 that the random variables $\xi_{1,n}$ and $\xi_{2,n}$ are interpreted as potential responses to competing treatments $\mathcal{T}_1$ and $\mathcal{T}_2$, whose distributions μ_1 and μ_2 depend on parameters $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, respectively. Let $\boldsymbol{\theta}=(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$. Now, letting f_1 and f_2 be two continuous functions, we recall that $\rho_1=f_1(\boldsymbol{\theta})$ and $\rho_2=f_2(\boldsymbol{\theta})$, and the adaptive thresholds are $\hat{\rho}_{1,n}:=f_1(\hat{\boldsymbol{\theta}}_n)$ and $\hat{\rho}_{2,n}:=f_2(\hat{\boldsymbol{\theta}}_n)$ for all $n\ge 1$, where $\hat{\boldsymbol{\theta}}_n$ is the adaptive estimator of $\boldsymbol{\theta}$ after the first n allocations.

The main goal of this section is to illustrate the asymptotic behavior of the allocation proportion $N_{1,n}/n$ and of the parameter estimator $\hat{\boldsymbol{\theta}}_{\boldsymbol{n}}$. Simulations are performed with $N=10^5$ independent urn processes, each that evolves following the model described in Section 2 with adaptive thresholds $\tilde{\rho}_{1,n}$ and $\tilde{\rho}_{2,n}$ that change at exponential times $\{q^j; j\ge 1\}$, with $q=1.25$ [see (2.4)]. For all the N urn processes, we used as initial composition $(y_{1,0},y_{2,0})=(2,2)$ and as sample size $n=200$. The functions f_1 and f_2 are chosen as in (2.10) with $p=0.75$. We analyze both Bernoulli and Gaussian responses.

5.1. *Bernoulli responses.* We assume responses to treatments $\mathcal{T}_1$ and $\mathcal{T}_2$ are Bernoulli distributed with parameters p_1 and p_2, respectively. In this case, $\boldsymbol{\theta}=(p_1,p_2)$. We examine two target allocations:

(a) $\eta(\boldsymbol{\theta})=(1-p_1)/(2-p_1-p_2)$, proposed by [23];

TABLE 1

Simulations of $N_{1,n}/n$ and $\hat{\boldsymbol{\theta}}_n$ are given for different designs, with mean square errors given in parentheses. The target allocation is $\rho_1 = (1-p)\cdot 1 + p\cdot\eta(\boldsymbol{\theta})$ with $p = 0.75$. Simulation used $N = 10^5$ ARRU processes with $n = 200$ and changes at times $\{q^j; j \geq 1\}$ with $q = 1.25$. Initial composition $(y_{1,0}, y_{2,0}) = (2, 2)$

Bernoulli responses					
$\boldsymbol{p_1}$	$\boldsymbol{p_2}$	$\boldsymbol{\rho_1}$	$\boldsymbol{N_{1,n}/n}$	$\boldsymbol{\hat{p}_{1,n}}$	$\boldsymbol{\hat{p}_{2,n}}$
		(a) $\eta = (1-p_1)/(2-p_1-p_2)$			
0.9	0.7	0.44	0.44 (0.07)	0.89 (0.03)	0.7 (0.04)
0.9	0.5	0.38	0.41 (0.06)	0.89 (0.03)	0.50 (0.05)
0.9	0.3	0.34	0.40 (0.07)	0.89 (0.03)	0.30 (0.04)
0.9	0.1	0.33	0.43 (0.12)	0.89 (0.03)	0.11 (0.03)
0.7	0.5	0.53	0.50 (0.07)	0.70 (0.05)	0.50 (0.05)
0.7	0.3	0.48	0.48 (0.05)	0.70 (0.05)	0.30 (0.04)
0.7	0.1	0.44	0.48 (0.06)	0.70 (0.05)	0.11 (0.03)
0.5	0.3	0.56	0.53 (0.06)	0.50 (0.05)	0.30 (0.05)
0.5	0.1	0.52	0.53 (0.04)	0.50 (0.05)	0.11 (0.03)
0.3	0.1	0.58	0.56 (0.05)	0.30 (0.04)	0.11 (0.03)
		(b) $\eta = \sqrt{p_1}/(\sqrt{p_1}+\sqrt{p_2})$			
0.9	0.7	0.65	0.57 (0.11)	0.89 (0.03)	0.69 (0.05)
0.9	0.5	0.68	0.63 (0.08)	0.89 (0.03)	0.50 (0.06)
0.9	0.3	0.73	0.69 (0.06)	0.89 (0.03)	0.30 (0.06)
0.9	0.1	0.81	0.76 (0.07)	0.89 (0.02)	0.11 (0.04)
0.7	0.5	0.66	0.58 (0.11)	0.69 (0.04)	0.50 (0.06)
0.7	0.3	0.70	0.66 (0.07)	0.70 (0.04)	0.30 (0.06)
0.7	0.1	0.79	0.74 (0.07)	0.70 (0.04)	0.12 (0.04)
0.5	0.3	0.67	0.60 (0.10)	0.50 (0.05)	0.30 (0.05)
0.5	0.1	0.77	0.70 (0.08)	0.50 (0.04)	0.11 (0.04)
0.3	0.1	0.73	0.64 (0.11)	0.30 (0.04)	0.11 (0.03)

(b) $\eta(\boldsymbol{\theta}) = \sqrt{p_1}/(\sqrt{p_1}+\sqrt{p_2})$, proposed by [21].

Hence, from (2.10) with $p = 0.75$, we have

$$\rho_1 = 0.25\cdot 1 + 0.75\cdot\eta(p_1, p_2), \quad \text{and} \quad \rho_2 = 0.25\cdot 0 + 0.75\cdot\eta(p_1, p_2).$$

In Table 1, we report simulation results on the mean and the standard error of the allocation proportion $N_{1,n}/n$ and of the estimators $\hat{p}_{1,n}$ and $\hat{p}_{2,n}$, defined as

$$\hat{p}_{1,n} = \frac{\sum_{i=1}^{n} X_i \xi_{1,i}}{N_{1,n}}, \quad \text{and} \quad \hat{p}_{2,n} = \frac{\sum_{i=1}^{n}(1-X_i)\xi_{2,i}}{N_{2,n}}.$$

5.2. *Gaussian responses.* We now assume responses to treatments $\mathcal{T}_1$ and $\mathcal{T}_2$ are normal distributed with parameters (m_1, σ_1^2) and (m_2, σ_2^2), respectively. In this case, $\boldsymbol{\theta} = (m_1, \sigma_1^2, m_2, \sigma_2^2)$. We examine two target allocations:

TABLE 1
(*Continued*)

Normal responses							
m_1	m_2	σ_1^2	σ_2^2	ρ_1	$N_{1,n}/n$	$\hat{\sigma}_{1,n}^2$	$\hat{\sigma}_{2,n}^2$
(c) $\eta = \sigma_1/(\sigma_1+\sigma_2)$							
10	5	1	1	0.63	0.61 (0.05)	1.01 (0.13)	1.01 (0.16)
8	5	1	1	0.63	0.59 (0.07)	1.01 (0.13)	1.01 (0.16)
6	5	1	1	0.63	0.55 (0.12)	1.01 (0.14)	1.01 (0.15)
10	5	4	1	0.75	0.73 (0.06)	4.00 (0.47)	1.01 (0.20)
8	5	4	1	0.75	0.71 (0.07)	4.00 (0.48)	1.01 (0.19)
6	5	4	1	0.75	0.66 (0.13)	4.03 (0.50)	1.01 (0.18)
10	5	1	4	0.50	0.49 (0.05)	1.01 (0.14)	4.00 (0.57)
8	5	1	4	0.50	0.48 (0.07)	1.01 (0.15)	4.03 (0.56)
6	5	1	4	0.50	0.43 (0.11)	1.01 (0.16)	4.03 (0.54)
(d) $\eta = \sigma_1\sqrt{m_2}/(\sigma_1\sqrt{m_2}+\sigma_2\sqrt{m_1})$							
10	5	1	1	0.56	0.55 (0.05)	1.01 (0.14)	1.01 (0.15)
8	5	1	1	0.58	0.55 (0.07)	1.01 (0.14)	1.01 (0.15)
6	5	1	1	0.61	0.53 (0.12)	1.01 (0.14)	1.01 (0.15)
10	5	4	1	0.69	0.67 (0.06)	4.03 (0.49)	1.01 (0.18)
8	5	4	1	0.71	0.67 (0.07)	4.03 (0.49)	1.01 (0.18)
6	5	4	1	0.73	0.65 (0.13)	4.03 (0.51)	1.01 (0.18)
10	5	1	4	0.45	0.44 (0.05)	1.01 (0.15)	4.03 (0.54)
8	5	1	4	0.46	0.44 (0.07)	1.01 (0.16)	4.03 (0.54)
6	5	1	4	0.48	0.42 (0.11)	1.01 (0.16)	4.03 (0.53)

(c) $\eta(\boldsymbol{\theta}) = \sigma_1/(\sigma_1+\sigma_2)$, used in [15];
(d) $\eta(\boldsymbol{\theta}) = \sigma_1\sqrt{m_2}/(\sigma_1\sqrt{m_2}+\sigma_2\sqrt{m_1})$, proposed by [24].

Hence, from (2.10) with $p = 0.75$, we have

$$\rho_1 = 0.25 \cdot 1 + 0.75 \cdot \eta(\boldsymbol{\theta}), \quad \text{and} \quad \rho_2 = 0.25 \cdot 0 + 0.75 \cdot \eta(\boldsymbol{\theta}).$$

In Table 1, we report simulation results on the mean and the standard error of the allocation proportion $N_{1,n}/n$ and the parameter estimators $\hat{\sigma}_{1,n}^2$ and $\hat{\sigma}_{2,n}^2$, defined as

$$\hat{\sigma}_{1,n}^2 = \frac{\sum_{i=1}^n X_i(\xi_{1,i} - \hat{m}_{1,n})^2}{N_{1,n}}, \qquad \hat{\sigma}_{2,n}^2 = \frac{\sum_{i=1}^n (1-X_i)(\xi_{2,i} - \hat{m}_{2,n})^2}{N_{2,n}},$$

where $\hat{m}_{1,n} = \sum_{i=1}^n X_i\xi_{1,i}/N_{1,n}$ and $\hat{m}_{2,n} = \sum_{i=1}^n (1-X_i)\xi_{2,i}/N_{2,n}$.

The results show that our methods target the true parameters effectively. In real clinical trials, further calibration may be performed to reduce small bias.

6. Extensions to multi-color urn models. It is important to note that all the results presented in this paper can be extended to the case of $K > 2$ colors, when there exists $j \in \{1, \dots, K\}$ such that $m_j > m_k$ for any $k \neq j$. In the context of clinical trials, the functions f_j should be interpreted as the target allocations for $N_{j,n}/n$ when $\mathcal{T}_j$ is the superior treatment, and the variables $W_{j,n}$ should be all defined as $\mathbf{1}_{\{Z_n \le \hat{\rho}_{j,n}\}}$.

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