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APPROXIMATIONS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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In this paper, we show that solutions of stochastic partial differential equations driven by Brownian motion can be approximated by stochastic partial differential equations forced by pure jump noise/random kicks. Applications to stochastic Burgers equations are discussed.

1. Introduction. Stochastic evolution equations and stochastic partial differential equations (SPDEs) are of great interest to many people. There exists a large amount of literature on the subject; see, for example, the monographs [5, 7, 8].

In this paper, we consider the following stochastic evolution equation:

$$dY_t = -AY_t\,dt + \big[b_1(Y_t) + b_2(Y_t)\big]\,dt + \sum_{i=1}^{m} \sigma_i(Y_t)\,dB_t^i, \tag{1.1}$$

$$Y_0 = h \in H, \tag{1.2}$$

in the framework of a Gelfand triple:

$$V \subset H \cong H^* \subset V^*, \tag{1.3}$$

where H, V are Hilbert spaces, H^*, V^* stand for the dual spaces of H and V, A is the infinitesimal generator of a strongly continuous semigroup, $b_1, \sigma_i, i = 1, \dots, m$ are measurable mappings from H into H, b_2 is a measurable mappings from V into V^*, $B_t = (B_t^1, \dots, B_t^m), t \geq 0$ is a m-dimensional Brownian motion. The solutions are considered to be weak solutions (in the PDE sense) in the space V and not as mild solutions in H. The stochastic evolution equations of this type driven by Wiener processes were first studied in [20] and subsequently in [18]. For stochastic equations with general Hilbert space valued semimartingales replacing the Brownian motion, we refer the reader to [12–14, 22] and also [1].

The aim of this paper is to study the approximations of stochastic evolution equations of the above type by solutions of stochastic evolution equations driven by pure jump processes, namely forced by random kicks. One of the motivations is to shine some light on numerical simulations of SPDEs driven by pure jump noise. To include interesting applications, the drift of equation (1.1) will consist of

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a "good" part b_1 and a "bad" part b_2. The crucial step of obtaining the approximation is to establish the tightness of the approximating equations in the space of Hilbert space-valued right continuous paths with left limits. This is tricky because of the nature of the infinite dimensions and the weak assumptions on the drift b_2. We first obtain the approximations assuming that the diffusion coefficients σ_i take values in the smaller space V and then remove this restriction by another layer of approximations. As far as we are aware of, this is the first paper to consider such approximations for SPDEs. The approximations of infinite activity Lévy processes were considered in [2]. Robustness of solutions of stochastic differential equations replacing the infinite activity of Lévy processes by Brownian motion was discussed in [3] and [6], and for the backward case in [9].

The rest of the paper is organized as follows. In Section 2, we lay down the precise framework. The main part is Section 3, where the approximations are established and the applications to stochastic Burgers equations are discussed.

2. Framework. Let V and H be two separable Hilbert spaces such that V is continuously, densely imbedded in H. Identifying H with its dual, we have

$$V \subset H \cong H^* \subset V^*, \tag{2.1}$$

where V^* stands for the topological dual of V. We assume that the imbedding $V \subset H$ is compact. Let A be a self-adjoint operator on the Hilbert space H satisfying the following coercivity hypothesis: There exist constants $\alpha_0 > 0$, $\alpha_1 > 0$ and $\lambda_0 \geq 0$ such that

$$\alpha_0 \|u\|_V^2 \leq 2\langle Au, u\rangle + \lambda_0 |u|_H^2 \leq \alpha_1 \|u\|_V^2 \qquad \text{for all } u \in V. \tag{2.2}$$

$\langle Au, u\rangle = Au(u)$ denotes the action of $Au \in V^*$ on $u \in V$.

We remark that A is generally not bounded as an operator from H into H. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $\{B_t = (B_t^1, \dots, B_t^m), t \geq 0\}$ be a m-dimensional $\mathcal{F}_t$-Brownian motion, $\nu(dx)$ a σ-finite measure on the measurable space $(R_0, \mathcal{B}(R_0))$, where $R_0 = R \setminus \{0\}$. We assume $0 < \int_{|x|\leq 1} x^2 \nu(dx) < \infty$. Let $p_i = (p_i(t)), t \in D_{p_i}, i = 1, \dots, m$ be mutually independent stationary $\mathcal{F}_t$-Poisson point processes on R_0 with characteristic measure ν. Here, D_{p_i} represents a countable (random) subset of $(0, \infty)$. See [15] for the details on Poisson point processes. Denote by $N_i(dt, dx)$ the Poisson counting measure associated with p_i, that is, $N_i(t, A) = \sum_{s\in D_{p_i}, s\leq t} I_A(p_i(s))$. Let $\tilde{N}_i(dt, dx) := N_i(dt, dx) - dt\, \nu(dx)$ be the compensated Poisson random measure. Let b_1, $\sigma_i, i = 1, \dots, m$ be measurable mappings from H into H, and $b_2(\cdot)$ a measurable mapping from V into V^*. Denote by $D([0, T], H)$ the space of all càdlàg paths from $[0, T]$ into H equipped with the Skorohod topology. Consider the stochastic evolution equation:

$$dX_t = -AX_t\, dt + \big[b_1(X_t) + b_2(X_t)\big]\, dt + \sum_{i=1}^m \sigma_i(X_t)\, dB_t^i, \tag{2.3}$$

$$X_0 = h \in H. \tag{2.4}$$

Let us introduce the following conditions:

(H.1) $b_1(\cdot), \sigma_i(\cdot) : H \to H$ are globally Lipschitz maps, that is, there exists a constant $C < \infty$ such that

$$\big|b_1(y_1) - b_1(y_2)\big|_H^2 + \sum_{i=1}^m \big|\sigma_i(y_1) - \sigma_i(y_2)\big|_H^2 \le C|y_1 - y_2|_H^2 \tag{2.5}$$
$$\text{for all } y_1, y_2 \in H.$$

(H.2) $b_2(\cdot)$ is a mapping from V into V^* that satisfies:

(i) $\langle b_2(u), u\rangle = 0$ for $u \in V$,
(ii) there exist constants C_1, $\beta < \frac{1}{2}$ such that

$$\begin{aligned}&\big\langle b_2(y_1) - b_2(y_2), y_1 - y_2\big\rangle \\ &\qquad \le \beta\alpha_0\|y_1 - y_2\|_V^2 + C_1|y_1 - y_2|_H^2\big(\|y_1\|_V^2 + \|y_2\|_V^2\big)\end{aligned} \tag{2.6}$$
$$\text{for all } y_1, y_2 \in V,$$

where α_0 is the constant appeared in (2.2).

(iii) There exists a constant $0 < \gamma < 1$ such that $\|b_2(u)\|_{V^*} \le C_2|u|_H^{2-\gamma}\|u\|_V^\gamma$ for $u \in V$.

DEFINITION 2.1. A continuous H-valued $(\mathcal{F}_t)$-adapted process $X = (X_t)_{t\ge 0}$ is said to be a solution to equation (2.3) if for any $T > 0$, $X \in L^2([0, T] \times \Omega; dt \times P, V)$ and P-a.s.

$$X_t = h - \int_0^t AX_s\,ds + \int_0^t \big[b_1(X_s) + b_2(X_s)\big]\,ds + \sum_{i=1}^m \int_0^t \sigma_i(X_s)\,dB_s^i.$$

Under the assumptions (H.1) and (H.2), it is known that equations (2.3) admits a unique solution (see, e.g., [19]).

We finish this section with two examples.

EXAMPLE 2.2. Let D be a bounded domain in $\mathbf{R}^d$. Set $H = L^2(D)$. Let $V = H_0^{1,2}(D)$ denote the Sobolev space of order one with homogenous boundary conditions. Denote by $a(x) = (a_{ij}(x))$ a symmetric matrix-valued function on D satisfying the uniform ellipticity condition:

$$\frac{1}{c} I_{d\times d} \le a(x) \le cI_{d\times d} \qquad \text{for some constant } c \in (0, \infty).$$

Define

$$Au = -\operatorname{div}\big(a(x)\nabla u(x)\big).$$

Then (2.2) is fulfilled for (H, V, A).

EXAMPLE 2.3. Let $A = -\Delta_\alpha$, where Δ_α denotes the generator of a symmetric α-stable process in R^d, $0 < \alpha < 2$. It is known that $\Delta_\alpha = -(-\Delta)^{\alpha/2}$, the fractional power of the Laplacian operator. The Dirichlet form associated with Δ_α in $L^2(\mathbf{R}^d)$ is given by

$$\mathcal{E}(u, v) = K(d, \alpha) \iint_{R^d \times R^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} \, dx \, dy,$$

$$D(\mathcal{E}) = \left\{ u \in L^2(R^d) : \iint_{R^d \times R^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+\alpha}} \, dx \, dy < \infty \right\},$$

where $K(d, \alpha) = \alpha 2^{\alpha-3} \pi^{-(d+2)/2} \sin(\frac{\alpha\pi}{2}) \Gamma(\frac{d+\alpha}{2}) \Gamma(\frac{\alpha}{2})$. The domain $D(\mathcal{E})$ can be identified as the fractional Sobolev space $H^{\alpha/2,2}$. See, for example, [11] for details about the fractional Laplace operator. To study equation (2.3), we choose $H = L^2(\mathbf{R}^d)$, and $V = D(\mathcal{E})$ with the inner product $\langle u, v \rangle = \mathcal{E}(u, v) + (u, v)_{L^2(R^d)}$. Then (2.2) is fulfilled for (H, V, A).

3. Approximations of SPDEs by pure jump type SPDEs.

Set, for $\varepsilon \in (0, 1)$,

$$\alpha(\epsilon) = \left(\int_{\{|x| \le \epsilon\}} x^2 \nu(dx) \right)^{1/2}.$$

Consider the following SPDE driven by pure jump noise:

$$\begin{aligned} X_t^\varepsilon = h - \int_0^t A X_s^\varepsilon \, ds + \int_0^t \left[b_1(X_s^\varepsilon) + b_2(X_s^\varepsilon) \right] ds \\ + \frac{1}{\alpha(\epsilon)} \sum_{i=1}^m \int_0^t \int_{|x| \le \varepsilon} \sigma_i(X_{s-}^\varepsilon) x \tilde{N}_i(ds, dx). \end{aligned} \tag{3.1}$$

DEFINITION 3.1. A H-valued $(\mathcal{F}_t)$-adapted process $X^\varepsilon = (X_t^\varepsilon)_{t \ge 0}$ is said to be a solution to equation (3.1) if:

(i) for any $T > 0$, $X^\varepsilon \in D([0, T], H) \cap L^2([0, T] \times \Omega; dt \times P, V)$,
(ii) for every $t \ge 0$, (3.1) holds P-almost surely.

The existence and uniqueness of the solution of equation (3.1) under the assumptions (H.1) and (H.2) can be found in [18, 19, 22]. Recall that X is the solution to the SPDE (2.3):

$$\begin{aligned} X_t = h - \int_0^t A X_s \, ds + \int_0^t \left[b_1(X_s) + b_2(X_s) \right] ds \\ + \sum_{i=1}^m \int_0^t \sigma_i(X_s) \, dB_s^i. \end{aligned} \tag{3.2}$$

Denote by μ_ε, μ, respectively, the laws of X^ε and X on the spaces $D([0, T], H)$ and $C([0, T], H)$ cf. Definitions 2.1 and 3.1. Consider the following conditions:

(H.3) For every $i \le m$, there exists a sequence of mappings $\sigma_n^i(\cdot): H \to V$ such that:

(i) $|\sigma_n^i(y_1) - \sigma_n^i(y_2)|_H \le c|y_1 - y_2|_H$, where c is a constant independent of n,
(ii) $|\sigma_n^i(y) - \sigma_i(y)|_H \to 0$ uniformly on bounded subsets of H as $n \to \infty$.

REMARK 3.2. In most of the cases, one simply chooses σ_n^i to be the finite-dimensional projections of σ_i into the subspaces of V.

(H.3)$'$ The maps $\sigma_i(\cdot), i = 1, \dots, m$ take the space V into itself and satisfy $\|\sigma_i(y)\|_V \le c(1 + \|y\|_V)$ for some constant c.
(H.4) There exists an orthonormal basis $\{e_k, k \ge 1\}$ of H such that $Ae_k = \lambda_k e_k$ and $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \to \infty$ as $n \to \infty$.

We first prepare some preliminary results needed for the proofs of the main theorems.
The following estimate holds for $\{X^\varepsilon, \varepsilon > 0\}$.

LEMMA 3.3. *Let X^ε be the solution of equation* (3.1). *If $\frac{\varepsilon}{\alpha(\epsilon)} \le C_0$ for some constant C_0, then we have for $p \ge 2$,*

$$\sup_{\varepsilon}\left\{E\Big[\sup_{0\le t\le T}|X_t^\varepsilon|_H^p\Big] + E\Big[\Big(\int_0^T \|X_s^\varepsilon\|_V^2\,ds\Big)^{p/2}\Big]\right\} < \infty. \tag{3.3}$$

PROOF. We prove the lemma for $p = 4$. Other cases are similar. In view of assumption (H.2), by Itô's formula, we have

$$\begin{aligned}|X_t^\varepsilon|_H^2 &= |h|_H^2 - 2\int_0^t \langle X_s^\varepsilon, AX_s^\varepsilon\rangle ds + 2\int_0^t \langle X_s^\varepsilon, b_1(X_s^\varepsilon)\rangle ds\\ &\quad + \sum_{i=1}^m \int_0^t \int_{|x|\le\varepsilon}\bigg(\bigg|\frac{1}{\alpha(\epsilon)}\sigma_i(X_{s-}^\varepsilon)x\bigg|_H^2\\ &\quad + 2\bigg\langle X_{s-}^\varepsilon, \frac{1}{\alpha(\epsilon)}\sigma_i(X_{s-}^\varepsilon)x\bigg\rangle\bigg)\tilde N_i(ds,dx)\\ &\quad + \sum_{i=1}^m \int_0^t \int_{|x|\le\varepsilon}\bigg|\frac{1}{\alpha(\epsilon)}\sigma_i(X_{s-}^\varepsilon)x\bigg|_H^2 ds\,\nu(dx).\end{aligned}\tag{3.4}$$

Let

$$\begin{aligned}M_t &= \sum_{i=1}^m \int_0^t \int_{|x|\le\varepsilon}\bigg(\bigg|\frac{1}{\alpha(\epsilon)}\sigma(X_{s-}^\varepsilon)x\bigg|_H^2\\ &\qquad + 2\bigg\langle X_{s-}^\varepsilon, \frac{1}{\alpha(\epsilon)}\sigma(X_{s-}^\varepsilon)x\bigg\rangle\bigg)\tilde N_i(ds,dx)\\ &:= \sum_{i=1}^m M_t^i.\end{aligned}\tag{3.5}$$

By Burkhölder's inequality, for $t \le T$, and a positive constant C, we have

$$
\begin{aligned}
E\Big[\sup_{0\le u\le t}|M_u|_H^2\Big] &\le C\sum_{i=1}^m E\Big[\sup_{0\le u\le t}|M_u^i|_H^2\Big] \le C\sum_{i=1}^m E\big[[M^i,M^i]_t\big]\\
&= C\sum_{i=1}^m E\bigg[\int_0^t\int_{|x|\le\varepsilon}\bigg(\bigg|\frac{1}{\alpha(\epsilon)}\sigma_i(X_{s-}^\varepsilon)x\bigg|_H^2\\
&\quad + 2\bigg\langle X_{s-}^\varepsilon,\frac{1}{\alpha(\epsilon)}\sigma_i(X_{s-}^\varepsilon)x\bigg\rangle\bigg)^2 N_i(ds,dx)\bigg]\\
&= C\sum_{i=1}^m E\bigg[\int_0^t\int_{|x|\le\varepsilon}\bigg(\bigg|\frac{1}{\alpha(\epsilon)}\sigma_i(X_s^\varepsilon)x\bigg|_H^2\\
&\quad + 2\bigg\langle X_s^\varepsilon,\frac{1}{\alpha(\epsilon)}\sigma_i(X_s^\varepsilon)x\bigg\rangle\bigg)^2 ds\,\nu(dx)\bigg]\\
&\le CE\bigg[\int_0^t(1+|X_s^\varepsilon|_H^4)\,ds\bigg],
\end{aligned}
\tag{3.6}
$$

where the linear growth condition on σ_i and the fact $\frac{\varepsilon}{\alpha(\epsilon)} \le C_0$ have been used. Use first (2.2) and then square both sides of the resulting inequality to obtain from (3.4) that

$$
\begin{aligned}
&|X_t^\varepsilon|_H^4 + \bigg(\int_0^t \|X_s^\varepsilon\|_V^2\,ds\bigg)^2\\
&\quad\le C_T|h|_H^4 + C_T\int_0^t(1+|X_s^\varepsilon|_H^4)\,ds + C_T M_t^2.
\end{aligned}
\tag{3.7}
$$

Take supremum over the interval $[0,t]$ in (3.7), use (3.6) to get

$$
\begin{aligned}
&E\Big[\sup_{0\le s\le t}|X_s^\varepsilon|_H^4\Big] + E\bigg[\bigg(\int_0^t\|X_s^\varepsilon\|_V^2\,ds\bigg)^2\bigg]\\
&\quad\le C|h|_H^4 + CE\bigg[\int_0^t(1+|X_s^\varepsilon|_H^4)\,ds\bigg].
\end{aligned}
\tag{3.8}
$$

Applying Gronwall's inequality proves the lemma. □

PROPOSITION 3.4. *Assume* (H.1), (H.2), (H.3)$'$, (H.4) *and* $\frac{\varepsilon}{\alpha(\epsilon)} \le C_0$ *for some constant* C_0. *Then the family* $\{X^\varepsilon, \varepsilon > 0\}$ *is tight on the space* $D([0,T],H)$.

PROOF. Write

$$
Y_t^\varepsilon = \frac{1}{\alpha(\epsilon)}\sum_{i=1}^m\int_0^t\int_{|x|\le\varepsilon}\sigma_i(X_{s-}^\varepsilon)x\tilde{N}_i(ds,dx),
\tag{3.9}
$$

and set $Z_t^\varepsilon = X_t^\varepsilon - Y_t^\varepsilon$. It suffices to prove that both $\{Y^\varepsilon, \varepsilon > 0\}$ and $\{Z^\varepsilon, \varepsilon > 0\}$ are tight. This is done in two steps.

Step 1. Prove that $\{Y^\varepsilon, \varepsilon > 0\}$ is tight.

In view of the assumption (H.3)$'$ on σ_i, we have $Y^\varepsilon \in D([0,T],V)$. Since the imbedding $V \subset H$ is compact, according to Theorem 3.1 in [16], it is sufficient to show that for every $e \in H$, $\{\langle Y^\varepsilon, e >, \varepsilon\rangle 0\}$ is tight in $D([0,T],R)$. Note that

$$
\begin{aligned}
&\sup_\varepsilon E\Big[\sup_{0\le t\le T}\langle Y_t^\varepsilon, e\rangle^2\Big]\\
&\quad\le \sup_\varepsilon E\Big[\sup_{0\le t\le T}|Y_t^\varepsilon|_H^2\Big]\\
&\quad\le C\sup_\varepsilon \frac{1}{\alpha(\epsilon)^2}\sum_{i=1}^m E\Big[\int_0^T\int_{|x|\le\varepsilon}|\sigma_i(X_s^\varepsilon)|_H^2 x^2\nu(dx)\,ds\Big]\\
&\quad= C\sum_{i=1}^m \sup_\varepsilon E\Big[\int_0^T|\sigma_i(X_s^\varepsilon)|_H^2\,ds\Big] < \infty,
\end{aligned}
\tag{3.10}
$$

and for any stopping times $\tau_\varepsilon \le T$ and any positive constants $\delta_\varepsilon \to 0$ we have

$$
\begin{aligned}
&E[|\langle Y_{\tau_\varepsilon}^\varepsilon, e\rangle - \langle Y_{\tau_\varepsilon+\delta_\varepsilon}^\varepsilon, e\rangle|^2]\\
&\qquad\le \frac{1}{\alpha(\epsilon)^2}\sum_{i=1}^m E\Big[\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\int_{|x|\le\varepsilon}|\sigma_i(X_s^\varepsilon)|_H^2 x^2\nu(dx)\,ds\Big]\\
&\qquad\le C\delta_\varepsilon \sup_\varepsilon E\Big[\Big(1+\sup_{0\le t\le T}|X_t^\varepsilon|_H^2\Big)\Big] \to 0,
\end{aligned}
\tag{3.11}
$$

as $\varepsilon \to 0$. By Theorem 3.1 in [16] (see also [4]), (3.10) and (3.11) yield the tightness of $\langle Y^\varepsilon, e\rangle, \varepsilon > 0$.

Step 2. Prove that $\{Z^\varepsilon, \varepsilon > 0\}$ is tight.

It is easy to see that Z^ε satisfies the equation:

$$
\begin{aligned}
Z_t^\varepsilon &= h - \int_0^t AZ_s^\varepsilon\,ds - \int_0^t AY_s^\varepsilon\,ds\\
&\quad + \int_0^t b_1(Z_s^\varepsilon + Y_s^\varepsilon)\,ds + \int_0^t b_2(Z_s^\varepsilon + Y_s^\varepsilon)\,ds.
\end{aligned}
\tag{3.12}
$$

Recall $\{e_k, k \ge 1\}$ is the orthonormal basis of H consisting of eigenvectors of A [see (H.4)]. We have

$$
\begin{aligned}
\langle Z_t^\varepsilon, e_k\rangle &= \langle h, e_k\rangle - \lambda_k\int_0^t\langle Z_s^\varepsilon, e_k\rangle\,ds - \lambda_k\int_0^t\langle Y_s^\varepsilon, e_k\rangle\,ds\\
&\quad + \int_0^t\langle b_1(Z_s^\varepsilon + Y_s^\varepsilon), e_k\rangle\,ds + \int_0^t\langle b_2(Z_s^\varepsilon + Y_s^\varepsilon), e_k\rangle\,ds.
\end{aligned}
\tag{3.13}
$$

By Corollary 5.2 in [16], to obtain the tightness of $\{Z^\varepsilon, \varepsilon > 0\}$ we need to show:

(i) $\{\langle Z^\varepsilon, e_k\rangle, \varepsilon > 0\}$ is tight in $D([0,T],R)$ for every k,

(ii) for any $\delta > 0$,

$$\lim_{N\to\infty} \sup_{\varepsilon} P\Big(\sup_{0\le t\le T} R_N^{\varepsilon}(t) > \delta\Big) = 0, \tag{3.14}$$

where

$$R_N^{\varepsilon}(t) = \sum_{k=N}^{\infty} \langle Z_t^{\varepsilon}, e_k\rangle^2.$$

The proof of (i) is similar to that of the tightness of $\langle Y^{\varepsilon}, e\rangle$, $\varepsilon > 0$. It is omitted. Let us prove (ii). By the chain rule, it follows that

$$\begin{aligned} \langle Z_t^{\varepsilon}, e_k\rangle^2 = \langle h, e_k\rangle^2 &- 2\lambda_k \int_0^t \langle Z_s^{\varepsilon}, e_k\rangle^2\, ds - 2\lambda_k \int_0^t \langle Y_s^{\varepsilon}, e_k\rangle \langle Z_s^{\varepsilon}, e_k\rangle\, ds \\ &+ 2\int_0^t \langle b_1(Z_s^{\varepsilon} + Y_s^{\varepsilon}), e_k\rangle \langle Z_s^{\varepsilon}, e_k\rangle\, ds \\ &+ 2\int_0^t \langle b_2(Z_s^{\varepsilon} + Y_s^{\varepsilon}), e_k\rangle \langle Z_s^{\varepsilon}, e_k\rangle\, ds. \end{aligned} \tag{3.15}$$

By the variation of constants formula, we have

$$\begin{aligned} \langle Z_t^{\varepsilon}, e_k\rangle^2 = e^{-2\lambda_k t} \langle h, e_k\rangle^2 &- 2\lambda_k \int_0^t e^{-2\lambda_k (t-s)} \langle Y_s^{\varepsilon}, e_k\rangle \langle Z_s^{\varepsilon}, e_k\rangle\, ds \\ &+ 2\int_0^t e^{-2\lambda_k (t-s)} \langle b_1(Z_s^{\varepsilon} + Y_s^{\varepsilon}), e_k\rangle \langle Z_s^{\varepsilon}, e_k\rangle\, ds \\ &+ 2\int_0^t e^{-2\lambda_k (t-s)} \langle b_2(Z_s^{\varepsilon} + Y_s^{\varepsilon}), e_k\rangle \langle Z_s^{\varepsilon}, e_k\rangle\, ds. \end{aligned} \tag{3.16}$$

Hence,

$$\begin{aligned} R_N^{\varepsilon}(t) &= \sum_{k=N}^{\infty} \langle Z_t^{\varepsilon}, e_k\rangle^2 \\ &= \sum_{k=N}^{\infty} e^{-2\lambda_k t} \langle h, e_k\rangle^2 - 2\int_0^t \sum_{k=N}^{\infty} \lambda_k e^{-2\lambda_k (t-s)} \langle Y_s^{\varepsilon}, e_k\rangle \langle Z_s^{\varepsilon}, e_k\rangle\, ds \\ &\quad + 2\int_0^t \sum_{k=N}^{\infty} e^{-2\lambda_k (t-s)} \langle b_1(Z_s^{\varepsilon} + Y_s^{\varepsilon}), e_k\rangle \langle Z_s^{\varepsilon}, e_k\rangle\, ds \\ &\quad + 2\int_0^t \sum_{k=N}^{\infty} e^{-2\lambda_k (t-s)} \langle b_2(Z_s^{\varepsilon} + Y_s^{\varepsilon}), e_k\rangle \langle Z_s^{\varepsilon}, e_k\rangle\, ds \\ &=: I_N^{(1)}(t) + I_N^{(2)}(t) + I_N^{(3)}(t) + I_N^{(4)}(t). \end{aligned} \tag{3.17}$$

Obviously,

$$I_N^{(1)}(t) \le \sum_{k=N}^{\infty} \langle h, e_k \rangle^2 \to 0, \tag{3.18}$$

as $N \to \infty$. For the third term on the right-hand side of (3.17), we have

$$\begin{aligned} |I_N^{(3)}(t)| &\le 2 \int_0^t e^{-2\lambda_N (t-s)} \sum_{k=N}^{\infty} |\langle b_1(Z_s^\varepsilon + Y_s^\varepsilon), e_k \rangle \langle Z_s^\varepsilon, e_k \rangle| \, ds \\ &\le 2 \int_0^t e^{-2\lambda_N (t-s)} \, ds \Bigl(\sup_{0 \le s \le T} |Z_s^\varepsilon|_H \Bigr) \Bigl(\sup_{0 \le s \le T} |b_1(Z_s^\varepsilon + Y_s^\varepsilon)|_H \Bigr) \\ &\le C \frac{1}{\lambda_N} \Bigl(1 + \sup_{0 \le s \le T} |Z_s^\varepsilon|_H^2 + \sup_{0 \le s \le T} |Y_s^\varepsilon|_H^2 \Bigr). \end{aligned} \tag{3.19}$$

Hence,

$$\begin{aligned} &\sup_\varepsilon E\Bigl[\sup_{0 \le t \le T} |I_N^{(3)}(t)| \Bigr] \\ &\quad \le C \frac{1}{\lambda_N} \Bigl(1 + \sup_\varepsilon E\Bigl[\sup_{0 \le s \le T} |Z_s^\varepsilon|_H^2 \Bigr] + \sup_\varepsilon E\Bigl[\sup_{0 \le s \le T} |Y_s^\varepsilon|_H^2 \Bigr] \Bigr) \\ &\quad \to 0 \qquad \text{as } N \to \infty. \end{aligned} \tag{3.20}$$

Let us turn to $I_N^{(2)}(t)$. By Hölder's inequality,

$$\begin{aligned} |I_N^{(2)}(t)| &\le 2 \int_0^t \Biggl(\sum_{k=N}^{\infty} e^{-4\lambda_k (t-s)} \lambda_k \langle Z_s^\varepsilon, e_k \rangle^2 \Biggr)^{1/2} \Biggl(\sum_{k=N}^{\infty} \lambda_k \langle Y_s^\varepsilon, e_k \rangle^2 \Biggr)^{1/2} ds \\ &\le 2 \int_0^t \Biggl(\sum_{k=N}^{\infty} e^{-4\lambda_k (t-s)} \lambda_k \langle Z_s^\varepsilon, e_k \rangle^2 \Biggr)^{1/2} (\langle A Y_s^\varepsilon, Y_s^\varepsilon \rangle^2)^{1/2} \, ds \\ &\le C \Bigl(\sup_{0 \le s \le T} \|Y_s^\varepsilon\|_V \Bigr) \\ &\qquad \times \int_0^t e^{-\lambda_N (t-s)} \Biggl(\sum_{k=N}^{\infty} e^{-2\lambda_k (t-s)} \lambda_k \langle Z_s^\varepsilon, e_k \rangle^2 \Biggr)^{1/2} ds \\ &\le C \Bigl(\sup_{0 \le s \le T} \|Y_s^\varepsilon\|_V \Bigr) \Bigl(\sup_{0 \le s \le T} |Z_s^\varepsilon|_H \Bigr) \int_0^t e^{-\lambda_N (t-s)} \frac{1}{\sqrt{t-s}} \, ds \\ &\le C \Bigl(\frac{1}{\sqrt{\lambda_N}} \int_0^\infty e^{-u} \frac{1}{\sqrt{u}} \, du \Bigr) \Bigl(\sup_{0 \le s \le T} \|Y_s^\varepsilon\|_V \Bigr) \Bigl(\sup_{0 \le s \le T} |Z_s^\varepsilon|_H \Bigr). \end{aligned} \tag{3.21}$$

In view of the assumption (H.2), the last term on the right-hand side of (3.17) can be estimated as follows:

$$
\begin{aligned}
\left|I_N^{(4)}(t)\right| &= \left|\int_0^t \sum_{k=N}^{\infty} e^{-2\lambda_k(t-s)}\langle b_2(X_s^\varepsilon), e_k\rangle\langle Z_s^\varepsilon, e_k\rangle\, ds\right| \\
&= \left|\int_0^t \sum_{k=N}^{\infty} e^{-2\lambda_k(t-s)}\sqrt{\lambda_0+\lambda_k}\langle (A+\lambda_0 I)^{-1/2} b_2(X_s^\varepsilon), e_k\rangle\langle Z_s^\varepsilon, e_k\rangle\, ds\right| \\
&\le C\int_0^t \left(\sum_{k=N}^{\infty} e^{-4\lambda_k(t-s)}\langle (A+\lambda_0 I)^{-1/2} b_2(X_s^\varepsilon), e_k\rangle^2\right)^{1/2} \\
&\qquad \times \left(\sum_{k=N}^{\infty}(\lambda_0+\lambda_k)\langle Z_s^\varepsilon, e_k\rangle^2\right)^{1/2} ds \qquad (3.22)\\
&\le C\int_0^t \|Z_s^\varepsilon\|_V e^{-2\lambda_N(t-s)}\left(\sum_{k=N}^{\infty}\langle (A+\lambda_0 I)^{-1/2} b_2(X_s^\varepsilon), e_k\rangle^2\right)^{1/2} ds \\
&\le C\int_0^t \|Z_s^\varepsilon\|_V e^{-2\lambda_N(t-s)}\|b_2(X_s^\varepsilon)\|_{V^*}\, ds \\
&\le C\int_0^t \|Z_s^\varepsilon\|_V e^{-2\lambda_N(t-s)}|X_s^\varepsilon|_H^{2-\gamma}\|X_s^\varepsilon\|_V^{\gamma}\, ds.
\end{aligned}
$$

This yields

$$
\begin{aligned}
\left|I_N^{(4)}(t)\right| &\le C\sup_{0\le s\le T}|X_s^\varepsilon|_H^{2-\gamma}\int_0^t \|Z_s^\varepsilon\|_V e^{-2\lambda_N(t-s)}\|X_s^\varepsilon\|_V^{\gamma}\, ds \\
&\le C\sup_{0\le s\le T}|X_s^\varepsilon|_H^{2-\gamma}\int_0^t e^{-2\lambda_N(t-s)}\left(\|X_s^\varepsilon\|_V^{1+\gamma}+\|X_s^\varepsilon\|_V^{\gamma}\|Y_s^\varepsilon\|_V\right) ds \\
&\le C\sup_{0\le s\le T}|X_s^\varepsilon|_H^{2-\gamma}\left(\int_0^t e^{-(4/(1-\gamma))\lambda_N(t-s)}\, ds\right)^{(1-\gamma)/2} \\
&\qquad \times \left(\int_0^T \left(\|X_s^\varepsilon\|_V^{1+\gamma}+\|X_s^\varepsilon\|_V^{\gamma}\|Y_s^\varepsilon\|_V\right)^{2/(1+\gamma)} ds\right)^{(1+\gamma)/2} \qquad (3.23)\\
&\le C\left(\frac{1}{\lambda_N}\right)^{(1-\gamma)/2} \\
&\qquad \times \sup_{0\le s\le T}|X_s^\varepsilon|_H^{2-\gamma}\left(\int_0^T \left(\|X_s^\varepsilon\|_V^{1+\gamma}+\|X_s^\varepsilon\|_V^{\gamma}\|Y_s^\varepsilon\|_V\right)^{2/(1+\gamma)} ds\right)^{(1+\gamma)/2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sup_\varepsilon E\Big[\sup_{0\le t\le T}\big|I_N^{(4)}(t)\big|\Big] \\
&\quad\le C\Big(\frac{1}{\lambda_N}\Big)^{(1-\gamma)/2}\sup_\varepsilon E\Big[\sup_{0\le s\le T}|X_s^\varepsilon|_H^{2-\gamma}\Big(\int_0^T\big(\|X_s^\varepsilon\|_V^{1+\gamma} \\
&\qquad + \|X_s^\varepsilon\|_V^{\gamma}\|Y_s^\varepsilon\|_V\big)^{2/(1+\gamma)}\,ds\Big)^{(1+\gamma)/2}\Big] \\
&\quad\le C\Big(\frac{1}{\lambda_N}\Big)^{(1-\gamma)/2} \\
&\qquad\times\sup_\varepsilon E\Big[\sup_{0\le s\le T}|X_s^\varepsilon|_H^{2-\gamma}\Big(\int_0^T\big(C\|X_s^\varepsilon\|_V^2+c\|Y_s^\varepsilon\|_V^2\big)\,ds\Big)^{(1+\gamma)/2}\Big] \\
&\quad\to 0\qquad\text{as } N\to\infty,
\end{aligned}\tag{3.24}
$$

where we used the fact that

$$
|ab|\le C\big(|a|^p+|b|^q\big),\qquad \frac{1}{p}+\frac{1}{q}=1.
$$

Putting together (3.17)–(3.24) and applying the Chebyshev inequality, we obtain (3.14). □

Let $\mathcal{D}$ denote the class of functions $f\in C_b^3(H)$ that satisfy (i) $\nabla f(z)\in D(A)$ and $|A\nabla f(z)|_H\le C(1+|z|_H)$ for some constant C, where $\nabla f(z)$ stands for the gradient of f at z, (ii) $f''(z)$, $f'''(z)$ are bounded, where f'', f''' denote the operators/multi-linear functionals associated with the second and the third derivatives of f.

For $f\in\mathcal{D}$, define

$$
\begin{aligned}
L^\varepsilon f(z) &= -\langle A\nabla f(z),z\rangle+\langle b_1(z),\nabla f(z)\rangle+\langle b_2(z),\nabla f(z)\rangle \\
&\quad+\sum_{i=1}^m\int_{|x|\le\varepsilon}\Big[f\Big(z+\frac{1}{\alpha(\epsilon)}\sigma_i(z)x\Big)-f(z) \\
&\quad-\Big\langle\nabla f(z),\frac{1}{\alpha(\epsilon)}\sigma_i(z)x\Big\rangle\Big]\nu(dx),
\end{aligned}\tag{3.25}
$$

and

$$
\begin{aligned}
Lf(z) &= -\langle A\nabla f(z),z\rangle+\langle b_1(z),\nabla f(z)\rangle+\langle b_2(z),\nabla f(z)\rangle \\
&\quad+\frac{1}{2}\sum_{i=1}^m\langle f''(z)\sigma_i(z),\sigma_i(z)\rangle.
\end{aligned}\tag{3.26}
$$

LEMMA 3.5. *Assume* $\lim_{\varepsilon\to 0}\frac{\varepsilon}{\alpha(\varepsilon)}=0$. *Then for* $f\in\mathcal{D}$, *it holds that*

$$L^{\varepsilon}f(z)\to Lf(z)\qquad \textit{uniformly on bounded subsets of } H \tag{3.27}$$

as $\varepsilon\to 0$.

PROOF. Note that for any $f\in\mathcal{D}$ we have

$$f(y+w)-f(y)-\langle\nabla f(y),w\rangle=\int_0^1 d\alpha\int_0^{\alpha}\langle f''(y+\beta w)w,w\rangle\,d\beta.$$

Thus, for every $z\in H$,

$$\begin{aligned}
&L^{\varepsilon}f(z)-Lf(z)\\
&\quad=\sum_{i=1}^m\biggl\{\int_{|x|\le\varepsilon}\int_0^1 d\alpha\int_0^{\alpha}d\beta\biggl\langle f''\biggl(z+\beta\frac{1}{\alpha(\epsilon)}\sigma_i(z)x\biggr)\\
&\qquad\times\frac{1}{\alpha(\epsilon)}\sigma_i(z)x,\frac{1}{\alpha(\epsilon)}\sigma_i(z)x\biggr\rangle\nu(dx)-\int_0^1 d\alpha\int_0^{\alpha}d\beta\langle f''(z)\sigma_i(z),\sigma_i(z)\rangle\biggr\}\\
&\quad=\frac{1}{\alpha(\epsilon)^2}\int_{|x|\le\varepsilon}x^2\nu(dx)\int_0^1 d\alpha\int_0^{\alpha}d\beta\sum_{i=1}^m\biggl[\biggl\langle f''\biggl(z+\beta\frac{1}{\alpha(\epsilon)}\sigma_i(z)x\biggr)\\
&\qquad\times\sigma_i(z),\sigma_i(z)\biggr\rangle-\langle f''(z)\sigma_i(z),\sigma_i(z)\rangle\biggr].
\end{aligned}\tag{3.28}$$

Hence, for $z\in B_N=\{z\in H;|z|_H\le N\}$ we have

$$\begin{aligned}
&|L^{\varepsilon}f(z)-Lf(z)|\\
&\quad\le C\frac{1}{\alpha(\epsilon)^2}\int_{|x|\le\varepsilon}x^2\nu(dx)\int_0^1 d\alpha\int_0^{\alpha}d\beta\,\beta|x|\frac{1}{\alpha(\epsilon)}\\
&\qquad\times\sum_{i=1}^m(|\sigma_i(z)|_H|\sigma_i(z)|_H^2)\\
&\quad\le C_N\frac{\varepsilon}{\alpha(\epsilon)}\to 0,
\end{aligned}\tag{3.29}$$

uniformly on B_N as $\varepsilon\to 0$, where we have used the local Lipschitz continuity of $f''(z)$. □

THEOREM 3.6. *Suppose* (H.1), (H.2), (H.3)′, (H.4) *hold and* $\lim_{\varepsilon\to 0}\frac{\varepsilon}{\alpha(\varepsilon)}=0$. *Then*, *for any* $T>0$, μ_{ε} *converges weakly to* μ, *for* $\varepsilon\to 0$, *on the space* $D([0,T],H)$ *equipped with the Skorohod topology.*

PROOF. Since the mappings σ_i take values in the space V, by Proposition 3.4, the family $\{\mu_{\varepsilon},\varepsilon>0\}$ is tight. Let μ_0 be the weak limit of any convergent sequence

$\{\mu_{\varepsilon_n}\}$ on the canonical space $(\Omega = D([0,T],H),\mathcal{F})$ as $\varepsilon_n \to 0$. We will show that $\mu_0 = \mu$. Denote by $X_t(\omega) = w(t), \omega \in \Omega$ the coordinate process. Set $J(X) = \sup_{0\le s\le T} |X_s - X_{s-}|_H$. Since

$$\begin{aligned} E^{\mu_\varepsilon}\big[J(X)\big] &= E\big[J\big(X^\varepsilon\big)\big] \\ &\le \frac{\varepsilon}{\alpha(\epsilon)}\sum_{i=1}^m E\Big[\sup_{0\le s\le T}\big|\sigma_i\big(X_s^\varepsilon\big)\big|_H\Big] \\ &\le C\frac{\varepsilon}{\alpha(\epsilon)}\Big(1 + E\Big[\sup_{0\le s\le T}\big|X_s^\varepsilon\big|_H\Big]\Big) \to 0, \end{aligned} \tag{3.30}$$

as $\varepsilon \to 0$, it follows from Theorem 13.4 in [4] that μ_0 is supported on the $C([0,T],H)$, the space of H-valued continuous functions on $[0,T]$. As a consequence, the finite-dimensional distributions of μ_{ε_n} converge to that of μ_0.

Let us fix $f \in \mathcal{D}$. Then by Itô's formula,

$$\begin{aligned} &f\big(X_t^\varepsilon\big) - f(h) - \int_0^t L^\varepsilon f\big(X_s^\varepsilon\big)\,ds \\ &\quad = \sum_{i=1}^m \int_0^t \int_{|x|\le\varepsilon}\Big\{f\Big(X_{s-}^\varepsilon + \frac{1}{\alpha(\epsilon)}\sigma_i\big(X_{s-}^\varepsilon\big)x\Big) - f\big(X_{s-}^\varepsilon\big)\Big\}\tilde{N}_i(ds,dx) \end{aligned} \tag{3.31}$$

is a martingale. Such an Itô formula for stochastic evolution equations/SPDEs driven by continuous martingales can be found in [20] (Theorem 3.2, page 147). In our pure jump case, due to the strong assumptions on the function f this Itô formula can be obtained by an approximation argument as follows. For $n \ge 1$, consider the finite-dimensional projection:

$$\overline{X}_t^{n,\varepsilon} = \sum_{i=1}^n \langle X_t^\varepsilon, e_i\rangle e_i,$$

where $e_i, i \ge 1$ are the eigenvectors of A [see (H.4)]. We first apply Itô's formula to the finite-dimensional process $\overline{X}_t^{n,\varepsilon}$ and the function $F_n(x_1,\dots,x_n) = f(\sum_{i=1}^n x_i e_i)$, and then take the limit as $n \to \infty$ to get (3.31).

By the martingale property, for any $s_0 < s_1 < \cdots < s_n \le s < t$ and $f_0, f_1, \dots, f_n \in C_b(H)$ it holds that

$$E^{\mu_\varepsilon}\Big[\Big(f(X_t) - f(X_s) - \int_s^t L^\varepsilon f(X_u)\,du\Big) f(X_{s_0})\cdots f(X_{s_n})\Big] = 0. \tag{3.32}$$

For any positive constant $M > 0$, by Lemma 3.5 we have

$$\lim_{n\to\infty} E^{\mu_{\varepsilon_n}}\Big[\int_s^t \big|L^{\varepsilon_n} f(X_u) - Lf(X_u)\big|\,du,\ \sup_{0\le u\le T}|X_u|_H \le M\Big] = 0. \tag{3.33}$$

On the other hand, in view of the assumptions on f we have

$$
\begin{aligned}
&\sup_n E^{\mu_{\varepsilon_n}}\left[\int_s^t \left|L^{\varepsilon_n} f(X_u) - Lf(X_u)\right| du, \sup_{0\le u\le T} |X_u|_H > M\right] \\
&\qquad \le C\frac{1}{M}\sup_n E^{\mu_{\varepsilon_n}}\left[\sup_{0\le u\le T} |X_u|_H^3\right] \\
&\qquad \le C'\frac{1}{M}.
\end{aligned}
\tag{3.34}
$$

Combining (3.33) with (3.34), we arrive at

$$
\lim_{n\to\infty} E^{\mu_{\varepsilon_n}}\left[\int_s^t \left|L^{\varepsilon_n} f(X_u) - Lf(X_u)\right| du\right] = 0. \tag{3.35}
$$

By the weak convergence of μ_{ε_n} and the convergence of finite distributions, it follows from (3.32) and (3.35) that

$$
\begin{aligned}
&E^{\mu_0}\left[\left(f(X_t) - f(X_s) - \int_s^t Lf(X_u)\,du\right) f(X_{s_0})\cdots f(X_{s_n})\right] \\
&\quad = \lim_{n\to\infty} E^{\mu_{\varepsilon_n}}\left[\left(f(X_t) - f(X_s) - \int_s^t Lf(X_u)\,du\right) f(X_{s_0})\cdots f(X_{s_n})\right] \\
&\quad = \lim_{n\to\infty} E^{\mu_{\varepsilon_n}}\left[\left(f(X_t) - f(X_s) - \int_s^t L^{\varepsilon_n} f(X_u)\,du\right) f(X_{s_0})\cdots f(X_{s_n})\right] \\
&\quad = 0.
\end{aligned}
\tag{3.36}
$$

Since $s_0 < s_1 < \cdots < s_n \le s < t$ are arbitrary, (3.36) implies that for any $f \in \mathcal{D}$,

$$
M_t^f := f(X_t) - f(h) - \int_0^t Lf(X_s)\,ds, \qquad t \ge 0,
$$

is a martingale under μ_0. In particular, let $f(z) = \langle e_k, z\rangle$ and $f(z) = \langle e_k, z\rangle\langle e_j, z\rangle$, respectively, to obtain that under μ_0

$$
\begin{aligned}
M_t^k :={}& \langle e_k, X_t\rangle - \langle e_k, h\rangle + \int_0^t \langle Ae_k, X_s\rangle\,ds - \int_0^t \langle b_1(X_s), e_k\rangle\,ds \\
&- \int_0^t \langle b_2(X_s), e_k\rangle\,ds
\end{aligned}
\tag{3.37}
$$

and

$$
\begin{aligned}
M_t^{k,j} :={}& \langle e_k, X_t\rangle\langle e_j, X_t\rangle - \langle e_k, h\rangle\langle e_j, h\rangle \\
&+ \int_0^t \{\langle Ae_k, X_s\rangle\langle e_j, X_s\rangle + \langle Ae_j, X_s\rangle\langle e_k, X_s\rangle\}\,ds \\
&- \int_0^t \langle b_1(X_s), e_k\langle e_j, X_s\rangle + e_j\langle e_k, X_s\rangle\rangle\,ds
\end{aligned}
\tag{3.38}
$$

$$- \int_0^t \langle b_2(X_s), e_k \langle e_j, X_s \rangle + e_j \langle e_k, X_s \rangle \rangle ds$$
$$- \sum_{i=1}^m \int_0^t \langle \sigma_i(X_s), e_k \rangle \langle \sigma_i(X_s), e_j \rangle ds$$

are martingales. This together with Itô's formula yields that

$$\langle M^k, M^j \rangle_t = \sum_{i=1}^m \int_0^t \langle \sigma_i(X_s), e_k \rangle \langle \sigma_i(X_s), e_j \rangle ds, \tag{3.39}$$

where $\langle M^k, M^j \rangle$ stands for the sharp bracket of the two martingales. Now by Theorem 18.12 in [17] (or Theorem 7.1′ in [15]), there exists a probability space $(\Omega', \mathcal{F}', P')$ with a filtration $\mathcal{F}'_t$ such that on the standard extension

$$(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \mathcal{F}_t \times \mathcal{F}'_t, \mu_0 \times P')$$

of $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ there exists a m-dimensional Brownian motion $B_t = (B_t^1, \dots, B_t^m)$, $t \geq 0$ such that

$$M_t^k = \sum_{i=1}^m \int_0^t \langle \sigma_i(X_s), e_k \rangle dB_s^i, \tag{3.40}$$

namely,

$$\begin{aligned} &\langle e_k, X_t \rangle - \langle e_k, h \rangle \\ &\quad = - \int_0^t \langle Ae_k, X_s \rangle ds + \int_0^t \langle b_1(X_s), e_k \rangle ds + \int_0^t \langle b_2(X_s), e_k \rangle ds \\ &\qquad + \sum_{i=1}^m \int_0^t \langle \sigma_i(X_s), e_k \rangle dB_s^i \end{aligned} \tag{3.41}$$

for any $k \geq 1$. Thus, under μ_0, $X_t, t \geq 0$ is a weak solution (both in the probabilistic and in PDE sense) of the SPDE

$$X_t = h - \int_0^t AX_s \, ds + \int_0^t b_1(X_s) \, ds + \int_0^t b_2(X_s) \, ds + \sum_{i=1}^m \int_0^t \sigma_i(X_s) \, dB_s^i.$$

By the uniqueness of the above equation, we conclude that $\mu_0 = \mu$ completing the proof of the theorem. □

THEOREM 3.7. *Suppose* (H.1), (H.2), (H.3) *and* (H.4) *hold and* $\lim_{\varepsilon \to 0} \frac{\varepsilon}{\alpha(\varepsilon)} = 0$. *Then, for any* $T > 0$, μ_ε *converges weakly to* μ, *for* $\varepsilon \to 0$, *on the space* $D([0, T], H)$ *equipped with the Skorohod topology.*

PROOF. Let $\sigma_n^i(\cdot)$ be the mappings specified in (H.3). Let $X^{n,\varepsilon}, X^n$ be the solutions of the SPDEs

$$
\begin{aligned}
X_t^{n,\varepsilon} &= h - \int_0^t AX_s^{n,\varepsilon}\,ds + \int_0^t b_1(X_s^{n,\varepsilon})\,ds + \int_0^t b_2(X_s^{n,\varepsilon})\,ds \\
&\quad + \frac{1}{\alpha(\epsilon)}\sum_{i=1}^m \int_0^t \int_{|x|\le\varepsilon} \sigma_n^i(X_{s-}^{n,\varepsilon})x\tilde{N}_i(ds,dx),
\end{aligned} \tag{3.42}
$$

$$
\begin{aligned}
X_t^{n} &= h - \int_0^t AX_s^{n}\,ds + \int_0^t b_1(X_s^{n})\,ds + \int_0^t b_2(X_s^{n})\,ds \\
&\quad + \sum_{i=1}^m \int_0^t \sigma_n^i(X_s^n)\,dB_s^i.
\end{aligned} \tag{3.43}
$$

We claim that for any $\delta > 0$,

$$\lim_{n\to\infty}\sup_{\varepsilon} P\Big(\sup_{0\le t\le T}|X_t^{n,\varepsilon} - X_t^{\varepsilon}| > \delta\Big) = 0, \tag{3.44}$$

$$\lim_{n\to\infty} P\Big(\sup_{0\le t\le T}|X_t^{n} - X_t|^2 > \delta\Big) = 0. \tag{3.45}$$

Let us only prove (3.44). The proof of (3.45) is simpler. As the proof of (3.3), we can show that

$$\sup_n\sup_\varepsilon\Big\{E\Big[\sup_{0\le t\le T}|X_t^{n,\varepsilon}|_H^2\Big] + E\Big[\int_0^T \|X_s^{n,\varepsilon}\|_V^2\,ds\Big]\Big\} < \infty, \tag{3.46}$$

$$\sup_n\Big\{E\Big[\sup_{0\le t\le T}|X_t^{n}|_H^2\Big] + E\Big[\int_0^T \|X_s^{n}\|_V^2\,ds\Big]\Big\} < \infty. \tag{3.47}$$

Applying Itô's formula first to $|X_t^{n,\varepsilon} - X_t^\varepsilon|_H^2$ and then applying the integration by parts formula, we obtain

$$
\begin{aligned}
&e^{-\gamma\int_0^t(\|X_s^{n,\varepsilon}\|_V^2+\|X_s^\varepsilon\|_V^2)\,ds}|X_t^{n,\varepsilon} - X_t^\varepsilon|_H^2 \\
&\quad = -\gamma\int_0^t e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2+\|X_u^\varepsilon\|_V^2)\,du}|X_s^{n,\varepsilon} - X_s^\varepsilon|_H^2\big(\|X_s^{n,\varepsilon}\|_V^2 + \|X_s^\varepsilon\|_V^2\big)\,ds \\
&\qquad - 2\int_0^t e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2+\|X_u^\varepsilon\|_V^2)\,du}\langle X_s^{n,\varepsilon} - X_s^\varepsilon, A(X_s^{n,\varepsilon} - X_s^\varepsilon)\rangle\,ds \\
&\qquad + 2\int_0^t e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2+\|X_u^\varepsilon\|_V^2)\,du}\langle X_s^{n,\varepsilon} - X_s^\varepsilon, b_1(X_s^{n,\varepsilon}) - b_1(X_s^\varepsilon)\rangle\,ds \\
&\qquad + 2\int_0^t e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2+\|X_u^\varepsilon\|_V^2)\,du}\langle X_s^{n,\varepsilon} - X_s^\varepsilon, b_2(X_s^{n,\varepsilon}) - b_2(X_s^\varepsilon)\rangle\,ds \\
&\qquad + \sum_{i=1}^m \int_0^t \int_{|x|\le\varepsilon} e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2+\|X_u^\varepsilon\|_V^2)\,du}
\end{aligned} \tag{3.48}
$$

$$\times \left(\left|\frac{1}{\alpha(\epsilon)}(\sigma_n^i(X_{s-}^{n,\varepsilon})x - \sigma_i(X_{s-}^{\varepsilon})x)\right|_H^2\right.$$
$$\left. + 2\left\langle (X_s^{n,\varepsilon} - X_s^{\varepsilon}), \frac{1}{\alpha(\epsilon)}(\sigma_n^i(X_{s-}^{n,\varepsilon})x - \sigma_i(X_{s-}^{\varepsilon})x)\right\rangle\right)\tilde{N}_i(ds, dx)$$
$$+ \sum_{i=1}^m \int_0^t \int_{|x|\le\varepsilon} e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2 + \|X_u^{\varepsilon}\|_V^2)\,du}$$
$$\times \left|\frac{1}{\alpha(\epsilon)}(\sigma_n^i(X_{s-}^{n,\varepsilon})x - \sigma_i(X_{s-}^{\varepsilon})x)\right|_H^2 ds\,\nu(dx)$$
$$:= \sum_{k=1}^6 I_k^{n,\varepsilon}(t).$$

Itô's formula for $|X_t^{n,\varepsilon} - X_t^{\varepsilon}|_H^2$ in the continuous setting can be found in [18], Theorem 3.1, or in [21], Theorem 4.2.5. In the current pure jump case, Itô's formula can be seen through finite-dimensional projections of $X_t^{n,\varepsilon} - X_t^{\varepsilon}$ and a limiting procedure. The expression in (3.48) is a result of further use of the integration by parts formula for the real-valued semimartingales $e^{-\gamma\int_0^t(\|X_s^{n,\varepsilon}\|_V^2 + \|X_s^{\varepsilon}\|_V^2)\,ds}$, $t \ge 0$ and $|X_t^{n,\varepsilon} - X_t^{\varepsilon}|_H^2$, $t \ge 0$.

In view of assumption (2.6), we see that

$$\begin{aligned} &I_1^{n,\varepsilon}(t) + I_2^{n,\varepsilon}(t) + I_4^{n,\varepsilon}(t) \\ &\quad \le -(1-2\beta)\alpha_0 \int_0^t e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2 + \|X_u^{\varepsilon}\|_V^2)\,du}\|X_s^{n,\varepsilon} - X_s^{\varepsilon}\|_V^2\,ds, \end{aligned} \tag{3.49}$$

if $\gamma \ge 2C_1$, where C_1 is the constant appeared in (2.6).

Similar to the proofs of (3.6), (3.8), using Burkhölder's inequality, we obtain from (3.48), (3.49) that for $t \le T$,

$$\begin{aligned} &E\Big[\sup_{0\le s\le t} e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2 + \|X_u^{\varepsilon}\|_V^2)\,du}|X_s^{n,\varepsilon} - X_s^{\varepsilon}|_H^2\Big] \\ &\qquad + E\Big[\int_0^t e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2 + \|X_u^{\varepsilon}\|_V^2)\,du}\|X_s^{n,\varepsilon} - X_s^{\varepsilon}\|_V^2\,ds\Big] \\ &\quad \le \frac{1}{4}E\Big[\sup_{0\le s\le t} e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2 + \|X_u^{\varepsilon}\|_V^2)\,du}|X_s^{n,\varepsilon} - X_s^{\varepsilon}|_H^2\Big] \\ &\qquad + CE\Big[\int_0^t e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2 + \|X_u^{\varepsilon}\|_V^2)\,du}|X_s^{n,\varepsilon} - X_s^{\varepsilon}|_H^2\,ds\Big] \\ &\qquad + C\sum_{i=1}^m E\Big[\int_0^t \int_{|x|\le\varepsilon} e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2 + \|X_u^{\varepsilon}\|_V^2)\,du} \\ &\qquad \times \left|\frac{1}{\alpha(\epsilon)}(\sigma_n^i(X_s^{n,\varepsilon})x - \sigma_n^i(X_s^{\varepsilon})x)\right|_H^2 ds\,\nu(dx)\Big] \end{aligned} \tag{3.50}$$

$$
\begin{aligned}
&+C\sum_{i=1}^m E\biggl[\int_0^t\int_{|x|\le\varepsilon} e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2+\|X_u^\varepsilon\|_V^2)\,du}\\
&\qquad\times\biggl|\frac{1}{\alpha(\epsilon)}\bigl(\sigma_n^i(X_s^\varepsilon)x-\sigma_i(X_s^\varepsilon)x\bigr)\biggr|_H^2\,ds\,\nu(dx)\biggr]\\
\le{}&\frac14 E\Bigl[\sup_{0\le s\le t} e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2+\|X_u^\varepsilon\|_V^2)\,du}\bigl|X_s^{n,\varepsilon}-X_s^\varepsilon\bigr|_H^2\Bigr]\\
&+CE\biggl[\int_0^t e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2+\|X_u^\varepsilon\|_V^2)\,du}\bigl|X_s^{n,\varepsilon}-X_s^\varepsilon\bigr|_H^2\,ds\biggr]\\
&+C\sum_{i=1}^m E\biggl[\int_0^t\int_{|x|\le\varepsilon} e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2+\|X_u^\varepsilon\|_V^2)\,du}\\
&\qquad\times\biggl|\frac{1}{\alpha(\epsilon)}\bigl(\sigma_n^i(X_s^\varepsilon)x-\sigma_i(X_s^\varepsilon)x\bigr)\biggr|_H^2\,ds\,\nu(dx)\biggr],
\end{aligned}
$$

where the uniform Lipschitz constant of σ_n^i in (H.3)(i) has been used. Applying the Gronwall's inequality, we obtain

$$
\begin{aligned}
&E\Bigl[\sup_{0\le s\le t} e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2+\|X_u^\varepsilon\|_V^2)\,du}\bigl|X_s^{n,\varepsilon}-X_s^\varepsilon\bigr|_H^2\Bigr]\\
&\qquad+E\biggl[\int_0^t e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2+\|X_u^\varepsilon\|_V^2)\,du}\bigl\|X_s^{n,\varepsilon}-X_s^\varepsilon\bigr\|_V^2\,ds\biggr]\\
&\quad\le C\sum_{i=1}^m E\biggl[\int_0^T\bigl|\sigma_n^i(X_s^\varepsilon)-\sigma_i(X_s^\varepsilon)\bigr|_H^2\,ds\biggr].
\end{aligned}\tag{3.51}
$$

For any $M>0$, we have

$$
\begin{aligned}
&\sum_{i=1}^m E\biggl[\int_0^T\bigl|\sigma_n^i(X_s^\varepsilon)-\sigma_i(X_s^\varepsilon)\bigr|_H^2\,ds\biggr]\\
&\quad=\sum_{i=1}^m E\biggl[\int_0^T\bigl|\sigma_n^i(X_s^\varepsilon)-\sigma_i(X_s^\varepsilon)\bigr|_H^2\,ds,\ \sup_{0\le s\le T}|X_s^\varepsilon|_H\le M\biggr]\\
&\qquad+\sum_{i=1}^m E\biggl[\int_0^T\bigl|\sigma_n^i(X_s^\varepsilon)-\sigma_i(X_s^\varepsilon)\bigr|_H^2\,ds,\ \sup_{0\le s\le T}|X_s^\varepsilon|_H> M\biggr]\\
&\quad\le T\sum_{i=1}^m\sup_{|z|\le M}\bigl|\sigma_n^i(z)-\sigma_i(z)\bigr|_H^2+CT\frac1M\Bigl(1+E\Bigl[\sup_{0\le s\le T}|X_s^\varepsilon|_H^3\Bigr]\Bigr)\\
&\quad\le T\sum_{i=1}^m\sup_{|z|\le M}\bigl|\sigma_n^i(z)-\sigma_i(z)\bigr|_H^2+CT\frac1M,
\end{aligned}\tag{3.52}
$$

where (3.3) has been used. Since M can be chosen as large as we wish, together with (3.51) and (H.3)(ii) we deduce that

$$\lim_{n\to\infty}\sup_{\varepsilon} E\Big[\sup_{0\le s\le T} e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2+\|X_u^{\varepsilon}\|_V^2)\,du}\big|X_s^{n,\varepsilon}-X_s^{\varepsilon}\big|_H^2\Big] = 0. \tag{3.53}$$

For any given $\delta_1>0$, in view of (3.46), (3.47), we can choose a positive constant M_1 such that

$$\begin{aligned}\sup_{n,\varepsilon} P\Big(&\sup_{0\le t\le T}\big|X_t^{n,\varepsilon}-X_t^{\varepsilon}\big|_H>\delta, \int_0^T(\|X_s^{n,\varepsilon}\|_V^2+\|X_s^{\varepsilon}\|_V^2)\,ds>M_1\Big)\\ &\le \sup_{n,\varepsilon} P\Big(\int_0^T(\|X_s^{n,\varepsilon}\|_V^2+\|X_s^{\varepsilon}\|_V^2)\,ds>M_1\Big)\le\frac{\delta_1}{2}.\end{aligned}\tag{3.54}$$

On the other hand,

$$\begin{aligned}&\sup_{\varepsilon} P\Big(\sup_{0\le t\le T}\big|X_t^{n,\varepsilon}-X_t^{\varepsilon}\big|_H^2>\delta, \int_0^T(\|X_s^{n,\varepsilon}\|_V^2+\|X_s^{\varepsilon}\|_V^2)\,ds\le M_1\Big)\\ &\quad\le \sup_{\varepsilon} P\Big(\sup_{0\le s\le T} e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2+\|X_u^{\varepsilon}\|_V^2)\,du}\big|X_s^{n,\varepsilon}-X_s^{\varepsilon}\big|_H^2\ge e^{-\gamma M_1}\delta^2\Big)\\ &\quad\le e^{\gamma M_1}\frac{1}{\delta^2}\sup_{\varepsilon} E\Big[\sup_{0\le s\le T} e^{-\gamma\int_0^s(\|X_u^{n,\varepsilon}\|_V^2+\|X_u^{\varepsilon}\|_V^2)\,du}\big|X_s^{n,\varepsilon}-X_s^{\varepsilon}\big|_H^2\Big].\end{aligned}\tag{3.55}$$

It follows from (3.53) and (3.55) that there exists $N>0$ such that for $n\ge N$,

$$\sup_{n,\varepsilon} P\Big(\sup_{0\le t\le T}\big|X_t^{n,\varepsilon}-X_t^{\varepsilon}\big|_H>\delta, \int_0^T(\|X_s^{n,\varepsilon}\|_V^2+\|X_s^{\varepsilon}\|_V^2)\,ds\le M_1\Big) \le\frac{\delta_1}{2}. \tag{3.56}$$

Combining (3.54) and (3.56) together yields (3.44).

Finally, we prove that μ^ε converges to μ. Let μ_n^ε, μ_n denote, respectively, the laws of $X^{n,\varepsilon}$ and X^n. Let G be a bounded, uniformly continuous function on $E:=D([0,T],H)$. For any $n\ge1$, we write

$$\begin{aligned}&\int_E G(w)\mu^\varepsilon(dw)-\int_E G(w)\mu(dw)\\ &\quad=\int_E G(w)\mu^\varepsilon(dw)-\int_E G(w)\mu_n^\varepsilon(dw)+\int_E G(w)\mu_n^\varepsilon(dw)\\ &\qquad-\int_E G(w)\mu_n(dw)+\int_E G(w)\mu_n(dw)-\int_E G(w)\mu(dw)\\ &\quad=E\big[G(X^\varepsilon)-G(X^{n,\varepsilon})\big]+\Big(\int_E G(w)\mu_n^\varepsilon(dw)-\int_E G(w)\mu_n(dw)\Big)\\ &\qquad+E\big[G(X^n)-G(X)\big].\end{aligned}\tag{3.57}$$

Give any $\delta > 0$. Since G is uniformly continuous, there exists $\delta_1 > 0$ such that

$$\left|E\Big[\big(G(X^{\varepsilon}) - G(X^{n,\varepsilon})\big), \sup_{0\le s\le T}|X_s^{n,\varepsilon} - X_s^{\varepsilon}|_H \le \delta_1\Big]\right| \le \frac{\delta}{4} \tag{3.58}$$

for all $n \ge 1$, $\varepsilon > 0$. In view of (3.44) and (3.45), there exists N_1,

$$\begin{aligned}&\sup_{\varepsilon}\left|E\Big[\big(G(X^{\varepsilon}) - G(X^{N_1,\varepsilon})\big), \sup_{0\le s\le T}|X_s^{N_1,\varepsilon} - X_s^{\varepsilon}|_H > \delta_1\Big]\right|\\ &\quad\le C\sup_{\varepsilon} P\Big(\sup_{0\le s\le T}|X_s^{N_1,\varepsilon} - X_s^{\varepsilon}|_H > \delta_1\Big) \le \frac{\delta}{4},\end{aligned} \tag{3.59}$$

and

$$\big|E\big[\big(G(X^{N_1}) - G(X)\big)\big]\big| \le \frac{\delta}{4}. \tag{3.60}$$

On the other hand, by Theorem 3.6, there exists $\varepsilon_1 > 0$ such that for $\varepsilon \le \varepsilon_1$,

$$\left|\int_E G(w)\mu_{N_1}^{\varepsilon}(dw) - \int_E G(w)\mu_{N_1}(dw)\right| \le \frac{\delta}{4}. \tag{3.61}$$

Putting (3.57)–(3.60) together we obtain that for $\varepsilon \le \varepsilon_1$,

$$\left|\int_E G(w)\mu^{\varepsilon}(dw) - \int_E G(w)\mu(dw)\right| \le \delta.$$

Since $\delta > 0$ is arbitrarily small, we deduce that

$$\lim_{\varepsilon\to 0}\int_E G(w)\mu^{\varepsilon}(dw) = \int_E G(w)\mu(dw)$$

completing the proof of the theorem. □

EXAMPLE 3.8. Approximations of stochastic Burgers equations.

Consider the stochastic Burgers equations on [0, 1]:

$$du(t,\xi) = \frac{\partial^2}{\partial\xi^2}u(t,\xi)\,dt + \frac{1}{2}\frac{\partial}{\partial\xi}[u^2(t,\xi)]\,dt + \sum_{i=1}^m \sigma_i\big(u(t,\xi)\big)\,dB_t^i, \tag{3.62}$$

$$u(t,0) = u(t,1) = 0, \qquad t > 0, \tag{3.63}$$

$$du^{\varepsilon}(t,\xi) = \frac{\partial^2}{\partial\xi^2}u^{\varepsilon}(t,\xi)\,dt + \frac{1}{2}\frac{\partial}{\partial\xi}\big[\big(u^{\varepsilon}\big)^2(t,\xi)\big]\,dt \tag{3.64}$$

$$+\frac{1}{\alpha(\epsilon)}\sum_{i=1}^m\int_{|x|\le\varepsilon}\sigma_i\big(u^{\varepsilon}(t-,\xi)\big)x\tilde{N}_i(dt,dx), \tag{3.65}$$

$$u^{\varepsilon}(t,0) = u^{\varepsilon}(t,1) = 0, \qquad t > 0, \tag{3.66}$$

where $\sigma_i(\cdot)$, $i = 1,\dots,m$ is a Lipschitz continuous functions with $\sigma_i(0) = 0$.

Let $V = H_0^1(0,1)$ with the norm

$$\|v\|_V := \left(\int_0^1 \left(\frac{\partial u(\xi)}{\partial \xi}\right)^2 d\xi\right)^{1/2} = \|v\|.$$

Let $H := L^2(0,1)$ be the L^2-space with inner product $(\cdot)$.

Set

$$Au = -\frac{\partial^2}{\partial \xi^2} u(\xi) \qquad \forall u \in D(A) = H^2(0,1) \cap V.$$

Define for $k \geq 1$,

$$e_k(\xi) = \sqrt{2}\sin(k\pi\xi), \qquad \xi \in [0,1].$$

Then $e_k, k \geq 1$ are eigenvectors of the operator A with eigenvalues $\lambda_k = \pi^2 k^2$, which forms an orthonormal basis of the Hilbert space H. For $u \in V$, define

$$B(u) := u(\xi)\frac{\partial}{\partial \xi} u(\xi), \qquad \sigma(u) := \sigma\big(u(\xi)\big).$$

Here, $B(u) \in H$ because $\|u\|_{L^\infty} < \infty$ for $u \in V$. By the Lipschitz continuity of σ_i, it is easily seen that

$$\big\|\sigma_i(u)\big\|_V \leq C\big(1 + \|u\|_V\big), \qquad i = 1, \ldots, m, \tag{3.67}$$

hence (H.3)$'$ holds. Now let us show that $B(u)$ satisfies condition (H.2). First, (H.2)(i),$\langle B(u), u\rangle = 0$, is well known (see, e.g., [8]). Note that $\bar{e}_k = \frac{1}{\sqrt{\lambda_k}} e_k, k \geq 1$ forms an orthonomal basis of V. Recall that any element $l \in V^*$ can be identified through the Riesz representation theorem as an element $\bar{l}$ in V, and moreover,

$$\|l\|_{V^*}^2 = \|\bar{l}\|_V^2 = \sum_{k=1}^\infty \langle \bar{l}, \bar{e}_k\rangle^2 = \sum_{k=1}^\infty \big(l(\bar{e}_k)\big)^2.$$

Thus, for $u \in V$, we have

$$\begin{aligned}
\big\|B(u)\big\|_{V^*}^2 &= \sum_{k=1}^\infty \big(B(u)(\bar{e}_k)\big)^2 \\
&= \sum_{k=1}^\infty \left(\frac{1}{2}\frac{1}{\sqrt{\lambda_k}} \int_0^1 \frac{\partial}{\partial \xi}[u^2(\xi)] e_k(\xi)\, d\xi\right)^2 \\
&= \sum_{k=1}^\infty \left(\frac{1}{2}\frac{1}{\sqrt{\lambda_k}} \int_0^1 u^2(\xi) \frac{\partial}{\partial \xi} e_k(\xi)\, d\xi\right)^2 \\
&= \sum_{k=1}^\infty \left(\frac{1}{2} \int_0^1 u(\xi)^2 \sqrt{2}\cos(k\pi\xi)\, d\xi\right)^2 \\
&\leq C\int_0^1 u^\varepsilon(s,\xi)^4\, d\xi = C|u|_{L^4}^4,
\end{aligned} \tag{3.68}$$

where we have used the fact that $\{\sqrt{2}\cos(k\pi\xi); k \geq 1\}$ also forms an orthonormal system of $L^2(0,1)$. By the Sobolev embedding theorem (see, e.g., Theorem 6, Chapter 5 in [10]), we have

$$L^4(0,1) \subset H^{1/4}(0,1),$$

where $H^{1/4}(0,1)$ is the usual Sobolev space of order $\frac{1}{4}$. Combing the above embedding theorem with the following well-known interpolation inequality (see, e.g., Section 4.3 in [23])

$$\|u\|_{H^{1/4}} \leq C|u|_H^{3/4}\|u\|_V^{1/4}, \tag{3.69}$$

we obtain from (3.68) that

$$\|B(u)\|_{V^*} \leq C|u|_H^{3/2}\|u\|_V^{1/2}$$

proving (H.2)(iii) with $\gamma = \frac{1}{2}$. Finally, we will check (H.2)(ii). Let $u, v \in V$. We have

$$\begin{aligned}
\langle B(u) - B(v), u - v\rangle &= \frac{1}{2}\int_0^1 \frac{\partial}{\partial\xi}\big[u^2(\xi) - v^2(\xi)\big]\big(u(\xi) - v(\xi)\big)\,d\xi \\
&= -\frac{1}{2}\int_0^1 \big(u^2(\xi) - v^2(\xi)\big)\frac{\partial}{\partial\xi}\big(u(\xi) - v(\xi)\big)\,d\xi \\
&\leq \frac{1}{2}\int_0^1 \left(\frac{\partial}{\partial\xi}\big(u(\xi) - v(\xi)\big)\right)^2 d\xi \\
&\quad + C\int_0^1 \big(u(\xi) - v(\xi)\big)^2\big(u(\xi) + v(\xi)\big)^2\,d\xi \\
&\leq \frac{1}{2}\|u - v\|_V^2 + C|u - v|_H^2\big(\|u\|_\infty^2 + \|v\|_\infty^2\big) \\
&\leq \frac{1}{2}\|u - v\|_V^2 + C|u - v|_H^2\big(\|u\|_V^2 + \|V\|_V^2\big),
\end{aligned} \tag{3.70}$$

which is (H.2)(ii).

Now we can apply Theorem 3.7 to obtain the following convergence of the solutions of stochastic Burgers equations.

THEOREM 3.9. *Let u^ε, u be solutions to the stochastic Burgers equations* (3.64) *and* (3.62). *Then u^ε converges weakly to u in the space $D([0,T]; H)$.*

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