Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2017, Vol. 53, No. 2, 658–678
DOI: 10.1214/15-AIHP729

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

A dynamical Curie–Weiss model of SOC: The Gaussian case

Matthias Gorny

Université Paris-Sud, Laboratoire de Mathématiques, Bât. 425, 91400 Orsay, France. E-mail: matthias.gorny@math.u-psud.fr

Received 16 July 2015; revised 30 October 2015; accepted 10 November 2015

Abstract. In this paper, we introduce a Markov process whose unique invariant distribution is the Curie–Weiss model of self-organized criticality (SOC) we designed and studied in (*Ann. Probab.* 44(1):444-478, 2016). In the Gaussian case, we prove rigorously that it is a dynamical model of SOC: the fluctuations of the sum $S_n(\cdot)$ of the process evolve in a time scale of order $\sqrt{n}$ and in a space scale of order $n^{3/4}$ and the limiting process is the solution of a "critical" stochastic differential equation.

Résumé. Dans cet article, nous introduisons un processus de Markov dont l'unique distribution invariante est le modèle d'Ising Curie–Weiss de criticalité auto-organisée que nous avons construit et étudié dans (*Ann. Probab.* 44(1):444-478, 2016). Dans le cas Gaussien, nous montrons rigoureusement qu'il s'agit d'un modèle dynamique de criticalité auto-organisée : les fluctuations de la somme $S_n(\cdot)$ du processus évoluent à une vitesse de temps d'ordre $\sqrt{n}$ et à une échelle spatiale d'ordre $n^{3/4}$ et le processus limite est la solution d'une équation différentielle stochastique « critique ».

MSC: 60J60; 60K35

Keywords: Ising Curie–Weiss; Self-organized criticality; Critical fluctuations; Langevin diffusion; Collapsing processes

1. Introduction

In [4] and [10], we introduced a Curie–Weiss model of self-organized criticality (SOC): we transformed the distribution associated to the generalized Ising Curie–Weiss model by implementing an automatic control of the inverse temperature which forces the model to evolve towards a critical state. It is the model given by an infinite triangular array of real-valued random variables $(X_n^k)_{1\le k\le n}$ such that, for all $n\ge 1$, $(X_n^1,\dots,X_n^n)$ has the distribution

$$\frac{1}{Z_n}\exp\left(\frac{1}{2}\frac{(x_1+\cdots+x_n)^2}{x_1^2+\cdots+x_n^2}\right)\mathbb{1}_{\{x_1^2+\cdots+x_n^2>0\}}\prod_{i=1}^n d\rho(x_i),$$

where ρ is a probability measure on $\mathbb{R}$ which is not the Dirac mass at 0, and where Z_n is the normalization constant. We extended the study of this model in [11,12] and [13]. For symmetric distributions satisfying some exponential moment condition, we proved that the sum S_n of the random variables behaves as in the typical critical generalized Ising Curie–Weiss model: the fluctuations are of order $n^{3/4}$ and the limiting law is $C\exp(-\lambda x^4)\,dx$ where C and λ are suitable positive constants. Moreover, by construction, the model does not depend on any external parameter. That is why we can conclude it exhibits the phenomenon of self-organized criticality (SOC). Our motivations for studying such a model are detailed in [4].

This model describes interacting elements in thermodynamic equilibrium. However self-organized criticality seems to be a dynamical phenomenon, as is highlighted by the archetype of SOC: the sandpile model introduced by Per Bak, Chao Tang and Kurt Wiesenfeld in their seminal 1987 paper [1]. That is why, in this paper, we try to design a dynamical Curie–Weiss model of SOC.

We choose to build a dynamical model as a Markov process whose unique invariant distribution is the law of (a modified version of) the Curie–Weiss model of SOC. One way of building such a process is to consider the associated Langevin diffusion (see [16] for example).

The model

Let φ be a C^2 function from $\mathbb{R}$ to $\mathbb{R}$ which is even and such that the function $\exp(2\varphi)$ is integrable over $\mathbb{R}$. We suppose that there exists $C > 0$ such that

$$\forall x \in \mathbb{R} \quad x\varphi'(x) \le C\big(1 + x^2\big).$$

We denote by ρ the probability measure with density

$$x \longmapsto \exp\big(2\varphi(x)\big)\left(\int_{\mathbb{R}} \exp\big(2\varphi(t)\big)\,dt\right)^{-1}$$

with respect to the Lebesgue measure on $\mathbb{R}$. We consider an infinite triangular array of stochastic processes $(X_n^k(t), t \ge 0)_{1\le k\le n}$ such that, for all $n \ge 1$,

$$\big(\big(X_n^1(t), \dots, X_n^n(t)\big), t \ge 0\big)$$

is the unique solution of the system of stochastic differential equations:

$$dX_n^j(t) = \varphi'\big(X_n^j(t)\big)\,dt + dB_j(t) + \frac{1}{2}\left(\frac{S_n(t)}{T_n(t)+1} - X_n^j(t)\left(\frac{S_n(t)}{T_n(t)+1}\right)^2\right)dt, \quad j \in \{1, \dots, n\}, \qquad (\Sigma_n^\varphi)$$

where $(B_1, \dots, B_n)$ is a standard n-dimensional Brownian motion and

$$\forall t \ge 0 \quad S_n(t) = X_n^1(t) + \cdots + X_n^n(t), \qquad T_n(t) = \big(X_n^1(t)\big)^2 + \cdots + \big(X_n^n(t)\big)^2.$$

In Section 2.3, we explain in details why we choose this drift. In this paper, we only prove a fluctuation theorem for the Gaussian case of this model:

Theorem 1. *Let $\sigma^2 > 0$. Assume that*

$$\forall x \in \mathbb{R} \quad \varphi(x) = -\frac{x^2}{4\sigma^2}$$

and that, for any $n \ge 1$, the random variables $X_n^1(0), \dots, X_n^n(0)$ are independent with common distribution $\rho = \mathcal{N}(0, \sigma^2)$. We denote $(\mathcal{U}(t), t \ge 0)$ the unique strong solution of the stochastic differential equation

$$dz(t) = -\frac{z^3(t)}{2\sigma^4}\,dt + dB(t), \quad z(0) = 0, \qquad (\mathcal{S}_\sigma)$$

where $(B(t), t \ge 0)$ is a standard Brownian motion. Then, for any $T > 0$,

$$\left(\frac{S_n(\sqrt{n}t)}{n^{3/4}}, 0 \le t \le T\right) \underset{n\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}} \big(\mathcal{U}(t), 0 \le t \le T\big),$$

in the sense of the convergence in distribution on $C([0, T], \mathbb{R})$.

This theorem suggests that, at least in the Gaussian case, our dynamical model exhibits self-organized criticality. Indeed it does not depend on any external parameter and the fluctuations of $S_n(\cdot)$ are critical: the processes evolve in a time scale of order $\sqrt{n}$ and in a space scale of $n^{3/4}$ and the limiting process is the solution of the "critical" stochastic differential equation ($\mathcal{S}_\sigma$). This is the same behaviour as in the critical case of the mean-field model studied by Donald A. Dawson in [7], see Section 3.1 for more details.

For any $n \geq 1$, we introduce $S_n^\star = \xi_n^1 + \cdots + \xi_n^n$ where $(\xi_n^1, \ldots, \xi_n^n)$ has the density proportional to

$$(x_1, \ldots, x_n) \longmapsto \exp\left(\frac{1}{2}\frac{(x_1 + \cdots + x_n)^2}{x_1^2 + \cdots + x_n^2 + 1} - \frac{x_1^2 + \cdots + x_n^2}{2\sigma^2}\right)$$

with respect to the Lebesgue measure on $\mathbb{R}^n$. In this paper, we also prove the following commutative diagram of convergences in distribution on $\mathbb{R}$:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\dfrac{S_n(\sqrt{n}t)}{n^{3/4}} \arrow[r, "(\mathcal{A}_1)", "t \to +\infty"'] \arrow[d, "(\mathcal{A}_4)"', "n \to +\infty"] & \dfrac{S_n^\star}{n^{3/4}} \arrow[d, "n \to +\infty"', "(\mathcal{A}_2)"] \\
\mathcal{U}(t) \arrow[r, "t \to +\infty", "(\mathcal{A}_3)"'] & \dfrac{\sqrt{2}}{\sigma}\Gamma\left(\dfrac{1}{4}\right)^{-1}\exp\left(-\dfrac{s^4}{4\sigma^4}\right)ds
\end{tikzcd}$$

In Section 2, we present some results on the general case of the model and we prove the convergences in distribution associated to the arrows ($\mathcal{A}_1$), ($\mathcal{A}_2$) and ($\mathcal{A}_3$) in the previous diagram. Next, in Section 3, we give the strategy for proving a fluctuation result for our model and we explain that the Gaussian case is special because it can be analyzed through a two-dimensional problem. Finally we prove Theorem 1 in Section 4, i.e., the convergence in distribution associated to the arrow ($\mathcal{A}_4$).

2. Results on the general case of the model

In this section, we first give general results on Langevin diffusions. Next we apply these results to prove existence and uniqueness of the solution of ($\mathcal{S}_\sigma$) and (Σ_n^φ). We also prove the convergences in distribution associated to the arrows ($\mathcal{A}_1$) and ($\mathcal{A}_3$). Finally we give a fluctuation theorem for an alternative version of the Curie–Weiss model of SOC.

2.1. *Langevin diffusions*

Let f be a probability density function on $\mathbb{R}^n$, $n \geq 1$. The Langevin diffusion associated to f is a stochastic process which is constructed so that, in continuous time, under suitable regularity conditions, it converges to $f(x)\,dx$, its unique invariant distribution.

Theorem 2. *Let f be a positive probability density function on $\mathbb{R}^n$, $n \geq 1$, such that $\ln f$ is C^2. We suppose that there exists $K > 0$ such that*

$$\forall x \in \mathbb{R}^n \quad \big\langle \nabla \ln f(x), x \big\rangle \leq K\big(1 + \|x\|^2\big).$$

If $(B(t), t \geq 0)$ is a standard n-dimensional Brownian motion and if ξ is a random variable in $\mathbb{R}^n$ satisfying $\mathbb{E}(\|\xi\|^2) < +\infty$, then there exists a unique strong solution to the stochastic differential equation

$$dY(t) = \frac{1}{2}\nabla \ln f\big(Y(t)\big) + dB(t), \qquad (\mathcal{S}_f)$$

with initial condition $Y(0)=\xi$. Moreover $(Y(t), t\geq 0)$ is a Markov diffusion process on $\mathbb{R}^n$ admitting $f(x)\,dx$ as unique invariant distribution and

$$\forall x\in\mathbb{R}^n \quad \lim_{t\to+\infty}\sup_{A\in\mathcal{B}_{\mathbb{R}^n}}\left|\mathbb{P}\big(Y(t)\in A|Y(0)=x\big)-\int_A f(z)\,dz\right|=0.$$

Proof. Theorems 3.7 and 3.11 of Chapter 5 of [9] imply that there exists a unique strong solution to $(\mathcal{S}_f)$ with initial condition ξ, that its sample path is continuous and that it is a solution of the martingale problem for (A_f,ξ), where

$$\forall g\in C^2\big(\mathbb{R}^n\big) \quad A_f g=\frac{1}{2}\sum_{i=1}^n\frac{\partial^2 g}{\partial x_i^2}+\sum_{i=1}^n\left(\frac{1}{2}\frac{\partial(\ln f)}{\partial x_i}\right)\frac{\partial g}{\partial x_i}.$$

Next, Theorems 4.1 and 4.2 of Chapter 4 of [9] imply that it is a Markov process and that its generator is $(A_f, D(A_f))$ with $C_c^\infty(\mathbb{R}^n)\subset D(A_f)$. Finally Theorem 2.1 of [16] gives us the uniqueness of the invariant distribution and the total variation norm convergence. □

Notice that this theorem is true if we remove the hypothesis that ξ has a finite second order moment, but the solution to $(\mathcal{S}_f)$ would be weak (see Theorem 3.10 of Chapter 5 of [9]).

2.2. *Solution of* $(\mathcal{S}_\sigma)$

Theorem 2 implies that $(\mathcal{S}_\sigma)$ admits a unique strong solution $(\mathcal{U}(t), t\geq 0)$ which is a Markov process whose unique invariant distribution is

$$\frac{\sqrt{2}}{\sigma}\Gamma\left(\frac{1}{4}\right)^{-1}\exp\left(-\frac{s^4}{4\sigma^4}\right)ds.$$

Moreover

$$\lim_{t\to+\infty}\sup_{A\in\mathcal{B}_{\mathbb{R}}}\left|\mathbb{P}\big(\mathcal{U}(t)\in A\big)-\frac{\sqrt{2}}{\sigma}\Gamma\left(\frac{1}{4}\right)^{-1}\int_A\exp\left(-\frac{s^4}{4\sigma^4}\right)ds\right|=0.$$

This is the convergence in distribution associated to the arrow $(\mathcal{A}_3)$ in the diagram on page 660.

2.3. *Solution of* (Σ_n^φ)

In this subsection, we prove that (Σ_n^φ) has a unique strong solution and that the convergence in distribution associated to $(\mathcal{A}_1)$ is true.

Let us define $\widetilde{\mu}^\star_{n,\rho}$, the probability measure with density

$$f^\star_{n,\rho}: y\in\mathbb{R}^n\longmapsto\frac{1}{Z_n^\star}\exp\left(\frac{1}{2}\frac{(y_1+\cdots+y_n)^2}{y_1^2+\cdots+y_n^2+1}+2\sum_{i=1}^n\varphi(y_i)\right) \tag{1}$$

with respect to the Lebesgue measure on $\mathbb{R}^n$, where $Z_n^\star$ is a normalization constant. Let us prove that (Σ_n^φ) admits a unique solution. For any $y\in\mathbb{R}^n$, we denote

$$S_n[y]=y_1+\cdots+y_n,\qquad T_n[y]=y_1^2+\cdots+y_n^2$$

and we notice that, for any $j\in\{1,\dots,n\}$,

$$\frac{\partial}{\partial y_j}\left(\frac{1}{2}\frac{(S_n[y])^2}{T_n[y]+1}+2\sum_{i=1}^n\varphi(y_i)\right)=\frac{S_n[y]}{T_n[y]+1}-y_j\left(\frac{S_n[y]}{T_n[y]+1}\right)^2+2\varphi'(y_j).$$

Therefore the system (Σ_n^φ) can be rewritten

$$dX_n(t) = \frac{1}{2}\nabla \ln f^\star_{n,\rho}\big(X_n(t)\big) + dB(t),$$

where $B = (B_1, \dots, B_n)$. As a consequence, the solution of (Σ_n^φ) (if it exists) is the Langevin diffusion associated to $f^\star_{n,\rho}$.

Let us introduce the operator L_n on $C^2(\mathbb{R}^n)$ such that, for any $f \in C^2(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$,

$$L_n f(y) = \frac{1}{2}\sum_{j=1}^n \frac{\partial^2 f(y)}{\partial y_j^2} + \sum_{j=1}^n \left(\frac{1}{2}\frac{S_n[y]}{T_n[y]+1} - \frac{y_j}{2}\left(\frac{S_n[y]}{T_n[y]+1}\right)^2 + \varphi'(y_j)\right)\frac{\partial f(y)}{\partial y_j}.$$

Theorem 3. *For any $n \ge 1$, there exists a unique strong solution*

$$\big(X_n(t), t \ge 0\big) = \big(\big(X_n^1(t), \dots, X_n^n(t)\big), t \ge 0\big)$$

to the system (Σ_n^φ) with initial condition $X_n(0)$ having a finite second moment. Moreover it is a Markov diffusion process on $\mathbb{R}^n$ with infinitesimal generator $(L_n, D(L_n))$, where $C_c^\infty(\mathbb{R}^n) \subset D(L_n)$, and whose unique invariant distribution is $\widetilde{\mu}^\star_{n,\rho}$. Finally

$$\forall x \in \mathbb{R}^n \quad \lim_{t\to+\infty} \sup_{A \in \mathcal{B}_{\mathbb{R}^n}} \big|\mathbb{P}\big(X_n(t) \in A \,|\, X_n(0) = x\big) - \widetilde{\mu}^\star_{n,\rho}(A)\big| = 0.$$

If we take $\varphi(x) = -x^2/(4\sigma^2)$ for any $x \in \mathbb{R}$, then Theorem 3 proves the convergence in distribution associated to the arrow $(\mathcal{A}_1)$ in the diagram on page 660.

Proof of Theorem 3. Let $n \ge 1$. By hypothesis, there exists $C > 0$ such that

$$\forall x \in \mathbb{R} \quad x\varphi'(x) \le C\big(1 + x^2\big).$$

Moreover φ is C^2 on $\mathbb{R}$ thus the function $\ln f^\star_{n,\rho}$ is C^2 on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
\big\langle \nabla \ln f^\star_{n,\rho}(x), x\big\rangle &= \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\frac{(S_n[x])^2}{T_n[x]+1} + 2\sum_{i=1}^n \varphi(x_i)\right) \\
&= \sum_{j=1}^n x_j \left(\frac{S_n[x]}{T_n[x]+1} - x_j\left(\frac{S_n[x]}{T_n[x]+1}\right)^2 + 2\varphi'(x_j)\right) \\
&= \frac{(S_n[x])^2}{T_n[x]+1} - \frac{T_n[x](S_n[x])^2}{(T_n[x]+1)^2} + 2\sum_{j=1}^n x_j\varphi'(x_j) \\
&\le \frac{(S_n[x])^2}{(T_n[x]+1)^2} + 2C\big(n + \|x\|^2\big).
\end{aligned}$$

Next the convexity of $t \longmapsto t^2$ on $\mathbb{R}$ implies that

$$\forall y \in \mathbb{R}^n \quad \frac{(S_n[y])^2}{(T_n[y]+1)^2} \le \frac{nT_n[y]}{(T_n[y]+1)^2} \le n,$$

since $T_n[\cdot] \le (T_n[\cdot]+1)^2$. Therefore $f^\star_{n,\rho}$ satisfies the hypothesis of Theorem 2 and Theorem 3 follows. □

Remark. *we have chosen to built our dynamical model so that $\widetilde{\mu}^{\star}_{n,\rho}$ is its unique invariant distribution. It is an alternative version of the Curie–Weiss model we designed in* [4], *given by the distribution*

$$d\widetilde{\mu}_{n,\rho}(x_1,\dots,x_n)=\frac{1}{Z_n}\exp\left(\frac{1}{2}\frac{(x_1+\cdots+x_n)^2}{x_1^2+\cdots+x_n^2}+2\sum_{i=1}^{n}\varphi(x_i)\right)dx_1\cdots dx_n,$$

where Z_n is a normalization constant. If we want to built the Langevin diffusion associated to the density of $\widetilde{\mu}_{n,\rho}$, we obtain the system of stochastic differential equations

$$dX_n^j(t)=\varphi'\big(X_n^j(t)\big)\,dt+dB_j(t)+\frac{1}{2}\left(\frac{S_n(t)}{T_n(t)}-X_n^j(t)\left(\frac{S_n(t)}{T_n(t)}\right)^2\right)dt,\quad j\in\{1,\dots,n\}.$$

In this case, the interaction function is not Lipschitz and we have to check first that $T_n(t)\neq 0$ for any $t\geq 0$: this would create technical difficulties to prove existence and uniqueness of a solution. In the next section, we give some results on the alternative version of the Curie–Weiss model of SOC (the model defined by the probability measure $\widetilde{\mu}^{\star}_{n,\rho}$ – see formula (1)).

2.4. *The alternative Curie–Weiss model of SOC*

Let ρ be a probability measure on $\mathbb{R}$. We consider an infinite triangular array of real-valued random variables $(\xi_n^k)_{1\leq k\leq n}$ such that for all $n\geq 1$, $(\xi_n^1,\dots,\xi_n^n)$ has the distribution

$$d\widetilde{\mu}^{\star}_{n,\rho}(x_1,\dots,x_n)=\frac{1}{Z_n^{\star}}\exp\left(\frac{1}{2}\frac{(x_1+\cdots+x_n)^2}{x_1^2+\cdots+x_n^2+1}\right)\prod_{i=1}^{n}d\rho(x_i),\tag{2}$$

where $Z_n^{\star}$ in the normalization constant. We define $S_n^{\star}=\xi_n^1+\cdots+\xi_n^n$.

We obtain the same fluctuation theorem as in [11]. We only present the case where ρ has a density:

Theorem 4. *Let ρ be a probability measure having an even density with respect to the Lebesgue measure on $\mathbb{R}$ and such that*

$$\exists v_0>0\quad\int_{\mathbb{R}}e^{v_0z^2}\,d\rho(z)<+\infty.$$

If σ^2 denotes the variance of ρ and μ_4 its fourth moment then, under $\widetilde{\mu}^{\star}_{n,\rho}$,

$$\frac{S_n^{\star}}{n^{3/4}}\xrightarrow[n\to+\infty]{\mathcal{L}}\left(\frac{4\mu_4}{3\sigma^8}\right)^{1/4}\Gamma\left(\frac{1}{4}\right)^{-1}\exp\left(-\frac{\mu_4s^4}{12\sigma^8}\right)ds.$$

The proof of this theorem is given in Section 18.b) of [12]. It is an adaptation of the proof of Theorem 1 of [11], which consists in replacing the function F by the function $(x,y)\longmapsto x^2/(2y+2/n)$.

If we take $\varphi(x)=-x^2/(4\sigma^2)$ for any $x\in\mathbb{R}$, then Theorem 4 implies the convergence in distribution associated to the arrow $(\mathcal{A}_2)$ in the diagram on page 660.

3. Strategy of proof

In this section, we first explain that the main ingredient for proving a fluctuation theorem for our dynamical model (in the case of a general function) will be the study of its associated empirical process. Next we will focus only on the Gaussian case, i.e., when $\varphi:x\longmapsto -x^2/(4\sigma^2)$ for some $\sigma^2>0$. Indeed we will see that the Gaussian case can be handled by studying the convergence of the process

$$\left(\left(\frac{S_n(\sqrt{n}t)}{n^{3/4}},n^{1/4}\left(\frac{T_n(\sqrt{n}t)}{n}-\sigma^2\right)\right),t\geq 0\right).$$

We compute the generator of this process in Section 3.2. Finally we give the sketch of proof of Theorem 1 in Section 3.3.

3.1. *The empirical process*

Let φ be such that Σ_n^φ has a unique strong solution $((X_n^1(t),\dots,X_n^n(t)),t\ge 0)$. As in the equilibrium case (i.e., the alternative Curie–Weiss model defined in formula (1) or (2)), we would like to study the process (S_n,T_n). However it is not Markov a priori, contrary to the empirical measure process M_n. It is the process taking its values on $\mathcal{M}_1(\mathbb{R})$ and defined by

$$\forall t\ge 0,\forall A\in\mathcal{B}_{\mathbb{R}}\quad M_n(t,A)=\frac{1}{n}\sum_{k=1}^n\delta_{X_n^k(t)}(A)=\frac{1}{n}\sum_{k=1}^n\mathbb{1}_A\big(X_n^k(t)\big),$$

where $((X_n^1(t),\dots,X_n^n(t)),t\ge 0)$ is the unique solution of (Σ_n^φ).

Lemma 5. *If the distribution of $X_n(0)$ is invariant under permutation of coordinates, then $(M_n(t,\cdot),t\ge 0)$ is a Markov diffusion process on $\mathcal{M}_1(\mathbb{R})$.*

This lemma has a similar proof than Lemma 2.3.1 of the article [7] – a paper by Donald A. Dawson about a mean-field model of cooperative behaviour. Dawson's model is defined through a Markov process which is solution of a system of stochastic differential equations. This process depends on two parameters and Dawson proves the existence of a critical curve in the space of the parameters. The critical fluctuations of the empirical measure process $M_n(\cdot)$ evolve in a time scale of order $\sqrt{n}$ and in a space scale of order $n^{3/4}$. We believe that our dynamical model has the same asymptotic behavior for the following reasons:

- ⋆ The invariant distribution of Dawson's process is a particular case of the law of the generalized Ising Curie–Weiss model, defined in [8].
- ⋆ The alternative Curie–Weiss model of SOC, defined in formula (1) or (2), has the same asymptotic behavior as the critical generalized Ising Curie–Weiss model (see Theorem 4).
- ⋆ The invariant distribution of our dynamical model is the law of the alternative Curie–Weiss model (see Theorem 3).

Let $n\ge 1$. As in Dawson's paper, we define the process U_n by

$$\forall t\ge 0,\forall A\in\mathcal{B}_{\mathbb{R}}\quad U_n(t,A)=n^{1/4}\bigg(M_n(\sqrt{n}t,A)-\int_A d\rho(x)\bigg).$$

It takes its values on $\mathcal{M}^\pm(\mathbb{R})$, the space of signed measures on $\mathbb{R}$.

The convergence of a sequence of Markov processes can be proved through the convergence of the sequence of their generators. Let us denote by G_n the infinitesimal generator of U_n. Let f and Φ belong to $C^2(\mathbb{R})$. We assume that Φ is ρ-integrable. We have

$$\forall t\ge 0\quad G_n f\bigg(\int_{\mathbb{R}}\Phi(z)U_n(t,dz)\bigg)=\sqrt{n}L_nF_{f,\Phi}\big(X_n^1(t),\dots,X_n^n(t)\big),$$

where

$$F_{f,\Phi}:x\in\mathbb{R}^n\longmapsto f\Bigg(n^{1/4}\bigg(\frac{1}{n}\sum_{k=1}^n\Phi(x_k)-\int_{\mathbb{R}}\Phi(z)\,d\rho(z)\bigg)\Bigg).$$

If $\Phi:z\longmapsto z$ then, for any $i\in\{1,\dots,n\}$ and $x\in\mathbb{R}^n$,

$$\frac{\partial F_{f,\Phi}}{\partial x_i}(x)=\frac{1}{n^{3/4}}F_{f',\Phi}(x)\quad\text{and}\quad\frac{\partial^2 F_{f,\Phi}}{\partial x_i^2}(x)=\frac{1}{n^{3/2}}F_{f'',\Phi}(x).$$

If $\Phi : z \longmapsto z^2$ then, for any $i \in \{1, \ldots, n\}$ and $x \in \mathbb{R}^n$,

$$\frac{\partial F_{f,\Phi}}{\partial x_i}(x) = \frac{2x_i}{n^{3/4}} F_{f',\Phi}(x) \quad \text{and} \quad \frac{\partial^2 F_{f,\Phi}}{\partial x_i^2}(x) = \frac{4x_i^2}{n^{3/2}} F_{f'',\Phi}(x) + \frac{2}{n^{3/4}} F_{f',\Phi}(x).$$

In both cases, if we suppose that $\varphi : z \longmapsto -z^2/(4\sigma^2)$, then we notice that, for any $x \in \mathbb{R}^n$, the term $L_n F_{f,\Phi}(x)$ only depends on n, $S_n[x]$ and $T_n[x]$. This suggests that, in the Gaussian case, in order to prove the convergence of the process $(S_n(\sqrt{n}t)/n^{3/4}, t \geq 0)$, we can turn the study of U_n (which is a problem in infinite dimensions) into a problem in only two dimensions. Indeed, we introduce the processes $\widetilde{S}_n$ and $\widetilde{T}_n$ defined by

$$\forall t \geq 0 \quad \widetilde{S}_n(t) = \frac{S_n(\sqrt{n}t)}{n^{3/4}} = \int_{\mathbb{R}} z U_n(t, dz)$$

and

$$\forall t \geq 0 \quad \widetilde{T}_n(t) = n^{1/4}\left(\frac{T_n(\sqrt{n}t)}{n} - \sigma^2\right) = \int_{\mathbb{R}} z^2 U_n(t, dz).$$

In the rest of the paper, we suppose that $\varphi(x) = -x^2/(4\sigma^2)$ for any $x \in \mathbb{R}$.

3.2. *Generator of $(\widetilde{S}_n, \widetilde{T}_n)$ in the Gaussian case*

Let $n \geq 1$ and $f \in C^2(\mathbb{R}^2)$. Let us define Ψ_f on $\mathbb{R}^n$ by

$$\forall x \in \mathbb{R}^n \quad \Psi_f(x) = f\left(\frac{S_n[x]}{n^{3/4}}, \frac{T_n[x]}{n^{3/4}} - n^{1/4}\sigma^2\right).$$

Proposition 6. *For any $n \geq 1$ and $f \in C^2(\mathbb{R}^2)$, we have*

$$\forall t \geq 0 \quad \sqrt{n} L_n \Psi_f\big(X_n^1(t), \ldots, X_n^n(t)\big) = \widetilde{G}_n f\big(\widetilde{S}_n(t), \widetilde{T}_n(t)\big),$$

where, for any $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}\widetilde{G}_n f(x, y) = {} & -\frac{\sqrt{n}y}{\sigma^2}\frac{\partial f}{\partial y}(x, y) - \frac{n^{1/4}xy}{2\sigma^4}\frac{\partial f}{\partial x}(x, y) \\ & + \frac{1}{2\sigma^6}\big(xy^2 - x^3\sigma^2\big)\frac{\partial f}{\partial x}(x, y) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x, y) + 2\sigma^2\frac{\partial^2 f}{\partial y^2}(x, y) + R_n^f(x, y)\end{aligned}$$

with

$$R_n^f(x, y) = \frac{\partial f}{\partial x}(x, y) R_n^{(1)}(x, y) + \frac{\partial f}{\partial y}(x, y) R_n^{(2)}(x, y) + \frac{2x}{n^{1/4}}\frac{\partial^2 f}{\partial x \partial y}(x, y) + \frac{2y}{n^{1/4}}\frac{\partial^2 f}{\partial y^2}(x, y),$$

where $(R_n^{(1)})_{n\geq 1}$ and $(R_n^{(2)})_{n\geq 1}$ are sequences of functions from $\mathbb{R}^2$ to $\mathbb{R}$ verifying

$$\forall k > 0 \quad \lim_{n\to+\infty} \sup_{\substack{(x,y)\in\mathbb{R}^2 \\ \|(x,y)\|\leq k}} \max\big(\big|R_n^{(1)}(x, y)\big|, \big|R_n^{(2)}(x, y)\big|\big) = 0.$$

Proof. Let us define Ψ_f on $\mathbb{R}^n$ by

$$\forall x \in \mathbb{R}^n \quad \Psi_f(x) = f\left(\frac{S_n[x]}{n^{3/4}}, \frac{T_n[x]}{n^{3/4}} - n^{1/4}\sigma^2\right).$$

Let $x \in \mathbb{R}^n$. For any $i \in \{1, \dots, n\}$, we have

$$\frac{\partial \Psi_f(x)}{\partial y_j} = \frac{1}{n^{3/4}} \frac{\partial f}{\partial x}(\cdots) + \frac{2x_j}{n^{3/4}} \frac{\partial f}{\partial y}(\cdots),$$

$$\frac{\partial^2 \Psi_f(x)}{\partial y_j^2} = \frac{1}{n^{3/2}} \frac{\partial^2 f}{\partial x^2}(\cdots) + \frac{4x_j}{n^{3/2}} \frac{\partial^2 f}{\partial x \partial y}(\cdots) + \frac{4x_j^2}{n^{3/2}} \frac{\partial^2 f}{\partial y^2}(\cdots) + \frac{2}{n^{3/4}} \frac{\partial f}{\partial y}(\cdots),$$

where we write

$$(\cdots) \quad \text{instead of} \quad \left(\frac{S_n[x]}{n^{3/4}}, \frac{T_n[x]}{n^{3/4}} - n^{1/4}\sigma^2 \right)$$

in order to simplify the notations. We have then

$$\begin{aligned} L_n \Psi_f(x) &= \frac{1}{2} \sum_{j=1}^{n} \left[\frac{\partial^2 \Psi_f(x)}{\partial x_j^2} + \left(\frac{S_n[x]}{T_n[x]+1} - x_j \left(\frac{S_n[x]}{T_n[x]+1} \right)^2 - \frac{x_j}{\sigma^2} \right) \frac{\partial \Psi_f(x)}{\partial x_j} \right] \\ &= \frac{1}{2\sqrt{n}} \frac{\partial^2 f}{\partial x^2}(\cdots) + \frac{2S_n[x]}{n^{3/2}} \frac{\partial^2 f}{\partial x \partial y}(\cdots) + \frac{2T_n[x]}{n^{3/2}} \frac{\partial^2 f}{\partial y^2}(\cdots) \\ &\quad + \left(n^{1/4} + \frac{S_n^2[x]}{n^{3/4}(1+T_n[x])^2} - \frac{T_n[x]}{n^{3/4}\sigma^2} \right) \frac{\partial f}{\partial y}(\cdots) \\ &\quad + \frac{1}{2} \left(\frac{n^{1/4} S_n[x]}{1+T_n[x]} - \frac{S_n^3[x]}{n^{3/4}(1+T_n[x])^2} - \frac{S_n[x]}{n^{3/4}\sigma^2} \right) \frac{\partial f}{\partial x}(\cdots). \end{aligned}$$

We obtain that

$$\sqrt{n} L_n \Psi_f(x) = \widetilde{G}_n f \left(\frac{S_n[x]}{n^{3/4}}, \frac{T_n[x]}{n^{3/4}} - n^{1/4}\sigma^2 \right),$$

where $\widetilde{G}_n f$ is defined on $\mathbb{R}^2$ by

$$\begin{aligned} \forall (x,y) \in \mathbb{R}^2 \quad \widetilde{G}_n f(x,y) &= \frac{2x}{n^{1/4}} \frac{\partial^2 f}{\partial x \partial y}(x,y) + \left(\frac{2y}{n^{1/4}} + 2\sigma^2 \right) \frac{\partial^2 f}{\partial y^2}(x,y) \\ &\quad + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x,y) + \left(-\frac{\sqrt{n}x}{2\sigma^2} \big(1 - h_n(y)\big) - \frac{x^3}{2\sigma^4} h_n(y)^2 \right) \frac{\partial f}{\partial x}(x,y) \\ &\quad + \left(-\frac{\sqrt{n}y}{\sigma^2} + \frac{x^2}{n^{3/4}\sigma^4} h_n(y)^2 \right) \frac{\partial f}{\partial y}(x,y), \end{aligned}$$

with

$$h_n : y \in]-\sigma^2 n^{1/4}, +\infty[\longmapsto \left(1 + \frac{y}{n^{1/4}\sigma^2} + \frac{1}{n\sigma^2} \right)^{-1}.$$

We introduce the functions $\varepsilon_n^{(1)}$ and $\varepsilon_n^{(2)}$ such that

$$\forall y > -\sigma^2 n^{1/4} \quad h_n(y) = 1 - \frac{y}{n^{1/4}\sigma^2} + \frac{y^2}{\sqrt{n}\sigma^4} + \frac{1}{\sqrt{n}} \varepsilon_n^{(1)}(y)$$

and $\varepsilon_n^{(2)}(y) = h_n(y)^2 - 1$. We obtain the formula of $\widetilde{G}_n f$ given in the proposition with

$$R_n^{(1)} : (x,y) \longmapsto \frac{x}{2\sigma^2} \varepsilon_n^{(1)}(y) - \frac{x^3}{2\sigma^4} \varepsilon_n^{(2)}(y) \quad \text{and} \quad R_n^{(2)} : (x,y) \longmapsto \frac{x^2 h_n(y)^2}{n^{3/4}\sigma^2}.$$

It is easy to see that $(R_n^{(1)})_{n\geq 1}$ and $(R_n^{(2)})_{n\geq 1}$ are sequences of functions which converge to 0 uniformly over any compact set in $\mathbb{R}^2$. □

3.3. *Sketch of proof of Theorem* 1

Let us denote by G_σ the infinitesimal generator of the Markov process which is solution of $(\mathcal{S}_\sigma)$. It is defined by

$$\forall f \in C^2(\mathbb{R}), \forall x \in \mathbb{R} \quad G_\sigma f(x) = \frac{1}{2} f''(x) - \frac{x^3}{2\sigma^4} f'(x).$$

Let $n \geq 1$ and $f \in C^2(\mathbb{R})$. By abuse of notation, we also write f for the function $(x, y) \in \mathbb{R}^2 \longmapsto f(x)$. The essential ingredient for the proof of Theorem 1 is the introduction of a suitable martingale problem. By Itô's formula (see [17]), we prove that

$$f\big(\widetilde{S}_n(t)\big) = f\big(\widetilde{S}_n(0)\big) + \int_0^t \widetilde{G}_n f\big(\widetilde{S}_n(s)\big)\, ds + \mathcal{M}_{n,f}(t),$$

where $\mathcal{M}_{n,f}$ is a local martingale. By Proposition 6, we have

$$\widetilde{G}_n f(\widetilde{S}_n) = \underbrace{\left(-\frac{n^{1/4}\widetilde{S}_n\widetilde{T}_n}{2\sigma^4} + \frac{\widetilde{S}_n\widetilde{T}_n^2}{2\sigma^6}\right) f'(\widetilde{S}_n)}_{\widetilde{A}_f(\widetilde{S}_n,\widetilde{T}_n)} + G_\sigma f(\widetilde{S}_n) + f'(\widetilde{S}_n) R_n^{(1)}(\widetilde{S}_n, \widetilde{T}_n),$$

where $(R_n^{(1)})_{n\geq 1}$ is a sequence of functions which converges to 0 uniformly over any compact set in $\mathbb{R}^2$.

Step 1: We notice that the term $\widetilde{A}_f(\widetilde{S}_n, \widetilde{T}_n)$ does not converge a priori. To solve this problem, we introduce a perturbation: we transform the function f into a function $F_{n,f}$ which converges to f as n goes to ∞, and which satisfies

$$\widetilde{G}_n F_{n,f}(\widetilde{S}_n, \widetilde{T}_n) = G_\sigma f(\widetilde{S}_n) + \text{a remainder}.$$

Notice that the perturbation theory and methodology was first introduced in [15].

Step 2: For any $k \geq 1$, we define the stopping time τ_n^k as the first exit time of a path of $(\widetilde{S}_n, \widetilde{T}_n)$ from the domain $[-k, k]^2$, and we prove that $\mathcal{M}_{n,f}^k = \mathcal{M}_{n,f}(\cdot \wedge \tau_n^k \wedge T)$ is a martingale which is bounded over L^2, for any $T > 0$ and $k \geq 1$.

Step 3: We prove that $\mathbb{P}(\tau_n^k \leq T)$, the probability that a path of $(\widetilde{S}_n, \widetilde{T}_n)$ exits $[-k, k]^2$ before the time T, goes to 0 when n and k goes to $+\infty$. We also use the concept of collapsing processes (see Appendix) in order to prove that the sequence of processes $(\widetilde{T}_n(t), t \geq 0)_{n\geq 1}$ converges to 0 in the following sense:

$$\forall \eta > 0 \qquad \lim_{n\to+\infty} \mathbb{P}\Big(\sup_{0\leq t\leq T} \big|\widetilde{T}_n(t)\big| > \eta\Big) = 0.$$

Step 4: We prove that the sequence $(\widetilde{S}_n(t), t \geq 0)_{n\geq 1}$ is tight in the Skorokhod space $\mathcal{D}([0, T], \mathbb{R})$.

Step 5: We deduce from the previous steps that there exists a subsequence $(\widetilde{S}_{m_n})_{n\geq 1}$ which converges in distribution to some process $\mathcal{U}$ on $\mathcal{D}([0, T], \mathbb{R})$. We prove then that, for any $k \geq 1$ and $t \in [0, T]$,

$$\mathcal{M}_{m_n,f}^k(t) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{M}_f(t) = f\big(\mathcal{U}(t\wedge T)\big) - f\big(\mathcal{U}(0)\big) - \int_0^{t\wedge T} G_\sigma f\big(\mathcal{U}(s)\big)\, ds,$$

and that $\mathcal{M}_f$ is a martingale. As a consequence $\mathcal{U}$ is uniquely determined as the unique solution of the martingale problem associated to G_σ. We conclude that $\mathcal{U}$ is the solution of $(\mathcal{S}_\sigma)$ and that $(\widetilde{S}_n)_{n\geq 1}$ converges in distribution to $\mathcal{U}$ on $\mathcal{D}([0, T], \mathbb{R})$, and thus on $C([0, T], \mathbb{R})$.

These steps are developed in detail in the next section.

4. Proof of Theorem 1

Step 1: *Perturbation*

Let $f \in C^2(\mathbb{R})$. We want to find functions H_f and K_f defined on $\mathbb{R}^2$ such that

$$F_{n,f} : (x,y) \longmapsto f(x) + \frac{1}{n^{1/4}} H_f(x,y) + \frac{1}{\sqrt{n}} K_f(x,y),$$

satisfies

$$\widetilde{G}_n F_{n,f} = G_\sigma f + \widetilde{R}_{n,f},$$

where $\widetilde{R}_{n,f}$ is a remainder term. Let us find necessary conditions. We suppose that we have built H_f and K_f and we assume that they are C^2. We have then, for any $(x,y) \in \mathbb{R}^2$,

$$\begin{aligned}\widetilde{G}_n F_{n,f}(x,y) = {} & n^{1/4}\left(-\frac{y}{\sigma^2}\frac{\partial H_f}{\partial y}(x,y) - \frac{xy}{2\sigma^4} f'(x)\right) - \frac{y}{\sigma^2}\frac{\partial K_f}{\partial y}(x,y) \\ & - \frac{xy}{2\sigma^4}\frac{\partial H_f}{\partial x}(x,y) + \frac{1}{2\sigma^6}\big(xy^2 - x^3\sigma^2\big) f'(x) + \frac{1}{2} f''(x) + \text{a remainder.}\end{aligned}$$

The function H_f should verify

$$\forall (x,y) \in \mathbb{R}^2 \quad -\frac{y}{\sigma^2}\frac{\partial H_f}{\partial y}(x,y) - \frac{xy}{2\sigma^4} f'(x) = 0.$$

We choose

$$H_f : (x,y) \longmapsto -\frac{xy}{2\sigma^2} f'(x).$$

Therefore the function K_f should satisfy, for all $(x,y) \in \mathbb{R}^2$,

$$\begin{aligned}\widetilde{G}_n F_{n,f}(x,y) &= -\frac{y}{\sigma^2}\frac{\partial K_f}{\partial y}(x,y) + \frac{xy^2}{4\sigma^6}\big(f'(x) + xf''(x)\big) + \frac{1}{2\sigma^6}\big(xy^2 - x^3\sigma^2\big) f'(x) \\ &\quad + \frac{1}{2} f''(x) + \text{the remainder} \\ &= -\frac{y}{\sigma^2}\frac{\partial K_f}{\partial y}(x,y) + \frac{xy^2}{4\sigma^6}\big(3f'(x) + xf''(x)\big) - \frac{x^3}{2\sigma^4} f'(x) + \frac{1}{2} f''(x) + \text{the remainder.}\end{aligned}$$

So that the variable y disappears in the leading term of $\widetilde{G}_n F_{n,f}(x,y)$, the function K_f should verify

$$\forall (x,y) \in \mathbb{R}^2 \quad -\frac{y}{\sigma^2}\frac{\partial K_f}{\partial y}(x,y) + \frac{xy^2}{4\sigma^6}\big(3f'(x) + xf''(x)\big) = 0.$$

We choose

$$K_f : (x,y) \longmapsto \frac{xy^2}{8\sigma^4}\big(3f'(x) + xf''(x)\big).$$

It is easy to see that these choices for H_f and K_f are sufficient for the variable y to disappear in the leading term of $\widetilde{G}_n F_{n,f}(x,y)$. The remainder term is then

$$\widetilde{R}_{n,f} = R_n^f + \frac{1}{n^{1/4}} R_n^{H_f} + \frac{1}{\sqrt{n}} R_n^{K_f}.$$

We notice that, so that the above computations are possible, it is necessary that f is C^4. Indeed, the first four derivatives of f appear in the remainder term. We also remark that, if $f \in C^4(\mathbb{R})$, then the functions H_f, K_f and their first and second derivatives are bounded over any compact set in $\mathbb{R}^2$. Finally let us recall that $(R_n^{(1)})_{n\geq 1}$ and $(R_n^{(2)})_{n\geq 1}$ are sequences of functions which converge to 0 when n goes to $+\infty$, uniformly over any compact set. As a consequence we have the following proposition:

Proposition 7. *Let $n \geq 1$ and $f \in C^4(\mathbb{R})$. We define H_f and K_f on $\mathbb{R}^2$ by*

$$\forall (x,y) \in \mathbb{R}^2 \quad H_f(x,y) = -\frac{xy}{2\sigma^2} f'(x), \qquad K_f(x,y) = \frac{xy^2}{8\sigma^4}\big(3f'(x) + xf''(x)\big).$$

Then the function

$$F_{n,f} : (x,y) \longmapsto f(x) + \frac{1}{n^{1/4}} H_f(x,y) + \frac{1}{\sqrt{n}} K_f(x,y),$$

verifies $\widetilde{G}_n F_{n,f} = G_\sigma f + \widetilde{R}_{n,f}$, with $\widetilde{R}_{n,f}$ a remainder term satisfying

$$\forall k > 0 \quad \lim_{n\to+\infty} \sup_{\substack{(x,y)\in\mathbb{R}^2 \\ \|(x,y)\|\leq k}} \big|\widetilde{R}_{n,f}(x,y)\big| = 0.$$

Step 2: *Introduction of a martingale problem*

We give ourselves $n \geq 1$ and $f \in C^4(\mathbb{R})$. For any $t \geq 0$, we have

$$f\left(\frac{S_n(\sqrt{n}t)}{n^{3/4}}\right) = f\big(\widetilde{S}_n(t)\big) = \left(F_{n,f} - \frac{1}{n^{1/4}} H_f - \frac{1}{\sqrt{n}} K_f\right)\big(\widetilde{S}_n(t), \widetilde{T}_n(t)\big).$$

We define the process $(\mathcal{M}_{n,f}(t), t \geq 0)$ by

$$\forall t \geq 0 \quad \mathcal{M}_{n,f}(t) = F_{n,f}\big(\widetilde{S}_n(t), \widetilde{T}_n(t)\big) - F_{n,f}\big(\widetilde{S}_n(0), \widetilde{T}_n(0)\big) - \int_0^t \widetilde{G}_n F_{n,f}\big(\widetilde{S}_n(s), \widetilde{T}_n(s)\big)\, ds.$$

By applying Itô's formula to the function

$$\Psi_{n,f} : (x_1, \dots, x_n) \in \mathbb{R}^n \longmapsto F_{n,f}\left(\frac{S_n[x]}{n^{3/4}}, \frac{T_n[x]}{n^{3/4}} - n^{1/4}\sigma^2\right),$$

we obtain

$$\forall t \geq 0 \quad \mathcal{M}_{n,f}(t) = n^{1/4} \sum_{j=1}^n \int_0^t \frac{\partial \Psi_{n,f}}{\partial x_j}\big(X_n(\sqrt{n}s)\big)\, dB_j(s).$$

It is a local martingale and

$$\forall t \geq 0 \quad \langle \mathcal{M}_{n,f}, \mathcal{M}_{n,f}\rangle_t = \sqrt{n} \sum_{j=1}^n \int_0^t \left(\frac{\partial \Psi_{n,f}}{\partial x_j}\right)^2 \big(X_n(\sqrt{n}s)\big)\, ds.$$

For any $k > 0$, we introduce the stopping time τ_n^k defined by

$$\tau_n^k = \inf_{t\geq 0}\big\{\big|\widetilde{S}_n(t)\big| \geq k \text{ or } \big|\widetilde{T}_n(t)\big| \geq k\big\}.$$

Let $T > 0$. We denote $\mathcal{M}_{n,f}^k(t) = \mathcal{M}_{n,f}(t \wedge \tau_n^k \wedge T)$ for any $t \geq 0$.

Lemma 8. *For all $k \geq 1$, $n \geq 1$ and $f \in C^4(\mathbb{R})$, the process $\mathcal{M}_{n,f}^k$ is a martingale which is bounded over* L^2. *Moreover*

$$\forall t \geq 0 \quad \sup_{n \geq 1} \mathbb{E}\big(\mathcal{M}_{n,f}^k(t)^2\big) < +\infty.$$

Proof. For any $t \geq 0$, we have

$$\big\langle \mathcal{M}_{n,f}^k, \mathcal{M}_{n,f}^k \big\rangle_t = \sqrt{n} \sum_{j=1}^n \int_0^{t \wedge \tau_n^k \wedge T} \left(\frac{\partial \Psi_{n,f}}{\partial x_j} \right)^2 \big(X_n(\sqrt{n} s) \big)\, ds.$$

Moreover, for all $i \in \{1, \dots, n\}$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} \frac{\partial \Psi_{n,f}}{\partial x_i}(x) = {} & \frac{1}{n^{3/4}} f'\left(\frac{S_n[x]}{n^{3/4}} \right) + \frac{1}{n^{3/4}} \left(\frac{1}{n^{1/4}} \frac{\partial H_f}{\partial x} + \frac{1}{n^{1/2}} \frac{\partial K_f}{\partial x} \right) \left(\frac{S_n[x]}{n^{3/4}}, \frac{T_n[x]}{n^{3/4}} - n^{1/4} \sigma^2 \right) \\ & + \frac{2x_i}{n^{3/4}} \left(\frac{1}{n^{1/4}} \frac{\partial H_f}{\partial y} + \frac{1}{n^{1/2}} \frac{\partial K_f}{\partial y} \right) \left(\frac{S_n[x]}{n^{3/4}}, \frac{T_n[x]}{n^{3/4}} - n^{1/4} \sigma^2 \right). \end{aligned} \tag{3}$$

By squaring these terms and by summing over all $i \in \{1, \dots, n\}$, we observe that there exists a constant $C_f^k > 0$ such that, for all $x \in \mathbb{R}^n$ verifying

$$\left| \frac{S_n[x]}{n^{3/4}} \right| < k \quad \text{and} \quad \left| \frac{T_n[x]}{n^{3/4}} - n^{1/4} \sigma^2 \right| < k,$$

we have

$$\sum_{j=1}^n \left(\frac{\partial \Psi_{n,f}}{\partial x_j} \right)^2 \leq \frac{C_f^k}{\sqrt{n}}.$$

As a consequence, for any $t \geq 0$,

$$\sup_{n \geq 1} \mathbb{E}\big(\big\langle \mathcal{M}_{n,f}^k, \mathcal{M}_{n,f}^k \big\rangle_t \big) \leq C_f^k T.$$

Therefore, for any $n \geq 1$, the process $\mathcal{M}_{n,f}^k$ is a martingale bounded over L^2 (see Theorem 4.8 of [14]) and

$$\forall t \geq 0 \quad \mathbb{E}\big(\mathcal{M}_{n,f}^k(t)^2 \big) = \mathbb{E}\big(\big\langle \mathcal{M}_{n,f}^k, \mathcal{M}_{n,f}^k \big\rangle_t \big) \leq C_f^k T.$$

This ends the proof of the lemma. □

Step 3: *Study of the asymptotic behavior* $(\tau_n^k)_{n \geq 1}$

Lemma 9. *For any $\varepsilon > 0$, there exist $n_\varepsilon \geq 1$ and $k_\varepsilon \geq 1$ such that*

$$\sup_{n \geq n_\varepsilon} \mathbb{P}\big(\tau_n^{k_\varepsilon} \leq T \big) \leq \varepsilon.$$

Moreover the process $(\widetilde{T}_n(t), t \geq 0)_{n \geq 1}$ collapses to zero, i.e.,

$$\forall \eta > 0 \qquad \lim_{n \to +\infty} \mathbb{P}\Big(\sup_{0 \leq t \leq T} \big| \widetilde{T}_n(t) \big| > \eta \Big) = 0.$$

Proof. Let $k, \varepsilon > 0$ and $n \geq 1$. We have

$$\mathbb{P}\big(\tau_n^k \leq T\big) \leq \mathbb{P}\bigg(\sup_{0\leq t\leq T\wedge\tau_n^k} \big|\widetilde{T}_n(t)\big| \geq \frac{k}{2}\bigg) + \mathbb{P}\bigg(\sup_{0\leq t\leq T\wedge\tau_n^k} \big|\widetilde{S}_n(t)\big| \geq \frac{k}{2}\bigg).$$

We denote $\mathbb{P}(A_n^k) + \mathbb{P}(B_n^k)$ the sum in the right side of this inequality.

Let us deal with the bound of $\mathbb{P}(A_n^k)$. To this end we would like to apply Proposition A.2 in Appendix to the positive semimartingale $(\xi_n(t), t \geq 0)_{n\geq 1}$ defined by

$$\forall n \geq 1, \forall t \geq 0 \quad \xi_n(t) = \widetilde{T}_n(t)^2.$$

By applying Itô's formula, we get

$$d\xi_n(t) = \widetilde{G}_n f_0\big(\widetilde{S}_n(t), \widetilde{T}_n(t)\big)\,dt + n^{1/4} \sum_{i=1}^{n} \frac{4X_n^i(\sqrt{n}t)}{n^{3/4}} \widetilde{T}_n(t)\,dB_i(t),$$

with $f_0 : (x, y) \longmapsto y^2$. With the notations of Proposition A.2, we have $\zeta_n(t) = \widetilde{G}_n f_0(\widetilde{S}_n(t), \widetilde{T}_n(t))$ and $Z_{n,i}(t) = 4n^{-1/2} X_n^i(\sqrt{n}t)\widetilde{T}_n(t)$ for all $t \geq 0$, $n \geq 1$ and $i \in \{1, \dots, n\}$. We have

$$\forall n \geq 1, \forall t \in \big[0, \tau_n^k\big] \quad \sum_{i=1}^{n} Z_{n,i}(t)^2 = 16\widetilde{T}_n(t)^2 \frac{1}{n} \sum_{i=1}^{n} X_n^i(\sqrt{n}t)^2 = 16\widetilde{T}_n(t)^2 \bigg(\sigma^2 + \frac{\widetilde{T}_n(t)}{n^{1/4}}\bigg).$$

Hence condition ($\mathcal{C}_4$) of Proposition A.2 is verified with $C_5 = 16k^2(\sigma^2 + k)$. Next, by Proposition 6, for any $n \geq 1$ and $t \in [0, \tau_n^k]$

$$\begin{aligned}
\zeta_n(t) &= -\frac{2\sqrt{n}}{\sigma^2}\widetilde{T}_n(t)^2 + 4\sigma^2 + 2\widetilde{T}_n(t) R_n^{(2)}\big(\widetilde{S}_n(t), \widetilde{T}_n(t)\big) + \frac{4}{n^{1/4}}\widetilde{T}_n(t) \\
&\leq -\frac{2\sqrt{n}}{\sigma^2}\xi_n(t) + 4\sigma^2 + 2k \sup_{\|(x,y)\|\leq k} \big|R_n^{(2)}(x, y)\big| + \frac{4k}{n^{1/4}}.
\end{aligned}$$

Condition ($\mathcal{C}_3$) is then verified with $\kappa_n = \sqrt{n}$ for any $n \geq 1$, $C_2 = 2/\sigma^2$,

$$C_4 = 4\sigma^2 + 2k \sup_{n\geq 1} \sup_{\|(x,y)\|\leq k} \big|R_n^{(2)}(x, y)\big| + 4k < +\infty$$

and C_3, $(\beta_n)_{n\geq 1}$ may be chosen arbitrarily. We choose $(\beta_n)_{n\geq 1}$ such that β_n/κ_n goes to 0 when n goes to $+\infty$.

Let us examine condition ($\mathcal{C}_2$): we denote $Y_n^i = (X_n^i(0))^2 - \sigma^2$ for any $i \in \{1, \dots, n\}$. Since the random variables $X_n^1(0), \dots, X_n^n(0)$ are independent with common distribution $\mathcal{N}(0, \sigma^2)$, we get that $Y_n^1, \dots, Y_n^n$ are independent identically distributed random variables which are centered and have finite moments of all orders. Theorem 2 of [3] implies that, for any $v \geq 2$, there exists $K_v > 0$ such that

$$\forall n \geq 1 \quad \mathbb{E}\big(\big|Y_n^1 + \cdots + Y_n^n\big|^v\big) \leq K_v n^{v/2}.$$

Hence, for all $d > 1$ and $n \geq 1$,

$$\mathbb{E}\big[\big(\xi_n(0)\big)^d\big] = \mathbb{E}\bigg[\bigg(\frac{1}{n^{3/4}}\big(Y_n^1 + \cdots + Y_n^n\big)\bigg)^{2d}\bigg] \leq K_{2d}\frac{n^d}{n^{3d/2}} = K_{2d} n^{-d/2}.$$

Condition ($\mathcal{C}_2$) is then satisfied for any $d > 1$, with $C_1 = K_{2d}$ and $\alpha_n \leq \sqrt{n}$ for all $n \geq 1$. So that condition ($\mathcal{C}_1$) is verified, we choose $d > 2$ and $\alpha_n = n^{1/4}$ for all $n \geq 1$. We have

$$\kappa_n^{1/d}\alpha_n^{-1} \vee \alpha_n\kappa_n^{-1} = n^{1/(2d)-1/4} \vee n^{-1/4} = n^{1/(2d)-1/4}.$$

As a consequence, Proposition A.2 implies that there exist $M>0$ and $n_1\geq 1$ such that

$$\sup_{n\geq n_1}\mathbb{P}\Big(\sup_{0\leq t\leq T\wedge\tau_n^k}\big|\widetilde{T}_n(t)\big|^2>Mn^{1/(2d)-1/4}\Big)\leq\frac{\varepsilon}{2}. \tag{4}$$

We increase the value of n_1 so that

$$\sup_{n\geq n_1}\mathbb{P}\Big(\sup_{0\leq t\leq T\wedge\tau_n^k}\big|\widetilde{T}_n(t)\big|^2>\frac{k}{2}\Big)\leq\frac{\varepsilon}{2}.$$

Let us deal now with the term $\mathbb{P}(B_n^k)$. In the rest of this proof, we assume that f is the function $(x,y)\longmapsto x^2$. We have

$$\forall n\geq 1\quad \widetilde{S}_n(t)^2=F_{n,f}\big(\widetilde{S}_n(t),\widetilde{T}_n(t)\big)+\frac{\widetilde{S}_n(t)^2\widetilde{T}_n(t)}{n^{1/4}\sigma^2}-\frac{\widetilde{S}_n(t)^2\widetilde{T}_n(t)^2}{\sqrt{n}\sigma^4},$$

thus

$$\forall n\geq 1\quad F_{n,f}\big(\widetilde{S}_n(t),\widetilde{T}_n(t)\big)=\widetilde{S}_n(t)^2\bigg(1-\frac{\widetilde{T}_n(t)}{n^{1/4}\sigma^2}+\frac{\widetilde{T}_n(t)^2}{\sqrt{n}\sigma^4}\bigg).$$

We obtain that, for n large enough,

$$\begin{aligned}\mathbb{P}\big(B_n^k\big)&=\mathbb{P}\Big(\sup_{0\leq t\leq T\wedge\tau_n^k}\big|\widetilde{S}_n(t)\big|^2>\frac{k^2}{4}\Big)\leq\mathbb{P}\Big(\sup_{0\leq t\leq T\wedge\tau_n^k}F_{n,f}\big(\widetilde{S}_n(t),\widetilde{T}_n(t)\big)>\frac{k^2}{8}\Big)\\&\leq\mathbb{P}\bigg(F_{n,f}\big(\widetilde{S}_n(0),\widetilde{T}_n(0)\big)>\frac{k^2}{24}\bigg)+\mathbb{P}\Big(\sup_{0\leq t\leq T\wedge\tau_n^k}\mathcal{M}_{n,f}(t)>\frac{k^2}{24}\Big)\\&\quad+\mathbb{P}\Big(\sup_{0\leq t\leq T\wedge\tau_n^k}\widetilde{G}_nF_{n,f}\big(\widetilde{S}_n(t),\widetilde{T}_n(t)\big)>\frac{k^2}{24T}\Big).\end{aligned}$$

For any $n\geq 1$, the random variables $X_n^1(0),\dots,X_n^n(0)$ are independent with common distribution $\mathcal{N}(0,\sigma^2)$ thus, by the Central Limit Theorem, we get $(\widetilde{S}_n(0))_{n\geq 1}$ and $(\widetilde{T}_n(0))_{n\geq 1}$ converge in distribution to 0. This implies that, for n large enough,

$$\mathbb{P}\bigg(F_{n,f}\big(\widetilde{S}_n(0),\widetilde{T}_n(0)\big)>\frac{k^2}{24}\bigg)\leq\frac{\varepsilon}{6}.$$

Next Proposition 7 gives us

$$\widetilde{G}_nF_{n,f}\big(\widetilde{S}_n(t),\widetilde{T}_n(t)\big)=1-\frac{\widetilde{S}_n(t)^4}{\sigma^4}+\widetilde{R}_{n,f}\big(\widetilde{S}_n(t),\widetilde{T}_n(t)\big)\leq 1+\big|\widetilde{R}_{n,f}\big(\widetilde{S}_n(t),\widetilde{T}_n(t)\big)\big|$$

and

$$\lim_{n\to+\infty}\sup_{\|(u,v)\|\leq k}\big|\widetilde{R}_{n,f}(u,v)\big|=0.$$

If we choose $k>\sqrt{24T}$ and n large enough, then

$$\mathbb{P}\Big(\sup_{0\leq t\leq T\wedge\tau_n^k}\widetilde{G}_nF_{n,f}\big(\widetilde{S}_n(t),\widetilde{T}_n(t)\big)>\frac{k^2}{24T}\Big)\leq\mathbb{P}\bigg(1+\sup_{\|(u,v)\|\leq k}\big|\widetilde{R}_{n,f}(u,v)\big|>\frac{k^2}{24T}\bigg)\leq\frac{\varepsilon}{6}.$$

Finally, by Lemma 8, $\mathcal{M}^k_{n,f}$ is a martingale thus Doob's maximal inequality implies

$$\mathbb{P}\Big(\sup_{0\le t\le T\wedge\tau_n^k}\mathcal{M}_{n,f}(t)>\frac{k^2}{24}\Big)\le\frac{\mathbb{E}(\mathcal{M}^k_{n,f}(T)^2)}{(k^2/24)^2}.$$

Lemma 8 also implies that $(\mathbb{E}(\mathcal{M}^k_{n,f}(T)^2))_{n\ge1}$ is a bounded sequence. Hence, for k large enough,

$$\mathbb{P}\Big(\sup_{0\le t\le T\wedge\tau_n^k}\mathcal{M}_{n,f}(t)>\frac{k^2}{24}\Big)\le\frac{\varepsilon}{6}.$$

As a consequence, there exist $n_2\ge1$ and $k_\varepsilon\ge1$ such that $\mathbb{P}(B_n^{k_\varepsilon})\le\varepsilon/2$ for all $n\ge n_2$. We denote $n_\varepsilon=n_1\vee n_2$. We have proved that

$$\forall n\ge n_\varepsilon\quad\mathbb{P}\big(\tau_n^{k_\varepsilon}\le T\big)\le\mathbb{P}\big(A_n^{k_\varepsilon}\big)+\mathbb{P}\big(B_n^{k_\varepsilon}\big)\le\varepsilon.$$

Let us prove the second assertion of the lemma: for any $\eta>0$, we have

$$\mathbb{P}\Big(\sup_{0\le t\le T}\big|\widetilde{T}_n(t)\big|>\eta\Big)\le\mathbb{P}\Big(\sup_{0\le t\le T\wedge\tau_n^{k_\varepsilon}}\big|\widetilde{T}_n(t)\big|^2>\eta^2\Big)+\mathbb{P}\big(\tau_n^{k_\varepsilon}\le T\big).$$

By formula (4), for n large enough,

$$\mathbb{P}\Big(\sup_{0\le t\le T}\big|\widetilde{T}_n(t)\big|>\eta\Big)\le\frac{\varepsilon}{2}+\mathbb{P}\big(\tau_n^{k_\varepsilon}\le T\big)\le\frac{3\varepsilon}{2}.$$

By letting ε goes to 0, we obtain that $(\widetilde{T}_n(t),t\ge0)_{n\ge1}$ collapses to zero. This ends the proof of the lemma. □

Step 4: *Tightness of* $(\widetilde{S}_n(t),t\ge0)_{n\ge1}$ *in* $\mathcal{D}([0,T],\mathbb{R})$

Since $(X_n(t),0\le t\le T)$, $n\ge1$, and the limiting process $(\mathcal{U}(t),0\le t\le T)$ belong to $C([0,T],\mathbb{R})$, it is enough to prove that $(\widetilde{S}_n(t),t\ge0)_{n\ge1}$ is relatively compact for the weak convergence in $\mathcal{D}([0,T],\mathbb{R})$, which is a Polish space (see Theorem 12.2 of [2]). Prohorov theorem (Theorem 5.1 of [2]) implies that it is enough to prove that $(\widetilde{S}_n(t),t\ge0)_{n\ge1}$ is a tight sequence. As in [6] and [5], we use the following tightness criterion:

Proposition 10. *A sequence* $(\xi_n(t),0\le t\le T)_{n\ge1}$ *on* $\mathcal{D}([0,T],\mathbb{R})$ *is tight if*

(a) *for any* $\varepsilon>0$, *there exists* $M>0$ *such that*

$$\sup_{n\ge1}\mathbb{P}\Big(\sup_{0\le t\le T}\big|\xi_n(t)\big|\ge M\Big)\le\varepsilon,$$

(b) *for any* $\varepsilon>0$ *and* $\eta>0$, *there exists* $\delta>0$ *such that*

$$\sup_{n\ge1}\sup_{\substack{\tau_1,\tau_2\in\mathcal{T}_n\\0\le\tau_1\le\tau_2\le(\tau_1+\delta)\wedge T}}\mathbb{P}\big(\big|\xi_n(\tau_2)-\xi_n(\tau_1)\big|\ge\eta\big)\le\varepsilon,$$

where, for any $n\ge1$, $\mathcal{T}_n$ *is the set of all the stopping times adapted to the filtration generated by the process* ξ_n.

Lemma 11. *The sequence* $(\widetilde{S}_n(t),0\le t\le T)_{n\ge1}$ *is relatively compact for the weak convergence on* $\mathcal{D}([0,T],\mathbb{R})$.

Proof. It is enough to prove that $(\widetilde{S}_n(t),0\le t\le T)_{n\ge1}$ verifies conditions (a) and (b) of Proposition 10. In the proof of Lemma 9, we proved that, for any $\alpha>0$, there exists $k_\alpha>0$ and $n_\alpha\ge1$ such that

$$\sup_{n\ge n_\alpha}\mathbb{P}\big(\tau_n^{k_\alpha}\le T\big)\le\alpha$$

and, for all $n \geq n_\alpha$,

$$\mathbb{P}\left(\sup_{0\leq t\leq T\wedge\tau_n^{k_\alpha}}\left|\widetilde{S}_n(t)\right| > \frac{k_\alpha}{2}\right) \leq \frac{\alpha}{2}.$$

We give ourselves $\varepsilon > 0$ and we denote $\alpha = 2\varepsilon/3$. We obtain that, for all $n \geq n_\alpha$,

$$\mathbb{P}\left(\sup_{0\leq t\leq T}\left|\widetilde{S}_n(t)\right| > \frac{k_\alpha}{2}\right) \leq \mathbb{P}\left(\sup_{0\leq t\leq T\wedge\tau_n^{k_\varepsilon}}\left|\widetilde{S}_n(t)\right| > \frac{k_\alpha}{2}\right) + \mathbb{P}\left(\tau_n^{k_\alpha} \leq T\right) \leq \frac{3\alpha}{2} = \varepsilon.$$

Hence condition (a) is verified.

We prove now condition (b): we give ourselves $n \geq 1$ and $\varepsilon, \eta, \delta > 0$. Let τ_1 and τ_2 be two stopping times adapted to the filtration generated by the process $\widetilde{S}_n$ and such that $0 \leq \tau_1 \leq \tau_2 \leq (\tau_1 + \delta) \wedge T$. Setting $\alpha = 2\varepsilon/3$, we have

$$\begin{aligned}\mathbb{P}\left(\left|\widetilde{S}_n(\tau_2) - \widetilde{S}_n(\tau_1)\right| \geq \eta\right) &\leq \mathbb{P}\left(\left|\widetilde{S}_n\left(\tau_2\wedge\tau_n^{k_\alpha}\right) - \widetilde{S}_n\left(\tau_1\wedge\tau_n^{k_\alpha}\right)\right| \geq \eta\right) + \mathbb{P}\left(\tau_n^{k_\alpha} \leq T\right) \\ &\leq \frac{1}{\eta}\mathbb{E}\left(\left|\widetilde{S}_n\left(\tau_2\wedge\tau_n^{k_\alpha}\right) - \widetilde{S}_n\left(\tau_1\wedge\tau_n^{k_\alpha}\right)\right|\right) + \alpha,\end{aligned}$$

where we used Markov's inequality. In the rest of this proof, we assume that f is the function $(x, y) \longmapsto x$. We have

$$\begin{aligned}\left|\widetilde{S}_n\left(\tau_2\wedge\tau_n^{k_\alpha}\right) - \widetilde{S}_n\left(\tau_1\wedge\tau_n^{k_\alpha}\right)\right| \leq& \int_{\tau_1\wedge\tau_n^{k_\alpha}}^{\tau_2\wedge\tau_n^{k_\alpha}}\left|\widetilde{G}_n F_{n,f}\left(\widetilde{S}_n(u), \widetilde{T}_n(u)\right)\right| du + \left|\mathcal{M}_{n,f}^{k_\alpha}(\tau_2) - \mathcal{M}_{n,f}^{k_\alpha}(\tau_1)\right| \\ &+ \frac{1}{n^{1/4}}\left|H_f\left(\widetilde{S}_n\left(\tau_2\wedge\tau_n^{k_\alpha}\right), \widetilde{T}_n\left(\tau_2\wedge\tau_n^{k_\alpha}\right)\right) - H_f\left(\widetilde{S}_n\left(\tau_1\wedge\tau_n^{k_\alpha}\right), \widetilde{T}_n\left(\tau_1\wedge\tau_n^{k_\alpha}\right)\right)\right| \\ &+ \frac{1}{\sqrt{n}}\left|K_f\left(\widetilde{S}_n\left(\tau_2\wedge\tau_n^{k_\alpha}\right), \widetilde{T}_n\left(\tau_2\wedge\tau_n^{k_\alpha}\right)\right) - K_f\left(\widetilde{S}_n\left(\tau_1\wedge\tau_n^{k_\alpha}\right), \widetilde{T}_n\left(\tau_1\wedge\tau_n^{k_\alpha}\right)\right)\right|.\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}\left(\left|\mathcal{M}_{n,f}^{k_\alpha}(\tau_2) - \mathcal{M}_{n,f}^{k_\alpha}(\tau_1)\right|\right)^2 &\leq \mathbb{E}\left(\left(\mathcal{M}_{n,f}^{k_\alpha}(\tau_2) - \mathcal{M}_{n,f}^{k_\alpha}(\tau_1)\right)^2\right) \\ &= \mathbb{E}\left(\mathbb{E}\left[\left(\mathcal{M}_{n,f}^{k_\alpha}(\tau_2) - \mathcal{M}_{n,f}^{k_\alpha}(\tau_1)\right)^2 |\mathcal{G}_{\tau_1}^n\right]\right),\end{aligned}$$

where $\mathcal{G}_t^n = \sigma(\mathcal{M}_{n,f}^{k_\alpha}(s), 0 \leq s \leq t)$ for all $t \geq 0$. By Lemma 8, $\mathcal{M}_{n,f}^{k_\alpha}$ is a martingale bounded over L^2 thus it is uniformly integrable. Martingale Stopping Theorem (Theorem 3.16 of [14]) implies that

$$\mathcal{M}_{n,f}^{k_\alpha}(\tau_1) = \mathbb{E}\left[\mathcal{M}_{n,f}^{k_\alpha}(\tau_2)|\mathcal{G}_{\tau_1}^n\right].$$

Hence

$$\begin{aligned}\mathbb{E}\left[\left(\mathcal{M}_{n,f}^{k_\alpha}(\tau_2) - \mathcal{M}_{n,f}^{k_\alpha}(\tau_1)\right)^2|\mathcal{G}_{\tau_1}^n\right] &= \mathbb{E}\left[\mathcal{M}_{n,f}^{k_\alpha}(\tau_2)^2|\mathcal{G}_{\tau_1}^n\right] + \mathcal{M}_{n,f}^{k_\alpha}(\tau_1)^2 - 2\mathcal{M}_{n,f}^{k_\alpha}(\tau_1)\mathbb{E}\left[\mathcal{M}_{n,f}^{k_\alpha}(\tau_2)|\mathcal{G}_{\tau_1}^n\right] \\ &= \mathbb{E}\left[\mathcal{M}_{n,f}^{k_\alpha}(\tau_2)^2|\mathcal{G}_{\tau_1}^n\right] - \mathcal{M}_{n,f}^{k_\alpha}(\tau_1)^2\end{aligned}$$

and thus

$$\begin{aligned}\mathbb{E}\left(\mathbb{E}\left[\left(\mathcal{M}_{n,f}^{k_\alpha}(\tau_2) - \mathcal{M}_{n,f}^{k_\alpha}(\tau_1)\right)^2|\mathcal{G}_{\tau_1}^n\right]\right) &= \mathbb{E}\left(\mathbb{E}\left[\left\langle\mathcal{M}_{n,f}^{k_\alpha}, \mathcal{M}_{n,f}^{k_\alpha}\right\rangle_{\tau_2} - \left\langle\mathcal{M}_{n,f}^{k_\alpha}, \mathcal{M}_{n,f}^{k_\alpha}\right\rangle_{\tau_1}|\mathcal{G}_{\tau_1}^n\right]\right) \\ &= \mathbb{E}\left(\sqrt{n}\sum_{j=1}^n\int_{\tau_1\wedge\tau_n^{k_\alpha}}^{\tau_2\wedge\tau_n^{k_\alpha}}\left(\frac{\partial F_{n,f}}{\partial x_j}\right)^2\left(X_n(\sqrt{n}u)\right) du\right) \leq C_f^{k_\alpha}\delta,\end{aligned}$$

where $C_f^{k_\alpha}$ is the constant introduced in the proof of Lemma 8 for $k = k_\alpha$. We get

$$\mathbb{E}\big(\big|\mathcal{M}_{n,f}^{k_\alpha}(\tau_2) - \mathcal{M}_{n,f}^{k_\alpha}(\tau_1)\big|\big) \le \sqrt{C_f^{k_\alpha}\delta}.$$

Next, since $f : (x, y) \longmapsto x$, Proposition 7 yields

$$\widetilde{G}_n F_{n,f}\big(\widetilde{S}_n(t), \widetilde{T}_n(t)\big) = -\frac{\widetilde{S}_n(t)^3}{2\sigma^4} + \widetilde{R}_{n,f}\big(\widetilde{S}_n(t), \widetilde{T}_n(t)\big)$$

and

$$\forall k > 0 \qquad \lim_{n\to+\infty} \sup_{\|(x,y)\|\le k} \big|\widetilde{R}_{n,f}(x, y)\big| = 0.$$

Therefore

$$\int_{\tau_1\wedge\tau_n^{k_\alpha}}^{\tau_2\wedge\tau_n^{k_\alpha}} \big|\widetilde{G}_n F_{n,f}\big(\widetilde{S}_n(u), \widetilde{T}_n(u)\big)\big|\, du \le \bigg(\frac{k_\alpha^3}{2\sigma^4} + \sup_{\|(x,y)\|\le k_\alpha} \big|\widetilde{R}_{n,f}(x, y)\big|\bigg)\delta.$$

Finally

$$\big|H_f\big(\widetilde{S}_n\big(\tau_2 \wedge \tau_n^{k_\alpha}\big), \widetilde{T}_n\big(\tau_2 \wedge \tau_n^{k_\alpha}\big)\big) - H_f\big(\widetilde{S}_n\big(\tau_1 \wedge \tau_n^{k_\alpha}\big), \widetilde{T}_n\big(\tau_1 \wedge \tau_n^{k_\alpha}\big)\big)\big| \le \frac{k_\alpha^2}{\sigma^2}$$

and

$$\big|K_f\big(\widetilde{S}_n\big(\tau_2 \wedge \tau_n^{k_\alpha}\big), \widetilde{T}_n\big(\tau_2 \wedge \tau_n^{k_\alpha}\big)\big) - K_f\big(\widetilde{S}_n\big(\tau_1 \wedge \tau_n^{k_\alpha}\big), \widetilde{T}_n\big(\tau_1 \wedge \tau_n^{k_\alpha}\big)\big)\big| \le \frac{3k_\alpha^3}{4\sigma^4}.$$

Hence, for n large enough and δ small enough,

$$\mathbb{E}\big(\big|\widetilde{S}_n\big(\tau_2 \wedge \tau_n^{k_\alpha}\big) - \widetilde{S}_n\big(\tau_1 \wedge \tau_n^{k_\alpha}\big)\big|\big) \le \frac{\eta\alpha}{2}.$$

We obtain

$$\mathbb{P}\big(\big|\widetilde{S}_n(\tau_2) - \widetilde{S}_n(\tau_1)\big| \ge \eta\big) \le \frac{3\alpha}{2} = \varepsilon.$$

Condition (b) of Proposition 10 is then satisfied and this ends the proof of the lemma. □

Step 5: *Identification of the limiting process and convergence*

Let us identify the limiting process. By Lemma 11, there exists a subsequence $(\widetilde{S}_{m_n}(t), t \ge 0)_{n\ge1}$ which converges in distribution to some process $(\mathcal{U}(t), t \ge 0)$ on $\mathcal{D}([0, T], \mathbb{R})$. By Lemma 9, $(\widetilde{T}_{m_n}(t), t \ge 0)_{n\ge1}$ converges in distribution to the null process on $\mathcal{D}([0, T], \mathbb{R})$.

For $k > 0$, we introduce the stopping time

$$\widetilde{\tau}_n^k = \min\Big(T, \inf_{t\ge0}\big\{\big|\widetilde{T}_n(t)\big| \ge k\big\}\Big).$$

If $t \ge T$ then $\mathbb{P}(\widetilde{\tau}_n^k \le t) = 1$ and, if $t < T$, then

$$\lim_{n\to+\infty} \mathbb{P}\big(\widetilde{\tau}_n^k \le t\big) \le \lim_{n\to+\infty} \mathbb{P}\Big(\sup_{0\le t\le T} \big|\widetilde{T}_n(t)\big| \ge k\Big) = 0,$$

by Lemma 9. As a consequence $(\widetilde{\tau}_n^k)_{n\ge1}$ converges in distribution to T.

We give ourselves $f \in C^4(\mathbb{R})$. For any $n \geq 1$ and $t \in [0, T]$,

$$F_{n,f}\big(\widetilde{S}_n(t), \widetilde{T}_n(t)\big) = f\big(\widetilde{S}_n(t)\big) + \bigg(\frac{1}{n^{1/4}} H_f + \frac{1}{n^{1/2}} K_f\bigg)\big(\widetilde{S}_n(t), \widetilde{T}_n(t)\big),$$

the functions H_f and K_f being continuous. Next, Proposition 7 implies that, for any $n \geq 1$ and $t \in [0, T]$,

$$\widetilde{G}_n F_{n,f}\big(\widetilde{S}_n(t), \widetilde{T}_n(t)\big) = G_\sigma f\big(\widetilde{S}_n(t)\big) + \widetilde{R}_{n,f}\big(\widetilde{S}_n(t), \widetilde{T}_n(t)\big),$$

where $\widetilde{R}_{n,f}$ is a continuous function on $\mathbb{R}^2$ such that

$$\forall k > 0 \quad \lim_{n \to +\infty} \sup_{\substack{(x,y) \in \mathbb{R}^2 \\ \|(x,y)\| \leq k}} \big|\widetilde{R}_{n,f}(x, y)\big| = 0.$$

Let $k > 0$. For any $t \geq 0$, we obtain

$$\mathcal{M}_{m_n,f}\big(t \wedge \widetilde{\tau}_n^k\big) \xrightarrow[n \to +\infty]{\mathcal{L}} \mathcal{M}_f(t) = f\big(\mathcal{U}(t \wedge T)\big) - f\big(\mathcal{U}(0)\big) - \int_0^{t \wedge T} G_\sigma f\big(\mathcal{U}(s)\big)\, ds.$$

For all $n \geq 1$ and $t \in [0, T]$, we have

$$\big\langle \mathcal{M}_{n,f}\big(\cdot \wedge \widetilde{\tau}_n^k\big), \mathcal{M}_{n,f}\big(\cdot \wedge \widetilde{\tau}_n^k\big)\big\rangle_t = \sqrt{n} \sum_{j=1}^n \int_0^{t \wedge \widetilde{\tau}_n^k} \bigg(\frac{\partial \Psi_{n,f}}{\partial x_j}\bigg)^2 \big(X_n(\sqrt{n}s)\big)\, ds,$$

and, using formula (3), we get

$$\begin{aligned}
&\sqrt{n} \sum_{j=1}^n \bigg(\frac{\partial \psi_{n,f}}{\partial x_j}\bigg)^2 \big(X_n(\sqrt{n}\cdot)\big) \\
&\quad = \bigg(f'(\widetilde{S}_n) + \bigg[\frac{1}{n^{1/4}} \frac{\partial H_f}{\partial x} + \frac{1}{n^{1/2}} \frac{\partial K_f}{\partial x}\bigg](\widetilde{S}_n, \widetilde{T}_n)\bigg)^2 \\
&\qquad + \frac{4\widetilde{S}_n}{n^{1/4}} \bigg(f'(\widetilde{S}_n) + \bigg[\frac{1}{n^{1/4}} \frac{\partial H_f}{\partial x} + \frac{1}{n^{1/2}} \frac{\partial K_f}{\partial x}\bigg](\widetilde{S}_n, \widetilde{T}_n)\bigg)\bigg(\bigg[\frac{1}{n^{1/4}} \frac{\partial H_f}{\partial y} + \frac{1}{n^{1/2}} \frac{\partial K_f}{\partial y}\bigg](\widetilde{S}_n, \widetilde{T}_n)\bigg) \\
&\qquad + \bigg(\frac{4\widetilde{T}_n}{n^{1/4}} + 4\sigma^2\bigg)\bigg(\bigg[\frac{1}{n^{1/4}} \frac{\partial H_f}{\partial y} + \frac{1}{n^{1/2}} \frac{\partial K_f}{\partial y}\bigg](\widetilde{S}_n, \widetilde{T}_n)\bigg)^2.
\end{aligned}$$

Assume that f has a compact support. Then we observe that there exists a constant $\widetilde{C}_f^k$ such that

$$\big|\widetilde{T}_n(t)\big| \leq k \quad \Longrightarrow \quad \sqrt{n} \sum_{j=1}^n \bigg(\frac{\partial \psi_{n,f}}{\partial x_j}\bigg)^2 \big(X_n(\sqrt{n}t)\big) \leq \widetilde{C}_f^k.$$

As a consequence $\mathcal{M}_{n,f}(\cdot \wedge \widetilde{\tau}_n^k)$ is a martingale and

$$\forall t \geq 0 \quad \sup_{n \geq 1} \mathbb{E}\big(\mathcal{M}_{n,f}\big(t \wedge \widetilde{\tau}_n^k\big)^2\big) \leq \widetilde{C}_f^k T < +\infty.$$

This implies that, for all $t \geq 0$, $(\mathcal{M}_{m_n,f}(t \wedge \widetilde{\tau}_{m_n}^k))_{n \geq 1}$ is an uniformly integrable family. Therefore $\mathcal{M}_f$ is a martingale.

Theorem 1.7 of Chapter 8 of [9] implies that the martingale problem associated to $\{(f, G_\sigma f) : f \in C_c^\infty(\mathbb{R})\}$ admits a unique solution: it is the strong solution of the differential stochastic equation

$$dz(t) = -\frac{z^3(t)}{2\sigma^4}\, dt + dB(t), \quad z(0) = 0,$$

where $(B(t), t \geq 0)$ is a standard Brownian motion. As a consequence the limiting process $(\mathcal{U}(t), 0 \leq t \leq T)$ is uniquely determined. Therefore

$$\left(\frac{S_n(\sqrt{n}t)}{n^{3/4}}, 0 \leq t \leq T\right)_{n\geq 1} = \big(\widetilde{S}_n(t), 0 \leq t \leq T\big)_{n\geq 1}$$

converges in distribution to $(\mathcal{U}(t), 0 \leq t \leq T)$ on $\mathcal{D}([0, T], \mathbb{R})$. Finally, since the sample paths of $(\mathcal{U}(t), 0 \leq t \leq T)$ are continuous, this convergence in distribution holds in $C([0, T], \mathbb{R})$. This ends the proof of Theorem 1.

Appendix: A proposition on collapsing processes

Definition A.1. *A sequence of real-valued stochastic processes $(\xi_n(t), t \geq 0)_{n\geq 1}$ collapses to zero if*

$$\forall \varepsilon > 0, \forall T > 0 \qquad \lim_{n\to+\infty} \mathbb{P}\Big(\sup_{0\leq t\leq T} \big|\xi_n(t)\big| > \varepsilon\Big) = 0.$$

The concept of collapsing processes has been developed by Francis Comets and Theodor Eisele in [6].

Proposition A.2. *Let $(\xi_n(t), t \geq 0)_{n\geq 1}$ be a sequence of positive semimartingales on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any $n \geq 1$, we give ourselves an integer $m_n \geq 1$ and independent standard Brownian motions $(B_i)_{1\leq i\leq m_n}$ which generate a filtration $(\mathcal{F}_t)_{t\geq 0}$. We assume that there exist $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $(Z_{n,i}(t), t \geq 0)_{1\leq i\leq m_n}$ and $(\zeta_n(t), t \geq 0)$ such that*

$$d\xi_n(t) = \zeta_n(t)\, dt + \sum_{i=1}^{m_n} Z_{n,i}(t)\, dB_i(t).$$

We suppose that there exist $d > 1$, positive constants $C_1, \dots, C_5$, increasing sequences $(\kappa_n)_{n\geq 1}$, $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 1}$ and a sequence $(\tau_n)_{n\geq 1}$ of stopping times verifying

$$\kappa_n^{1/d}\alpha_n^{-1} \underset{n\to+\infty}{\longrightarrow} 0, \qquad \kappa_n^{-1}\alpha_n \underset{n\to+\infty}{\longrightarrow} 0, \qquad \kappa_n^{-1}\beta_n \underset{n\to+\infty}{\longrightarrow} 0, \tag{$\mathcal{C}_1$}$$

$$\forall n \geq 1 \quad \mathbb{E}\big[\big(\xi_n(0)\big)^d\big] \leq C_1\alpha_n^{-d}, \tag{$\mathcal{C}_2$}$$

$$\forall n \geq 1, \forall t \in [0, \tau_n] \quad \zeta_n(t) \leq -\kappa_n C_2\xi_n(t) + \beta_n C_3 + C_4, \tag{$\mathcal{C}_3$}$$

and

$$\forall n \geq 1, \forall t \in [0, \tau_n] \quad \sum_{i=1}^{m_n} Z_{n,i}(t)^2 \leq C_5. \tag{$\mathcal{C}_4$}$$

Then, for any $\varepsilon > 0$ and $T > 0$, there exist $M > 0$ and $n_0 \geq 1$ such that

$$\sup_{n\geq n_0} \mathbb{P}\Big(\sup_{0\leq t\leq T\wedge\tau_n} \xi_n(t) > M\big(\kappa_n^{1/d}\alpha_n^{-1} \vee \alpha_n\kappa_n^{-1}\big)\Big) \leq \varepsilon.$$

This is Proposition 4.2 of [5]. It is a simple adaptation of the Proposition in appendix of [6].

Acknowledgement

The author would like to thank the anonymous referee for his careful review and useful comments which helped to improve the presentation of the paper.

References

[1] P. Bak, C. Tang and K. Wiesenfeld. Self-organized criticality: An explanation of $1/f$ noise. *Phys. Rev. Lett.* **59** (1987) 381–384. MR0949160

[2] P. Billingsley. *Convergence of Probability Measures*, 2nd edition. *Wiley Series in Probability and Statistics: Probability and Statistics*. Wiley, New York, 1999. MR1700749

[3] D. R. Brillinger. A note on the rate of convergence of a mean. *Biometrika* **49** (1962) 574–576. MR0156372

[4] R. Cerf and M. Gorny. A Curie–Weiss model of self-organized criticality. *Ann. Probab.* **44** (1) (2016) 444–478. MR3456343

[5] F. Collet and P. Dai Pra. The role of disorder in the dynamics of critical fluctuations of mean field models. *Electron. J. Probab.* **17** (26) (2012) 1–40. MR2912503

[6] F. Comets and T. Eisele. Asymptotic dynamics, noncritical and critical fluctuations for a geometric long-range interacting model. *Comm. Math. Phys.* **118** (4) (1988) 531–567. MR0962487

[7] D. A. Dawson. Critical dynamics and fluctuations for a mean-field model of cooperative behavior. *J. Stat. Phys.* **31** (1) (1983) 29–85. MR0711469

[8] R. S. Ellis and C. M. Newman. Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Z. Wahrsch. Verw. Gebiete* **44** (2) (1978) 117–139. MR0503333

[9] S. N. Ethier and T. G. Kurtz. *Markov Processes: Characterization and Convergence*. *Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. Wiley, New York, 1986. MR0838085

[10] M. Gorny. A Curie–Weiss model of self-organized criticality: The Gaussian case. *Markov Process. Related Fields* **20** (3) (2014) 563–576. MR3289133

[11] M. Gorny. The Cramér condition for the Curie-Weiss model of SOC. *Braz. J. Probab. Stat.* **30** (3) (2016) 401–431. MR3531691

[12] M. Gorny. Un modèle d'Ising Curie–Weiss de criticalité auto-organisée. Ph.D. thesis, Université Paris Sud, 2015.

[13] M. Gorny and S. R. Srinivasa Varadhan. Fluctuations of the self-normalized sum in the Curie–Weiss model of SOC. *J. Stat. Phys.* **160** (3) (2015) 513–518. MR3366089

[14] J.-F. Le Gall. *Mouvement Brownien, Martingales et Calcul Stochastique*. *Mathématiques et Applications*. Springer, Heidelberg, 2012. MR3184878

[15] G. C. Papanicolaou, D. W. Stroock and S. R. Srinivasa Varadhan. *Martingale Approach to Some Limit Theorems*. *Papers from the Duke Turbulence Conference*. *Duke Univ. Math. Ser.* **III**. Duke Univ., Durham, NC, 1977. Paper No. 6, pages ii+120. MR0461684

[16] G. O. Roberts and R. L. Tweedie. Exponential convergence of Langevin distributions and their discrete approximations. *Bernoulli* **2** (4) (1996) 341–363. MR1440273

[17] D. W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional Diffusion Processes*. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **233**. Springer, Berlin, 1979. MR0532498