

Uniform central limit theorems for the Grenander estimator

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Abstract: We consider the Grenander estimator that is the maximum likelihood estimator for non-increasing densities. We prove uniform central limit theorems for certain subclasses of bounded variation functions and for Hölder balls of smoothness $s > 1/2$. We do not assume that the density is differentiable or continuous. The proof can be seen as an adaptation of the method for the parametric maximum likelihood estimator to the nonparametric setting. Since nonparametric maximum likelihood estimators lie on the boundary, the derivative of the likelihood cannot be expected to equal zero as in the parametric case. Nevertheless, our proofs rely on the fact that the derivative of the likelihood can be shown to be small at the maximum likelihood estimator.

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1. Introduction

A fundamental approach to statistical estimation is finding the probability measure which renders the observation most likely. This principle of maximum likelihood estimation has proved very successful in parametric estimation but leads to difficulties in nonparametric problems since the likelihood is typically unbounded so that no maximum is attained. However, in nonparametric problems with shape constraints the maximum likelihood estimator is often well-defined and thus the maximum likelihood approach can be extended to these situations. Examples include non-increasing, k -monotone, convex, concave and log-concave functions.

The classical parametric maximum likelihood theory is based on the estimator $\hat{\theta}_n$ being in the interior of the parameter space and on the resulting fact

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that the derivative of the likelihood vanishes at $\hat{\theta}_n$. The theory of nonparametric maximum likelihood estimation is quite different from the parametric theory since the estimator lies on the boundary of the parameter space and thus in general the derivative of the likelihood will not be zero. But in some nonparametric situations the derivative of the likelihood can be shown to be sufficiently small, thus enabling a proof strategy paralleling the one in the classical parametric theory. Nickl (2007) considered the nonparametric maximum likelihood estimator for estimating a density in a Sobolev ball and proved uniform central limit theorems using this approach.

We pursue this method of proof in the problem of estimating a non-increasing density p_0 . The maximum likelihood estimator \hat{p}_n is called the Grenander estimator in this situation since it was first derived by Grenander (1956). It is well known to be the left-derivative of the least concave majorant of the empirical distribution function. The main results will be uniform central limit theorems for the Grenander estimator that in particular imply for functions f

$$\sqrt{n} \int_0^\infty (\hat{p}_n(x) - p_0(x))f(x)dx \rightarrow^d N(0, \|f - P_0f\|_{L^2(P_0)}^2),$$

where P_0 is the probability measure of the non-increasing density p_0 and $P_0f \equiv \int_0^\infty f(x)dP_0(x)$. We do not assume the density p_0 to be differentiable or continuous. Our results are uniform for f varying in a class of functions and we cover two different types of classes. The first type is a subclass of the bounded variation functions and for each point of discontinuity t of p_0 the indicator function $\mathbb{1}_{[0,t]}$ is contained in such a class. The second type of classes is given by balls in Hölder spaces C^s of order $s > 1/2$.

Under a strict curvature condition and for continuously differentiable p_0 , Kiefer and Wolfowitz (1976) proved that the difference between the distribution function of the Grenander estimator and the empirical distribution function in supremum norm is with probability one of order $n^{-2/3} \log(n)$. On the one hand this means that the two distribution functions are close and the distribution function of the Grenander estimator does essentially not improve on the empirical distribution function. On the other hand it shows that the distribution function of the Grenander estimator enjoys many optimality properties of the empirical distribution function. The Kiefer–Wolfowitz theorem can be used to prove that the distribution function of the Grenander estimator is an asymptotically minimax estimator for concave distribution functions. It further implies a uniform central limit theorem for the Grenander estimator over the class of all indicator functions $\mathbb{1}_{[0,t]}$, $t \geq 0$. The Kiefer–Wolfowitz theorem was used by Sen et al. (2010) to study consistency and inconsistency of bootstrap methods when estimating a non-increasing density.

Results similar to the Kiefer–Wolfowitz theorem hold under other shape constraints as well. In addition to giving an updated proof of the Kiefer–Wolfowitz theorem, Balabdaoui and Wellner (2007) showed such a theorem in the case where the density is assumed to be convex decreasing and where the maximum likelihood estimator is replaced by the least squares estimator. Dümbgen and Rufibach (2009) derived the rate of estimation for log-concave densities in

supremum norm and showed that the difference between empirical and estimated distribution function is $o(n^{-1/2})$. Durot and Lopuhaä (2014) showed a general Kiefer–Wolfowitz type theorem which covers the estimation of monotone regression curves, monotone densities and monotone failure rates.

Jankowski (2014) studied the local convergence rates of the Grenander estimator in situations where it is misspecified and derived the asymptotic distribution of linear functionals under possible misspecification. She distinguishes between curved and flat parts regarding the density p_0 . On the curved parts our results have the advantage that we do not need to assume that p_0 and f are differentiable, whereas on the flat parts Jankowski (2014) is more general by treating L^p functions f . We will discuss this in detail in Section 3. Beyond asymptotic normality for single functionals our work includes a uniformity in the underlying functional, which is important to apply the results by Nickl (2009) concerning convolutions of density estimators and to show the “plug-in property” introduced by Bickel and Ritov (2003). Estimators with this property are rate optimal density estimators that simultaneously lead to the efficient estimation of functionals with uniform convergence over a class of functionals. We will elaborate on these applications in Section 3. Nickl (2007) discusses applications of uniform central limit theorems in the context of the maximum likelihood estimator over a Sobolev ball. Uniform central limit theorems were also shown for kernel density estimators and for wavelet density estimators by Giné and Nickl (2008, 2009).

The method of proof presented in this paper is of general nature and its relevance extends to maximum likelihood estimators in other problems with shape constraints. For example one could consider the estimation of classes of convex decreasing densities or, more general, of k -monotone densities or of other shape constrained densities as long as the classes are convex and closed with respect to the supremum norm.

This paper is organised as follows. In Section 2 we state the uniform central limit theorems for a subclass of the bounded variation functions and for Hölder balls. Section 3 provides a discussion and applications of our results. In Section 4 we explain the general approach. In Section 5 we derive upper and lower bounds in probability for the Grenander estimator and recall the L^2 -convergence rate. Section 6 develops the approach further and contains the proofs of the main results.

2. Main results

Let X_1, \dots, X_n be i.i.d. on $[0, \infty)$ with law P_0 and distribution function $F_0(x) = \int_0^x dP_0, x \in [0, \infty)$. In order to state the main results we introduce some notation. We define the empirical measure $P_n = n^{-1} \sum_{i=1}^n \delta_{X_i}$, the empirical cumulative distribution function $F_n(x) = \int_0^x dP_n, x \in [0, \infty)$ and the log-likelihood function

$$\ell_n(p) = \frac{1}{n} \sum_{i=1}^n \log p(X_i). \quad (1)$$

Under the assumption $E_{P_0} |\log p(X)| < \infty$ for all $p \in \mathcal{P}$, we can define the limiting log-likelihood function

$$\ell(p) = \int_0^\infty \log p(x) dP_0(x). \tag{2}$$

If \mathcal{P} is known to have a monotone decreasing density p then the associated maximum likelihood estimator \hat{p}_n maximises the log-likelihood function $\ell_n(p)$ over

$$\mathcal{P} \equiv \mathcal{P}^{mon} = \left\{ p : [0, \infty) \rightarrow [0, \infty), \int_0^\infty p(x) dx = 1, p \text{ is non-increasing} \right\},$$

that is,

$$\max_{p \in \mathcal{P}^{mon}} \ell_n(p) = \ell_n(\hat{p}_n). \tag{3}$$

The maximum likelihood estimator \hat{p}_n is known to be the left-derivative of the least concave majorant \hat{F}_n of the empirical distribution function F_n . Let \hat{P}_n be the probability measure corresponding to the density \hat{p}_n . For a set T let $\ell^\infty(T)$ denote the space of bounded real-valued functions on T with the usual supremum norm $\|\cdot\|_\infty$. Throughout we will denote by \rightarrow^d the convergence in distribution as in Chapter 1 in van der Vaart and Wellner (1996). The P_0 -Brownian bridge G_{P_0} is defined as tight Gaussian random variable arising from the centred Gaussian process with covariance

$$\mathbb{E}[G_{P_0}(f)G_{P_0}(g)] = P_0(fg) - P_0fP_0g,$$

where $P_0f \equiv \int_0^\infty f(x)dP_0(x)$.

The first main result is a uniform central limit theorem for a subclass of the bounded variation functions. We start with the general result in Theorem 1 and consider its consequences in Corollary 1 and Theorem 2. Let $f, p_0 \in L^1([0, \infty))$ and assume that the weak derivatives of $f|_{(0, \infty)}$ and $p_0|_{(0, \infty)}$ in the sense of regular Borel signed measures exist and denote them by Df and Dp_0 , respectively; cf., e.g., p. 42 in Ziemer (1989). We define $BV[0, \infty) \equiv \{f \in L^1([0, \infty)) : \|f\|_1 + |Df|(0, \infty) < \infty\}$, where $|Df|$ is total variation of the signed measure Df . In the following theorem it will be important that Df is absolutely continuous with respect to Dp_0 since we want to ensure that the perturbations $p_0 \pm \eta f$ with $|\eta|$ small (or slightly modified perturbations) are decreasing functions. To this end we denote the Radon–Nikodym derivative of Df with respect to Dp_0 by Df/Dp_0 and assume that its essential supremum with respect to Dp_0 , denoted by $\|Df/Dp_0\|_{\infty, Dp_0}$, is bounded. We will consider decreasing densities p_0 with bounded support S_0 . Then we can write $S_0 = [0, \alpha_1]$ for some $\alpha_1 > 0$. For the statement of the theorem the values of f are only important on the open interval $(0, \alpha_1)$ so that we can restrict f to this interval for the assumptions.

Theorem 1. *Suppose $p_0 \in \mathcal{P}$ is bounded, has bounded support $[0, \alpha_1]$ and that $p_0(x) \geq \zeta > 0$ for all $x \in [0, \alpha_1]$. Then for*

$$\mathcal{B} = \{f \in BV[0, \infty) : f|_{(0, \alpha_1)} = g, \|g\|_\infty + \|Dg/Dp_0\|_{\infty, Dp_0} \leq B\}$$

we have

$$\sup_{f \in \mathcal{B}} |(\hat{P}_n - P_n)(f)| = O_{P_0^n}(Bn^{-2/3})$$

and consequently

$$\sqrt{n}(\hat{P}_n - P_0) \rightarrow^d G_{P_0} \text{ in } \ell^\infty(\mathcal{B}).$$

The proof of Theorem 1 is deferred to the end of the paper. Let us consider one particular example as a corollary. We can take for points $t > 0$ where p_0 is discontinuous the indicator function $f = \mathbb{1}_{[0,t]}$ in Theorem 1. We have $Df = -\delta_t$. Say p_0 has a discontinuity of size $\Delta \equiv \lim_{s \nearrow t} p_0(s) - \lim_{s \searrow t} p_0(s) > 0$ at t . Then we have $Dp_0 = -\Delta\delta_t - \mu$ for a positive measure μ . In this case we obtain $\|Df/Dp_0\|_{\infty, Dp_0} = 1/\Delta$ leading to the following corollary.

Corollary 1. *Suppose $p_0 \in \mathcal{P}$ is bounded, has bounded support S_0 and that $p_0(x) \geq \zeta > 0$ for all $x \in S_0$. Then for each $t > 0$ where p_0 is discontinuous we have*

$$|\hat{F}_n(t) - F_n(t)| = O_{P_0^n}(n^{-2/3})$$

and consequently

$$\sqrt{n}(\hat{F}_n(t) - F_0(t)) \rightarrow^d N(0, F_0(t) - F_0(t)^2).$$

Here for $t > 0$ such that $F_0(t) = 1$ we understand $N(0, 0)$ to be δ_0 . To formulate the next theorem we define for $s > 0$ the Hölder spaces

$$C^s([0, \infty)) \equiv \left\{ f \in C([0, \infty)) : \|f\|_{C^s} = \sum_{j=0}^{[s]} \|D^j f\|_\infty + \sup_{x \neq y} \frac{|D^{[s]} f(x) - D^{[s]} f(y)|}{|x - y|^{s-[s]}} < \infty \right\},$$

where $[s]$ denotes the integer part of s and $C([0, \infty))$ are the bounded real-valued continuous functions on $[0, \infty)$. We will see that under a strict curvature condition the set \mathcal{B} in Theorem 1 contains C^1 -Hölder balls. By strict curvature condition we have in mind that p'_0 is bounded away from zero, that is $\inf_{x \in S_0} |p'_0(x)| \geq \xi > 0$ or equivalently $\sup_{x \in S_0} 1/|p'_0(x)| \leq 1/\xi$. We do not want to assume that p'_0 exists classically. To allow for discontinuities of p_0 and to stay in the general setting of weak derivatives we assume that the Lebesgue measure on S_0 denoted by λ is absolutely continuous with respect to Dp_0 and replace the assumption $\sup_{x \in S_0} 1/|p'_0(x)| \leq 1/\xi$ by the weaker assumption $\|\lambda/Dp_0\|_\infty \leq 1/\xi$, where λ/Dp_0 is the Radon–Nikodym derivative of λ with respect to Dp_0 . We remark that $\|\lambda/Dp_0\|_{\infty, Dp_0} = \|\lambda/Dp_0\|_\infty$. Let \mathcal{F} be a C^1 -Hölder ball and $g = f|_{(0, \alpha_1)}$ with $f \in \mathcal{F}$. Then $\|Dg/Dp_0\|_{\infty, Dp_0} = \|Dg/\lambda \cdot \lambda/Dp_0\|_{\infty, Dp_0} \leq (1/\xi)\|g'\|_\infty$ and we see that the C^1 -Hölder ball \mathcal{F} is contained in \mathcal{B} for some B . This special case of Theorem 1 with \mathcal{F} instead of \mathcal{B} can be generalized to balls in Hölder spaces $C^s([0, \infty))$ of order $s > 1/2$.

Theorem 2. *Suppose $p_0 \in \mathcal{P}$ is bounded, has bounded support S_0 and that $p_0(x) \geq \zeta > 0$ for all $x \in S_0$. Denote by λ the Lebesgue measure on S_0 and let λ*

be absolutely continuous with respect to Dp_0 and $\|\lambda/Dp_0\|_\infty < \infty$. Let \mathcal{F} be a ball in the s -Hölder space $C^s([0, \infty))$ of order $s > 1/2$. Then

$$\sup_{\mathcal{F}} |(\hat{P}_n - P_n)(f)| = o_{P_0^n}(1/\sqrt{n})$$

as $n \rightarrow \infty$ and thus

$$\sqrt{n}(\hat{P}_n - P_0) \rightarrow^d G_{P_0} \text{ in } \ell^\infty(\mathcal{F}).$$

In particular, for any $f \in C^s([0, \infty))$ with $s > 1/2$ we have

$$\sqrt{n} \int_0^\infty (\hat{p}_n(x) - p_0(x))f(x)dx \rightarrow^d N(0, \|f - P_0 f\|_{L^2(P_0)}^2).$$

The proof of Theorem 2 will be given at the end of the paper.

3. Discussion and applications

The $n^{2/3}$ -rate appearing in Theorem 1 and Corollary 1 is the pointwise rate at which the least concave majorant \hat{F}_n converges to the empirical distribution function F_n . Wang (1994) derived the pointwise limit theorem with a $n^{2/3}$ -rate for $\hat{F}_n(t_0)$ at a point $t_0 > 0$ where p_0 has a negative derivative. Although our statement is of uniform nature we obtain the same rate in Theorem 1 and no additional logarithmic factor needs to be paid for the uniformity. A possible explanation is that the class of functions is adapted to the density p_0 . We also note that Theorem 1 yields uniformity only over finitely many indicator functions even if p_0 has infinitely many discontinuities. The rate in the Kiefer–Wolfowitz theorem is $(n/\log n)^{2/3}$, which differs from our rate by a logarithmic factor. Indeed for bounding $\|\hat{F}_n - F_n\|_\infty$ this additional factor seems necessary at least it was shown by Durot and Tocquet (2003) that it is necessary in the monotone regression framework. Our results further differ from the Kiefer–Wolfowitz theorem in the sense that the convergence is in probability, whereas the Kiefer–Wolfowitz theorem yields almost sure convergence. Our assumptions are weaker than in the Kiefer–Wolfowitz theorem since p_0 is neither assumed to be differentiable nor continuous. We require a strict curvature condition only in Theorem 2, but not in Theorem 1 or Corollary 1.

Linear functionals of the Grenander estimator have also been studied by Jankowski (2014) so let us discuss the differences in scope and in the assumptions between the results. A distinct feature of Theorems 1 and 2 is that they are not for a fixed function f but they are uniform in f over classes of functions. Jankowski (2014) takes a different perspective and emphasises the problem of possible misspecification, meaning that the true density p_0 does not necessarily need to be non-increasing. She distinguishes between curved and flat parts of p_0 (or in case of misspecification of its Kullback–Leibler projection) and assumes on the portion of support where p_0 is curved that p_0 is continuously differentiable and that $|p'_0|$ is bounded, which is used for the application of the Kiefer–Wolfowitz theorem in the proof. The assumption that p_0 is continuously

differentiable is widely used in the literature on the Grenander estimator so it is worthwhile to remark that Theorems 1 and 2 do not require p_0 to be differentiable nor to be continuous. The function f defining the functional is assumed by Jankowski (2014) to be differentiable on the curved parts, whereas Theorem 1 allows for discontinuities of f at points where p_0 is discontinuous and Theorem 2 only requires Hölder smoothness of order $s > 1/2$ on the curved parts. Jankowski (2014) assumes the function f to be in L^p , $p > 2$, on the flat parts, while Theorem 1 assumes f to be constant on the flat parts which in view of the results by Jankowski (2014) is the natural condition to ensure a Gaussian limit. Theorem 2 excludes flat parts by a strict curvature condition. To summarise the comparison with Jankowski's results we can say that our approach has the advantage of providing uniform results under low regularity assumptions on p_0 and requires stronger assumptions on f on the flat parts while needing weaker assumptions on the curved parts.

As an application we present the estimation of sums of independent random variables X and Y with densities by p_0 and q_0 , respectively. Let $X + Y = Z$ and the aim is the estimation of the density of Z . We observe either independent i.i.d. samples X_1, \dots, X_n and Y_1, \dots, Y_n of X and Y , respectively, or in the special case, where X and Y have the same distribution, we only need one sample and can set $X_j = Y_j$ for $j = 1, \dots, n$. The random variable Z has the density $p_0 * q_0$ and the canonical estimator is $\hat{p}_n * \hat{q}_n$, where \hat{p}_n and \hat{q}_n are estimators of p_0 and q_0 , respectively. For kernel estimators the convergence of $\hat{p}_n * \hat{q}_n - p_0 * q_0$ in distribution in $L^1(\mathbb{R})$ was shown by Schick and Wefelmeyer (2004, 2007) as well as Giné and Mason (2007). Nickl (2009) derives general conditions for the convergence of convolutions of density estimators which we verify by our uniform central limit theorems for the Grenander estimator. To this end let $p_0 : \mathbb{R} \rightarrow [0, \infty)$ and $q_0 : \mathbb{R} \rightarrow [0, \infty)$ be densities of random variables such that $p_0|_{[0, \infty)}$ and $q_0|_{[0, \infty)}$ satisfy the assumptions of Theorem 2. We denote by \hat{p}_n and \hat{q}_n the respective Grenander estimators. We decompose

$$\begin{aligned} \sqrt{n}(\hat{p}_n * \hat{q}_n - p_0 * q_0) &= \sqrt{n}(\hat{p}_n - p_0) * q_0 + \sqrt{n}(\hat{q}_n - q_0) * p_0 \\ &\quad + \sqrt{n}(\hat{p}_n - p_0) * (\hat{q}_n - q_0). \end{aligned} \tag{4}$$

By Theorem 2 we have uniform central limit theorems for \hat{P}_n and \hat{Q}_n in $\ell^\infty(\mathcal{F})$ for balls \mathcal{F} in the Hölder space C^s , $s > 1/2$. By Lemma 8(b) in Giné and Nickl (2008) we infer from p_0, q_0 being of bounded variation and in $L^1(\mathbb{R})$ that $p_0, q_0 \in B_{1\infty}^1(\mathbb{R}) \hookrightarrow B_{11}^s(\mathbb{R})$ for $s < 1$, where $B_{pq}^s(\mathbb{R})$ are Besov spaces. Together these two statements yield the convergence of $\sqrt{n}(\hat{p}_n - p_0) * q_0$ in distribution in $L^1(\mathbb{R})$ by Theorem 2 in Nickl (2009) and likewise for the term $\sqrt{n}(\hat{q}_n - q_0) * p_0$. To bound the last term in (4) we use Young's inequality, the bounded support of p_0 and q_0 as well as Proposition 2 below

$$\begin{aligned} \|(\hat{p}_n - p_0) * (\hat{q}_n - q_0)\|_1 &\leq \|\hat{p}_n - p_0\|_1 \|\hat{q}_n - q_0\|_1 \leq C \|\hat{p}_n - p_0\|_2 \|\hat{q}_n - q_0\|_2 \\ &= O_{P_0^N}(n^{-2/3}) = o_{P_0^N}(n^{-1/2}), \end{aligned}$$

where $C > 0$ is some constant. We conclude that

$$\sqrt{n}(\hat{p}_n * \hat{q}_n - p_0 * q_0)$$

converges in distribution in $L^1(\mathbb{R})$. By Remark 2 in Nickl (2009) the limiting random variable can be determined by calculating for every $h \in L^\infty = (L^1)^*$ the limits of

$$\sqrt{n} \int h(x) d\left((\hat{P}_n - P_0) * Q_0 + (\hat{Q}_n - Q_0) * P_0\right)(x).$$

Moreover, it follows from our Theorem 2 and from the continuous linear mapping established in the proof of Theorem 2 in Nickl (2009) that the limiting random variable may likewise be determined by calculating for all $h \in L^\infty = (L^1)^*$ the limits of

$$\begin{aligned} & \sqrt{n} \int h(x) d\left((P_n - P_0) * Q_0 + (Q_n - Q_0) * P_0\right)(x) \\ &= \sqrt{n} \left(\frac{1}{n} \sum_{j=1}^n g(X_j) - E_{P_0} g \right) + \sqrt{n} \left(\frac{1}{n} \sum_{j=1}^n f(Y_j) - E_{Q_0} f \right) \end{aligned}$$

with $g = q_0(-\cdot) * h$ and $f = p_0(-\cdot) * h$. So the limit is a mean zero Gaussian random variable with an explicitly given covariance structure.

The uniform results can be interpreted in the context of Bickel and Ritov (2003). They coin the expression “plug-in estimator” for a rate optimal estimator of a density which simultaneously leads to the efficient estimation of functionals with uniform convergence over a class of functionals. The Grenander estimator attains the optimal $n^{1/3}$ -rate for non-increasing densities. Since \mathcal{P} is a nonparametric model, the empirical distribution function is an asymptotically efficient estimator of P_0 considered as an element $f \mapsto P_0 f$ of the space $\ell^\infty(\mathcal{F})$ for a Donsker class \mathcal{F} , see van der Vaart and Wellner (1996, p. 420). By our results \hat{P}_n and P_n are closer than P_n and P_0 so that \hat{P}_n is an asymptotically efficient estimator as well. In summary our results show that the Grenander estimator is a plug-in estimator for a subclass of the bounded variation functions and for Hölder balls of smoothness $s > 1/2$.

4. The derivative of the likelihood function

Many classical properties of maximum likelihood estimators $\hat{\theta}_n$ of regular parameters $\theta \in \Theta \subset \mathbb{R}^p$, such as asymptotic normality, are derived from the fact that the derivative of the log-likelihood function vanishes at $\hat{\theta}_n$,

$$\frac{\partial}{\partial \theta} \ell_n(\theta) \Big|_{\hat{\theta}_n} = 0. \tag{5}$$

This typically relies on the assumption that the true parameter θ_0 is interior to Θ so that by consistency $\hat{\theta}_n$ will then eventually also be. In the infinite-dimensional setting, even if one can define an appropriate notion of derivative, this approach is usually not viable since \hat{p}_n is *never* an interior point in the parameter space even when p_0 is.

We now investigate these matters in more detail in the setting where \mathcal{P} consists of bounded probability densities. In this case we can compute the

Fréchet derivatives of the log-likelihood function on the space $L^\infty = L^\infty([0, \infty))$ equipped with the $\|\cdot\|_\infty$ -norm. Recall that a real-valued function $L : U \rightarrow \mathbb{R}$ defined on an open subset U of a Banach space B is Fréchet differentiable at $f \in U$ if

$$\lim_{\|h\|_B \rightarrow 0} \frac{|L(f+h) - L(f) - DL(f)[h]|}{\|h\|_B} = 0 \quad (6)$$

for some linear continuous map $DL(f) : B \rightarrow \mathbb{R}$. If $g \in U$ is such that the line segment $(1-t)f + tg, t \in (0, 1)$, joining f and g lies in U (for instance if U is convex) then the directional derivative of L at f in the direction g equals precisely

$$\lim_{t \rightarrow 0^+} \frac{L(f + t(g-f)) - L(f)}{t} = DL(f)[g-f].$$

The second order Fréchet derivatives are defined by taking the Fréchet derivative of $DL(f)[h]$ for a fixed direction h and likewise higher order Fréchet derivatives are defined. The following proposition shows that the log-likelihood function ℓ_n is Fréchet differentiable on the open convex subset of L^∞ consisting of functions that are positive at the sample points. A similar result holds for ℓ if one restricts to functions that are bounded away from zero on the support S_0 of p_0 . We recall here these results of Proposition 3 in Nickl (2007).

Proposition 1. *For any finite set of points $x_1, \dots, x_n \in [0, \infty)$ define*

$$\mathcal{U}(x_1, \dots, x_n) = \left\{ f \in L^\infty([0, \infty)) : \min_{1 \leq i \leq n} f(x_i) > 0 \right\}$$

and

$$\mathcal{U} = \left\{ f \in L^\infty([0, \infty)) : \inf_{x \in S_0} f(x) > 0 \right\}.$$

Then $\mathcal{U}(x_1, \dots, x_n)$ and \mathcal{U} are open subsets of $L^\infty([0, \infty))$.

Let ℓ_n be the log-likelihood function from (1) based on $X_1, \dots, X_n \sim^{i.i.d.} P_0$, and denote by P_n the empirical measure associated with the sample. Let ℓ be as in (2). For $\alpha \in \mathbb{N}$ and $f_1, \dots, f_\alpha \in L^\infty([0, \infty))$ the α -th Fréchet derivatives of $\ell_n : \mathcal{U}(X_1, \dots, X_n) \rightarrow \mathbb{R}$, $\ell : \mathcal{U} \rightarrow \mathbb{R}$ at a point $f \in \mathcal{U}(X_1, \dots, X_n)$, $f \in \mathcal{U}$, respectively, are given by

$$D^\alpha \ell_n(f)[f_1, \dots, f_\alpha] \equiv (-1)^{\alpha-1} (\alpha-1)! P_n(f^{-\alpha} f_1 \cdots f_\alpha), \quad (7)$$

$$D^\alpha \ell(f)[f_1, \dots, f_\alpha] \equiv (-1)^{\alpha-1} (\alpha-1)! P_0(f^{-\alpha} f_1 \cdots f_\alpha). \quad (8)$$

We deduce from the above proposition the intuitive fact that the limiting log-likelihood function has a derivative at the true point $p_0 > 0$ that is zero in all ‘tangent space’ directions h in

$$\mathcal{H} \equiv \left\{ h : \int_{S_0} h = 0 \right\} \quad (9)$$

since

$$D\ell(p_0)[h] = \int_{S_0} p_0^{-1} h dP_0 = \int_{S_0} h = 0. \quad (10)$$

However, in the infinite-dimensional setting the empirical counterpart of (10),

$$D\ell_n(\hat{p}_n)[h] = 0 \tag{11}$$

for $h \in \mathcal{H}$ and \hat{p}_n the nonparametric maximum likelihood estimator is not true in general. Even if the set \mathcal{P} the likelihood was maximised over is contained in $\mathcal{U}(X_1, \dots, X_n)$ it will itself in typical nonparametric situations have empty interior in L^∞ , and the maximiser \hat{p}_n will lie at the boundary of \mathcal{P} . As a consequence we cannot expect that \hat{p}_n is a zero of $D\ell_n$.

Following ideas in Nickl (2007) we can circumvent this problem in some situations: if the true value p_0 lies in the ‘interior’ of \mathcal{P} in the sense that local L^∞ -perturbations of p_0 are contained in $\mathcal{P} \cap \mathcal{U}(X_1, \dots, X_n)$, then we can bound $D\ell_n$ at \hat{p}_n .

Lemma 1. *Let \hat{p}_n be as in (3) and suppose that for some $h \in L^\infty([0, \infty))$, $\eta > 0$, the line segment joining \hat{p}_n and $p_0 \pm \eta h$ is contained in $\mathcal{P} \cap \mathcal{U}(X_1, \dots, X_n)$. Then*

$$|D\ell_n(\hat{p}_n)[h]| \leq (1/\eta)|D\ell_n(\hat{p}_n)[\hat{p}_n - p_0]|. \tag{12}$$

Proof. Since \hat{p}_n is a maximiser over \mathcal{P} we deduce from differentiability of ℓ_n on $\mathcal{U}(X_1, \dots, X_n)$ that the derivative at \hat{p}_n in the direction $p_0 + \eta h \in \mathcal{P} \cap \mathcal{U}(X_1, \dots, X_n)$ necessarily has to be nonpositive, that is

$$D\ell_n(\hat{p}_n)[p_0 + \eta h - \hat{p}_n] = \lim_{t \rightarrow 0^+} \frac{\ell_n(\hat{p}_n + t(p_0 + \eta h - \hat{p}_n)) - \ell_n(\hat{p}_n)}{t} \leq 0 \tag{13}$$

or, by linearity of $D\ell_n(\hat{p}_n)[\cdot]$,

$$D\ell_n(\hat{p}_n)[\eta h] \leq D\ell_n(\hat{p}_n)[\hat{p}_n - p_0]. \tag{14}$$

Applying the same reasoning with $-\eta$ we see

$$|D\ell_n(\hat{p}_n)[\eta h]| \leq D\ell_n(\hat{p}_n)[\hat{p}_n - p_0] = |D\ell_n(\hat{p}_n)[\hat{p}_n - p_0]|. \tag{15}$$

Divide by η to obtain the result. □

The above lemma is interesting if we are able to show that

$$D\ell_n(\hat{p}_n)[\hat{p}_n - p_0] = o_{P_0^n}(1/\sqrt{n}),$$

as then the same rate bound carries over to $D\ell_n(\hat{p}_n)[h]$. This can in turn be used to mimic the finite-dimensional asymptotic normality proof of maximum likelihood estimators, which does not require (5) but only that the score is of smaller stochastic order of magnitude than $1/\sqrt{n}$. As a consequence we will be able to obtain the asymptotic distribution of linear integral functionals of \hat{p}_n , and more generally, for \hat{P}_n the probability measure associated with \hat{p}_n , central limit theorems for $\sqrt{n}(\hat{P}_n - P)$ in ‘empirical process - type’ spaces $\ell^\infty(\mathcal{F})$. To understand this better we notice that Proposition 1 implies the following relationships: If we define the following projection of $f \in L^\infty$ onto \mathcal{H} ,

$$\pi_0(f) \equiv (f - P_0 f)p_0 \in \mathcal{H}, \quad P_0(f) = \int_0^\infty f dP_0, \tag{16}$$

and if we assume $p_0 > 0$ on S_0 then

$$\int_0^\infty (\hat{p}_n - p_0) f dx = \int_{S_0} p_0^{-2} (\hat{p}_n - p_0) (f - P_0 f) p_0 dP_0 = -D^2 \ell(p_0) [\hat{p}_n - p_0, \pi_0(f)]$$

and

$$D \ell_n(p_0) [\pi_0(f)] = (P_n - P_0) f$$

so that:

Lemma 2. *Suppose $p_0 > 0$ on S_0 . Let \hat{p}_n be as in (3) and let \hat{P}_n be the random probability measure induced by \hat{p}_n . For any $f \in L^\infty([0, \infty))$ and P_n the empirical measure we have*

$$\begin{aligned} |(\hat{P}_n - P_n)(f)| &= \left| \int_0^\infty f d(\hat{P}_n - P_n) \right| \\ &= |D \ell_n(p_0) [\pi_0(f)] + D^2 \ell(p_0) [\hat{p}_n - p_0, \pi_0(f)]|. \end{aligned} \tag{17}$$

Heuristically the right hand side equals, up to second order

$$D \ell_n(\hat{p}_n) [\pi_0(f)] - D^2 \ell_n(p_0) [\hat{p}_n - p_0, \pi_0(f)] + D^2 \ell(p_0) [\hat{p}_n - p_0, \pi_0(f)]. \tag{18}$$

Control of (12) at a rate $o_{P_0^{\mathbb{N}}}(1/\sqrt{n})$ combined with stochastic bounds on the second centred log-likelihood derivatives and convergence rates for $\hat{p}_n - p_0 \rightarrow 0$ thus give some hope that one may be able to prove

$$(\hat{P}_n - P_0 - P_n + P_0)(f) = (\hat{P}_n - P_n)(f) = o_{P_0^{\mathbb{N}}}(1/\sqrt{n})$$

and that thus, by the central limit theorem for $(P_n - P_0)f$,

$$\sqrt{n} \int_0^\infty (\hat{p}_n - p_0) f dx \rightarrow^d N(0, P_0(f - P_0 f)^2)$$

as $n \rightarrow \infty$.

5. Bounding the estimator and L^2 -convergence rate

We establish some first probabilistic properties of \hat{p}_n that will be useful below: If p_0 is bounded away from zero on S_0 then so is \hat{p}_n on the interval $[0, X_{(n)}]$, where $X_{(n)}$ is the last order statistic. Similarly if p_0 is bounded above then so is \hat{p}_n with high probability.

Lemma 3. *a) Suppose the true density p_0 has compact support S_0 and that $\inf_{x \in S_0} p_0(x) \geq \zeta > 0$. Then, for every $\epsilon > 0$, there exists $\xi > 0$ and a finite index $N(\epsilon)$ such that, for all $n \geq N(\epsilon)$,*

$$P_0^{\mathbb{N}} \left(\inf_{x \in [0, X_{(n)}]} \hat{p}_n(x) < \xi \right) = \Pr(\hat{p}_n(X_{(n)}) < \xi) < \epsilon.$$

b) Suppose the true density p_0 satisfies $p_0(x) \leq K < \infty$ for all $x \geq 0$. Then, for every $\epsilon > 0$, there exists $0 < k < \infty$ such that for all $n \in \mathbb{N}$

$$P_0^{\mathbb{N}} \left(\sup_{x \geq 0} \hat{p}_n(x) > k \right) = \Pr(\hat{p}_n(0) > k) < \epsilon.$$

Proof. a) The first equality is obvious, since \hat{p}_n is monotone decreasing. Let $X_{(1)}, \dots, X_{(n)}$ denote the order statistic of X_1, \dots, X_n . On each of the intervals

$(X_{(j-1)}, X_{(j)})$, f_n is the slope of the least concave majorant of F_n . The least concave majorant connects $(X_{(n)}, 1)$ and at least one other order statistic (possibly $(X_{(0)}, 0) \equiv (0, 0)$), so that

$$\{\hat{p}_n(X_{(n)}) < \xi\} \subseteq \{X_{(n)} - X_{(n-j)} > j/(\xi n) \text{ for some } j = 1, \dots, n\}.$$

Note next that since F_0 is strictly monotone on S_0 we have $X_i = F_0|_{S_0}^{-1}F_0(X_i)$ and

$$\begin{aligned} F_0|_{S_0}^{-1}F_0(X_{(n)}) - F_0|_{S_0}^{-1}F_0(X_{(n-j)}) &\leq (\inf_{x \in S_0} p_0(x))^{-1} (F_0(X_{(n)}) - F_0(X_{(n-j)})) \\ &\leq \zeta^{-1} (U_{(n)} - U_{(n-j)}), \end{aligned}$$

where the $U_{(i)}$'s are distributed as the order statistics of a sample of size n of a uniform random variable on $[0, 1]$, and where $U_{(0)} = 0$ by convention. Hence it suffices to bound

$$\Pr \left(U_{(n)} - U_{(n-j)} > \frac{\zeta j}{\xi n} \text{ for some } j = 1, \dots, n \right). \tag{19}$$

By a standard computation involving order statistics, the joint distribution of $U_{(i)}$, $i = 1, \dots, n$, is the same as the one of Z_i/Z_{n+1} where $Z_n = \sum_{l=1}^n W_l$ and where W_l are independent standard exponential random variables. Consequently, for $\delta > 0$, the probability in (19) is bounded by

$$\begin{aligned} &\Pr \left(\frac{W_{n-j+1} + \dots + W_n}{Z_{n+1}} > \frac{\zeta j}{\xi n} \text{ for some } j \right) \\ &= \Pr \left(\frac{n}{Z_{n+1}} \frac{W_{n-j+1} + \dots + W_n}{n} > \frac{\zeta j}{\xi n} \text{ for some } j \right) \\ &\leq \Pr(n/Z_{n+1} > 1 + \delta) + \Pr \left(\frac{W_{n-j+1} + \dots + W_n}{n} > \frac{\zeta j}{\xi n(1 + \delta)} \text{ for some } j \right) \\ &= A + B. \end{aligned}$$

To bound A, note that it is equal to

$$\Pr \left(\frac{1}{n+1} \sum_{l=1}^{n+1} (W_l - EW_l) < \frac{-\delta - (1 + \delta)/n}{1 + \delta} \frac{n}{n+1} \right),$$

which, since $\delta > 0$, is less than $\epsilon/2 > 0$ arbitrary, from some n onwards, by the law of large numbers. For the term B we have, for ξ small enough and by Markov's inequality

$$\begin{aligned} &\Pr \left(W_{n-j+1} + \dots + W_n > \frac{\zeta j}{\xi(1 + \delta)} \text{ for some } j \right) \\ &\leq \sum_{j=1}^n \Pr \left(W_{n-j+1} + \dots + W_n > \frac{\zeta j}{\xi(1 + \delta)} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^n \Pr \left(\sum_{l=1}^j (W_{n-l+1} - EW_{n-l+1}) > \frac{\zeta j}{\xi(1+\delta)} - j \right) \\
&\leq \sum_{j=1}^n \frac{\xi^4 E(\sum_{l=1}^j (W_{n-l+1} - EW_{n-l+1}))^4}{j^4 C(\delta, \zeta)} \\
&\leq \xi^4 C'(\delta, \zeta) \sum_{j=1}^n j^{-2} \leq \xi^4 C''(\delta, \zeta) < \epsilon/2,
\end{aligned}$$

since, for $Y_l = W_{n-l+1} - EW_{n-l+1}$, by Hoffmann-Jørgensen's inequality (de la Peña and Giné, 1999, Corollary 1.2.7)

$$\left\| \sum_{l=1}^j Y_l \right\|_{4,P} \leq K \left[\left\| \sum_{l=1}^j Y_l \right\|_{2,P} + \left\| \max_l Y_l \right\|_{4,P} \right] \leq K' \left(\sqrt{j} + j^{1/4} \right),$$

using the fact that $EW_1^p = p!$ and $\text{Var}(Y_1) = 1$.

b) Since \hat{p}_n is the left derivative of the least concave majorant of the empirical distribution F_n ,

$$\|\hat{p}_n\|_\infty = \hat{p}_n(0) > M \iff F_n(t) > Mt \text{ for some } t.$$

Since F_0 is concave and continuous it maps $[0, \infty)$ onto $[0, 1]$ and satisfies $F_0(t) \leq p_0(0)t \leq t\|p_0\|_\infty$ so that we obtain

$$\begin{aligned}
P_0^{\mathbb{N}}(\|\hat{p}_n\|_\infty > M) &\leq P_0^{\mathbb{N}} \left(\sup_{t>0} \frac{F_n(t)}{F_0(t)} > M/\|p_0\|_\infty \right) \\
&= P_0^{\mathbb{N}} \left(\sup_{t>0} \frac{F_n^U(t)}{t} > M/\|p_0\|_\infty \right),
\end{aligned} \tag{20}$$

where F_n^U is the empirical distribution function based on a sample of size n from the uniform distribution. The density of the order statistic of n uniform $U(0, 1)$ random variables is $n!$ on the set of all $0 \leq x_1 < \dots < x_n \leq 1$. Let $M \geq \|p_0\|_\infty$ and set $C \equiv M/\|p_0\|_\infty$ then the complement of the event in (20) has the probability

$$\begin{aligned}
&P_0^{\mathbb{N}} \left(\frac{F_n^U(t)}{t} \leq C \quad \forall t \in [0, 1] \right) \\
&= n! \int_{1/C}^1 \int_{(n-1)/(nC)}^{x_n} \dots \int_{1/(nC)}^{x_2} dx_1 \dots dx_{n-1} dx_n \\
&= n! \int_{1/C}^1 \dots \int_{j/(nC)}^{x_{j+1}} \frac{1}{(j-1)!} x_j^{j-1} - \frac{1}{nC} \frac{1}{(j-2)!} x_j^{(j-2)} dx_j \dots dx_n \\
&= 1 - \frac{1}{C},
\end{aligned}$$

where $j = 2, \dots, n-1$. In particular, the probability in (20) equals $\|p_0\|_\infty/M$ and can be made small by choosing M large. \square

We can now derive the rate of convergence of the maximum likelihood estimator of a monotone density. The rate corresponds to functions that are once differentiable in an L^1 -sense, which is intuitively correct since a monotone decreasing function has a weak derivative that is a finite signed measure. The following convergence rate in Hellinger distance is given in Example 7.4.2 by van de Geer (2000). It is used here to derive the L^2 -convergence rate. Kulikov and Lopuhaä (2005) prove a much finer result by deriving the asymptotic distribution of the L^p -errors under stronger smoothness assumptions. Gao and Wellner (2009) consider the maximum likelihood estimator of a k -monotone density on a bounded interval and extend Lemma 3b) and the Hellinger convergence rate in the following proposition to this setting. We recall that the Hellinger distance between two Lebesgue densities p and q is defined by

$$h^2(p, q) = \frac{1}{2} \int \left(p^{1/2}(x) - q^{1/2}(x) \right)^2 dx.$$

Proposition 2. *Suppose $p_0 \in \mathcal{P}^{mon}$ and that p_0 is bounded and has bounded support. Let \hat{p}_n satisfy (3). Then*

$$h(\hat{p}_n, p_0) = O_{P_0^n}(n^{-1/3}) \quad (21)$$

and also

$$\|\hat{p}_n - p_0\|_2 = O_{P_0^n}(n^{-1/3}). \quad (22)$$

Proof. Since p_0 is bounded and has bounded support the statement for the Hellinger distance is contained in Example 7.4.2 by van de Geer (2000). The density p_0 is bounded by assumption and we have $\|\hat{p}_n\|_\infty = O_{P_0^n}(1)$ by Lemma 3b). Then the result in L^2 -distance follows by the bound

$$\begin{aligned} \|\hat{p}_n - p_0\|_2^2 &\leq \int \left(\hat{p}_n^{1/2}(x) - p_0^{1/2}(x) \right)^2 \left(\hat{p}_n^{1/2}(x) + p_0^{1/2}(x) \right)^2 dx \\ &\leq 2(\|\hat{p}_n\|_\infty^{1/2} + \|p_0\|_\infty^{1/2})^2 h^2(\hat{p}_n, p_0) \\ &\leq 4(\|\hat{p}_n\|_\infty + \|p_0\|_\infty) h^2(\hat{p}_n, p_0). \quad \square \end{aligned}$$

6. Putting things together

The maximiser \hat{p}_n is in some sense an object that lives on the boundary of \mathcal{P} – it is piecewise constant with step-discontinuities at the observation points, exhausting the possible ‘roughness’ of a monotone function.

We can construct line segments in the parameter space through p_0 , following the philosophy of Lemma 1. Let p_0 be a non-increasing density with compact support S_0 , $\inf_{x \in S_0} p_0(x) \geq \zeta > 0$ and weak derivative Dp_0 . In order to ensure that the perturbed function lies again in \mathcal{P} we will perturb p_0 by ηh where $h \in L^\infty$, $\text{supp}(h) \subseteq S_0$, $\int h = 0$ and Dh is absolutely continuous with respect to Dp_0 such that the Radon–Nikodym density satisfies $\|Dh/Dp_0\|_{\infty, Dp_0} < \infty$. Then we have indeed for η of absolute value small enough

$$\inf_{x \in S_0} (p_0 + \eta h)(x) \geq \zeta - \eta \|h\|_\infty > 0, \quad \int_0^1 (p_0 + \eta h) = 1, \quad (23)$$

and that $D(p_0 + \eta h) = Dp_0 + \eta Dh$ is a negative measure. We change $p_0 + \eta h$ on a nullset so that it is equal to the integral of $Dp_0 + \eta Dh$ everywhere and thus is a non-increasing function. Similar statements hold if we replace h by $\pi_0(f)$ defined in (16) when $\|f\|_\infty + \|Df/Dp_0\|_{\infty, Dp_0}$ is finite. We possibly modify $p_0 + \eta \pi_0(f)$ on a nullset so that it equals the integral of its weak derivative.

Lemma 4. *Let p_0 be non-increasing and have bounded support S_0 with $K \geq p_0(x) \geq \zeta > 0$ for all $x \in S_0$. Let f be such that $\|f\|_\infty + \|Df/Dp_0\|_{\infty, Dp_0}$ is finite. Then we have $p_0 + \eta \pi_0(f) \in \mathcal{P} \cap \mathcal{U}$ for $|\eta| \leq c(\|f\|_\infty + \|Df/Dp_0\|_{\infty, Dp_0})^{-1}$, where $c > 0$ depends on K and ζ only.*

Proof. $\pi_0(f) = (f - P_0 f)p_0$ is bounded by $2K\|f\|_\infty$. The assumption $p_0(x) \geq \zeta > 0$ for all $x \in S_0$ yields $p_0 + \eta \pi_0(f) \in \mathcal{U}$ for $|\eta| < \zeta/(2K\|f\|_\infty)$. In addition to $|\eta| < \zeta/(2K\|f\|_\infty)$ we will choose η small enough such that

$$D(p_0 + \eta \pi_0(f)) = (1 - \eta P_0 f + \eta f)Dp_0 + \eta p_0 Df$$

is a negative measure. This is the case if

$$\frac{K|\eta|\|Df/Dp_0\|_{\infty, Dp_0}}{1 - 2|\eta|\|f\|_\infty} \leq 1 \quad \Leftrightarrow \quad (K\|Df/Dp_0\|_{\infty, Dp_0} + 2\|f\|_\infty)|\eta| \leq 1,$$

which holds for $|\eta| \leq (\max(2, K)(\|f\|_\infty + \|Df/Dp_0\|_{\infty, Dp_0}))^{-1}$. □

For p_0 and f as above we can apply Lemma 1 with $h = \pi_0(f)$, where the line segment between \hat{p}_n and $p_0 \pm \eta \pi_0(f)$ being in $\mathcal{P} \cap \mathcal{U}(X_1, \dots, X_n)$ is guaranteed by Lemma 3a) provided $p_0 \pm \eta \pi_0(f) \in \mathcal{P} \cap \mathcal{U}(X_1, \dots, X_n)$. We thus obtain that on events of probability as close to one as desired and for n large enough,

$$|D\ell_n(\hat{p}_n)[\pi_0(f)]| \leq C(\|f\|_\infty + \|Df/Dp_0\|_{\infty, Dp_0})|D\ell_n(\hat{p}_n)[\hat{p}_n - p_0]| \quad (24)$$

for some constant C that depends on K and ζ only.

We next need to derive stochastic bounds of the likelihood derivative at \hat{p}_n in the direction of p_0 .

Lemma 5. *Suppose p_0 is bounded, has bounded support $[0, \alpha_1]$ and satisfies $\inf_{x \in [0, \alpha_1]} p_0(x) > 0$. For \hat{p}_n satisfying (3) we have*

$$|D\ell_n(\hat{p}_n)[\hat{p}_n - p_0]| = O_{P_0^n}(n^{-2/3})$$

Proof. By Lemma 3 we can restrict to an event where

$$0 < \xi \leq \inf_{x \in [0, X(n)]} \hat{p}_n(x) \leq \sup_{x \in [0, \infty)} \hat{p}_n(x) \leq k < \infty$$

and by (22) further to an event where

$$\|\hat{p}_n - p_0\|_{2, P_0} \leq \|p_0\|_\infty^{1/2} \|\hat{p}_n - p_0\|_2 \leq \|p_0\|_\infty^{1/2} M n^{-1/3}$$

for some finite constant M . For any $\delta_n \rightarrow 0$ with $n\delta_n \rightarrow \infty$ and some $c > 0$

$$\begin{aligned} \Pr((\alpha_1 - X_{(n)}) > \delta_n) &= \Pr((\alpha_1 - \delta_n) > X_{(n)}) \\ &= (F_0(\alpha_1 - \delta_n))^n \leq (1 - c\delta_n)^n \rightarrow 0, \end{aligned}$$

in particular we obtain for $\delta_n = \log n/n$ that

$$\alpha_1 - X_{(n)} = O_{P_0^n} \left(\frac{\log n}{n} \right). \tag{25}$$

Let us define the random function $\tilde{p}_n^{-1} \equiv \hat{p}_n^{-1}$ on $[0, X_{(n)}]$ and zero on $(X_{(n)}, \infty)$. By $D\ell_n(\tilde{p}_n)$ and $D\ell(\tilde{p}_n)$ we denote the corresponding right hand sides in (7) and (8). We observe that $D\ell_n(\hat{p}_n) = D\ell_n(\tilde{p}_n)$. The function $h \equiv \tilde{p}_n^{-1}(\hat{p}_n - p_0)$ on $[0, X_{(n)}]$ and $h \equiv 0$ elsewhere is of bounded variation with norm $\|h\|_{BV} \equiv \|h\|_1 + |Dh|(\mathbb{R})$ bounded by a fixed constant C that depends only on $k, \xi, \|p_0\|_\infty$ and α_1 . We observe that $D\ell(p_0)[\hat{p}_n - p_0] = 0$ by (8) and obtain

$$\begin{aligned} &|D\ell_n(\hat{p}_n)[\hat{p}_n - p_0]| \\ &= |D\ell_n(\tilde{p}_n)[\hat{p}_n - p_0] - D\ell(\tilde{p}_n)[\hat{p}_n - p_0] + (D\ell(\tilde{p}_n) - D\ell(p_0))[\hat{p}_n - p_0]| \\ &\lesssim \sup_{h: \|h\|_{BV} \leq C, \|h\|_2, P_0 \leq \bar{M}n^{-1/3}} |(P_n - P_0)(h)| + \|\hat{p}_n - p_0\|_2^2 + \int_{X_{(n)}}^{\alpha_1} |\hat{p}_n - p_0| \\ &= O_{P_0^n} \left(n^{-1/2}n^{-1/6} + n^{-2/3} + \frac{\log n}{n} \right), \end{aligned} \tag{26}$$

where we have used Theorem 3.1 in Giné and Koltchinskii (2006) with

$$H = id, \sigma = \bar{M}n^{-1/3}, F = const$$

combined with the bracketing entropy bound for monotone functions (van der Vaart and Wellner, 1996, Theorem 2.7.5) and its straight forward generalisation to bounded variation functions to control the supremum of the empirical process, (22) to control the second term, and (25) for the last integral. \square

Proposition 3. *Suppose $p_0 \in \mathcal{P}$ is bounded, has bounded support S_0 and satisfies $p_0(x) \geq \zeta > 0$ for all $x \in S_0$. Let $f \in L^\infty$ be such that $\|Df/Dp_0\|_{\infty, Dp_0}$ is finite. Then*

$$|D\ell_n(\hat{p}_n)[\pi_0(f)]| = O_{P_0^n} \left((\|f\|_\infty + \|Df/Dp_0\|_{\infty, Dp_0})n^{-2/3} \right).$$

Proof. By Lemma 4 we have that $p_0 + \eta\pi_0(f) \in \mathcal{P} \cap \mathcal{U}$ for η a small multiple of $\|f\|_\infty + \|Df/Dp_0\|_{\infty, Dp_0}$. The claim of the proposition then follows from (24) and Lemma 5. \square

We are now ready to prove Theorem 1 and Theorem 2.

Proof of Theorem 1. Without loss of generality we can set f equal to zero outside of $(0, \alpha_1)$. We use Lemma 2, Proposition 1, $\hat{p}_n, p_0 \in \mathcal{U}(X_1, \dots, X_n)$ by Lemma 3 and a Taylor expansion up to second order to see

$$|(\hat{P}_n - P_n)(f)| = |D\ell_n(p_0)[\pi_0(f)] + D^2\ell(p_0)[\hat{p}_n - p_0, \pi_0(f)]|$$

$$\begin{aligned} &\leq |D\ell_n(\hat{p}_n)[\pi_0(f)]| + |(D^2\ell_n(p_0) - D^2\ell(p_0))[\hat{p}_n - p_0, \pi_0(f)]| \\ &\quad + \frac{1}{2}|(D^3\ell_n(\bar{p}_n) - D^3\ell(\bar{p}_n))[\hat{p}_n - p_0, \hat{p}_n - p_0, \pi_0(f)]| \\ &\quad + \frac{1}{2}|D^3\ell(\bar{p}_n)[\hat{p}_n - p_0, \hat{p}_n - p_0, \pi_0(f)]|, \end{aligned}$$

where \bar{p}_n equals, on $[0, X_{(n)}]$, some mean values between \hat{p}_n and p_0 , and \bar{p}_n^{-1} is zero otherwise by convention. Here again $D^3\ell_n(\bar{p}_n)$ and $D^3\ell(\bar{p}_n)$ stand for the corresponding right hand sides in (7) and (8). The first term is bounded using Proposition 3, giving the bound $Bn^{-2/3}$ in probability. We define $h \equiv p_0^{-1}(\hat{p}_n - p_0)(f - P_0f)$ on $[0, \alpha_1]$ and $h \equiv 0$ elsewhere so that the second term equals $|(P_n - P_0)h|$. With probability arbitrarily close to one we have $\|h\|_{BV} \lesssim \|f\|_\infty + \|f\|_{BV} \lesssim \|f\|_\infty + \|Df/Dp_0\|_{\infty, Dp_0}$ and $\|h\|_{2, P_0} \lesssim \|f\|_\infty n^{-1/3}$. The second term is bounded similarly as in (26) above by

$$\sup_{h: \|h\|_{BV} \leq \tilde{C}B, \|h\|_{2, P_0} \leq \tilde{M}Bn^{-1/3}} |(P_n - P_0)(h)| = O_{P_0^{\mathbb{N}}}(Bn^{-2/3}).$$

The third term is bounded the same way, using $\|\hat{p}_n - p_0\|_{BV} = O_{P_0^{\mathbb{N}}}(1)$, and noting that \bar{p}_n as a convex combination of \hat{p}_n, p_0 has variation bounded by a fixed constant on $[0, X_{(n)}]$, so that we can estimate the term by the supremum of the empirical process over a fixed BV -ball, and using again Lemma 3 to bound \bar{p}_n from below on $[0, X_{(n)}]$. Using the last fact the fourth term is also seen to be of order

$$\|f\|_\infty \|\hat{p}_n - p_0\|_2^2 = O_{P_0^{\mathbb{N}}}(Bn^{-2/3})$$

in view of (22) completing the proof the first claim. The second claim follows from the fact that \mathcal{B} is a bounded set in the space of bounded variation functions and thus a Donsker class, which follows from Theorem 2.7.5 in van der Vaart and Wellner (1996). \square

Proof of Theorem 2. It is sufficient to prove the result for $1/2 < s < 1$ since the Hölder spaces are nested. Let $[a, b]$ be a compact interval. In order to define Besov spaces $B_{pq}^s([a, b])$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $0 < s < S$, we consider a boundary corrected Daubechies wavelet basis of regularity S and such that $\phi, \psi \in C^S([a, b])$, see Cohen et al. (1993). We define Besov spaces as in (Giné and Nickl, 2015) by the wavelet characterisation

$$B_{pq}^s([a, b]) \equiv \begin{cases} \{f \in L^p([a, b]) : \|f\|_{B_{p,q}^s} < \infty\}, & 1 \leq p < \infty, \\ \{f \in C([a, b]) : \|f\|_{B_{p,q}^s} < \infty\}, & p = \infty, \end{cases}$$

with norms given by

$$\|f\|_{B_{pq}^s([a, b])} \equiv \begin{cases} \left(\left(\sum_{k=0}^{2^J-1} |\langle f, \phi_{Jk} \rangle|^p \right)^{\frac{1}{p}} + \left(\sum_{l=J}^{\infty} 2^{ql(s+\frac{1}{2}-\frac{1}{p})} \left(\sum_{m=0}^{2^l-1} |\langle f, \psi_{lm} \rangle|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}, & p < \infty, \\ \max_k |\langle f, \phi_{Jk} \rangle| + \left(\sum_{l=J}^{\infty} 2^{ql(s+\frac{1}{2})} \left(\max_m |\langle f, \psi_{lm} \rangle| \right)^q \right)^{\frac{1}{q}}, & p = \infty, \end{cases}$$

where in the case $q = \infty$ the ℓ_q -sequence norm has to be replaced by the supremum norm $\|\cdot\|_\infty$.

Without loss of generality we consider a ball \mathcal{F} in the Hölder space $C^s(S_0)$. We decompose the functions f in a ball \mathcal{F} of $C^s(S_0)$ by using the projection $\pi_{V_j}(f)$ onto the span of the wavelets up to resolution level j ,

$$\begin{aligned} \sup_{f \in \mathcal{F}} |(\hat{P}_n - P_n)(f)| &\leq \sup_{f \in \mathcal{F}} \left| \int_{S_0} (\hat{p}_n - p_0)(f - \pi_{V_j}(f)) \right| \\ &\quad + \sup_{f \in \mathcal{F}} |(\hat{P}_n - P_n)(\pi_{V_j}(f))| \\ &\quad + \sup_{f \in \mathcal{F}} |(P_n - P_0)(f - \pi_{V_j}(f))|. \end{aligned} \tag{27}$$

Since $C^s(S_0) = B_{\infty\infty}^s(S_0)$ for $s \notin \mathbb{N}$ and since the C^1 -norm is bounded by the $B_{\infty 1}^1$ -norm, we have for the wavelet partial sum $\pi_{V_j}(f)$ of $f \in C^s(S_0)$ using the unified notation $\psi_{-1,k} = \phi_{l_0,k}$

$$\begin{aligned} \|\pi_{V_j}(f)\|_{C^1} &\lesssim \sum_{l \leq j} 2^{3l/2} \max_k |\langle f, \psi_{lk} \rangle| \lesssim 2^{j(1-s)} \max_{l \leq j} 2^{l(s+1/2)} \max_{l \leq j,k} |\langle f, \psi_{lk} \rangle| \\ &\leq 2^{j(1-s)} \|f\|_{B_{\infty\infty}^s}. \end{aligned}$$

Thus taking $2^j \sim n^{1/3}$ we have by Proposition 3

$$\sup_{f \in \mathcal{F}} |(\hat{P}_n - P_n)(\pi_{V_j}(f))| = O_{P_0^{\mathbb{N}}}(n^{-2/3} n^{(1-s)/3}) = o_{P_0^{\mathbb{N}}}(1/\sqrt{n})$$

since $s > 1/2$. Moreover, by Parseval's identity

$$\sup_{f \in \mathcal{F}} \|\pi_{V_j}(f) - f\|_2 = O(2^{-js}).$$

Also, using the L^2 -convergence rate in (22) and the Cauchy–Schwarz inequality

$$\sup_{f \in \mathcal{F}} \left| \int_0^1 (\hat{p}_n - p_0)(f - \pi_{V_j}(f)) \right| = O_{P_0^{\mathbb{N}}}(n^{-1/3} n^{-s/3}) = o_{P_0^{\mathbb{N}}}(1/\sqrt{n})$$

and since the class $\{f - \pi_{V_j}(f)\}$ is contained in the fixed s -Hölder ball \mathcal{F} , which is a P_0 -Donsker class for $s > 1/2$ in view of Corollary 5 in Nickl and Pötscher (2007), and has envelopes that converge to zero we see that the third term in (27) is also $o_{P_0^{\mathbb{N}}}(1/\sqrt{n})$ (since the empirical process is tight and has a degenerate Gaussian limit). The remaining claims follow from the fact that \mathcal{F} is a P_0 -Donsker class. \square

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