The Annals of Applied Probability
2014, Vol. 24, No. 3, 1199–1225
DOI: 10.1214/13-AAP945

LIMITING SPECTRAL DISTRIBUTION OF A SYMMETRIZED AUTO-CROSS COVARIANCE MATRIX

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This paper studies the limiting spectral distribution (LSD) of a symmetrized auto-cross covariance matrix. The auto-cross covariance matrix is defined as $\mathbf{M}_\tau = \frac{1}{2T}\sum_{j=1}^{T}(\mathbf{e}_j\mathbf{e}_{j+\tau}^* + \mathbf{e}_{j+\tau}\mathbf{e}_j^*)$, where $\mathbf{e}_j$ is an N dimensional vectors of independent standard complex components with properties stated in Theorem 1.1, and τ is the lag. $\mathbf{M}_0$ is well studied in the literature whose LSD is the Marčenko–Pastur (MP) Law. The contribution of this paper is in determining the LSD of $\mathbf{M}_\tau$ where $\tau \geq 1$. It should be noted that the LSD of the $\mathbf{M}_\tau$ does not depend on τ. This study arose from the investigation of and plays an key role in the model selection of any large dimensional model with a lagged time series structure, which is central to large dimensional factor models and singular spectrum analysis.

1. Introduction. Over the last decade and as a result of new sources of large data, the analysis of high-dimensional statistical models has received renewed attention. These models are currently being analyzed within the context of random matrix theory (RMT) in many areas such as statistics [Bai and Silverstein (2010)], economics [Harding (2012), Onatski (2009, 2012)] and engineering [Rao and Edelman (2008), Tulino and Verdu (2004)]. The asymptotic framework assumes that both the dimension corresponding to the number of individual units, N and the number of samples T are large.

Suppose $\mathbf{A}_n$ is an $n \times n$ random Hermitian matrix with eigenvalues λ_j, $j = 1, 2, \ldots, n$. Define a one-dimensional distribution function of the eigenvalues

$$F^{\mathbf{A}_n}(x) = \frac{1}{n}\sharp\{j \leq n : \lambda_j \leq x\},$$

Received October 2012; revised June 2013.
[1]Supported by NSF China Young Scientist Grant 11101397.
[2]Supported by NSF China 11171057, the Program for Changjiang Scholars and the Innovative Research Team in University, the Fundamental Research Funds for the Central Universities and the NUS Grant R-155-000-141-112.
[3]Supported by the Stanford Presidential Fund for Innovation in International Studies.
MSC2010 subject classifications. Primary 60F15, 15A52, 62H25; secondary 60F05, 60F17.
Key words and phrases. Auto-cross covariance, factor analysis, Marčenko–Pastur law, limiting spectral distribution, order detection, random matrix theory, Stieltjes transform.

and call $F^{\mathbf{A}_n}(x)$ the empirical spectral distribution (ESD) of matrix $\mathbf{A}_n$. Here $\sharp E$ denotes the cardinality of the set E. The limit distribution of $\{F^{\mathbf{A}_n}\}$ for a given sequence of random matrices $\{\mathbf{A}_n\}$ is called the limiting spectral distribution (LSD). For any function of bounded variation G, the *Stieltjes transform* of G is defined as $m_G(z) \equiv \int \frac{1}{\lambda - z} dG(\lambda)$ where $z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} : \Im z > 0\}$. For any $n \times n$ matrix $\mathbf{A}_n$ with real eigenvalues $\lambda_1, \ldots, \lambda_n$, the Stieltjes transform of $F^{\mathbf{A}_n}$ is

$$m_{F^{\mathbf{A}_n}}(z) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\lambda_j - z} = \frac{1}{n} \operatorname{tr}(\mathbf{A}_n - z\mathbf{I})^{-1}.$$

Similar to the Fourier transformation in probability theory, there is also a one-to-one correspondence between the distributions and their Stieltjes transforms via the *inversion formula*: for any continuity points $a < b$ of G,

$$G\{[a, b]\} = \frac{1}{\pi} \lim_{\eta \to 0^+} \int_a^b \Im m_G(\xi + i\eta)\, d\xi.$$

Moreover, the continuity theorem holds, that is, a sequence of distributions tends to a weak limit, if and only if their Stieltjes transforms tends to that of the limiting distribution. Therefore, to find the limiting distribution, one can work on finding the limiting Stieltjes transform and use the inversion formula to obtain the limiting distribution.

Research on the LSD of large dimensional random matrices dates back to Wigner (1955, 1958). In these studies, he established that the ESD of a large dimensional Wigner matrix tends to the so-called semicircular law. The LSD of large dimensional sample covariance matrices was studied by Marčenko and Pastur (1967), and the limiting distribution is referred to as the MP law. Further research efforts were conducted to estimate the LSD of a product of two random matrices. To this end, pioneering work was done by Wachter (1980), who considered the LSD of the multivariate F-matrix, the explicit form of which was derived by Bai, Yin and Krishnaiah (1986) and Silverstein (1995). The existence of the LSD of the matrix sequence $\{\mathbf{S}_n\mathbf{T}_n\}$ was established by Yin and Krishnaiah (1983) where $\mathbf{S}_n$ is a standard Wishart matrix, and $\mathbf{T}_n$ is a positive definite matrix. Bai, Miao and Jin (2007) proved the existence of the LSD of $\{\mathbf{S}_n\mathbf{T}_n\}$ where $\mathbf{S}_n$ is a sample covariance matrix, and $\mathbf{T}_n$ is an arbitrary Hermitian matrix. In particular, Bai, Miao and Jin (2007) established the explicit form of LSD of $\{\mathbf{S}_n\mathbf{T}_n\}$ where $\mathbf{S}_n$ is a sample covariance matrix, and $\mathbf{T}_n$ is Wigner matrix. Random matrices of the form $\mathbf{A}_n + \mathbf{X}_n^*\mathbf{T}_n\mathbf{X}_n$ where $\mathbf{A}_n$ is Hermitian matrix, $\mathbf{T}_n$ is diagonal and $\mathbf{X}_n$ consists of i.i.d. (independently and identically distributed) entries, was extensively investigated by many researchers, including Marčenko and Pastur (1967), Grenander and Silverstein (1977), Wachter (1978), Jonsson (1982) and Silverstein and Bai (1995). Furthermore, the LSD of a circulant random matrix was derived by Bose and Mitra (2002) and the LSD of sample correlation matrices was studied by Jiang (2004). Bai and Zhou (2008) considered the LSD of a large-dimensional sample matrix where the assumption of column independence has been relaxed. A large-dimensional vector autoregressive moving average models (LDVARMA) is a special case of the random matrix framework considered by Bai and Zhou (2008). Jin et al. (2009) established the explicit forms of the LSD of covariance matrices of LDVAR(1) and LDVMA(1). Wang, Jin and Miao (2011) established the relationship between the power spectral density function and LSD of covariance matrices of LDVARMA(p, q). A detailed exposition of spectral properties of random matrices is presented in Bai and Silverstein (2004, 2010) and Zheng (2012).

1.1. *Motivation and main result.* In this paper, we will focus our attention on the LSD of a symmetrized auto-cross covariance matrix $\mathbf{M}_\tau = \sum_{k=1}^{T}(\boldsymbol{\gamma}_k\boldsymbol{\gamma}_{k+\tau}^* + \boldsymbol{\gamma}_{k+\tau}\boldsymbol{\gamma}_k^*)$, where $\boldsymbol{\gamma}_k = \frac{1}{\sqrt{2T}}\mathbf{e}_k$, $\mathbf{e}_k = (\varepsilon_{1k}, \dots, \varepsilon_{Nk})'$ and $\{\varepsilon_{it}\}$ are independent random variables with mean 0 and variance σ^2. Here, $\tau \ge 1$ denotes the number of lags. The motivation of this paper comes from any large dimensional model with a lagged time series structure which are central to large dimensional dynamic factor models [Forni and Lippi (2001)] and singular spectrum analysis [Vautard, Tiou and Ghil (1992), Zhigljavsky (2012)].

Consider the framework of a large dimensional dynamic k-factor model with lag q to understand the underlying motivation of this work. This takes the following form:

$$\mathbf{R}_t = \sum_{i=0}^{q} \boldsymbol{\Lambda}_i \mathbf{F}_{t-i} + \mathbf{e}_t, \qquad t = 1, \dots, T,$$

where $\boldsymbol{\Lambda}_i$'s are $N \times k$ nonrandom matrices with full rank. For $t = 1, \dots, T$, $\mathbf{F}_t$'s are k dimensional vectors of i.i.d. standard complex components with finite fourth moment and $\mathbf{e}_t$'s are N dimensional vectors of i.i.d. standard complex components with finite second moment, independent of $\mathbf{F}_t$. This model can be viewed as a large dimensional *information-plus-noise* type model [Dozier and Silverstein (2007a, 2007b), Bai and Silverstein (2012)], with information contained in the summation part and noise in $\mathbf{e}_t$'s. Here "large dimension" refers to N and T, while the number of factors k and the number of lags q are small and fixed. Under this high-dimensional setting, an important statistical problem is the estimation of k and q [Bai and Ng (2002), Harding (2012)]. Let τ be a nonnegative integer. For $j = 1, \dots, T$, define

$$\boldsymbol{\Phi}(\tau) = \frac{1}{2T}\sum_{j=1}^{T}\big(\mathbf{R}_j\mathbf{R}_{j+\tau}^* + \mathbf{R}_{j+\tau}\mathbf{R}_j^*\big)$$

and

$$\mathbf{M}_\tau = \sum_{j=1}^{T}(\boldsymbol{\gamma}_j\boldsymbol{\gamma}_{j+\tau}^* + \boldsymbol{\gamma}_{j+\tau}\boldsymbol{\gamma}_j^*) \qquad \text{where } \boldsymbol{\gamma}_j = \frac{1}{\sqrt{2T}}\mathbf{e}_j.$$

Note that essentially, $\mathbf{M}_\tau$ and $\mathbf{\Phi}(\tau)$ are symmetrized auto-cross covariance matrices at lag τ and generalize the usual sample covariance matrices $\mathbf{M}_0$ and $\mathbf{\Phi}(0)$. The matrix $\mathbf{M}_0$ is well studied in the literature, and it is well known that the limiting spectral distribution (LSD) has an MP law [Marcenko and Pastur (1967)]. Moreover, when $\tau = 0$ and assuming that $\mathrm{Cov}(\mathbf{F}_t) = \mathbf{\Sigma}_f$, the population covariance matrix of $\mathbf{R}_t$ has the same eigenvalues as those of

$$\begin{pmatrix} \sigma^2\mathbf{I} + \mathbf{\Lambda}^*\mathbf{\Sigma}_f\mathbf{\Lambda} & 0 \\ 0 & \sigma^2\mathbf{I} \end{pmatrix}$$

with the two diagonal blocks of size $k(q+1) \times k(q+1)$ and $(N - k(q+1)) \times (N - k(q+1))$, respectively. Therefore, we have the *spiked population model framework* [Johnstone (2001), Baik and Silverstein (2006), Bai and Yao (2008)]. In fact, under certain conditions, we can estimate $k(q+1)$ by counting the number of eigenvalues of $\mathbf{\Phi}(0)$ that are larger than $\sigma^2(1+\sqrt{c})^2$, where c is the limiting ratio of N/T. However, to estimate the values of k and q separately, we need to study the LSD of $\mathbf{M}_\tau$ for at least one $\tau \geq 1$.

It is interesting to note that for $\tau \geq 1$ (τ being a fixed integer), the LSD of $\mathbf{M}_\tau$ does not depend on τ; see Theorem 1.1 for details. However, the number of eigenvalues of $\mathbf{\Phi}(\tau)$ that lie outside the support of the LSD of $\mathbf{M}_\tau$ at lags $1 \leq \tau \leq q$ is dependent on the lag τ and is different with those obtained at lags $\tau > q$. This is mainly because of the contribution of eigenvalues of the terms containing factor and error components are nonzero for $\tau \geq 1$. Thus, we can separate the estimates of k and q by counting the number of eigenvalues of $\Phi(\tau)$ that lie outside the support of the LSD of $\mathbf{M}_\tau$ from $\tau = 0, 1, 2, \dots, q, q+1, \dots$.

Unlike the case $\tau = 0$, not much is known in the literature for $\mathbf{M}_\tau$ as $\tau \geq 1$. The goal of this paper is to derive the LSD of $\mathbf{M}_\tau$ denoted as F_τ, that is, $F_\tau(x) = \lim_{N\to\infty} F^{\mathbf{M}_\tau}(x)$. In our derivation of the LSD of $\mathbf{M}_\tau$, a recursive method is created to solve a disturbed difference equations of order 2.

The main result of this paper is the following theorem.

THEOREM 1.1. *Assume*:

(a) $\tau \geq 1$ *is a fixed integer*;

(b) $\mathbf{e}_k = (\varepsilon_{1k}, \dots, \varepsilon_{Nk})'$, $k = 1, 2, \dots, T+\tau$, *are* N *dimensional vectors of independent standard complex components with* $\sup_{1\leq i\leq N, 1\leq t\leq T+\tau} \mathrm{E}|\varepsilon_{it}|^{2+\delta} \leq M < \infty$ *for some* $\delta \in (0, 2]$, *and for any* $\eta > 0$,

$$\frac{1}{\eta^{2+\delta}NT}\sum_{i=1}^{N}\sum_{t=1}^{T+\tau}\mathrm{E}\big(|\varepsilon_{it}|^{2+\delta}I\big(|\varepsilon_{it}| \geq \eta T^{1/(2+\delta)}\big)\big) = o(1); \tag{1.1}$$

(c) $N/(T+\tau) \to c > 0$ *as* $N, T \to \infty$;
(d) $\mathbf{M}_\tau = \sum_{k=1}^{T}(\boldsymbol{\gamma}_k \boldsymbol{\gamma}_{k+\tau}^* + \boldsymbol{\gamma}_{k+\tau}\boldsymbol{\gamma}_k^*)$, *where* $\boldsymbol{\gamma}_k = \frac{1}{\sqrt{2T}}\mathbf{e}_k$.

Then as $N, T \to \infty$, $F^{\mathbf{M}_\tau} \xrightarrow{D} F_\tau$ *a.s., and* F_τ *has a density function given by*

$$\phi_c(x) = \frac{1}{2c\pi}\sqrt{\frac{y_0^2}{1+y_0} - \left(\frac{1-c}{|x|} + \frac{1}{\sqrt{1+y_0}}\right)^2}, \qquad |x| \le a,$$

where

$$a = \begin{cases} \dfrac{(1-c)\sqrt{1+y_1}}{y_1 - 1}, & c \neq 1, \\ 2, & c = 1, \end{cases}$$

y_0 *is the largest real root of the equation* $y^3 - \frac{(1-c)^2 - x^2}{x^2}y^2 - \frac{4}{x^2}y - \frac{4}{x^2} = 0$, *and* y_1 *is the only real root of the equation*

$$\big((1-c)^2 - 1\big)y^3 + y^2 + y - 1 = 0 \tag{1.2}$$

such that $y_1 > 1$ *if* $c < 1$ *and* $y_1 \in (0,1)$ *if* $c > 1$. *Further, if* $c > 1$, *then* F_τ *has a point mass* $1 - 1/c$ *at the origin.*

REMARK 1.1. Notice that as long as $\tau \ge 1$, F_τ is the same as τ takes different values other than 0. However, F_τ is different from and cannot be reduced to F_0, the distribution of MP law.

REMARK 1.2. When $\tau = o(T)$, the conclusion still holds; see the remark after the proof of Lemma B.3 for details. When $\frac{\tau}{T} \to d$ for some $d > 0$, we conjecture that the LSD will depend on d as well and leave it as future research.

Figure 1 displays the density functions $\phi_c(x)$ with $c = 0.2, 0.5$ and 0.7. Figure 2 displays the density functions $\phi_c(x)$ with $c = 1.5, 2$ and 2.5. It is shown from these two figures that as c increases, the support of $\phi_c(x)$ becomes wider, and $\phi_c(x)$ has the maximum at $x = 0$ which is sharper as c gets closer to 1.

The rest of this paper is organized as follows. The truncation and centralization steps are provided in Section 2. Section 3 outlines the proof of the main theorem Theorem 1.1. Justification of variable truncation, centralization and standardization is provided in Appendix A and some technical lemmas used for the derivation of Theorem 1.1 are presented in Appendix B.

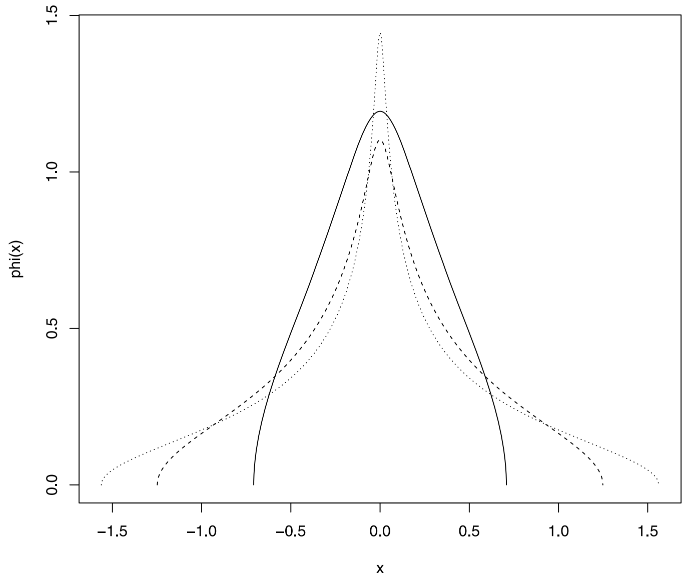

FIG. 1. *Density functions* $\phi_c(x)$ *of the LSD of* $\mathbf{M}_\tau$ *with* $c = 0.2$ (*the solid line*), $c = 0.5$ (*the dashed line*) *and* $c = 0.7$ (*the dotted line*).

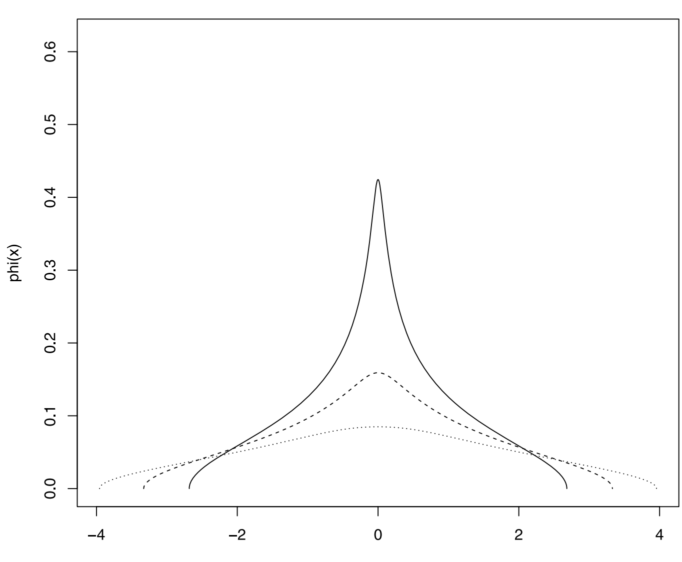

FIG. 2. *Density functions* $\phi_c(x)$ *of the LSD of* $\mathbf{M}_\tau$ *with* $c = 1.5$ (*the solid line*), $c = 2$ (*the dashed line*) *and* $c = 2.5$ (*the dotted line*). *Note that the area under each density function curve is* $1/c$.

2. Truncation, centralization and standardization. First, we can select a sequence $\eta_N \downarrow 0$ such that (1.1) remains true when η is replaced by η_N.

After truncation at $\eta_N T^{1/(2+\delta)}$, centralization and standardization, in what follows, we may assume that

$$|\varepsilon_{ij}| \le \eta_N T^{1/(2+\delta)}, \qquad \mathrm{E}\varepsilon_{ij} = 0, \qquad \mathrm{E}|\varepsilon_{ij}|^2 = 1, \qquad \mathrm{E}|\varepsilon_{ij}|^{2+\delta} < M.$$

The details of verification is provided in Appendix A.

3. Derivation of the LSD of $\mathbf{M}_\tau$. In this section, we will provide the proof for the derivation of Theorem 1.1. To this end, we start with a section on notation followed by the proof.

3.1. *Notation.* Let the Stieltjes transform of $\mathbf{M}_\tau$ be denoted by $m_N(z) = \frac{1}{N}\operatorname{tr}(\mathbf{M}_\tau - z\mathbf{I}_N)^{-1}$ where $z = u + iv$, $v > 0$. We shall prove that $m_N(z) \to m(z)$ for some $m(z)$. It follows that the LSD of $\mathbf{M}_\tau$ exists and has a probability density function $\lim_{v\to 0}\frac{1}{\pi}\Im(m(x+iv))$.

Define

$$\begin{aligned}
\mathbf{A} &= \mathbf{M}_\tau - z\mathbf{I}_N,\\
\mathbf{A}_k &= \mathbf{A} - \boldsymbol{\gamma}_k(\boldsymbol{\gamma}_{k+\tau} + \boldsymbol{\gamma}_{k-\tau})^* - (\boldsymbol{\gamma}_{k+\tau} + \boldsymbol{\gamma}_{k-\tau})\boldsymbol{\gamma}_k^*,\\
\mathbf{A}_{k,k+\tau,\dots,k+n\tau} &= \mathbf{A}_{k,k+\tau,\dots,k+(n-1)\tau} - \boldsymbol{\gamma}_{k+(n+1)\tau}\boldsymbol{\gamma}_{k+n\tau}^* - \boldsymbol{\gamma}_{k+n\tau}\boldsymbol{\gamma}_{k+(n+1)\tau}^*,\\
&\qquad n \ge 1
\end{aligned}$$

for $k \in [\tau+1, T]$. Note that $\mathbf{A}_{k,k+\tau,\dots,k+n\tau}$ is independent of $\boldsymbol{\gamma}_k, \dots, \boldsymbol{\gamma}_{k+n\tau}$. For $k \le \tau$ or $k > T$, we still use the definition of $\mathbf{A}_k$ with the convention that $\boldsymbol{\gamma}_l = \mathbf{0}$ for $l \le 0$ or $l > T + \tau$.

3.2. *Derivation.* By

$$\mathbf{A} = \sum_{k=1}^{T}(\boldsymbol{\gamma}_k\boldsymbol{\gamma}_{k+\tau}^* + \boldsymbol{\gamma}_{k+\tau}\boldsymbol{\gamma}_k^*) - z\mathbf{I}_N$$

we have

$$\mathbf{I}_N = \sum_{k=1}^{T}(\mathbf{A}^{-1}\boldsymbol{\gamma}_k\boldsymbol{\gamma}_{k+\tau}^* + \mathbf{A}^{-1}\boldsymbol{\gamma}_{k+\tau}\boldsymbol{\gamma}_k^*) - z\mathbf{A}^{-1}.$$

Taking trace and dividing by N, we obtain

$$1 + zm_N(z) = \frac{1}{N}\sum_{k=1}^{T}(\boldsymbol{\gamma}_{k+\tau}^*\mathbf{A}^{-1}\boldsymbol{\gamma}_k + \boldsymbol{\gamma}_k^*\mathbf{A}^{-1}\boldsymbol{\gamma}_{k+\tau}). \tag{3.1}$$

Applying the identity

$$(\mathbf{B} + \boldsymbol{\alpha}\boldsymbol{\gamma}^*)^{-1} = \mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}\boldsymbol{\alpha}\boldsymbol{\gamma}^*\mathbf{B}^{-1}}{1 + \boldsymbol{\gamma}^*\mathbf{B}^{-1}\boldsymbol{\alpha}} \tag{3.2}$$

for any nonsingular matrix $\mathbf{B}$, we have

$$\begin{aligned}\boldsymbol{\gamma}_k^*\mathbf{A}^{-1}(\boldsymbol{\gamma}_{k+\tau}+\boldsymbol{\gamma}_{k-\tau}) &= \frac{\boldsymbol{\gamma}_k^*\tilde{\mathbf{A}}_k^{-1}(\boldsymbol{\gamma}_{k+\tau}+\boldsymbol{\gamma}_{k-\tau})}{1+\boldsymbol{\gamma}_k^*\tilde{\mathbf{A}}_k^{-1}(\boldsymbol{\gamma}_{k+\tau}+\boldsymbol{\gamma}_{k-\tau})}\\ &= 1-\frac{1}{1+\boldsymbol{\gamma}_k^*\tilde{\mathbf{A}}_k^{-1}(\boldsymbol{\gamma}_{k+\tau}+\boldsymbol{\gamma}_{k-\tau})},\end{aligned}$$

where $\tilde{\mathbf{A}}_k = \mathbf{A} - (\boldsymbol{\gamma}_{k+\tau}+\boldsymbol{\gamma}_{k-\tau})\boldsymbol{\gamma}_k^*$ and we have used the previously made convention that $\boldsymbol{\gamma}_l = \mathbf{0}$ for $l \le 0$ or $l > T+\tau$. Note that $\mathbf{A}_k = \tilde{\mathbf{A}}_k - \boldsymbol{\gamma}_k(\boldsymbol{\gamma}_{k+\tau}+\boldsymbol{\gamma}_{k-\tau})^*$. Using (3.2) again, we have

$$\begin{aligned}&\boldsymbol{\gamma}_k^*\tilde{\mathbf{A}}_k^{-1}(\boldsymbol{\gamma}_{k+\tau}+\boldsymbol{\gamma}_{k-\tau})\\ &\quad= \boldsymbol{\gamma}_k^*\mathbf{A}_k^{-1}(\boldsymbol{\gamma}_{k+\tau}+\boldsymbol{\gamma}_{k-\tau}) - \frac{\boldsymbol{\gamma}_k^*\mathbf{A}_k^{-1}\boldsymbol{\gamma}_k(\boldsymbol{\gamma}_{k+\tau}^*+\boldsymbol{\gamma}_{k-\tau}^*)\mathbf{A}_k^{-1}(\boldsymbol{\gamma}_{k+\tau}+\boldsymbol{\gamma}_{k-\tau})}{1+(\boldsymbol{\gamma}_{k+\tau}^*+\boldsymbol{\gamma}_{k-\tau}^*)\mathbf{A}_k^{-1}\boldsymbol{\gamma}_k}.\end{aligned}$$

By Lemmas B.1 and B.2, we have

$$\boldsymbol{\gamma}_k^*\tilde{\mathbf{A}}_k^{-1}(\boldsymbol{\gamma}_{k+\tau}+\boldsymbol{\gamma}_{k-\tau}) = -\frac{c}{2}m_N(z)(\boldsymbol{\gamma}_{k+\tau}^*+\boldsymbol{\gamma}_{k-\tau}^*)\mathbf{A}_k^{-1}(\boldsymbol{\gamma}_{k+\tau}+\boldsymbol{\gamma}_{k-\tau}) + o_{\text{a.s.}}(1).$$

Consequently,

$$\begin{aligned}&\boldsymbol{\gamma}_k^*\mathbf{A}^{-1}(\boldsymbol{\gamma}_{k+\tau}+\boldsymbol{\gamma}_{k-\tau})\\ &\quad= 1-\frac{1}{1-(c/2)m_N(z)(\boldsymbol{\gamma}_{k+\tau}^*+\boldsymbol{\gamma}_{k-\tau}^*)\mathbf{A}_k^{-1}(\boldsymbol{\gamma}_{k+\tau}+\boldsymbol{\gamma}_{k-\tau})} + o_{\text{a.s.}}(1).\end{aligned}$$

Write $\mathbf{A}_{k,k+\tau} = \mathbf{A}_k - \boldsymbol{\gamma}_{k+2\tau}\boldsymbol{\gamma}_{k+\tau}^* - \boldsymbol{\gamma}_{k+\tau}\boldsymbol{\gamma}_{k+2\tau}^*$ which is independent of $\boldsymbol{\gamma}_{k+\tau}$. Then, using (3.2) again, we obtain

$$\begin{aligned}&\boldsymbol{\gamma}_{k+\tau}^*\mathbf{A}_k^{-1}\boldsymbol{\gamma}_{k+\tau}\\ &\quad= \frac{\boldsymbol{\gamma}_{k+\tau}^*(\mathbf{A}_{k,k+\tau}+\boldsymbol{\gamma}_{k+2\tau}\boldsymbol{\gamma}_{k+\tau}^*)^{-1}\boldsymbol{\gamma}_{k+\tau}}{1+\boldsymbol{\gamma}_{k+2\tau}^*(\mathbf{A}_{k,k+\tau}+\boldsymbol{\gamma}_{k+2\tau}\boldsymbol{\gamma}_{k+\tau}^*)^{-1}\boldsymbol{\gamma}_{k+\tau}}\\ &\quad= \big(\boldsymbol{\gamma}_{k+\tau}^*\mathbf{A}_{k,k+\tau}^{-1}\boldsymbol{\gamma}_{k+\tau} - (\boldsymbol{\gamma}_{k+\tau}^*\mathbf{A}_{k,k+\tau}^{-1}\boldsymbol{\gamma}_{k+2\tau}\boldsymbol{\gamma}_{k+\tau}^*\mathbf{A}_{k,k+\tau}^{-1}\boldsymbol{\gamma}_{k+\tau})\\ &\qquad\qquad /(1+\boldsymbol{\gamma}_{k+\tau}^*\mathbf{A}_{k,k+\tau}^{-1}\boldsymbol{\gamma}_{k+2\tau})\big)\\ &\qquad /\big(1+\boldsymbol{\gamma}_{k+2\tau}^*\mathbf{A}_{k,k+\tau}^{-1}\boldsymbol{\gamma}_{k+\tau} - (\boldsymbol{\gamma}_{k+2\tau}^*\mathbf{A}_{k,k+\tau}^{-1}\boldsymbol{\gamma}_{k+2\tau}\boldsymbol{\gamma}_{k+\tau}^*\mathbf{A}_{k,k+\tau}^{-1}\boldsymbol{\gamma}_{k+\tau})\\ &\qquad\qquad /(1+\boldsymbol{\gamma}_{k+\tau}^*\mathbf{A}_{k,k+\tau}^{-1}\boldsymbol{\gamma}_{k+2\tau})\big)\\ &\quad= \frac{(c/2)m_N(z)}{1-(c/2)m_N(z)\boldsymbol{\gamma}_{k+2\tau}^*\mathbf{A}_{k,k+\tau}^{-1}\boldsymbol{\gamma}_{k+2\tau}} + o_{\text{a.s.}}(1).\end{aligned}\tag{3.3}$$

By the same reasoning,

$$\boldsymbol{\gamma}_{k-\tau}^*\mathbf{A}_k^{-1}\boldsymbol{\gamma}_{k-\tau} = \frac{(c/2)m_N(z)}{1-(c/2)m_N(z)\boldsymbol{\gamma}_{k-2\tau}^*\mathbf{A}_{k,k-\tau}^{-1}\boldsymbol{\gamma}_{k-2\tau}} + o_{\text{a.s.}}(1).\tag{3.4}$$

Next, we consider the cross terms. We have

$$
\begin{aligned}
&\boldsymbol{\gamma}_{k-\tau}^* \mathbf{A}_k^{-1} \boldsymbol{\gamma}_{k+\tau} \\
&\quad= \frac{\boldsymbol{\gamma}_{k-\tau}^* (\mathbf{A}_{k,k+\tau} + \boldsymbol{\gamma}_{k+2\tau}\boldsymbol{\gamma}_{k+\tau}^*)^{-1} \boldsymbol{\gamma}_{k+\tau}}{1 + \boldsymbol{\gamma}_{k+2\tau}^* (\mathbf{A}_{k,k+\tau} + \boldsymbol{\gamma}_{k+2\tau}\boldsymbol{\gamma}_{k+\tau}^*)^{-1} \boldsymbol{\gamma}_{k+\tau}} \\
&\quad= \big(\boldsymbol{\gamma}_{k-\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \boldsymbol{\gamma}_{k+\tau} - (\boldsymbol{\gamma}_{k-\tau} \mathbf{A}_{k,k+\tau}^{-1} \boldsymbol{\gamma}_{k+2\tau} \boldsymbol{\gamma}_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \boldsymbol{\gamma}_{k+\tau}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad / \big(1 + \boldsymbol{\gamma}_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \boldsymbol{\gamma}_{k+2\tau}\big)\big) \qquad (3.5) \\
&\qquad / \big(1 + \boldsymbol{\gamma}_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \boldsymbol{\gamma}_{k+\tau} - (\boldsymbol{\gamma}_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \boldsymbol{\gamma}_{k+2\tau} \boldsymbol{\gamma}_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \boldsymbol{\gamma}_{k+\tau}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad / \big(1 + \boldsymbol{\gamma}_{k+\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \boldsymbol{\gamma}_{k+2\tau}\big)\big) \\
&\quad= \frac{-(c/2) m_N(z) \boldsymbol{\gamma}_{k-\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \boldsymbol{\gamma}_{k+2\tau}}{1 - (c/2) m_N(z) \boldsymbol{\gamma}_{k+2\tau}^* \mathbf{A}_{k,k+\tau}^{-1} \boldsymbol{\gamma}_{k+2\tau}} + o_{\text{a.s.}}(1).
\end{aligned}
$$

Suppose that $m_N(z)$ converges to $m(z)$ along some subsequence $N = n'$, by Lemmas B.3 and B.4, (3.1) will converge to

$$
c + czm(z) = 1 - \frac{1}{1 - c^2 m^2(z)/(2x_1)}, \tag{3.6}
$$

where x_1 is the root of the equation $x^2 = x - \frac{c^2 m^2(z)}{4}$ with the larger absolute value. Substituting the expression of x_1, we obtain

$$
\big(1 - c^2 m^2(z)\big)\big(c + czm(z) - 1\big)^2 = 1. \tag{3.7}
$$

This can be further simplified to

$$
\begin{aligned}
&\big(cm(z)\big)^4 - \frac{2(1-c)}{z}\big(cm(z)\big)^3 + \frac{(1-c)^2 - z^2}{z^2}\big(cm(z)\big)^2 \\
&\qquad + \frac{2(1-c)}{z} cm(z) + \frac{1 - (1-c)^2}{z^2} = 0.
\end{aligned} \tag{3.8}
$$

Now, we shall employ the method developed in Bai, Miao and Jin (2007) to solve the 4th degree polynomial equation and identify the unique solution of the limiting spectral distribution. Rewrite equation (3.8) as

$$
\begin{aligned}
&\left(\big(cm(z)\big)^2 - \frac{(1-c)}{z} cm(z) + \frac{y}{2}\right)^2 \\
&\qquad = (1+y)\big(cm(z)\big)^2 - \frac{(1-c)}{z}(y+2) cm(z) + \frac{y^2}{4} - \frac{1-(1-c)^2}{z^2}.
\end{aligned} \tag{3.9}
$$

Let y_0 be the root with largest real part of the equation

$$
y^3 - \frac{(1-c)^2 - z^2}{z^2} y^2 - \frac{4}{z^2} y - \frac{4}{z^2} = 0.
$$

Let $f(y) = y^3 - \frac{(1-c)^2 - z^2}{z^2} y^2 - \frac{4}{z^2} y - \frac{4}{z^2}$. For $f(+\infty) > 0$, $f(0) < 0$, we have $y_0 > 0$. Further if $z \to 0$, then $y_0 \to \infty$ and $z^2 y_0 \to (1-c)^2$. If we replace y by y_0 in equation (3.9), the solutions to (3.8) will be those to equations

$$\begin{aligned}(cm(z))^2 - \frac{1-c}{z} cm(z) + \frac{1}{2} y_0 \\ = \pm\sqrt{1+y_0}\left(cm(z) - \frac{(y_0+2)(1-c)}{2z(1+y_0)}\right),\end{aligned}$$

from which we get the following four roots:

$$m_1(z) = \frac{((1-c)/z + \sqrt{1+y_0}) + \sqrt{((1-c)/z - 1/\sqrt{1+y_0})^2 - y_0^2/(1+y_0)}}{2c},$$

$$m_2(z) = \frac{((1-c)/z + \sqrt{1+y_0}) - \sqrt{((1-c)/z - 1/\sqrt{1+y_0})^2 - y_0^2/(1+y_0)}}{2c},$$

$$m_3(z) = \frac{((1-c)/z - \sqrt{1+y_0}) + \sqrt{((1-c)/z + 1/\sqrt{1+y_0})^2 - y_0^2/(1+y_0)}}{2c},$$

$$m_4(z) = \frac{((1-c)/z - \sqrt{1+y_0}) - \sqrt{((1-c)/z + 1/\sqrt{1+y_0})^2 - y_0^2/(1+y_0)}}{2c}.$$

Now, we claim that the point mass of F_τ at the origin $\lim_{z\to 0} -zm(z)$ satisfies

$$\lim_{z\to 0} -zm(z) = \begin{cases} 1 - 1/c, & c > 1, \\ 0, & c \le 1. \end{cases} \tag{3.10}$$

To show our claim, first by equation (3.7), we have

$$(z^2 - c^2 z^2 m^2(z))(czm(z) - (1-c))^2 = z^2.$$

This means $zm(z)$ must be bounded as $z \to 0$. Otherwise, the LHS of the equation above is unbounded while the RHS tends to 0, which is a contradiction. Hence, the equation above can be simplified as

$$(czm(z))^2 (czm(z) - (1-c))^2 = 0.$$

This means there exists a convergent subsequence $\{z_k m(z_k)\}$ such that its limit $\lim_{z_k\to 0} -z_k m(z_k)$ can only be either 0 or $1 - \frac{1}{c}$. Notice that $\lim_{z\to 0} -zm(z)$ is the point mass of F_τ at 0, which is nonnegative. Therefore, as $c < 1$, $\lim_{z_k\to 0} -z_k m(z_k) \ne 1 - \frac{1}{c}$. Hence $\lim_{z_k\to 0} -z_k m(z_k) = 0$ and the second part of our claim is proved. When $c > 1$, assume $\lim_{z_k\to 0} -z_k m(z_k) \ne 1 - \frac{1}{c}$, that is, $\lim_{z_k\to 0} -z_k m(z_k) = 0$, and then (3.6) becomes

$$c = 1 - \frac{1}{1 - c^2 m^2(z)/(2x_1)}.$$

Solve this for x_1, and we have

$$x_1 = \frac{c^2 m^2(z)(c-1)}{2c} = \frac{c-2}{2(c-1)}.$$

Here the last equality is due to the fact $1 - c^2 m^2(z) = \frac{1}{(1-c)^2}$ which can be derived from (3.7) and our assumption that $\lim_{z\to 0} -zm(z) = 0$. However, solve the equation $x^2 = x - \frac{c^2 m^2(z)}{4}$, and use the fact $1 - c^2 m^2(z) = \frac{1}{(1-c)^2}$ again, and we have

$$x_1 = \frac{1}{2}\left(1 + \frac{1}{c-1}\right) = \frac{c}{2(c-1)},$$

which contradicts our last expression of x_1. Hence the first part of the claim is proved.

By (3.10), $\lim_{z\to 0^+} z\sqrt{y} \to |1-c|$, $\lim_{z\to 0^-} z\sqrt{y} \to -|1-c|$ and $\phi_c(x) = \lim_{v\to 0} \frac{1}{\pi}\Im(m(x+iv)) > 0$, and we get

$$m(z) = \begin{cases} m_1(z), & z < 0, \\ m_3(z), & z > 0. \end{cases}$$

Therefore, we have

$$\phi_c(x) = \lim_{v\to 0} \frac{1}{\pi}\Im\big(m(x+iv)\big) = \frac{1}{2c\pi}\sqrt{\frac{y_0^2}{1+y_0} - \left(\frac{1-c}{|x|} + \frac{1}{\sqrt{1+y_0}}\right)^2}, \qquad |x| \le a,$$

where y_0 is the largest real root of the equation $y^3 - \frac{(1-c)^2 - x^2}{x^2} y^2 - \frac{4}{x^2} y - \frac{4}{x^2} = 0$, and a satisfies equations

$$\begin{cases} \dfrac{y^2}{1+y} - \left(\dfrac{1-c}{a} + \dfrac{1}{\sqrt{1+y}}\right)^2 = 0, \\ y^3 - \dfrac{(1-c)^2 - a^2}{a^2} y^2 - \dfrac{4}{a^2} y - \dfrac{4}{a^2} = 0. \end{cases}$$

Solving these equations under the condition $a > 0$, we have

$$a = \begin{cases} \dfrac{(1-c)\sqrt{1+y_1}}{y_1 - 1}, & c \ne 1, \\ 2, & c = 1, \end{cases}$$

where y_1 can be chosen as a real root of the equation

$$\big((1-c)^2 - 1\big)y^3 + y^2 + y - 1 = 0 \tag{3.11}$$

such that $y_1 > 1$ if $c < 1$ and $y_1 \in (0, 1)$ if $c > 1$.

To show the unique existence of y_1, let $f(y) = ((1-c)^2 - 1)y^3 + y^2 + y - 1$. If $c < 1$, by $f(-\infty) > 0$, $f(0) < 0$, $f(1) > 0$ and $f(\infty) < 0$, there are three real roots $y_1 > 1$, $y_2 \in (0,1)$ and $y_3 < 0$ of (3.11). Similarly, if $1 < c < 2$, there are three real roots $y_1 \in (0,1)$, $y_2 > 1$ and $y_3 < 0$ of (3.11). If $c = 2$, it is easy to see that there are two real roots of (3.11): $y_1 = (\sqrt{5}-1)/2 \in (0,1)$ and $y_2 = (-\sqrt{5}-1)/2 < 0$. If $c > 2$, by $f(0) < 0$ and $f(1) > 0$, there is a real root $y_1 \in (0,1)$. If there is more than one real root in the interval $(0,1)$ when $c > 2$, then by the continuity of $f(y)$, the three roots y_1, y_2, y_3 of $f(y)$ are all in the interval $(0,1)$, that would contradict $y_1 + y_2 + y_3 = -1/((1-c)^2 - 1) < 0$. Thus there is only one real root $y_1 \in (0,1)$ if $c > 2$. The proof of Theorem 1.1 is complete.

APPENDIX A: JUSTIFICATION OF TRUNCATION, CENTRALIZATION AND STANDARDIZATION

Note that $\operatorname{rank}(\mathbf{AB} - \mathbf{CD}) \le \operatorname{rank}(\mathbf{A} - \mathbf{C}) + \operatorname{rank}(\mathbf{B} - \mathbf{D})$ because

$$\mathbf{AB} - \mathbf{CD} = (\mathbf{A} - \mathbf{C})\mathbf{B} + \mathbf{C}(\mathbf{B} - \mathbf{D}).$$

Let $\tilde{\varepsilon}_{it} = \varepsilon_{it} I(|\varepsilon_{it}| < \eta_N T^{1/(2+\delta)})$, $\tilde{\boldsymbol{\gamma}}_k = \frac{1}{\sqrt{2T}}(\tilde{\varepsilon}_{1k}, \dots, \tilde{\varepsilon}_{Nk})'$ and $\tilde{\mathbf{M}}_\tau = \sum_{k=1}^T (\tilde{\boldsymbol{\gamma}}_k \tilde{\boldsymbol{\gamma}}_{k+\tau}^* + \tilde{\boldsymbol{\gamma}}_{k+\tau} \tilde{\boldsymbol{\gamma}}_k^*)$.

By Theorem A.43 of Bai and Silverstein (2010),

$$\begin{aligned} \|F^{\mathbf{M}_\tau} - F^{\tilde{\mathbf{M}}_\tau}\| &\le \frac{1}{N} \operatorname{rank}(\mathbf{M}_\tau - \tilde{\mathbf{M}}_\tau) \\ &\le \frac{2}{N} \operatorname{rank}\left(\sum_{k=1}^T (\boldsymbol{\gamma}_k \boldsymbol{\gamma}_{k+\tau}^*) - \sum_{k=1}^T (\tilde{\boldsymbol{\gamma}}_k \tilde{\boldsymbol{\gamma}}_{k+\tau}^*) \right) \\ &\le \frac{4}{N} \sum_{i=1}^N \sum_{t=1}^{T+\tau} I\big(|\varepsilon_{it}| \ge \eta_N T^{1/(2+\delta)}\big). \end{aligned}$$

By (1.1) we have

$$\begin{aligned} &\mathrm{E}\left(\frac{1}{N} \sum_{i=1}^N \sum_{t=1}^{T+\tau} I\big(|\varepsilon_{it}| \ge \eta T^{1/(2+\delta)}\big) \right) \\ &\quad \le \frac{1}{\eta^{2+\delta} NT} \sum_{i=1}^N \sum_{t=1}^{T+\tau} \mathrm{E}\big(|\varepsilon_{it}|^{(2+\delta)} I\big(|\varepsilon_{it}| \ge \eta T^{1/(2+\delta)}\big)\big) = o(1) \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} &\operatorname{Var}\left(\frac{1}{N} \sum_{i=1}^N \sum_{t=1}^{T+\tau} I\big(|\varepsilon_{it}| \ge \eta T^{1/(2+\delta)}\big) \right) \\ &\quad \le \frac{1}{\eta^{2+\delta} N^2 T} \sum_{i=1}^N \sum_{t=1}^{T+\tau} \mathrm{E}\big(|\varepsilon_{it}|^{2+\delta} I\big(|\varepsilon_{it}| \ge \eta T^{1/(2+\delta)}\big)\big) = o(1/N). \end{aligned}$$

Applying Bernstein's inequality, for all small $\varepsilon > 0$ and large N, we have

$$\mathrm{P}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T+\tau} I\big(|\varepsilon_{it}| \geq \eta T^{1/(2+\delta)}\big) \geq \varepsilon\right) \leq 2e^{-(1/2)\varepsilon^2 N}.$$

By the Borel–Cantelli lemma, with probability 1, we have

$$\|F^{\mathbf{M}_\tau} - F^{\tilde{\mathbf{M}}_\tau}\| \to 0.$$

Let $\hat{\varepsilon}_{it} = \tilde{\varepsilon}_{it} - \mathrm{E}\tilde{\varepsilon}_{it}$, $\hat{\boldsymbol{\gamma}}_k = \frac{1}{\sqrt{2T}}(\hat{\varepsilon}_{1k}, \dots, \hat{\varepsilon}_{Nk})'$, and $\hat{\mathbf{M}}_\tau = \sum_{k=1}^{T}(\hat{\boldsymbol{\gamma}}_k\hat{\boldsymbol{\gamma}}_{k+\tau}^* + \hat{\boldsymbol{\gamma}}_{k+\tau}\hat{\boldsymbol{\gamma}}_k^*)$.

By Theorem A.46 of Bai and Silverstein (2010),

$$\begin{aligned}
&L\big(F^{\tilde{\mathbf{M}}_\tau}, F^{\hat{\mathbf{M}}_\tau}\big) \\
&\quad\leq \max_k\big|\lambda_k(\tilde{\mathbf{M}}_\tau) - \lambda_k(\hat{\mathbf{M}}_\tau)\big| \leq \|\tilde{\mathbf{M}}_\tau - \hat{\mathbf{M}}_\tau\| \\
&\quad\leq 2\left\|\sum_{k=1}^{T}(\hat{\boldsymbol{\gamma}}_k\mathrm{E}\tilde{\boldsymbol{\gamma}}_{k+\tau}^* + \hat{\boldsymbol{\gamma}}_{k+\tau}\mathrm{E}\tilde{\boldsymbol{\gamma}}_k^*)\right\| + \left\|\sum_{k=1}^{T}(\mathrm{E}\tilde{\boldsymbol{\gamma}}_k\mathrm{E}\tilde{\boldsymbol{\gamma}}_{k+\tau}^* + \mathrm{E}\tilde{\boldsymbol{\gamma}}_{k+\tau}\mathrm{E}\tilde{\boldsymbol{\gamma}}_k^*)\right\|,
\end{aligned}$$

where L is the Lévy distance between two distribution functions. For the second part, we have

$$\begin{aligned}
&\left\|\sum_{k=1}^{T}(\mathrm{E}\tilde{\boldsymbol{\gamma}}_k\mathrm{E}\tilde{\boldsymbol{\gamma}}_{k+\tau}^* + \mathrm{E}\tilde{\boldsymbol{\gamma}}_{k+\tau}\mathrm{E}\tilde{\boldsymbol{\gamma}}_k^*)\right\| \\
&\quad\leq \frac{1}{T}\sum_{k=1}^{T}\sum_{i=1}^{N}\big|\mathrm{E}\big(\varepsilon_{ik}I\big(|\varepsilon_{ik}| \geq \eta T^{1/(2+\delta)}\big)\big)\mathrm{E}\big(\varepsilon_{i(k+\tau)}I\big(|\varepsilon_{i(k+\tau)}| \geq \eta T^{1/(2+\delta)}\big)\big)\big| \\
&\quad\leq \frac{C}{T^2}\sum_{k=1}^{T+\tau}\sum_{i=1}^{N}\mathrm{E}\big(|\varepsilon_{ik}|^{2+\delta}I\big(|\varepsilon_{ik}| \geq \eta T^{1/(2+\delta)}\big)\big) = o(1).
\end{aligned}$$

For the first part, notice that

$$\left\|\sum_{k=1}^{T}(\hat{\boldsymbol{\gamma}}_k\mathrm{E}\tilde{\boldsymbol{\gamma}}_{k+\tau}^* + \hat{\boldsymbol{\gamma}}_{k+\tau}\mathrm{E}\tilde{\boldsymbol{\gamma}}_k^*)\right\|^2 \leq 2\left(\left\|\sum_{k=1}^{T}\hat{\boldsymbol{\gamma}}_k\mathrm{E}\tilde{\boldsymbol{\gamma}}_{k+\tau}^*\right\|^2 + \left\|\sum_{k=1}^{T}\hat{\boldsymbol{\gamma}}_{k+\tau}\mathrm{E}\tilde{\boldsymbol{\gamma}}_k^*\right\|^2\right),$$

$$\begin{aligned}
&\left\|\sum_{k=1}^{T}\hat{\boldsymbol{\gamma}}_k\mathrm{E}\tilde{\boldsymbol{\gamma}}_{k+\tau}^*\right\|^2 \\
&\quad\leq \frac{1}{4T^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\sum_{k=1}^{T}\hat{\varepsilon}_{ki}\mathrm{E}\tilde{\varepsilon}_{(k+\tau)j}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k_1=1}^{T} \sum_{k_2=1}^{T} (\hat{\varepsilon}_{k_1 i} \hat{\varepsilon}_{k_2 i} \mathrm{E}\tilde{\varepsilon}_{(k_1+\tau)j} \mathrm{E}\tilde{\varepsilon}_{(k_2+\tau)j}) \\
&= \frac{1}{4T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\sum_{k_1=1}^{T} \hat{\varepsilon}_{k_1 i}^2 (\mathrm{E}\tilde{\varepsilon}_{(k_1+\tau)j})^2 + \sum_{k_1 \neq k_2} \hat{\varepsilon}_{k_1 i} \hat{\varepsilon}_{k_2 i} \mathrm{E}\tilde{\varepsilon}_{(k_1+\tau)j} \mathrm{E}\tilde{\varepsilon}_{(k_2+\tau)j} \right) \\
&\equiv J_{11} + J_{12}.
\end{aligned}
$$

For $\mathrm{E}\hat{\varepsilon}_{k_1 i}^2 < \infty$ and $\mathrm{E}\tilde{\varepsilon}_{(k_1+\tau)j}^{2+\delta} < \infty$, there exist constants C_1, C_2 and C_3 such that

$$
\begin{aligned}
\mathrm{E}J_{11} &= \frac{1}{4T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k_1=1}^{T} \mathrm{E}\hat{\varepsilon}_{k_1 i}^2 (\mathrm{E}\tilde{\varepsilon}_{(k_1+\tau)j})^2 \\
&\le \frac{C_1}{4T^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k_1=1}^{T} \bigl(\mathrm{E}\bigl(|\varepsilon_{jk}| I\bigl(|\varepsilon_{jk}| \ge \eta T^{1/(2+\delta)}\bigr)\bigr)\bigr)^2 \\
&\le \frac{C_1}{4T^2 \eta^{2(1+\delta)} T^{2(1+\delta)/(2+\delta)}} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k_1=1}^{T} \bigl(\mathrm{E}\bigl(|\varepsilon_{jk}|^{2+\delta} I\bigl(|\varepsilon_{jk}| \ge \eta T^{1/(2+\delta)}\bigr)\bigr)\bigr)^2 \\
&= O\bigl(T^{-\delta/(2+\delta)}\bigr)
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{Var} J_{11} &= \frac{1}{4^2 T^4} \sum_{i=1}^{N} \sum_{k_1=1}^{T} \mathrm{E}\bigl(\hat{\varepsilon}_{k_1 i}^2 - \mathrm{E}\hat{\varepsilon}_{k_1 i}^2\bigr)^2 \left(\sum_{j=1}^{N} (\mathrm{E}\tilde{\varepsilon}_{(k_1+\tau)j})^2 \right)^2 \\
&\le \frac{C_2}{4^2 T^4} \sum_{i=1}^{N} \sum_{k_1=1}^{T} \mathrm{E}\tilde{\varepsilon}_{k_1 i}^4 \left(N \frac{1}{\eta^{2(1+\delta)} T^{2(1+\delta)/(2+\delta)}} \right)^2 \\
&= O\bigl(T^{-1-4\delta/(2+\delta)}\bigr).
\end{aligned}
$$

The previous two equations imply that $J_{11} \to 0$, a.s.

Furthermore, we have

$$
\begin{aligned}
\operatorname{Var} J_{12} &= \frac{1}{4^2 T^4} \sum_{i=1}^{N} \sum_{k_1 \neq k_2} \mathrm{E}\hat{\varepsilon}_{k_1 i}^2 \mathrm{E}\hat{\varepsilon}_{k_2 i}^2 \left(\sum_{j=1}^{N} \mathrm{E}\tilde{\varepsilon}_{(k_1+\tau)j} \mathrm{E}\tilde{\varepsilon}_{(k_2+\tau)j} \right)^2 \\
&\le \frac{C_3}{4^2 T^4} \sum_{i=1}^{N} \sum_{k_1 \neq k_2} \left(\sum_{j=1}^{N} \frac{1}{\eta^{2(1+\delta)} T^{2(1+\delta)/(2+\delta)}} \right)^2 \\
&= O\bigl(T^{-1-2\delta/(2+\delta)}\bigr),
\end{aligned}
$$

which implies $J_{12} \to 0$, a.s. Hence, we have $\| \sum_{k=1}^{T} \hat{\boldsymbol{\gamma}}_k \mathrm{E}\tilde{\boldsymbol{\gamma}}_{k+\tau}^* \|^2 \to 0$, a.s. Similarly $\| \sum_{k=1}^{T} \hat{\boldsymbol{\gamma}}_{k+\tau} \mathrm{E}\tilde{\boldsymbol{\gamma}}_k^* \|^2 \to 0$ a.s. Thus $L(F^{\tilde{\mathbf{M}}_\tau}, F^{\hat{\mathbf{M}}_\tau}) \to 0$, a.s.

Now, we want to rescale the variables.

Let $\sigma_{ij}^2 = \mathrm{E}|\hat{\varepsilon}_{ij}|^2 = \mathrm{E}|\tilde{\varepsilon}_{ij} - \mathrm{E}\tilde{\varepsilon}_{ij}|^2$. Define $E \equiv \{(i,j) : \sigma_{ij}^2 < 1 - \Delta\}$ and

$$\check{\varepsilon}_{it} = \begin{cases} X_{it}, & (i,t) \in E, \\ \dfrac{\hat{\varepsilon}_{it}}{\sigma_{it}}, & \text{otherwise.} \end{cases}$$

Here $\Delta = T^{-\delta/(4+2\delta)}$ and X_{it}'s are i.i.d. random variables taking values 1 and -1, each with probability $\frac{1}{2}$. Note that $\mathrm{E}\check{\varepsilon}_{it} = 0$ and $\mathrm{Var}(\check{\varepsilon}_{it}) = 1$.

Let $\check{\boldsymbol{\gamma}}_k = \frac{1}{\sqrt{2T}}(\check{\varepsilon}_{1k}, \ldots, \check{\varepsilon}_{Nk})'$ and $\check{\mathbf{M}}_\tau = \sum_{k=1}^T (\check{\boldsymbol{\gamma}}_k \check{\boldsymbol{\gamma}}_{k+\tau}^* + \check{\boldsymbol{\gamma}}_{k+\tau}\check{\boldsymbol{\gamma}}_k^*)$. For simplicity, denote $\mathbf{A} = (\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2, \ldots, \hat{\boldsymbol{\gamma}}_T)$, $\mathbf{B} = (\hat{\boldsymbol{\gamma}}_{1+\tau}, \hat{\boldsymbol{\gamma}}_{2+\tau}, \ldots, \hat{\boldsymbol{\gamma}}_{T+\tau})$, $\check{\mathbf{A}} = (\check{\boldsymbol{\gamma}}_1, \check{\boldsymbol{\gamma}}_2, \ldots, \check{\boldsymbol{\gamma}}_T)$ and $\check{\mathbf{B}} = (\check{\boldsymbol{\gamma}}_{1+\tau}, \check{\boldsymbol{\gamma}}_{2+\tau}, \ldots, \check{\boldsymbol{\gamma}}_{T+\tau})$. Then by Corollary A.41 of Bai and Silverstein (2010), we have

$$\begin{aligned}
&L^3\big(F^{\hat{\mathbf{M}}_\tau}, F^{\check{\mathbf{M}}_\tau}\big) \\
&\quad = L^3\big(F^{\mathbf{A}\mathbf{B}^* + \mathbf{B}\mathbf{A}^*}, F^{\check{\mathbf{A}}\check{\mathbf{B}}^* + \check{\mathbf{B}}\check{\mathbf{A}}^*}\big) \\
&\quad \le \frac{1}{N}\operatorname{tr}\big[\big(\mathbf{A}\mathbf{B}^* + \mathbf{B}\mathbf{A}^* - (\check{\mathbf{A}}\check{\mathbf{B}}^* + \check{\mathbf{B}}\check{\mathbf{A}}^*)\big)\big(\mathbf{A}\mathbf{B}^* + \mathbf{B}\mathbf{A}^* - (\check{\mathbf{A}}\check{\mathbf{B}}^* + \check{\mathbf{B}}\check{\mathbf{A}}^*)\big)^*\big] \\
&\quad \le \frac{2}{N}\operatorname{tr}\big[(\mathbf{A}\mathbf{B}^* - \check{\mathbf{A}}\check{\mathbf{B}}^*)(\mathbf{A}\mathbf{B}^* - \check{\mathbf{A}}\check{\mathbf{B}}^*)^*\big] \\
&\qquad + \frac{2}{N}\operatorname{tr}\big[(\mathbf{B}\mathbf{A}^* - \check{\mathbf{B}}\check{\mathbf{A}}^*)(\mathbf{B}\mathbf{A}^* - \check{\mathbf{B}}\check{\mathbf{A}}^*)^*\big]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{N}\operatorname{tr}\big[(\mathbf{A}\mathbf{B}^* - \check{\mathbf{A}}\check{\mathbf{B}}^*)(\mathbf{A}\mathbf{B}^* - \check{\mathbf{A}}\check{\mathbf{B}}^*)^*\big] \\
&\quad = \frac{1}{N}\operatorname{tr}\big[\big((\mathbf{A} - \check{\mathbf{A}})\mathbf{B}^* + \check{\mathbf{A}}(\mathbf{B} - \check{\mathbf{B}})^*\big)\big((\mathbf{A} - \check{\mathbf{A}})\mathbf{B}^* + \check{\mathbf{A}}(\mathbf{B} - \check{\mathbf{B}})^*\big)^*\big] \\
&\quad \le \frac{2}{N}\operatorname{tr}\big[\big((\mathbf{A} - \check{\mathbf{A}})\mathbf{B}^*\big)\big((\mathbf{A} - \check{\mathbf{A}})\mathbf{B}^*\big)^* + \big(\check{\mathbf{A}}(\mathbf{B} - \check{\mathbf{B}})^*\big)\big(\check{\mathbf{A}}(\mathbf{B} - \check{\mathbf{B}})^*\big)^*\big].
\end{aligned}$$

Define $J_2 = \frac{1}{N}\operatorname{tr}((\mathbf{A} - \check{\mathbf{A}})\mathbf{B}^*)((\mathbf{A} - \check{\mathbf{A}})\mathbf{B}^*)^*$, and then we have

$$\begin{aligned}
J_2 &= \frac{1}{N}\operatorname{tr}\big((\mathbf{A} - \check{\mathbf{A}})^*(\mathbf{A} - \check{\mathbf{A}})\big)\big(\mathbf{B}^*\mathbf{B}\big) \\
&= \frac{C}{T^3}\sum_{i=1}^N\sum_{j=1}^N \left|\sum_{k=1}^T (\check{\varepsilon}_{ik} - \hat{\varepsilon}_{ik})\bar{\hat{\varepsilon}}_{j(k+\tau)}\right|^2 \\
&= \frac{C}{T^3}\sum_{i=1}^N\sum_{j=1}^N \left(\sum_{k_1=1}^T (\check{\varepsilon}_{ik_1} - \hat{\varepsilon}_{ik_1})\bar{\hat{\varepsilon}}_{j(k_1+\tau)}\right)\left(\sum_{k_2=1}^T (\bar{\check{\varepsilon}}_{ik} - \bar{\hat{\varepsilon}}_{ik})\hat{\varepsilon}_{j(k_2+\tau)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{C}{T^3}\sum_{i=1}^{N}\sum_{j=1}^{N}\Bigg[\sum_{k=1}^{T}|\check{\varepsilon}_{ik}-\hat{\varepsilon}_{ik}|^2|\hat{\varepsilon}_{j(k+\tau)}|^2 \\
&\qquad\qquad + \sum_{k_1,k_2=1,k_1>k_2}^{T}(\check{\varepsilon}_{ik_1}-\hat{\varepsilon}_{ik_1})(\bar{\check{\varepsilon}}_{ik_2}-\bar{\hat{\varepsilon}}_{ik_2})\bar{\hat{\varepsilon}}_{j(k_1+\tau)}\hat{\varepsilon}_{j(k_2+\tau)} \\
&\qquad\qquad + \sum_{k_1,k_2=1,k_1<k_2}^{T}(\check{\varepsilon}_{ik_1}-\hat{\varepsilon}_{ik_1})(\bar{\check{\varepsilon}}_{ik_2}-\bar{\hat{\varepsilon}}_{ik_2})\bar{\hat{\varepsilon}}_{j(k_1+\tau)}\hat{\varepsilon}_{j(k_2+\tau)}\Bigg] \\
&\equiv J_{21}+J_{22}+J_{23}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
J_{21} &= \frac{C}{T^3}\sum_{j=1}^{N}\Bigg[\sum_{i=1,\dots,N,k=1,\dots,T,(i,k)\in E}|\check{\varepsilon}_{ik}-\hat{\varepsilon}_{ik}|^2|\hat{\varepsilon}_{j(k+\tau)}|^2 \\
&\qquad + \sum_{i=1,\dots,N,k=1,\dots,T,(i,k)\notin E}|\check{\varepsilon}_{ik}-\hat{\varepsilon}_{ik}|^2|\hat{\varepsilon}_{j(k+\tau)}|^2\Bigg] \\
&\equiv J_{211}+J_{212}.
\end{aligned}
$$

By definition of E, we have $\frac{1-\sigma_{ik}^2}{\Delta}>1$ for any $(i,k)\in E$ and therefore

$$
\begin{aligned}
\mathrm{E}J_{211} &= \frac{C}{T^3}\sum_{j=1}^{N}\sum_{i=1,\dots,N,k=1,\dots,T,(i,k)\in E}\mathrm{E}|\check{\varepsilon}_{ik}-\hat{\varepsilon}_{ik}|^2\mathrm{E}|\hat{\varepsilon}_{j(k+\tau)}|^2 \\
&\le \frac{C}{T^3}\sum_{j=1}^{N}\sum_{i=1,\dots,N,k=1,\dots,T,(i,k)\in E}\mathrm{E}|\hat{\varepsilon}_{j(k+\tau)}|^2 \\
&\le \frac{C}{T^3}\sum_{j=1}^{N}\sum_{i=1,\dots,N,k=1,\dots,T,(i,k)\in E}\frac{1-\sigma_{ik}^2}{\Delta}\mathrm{E}|\hat{\varepsilon}_{j(k+\tau)}|^2 \\
&\le \frac{C}{T^3}\sum_{j=1}^{N}\sum_{i=1}^{N}\sum_{k=1}^{T}T^{-\delta/(4+2\delta)} \\
&= O\big(T^{-\delta/(4+2\delta)}\big).
\end{aligned}
$$

For any $(i,k)\notin E$, we have

$$
\big(1-\sigma_{ik}^{-1}\big)^2=\frac{(\sigma_{ik}-1)^2}{\sigma_{ik}^2}=\frac{(1-\sigma_{ik}^2)^2}{\sigma_{ik}^2(1+\sigma_{ik})^2}\le C\big(1-\sigma_{ik}^2\big)^2\le C\eta^{-2\delta}T^{-2\delta/(2+\delta)}.
$$

Here and in what follows, we assume that $\eta \to 0$ slow enough such that the above upper bound tends to 0 as $T \to \infty$. This together with $\mathrm{E}\hat{\varepsilon}_{ij}^2 < \infty$ implies

$$\begin{aligned}
\mathrm{E}J_{212} &= \frac{C}{T^3}\sum_{j=1}^{N}\sum_{i=1,\dots,N,k=1,\dots,T,(i,k)\notin E}(1-\sigma_{ik}^{-1})^2|\mathrm{E}\hat{\varepsilon}_{ik}|^2|\mathrm{E}\hat{\varepsilon}_{j(k+\tau)}|^2\\
&\le \frac{C}{T^3}\sum_{j=1}^{N}\sum_{i=1,\dots,N,k=1,\dots,T,(i,k)\notin E}(1-\sigma_{ik}^{-1})^2\\
&= O\bigl(T^{-2\delta/(2+\delta)}\bigr).
\end{aligned}$$

Note that summands in J_{22} and J_{23} are pairwise orthogonal; hence we have $\mathrm{E}J_{22} = \mathrm{E}J_{23} = 0$. Therefore, we have $\mathrm{E}J_2 \to 0$.

Now, we want to compute $\operatorname{Var} J_2$. First, we have

$$\begin{aligned}
\operatorname{Var}(J_{211}) &= \frac{C}{T^6}\sum_{j=1}^{N}\sum_{i=1,\dots,N,k=1,\dots,T,(i,k)\in E}\mathrm{E}|\check{\varepsilon}_{ik}-\hat{\varepsilon}_{ik}|^4\mathrm{E}|\hat{\varepsilon}_{j(k+\tau)}|^4\\
&\le \frac{C}{T^6}\sum_{j=1}^{N}\sum_{i=1,\dots,N,k=1,\dots,T,(i,k)\in E}\bigl(\mathrm{E}|\check{\varepsilon}_{ik}|^4+\mathrm{E}|\hat{\varepsilon}_{ik}|^4\bigr)\mathrm{E}|\hat{\varepsilon}_{j(k+\tau)}|^4\\
&\le \frac{C}{T^6}\sum_{j=1}^{N}\sum_{i=1,\dots,N,k=1,\dots,T,(i,k)\in E}\mathrm{E}|\hat{\varepsilon}_{ik}|^4\mathrm{E}|\hat{\varepsilon}_{j(k+\tau)}|^4\\
&= O\bigl(T^{-1-4\delta/(2+\delta)}\bigr).
\end{aligned}$$

For simplicity, write

$$\begin{aligned}
J_{212} &= \sum_{j=1}^{N}\sum_{i=1,\dots,N,k=1,\dots,T,(i,k)\notin E}(1-\sigma_{ik}^{-1})^2\bigl[\bigl(|\hat{\varepsilon}_{ik}|^2-\sigma_{ik}^2\bigr)\bigl(|\hat{\varepsilon}_{j(k+\tau)}|^2-\sigma_{j(k+\tau)}^2\bigr)\\
&\qquad\qquad + \sigma_{ik}^2\bigl(|\hat{\varepsilon}_{j(k+\tau)}|^2-\sigma_{j(k+\tau)}^2\bigr)\\
&\qquad\qquad + \sigma_{j(k+\tau)}^2\bigl(|\hat{\varepsilon}_{ik}|^2-\sigma_{ik}^2\bigr)+\sigma_{ik}^2\sigma_{j(k+\tau)}^2\bigr]\\
&\equiv J_{2121}+J_{2122}+J_{2123}+J_{2124},\\
J_{22} &= \frac{C}{T^3}\Biggl[\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k_1,k_2=1,k_1>k_2,k_1\ne k_2+\tau}^{T}(\check{\varepsilon}_{ik_1}-\hat{\varepsilon}_{ik_1})(\bar{\check{\varepsilon}}_{ik_2}-\bar{\hat{\varepsilon}}_{ik_2})\bar{\hat{\varepsilon}}_{j(k_1+\tau)}\hat{\varepsilon}_{j(k_2+\tau)}\\
&\qquad + \sum_{i,j=1,i\ne j}^{N}\sum_{k_2=1}^{T}(\check{\varepsilon}_{i(k_2+\tau)}-\hat{\varepsilon}_{i(k_2+\tau)})(\bar{\check{\varepsilon}}_{ik_2}-\bar{\hat{\varepsilon}}_{ik_2})\bar{\hat{\varepsilon}}_{j(k_2+2\tau)}\hat{\varepsilon}_{j(k_2+\tau)}\\
&\qquad + \sum_{i=1}^{N}\sum_{k_2=1}^{T}(\check{\varepsilon}_{i(k_2+\tau)}-\hat{\varepsilon}_{i(k_2+\tau)})(\bar{\check{\varepsilon}}_{ik_2}-\bar{\hat{\varepsilon}}_{ik_2})\bar{\hat{\varepsilon}}_{i(k_2+2\tau)}\hat{\varepsilon}_{i(k_2+\tau)}\Biggr]\\
&\equiv J_{221}+J_{222}+J_{223}.
\end{aligned}$$

Note that in all expressions except J_{2124}, components are orthogonal to each other. In addition, as a constant, J_{2124} does not contribute to $\operatorname{Var} J_2$. Therefore, we have

$$\begin{aligned}
\operatorname{Var} J_{2121} &\le \frac{C}{T^6}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{T}(1-\sigma_{ik}^{-1})^4(\mathrm{E}|\hat{\varepsilon}_{ik}|^4)^2 \\
&\le \frac{C}{T^6}T^3T^{-4\delta/(2+\delta)}T^{2(2-\delta)/(2+\delta)} \\
&= O\bigl(T^{-1-8\delta/(2+\delta)}\bigr), \\
\operatorname{Var} J_{2122} &= \operatorname{Var} J_{2123} \\
&\le \frac{C}{T^6}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{T}(1-\sigma_{ik}^{-1})^4\sigma_{ik}^4\mathrm{E}|\hat{\varepsilon}_{ik}|^4 \\
&\le \frac{C}{T^6}T^3T^{-4\delta/(2+\delta)}T^{(2-\delta)/(2+\delta)} \\
&= O\bigl(T^{-2-6\delta/(2+\delta)}\bigr), \\
\operatorname{Var} J_{221} &\le \frac{C}{T^6}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k_1,k_2=1,k_1>k_2,k_1\ne k_2+\tau}^{T}\mathrm{E}|\check{\varepsilon}_{ik_1}-\hat{\varepsilon}_{ik_1}|^2\mathrm{E}|\bar{\check{\varepsilon}}_{ik_2}-\bar{\hat{\varepsilon}}_{ik_2}|^2 \\
&\qquad\times \mathrm{E}|\bar{\hat{\varepsilon}}_{j(k_1+\tau)}|^2\mathrm{E}|\hat{\varepsilon}_{j(k_2+\tau)}|^2 \\
&\le \frac{C}{T^6}T^4 \\
&= O\bigl(T^{-2}\bigr), \\
\operatorname{Var} J_{222} &\le \frac{C}{T^6}\sum_{i,j=1,i\ne j}^{N}\sum_{k_2=1}^{T}\mathrm{E}|\check{\varepsilon}_{i(k_2+\tau)}-\hat{\varepsilon}_{i(k_2+\tau)}|^2\mathrm{E}|\bar{\check{\varepsilon}}_{ik_2}-\bar{\hat{\varepsilon}}_{ik_2}|^2 \\
&\qquad\times \mathrm{E}|\bar{\hat{\varepsilon}}_{j(k_2+2\tau)}|^2\mathrm{E}|\hat{\varepsilon}_{j(k_2+\tau)}|^2 \\
&\le \frac{C}{T^6}T^3 \\
&= O\bigl(T^{-3}\bigr), \\
\operatorname{Var} J_{223} &\le \frac{C}{T^6}\sum_{i=1}^{N}\sum_{k_2=1}^{T}\mathrm{E}\bigl|(\check{\varepsilon}_{i(k_2+\tau)}-\hat{\varepsilon}_{i(k_2+\tau)})\hat{\varepsilon}_{i(k_2+\tau)}\bigr|^2\mathrm{E}|\bar{\check{\varepsilon}}_{ik_2}-\bar{\hat{\varepsilon}}_{ik_2}|^2 \\
&\qquad\times \mathrm{E}|\bar{\hat{\varepsilon}}_{i(k_2+2\tau)}|^2 \\
&\le \frac{C}{T^6}\sum_{i=1}^{N}\sum_{k_2=1}^{T}\mathrm{E}|\hat{\varepsilon}_{i(k_2+\tau)}|^4
\end{aligned}$$

$$\leq \frac{C}{T^6} T^2 T^{(2-\delta)/(2+\delta)}$$
$$= O\big(T^{-3-2\delta/(2+\delta)}\big).$$

Therefore, we have $\operatorname{Var} J_2 = O(T^{-1-\varepsilon})$ for some $\varepsilon > 0$. Thus $J_2 \to 0$ a.s.

Similarly, we have

$$\mathrm{E}\frac{1}{N}\operatorname{tr}\big(\check{\mathbf{A}}(\mathbf{B}-\check{\mathbf{B}})^*\big)\big(\check{\mathbf{A}}(\mathbf{B}-\check{\mathbf{B}})^*\big)^* \to 0$$

and

$$\operatorname{Var}\frac{1}{N}\operatorname{tr}\big(\check{\mathbf{A}}(\mathbf{B}-\check{\mathbf{B}})^*\big)\big(\check{\mathbf{A}}(\mathbf{B}-\check{\mathbf{B}})^*\big)^* = O\big(T^{-1-\varepsilon'}\big)$$

for some $\varepsilon' > 0$. Hence, we have $\frac{1}{N}\operatorname{tr}[(\mathbf{A}\mathbf{B}^* - \check{\mathbf{A}}\check{\mathbf{B}}^*)(\mathbf{A}\mathbf{B}^* - \check{\mathbf{A}}\check{\mathbf{B}}^*)^*] \to 0$ a.s. By interchanging $\mathbf{A}$ and $\mathbf{B}$, we have $\frac{1}{N}\operatorname{tr}[(\mathbf{B}\mathbf{A}^* - \check{\mathbf{B}}\check{\mathbf{A}}^*)(\mathbf{B}\mathbf{A}^* - \check{\mathbf{B}}\check{\mathbf{A}}^*)^*] \to 0$ a.s. Therefore, $L^3(F^{\hat{\mathbf{M}}_\tau}, F^{\check{\mathbf{M}}_\tau}) \to 0$ a.s.

APPENDIX B: SOME TECHNICAL LEMMAS

LEMMA B.1. *Under the assumptions of Theorem 1.1, $\boldsymbol{\gamma}_k^* \mathbf{A}_k^{-1} \boldsymbol{\gamma}_k - \frac{c}{2} m_N(z) \to 0$ almost surely and uniformly in $k \leq T + \tau$, where $m_N(z) = \frac{1}{N}\operatorname{tr}\mathbf{A}^{-1}$.*

The proof of Lemma B.1 is similar to the proof of Lemma 9.1 of Bai and Silverstein (2010).

PROOF OF LEMMA B.1. Write $\mathbf{A}_k^{-1} = (a_{ij})$. For any given $r \geq 1$, we have

$$\mathrm{E}\left|\boldsymbol{\gamma}_k^* \mathbf{A}_k^{-1} \boldsymbol{\gamma}_k - \frac{1}{2T}\operatorname{tr}\mathbf{A}_k^{-1}\right|^{2r} \leq 2^{2r-1}\big(\mathrm{E}|S_1|^{2r} + \mathrm{E}|S_2|^{2r}\big),$$

where $S_1 = \frac{1}{2T}\sum_{i=1}^N a_{ii}(\varepsilon_{ki}^2 - 1)$ and $S_2 = \frac{1}{2T}\sum_{1\leq i\neq j\leq N} a_{ij}\varepsilon_{ki}\varepsilon_{kj}$.

By noting $|a_{ii}| \leq \|\mathbf{A}_k^{-1}\| \leq v^{-1}$ and $\mathrm{E}|\varepsilon_{ki}^2 - 1|^{(2+\delta)/2} \equiv M < \infty$, we get

$$\mathrm{E}|S_1|^{2r} = \frac{1}{4^r T^{2r}}\mathrm{E}\left(\left|\sum_{i=1}^N (\varepsilon_{ki}^2 - 1)a_{ii}\right|^{2r}\right)$$

$$\leq \frac{1}{4^r T^{2r}}\sum_{l=1}^r \sum_{1\leq j_1<\cdots<j_l\leq N} \sum_{\substack{i_1+\cdots+i_l=2r,\\ i_1\geq 2,\ldots,i_l\geq 2}} (2r)!\prod_{t=1}^l \frac{\mathrm{E}|\varepsilon_{kj_t}^2 - 1|^{i_t}|a_{j_tj_t}|^{i_t}}{i_t!} \tag{B.1}$$

$$\leq \frac{\eta^{4r}}{4^r v^{2r} T^{2\delta r/(2+\delta)}}\sum_{l=1}^r \eta^{-(2+\delta)l} T^{-l} N^l M^l l^{2r}.$$

Next, let us consider

$$\mathrm{E}|S_2|^{2r} = \frac{1}{4^r T^{2r}} \sum a_{i_1 j_1} \bar{a}_{t_1 \ell_1} \cdots a_{i_r j_r} \bar{a}_{t_r \ell_r} \mathrm{E}(\varepsilon_{ki_1} \varepsilon_{kt_1} \varepsilon_{kj_1} \varepsilon_{k\ell_1} \cdots \varepsilon_{k\ell_r} \varepsilon_{kt_r} \varepsilon_{kj_r} \varepsilon_{k\ell_r}).$$

Draw a directional graph G of $2r$ edges that link i_s to j_s and ℓ_s to t_s, $s = 1, \ldots, r$. Note that if G has a vertex whose degree is 1, then the graph corresponds to a term with expectation 0. That is, for any nonzero term, the vertex degrees of the graph are not less than 2. Write the noncoincident vertices as $v_1, \ldots, v_m$ with degrees $p_1, \ldots, p_m$ greater than 1. We have $m \le r$. By assumption, we have

$$\left|\mathrm{E}(\varepsilon_{ki_1} \varepsilon_{kt_1} \varepsilon_{kj_1} \varepsilon_{k\ell_1} \cdots \varepsilon_{k\ell_r} \varepsilon_{kt_r} \varepsilon_{kj_r} \varepsilon_{k\ell_r})\right| \le \left(\eta^2 T^{2/(2+\delta)}\right)^{r-m}.$$

Now, suppose that the graph consists of q connected components $G_1, \ldots, G_q$ with $m_1, \ldots, m_q$ noncoincident vertices, respectively. Let us consider the contribution by G_1 to $\mathrm{E}|S_2|^r$. Assume that G_1 has s_1 edges, $e_1, \ldots, e_{s_1}$. Choose a tree G_1' from G_1, and assume its edges are e_1, e_{m_1-1}, without loss of generality. Note that

$$\sum_{v_1,\ldots,v_{m_1} \le N} \prod_{t=1}^{m_1-1} |a_{e_t}|^2 \le \|\mathbf{A}_k^{-1}\|^{2m_1-2} N \le \frac{N}{v^{2m_1-2}}$$

and

$$\sum_{v_1,\ldots,v_{m_1} \le N} \prod_{t=m_1}^{s_1} |a_{e_t}|^2 \le \frac{N^{m_1-1}}{v^{2s_1-2m_1+2}}.$$

Here, the first inequality follows from the fact that $\sum_{v_1} |a_{v_1 v_2}|^2 \le \|\mathbf{A}_k^{-1}\|^2 \le v^{-2}$ since it is a diagonal element of $\mathbf{A}_k^{-1}(\mathbf{A}_k^{-1})^*$. The second inequality follows from the fact that $\sum_{v_1} |a_{v_1 v_2}|^\ell \le v^{-\ell}$ for any $\ell \ge 2$ and that $s_1 \ge m_1$ since all vertices have degrees not less than 2. Therefore, the contribution of G_1 is bounded by

$$\begin{aligned}\sum_{v_1,\ldots,v_{m_1} \le N} \prod_{t=1}^{s_1} |a_{e_t}| &\le \left(\sum_{v_1,\ldots,v_{m_1} \le N} \prod_{t=1}^{m_1-1} |a_{e_t}|^2 \sum_{v_1,\ldots,v_{m_1} \le N} \prod_{t=m_1}^{s_1} |a_{e_t}|^2\right)^{1/2} \\ &\le \frac{N^{m_1/2}}{v^{s_1}}.\end{aligned}$$

Noting that $m_1 + \cdots + m_q = m$ and $s_1 + \cdots + s_q = 2r$, eventually we obtain that the contribution of the isomorphic class for a given canonical graph is $\frac{N^{m/2}}{v^{2r}}$. Because the two vertices of each edge cannot coincide, we have $q \le m/2$. The number of canonical graphs is less than $\binom{m}{2}^{2r} \le m^{4r}$. We finally obtain

$$\begin{aligned}\mathrm{E}|S_2|^{2r} &\le \frac{1}{4^r v^{2r} T^{2r}} \sum_{m=2}^{r} N^{m/2}\left(\eta^2 T^{2/(2+\delta)}\right)^{2r-m} m^{4r} \\ &\le \frac{1}{4^r v^{2r} T^{2\delta r/(2+\delta)}} \sum_{m=2}^{r} \left(\frac{N^{1/2}}{\eta^2 T^{2/(2+\delta)}}\right)^m m^{4r}.\end{aligned} \tag{B.2}$$

Using (B.1) and (B.2), for any $t > 0$, there exists $r > t/\delta + t/2$ such that $\mathrm{E}|\boldsymbol{\gamma}_k^* \mathbf{A}_k^{-1} \boldsymbol{\gamma}_k - \frac{1}{2T} \operatorname{tr} \mathbf{A}_k^{-1}|^{2r} = O(T^{-t})$. Therefore, by the Borel–Cantelli lemma,

$$\boldsymbol{\gamma}_k^* \mathbf{A}_k^{-1} \boldsymbol{\gamma}_k - \frac{1}{2T} \operatorname{tr} \mathbf{A}_k^{-1} \to 0 \tag{B.3}$$

almost surely and uniformly in $k \le T + \tau$.

Let F_N denote the ESD of $\mathbf{M}_\tau$ and F_{Nk} the ESD of $\mathbf{M}_\tau - \boldsymbol{\gamma}_k(\boldsymbol{\gamma}_{k+\tau} + \boldsymbol{\gamma}_{k-\tau})^* - (\boldsymbol{\gamma}_{k+\tau} + \boldsymbol{\gamma}_{k-\tau})\boldsymbol{\gamma}_k^*$.

By Theorem A.43 of Bai and Silverstein (2010), we have

$$\|F_N - F_{Nk}\| \le \frac{4}{N},$$

where $\|f\| = \sup_x |f(x)|$. Thus

$$\begin{aligned}\left|\frac{1}{N}(\operatorname{tr}(\mathbf{A}^{-1}) - \operatorname{tr}(\mathbf{A}_k^{-1}))\right| &= \left|\int \frac{1}{x - u - iv}\, d(F_N - F_{Nk})\right| \\ &\le \frac{1}{v}\|F_N - F_{Nk}\| \le \frac{4}{vN}.\end{aligned}$$

This implies that

$$\frac{1}{N}(\operatorname{tr}(\mathbf{A}^{-1}) - \operatorname{tr}(\mathbf{A}_k^{-1})) \to 0 \qquad \text{a.s.}$$

uniformly in $k \le T + \tau$. Substituting the above into (B.3), the proof of the lemma is complete. $\square$

LEMMA B.2. *Under the assumptions of Theorem 1.1, we have* $\boldsymbol{\gamma}_k^* \mathbf{A}_k^{-1} \boldsymbol{\gamma}_l \to 0$, *almost surely and uniformly in* $k \ne l$.

PROOF. Let $\mathbf{A}_k^{-1}\boldsymbol{\gamma}_l = \mathbf{b} = (b_1, \dots, b_N)'$. Noting $\varepsilon_{lj} < \eta_N T^{1/(2+\delta)}$ and $\mathrm{E}|\varepsilon_{lj}|^{2+\delta} = v_{2+\delta} < \infty$, we have

$$\begin{aligned}
&\mathrm{E}(\boldsymbol{\gamma}_k^* \mathbf{A}_k^{-1} \boldsymbol{\gamma}_l)^{2r} \\
&\quad= \frac{1}{2^r T^r}\mathrm{E}\left(\left|\sum_{i=1}^N \varepsilon_{ki} b_i\right|^{2r}\right) \\
&\quad\le \frac{1}{2^r T^r}\mathrm{E}\sum_{l=1}^r \sum_{1 \le j_1 < \cdots < j_l \le N} \sum_{i_1 + \cdots + i_l = 2r} \frac{(2r)!}{i_1! \cdots i_l!} |\varepsilon_{kj_1}^{i_1} b_{j_1}^{i_1} \cdots \varepsilon_{kj_l}^{i_l} b_{j_1}^{i_l}| \\
&\quad\le \frac{\eta^{2r}}{2^r T^{\delta r/(2+\delta)}}\mathrm{E}\sum_{l=1}^r \eta^{-(2+\delta)l} T^{-l} v_{2+\delta}^l \\
&\qquad\qquad \times \sum_{1 \le j_1 < \cdots < j_l < N} \sum_{\substack{i_1 + \cdots + i_l = 2r, \\ i_1 \ge 2, \dots, i_l \ge 2}} \frac{(2r)!}{i_1! \cdots i_l!} |b_{j_1}|^{i_1} \cdots |b_{j_l}|^{i_l}.
\end{aligned}$$

By $\sum_{j=1}^{N}|b_j|^2 = \|\mathbf{A}_k^{-1}\boldsymbol{\gamma}_l\|^2$ and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\sum_{1\le j_1<\cdots<j_l\le N}\ \sum_{\substack{i_1+\cdots+i_l=2r,\\ i_1\ge 2,\dots,i_l\ge 2}} \frac{(2r)!}{i_1!\cdots i_l!}|b_{j_1}|^{i_1}\cdots|b_{j_l}|^{i_l}\\
&\quad\le \sum_{\substack{i_1+\cdots+i_l=2r,\\ i_1\ge 2,\dots,i_l\ge 2}} \frac{(2r)!}{i_1!\cdots i_l!}\left(\sum_{j=1}^{N}|b_j|^2\right)^r\\
&\quad\le l^{2r}\|\mathbf{A}_k^{-1}\|^{2r}\|\boldsymbol{\gamma}_l\|^{2r}\\
&\quad\le \frac{l^{2r}}{v^{2r}}\|\boldsymbol{\gamma}_l\|^{2r}.
\end{aligned}$$

Noting $\varepsilon_{lj} < \eta_N T^{1/(2+\delta)}$ and $\mathrm{E}|\varepsilon_{lj}|^{2+\delta} = \upsilon_{2+\delta} < \infty$, we get

$$\begin{aligned}
\mathrm{E}\|\boldsymbol{\gamma}_l\|^{2r} &= \frac{1}{2^r T^r}\mathrm{E}\left(\sum_{j=1}^{N}\varepsilon_{lj}^2\right)^r\\
&= \mathrm{E}\frac{1}{2^r T^r}\sum_{l=1}^{r}\sum_{1\le j_1<\cdots<j_l\le N}\ \sum_{i_1+\cdots+i_l=r}\frac{r!}{i_1!\cdots i_l!}\varepsilon_{lj_1}^{2i_1}\cdots\varepsilon_{lj_l}^{2i_l}\\
&\le \mathrm{E}\frac{1}{2^r}\sum_{l=1}^{r}\upsilon_{2+\delta}^{l}\eta^{2r-(2+\delta)l}T^{-\delta r/(2+\delta)-l}\sum_{1\le j_1<\cdots<j_l\le N}\ \sum_{i_1+\cdots+i_l=r}\frac{r!}{i_1!\cdots i_l!}\\
&\le \frac{\eta^{2r}}{2^r T^{\delta r/(2+\delta)}}\sum_{l=1}^{r}\left(\frac{\eta^{2+\delta}T}{N\upsilon_{2+\delta}}\right)^{-l} l^r.
\end{aligned}$$

For any $t>0$, there exists $r > 2t/\delta + t$ such that $\mathrm{E}|\boldsymbol{\gamma}_k^*\mathbf{A}_k^{-1}\boldsymbol{\gamma}_l|^r = O(T^{-t})$. Therefore by the Borel–Cantelli lemma,

$$\boldsymbol{\gamma}_k^*\mathbf{A}_k^{-1}\boldsymbol{\gamma}_l \to 0$$

almost surely and uniformly in $k \ne l$. The proof of the lemma is complete. □

In the next lemma, we find the limit of $\boldsymbol{\gamma}_{k+\tau}^*\mathbf{A}_k^{-1}\boldsymbol{\gamma}_{k+\tau}$ when $T-k\to\infty$ and that of $\boldsymbol{\gamma}_{k-\tau}^*\mathbf{A}_k^{-1}\boldsymbol{\gamma}_{k-\tau}$ when $k\to\infty$.

LEMMA B.3. *Assume that* $\frac{1}{N}\operatorname{tr}(\mathbf{A}-z\mathbf{I})^{-1}\to m = m(z)$. *When* $T-k\to\infty$,

$$\boldsymbol{\gamma}_{k+\tau}^*\mathbf{A}_k^{-1}\boldsymbol{\gamma}_{k+\tau} \to \frac{(c/2)m}{x_1},$$

where x_1 *is the root of the quadratic equation* $x^2 - x + \frac{1}{4}c^2m^2 = 0$ *with the larger absolute value.*

When $k \to \infty$,

$$\boldsymbol{\gamma}_{k-\tau}^* \mathbf{A}_k^{-1} \boldsymbol{\gamma}_{k-\tau} \to \frac{(c/2)m}{x_1},$$

where x_1 is the same as above.

PROOF. Write $a = \frac{c}{2}m$, $W_k = \boldsymbol{\gamma}_{k+\tau}^* \mathbf{A}_k^{-1} \boldsymbol{\gamma}_{k+\tau}$ and $W_{k,k+\tau,\dots,k+\ell\tau} = \boldsymbol{\gamma}_{k+(\ell+1)\tau}^* \mathbf{A}_{k,k+\tau,\dots,k+\ell\tau}^{-1} \boldsymbol{\gamma}_{k+(\ell+1)\tau}$. Then by (3.3), we have

$$W_k = \frac{a + r(k)}{1 - aW_{k,k+\tau}},$$

where $r(k) = o_{\text{a.s.}}(1)$, uniformly in $k \le T + \tau$. Using this relation again, we obtain

$$\begin{aligned} W_k &= \frac{a + r(k)}{1 - a(a + r(k+\tau))/(1 - aW_{k,k+\tau,k+2\tau})} \\ &= \frac{(a + r(k))(1 - aW_{k,k+\tau,k+2\tau})}{1 - aW_{k,k+\tau,k+2\tau} - a(a + r(k+\tau))}. \end{aligned} \tag{B.4}$$

Applying this relation ℓ times, we may express W_k in the following form:

$$W_k = \frac{(a + r(k))(\alpha_{k,\ell-1} - a\alpha_{k,\ell-2} W_{k,k+\tau,\dots,k+\ell\tau})}{\alpha_{k,\ell} - a\alpha_{k,\ell-1} W_{k,k+\tau,\dots,k+\ell\tau}},$$

where the coefficients satisfy the recursive relation

$$\alpha_{k,\ell} = \alpha_{k,\ell-1} - a\big(a + r(k+\ell\tau)\big)\alpha_{k,\ell-2}, \qquad \alpha_{k,1} = 1, \qquad \alpha_{k,0} = 1. \tag{B.5}$$

Define x_1 and x_0 as the roots of the equation $x^2 = x - a^2$ with $|x_1| > |x_0|$. [Note that the equal sign happens only when $a = \pm\frac{1}{2}$ which is impossible for $\Im(z) > 0$ because a is the Stieltjes transform of a distribution function.] Then we have

$$x_1 = \begin{cases} \frac{1}{2}(1 - \sqrt{1 - 4a^2}), & \text{if } \Im(a^2) > 0, \\ \frac{1}{2}(1 + \sqrt{1 - 4a^2}), & \text{if } \Im(a^2) < 0, \end{cases}$$

$$x_0 = \begin{cases} \frac{1}{2}(1 + \sqrt{1 - 4a^2}), & \text{if } \Im(a^2) > 0, \\ \frac{1}{2}(1 - \sqrt{1 - 4a^2}), & \text{if } \Im(a^2) < 0. \end{cases}$$

Similarly, define $\nu_{k,1}$ and $\nu_{k,0}$ as the roots of the equation $x^2 = x - a(a + r(k))$, with $|\nu_{k,1}| > |\nu_{k,0}|$. By this definition, we have

$$\nu_{k,1} = \begin{cases} \frac{1}{2}\big(1 - \sqrt{1 - 4a\big(a + r(k)\big)}\big), & \text{if } \Im\big(a\big(a + r(k)\big)\big) > 0, \\ \frac{1}{2}\big(1 + \sqrt{1 - 4a\big(a + r(k)\big)}\big), & \text{if } \Im\big(a\big(a + r(k)\big)\big) < 0. \end{cases}$$

Further, define α such that $\alpha\nu_{k,1} + (1 - \alpha)\nu_{k,0} = 1$. Then, define $\nu_{k,1,j} = \nu_{k,j}$, for $j = 0, 1$. For $t \ge 1$, define

$$\nu_{k,t+1,j} = 1 - \frac{a(a + r(k + (t+1)\tau)}{\nu_{k,t,j}} \qquad \text{for } j = 0, 1. \tag{B.6}$$

From this, we have

$$\alpha \nu_{k,1,1} + (1-\alpha)\nu_{k,1,0} = 1 = \alpha_{k,1}.$$

And

$$\begin{aligned}\alpha \nu_{k,2,1}\nu_{k,1,1} &+ (1-\alpha)\nu_{k,2,0}\nu_{k,1,0} \\ &= \alpha\bigl(\nu_{k,1,1} - a\bigl(a + r(k+2\tau)\bigr)\bigr) + (1-\alpha)\bigl(\nu_{k,1,0} - a\bigl(a + r(k+2\tau)\bigr)\bigr) \\ &= \alpha_{k,1} - a\bigl(a + r(k+2\tau)\bigr) = \alpha_{k,2}.\end{aligned}$$

Using (B.5) and (B.6), we may prove by induction that

$$\alpha \prod_{t=1}^{\ell} \nu_{k,t,1} + (1-\alpha)\prod_{t=1}^{\ell} \nu_{k,t,0} = \alpha_{k,\ell}.$$

Our next goal is to estimate the difference between $\nu_{k,t+1,j}$ and x_j. First, by noticing the definition of $\nu_{k,j}$, we have

$$\nu_{k,j} - x_j = \nu_{k,1,j} - x_j = o_{\mathrm{a.s.}}(1),$$

where, and in what follows, the remainder term $o_{\mathrm{a.s.}}(1)$ is uniform in k and ℓ. Then, by $\nu_{k,1,j} = \nu_{k,1,j}^2 + a(a + r(k))$ we have

$$\begin{aligned}\nu_{k,2,j} - x_j &= \frac{\nu_{k,1,j} - a(a + r(k+2\tau))}{\nu_{k,1,j}} - x_j \\ &= \frac{a(r(k) - r(k+2\tau))}{\nu_{k,1,j}} + \nu_{k,1,j} - x_j = o_{\mathrm{a.s.}}(1).\end{aligned}$$

By induction, we can prove that

$$\nu_{k,t+1,j} - x_j = o_{\mathrm{a.s.}}(1),$$

provided that t is bounded by a fixed amount M. Therefore, for any given $\eta > 0$, when N is large, we have

$$\prod_{t=1}^{M} \left|\frac{\nu_{k,t+1,0}}{\nu_{k,t+1,1}}\right| \le \left(\frac{|x_0|}{|x_1|} + \eta\right)^M.$$

Note that $|x_0| < |x_1|$. Thus, for any given $\varepsilon > 0$, we may choose $\eta > 0$ and $\ell < \infty$, such that

$$\prod_{t=1}^{\ell} \left|\frac{\nu_{k,t+1,0}}{\nu_{k,t+1,1}}\right| \le \varepsilon.$$

That means, when $\ell \to \infty$ slowly, we have $\alpha_{k,\ell+1} = x_1\alpha_{k,\ell}(1 + o_{\mathrm{a.s.}}(1))$. Consequently,

$$\begin{aligned}W_k &= \frac{(a + r(k))(\alpha_{k,\ell} - a\alpha_{k,\ell-1}W_{k,k+\tau,\dots,k+\ell\tau})}{\alpha_{k,\ell+1} - a\alpha_{k,\ell}W_{k,k+\tau,\dots,k+\ell\tau}} \\ &= \frac{a}{x_1}\bigl(1 + o_{\mathrm{a.s.}}(1)\bigr).\end{aligned}$$

The first conclusion of the lemma is proved. By duality, the second conclusion follows. $\square$

REMARK B.1. Note that one of the key steps in the above proof is to let $\ell \to \infty$ slowly. This may not be possible when $\tau \to \infty$, as in this case we may have $\ell\tau > T-k$ and $\boldsymbol{\gamma}_{k+\ell\tau}$ does not exist. However, such ℓ exists when $\tau = o(T)$. Therefore, the proof is still valid in this case.

LEMMA B.4. *Assume the conditions of Lemma B.3 hold. For all $k \in [1, T+\tau]$, we have*

$$\boldsymbol{\gamma}_{k-\tau}^* \mathbf{A}_k^{-1} \boldsymbol{\gamma}_{k+\tau} \to 0 \qquad a.s.,$$

where the convergence is uniform in k.

PROOF. Obviously, when $\tau < k \le 2\tau$, the lemma is true because $\boldsymbol{\gamma}_{k-\tau}$ is independent of $\mathbf{A}_k$. Similarly, the lemma is true when $T-\tau < k \le T$.

When $2\tau < k \le T-\tau$, by (3.5) and what is proved in the last lemma,

$$\begin{aligned}\boldsymbol{\gamma}_{k-\tau} \mathbf{A}_k^{-1} \boldsymbol{\gamma}_{k+\tau} &= \frac{-a}{1 - a^2/x_1 + o_{\text{a.s.}}(1)} \boldsymbol{\gamma}_{k-\tau} \mathbf{A}_{k,k+\tau}^{-1} \boldsymbol{\gamma}_{k+2\tau} \\ &= \frac{-a}{x_1 + o_{\text{a.s.}}(1)} \boldsymbol{\gamma}_{k-\tau} \mathbf{A}_{k,k+\tau}^{-1} \boldsymbol{\gamma}_{k+2\tau}.\end{aligned}$$

For $x_1 x_0 = a^2$ and $|x_1| > |x_0|$, we have $|a| < |x_1|$. Thus

$$|\boldsymbol{\gamma}_{k-\tau} \mathbf{A}_k^{-1} \boldsymbol{\gamma}_{k+\tau}| \le \left(\frac{|a|}{|x_1|} + \eta\right) |\boldsymbol{\gamma}_{k-\tau} \mathbf{A}_{k,k+\tau}^{-1} \boldsymbol{\gamma}_{k+2\tau}|$$

for some $\eta > 0$ such that $|a/x_1| + \eta < 1$. Now, the lemma can be proved by induction. $\square$

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