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Exit times for integrated random walks[1]

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Abstract. We consider a centered random walk with finite variance and investigate the asymptotic behaviour of the probability that the area under this walk remains positive up to a large time n. Assuming that the moment of order $2+\delta$ is finite, we show that the exact asymptotics for this probability is $n^{-1/4}$. To show this asymptotics we develop a discrete potential theory for integrated random walks.

Résumé. Nous considérons une marche aléatoire centrée de variance finie et étudions le comportement asymptotique de la probabilité que l'aire sous la marche reste positive jusqu'à un grand temps n. Si le moment d'ordre $2+\delta$ est fini, nous montrons que cette probabilité décroit comme $n^{-1/4}$. Pour prouver ce comportement asymptotique, nous développons une théorie du potentiel discrète pour des marches aléatoires intégrées.

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1. Introduction, main results and discussion

1.1. *Background and motivation*

Let $X, X_1, X_2, \dots$ be independent identically distributed random variables with $\mathbf{E}[X]=0$. For every starting point (x,y) define

$$S_n = y + X_1 + X_2 + \cdots + X_n, \quad n \ge 0$$

and

$$S_n^{(2)} = x + S_1 + S_2 + \cdots + S_n = x + ny + nX_1 + (n-1)X_2 + \cdots + X_n.$$

Sinai [13] initiated the study of asymptotics of the probability of the event

$$A_n := \{S_k^{(2)} > 0 \text{ for all } k \le n \mid S_0 = S_0^{(2)} = 0\}.$$

Assuming that S_n is a simple symmetric random walk he showed that

$$C_1 n^{-1/4} \le \mathbf{P}(A_n) \le C_2 n^{-1/4}. \tag{1}$$

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The same bounds were obtained for some other special cases in [14].

Aurzada and Dereich [2] have shown that if $\mathbf{E}e^{\beta|X|} < \infty$ for some positive β then

$$C_* n^{-1/4} \log^{-4} n \le \mathbf{P}(A_n) \le C^* n^{-1/4} \log^4 n \tag{2}$$

with some positive constants C_* and C^*. The bounds in (2) are just a special case of the results in [2] for q-times integrated random walks and Levy processes. Dembo, Ding and Gao [5] have recently shown that (1) is valid for all random walks with finite second moment.

Exact asymptotics for $\mathbf{P}(A_n)$ are known only in some special cases. Vysotsky [15] has shown that if, in addition to the second moment assumption, S_n is either right-continuous or right-exponential then

$$\mathbf{P}(A_n) \sim C n^{-1/4}. \tag{3}$$

(Here and throughout $a_n \sim b_n$ means that $\frac{a_n}{b_n} \to 1$ as $n \to \infty$.)

It is natural to expect that (3) holds for all driftless random walks with finite variance.

If the second moment condition is replaced by the assumption that X belongs to the domain of normal attraction of the spectrally positive α-stable law with some $\alpha \in (1, 2]$, then (1) and (3) remain valid with $n^{-(\alpha-1)/2\alpha}$ instead of $n^{-1/4}$, see [5] and [15].

The methods used in the above mentioned papers are quite different. It is not clear what the most natural tool is for this problem. Here we propose another approach to this problem. More precisely, we develop a potential theory for integrated random walks, which allows one to determine the exact asymptotic behaviour of $\mathbf{P}(A_n)$. It can be seen as a continuation of our studies of exit times of multi-dimensional random walks, see [6,7].

It is clear that the sequence $\{S_n^{(2)}\}_{n\ge 1}$ is non-Markovian. This fact complicates the analysis of the integrated random walk. However, it is possible to obtain the Markovian property by increasing the dimension of the process. More precisely, we consider the process

$$Z_n := \big(S_n^{(2)}, S_n\big).$$

Then, the first time when $S_n^{(2)}$ is not positive coincides with the following exit time of Z_n

$$\tau := \min\{k \ge 1:\ Z_k \notin \mathbb{R}_+ \times \mathbb{R}\}.$$

In our recent paper [7] we suggested a method of studying random walks conditioned to stay in a cone. Similarly in the case of the integrated random walks we have a (quite simple) cone $\mathbb{R}_+ \times \mathbb{R}$, but the process Z_n is "really" Markov, i.e. the increments are not independent. We show that the method from [7] can be adapted to the case of Markov chain Z_n, and this adaptation allows one to find asymptotics of $\mathbf{P}_z(\tau > n)$ for every starting point $z = (x, y)$.

1.2. *Main result*

We start with results and notation for the integrated Brownian motion. This process is also known as the Kolmogorov diffusion.

Let B_t be a standard Brownian motion and consider a two-dimensional process $(\int_0^t B_s\,\mathrm{d}s, B_t)$. Since this process is Gaussian, one can obtain, by computing correlations, that the transition density of $(\int_0^t B_s\,\mathrm{d}s, B_t)$ is given by

$$p_t(x, y; u, v) = \frac{\sqrt{3}}{\pi t^2} \exp\left\{-\frac{6(u - x - ty)^2}{t^3} + \frac{6(u - x - ty)(v - y)}{t^2} - \frac{2(v - y)^2}{t}\right\}.$$

Let

$$\tau^{bm} := \min\left\{t > 0:\ x + yt + \int_0^t B_s\,\mathrm{d}s \le 0\right\}.$$

The behaviour of $(\int_0^t B_s\,\mathrm{d}s, B_t)$ killed upon $\mathbb{R}_+ \times \mathbb{R}$ was studied by many authors. Here we will follow a paper by Groeneboom, Jongbloed and Wellner [9], where one can also find a history of the subject and corresponding references. In particular they found the positive harmonic function for this process, which is given by the following relations:

$$h(x,y)=\begin{cases}(\frac{2}{9})^{1/6}\frac{y}{x^{1/6}}U(\frac{1}{6},\frac{4}{3},\frac{2y^3}{9x}), & y\ge 0,\\ -(\frac{2}{9})^{1/6}\frac{1}{6}\frac{y}{x^{1/6}}\mathrm{e}^{2y^3/9x}U(\frac{7}{6},\frac{4}{3},-\frac{2y^3}{9x}), & y<0,\end{cases}\tag{4}$$

where U is the confluent hypergeometric function:

$$U(a,b,w)=\frac{\Gamma(1-b)}{\Gamma(a-b+1)}M(a,b,w)+\frac{\Gamma(b-1)}{\Gamma(a)}w^{1-b}M(a-b+1,2-b,w)$$

with

$$M(a,b,w)=\sum_{n=0}^{\infty}\frac{w^n}{n!}\prod_{j=0}^{n-1}\left(\frac{a+j}{b+j}\right).$$

Function $h(x,y)$ is harmonic in the sense that $\mathcal{D}h=0$ on $\mathbb{R}_+\times\mathbb{R}$, where $\mathcal{D}=y\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial^2}{\partial y^2}$ is the generator of $(\int_0^t B_s\,\mathrm{d}s, B_t)$. Using the explicit density of $\mathbf{P}_{(0,1)}(\tau^{bm}>t)$ found in [12], they derived asymptotics

$$\mathbf{P}_{(x,y)}\bigl(\tau^{bm}>t\bigr)\sim\varkappa\frac{h(x,y)}{t^{1/4}},\quad t\to\infty,\tag{5}$$

where $\varkappa=\frac{3\Gamma(1/4)}{2^{3/4}\pi^{3/2}}$.

The function h defined in (4) is harmonic for the killed integrated Brownian motion, that is,

$$\mathbf{E}_{(x,y)}\left[h\left(\int_0^t B_s\,\mathrm{d}s, B_t\right);\tau^{bm}>t\right]=h(x,y),\quad x\in\mathbb{R}_+, y\in\mathbb{R}, t>0.$$

In other words, $h(\int_0^t B_s\,\mathrm{d}s, B_t)1\{\tau^{bm}>t\}$ is a non-negative martingale.

Our approach relies on the construction of a harmonic function for the killed integrated random walk. More precisely, we are looking for a positive function V such that the sequence $V(Z_n)1\{\tau>n\}$ is a martingale or, equivalently,

$$V(z)=\mathbf{E}_z\bigl[V(Z_1);\tau>1\bigr],\quad z\in\mathbb{R}_+\times\mathbb{R}.\tag{6}$$

Our main result is the following theorem.

Theorem 1. *Assume that* $\mathbf{E}X=0$, $\mathbf{E}[X^2]=1$ *and* $\mathbf{E}|X|^{2+\delta}<\infty$ *for some* $\delta>0$. *Then the function*

$$V(z):=\lim_{n\to\infty}\mathbf{E}_z\bigl[h(Z_n);\tau>n\bigr]$$

is well-defined, strictly positive on

$$K_+:=\bigl\{z\in\mathbb{R}_+\times\mathbb{R}\colon \mathbf{P}_z(Z_n\in\mathbb{R}_+\times\mathbb{R}_+,\tau>n)>0 \text{ for some } n\ge 0\bigr\}$$

and satisfies (6), *i.e. it is harmonic for the killed integrated random walk. Moreover,*

$$\mathbf{P}_z(\tau>n)\sim\varkappa\frac{V(z)}{n^{1/4}}\quad \text{as } n\to\infty\tag{7}$$

and

$$\mathbf{P}_z\left(\left(\frac{S_n^{(2)}}{n^{3/2}},\frac{S_n}{n^{1/2}}\right)\in\cdot\,\middle|\,\tau>n\right)\to\mu\quad\text{weakly,}\tag{8}$$

where μ has density

$$\overline{h}(x,y)=\frac{2^{9/4}\sqrt{\pi}}{\sqrt{3}\Gamma(1/4)}\int_0^1\int_0^\infty w^{3/2}s^{-1/2}\exp\{-2w^2/s\}q_{1-s}(x,-y;0,-w)\,\mathrm{d}s\,\mathrm{d}w$$

and

$$q_t(x,y;u,v)=p_t(x,y;u,v)-p_t(x,y;u,-v).$$

Remark 2. *One should notice that $\overline{h}(x,y)=\varkappa^{-1}\overline{h}(1,x,-y)$, where $\overline{h}(t,x,y)$ is defined in* (2.24) *of* [9].

From (7) and the total probability formula we obtain

Corollary 3. *For every random walk satisfying the conditions of Theorem* 1

$$\mathbf{P}(A_n)\sim\frac{C}{n^{1/4}}$$

with

$$C=\varkappa\mathbf{E}\big[V\big((X,X)\big),X>0\big].$$

It should be noted that the function V constructed in Theorem 1 is very hard to compute. We did not find any example, where one can give an explicit expression for V. For numerical calculations, e.g. Monte-Carlo simulations, the definition in Theorem 1 is not very helpful, since it contains a limit. For that reason we derive an alternative representation for the harmonic function.

As the function h is defined only for $z\in\mathbb{R}_+\times\mathbb{R}$, we extend it to $\mathbb{R}^2$ by putting $h=0$ outside $\mathbb{R}_+\times\mathbb{R}$ and introduce a corrector function

$$f(z)=\mathbf{E}_z h\big(Z(1)\big)-h(z),\quad z\in\mathbb{R}^2. \tag{9}$$

This function is well defined since we have extended h to the whole plane.

Proposition 4. *Under the assumptions of Theorem* 1,

$$V(z)=h(z)+\mathbf{E}_z\sum_{k=0}^{\tau-1}f(Z_k),\quad z\in\mathbb{R}_+\times\mathbb{R}. \tag{10}$$

1.3. *Local asymptotics for integrated random walks*

Caravenna and Deuschel [4] have proven a local limit theorem for Z_n under the assumption that the distribution of X is absolutely continuous. Using similar arguments one can show that if X is $\mathbb{Z}$-valued and aperiodic then

$$\sup_{\tilde{z}}\left|n^2\mathbf{P}_z(Z_n=\tilde{z})-p_1\left(0,0;\frac{\tilde{x}}{n^{3/2}},\frac{\tilde{y}}{n^{1/2}}\right)\right|\to 0. \tag{11}$$

Combining this unconditioned local limit theorem with (8) one can derive a conditional local limit theorem:

$$\sup_{\tilde{z}}\left|n^{2+1/4}\mathbf{P}_z(Z_n=\tilde{z},\tau>n)-\varkappa V(z)h\left(\frac{\tilde{x}}{n^{3/2}},\frac{\tilde{y}}{n^{1/2}}\right)p_1\left(0,0;\frac{\tilde{x}}{n^{3/2}},\frac{\tilde{y}}{n^{1/2}}\right)\right|\to 0. \tag{12}$$

Furthermore, for every fixed $\tilde{z}\in K_+$,

$$\lim_{n\to\infty}n^{2+1/2}\mathbf{P}_z(Z_n=\tilde{z},\tau>n)=V(z)V'(z) \tag{13}$$

with some positive function V'.

The proof of (12) and (13) repeats virtually word by word the proof of local asymptotics in [7], see Section 1.4 and Section 6 there. For this reason we do not give a proof of these statements.

Having (13) one can easily show that

$$\mathbf{P}_0(A_n|Z_{n+2}=0)\sim\frac{C}{n^{1/2}}$$

with some positive constant C. A slightly weaker form of this relation was conjectured by Caravenna and Deuschel [4], Eq. (1.22).

Aurzada, Dereich and Lifshits [3] have recently obtained lower and upper bounds for the integrated simple random walk,

$$cn^{-1/2}\le\mathbf{P}_0\big(S_1^{(2)}\ge 0,\dots,S_{4n}^{(2)}\ge 0|S_{4n}=0,S_{4n}^{(2)}=0\big)\le Cn^{-1/2}.$$

1.4. *Organisation of the paper*

In [7] we have suggested a method of investigating exit times from cones for random walks. In the present paper we have a Markov chain instead of a random walk with independent increments. But it turns out that this fact is not important, and the method from [7] works also for Markov processes.

The first step consists in construction of a harmonic function $V(z)$. As in [7] we start from the harmonic function for the corresponding limiting process. Obviously,

$$\left(\frac{S_{[nt]}^{(2)}}{n^{3/2}},\frac{S_{[nt]}}{n^{1/2}}\right)\Rightarrow\left(\int_0^t B_s\,\mathrm{d}s,B_t\right).$$

We then define for every $z\in\mathbb{R}_+\times\mathbb{R}$

$$V(z)=\lim_{n\to\infty}\mathbf{E}_z\big[h(Z_n),\tau>n\big].\tag{14}$$

The justification of this formal definition is the most technical part of our approach. It is worth mentioning that we cannot just repeat the proof from [7]. There we used a certain a-priori information on the behaviour of first exit times. (It was some moment inequalities, which were already known in the literature.) For integrated random walks we do not have such information and, therefore, should find an alternative way of justification of (14). Here we perform the following steps:

(1) We show that if Z_n stays in $\mathbb{R}_+\times\mathbb{R}$ for a long time, then its first coordinate becomes large quite quickly, see Lemma 12.
(2) Furthermore, for starting points z with big first coordinate we derive recursive upper and lower estimates for $\mathbf{E}_z[h(Z_n),\tau>n]$, see Lemma 11. This step requires estimates for $h(x,y)$ and its derivatives, and for the corrector function $f(x,y)$. These bounds are obtained in Section 2.1.
(3) Finally, using recursion we show the existence of $\lim_{n\to\infty}\mathbf{E}_z[h(Z_n),\tau>n]$, see Proposition 14.

Having constructed $V(x)$ we follow our approach in [6,7]. More precisely, we apply the KMT-coupling to obtain the asymptotics for τ. (This explains our moment condition in Theorem 1.) It is worth mentioning that we apply this strong approximation at a stopping time, where the first coordinate of Z_n becomes sufficiently large. Note that, according to Lemma 12, this stopping time is relatively small. The harmonic function is required to come from this stopping time back to the origin. This final step is performed in Lemmas 20 and 21.

For integrated random walks a strong approximation was used in Aurzada and Dereich [2]. They apply KMT-coupling at a deterministic moment to obtain (2). This formula shows that a direct, without use of potential theory, application of coupling produces superfluous logarithmic terms even under the exponential moment assumption.

1.5. *Conclusion*

In our previous works [6,7] we showed that Brownian asymptotics for exit times can be transferred to exit times for multidimensional random walks. In the present work we consider an integrated random walk which can be viewed as a two-dimensional Markov chain. We study exit times from a half-space and transfer the corresponding results for the Kolmogorov diffusion. These examples make plausible the following hypothesis.

Let X_n be a Markov chain, D be an unbounded domain and $\tau_D := \min\{n \ge 1\colon\ X_n \notin D\}$. Assume that this Markov chain, properly scaled, converges as a process to a diffusion $Y_t, t \ge 0$. Assume also that the exit time of this diffusion $T_D := \min\{t \ge 0\colon\ Y_t \notin D\}$ has the following asymptotics

$$\mathbf{P}_y(T_D > t) \sim \frac{h(y)}{t^p}, \quad t \to \infty,$$

where $h(y)$ is the corresponding harmonic function of the killed diffusion $Y_{t\wedge T_D}$. Then, there exists a positive harmonic function $V(x)$ for the killed Markov chain $X_{n\wedge\tau_D}$ such that

$$\mathbf{P}_x(\tau_D > n) \sim \frac{V(x)}{n^p}, \quad n \to \infty.$$

Naturally, this general theorem will require some moment assumptions and some assumptions on the smoothness of the unbounded domain D. Since we have convergence of processes the domain D should have certain scaling properties. Hence it seems natural for the domain D to be a cone, at least asymptotically.

2. Construction of harmonic function

2.1. *Preliminary estimates for $h(x, y)$ and $f(x, y)$*

The main result of this subsection is an upper bound for f which is stated in Lemma 7. In the proof we use the Taylor formula, and for that reason we need information on h and its derivatives which are proven in following two lemmas.

Lemma 5. *Function h has the following partial derivatives,*

$$\frac{\partial^i h(x,y)}{\partial x^i} = \begin{cases} C_i(\frac{2}{9})^{1/6}\frac{y}{x^{1/6+i}}U(\frac{1}{6}+i, \frac{4}{3}, \frac{2y^3}{9x}), & x \ge 0, y \ge 0, \\ -(\frac{2}{9})^{1/6}\frac{1}{6}\frac{y}{x^{1/6+i}}e^{2y^3/9x}U(\frac{7}{6}-i, \frac{4}{3}, -\frac{2y^3}{9x}), & x \ge 0, y < 0 \end{cases} \tag{15}$$

for $i \ge 0$ and

$$\frac{\partial^{i+1} h(x,y)}{\partial x^i\,\partial y} = \begin{cases} \frac{-3}{i-1/6}C_i(\frac{2}{9})^{1/6}\frac{1}{x^{1/6+i}}U(\frac{1}{6}+i, \frac{1}{3}, \frac{2y^3}{9x}), & x \ge 0, y \ge 0, \\ (\frac{2}{9})^{1/6}\frac{1}{2}\frac{1}{x^{1/6+i}}e^{2y^3/9x}U(\frac{7}{6}-i-1, \frac{1}{3}, -\frac{2y^3}{9x}), & x \ge 0, y < 0. \end{cases} \tag{16}$$

Here, $C_0 = 1$ and $C_{i+1} = -C_i(i+1/6)(i-1/6)$ for $i \ge 0$.

Proof. We will prove (15) by induction. The base of induction $i = 0$ corresponds to the definition of h. Now suppose that (15) is true for i and prove it for $i+1$.

Consider first $y \ge 0$. By the induction hypothesis,

$$\begin{aligned}\frac{\partial^{i+1} h(x,y)}{\partial x^{i+1}} &= C_i\left(\frac{2}{9}\right)^{1/6}\frac{\partial}{\partial x}\left[\frac{y}{x^{1/6+i}}U\left(\frac{1}{6}+i, \frac{4}{3}, \frac{2y^3}{9x}\right)\right] \\ &= -C_i\left(\frac{2}{9}\right)^{1/6}\frac{y}{x^{1/6+i+1}}\left(\left(\frac{1}{6}+i\right)U\left(\frac{1}{6}+i, \frac{4}{3}, \frac{2y^3}{9x}\right)\right.\end{aligned}$$

$$+ \frac{2y^3}{9x}U'\left(\frac{1}{6}+i, \frac{4}{3}, \frac{2y^3}{9x}\right)\Bigg)$$

$$= -C_i\left(\frac{2}{9}\right)^{1/6}\frac{y}{x^{1/6+i+1}}\left(i+\frac{1}{6}\right)\left(i-\frac{1}{6}\right)U\left(\frac{1}{6}+i+1, \frac{4}{3}, \frac{2y^3}{9x}\right),$$

where we applied (13.4.23) of [1] in the last step. Recalling the definition of C_{i+1} we see that (15) holds for $i+1$ and positive y.

Consider second the case $y < 0$. By the induction hypothesis,

$$\frac{\partial^{i+1}h(x,y)}{\partial x^{i+1}} = -\left(\frac{2}{9}\right)^{1/6}\frac{1}{6}\frac{\partial}{\partial x}\left[e^{2y^3/9x}\frac{y}{x^{1/6+i}}U\left(\frac{7}{6}-i, \frac{4}{3}, -\frac{2y^3}{9x}\right)\right]$$

$$= -\left(\frac{2}{9}\right)^{1/6}\frac{1}{6}\frac{y}{x^{1/6+i+1}}e^{2y^3/9x}\Bigg(-\left(\frac{1}{6}+i\right)U\left(\frac{7}{6}-i, \frac{4}{3}, -\frac{2y^3}{9x}\right)$$

$$- \frac{2y^3}{9x}U\left(\frac{7}{6}-i, \frac{4}{3}, -\frac{2y^3}{9x}\right) + \frac{2y^3}{9x}U'\left(\frac{7}{6}-i, \frac{4}{3}, -\frac{2y^3}{9x}\right)\Bigg)$$

$$= -\left(\frac{2}{9}\right)^{1/6}\frac{1}{6}\frac{y}{x^{1/6+i+1}}e^{2y^3/9x}U\left(\frac{7}{6}-i-1, \frac{4}{3}, -\frac{2y^3}{9x}\right),$$

where we applied (13.4.26) of [1] in the final step. This proves (15) for negative values of y.

To prove (16) we differentiate expressions for $\frac{\partial^i h}{\partial x^i}$ we just obtained. First we consider the case $y \ge 0$. Using (15),

$$\frac{\partial^{i+1}h(x,y)}{\partial x^i\,\partial y} = C_i\left(\frac{2}{9}\right)^{1/6}\frac{\partial}{\partial y}\left[\frac{y}{x^{1/6+i}}U\left(\frac{1}{6}+i, \frac{4}{3}, \frac{2y^3}{9x}\right)\right]$$

$$= C_i\left(\frac{2}{9}\right)^{1/6}\frac{-3}{x^{1/6+i}}\Bigg(-\frac{1}{3}U\left(\frac{1}{6}+i, \frac{4}{3}, \frac{2y^3}{9x}\right)$$

$$- \frac{2y^3}{9x}U'\left(\frac{1}{6}+i, \frac{4}{3}, \frac{2y^3}{9x}\right)\Bigg)$$

$$= C_i\left(\frac{2}{9}\right)^{1/6}\frac{-3}{(i-1/6)x^{1/6+i}}U\left(\frac{1}{6}+i, \frac{1}{3}, \frac{2y^3}{9x}\right),$$

this time we used (13.4.24) of [1]. Finally, for $y < 0$,

$$\frac{\partial^{i+1}h(x,y)}{\partial x^i\,\partial y} = -\left(\frac{2}{9}\right)^{1/6}\frac{1}{6}\frac{\partial}{\partial y}\left[\frac{y}{x^{1/6+i}}e^{2y^3/9x}U\left(\frac{7}{6}-i, \frac{4}{3}, -\frac{2y^3}{9x}\right)\right]$$

$$= -\left(\frac{2}{9}\right)^{1/6}\frac{1}{6}\frac{-3}{x^{1/6+i}}e^{2y^3/9x}\Bigg(\left(-\frac{1}{3}-\frac{2y^3}{9x}\right)U\left(\frac{7}{6}-i, \frac{4}{3}, -\frac{2y^3}{9x}\right)$$

$$+ \frac{2y^3}{9x}U'\left(\frac{7}{6}-i, \frac{4}{3}, -\frac{2y^3}{9x}\right)\Bigg)$$

$$= \left(\frac{2}{9}\right)^{1/6}\frac{1}{2}\frac{1}{x^{1/6+i}}e^{2y^3/9x}U\left(\frac{7}{6}-i-1, \frac{1}{3}, -\frac{2y^3}{9x}\right),$$

where we used (13.4.27) of [1] in the last line. □

Let

$$\alpha(x,y) = \max\left(|x|^{1/3}, |y|\right). \tag{17}$$

Lemma 6. *There exist positive constants c and C such that*

$$c\sqrt{\alpha(z)} \le h(z) \le C\sqrt{\alpha(z)}, \quad z \in \mathbb{R}^2_+. \tag{18}$$

Furthermore, the upper bound is valid for all z. Function h is at least C^3 continuous except the half-line $\{z\colon x = 0, y \ge 0\}$. Furthermore, for all $i + j \le 3$ and all $(x, y) \in \mathbb{R}^2 \setminus \{z\colon x = 0, y \ge 0\}$,

$$\left|\frac{\partial^{i+j} h}{\partial x^i\, \partial y^j}(x, y)\right| \le C\alpha(x, y)^{1/2-3i-j}.$$

Here and throughout the text we denote as C, c some generic constants.

Proof of Lemma 6. The estimates will follow from Lemma 5 and the following properties of the confluent hypergeometric function, see (13.1.8), (13.5.8) and (13.5.10) of [1],

$$U(a, b, s) \sim s^{-a}, \quad s \to \infty, \tag{19}$$

$$U(a, b, s) \sim \frac{\Gamma(b-1)}{\Gamma(a)} s^{1-b}, \quad s \to 0, b \in (1, 2), \tag{20}$$

$$U(a, b, s) \sim \frac{\Gamma(1-b)}{\Gamma(1+a-b)}, \quad s \to 0, b \in (0, 1). \tag{21}$$

Asymptotics (19), (20) and the definition of h immediately imply (18).

Function h is obviously infinitely differentiable when $x < 0$ or $x > 0$. The only problematic zone is $x = 0$, $y \le 0$. (Recall that we extend h through this half-line.) Since $h(x, y) = 0$ for $x < 0$, $y < 0$ all derivatives are equal to 0. Using the expressions for derivatives found in Lemma 5 one can immediately see that derivatives of $h(x, y)$ go to 0 as $x \to 0$ for $y < 0$ thanks to the exponent $\mathrm{e}^{2y^3/9x}$.

We continue with partial derivatives with respect to x, that is, $j = 0$, $i = 1, 2, 3$. First, using (15) and (19) for sufficiently large $A > 0$ and $y^3/x > A$,

$$\left|\frac{\partial^i h(x, y)}{\partial x^i}\right| \le C \frac{y}{x^{1/6+i}} \left(\frac{2y^3}{9x}\right)^{-1/6-i} \le C y^{1/2-3i}, \quad i \ge 0.$$

For $y < 0$, sufficiently large A and $-y^3/9x > A$, the same inequality hold since $\mathrm{e}^{2y^3/9x}$ is decreasing much faster than any power function as $y^3/x \to -\infty$. Next, using (15) and (20) for sufficiently small $\varepsilon > 0$ and y: $|y|^3/x \le \varepsilon$,

$$\left|\frac{\partial^i h(x, y)}{\partial x^i}\right| \le C \frac{y}{x^{1/6+i}} \left(\frac{2y^3}{9x}\right)^{-1/3} \le C x^{1/6-i}.$$

Finally, when $y^3/x \in (\varepsilon, A)$,

$$\left|\frac{\partial^i h(x, y)}{\partial x^i}\right| \le C.$$

We can summarise this in one formula

$$\left|\frac{\partial^i h(x, y)}{\partial x^i}\right| \le C\alpha(x, y)^{1/2-3i},$$

where $\alpha(x, y)$ is defined in (17). This proves the statement.

The proof for $j = 1$, $i = 0, 1, 2$ goes the same line and we omit it.

For $j = 2$ and $j = 3$ we shall use the fact that $h_{yy} + 2yh_x = 0$ (recall that h solves $\mathcal{D}h = 0$). Hence,

$$\left|h_{yy}(x, y)\right| \le C|y|\left|h_x(x, y)\right| \le C\alpha(x, y)\alpha(x, y)^{-2.5} \le C\alpha(x, y)^{-1.5}.$$

Next,

$$\left|h_{yyx}(x,y)\right| \le C|y|\left|h_{xx}(x,y)\right| \le C\alpha(x,y)\alpha(x,y)^{-5.5} \le C\alpha(x,y)^{-4.5}.$$

Finally,

$$\begin{aligned}\left|h_{yyy}(x,y)\right| &\le C\left|h_x(x,y)\right| + C|y|\left|h_{xy}(x,y)\right| \\ &\le C\alpha(x,y)^{-2.5} + C\alpha(x,y)\alpha(x,y)^{-3.5} \le C\alpha(x,y)^{-2.5}.\end{aligned}$$

The proof is complete. □

Next we derive an upper bound for $f(x,y)$.

Lemma 7. *Assume that* $\mathbf{E}X=0$, $\mathbf{E}X^2=1$ *and* $\mathbf{E}|X|^{2+\delta}<\infty$. *Then*

$$\left|f(x,y)\right| \le C\min\bigl(1,\alpha(x,y)^{-3/2-\delta}\bigr), \quad (x,y)\in\mathbb{R}_+\times\mathbb{R}.$$

Proof. Let $A \ge 2$ be fixed. Then for (x,y) such that $\alpha(x,y)\le A$ using the fact that function h is bounded on any compact we have $|f(x,y)|\le C$. In the rest of the proof we consider the case $\alpha(x,y)>A$.

According to Lemma 6 function h is at least C^3 smooth except the line $(x=0, y\ge 0)$. Then, for t: $|t|\le \frac{1}{2}\alpha(x,y)$, by the Taylor formula,

$$\begin{aligned}&\biggl|h(x+y+t,y+t)-h(x,y)\\ &\qquad -\left((y+t)h_x(x,y)+th_y(x,y)+\frac{1}{2}h_{xx}(x,y)(y+t)^2+h_{xy}(x,y)(y+t)t+\frac{1}{2}h_{yy}(x,y)t^2\right)\biggr|\\ &\quad\le \sum_{i+j=3}\max_{\theta:|\theta|\le(1/2)\alpha(x,y)}\left|\frac{\partial^{i+j}h(x+y+\theta,y+\theta)}{\partial x^i\,\partial y^j}(y+t)^i t^j\right| := r(x,y,t).\end{aligned}$$

To ensure that the Taylor formula is applicable we need to check that the set $\{(x+y+t,y+t)\colon |t|\le\frac{1}{2}\alpha(x,y)\}$ does not intersect with the half-line $\{x=0, y>0\}$, where the derivatives of the function $h(x,y)$ are discontinuous. First, if $y<0$ and $\alpha(x,y)=|y|$, then $y+t\le-\frac{1}{2}|y|$ for any t with $|t|\le\frac{1}{2}|y|$. Therefore $|y+t|\ge\frac{1}{2}A$ in this case. Second, if $y>0$ and $\alpha(x,y)=y$, then $x+y+t\ge x+y/2>\frac{1}{2}A$. Third, if $\alpha(x,y)=x^{1/3}$, then $|x+y+t|\ge|x|-\frac{3}{2}|x|^{1/3}\ge\frac{1}{2}A$ for all $A\ge 2$. This shows that the Taylor formula is valid.

Then,

$$\begin{aligned}&\bigl|\mathbf{E}h(x+y+X,y+X)-h(x,y)\bigr|\\ &\quad\le\left|\mathbf{E}\left[h(x+y+X,y+X)-h(x,y); |X|>\frac{1}{2}\alpha(x,y)\right]\right|\\ &\qquad+\left|\mathbf{E}\left[h(x+y+X,y+X)-h(x,y); |X|\le\frac{1}{2}\alpha(x,y)\right]\right|.\end{aligned}$$

We can estimate the second term in the right-hand side using the Taylor formula above,

$$\begin{aligned}&\left|\mathbf{E}\left[h(x+y+X,y+X)-h(x,y); |X|\le\frac{1}{2}\alpha(x,y)\right]\right|\\ &\quad\le\left|\mathbf{E}\left[(y+X)h_x(x,y)+Xh_y(x,y)+\frac{1}{2}h_{xx}(x,y)(y+X)^2\right.\right.\\ &\qquad\left.\left.+h_{xy}(x,y)(y+X)X+\frac{1}{2}h_{yy}(x,y)X^2\right]\right|\end{aligned}$$

$$
\begin{aligned}
&+\left|\mathbf{E}\left[(y+X)h_x(x,y)+Xh_y(x,y)+\frac{1}{2}h_{xx}(x,y)(y+X)^2\right.\right.\\
&\left.\left.+h_{xy}(x,y)(y+X)X+\frac{1}{2}h_{yy}(x,y)X^2;|X|>\frac{1}{2}\alpha(x,y)\right]\right|\\
&+\mathbf{E}\left[r(x,y,X);|X|\le\frac{1}{2}\alpha(x,y)\right]\\
:=\ &E_1(x,y)+E_2(x,y)+E_3(x,y).
\end{aligned}
$$

First, we can simplify the first term $E_1(x,y)$ using the assumption $\mathbf{E}X=0,\mathbf{E}X^2=1$. Then,

$$
E_1(x,y)=\left|yh_x(x,y)+\frac{1}{2}h_{yy}(x,y)+\frac{1}{2}h_{xx}(x,y)\big(y^2+1\big)+h_{xy}(x,y)\right|.
$$

Recalling that $yh_x+\frac{1}{2}h_{yy}=0$, we obtain

$$
E_1(x,y)=\left|\frac{1}{2}h_{xx}(x,y)\big(y^2+1\big)+h_{xy}(x,y)\right|.
$$

Applying Lemma 6, we finally get

$$
E_1(x,y)\le C\alpha(x,y)^{-5.5}\alpha(x,y)^2+C\alpha(x,y)^{-3.5}\le C\alpha(x,y)^{-3.5}. \tag{22}
$$

Second, noting that $|X|>|y|/2$ on the event $|X|>\frac{1}{2}\alpha(x,y)$ and applying the Chebyshev inequality, we obtain

$$
\begin{aligned}
E_2(x,y)&\le C\mathbf{E}\left[|X|\big(h_x(x,y)+h_y(x,y)\big)+X^2\big(h_{xx}(x,y)+h_{xy}(x,y)+h_{yy}(x,y)\big);|X|>\frac{1}{2}\alpha(x,y)\right]\\
&\le C\frac{|h_x(x,y)|+|h_y(x,y)|}{\alpha(x,y)^{1+\delta}}+C\frac{|h_{xx}(x,y)|+|h_{xy}(x,y)|+|h_{yy}(x,y)|}{\alpha(x,y)^{\delta}}.
\end{aligned}
$$

Applying Lemma 6, we obtain

$$
E_2(x,y)\le C\alpha(x,y)^{-1.5-\delta}. \tag{23}
$$

Third, applying Lemma 6 once again,

$$
\begin{aligned}
E_3(x,y)&\le C\max_{\theta:|\theta|\le(1/2)\alpha(x,y)}\big|h_{xxx}(x+y+\theta,y+\theta)\big|\mathbf{E}\left[|y+X|^3;|X|\le\frac{1}{2}\alpha(x,y)\right]\\
&\quad+C\max_{\theta:|\theta|\le(1/2)\alpha(x,y)}\big|h_{xxy}(x+y+\theta,y+\theta)\big|\mathbf{E}\left[|y+X|^2|X|;|X|\le\frac{1}{2}\alpha(x,y)\right]\\
&\quad+C\max_{\theta:|\theta|\le(1/2)\alpha(x,y)}\big|h_{xyy}(x+y+\theta,y+\theta)\big|\mathbf{E}\left[|y+X||X|^2;|X|\le\frac{1}{2}\alpha(x,y)\right]\\
&\quad+C\max_{\theta:|\theta|\le(1/2)\alpha(x,y)}\big|h_{yyy}(x+y+\theta,y+\theta)\big|\mathbf{E}\left[|X|^3;|X|\le\frac{1}{2}\alpha(x,y)\right]\\
&\le C\alpha(x,y)^{-8.5}\alpha(x,y)\mathbf{E}X^2+C\alpha(x,y)^{-6.5}\alpha(x,y)\mathbf{E}X^2\\
&\quad+C\alpha(x,y)^{-4.5}\alpha(x,y)\mathbf{E}X^2+C\alpha(x,y)^{-2.5}\alpha(x,y)^{1-\delta}\mathbf{E}|X|^{2+\delta}\\
&\le C\alpha(x,y)^{-1.5-\delta}.
\end{aligned}
$$

We are left to estimate

$$\left|\mathbf{E}\left[h(x+y+X, y+X) - h(x,y); |X| > \frac{1}{2}\alpha(x,y)\right]\right|$$
$$\le C\mathbf{E}\left[\alpha\big(|x+y+X|, |y+X|\big)^{0.5}; |X| > \frac{1}{2}\alpha(x,y)\right]$$
$$+ h(x,y)\mathbf{P}\left(|X| > \frac{1}{2}\alpha(x,y)\right)$$
$$\le C\mathbf{E}\left[|X|^{0.5}; |X| > \frac{1}{2}\alpha(x,y)\right] + C\alpha(x,y)^{0.5}\mathbf{P}\left(|X| > \frac{1}{2}\alpha(x,y)\right)$$
$$\le C\alpha(x,y)^{-1.5-\delta}\mathbf{E}|X|^{2+\delta},$$

where we applied the Chebyshev inequality in the last step and Lemma 6 in the first step. This proves the statement of the lemma. □

2.2. *Concentration bounds for Z_n*

In this paragraph we are going to derive a concentration bound for the two-dimensional process Z_n which will play a crucial role in the proof of our main results.

We start with a simple arithmetical estimate which is required to apply a concentration result by Friedland and Sodin [8].

Lemma 8. *There exist absolute positive constants a, b such that*

$$\Sigma_n := \min_{m_1,m_2,\dots,m_n\in\mathbb{Z}} \sum_{k=1}^{n} (\eta_1 k + \eta_2 - m_k)^2 \ge (an-b)^+$$

for all $\eta = (\eta_1, \eta_2)$ satisfying $|\eta_i| \le 1/10$ and $\max_{k\le n} |\eta_1 k + \eta_2| \ge 1/2$.

Proof. In view of the symmetry we may assume that $\eta_1 > 0$. Then the condition $\max_{k\le n} |\eta_1 k + \eta_2| \ge 1/2$ simplifies to $\eta_1 n + \eta_2 \ge 1/2$.

For every $j \ge 0$ we define

$$k_j := \min\{k\colon\ \eta_1 k + \eta_2 \ge j + 1/2\}.$$

First we note that $k_2 > n$ means that $\eta_1 n + \eta_2 < 5/2$. Therefore, $\eta_1 < (5/2 + 1/10)/n = 26/10n$. If k is such that $\eta_1 k + \eta_2 \in [1/4, 1/2]$, then $\min_{m_k\in\mathbb{Z}}(\eta_1 k + \eta - m_k)^2 = (\eta_1 k + \eta_2)^2 \ge 1/16$. Summing over these special values of k only, we obtain the following lower bound:

$$\Sigma_n \ge \frac{1}{16}\left(\left[\frac{1/2-\eta_2}{\eta_1}\right] - \left[\frac{1/4-\eta_2}{\eta_1}\right]\right) \ge \frac{1}{16}\left(\frac{1}{4\eta_1} - 1\right) \ge \frac{1}{16}\left(\frac{5}{52}n - 1\right).$$

Assume now that $k_2 \le n$. If k is such that $\eta_1 k + \eta_2 \in [j + 1/2, j + 3/4]$, then we have $\min_{m_k\in\mathbb{Z}}(\eta_1 k + \eta - m_k)^2 = (\eta_1 k + \eta_2 - j - 1)^2 \ge 1/16$. The sum over these indices is the greater than $\frac{1}{16}(1/4\eta_1 - 1) = \frac{1}{64\eta_1}(1 - 4\eta_1) \ge \frac{3}{320\eta_1}$. Consequently,

$$\Sigma_n \ge \frac{3}{320\eta_1}\max\{j\colon\ k_j \le n\}.$$

Noting that $\max\{j\colon\ k_j \le n\} = [\eta_1 n + \eta_2 - 1/2] \ge 2$ implies that $\max\{j\colon\ k_j \le n\} \ge \eta_1 n$, we obtain

$$\Sigma_n \ge \frac{3}{320}n.$$

Thus, the proof is completed. □

Lemma 9. *There exists a constant C such that*

$$\sup_{x,y}\mathbf{P}\big(\big|S_n^{(2)}-x\big|\le 1, |S_n-y|\le 1\big)\le \frac{C}{n^2},\quad n\ge 1$$

and

$$\sup_{x}\mathbf{P}\big(\big|S_n^{(2)}-x\big|\le 1\big)\le \frac{C}{n^{3/2}},\quad n\ge 1.$$

Proof. In order to prove the first statement we apply Theorem 1.2 from Friedland and Sodin [8] with $\vec{a}_k=(k,1)$:

$$\sup_{x,y}\mathbf{P}\big(\big|S_n^{(2)}-x\big|\le 1, |S_n-y|\le 1\big)\le C\left(\exp\{-c\alpha^2\}+\left(\det\left[\sum_{k=1}^n \vec{a}_k\otimes\vec{a}_k\right]\right)^{-1/2}\right),$$

where α^2 is such that $\sum_{k=1}^n(\eta_1 k+\eta_2-m_k)^2\ge\alpha^2$ for all $m_k\in\mathbb{Z}$, $\eta=(\eta_1,\eta_2)$ with $|\eta_i|\le 1/10$ and $\max_{k\le n}|\eta_1 k+\eta_2|\ge 1/2$.

According to Lemma 8, we may take $\alpha^2=(an-b)^+$. Furthermore, one can easily check that

$$\det\left[\sum_{k=1}^n \vec{a}_k\otimes\vec{a}_k\right]\sim\frac{n^4}{12}\quad\text{as } n\to\infty.$$

Thus, the first bound is proved.

The second inequality follows from Theorem 1.1 of [8]. □

2.3. *Construction of harmonic function*

Let

$$\begin{aligned}
&Y_0=h(z),\\
&Y_{n+1}=h(Z_{n+1})-\sum_{k=0}^n f(Z_k),\quad n\ge 0.
\end{aligned}\tag{24}$$

Lemma 10. *The sequence Y_n defined in (24) is a martingale.*

Proof. Clearly,

$$\begin{aligned}
\mathbf{E}_z[Y_{n+1}-Y_n|\mathcal{F}_n]&=\mathbf{E}_z\big[\big(h(Z_{n+1})-h(Z_n)-f(Z_n)\big)|\mathcal{F}_n\big]\\
&=-f\big(Z(n)\big)+\mathbf{E}_z\big[\big(h(Z_{n+1})-h(Z_n)\big)|Z_n\big]\\
&=-f(Z_n)+f(Z_n)=0,
\end{aligned}$$

where we used the definition of the function f in (9). □

Let

$$K_{n,\varepsilon}=\big\{(x,y):\ y>0, x\ge n^{3/2-3\varepsilon}\big\}.$$

The next lemma gives us a possibility to control $E_z[h(Z_k);\tau>k]$ for z sufficiently far from the boundary.

Lemma 11. *For any sufficiently small $\varepsilon > 0$ there exists $\gamma > 0$ such that for $k \le n$ the following inequalities hold*

$$\mathbf{E}_z\big[h(Z_k); \tau > k\big] \le \left(1 + \frac{C}{n^\gamma}\right)h(z), \quad z \in K_{n,\varepsilon}, \tag{25}$$

$$\mathbf{E}_z\big[h(Z_k); \tau > k\big] \ge \left(1 - \frac{C}{n^\gamma}\right)h(z), \quad z \in K_{n,\varepsilon}. \tag{26}$$

Proof. First, using (24) we obtain,

$$\begin{aligned}\mathbf{E}_z\big[h(Z_k); \tau > k\big] &= \mathbf{E}_z[Y_k; \tau > k] + \sum_{l=0}^{k-1}\mathbf{E}_z\big[f(Z_l); \tau > k\big]\\ &= \mathbf{E}_z[Y_k] - \mathbf{E}_z[Y_k; \tau \le k] + \sum_{l=0}^{k-1}\mathbf{E}_z\big[f(Z_l); \tau > k\big]. \end{aligned}\tag{27}$$

Since Y_k is a martingale and $\tau \wedge k$ is a bounded stopping time,

$$\mathbf{E}_z[Y_k] = \mathbf{E}_z[Y_{\tau\wedge k}] = \mathbf{E}_z[Y_0] = h(z). \tag{28}$$

From the first equality in this chain and $\mathbf{E}_z[Y_{\tau\wedge k}] = \mathbf{E}_z[Y_\tau; \tau \le k] + \mathbf{E}_z[Y_k; \tau > k]$ we infer that

$$\mathbf{E}_z[Y_k; \tau \le k] = \mathbf{E}_z[Y_\tau; \tau \le k]. \tag{29}$$

Applying (28) and (29) to the corresponding terms in (27) and using the definition of Y_k once again, we arrive at

$$\begin{aligned}\mathbf{E}_z\big[h(Z_k); \tau > k\big] &= h(z) - \mathbf{E}_z\big[h(Z_\tau), \tau \le k\big]\\ &\quad + \mathbf{E}_z\left[\sum_{l=0}^{\tau-1} f(Z_l); \tau \le k\right] + \sum_{l=0}^{k-1}\mathbf{E}_z\big[f(Z_l); \tau > k\big]\\ &= h(z) + \mathbf{E}_z\left[\sum_{l=0}^{\tau-1} f(Z_l); \tau \le k\right] + \sum_{l=0}^{k-1}\mathbf{E}_z\big[f(Z_l); \tau > k\big], \end{aligned}\tag{30}$$

since $h(Z_\tau) = 0$.

For $k \le n$ we have

$$\mathbf{E}_z\left[\sum_{l=0}^{\tau-1} f(Z_l); \tau \le k\right] + \sum_{l=0}^{k-1}\mathbf{E}_z\big[f(Z_l); \tau > k\big] \le \sum_{l=0}^{n-1}\mathbf{E}_z\big[\big|f(Z_l)\big|\big]. \tag{31}$$

We split the sum in (31) into three parts,

$$\begin{aligned}\sum_{l=0}^{n-1}\mathbf{E}_z\big[\big|f(Z_l)\big|\big] = {} & f(z) + \mathbf{E}_z\sum_{l=1}^{n-1}\big[\big|f(Z_l)\big|; \max\big(\big|S_l^{(2)}\big|, |S_l|\big) \le 1\big]\\ &+ \mathbf{E}_z\sum_{l=1}^{n-1}\big[\big|f(Z_l)\big|; \big|S_l^{(2)}\big|^{1/3} > |S_l|, \max\big(\big|S_l^{(2)}\big|, |S_l|\big) > 1\big]\\ &+ \mathbf{E}_z\sum_{l=1}^{n-1}\big[\big|f(Z_l)\big|; \big|S_l^{(2)}\big|^{1/3} \le |S_l|, \max\big(\big|S_l^{(2)}\big|, |S_l|\big) > 1\big]\\ =: {} & f(z) + \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned}$$

First, using the fact that $|f(x,y)| \le C$ for $|x|, |y| \le 1$ and Lemma 9, we obtain

$$\Sigma_1 \le C \sum_{l=1}^{\infty} \mathbf{P}_z\big(\big|S_l^{(2)}\big|, |S_l| \le 1\big) \le C \sum_{l=1}^{\infty} l^{-2} < C.$$

Second, by Lemma 7,

$$\begin{aligned}\Sigma_2 &\le C \sum_{l=1}^{n-1} \mathbf{E}_z\big[\big|S_l^{(2)}\big|^{-1/2-\delta/3}\big] \\ &\le C \sum_{l=1}^{n-1} \sum_{j=1}^{\infty} \mathbf{E}_z\big[\big|S_l^{(2)}\big|^{-1/2-\delta/3}; j \le \big|S_l^{(2)}\big| \le j+1\big] \\ &\le C \sum_{l=1}^{n-1} \left(l^{3/2(-1/2-\delta/3)} \mathbf{P}_z\big(\big|S_l^{(2)}\big| > l^{3/2}\big) + \sum_{j=1}^{l^{3/2}} j^{-1/2-\delta/3} \mathbf{P}_z\big(j \le \big|S_l^{(2)}\big| \le j+1\big) \right).\end{aligned}$$

Now we use the second concentration inequality from Lemma 9 to get an estimate

$$\mathbf{P}_z\big(j \le \big|S_l^{(2)}\big| \le j+1\big) \le C l^{-3/2}.$$

Then,

$$\Sigma_2 \le C \sum_{l=1}^{n-1} \left(l^{-3/4-\delta/2} + l^{-3/2} \sum_{j=1}^{l^{3/2}} j^{-1/2-\delta/3} \right) \le C \sum_{l=1}^{n-1} l^{-3/4-\delta/2} \le C n^{1/4-\delta/2}.$$

Similarly,

$$\begin{aligned}\Sigma_3 &\le C \sum_{l=1}^{n-1} \mathbf{E}_z\big[|S_l|^{-3/2-\delta}; |S_l| \ge 1; |S_l| \ge \big|S_l^{(2)}\big|^{1/3}\big] \\ &\le C \sum_{l=1}^{n-1} \sum_{j=1}^{\infty} \mathbf{E}_z\big[|S_l|^{-3/2-\delta}; j \le |S_l| \le j+1; \big|S_l^{(2)}\big| \le (j+1)^3\big] \\ &\le C \sum_{l=1}^{n-1} \left(l^{-3/4-\delta/2} \mathbf{P}_z\big(|S_l| > l^{1/2}\big) + \sum_{j=1}^{l^{1/2}} j^{-3/2-\delta} \mathbf{P}_z\big(j \le |S_l| \le j+1; \big|S_l^{(2)}\big| \le (j+1)^3\big) \right).\end{aligned}$$

Using Lemma 9 once again, we get an estimate

$$\begin{aligned}\mathbf{P}_z\big(j \le |S_l| \le j+1; \big|S_l^{(2)}\big| \le (j+1)^3\big) &\le C \sum_{i=1}^{(j+1)^3} \mathbf{P}_z\big(j \le |S_l| \le j+1; \big|S_l^{(2)}\big| \in (i, i+1)\big) \\ &\le C l^{-2} j^3.\end{aligned}$$

Then,

$$\begin{aligned}\Sigma_3 &\le C \sum_{l=1}^{n-1} \left(l^{-3/4-\delta/2} + \sum_{j=1}^{l^{1/2}} j^{-3/2-\delta} l^{-2} j^3 \right) \\ &\le C \sum_{l=1}^{n-1} \big(l^{-2} l^{5/4-\delta/2} + l^{-3/4-\delta/2}\big) \le C n^{1/4-\delta/2}.\end{aligned}$$

Therefore,

$$\sum_{l=0}^{n-1}\mathbf{E}_z\big[\big|f(Z_l)\big|\big]\le f(z)+Cn^{1/4-\delta/2}$$

and, consequently,

$$\big|\mathbf{E}\big[h(Z_n);\tau>n\big]-h(z)\big|\le f(z)+Cn^{1/4-\delta/2},\quad z\in\mathbb{R}_+\times\mathbb{R}. \tag{32}$$

Now we use the assumption that $z\in K_{n,\varepsilon}$. Combining Lemma 6 and Lemma 7, we get

$$\big|f(z)\big|\le C\max\big(1,\alpha(z)\big)^{-3/2-\delta}\le C\frac{h(z)}{(\alpha(z))^{2+\delta}}.$$

Applying now the lower bound from Lemma 6, we see that

$$h(z)\ge c\big(\alpha(z)\big)^{1/2}\ge cn^{1/4-\varepsilon/2},\quad z\in K_{n,\varepsilon}.$$

From these estimates we infer that

$$\begin{aligned} f(z)+Cn^{1/4-\delta/2}&\le C\frac{h(z)}{n^{1-2\varepsilon+\delta(1/2-\varepsilon)}}+Ch(z)\frac{n^{1/4-\delta/2}}{n^{1/4-\varepsilon/2}}\\ &\le Ch(z)n^{-\gamma}, \end{aligned} \tag{33}$$

where γ is positive for sufficiently small ε. Combining (32) and (33), we complete the proof. □

We now prove a result which shows that Z_n confined to $\mathbb{R}_+\times\mathbb{R}$ cannot stay near the boundary. As we mentioned in the Introduction this is one of the crucial steps in our construction.

Lemma 12. *There exist a positive constant r such that for*

$$\sup_{z\in\mathbb{R}_+\times\mathbb{R}}\mathbf{P}_z\big(\nu_n\ge n^{1-\varepsilon},\tau>n^{1-\varepsilon}\big)\le\exp\big\{-rn^{\varepsilon}\big\},$$

where

$$\nu_n:=\min\{k\ge 0:\ Z_k\in K_{n,\varepsilon}\}.$$

Proof. Fix some integer $A>0$ and put $b_n:=A[n^{1-2\varepsilon}]$. Define also $R_n:=[n^{1-\varepsilon}/b_n]$. It is clear that

$$\mathbf{P}_z\big(\nu_n>n^{1-\varepsilon},\tau>n^{1-\varepsilon}\big)\le\mathbf{P}_z\big(S^{(2)}_{jb_n}\in\big[0,n^{3/2-3\varepsilon}\big]\text{ for all }j\le R_n\big).$$

It follows from the definition of $S^{(2)}_n$ that

$$S^{(2)}_{(j+1)b_n}=S^{(2)}_{jb_n}+b_nS_{jb_n}+\tilde S^{(2)}_{b_n},$$

where $\tilde S^{(2)}_n$ is an independent copy of $S^{(2)}_n$ with starting point $(0,0)$. From this representation and the Markov property we conclude that

$$\begin{aligned} &\mathbf{P}_z\big(S^{(2)}_{jb_n}\in\big[0,n^{3/2-3\varepsilon}\big]\text{ for all }j\le R_n\big)\\ &\quad\le\mathbf{P}_z\big(S^{(2)}_{jb_n}\in\big[0,n^{3/2-3\varepsilon}\big]\text{ for all }j\le R_n-1\big)Q_{b_n}\big(n^{3/2-3\varepsilon}\big)\le\cdots\\ &\quad\le\big(Q_{b_n}\big(n^{3/2-3\varepsilon}\big)\big)^{R_n}, \end{aligned}$$

where

$$Q_k(\lambda) := \sup_{x\in\mathbb{R}} \mathbf{P}_{(0,0)}\big(S_k^{(2)} \in [x, x+\lambda]\big).$$

Using the second inequality in Lemma 9, we get

$$Q_{b_n}\big(n^{3/2-3\varepsilon}\big) \le \frac{Cn^{3/2-3\varepsilon}}{A^{3/2}(n^{1-2\varepsilon})^{3/2}} = \frac{C}{A^{3/2}}.$$

Choosing A so large that $\frac{C}{A^{3/2}} \le \frac{1}{2}$, we obtain

$$\mathbf{P}_z\big(\nu_n > n^{1-\varepsilon}, \tau > n^{1-\varepsilon}\big) \le \left(\frac{1}{2}\right)^{R_n}.$$

Thus, the proof is finished. □

Lemma 13. *There exist a constant C such that for $k \ge n^{1-\varepsilon}$,*

$$\mathbf{E}_z\big[h(Z_n), \nu_n \ge k, \tau > n^{1-\varepsilon}\big] \le Cn^{1/4}\big(1+\alpha(z)\big)^{1/2} \exp\big\{-rn^{\varepsilon}/2\big\}.$$

Proof. Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\mathbf{E}_z\big[h(Z_n), \tau > n^{1-\varepsilon}, \nu_n \ge k\big] \\
&\quad\le \big(\mathbf{E}_z\big[h^2(Z_n), \tau > n^{1-\varepsilon}\big]\big)^{1/2}\big(\mathbf{P}_z\big(\nu_n \ge k, \tau > n^{1-\varepsilon}\big)\big)^{1/2} \\
&\quad\le \big(\mathbf{E}_z\big[h^2(Z_n), \tau > n^{1-\varepsilon}\big]\big)^{1/2}\big(\mathbf{P}_z\big(\nu_n \ge n^{1-\varepsilon}, \tau > n^{1-\varepsilon}\big)\big)^{1/2}.
\end{aligned}$$

Recalling that $h(z) \le C(\alpha(z))^{1/2}$ for all $z \in \mathbb{R}_+ \times \mathbb{R}$, one can easy obtain the inequality

$$\begin{aligned}
\mathbf{E}_z\big[h^2(Z_n), \tau > n^{1-\varepsilon}\big] &\le C\mathbf{E}_z\big[\alpha(Z_n)\big] \le C\big(\alpha(z) + \mathbf{E}_0 \max\big\{\big((M_n+y)n\big)^{1/3}, M_n\big\}\big) \\
&\le C\big(1+\alpha(z)\big)n^{1/2},
\end{aligned}$$

where $M_n = \max_{0\le i\le n} S_i$. Combining this with Lemma 12, we complete the proof. □

We are now in position to state the main result of the present section. Its proof uses a recursion procedure and estimates from Lemmas 11 and 12.

Proposition 14. *For any starting point z there exists a limit*

$$V(z) = \lim_{n\to\infty} \mathbf{E}_z\big[h(Z_n); \tau > n\big]. \tag{34}$$

Moreover, this limit is harmonic and strictly positive on K_+.

Proof. Fix a large integer $n_0 > 0$ and put, for $m \ge 1$,

$$n_m = \big[n_0^{((1-\varepsilon)^{-m})}\big],$$

where $[r]$ denotes the integer part of r. Let n be any integer. There exists unique m such that $n \in (n_m, n_{m+1}]$. We first split the expectation into 2 parts,

$$\begin{aligned}
\mathbf{E}_z\big[h(Z_n); \tau > n\big] &= E_1(z) + E_2(z) \\
&:= \mathbf{E}_z\big[h(Z_n); \tau > n, \nu_n \le n_m\big] + \mathbf{E}_z\big[h(Z_n); \tau > n, \nu_n > n_m\big].
\end{aligned}$$

By Lemma 13, since $n_m \ge n^{1-\varepsilon}$, the second term on the right hand side is bounded by

$$E_2(z) \le \mathbf{E}_z\big[h(Z_n); \tau > n_m, \nu_n > n_m\big] \le C\big(1+\alpha(z)\big)^{1/2} n_m^{1/4} \exp\big\{-r n_m^{\varepsilon}/2\big\}.$$

For the first term we have

$$E_1(z) = \sum_{i=1}^{n_m} \int_{K_{n,\varepsilon}} \mathbf{P}_z\big\{\nu_n = i, \tau > i, S_i^{(2)} \in \mathrm{d}a, S_i \in \mathrm{d}b\big\} \mathbf{E}_{(a,b)}\big[h(Z_{n-i}); \tau > n-i\big].$$

Then, by (25),

$$E_1(z) \le \left(1 + \frac{C}{n^{\gamma}}\right) \sum_{i=1}^{n_m} \int_{K_{n,\varepsilon}} \mathbf{P}_z\big\{\nu_n = i, \tau > i, S_i^{(2)} \in \mathrm{d}a, S_i \in \mathrm{d}b\big\} h(a,b).$$

Now noting that $K_{n,\varepsilon} \subset K_{n_m,\varepsilon}$, we apply (26) to obtain

$$\begin{aligned}
E_1(z) &\le \frac{(1+C/n^{\gamma})}{(1-C/n_m^{\gamma})} \sum_{i=1}^{n_m} \int_{K_{n,\varepsilon}} \mathbf{P}_z\big\{\nu_n = i, \tau > i, S_i^{(2)} \in \mathrm{d}a, S_i \in \mathrm{d}b\big\} \\
&\quad \times \mathbf{E}_{(a,b)}\big[h(Z_{n_m-i}); \tau > n_m - i\big] \\
&= \frac{(1+C/n_m^{\gamma})}{(1-C/n_m^{\gamma})} \mathbf{E}_z\big[h(Z_{n_m}); \tau > n_m, \nu_n \le n_m\big].
\end{aligned}$$

As a result we have

$$\mathbf{E}_z\big[h(Z_n); \tau > n\big] \le \frac{(1+C/n_m^{\gamma})}{(1-C/n_m^{\gamma})} \mathbf{E}_z\big[h(Z_{n_m}); \tau > n_m\big] + C\big(1+\alpha(z)\big)^{1/2} n_m^{1/4} \exp\big\{-r n_m^{\varepsilon}/2\big\}. \tag{35}$$

Iterating this procedure m times, we obtain

$$\begin{aligned}
\mathbf{E}_z\big[h(Z_n); \tau > n\big] &\le \prod_{j=0}^{m} \frac{(1+C/n_m^{\gamma(1-\varepsilon)^j})}{(1-C/n_m^{\gamma(1-\varepsilon)^j})} \\
&\quad \times \left(\mathbf{E}_z\big[h(Z_{n_0}); \tau > n_0\big] + C\big(1+\alpha(z)\big)^{1/2} \sum_{j=0}^{m} n_{m-j}^{1/4} \exp\big\{-r n_{m-j}^{\varepsilon}/2\big\}\right).
\end{aligned} \tag{36}$$

First of all we immediately obtain that

$$\sup_n \mathbf{E}_z\big[h(Z_n); \tau > n\big] \le C(z) < \infty. \tag{37}$$

An identical procedure gives a lower bound

$$\begin{aligned}
\mathbf{E}_z\big[h(Z_n); \tau > n\big] &\ge E_1(z) \ge \prod_{j=0}^{m} \frac{(1-C/n_m^{\gamma(1-\varepsilon)^j})}{(1+C/n_m^{\gamma(1-\varepsilon)^j})} \mathbf{E}_z\big[h(Z_{n_0}); \tau > n_0, \nu_{n_0} \le n_0\big] \\
&= \prod_{j=0}^{m} \frac{(1-C/n_m^{\gamma(1-\varepsilon)^j})}{(1+C/n_m^{\gamma(1-\varepsilon)^j})} \big(\mathbf{E}_z\big[h(Z_{n_0}); \tau > n_0, \nu_{n_1} \le n_0\big] - \mathbf{E}_z\big[h(Z_{n_0}); \tau > n_0, \nu_{n_1} > n_0\big]\big) \\
&\ge \prod_{j=0}^{m} \frac{(1-C/n_m^{\gamma(1-\varepsilon)^j})}{(1+C/n_m^{\gamma(1-\varepsilon)^j})} \big(\mathbf{E}_z\big[h(Z_{n_0}); \tau > n_0\big] - C\big(1+\alpha(z)\big)^{1/2} n_0^{1/4} \exp\big\{-r n_0^{\varepsilon}/2\big\}\big).
\end{aligned} \tag{38}$$

For every positive δ we can choose $n_0 = n_0(\delta)$ such that

$$\left|\prod_{j=0}^{m} \frac{(1 - C/n_m^{\gamma(1-\varepsilon)^j})}{(1 + C/n_m^{\gamma(1-\varepsilon)^j})} - 1\right| \le \delta \quad \text{and} \quad \sum_{j=0}^{m} n_{m-j}^{1/4} \exp\{-r n_{m-j}^{\varepsilon}/2\} \le \delta.$$

Then, for this value of n_0 and all $z \in \mathbb{R}_+ \times \mathbb{R}$,

$$\sup_{n>n_0} \mathbf{E}_z\big[h(Z_n); \tau > n\big] \le (1+\delta)\big(\mathbf{E}_z\big[h(Z_{n_0}); \tau > n_0\big] + C\big(1+\alpha(z)\big)^{1/2}\delta\big)$$

and

$$\inf_{n>n_0} \mathbf{E}_z\big[h(Z_n); \tau > n\big] \ge (1-\delta)\big(\mathbf{E}_z\big[h(Z_{n_0}); \tau > n_0\big] - C\big(1+\alpha(z)\big)^{1/2}\delta\big).$$

Consequently,

$$\sup_{n>n_0} \mathbf{E}_z\big[h(Z_n); \tau > n\big] - \inf_{n>n_0} \mathbf{E}_z\big[h(Z_n); \tau > n\big]$$
$$\le \delta \mathbf{E}_z\big[h(Z_{n_0}); \tau > n_0\big] + 2C\big(1+\alpha(z)\big)^{1/2}\delta.$$

Taking into account (37) and that δ can be made arbitrarily small we arrive at the conclusion that the limit in (34) exists.

To prove harmonicity of V_0 note that by the Markov property

$$\mathbf{E}_z\big[h(Z_{n+1}); \tau > n+1\big] = \int_{\mathbb{R}_+\times\mathbb{R}} \mathbf{P}\big(z + Z \in \mathrm{d}z'\big)\mathbf{E}_{z'}\big[h(Z_n); \tau > n\big].$$

Letting n to infinity we obtain

$$V(z) = \mathbf{E}_z\big[V(Z_1); \tau > 1\big].$$

The existence of the limit in the right hand side is justified by the dominated convergence theorem and the above estimates for $\sup_{n>n_0} \mathbf{E}_z[h(Z_n); \tau > n]$.

Function V has the following monotonicity property: if $x' \ge x$ and $y' \ge y$ then $V(x', y') \ge V(x, y)$. Indeed, first the function h satisfies this property since $h_x \ge 0, h_y \ge 0$, see Lemma 5. Second it clear that the exit time $\tau' \ge \tau$, where τ' is the exit of time the integrated random walk started from (x', y') and τ is the exit of time the integrated random walk started from (x, y). Third,

$$\tilde{S}_n = y' + X_1 + X_2 + \cdots + X_n \ge y + X_1 + X_2 + \cdots + X_n = S_n,$$
$$\tilde{S}_n^{(2)} = x' + \tilde{S}_1 + \tilde{S}_2 + \cdots + \tilde{S}_n \ge S_n^{(2)}.$$

Therefore, for any n,

$$\mathbf{E}_{(x',y')}\big[h(Z_n); \tau > n\big] \ge \mathbf{E}_{(x,y)}\big[h(Z_n); \tau > n\big].$$

Letting n go to infinity, we obtain $V(x', y') \ge V(x, y)$.

It remains to show that V is strictly positive on K_+. As we have already shown, for every $\delta > 0$ there exists n_0 such that

$$\inf_{n\ge n_0} \mathbf{E}_z\big[h(Z_n); \tau > n\big] \ge (1-\delta)\big(h(z) - C\big(1+\alpha(z)\big)^{1/2}\delta\big)$$

for all z. Furthermore, for every fixed n we have $\mathbf{E}_{(x,y)}[h(Z_n); \tau > n] \sim h(x, y)$ as $x, y \to \infty$. Thus there exist x_0, y_0 such that $\mathbf{E}_{(x_0,y_0)}[h(Z_{n_0}); \tau > n_0] > (1-\delta)h(x_0, y_0)$. Taking into account (18), we conclude that $V(z)$ is positive for all z with $x > x_{n_0},\ y > y_{n_0}$. From every starting point $z \in \mathbb{R}_+^2$ our process visits the set $x > x_{n_0},\ y > y_{n_0}$ before τ with positive probability. Recalling that $V(z) = \mathbf{E}_z[V(Z_1), \tau > k]$ for all $k \ge 1$, we conclude that $V(z) > 0$. The same argument shows that V is strictly positive on K_+. □

3. Proof of Theorem 1

3.1. *On integrated Brownian motion*

We first mention some properties of the integrated Brownian motion which we shall use in the proof of our main theorem.

Lemma 15. *There exists a finite constant* C *such that*

$$\mathbf{P}_{(x,y)}\big(\tau^{bm} > t\big) \le C\frac{h(x,y)}{t^{1/4}}, \quad x, y > 0. \tag{39}$$

Moreover,

$$\mathbf{P}_{(x,y)}\big(\tau^{bm} > t\big) \sim \varkappa\frac{h(x,y)}{t^{1/4}} \quad \text{as } t \to \infty, \tag{40}$$

uniformly in $x, y > 0$ *satisfying* $\max(x^{1/6}, y^{1/2}) \le \theta_t t^{1/4}$ *with some* $\theta_t \to 0$.

Proof. To prove this lemma we are going to use the scaling property of the Brownian motion, which immediately gives

$$\mathbf{P}_{(x,y)}\big(\tau^{bm} > t\big) = \mathbf{P}_{(\lambda^3 x,\lambda y)}\big(\tau^{bm} > t\lambda^2\big), \quad \lambda > 0. \tag{41}$$

We start with (40). Consider first the case $x^{1/3} \ge y$. Putting $\lambda = x^{-1/3}$ in (41) we obtain

$$\mathbf{P}_{(x,y)}\big(\tau^{bm} > t\big) = \mathbf{P}_{(1,yx^{-1/3})}\big(\tau^{bm} > tx^{-2/3}\big).$$

In view of our assumption $tx^{-2/3} \ge \theta_t^{-1/4} \to \infty$. We use the continuity of $h(1,u)$ in $u \in [0,1]$ and immediately obtain that the asymptotics

$$\mathbf{P}_{(1,yx^{-1/3})}\big(\tau^{bm} > tx^{-2/3}\big) \sim \varkappa\frac{h(1,yx^{-1/3})}{(tx^{-2/3})^{1/4}}$$

hold uniformly in $yx^{-1/3} \in [0,1]$. Then,

$$\mathbf{P}_{(x,y)}\big(\tau^{bm} > t\big) \sim \varkappa\frac{h(1,yx^{-1/3})}{(tx^{-2/3})^{1/4}} = \varkappa\frac{h(x,y)}{t^{1/4}}.$$

If $x^{1/3} \le y$ then, choosing $\lambda = y^{-1}$ in (41), we obtain

$$\mathbf{P}_{(x,y)}\big(\tau^{bm} > t\big) = \mathbf{P}_{(xy^{-3},1)}\big(\tau^{bm} > ty^{-2}\big).$$

The rest of the proof goes exactly the same way.

To prove (39) first notice that the above proof showed that for sufficiently small $\varepsilon > 0$ and $t^{1/2} > \varepsilon^{-1}\max(x^{1/3}, y)$ the bound (39) holds. Hence, it is sufficient to consider $t^{1/2} \le \varepsilon^{-1}\max(x^{1/3}, y)$. Using the lower bound in (18), we see that

$$\frac{h(x,y)}{t^{1/4}} \ge \frac{c\max(x^{1/6}, y^{1/2})}{(\varepsilon^{-1}\max(x^{1/3}, y))^{1/2}} = c\varepsilon^2 > 0$$

for $t^{1/2} \le \varepsilon^{-1}\max(x^{1/3}, y)$. Therefore,

$$\mathbf{P}_{(x,y)}\big(\tau^{bm} > t\big) \le 1 \le \frac{1}{c\varepsilon^2}\frac{h(x,y)}{t^{1/4}}.$$

This proves (39). $\square$

Lemma 16. *Let $\overline{p}_t(x,y;u,v)$ denote the transition density of $(\int_0^t B_s\,\mathrm{d}s, B_t)$ killed at leaving $\mathbb{R}_+\times\mathbb{R}$. Then, for $x,y\to 0$,*

$$\frac{\overline{p}_1(x,y;u,v)}{h(x,y)}\to \varkappa\overline{h}(u,v).$$

The proof of this lemma for $x=0$ is given in [9], see the proof of Theorem 4.1. For a general starting point (x,y) one can obtain the statement from the equalities

$$\overline{p}_t(x,y;u,v)=p_t(x,y;u,v)-\int_0^t\int_0^\infty p_{t-s}(0,-w;u,v)\mathbf{P}_{(x,y)}\big(\tau^{bm}\in \mathrm{d}s, B_{\tau^{bm}}\in -\mathrm{d}w\big),$$
$$\overline{p}_t(x,y;u,v)=\overline{p}_t(u,-v,y;x,-y),$$

see relations (3) and (4) in Lachal [10].

3.2. *Coupling*

In this paragraph we derive some asymptotical results for integrated random walks from the corresponding statements for the integrated Brownian motion. For that we use the following classical result (see, for example, [11]) on the quality of the normal approximation.

Lemma 17. *If $\mathbf{E}|X|^{2+\delta}<\infty$ for some $\delta\in(0,1)$, then for every $n\ge 1$ one can define a Brownian motion B_t on the same probability space such that, for any γ satisfying $0<\gamma<\frac{\delta}{2(2+\delta)}$,*

$$\mathbf{P}\Big(\sup_{u\le n}|S_{[u]}-B_u|\ge n^{1/2-\gamma}\Big)=\mathrm{o}\big(n^{2\gamma+\gamma\delta-\delta/2}\big). \tag{42}$$

As a first consequence of this coupling we derive asymptotics for $\mathbf{P}_z(\tau>n)$ with sufficiently large z.

Lemma 18. *For all sufficiently small $\varepsilon>0$,*

$$\mathbf{P}_z(\tau>n)=\varkappa h(z)n^{-1/4}\big(1+\mathrm{o}(1)\big)\quad \text{as } n\to\infty \tag{43}$$

uniformly in $z\in K_{n,\varepsilon}$ such that $\max\{x^{1/3},y\}\le\theta_n\sqrt{n}$ for some $\theta_n\to 0$. Moreover, there exists a constant C such that

$$\mathbf{P}_z(\tau>n)\le C\frac{h(z)}{n^{1/4}}, \tag{44}$$

uniformly in $z\in K_{n,\varepsilon}$, $n\ge 1$.

Proof. For every $z=(x,y)\in K_{n,\varepsilon}$ denote

$$z^{\pm}=\big(x\pm n^{3/2-\gamma},y\big).$$

Define

$$A_n=\Big\{\sup_{u\le n}|S_{[u]}-B_u|\le n^{1/2-\gamma}\Big\},$$

where B is the Brownian motion constructed in Lemma 17. Then, using (42), we obtain

$$\begin{aligned}\mathbf{P}_z(\tau>n)&=\mathbf{P}_z(\tau>n,A_n)+\mathrm{o}\big(n^{-r}\big)\\&\le \mathbf{P}_{z^+}\big(\tau^{bm}>n,A_n\big)+\mathrm{o}\big(n^{-r}\big)\\&=\mathbf{P}_{z^+}\big(\tau^{bm}>n\big)+\mathrm{o}\big(n^{-r}\big),\end{aligned} \tag{45}$$

where $r = r(\delta, \gamma) = \delta/2 - 2\gamma - \gamma\delta$. In the same way one can get

$$\mathbf{P}_{z^-}\big(\tau^{bm} > n\big) \le \mathbf{P}_z(\tau > n) + \mathrm{o}\big(n^{-r}\big). \tag{46}$$

Note that if we take $\gamma > 3\varepsilon$, then $z^{\pm} \in K_{n,\varepsilon'}$ for any $\varepsilon' > \varepsilon$ and all sufficiently large n. Therefore, we may apply Lemma 15:

$$\mathbf{P}_{z^{\pm}}\big(\tau^{bm} > n\big) \sim \varkappa h\big(z^{\pm}\big)n^{-1/4}.$$

It follows from the Taylor formula and Lemma 6 that

$$\big|h\big(z^{\pm}\big) - h(z)\big| \le Cn^{3/2-\gamma}\big(\alpha\big(x \pm n^{3/2-\gamma}, y\big)\big)^{-5/2} \le Cn^{1/4+5\varepsilon/6-\gamma}. \tag{47}$$

Furthermore, in view of (18),

$$h(z) > cn^{1/4-\varepsilon/2}, \quad z \in K_{n,\varepsilon}. \tag{48}$$

From this bound and (47) we infer that

$$h\big(z^{\pm}\big) = h(z)\big(1 + \mathrm{o}(1)\big), \quad z \in K_{n,\varepsilon}. \tag{49}$$

Therefore, we have

$$\mathbf{P}_{z^{\pm}}\big(\tau^{bm} > n\big) = \varkappa h(z)n^{-1/4}\big(1 + \mathrm{o}(1)\big).$$

From this relation and bounds (45) and (46) we obtain

$$\mathbf{P}_z(\tau > n) = \varkappa h(z)n^{-1/4}\big(1 + \mathrm{o}(1)\big) + \mathrm{o}\big(n^{-r}\big).$$

Using (48), we see that $n^{-r} = \mathrm{o}(h(z)n^{-1/4})$ for all ε satisfying $r = \delta/2 - 2\gamma - 2\gamma\delta > \varepsilon/6$. This proves (43). To prove (44) it is sufficient to substitute (39) into (45). □

Lemma 19. *For all sufficiently small $\varepsilon > 0$ and all rectangles $D = [a, b] \times [c, d]$ with positive a,*

$$\mathbf{P}_z\left(\left(\frac{S_n^{(2)}}{n^{3/2}}, \frac{S_n}{\sqrt{n}}\right) \in D, \tau > n\right) = \varkappa h(z)n^{-1/4}\int_D \overline{h}(u, v)\,\mathrm{d}u\,\mathrm{d}v\big(1 + \mathrm{o}(1)\big) \quad \text{as } n \to \infty \tag{50}$$

uniformly in $z \in K_{n,\varepsilon}$ such that $\max\{x^{1/3}, y\} \le \theta_n\sqrt{n}$ *for some $\theta_n \to 0$.*

Proof. Define two sets,

$$D^+ = \big[a - n^{-\gamma}, b + n^{-\gamma}\big] \times [c, d],$$
$$D^- = \big[a + n^{-\gamma}, b - n^{-\gamma}\big] \times [c, d].$$

Clearly $D^- \subset D \subset D^+$. Then, arguing as in the proof of the previous lemma, we get

$$\begin{aligned}
\mathbf{P}_z\left(\left(\frac{S_n^{(2)}}{n^{3/2}}, \frac{S_n}{\sqrt{n}}\right) \in D, \tau > n\right) &\le \mathbf{P}_z\left(\left(\frac{S_n^{(2)}}{n^{3/2}}, \frac{S_n}{\sqrt{n}}\right) \in D, \tau > n, A_n\right) + \mathrm{o}\big(n^{-r}\big) \\
&\le \mathbf{P}_{z^+}\left(\left(\frac{\int_0^n B_s\,\mathrm{d}s}{n^{3/2}}, \frac{B_n}{\sqrt{n}}\right) \in D, \tau^{bm} > n, A_n\right) + \mathrm{o}\big(n^{-r}\big) \\
&\le \mathbf{P}_{z^+}\left(\left(\frac{\int_0^n B_s\,\mathrm{d}s}{n^{3/2}}, \frac{B_n}{\sqrt{n}}\right) \in D, \tau^{bm} > n\right) + \mathrm{o}\big(n^{-r}\big).
\end{aligned} \tag{51}$$

Similarly,

$$\mathbf{P}_z\left(\left(\frac{S_n^{(2)}}{n^{3/2}}, \frac{S_n}{\sqrt{n}}\right) \in D, \tau > n\right) \ge \mathbf{P}_{z^-}\left(\left(\frac{\int_0^n B_s\,\mathrm{d}s}{n^{3/2}}, \frac{B_n}{\sqrt{n}}\right) \in D, \tau^{bm} > n\right) + \mathrm{o}\big(n^{-r}\big). \tag{52}$$

Using the scaling property of the Brownian motion and applying Lemma 16, we obtain

$$\mathbf{P}_{z^\pm}\left(\left(\frac{\int_0^n B_s\,\mathrm{d}s}{n^{3/2}}, \frac{B_n}{\sqrt{n}}\right) \in D, \tau^{bm} > n\right) \sim \varkappa h\left(\frac{x^\pm}{n^{3/2}}, \frac{y}{n^{1/2}}\right) \int_{D^\pm} \overline{h}(u,v)\,\mathrm{d}u\,\mathrm{d}v.$$

It is sufficient to note now that

$$h\left(\frac{x^\pm}{n^{3/2}}, \frac{y}{n^{1/2}}\right) \sim h\left(\frac{x}{n^{3/2}}, \frac{y}{n^{1/2}}\right) \quad \text{and} \quad \int_{D^\pm} \overline{h}(u,v)\,\mathrm{d}u\,\mathrm{d}v \to \int_D \overline{h}(u,v)\,\mathrm{d}u\,\mathrm{d}v$$

as $n \to \infty$. Thus, from bounds (51), (52) and relations (48), (49) we obtain the desired conclusion. □

3.3. *Asymptotic behaviour of* τ: *Proof of* (7)

Applying Lemma 12, we obtain

$$\begin{aligned}\mathbf{P}_z(\tau > n) &= \mathbf{P}_z\big(\tau > n, \nu_n < n^{1-\varepsilon}\big) + \mathbf{P}_z\big(\tau > n, \nu_n \ge n^{1-\varepsilon}\big)\\ &= \mathbf{P}_z\big(\tau > n, \nu_n < n^{1-\varepsilon}\big) + \mathrm{O}\big(\mathrm{e}^{-rn^\varepsilon}\big).\end{aligned} \tag{53}$$

Using the strong Markov property, we get for the first term the following estimates

$$\begin{aligned}&\int_{K_{n,\varepsilon}} \mathbf{P}_z\big(Z_{\nu_n} \in \mathrm{d}\tilde{z}, \tau > \nu_n, \nu_n < n^{1-\varepsilon}\big)\mathbf{P}_{\tilde{z}}(\tau > n)\\ &\quad\le \mathbf{P}_z\big(\tau > n, \nu_n < n^{1-\varepsilon}\big)\\ &\quad\le \int_{K_{n,\varepsilon}} \mathbf{P}_z\big(Z_{\nu_n} \in \mathrm{d}\tilde{z}, \tau > \nu_n, \nu_n < n^{1-\varepsilon}\big)\mathbf{P}_{\tilde{z}}\big(\tau > n - n^{1-\varepsilon}\big).\end{aligned} \tag{54}$$

Applying now Lemma 18, we obtain

$$\begin{aligned}&\mathbf{P}_z\big(\tau > n; \nu_n < n^{1-\varepsilon}\big)\\ &\quad= \frac{\varkappa + \mathrm{o}(1)}{n^{1/4}}\mathbf{E}_z\big[h(Z_{\nu_n}); \tau > \nu_n, M_{\nu_n} \le \theta_n\sqrt{n}, \nu_n < n^{1-\varepsilon}\big]\\ &\qquad+ \mathrm{O}\left(\frac{1}{n^{1/4}}\mathbf{E}_z\big[h(Z_{\nu_n}); \tau > \nu_n, M_{\nu_n} > \theta_n\sqrt{n}, \nu_n < n^{1-\varepsilon}\big]\right)\\ &\quad= \frac{\varkappa + \mathrm{o}(1)}{n^{1/4}}\mathbf{E}_z\big[h(Z_{\nu_n}); \tau > \nu_n, \nu_n < n^{1-\varepsilon}\big]\\ &\qquad+ \mathrm{O}\left(\frac{1}{n^{1/4}}\mathbf{E}_z\big[h(Z_{\nu_n}); \tau_x > \nu_n, M_{\nu_n} > \theta_n\sqrt{n}, \nu_n < n^{1-\varepsilon}\big]\right),\end{aligned} \tag{55}$$

where $M_k := \max_{j \le k} |S_j|$.

We now show that the first expectation converges to $V(z)$ and that the second expectation is negligibly small.

Lemma 20. *Under the assumptions of Theorem* 1,

$$\lim_{n\to\infty} \mathbf{E}_z\big[h(Z_{\nu_n}); \tau > \nu_n, \nu_n < n^{1-\varepsilon}\big] = V(z).$$

Proof. Put $T = \tau \wedge n^{1-\varepsilon}$. Since T is a bounded stopping time and Y_k is a martingale,

$$\mathbf{E}_z[Y_T] = \mathbf{E}_z[Y_{\nu_n \wedge T}] = \mathbf{E}_z[Y_{\nu_n}, \nu_n < T] + \mathbf{E}_z[Y_T, \nu_n \geq T]$$

and, consequently,

$$\mathbf{E}_z[Y_{\nu_n}, \nu_n < T] = \mathbf{E}_z[Y_T, \nu_n < T].$$

Using the definition of Y_k, we infer from the last equality that

$$\mathbf{E}_z\big[h(Z_{\nu_n}), \nu_n < T\big] = \mathbf{E}_z\big[h(Z_T), \nu_n < T\big] - \mathbf{E}_z\left[\sum_{k=\nu_n}^{T-1} f(Z_k), \nu_n < T\right].$$

Conditioning on Z_{ν_n} and applying (33), we obtain

$$\begin{aligned}\left|\mathbf{E}_z\left[\sum_{k=\nu_n}^{T-1} f(Z_k), \nu_n < T\right]\right| &\leq \mathbf{E}_z\left[1\{T > \nu_n\}\mathbf{E}_{Z_{\nu_n}}\left[\sum_{k=0}^{T-\nu_n}\big|f(Z_k)\big|\right]\right] \\ &\leq \frac{C}{n^{\gamma(1-\varepsilon)}}\mathbf{E}_z\big[h(Z_{\nu_n}), \nu_n < \tau\big].\end{aligned}$$

From this inequality and Lemma 13 we conclude

$$\mathbf{E}_z\big[h(Z_{\nu_n}), \nu_n < T\big] = \big(1 + \mathrm{o}(1)\big)\mathbf{E}_z\big[h(Z_T), \nu_n < T\big] \quad \text{as } n \to \infty. \tag{56}$$

Since $h(Z_\tau) = 0$, we have $h(Z_T) = h(Z_{n^{1-\varepsilon}})1\{\tau > n^{1-\varepsilon}\}$. Noting that Lemma 13 remains valid with $h(Z_{n^{1-\varepsilon}})$ instead of $h(Z_n)$, we get

$$\begin{aligned}\mathbf{E}_z\big[h(Z_T), \nu_n < T\big] &= \mathbf{E}_z\big[h(Z_{n^{1-\varepsilon}}), \nu_n < n^{1-\varepsilon}, \tau > n^{1-\varepsilon}\big] \\ &= \mathbf{E}_z\big[h(Z_{n^{1-\varepsilon}}), \tau > n^{1-\varepsilon}\big] + \mathrm{O}\big(n^{1/4}\mathrm{e}^{-rn^{\varepsilon}/2}\big).\end{aligned}$$

And in view of Proposition 14,

$$\lim_{n\to\infty}\mathbf{E}_z\big[h(Z_T), \nu_n < T\big] = V(z).$$

Combining this relation with (56), we get the desired result. □

Lemma 21. *As* $n \to \infty$,

$$\mathbf{E}_z\big[h(Z_{\nu_n}), \tau > \nu_n, \nu_n < n^{1-\varepsilon}, M_{\nu_n} > \theta_n\sqrt{n}\big] \to 0.$$

Proof. On the event $\nu_n \leq n^{1-\varepsilon}$,

$$h(Z_{\nu_n}) \leq C\alpha(z) + C\Big(\max\big\{\big(n^{1-\varepsilon}(y + M_{n^{1-\varepsilon}})\big)^{1/3}, M_{n^{1-\varepsilon}}\big\}\Big)^{1/2}$$

and, consequently,

$$\mathbf{E}_z\big[h(Z_{\nu_n}), \tau > \nu_n, \nu_n < n^{1-\varepsilon}, M_{\nu_n} > \theta_n\sqrt{n}\big] \leq C\alpha(z)\mathbf{P}(M_{n^{1-\varepsilon}} > \theta_n\sqrt{n}) + C\mathbf{E}\big[M_{n^{1-\varepsilon}}^{1/2}, M_{n^{1-\varepsilon}} > \theta_n\sqrt{n}\big]. \tag{57}$$

Here we used the fact that if $\theta_n \to 0$ sufficiently slow, then

$$\max\big\{\big(n^{1-\varepsilon}M_{n^{1-\varepsilon}}\big)^{1/3}, M_{n^{1-\varepsilon}}\big\} = M_{n^{1-\varepsilon}}$$

on the set $\{M_{n^{1-\varepsilon}} > \theta_n\sqrt{n}\}$.

Using now the Kolmogorov inequality, one can easily conclude that both summands on the right hand side of (57) vanish as $n \to \infty$. □

3.4. *Proof of weak convergence* (8)

It suffices to show that, for any rectangle $D \subset \mathbb{R}_+ \times \mathbb{R}$,

$$\frac{\mathbf{P}_z((S_n^{(2)}/n^{3/2}, S_n/n^{1/2}) \in D, \tau > n)}{\mathbf{P}_z(\tau > n)} \to \int_D \overline{h}(u,v)\,\mathrm{d}u\,\mathrm{d}v. \tag{58}$$

Take θ_n which goes to zero slower than any power function. First note that, in view of Lemma 12,

$$\mathbf{P}_z\left(\left(\frac{S_n^{(2)}}{n^{3/2}}, \frac{S_n}{n^{1/2}}\right) \in D, \tau > n\right)$$
$$= \mathbf{P}_z\left(\left(\frac{S_n^{(2)}}{n^{3/2}}, \frac{S_n}{n^{1/2}}\right) \in D, \tau > n, \nu_n \le n^{1-\varepsilon}\right) + \mathrm{O}\big(\mathrm{e}^{-rn^{\varepsilon}}\big).$$

Repeating the arguments from the derivation of (55) and applying Lemma 21, we get

$$\mathbf{P}_z\left(\left(\frac{S_n^{(2)}}{n^{3/2}}, \frac{S_n}{n^{1/2}}\right) \in D, \tau > n, \nu_n \le n^{1-\varepsilon}\right)$$
$$= \mathbf{P}_z\left(\left(\frac{S_n^{(2)}}{n^{3/2}}, \frac{S_n}{n^{1/2}}\right) \in D, \tau > n, \nu_n \le n^{1-\varepsilon}, M_{\nu_n} \le \theta_n\sqrt{n}\right) + \mathrm{o}\big(\mathbf{P}_z(\tau > n)\big).$$

Next,

$$\mathbf{P}_z\left(\left(\frac{S_n^{(2)}}{n^{3/2}}, \frac{S_n}{n^{1/2}}\right) \in D, \tau > n, \nu_n \le n^{1-\varepsilon}, M_{\nu_n} \le \theta_n n^{1/2}\right)$$
$$= \sum_{k=1}^{n^{1-\varepsilon}} \int_{K_{n,\varepsilon}\cap\{|y|\le\theta_n n^{1/2}\}} \mathbf{P}_z\big(\tau > k, Z_k \in (\mathrm{d}x, \mathrm{d}y), \nu_n = k\big)$$
$$\times \mathbf{P}_{(x,y)}\left(\tau > n-k, \left(\frac{S_{n-k}^{(2)}}{n^{3/2}}, \frac{S_{n-k}}{n^{1/2}}\right) \in D\right).$$

Using the coupling and arguing as in Lemma 19, one can show that

$$\mathbf{P}_{(x,y)}\left(\tau > n-k, \left(\frac{S_{n-k}^{(2)}}{n^{3/2}}, \frac{S_{n-k}}{n^{1/2}}\right) \in D\right) \sim \varkappa h(x,y) n^{-1/4} \int_D \overline{h}(u,v)\,\mathrm{d}u\,\mathrm{d}v$$

uniformly in $k \le n^{1-\varepsilon}$ and $z \in K_{n,\varepsilon}$. As a result we obtain

$$\mathbf{P}_z\left(\left(\frac{S_n^{(2)}}{n^{3/2}}, \frac{S_n}{n^{1/2}}\right) \in D, \tau > n, \nu_n \le n^{1-\varepsilon}, M_{\nu_n} \le \theta_n\sqrt{n}\right)$$
$$\sim \varkappa n^{-1/4}\left(\int_D \overline{h}(u,v)\,\mathrm{d}u\,\mathrm{d}v\right)\mathbf{E}_z\big[h(Z_{\nu_n}); \tau > \nu_n, M_{\nu_n} \le \theta_n\sqrt{n}, \nu_n < n^{1-\varepsilon}\big].$$

Using now Lemma 20 and Lemma 21, we get

$$\mathbf{P}_z\left(\left(\frac{S_n^{(2)}}{n^{3/2}}, \frac{S_n}{n^{1/2}}\right) \in D, \tau > n, \nu_n \le n^{1-\varepsilon}, M_{\nu_n} \le \theta_n\sqrt{n}\right)$$
$$\sim \varkappa n^{-1/4} \int_D \overline{h}(u,v)\,\mathrm{d}u\,\mathrm{d}v\, V(z).$$

This completes the proof of (58).

3.5. *Proof of Proposition* 4

The proof follows closely the proof of Lemma 11. Recalling (30), we get

$$\mathbf{E}_z\big[h(Z_n);\tau>n\big]=h(z)+\mathbf{E}_z\left[\sum_{l=0}^{\tau-1}f(Z_l);\tau\le n\right]+\sum_{l=0}^{n-1}\mathbf{E}_z\big[f(Z_l);\tau>n\big].$$

Thus, it is sufficient to prove that

$$\mathbf{E}_z\left[\sum_{l=1}^{\tau-1}\big|f\big(Z(l)\big)\big|\right]<\infty. \tag{59}$$

Indeed, the dominated convergence theorem then implies that

$$\mathbf{E}_z\left[\sum_{l=0}^{\tau-1}f\big(Z(l)\big);\tau_x\le n\right]\to\mathbf{E}_z\left[\sum_{l=0}^{\tau-1}f\big(Z(l)\big)\right]$$

and

$$\left|\sum_{l=0}^{n-1}\mathbf{E}_z\big[f\big(Z(l)\big);\tau>n\big]\right|\le\mathbf{E}_z\left[\sum_{l=0}^{\tau-1}\big|f\big(Z(l)\big)\big|;\tau>n\right]\to 0$$

since τ is finite a.s. Then, as $n\to\infty$,

$$\mathbf{E}_z\big[h(Z_n);\tau>n\big]\to h(z)+\mathbf{E}_z\sum_{l=0}^{\tau-1}f(Z_l)=V(z),$$

which proves the desired representation.

To prove (59) we use the fact that we have already proved that

$$\mathbf{P}_z(\tau>n)\sim V(z)n^{-1/4}.$$

We split (59) into three parts,

$$\begin{aligned}\mathbf{E}_z\sum_{l=0}^{\tau-1}\big|f(Z_l)\big| &= f(z)+\sum_{l=1}^{\infty}\mathbf{E}_z\big[\big|f(Z_l)\big|;\tau>l\big]\\ &= f(z)+\sum_{l=1}^{\infty}\mathbf{E}_z\big[\big|f(Z_l)\big|;\big|S_l^{(2)}\big|,|S_l|\le 1,\tau>l\big]\\ &\quad+\sum_{l=1}^{\infty}\mathbf{E}_z\big[\big|f(Z_l)\big|;\big|S_l^{(2)}\big|^{1/3}>|S_l|,\tau>l\big]\\ &\quad+\sum_{l=1}^{\infty}\mathbf{E}_z\big[\big|f(Z_l)\big|;\big|S_l^{(2)}\big|^{1/3}\le|S_l|,\tau>l\big]\\ &=: f(z)+\Sigma_1+\Sigma_2+\Sigma_3.\end{aligned}$$

Repeating the arguments from the proof of Lemma 11, one obtains easily

$$\Sigma_1+\Sigma_3\le C(z).$$

Further, by Lemma 7,

$$\begin{aligned}
\Sigma_2 &\le C\sum_{l=1}^{\infty}\mathbf{E}_z\big[\big|S_l^{(2)}\big|^{-1/2-\delta/3}, \tau > l\big]\\
&\le C\sum_{l=1}^{\infty}\mathbf{P}_z(\tau > l/2)\sup_{\tilde z}\mathbf{E}_{\tilde z}\big[\big|S_{l/2}^{(2)}\big|^{-1/2-\delta/3}\big]\\
&\le C(z)\sum_{l=1}^{\infty}l^{-1/4}\sum_{j=1}^{\infty}\sup_{\tilde z}\mathbf{E}_{\tilde z}\big[\big|S_{l/2}^{(2)}\big|^{-1/2-\delta/3}; j\le \big|S_{l/2}^{(2)}\big|\le j+1\big]\\
&\le C(z)\sum_{l=1}^{\infty}l^{-1/4}\Bigg(\sum_{j=1}^{l^{3/2}}j^{-1/2-\delta/3}\sup_{\tilde z}P_{\tilde z}\big(j\le \big|S_{l/2}^{(2)}\big|\le j+1\big)\\
&\quad + l^{3/2(-1/2-\delta/3)}\sup_{\tilde z}P_{\tilde z}\big(\big|S_{l/2}^{(2)}\big| > l^{3/2}\big)\Bigg).
\end{aligned}$$

Now we use the second concentration inequality from Lemma 9 to get an estimate

$$\sup_{\tilde z}P_{\tilde z}\big(j\le \big|S_l^{(2)}\big|\le j+1\big)\le Cl^{-3/2}.$$

Then,

$$\begin{aligned}
\Sigma_2 &\le C(z)\sum_{l=1}^{\infty}l^{-1/4}\Bigg(l^{-3/2}\sum_{j=1}^{l^{3/2}}j^{-1/2-\delta/3}+l^{-3/4-\delta/2}\Bigg)\\
&\le C(z)\sum_{l=1}^{\infty}l^{-1-\delta/2}\le C(z).
\end{aligned}$$

This proves that the sum (59) is finite.

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References

[1] M. Abramowitz and I. A. Stegun, eds. *Handbook of Mathematical Functions Formulas, Graphs and Mathematical Tables*. Dover Publications Inc., New York. 1992. Reprint of the 1972 edition.

[2] F. Aurzada and S. Dereich. Universality of asymptotics of the one-sided exit problem for integrated processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** (2013) 236–251. MR3060155

[3] F. Aurzada, S. Dereich and M. Lifshits. Persistence probabilities for an integrated random walk bridge. Available at arXiv:1205.2895, 2012.

[4] F. Caravenna and J.-D. Deuschel. Pinning and wetting transition in $(1+1)$-dimensional fields with Laplacian interaction. *Ann. Probab.* **36** (2008) 2388–2433. MR2478687

[5] A. Dembo, J. Ding and F. Gao. Persistence of iterated partial sums. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** (2013) 873–884. Available at arXiv:1205.5596.

[6] D. Denisov and V. Wachtel. Conditional limit theorems for ordered random walks. *Electron. J. Probab.* **15** (2010) 292–322. MR2609589

[7] D. Denisov and V. Wachtel. Random walks in cones. *Ann. Probab.* To appear. Available at arXiv:1110.1254, 2011.

[8] O. Friedland and S. Sodin. Bounds on the concentration function in terms of the Diophantine approximation. *C. R. Math. Acad. Sci. Paris* **345** (2007) 513–518. MR2375113

[9] P. Groeneboom, G. Jongbloed and J. A. Wellner. Integrated Brownian motion, conditioned to be positive. *Ann. Probab.* **27** (1999) 1283–1303. MR1733148

[10] A. Lachal. Sur les excursions de l'intégrale du mouvement brownien. *C. R. Acad. Sci. Paris Sér. I Math.* **314** (1992) 1053–1056. MR1168534
[11] P. Major. The approximation of partial sums of RV's. *Z. Wahrsch. verw. Gebiete* **35** (1976) 213–220. MR0415743
[12] H. P. McKean. Jr. A winding problem for a resonator driven by a white noise. *J. Math. Kyoto Univ.* **2** (1963) 227–235. MR0156389
[13] Ya. G. Sinai. Distribution of some functionals of the integral of a random walk. *Theor. Math. Phys.* **90** (1992) 219–241. MR1182301
[14] V. Vysotsky. On the probability that integrated random walks stay positive. *Stochastic Process. Appl.* **120** (2010) 1178–1193. MR2639743
[15] V. Vysotsky. Positivity of integrated random walks. *Ann. Inst. Henri Poincaré Probab. Stat.* **50** (2014) 195–213. MR3161528