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Nonparametric estimation for a two-dimensional renewal process

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Abstract: This paper introduces nonparametric estimation methods for spatial point processes satisfying a particular renewal property. Martingale methods yield a unified approach for renewal processes in both one and two dimensions, and can be used for both synchronous and asynchronous data. In each case, we obtain martingale estimators of the avoidance probabilities that characterize the renewal process. Asymptotic properties of the estimators are studied analytically and empirically.

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1. Introduction

Statistical inference of point processes arises in diverse areas of application, such as the biological and life sciences, astronomy, geology, ecology and econometrics, to name a few. For point processes on $\Re$, both parametric and nonparametric methods have been well-developed for various examples, including the two archetypal models: Poisson point processes and renewal processes. Poisson point processes are easily extended to higher dimensions, but the renewal model has proved to be more of a challenge. On $\Re_+$, renewal times can be expressed as

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a sum of independent and identically distributed (i.i.d.) non-negative random variables, and so the renewal model includes the Poisson process as a special case. However, in higher dimensions, a sum of i.i.d non-negative random vectors gives rise to a process with totally ordered jumps; this of course cannot include the Poisson process.

A different approach to defining renewal properties on $\Re^2_+$ was taken in [9]; this more general model includes the Poisson process as well as the sum of i.i.d. random vectors as special cases. The relation between the renewal model and an inhomogeneous Poisson process was established in [7] and used to develop a simple algorithm for simulations. In particular, certain "avoidance probabilities" are shown to be the multidimensional analogue of the interarrival distribution and therefore, to characterize the process. In [4], it was shown how to apply the renewal model to environmental and other forms of spatial point process data. For particular parametric models, maximum likelihood estimation of the parameters defining the avoidance probabilities was described and applied to two data sets. The model was further extended in [5] to a Markov renewal mechanism that can be spatially dependent.

The goal of this paper is to develop nonparametric techniques for estimating the avoidance probabilities of a spatial renewal process for both synchronous and asynchronous data. (Synchronous data consists of a fixed number of renewals, whereas asynchronous data consists of a single realization of the renewal process over a fixed interval of time or region of space.) We shall make use of the theory of multiparameter martingales to provide a unified approach to the analysis of both types of data in one and two dimensions. In the case of synchronous data on $\Re_+$, the martingale approach yields the usual empirical estimator of the interarrival distribution. However, for synchronous data on $\Re^2_+$, the empirical and martingale estimators of the avoidance probabilities are different and we will see that the martingale estimator performs better than the empirical estimator. For asynchronous data, successive renewals are censored by values that depend on prior renewals (this is true in any dimension), and we will see that martingale techniques adapt readily to this form of censored data; the resulting estimators are new in both one and two dimensions.

We proceed as follows: in §2 we begin with generalities on point processes and their characterization via random sets. We then move to the basic building block of the spatial renewal model: the single line process. We consider its characterization by avoidance probabilities and its martingale properties. Finally we introduce the general renewal model and see that it is characterized by the law of a single line process and the associated avoidance probabilities. In §3 and §4 we use multiparameter martingale methods to produce nonparametric estimators of the avoidance probabilities of the renewal process, first for synchronous data (§3) and then for asynchronous data (§4). The asymptotic properties of the estimators are demonstrated analytically and empirically. We conclude with some comments about future research directions in §5. Most of the empirical results appear in the Appendix.

2. Preliminaries on renewal point processes

2.1. Generalities on point processes

There are various characterizations of a point (or counting) process N on $\Re_+^d$, the positive quadrant of d-dimensional Euclidean space. We will always assume that our point processes have no accumulation points, and so must be finite on any bounded set. Likewise, it is assumed that point processes on $\Re_+^2$ are *strictly simple*: i.e. all of the jump points are distinct and no two points fall on the same horizontal or vertical line.

In what follows, $\Re_+^d$ is equipped with the usual partial order: $(s_1, \ldots, s_d) = s \le t = (t_1, \ldots, t_d)$ if and only if $s_i \le t_i \ \forall i = 1, \ldots, d$. Two points $s, t \in \Re_+^d$ are incomparable if $s \not\le t$ and $t \not\le s$. Two sets A and B are incomparable if s and t are incomparable $\forall s \in A, \forall t \in B$.

A point process N on $\Re_+^d$ can be defined in terms of the random set $\Delta(N)$ of its jump points. N can be viewed as a random measure on the Borel sets $\mathcal{B}$:

$$N(B) := \sum_{\tau \in \Delta(N)} I\{\tau \in B\}, \ B \in \mathcal{B}$$

or equivalently as a stochastic process on $\Re_+^d$:

$$N(t) := N(A_t) = \sum_{\tau \in \Delta(N)} I\{\tau \le t\}, \ t \in \Re_+^d,$$

where A_t denotes the rectangle $[0, t] = \{s \in \Re_+^d : s \le t\}$ (the strict "past" of t). Since no ambiguity arises, we use the notation N for both the random measure and the stochastic process. The law of N is determined by its finite dimensional distributions; those of the stochastic process determine those of the random measure and vice versa. Finally, N can also be characterized by the decreasing sequence ($\Re_+^d = \zeta_0 \supseteq \zeta_1 \supseteq \zeta_2 \supseteq \cdots$) of random sets

$$\zeta_k(N) := \{t \in \Re_+^d : N(t) \ge k\}, \ k = 0, 1, 2, \ldots, \tag{1}$$

and it is this representation of N that leads to the generalization of the renewal property. The law of the random set $\zeta_k(N)$ is determined by its finite dimensional distributions: for $t_1, \ldots, t_n \in \Re_+^d, \ n \ge 1$,

$$P(t_1, \ldots, t_n \in \zeta_k(N)) = P(N(t_1) \ge k, \ldots, N(t_n) \ge k). \tag{2}$$

It is important to note the following: if

$$\min(B) = \{t \in B : s \not\le t \ \forall s \in B \text{ such that } s \ne t\}$$

for any Borel set B, then $\zeta_k(N)$ is determined by $\min \zeta_k(N)$:

$$\zeta_k(N) = \cup_{\epsilon \in \min \zeta_k(N)} E_\epsilon,$$

where for $t \in \Re^d_+$, $E_t := \{s \in T : t \leq s\}$ (this is the strict "future" of t). Also,

$$\min(\zeta_1(N)) \subseteq \Delta(N),$$

but the same is not true of $\min(\zeta_k(N))$ for $k > 1$. Each point in $\min(\zeta_k(N))$ is the supremum of exactly k points in $\Delta(N)$, and so

$$\Delta(N) \subseteq \cup_{k \geq 1} \min(\zeta_k(N)),$$

with equality if and only if the jump points $\Delta(N)$ are totally ordered.

To clarify the relationship between $\Delta(N)$ and the sets ζ_k in $\Re^2_+$, note that $\zeta_k(N) \supseteq \zeta_{k+1}(N) \supseteq \zeta_k^+(N)$, where

$$\zeta_k^+(N) := \cup_{\epsilon, \epsilon' \in \min(\zeta_k), \epsilon \neq \epsilon'} E_{\epsilon \vee \epsilon'}.$$

If $\zeta_k(N) = \emptyset$ or if $\min(\zeta_k(N))$ consists of a single point, then $\zeta_k^+(N) = \emptyset$. In $\Re^2_+$, $\zeta_k(N) \setminus \zeta_k^+(N)$ is a collection of disjoint, incomparable rectangles, one corresponding to each point in $\min(\zeta_k(N))$ (this point is the lower left corner of the rectangle). Furthermore, since N is strictly simple, each point in $\min(\zeta_k^+(N))$ is a point in $\min(\zeta_{k+1}(N))$, but is not in $\Delta(N)$. In fact, it can be seen that in general

$$\min(\zeta_{k+1}(N)) = \min(\zeta_k^+(N)) \cup \min(\Delta(N) \cap (\zeta_k(N) \setminus \zeta_k^+(N))^o). \tag{3}$$

("$(\cdot)^o$" indicates the interior of a set.)

These ideas are illustrated in Figure 1. Actual jump points in $\Delta(N)$ are indicated with "●", and the lower boundaries of $\zeta_1(N)$ and $\zeta_2(N)$ are depicted with

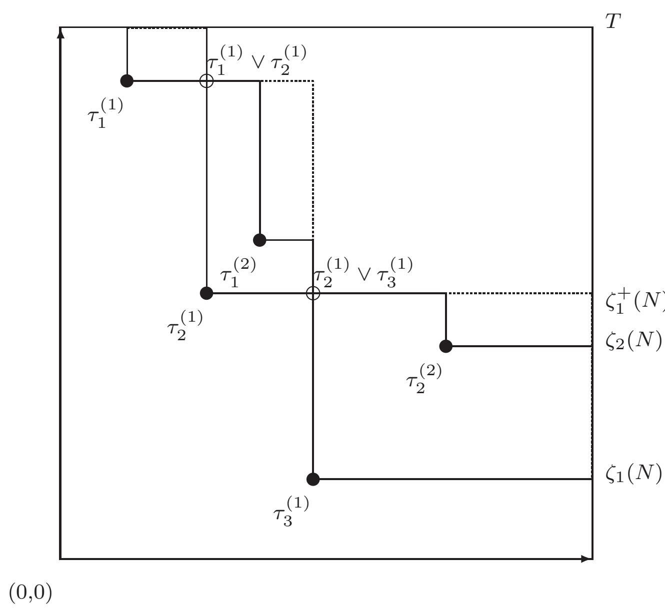

Fig 1. *Lower boundaries of the random sets $\zeta_1(N), \zeta_1^+(N)$ and $\zeta_2(N)$.*

solid lines. Note that the minimal points of $\zeta_1(N)$ are $\tau_1^{(1)}, \tau_2^{(1)}, \tau_3^{(1)} \in \Delta(N)$. The minimal points of $\zeta_1^+(N)$ are indicated with "$\circ$" (they are, respectively, $\tau_1^{(1)} \vee \tau_2^{(1)}, \tau_2^{(1)} \vee \tau_3^{(1)}$, but are not elements of $\Delta(N)$) and the lower boundary of $\zeta_1^+(N)$ is marked with a dotted line. Finally, the minimal points of $\zeta_2(N)$ include $\min(\zeta_1^+(N))$ as well as two new jump points $\tau_1^{(2)}, \tau_2^{(2)} \in (\Delta(N) \cap (\zeta_1(N) \setminus \zeta_1^+(N))^o)$.

2.2. Single line point processes

In generalizing the renewal property from one dimension to two, in [9] and [7] it is noted that the most natural analogue of a single jump point process on $\Re_+$ is a *single line process* on $\Re_+^2$, and this is the key to extending the renewal property. To be specific, a single jump point process M on $\Re_+$ is defined by a random variable $\tau \geq 0$, and we have

$$M(t) = I(\tau \leq t),\ t \in \Re_+.$$

In this case, $\zeta_1(M) = [\tau, \infty)$ and $\zeta_k(M) = \emptyset,\ \forall k \geq 2$.

A single line process M on $\Re_+^2$ is defined by a collection $\Delta(M) = \{\tau_1, \tau_2, \ldots\} \subset \Re_+^2$ of *incomparable* random points, and the corresponding point process is defined by

$$M(t) = \sum_{i=1}^{\infty} I(\tau_i \leq t),\ t \in \Re_+^2.$$

Although in $\Re_+^2$ the sets $\zeta_k(M)$ are no longer necessarily empty for $k \geq 2$, M is characterized by $\zeta_1(M)$, as it is in one dimension. In particular, it is easily seen that

$$\Delta(M) = \{\tau_1, \tau_2, \ldots\} = \min(\zeta_1(M)).$$

The law of a single line process M on $\Re_+^2$ is determined by the law of $\zeta_1(M)$, or equivalently its complement - this is the collection of probabilities

$$P(t_1, \ldots, t_n \in \zeta_1(M)^c) = P(M(t_1) = 0, \ldots, M(t_n) = 0),\ t_1, \ldots, t_n \in \Re_+^2, n \geq 1. \tag{4}$$

This characterization can be greatly simplified in the following situation. For $s = (s_1, s_2), t = (t_1, t_2) \in \Re_+^2$, define the following σ-fields:

$$\begin{aligned}
\mathcal{F}(t) &= \sigma\{M(s) : s \in A_t\} = \sigma\{M(s) : s_1 \leq t_1 \text{ and } s_2 \leq t_2\} \\
\mathcal{F}^1(t) &= \sigma\{M(s) : s_1 \leq t_1\}, \\
\mathcal{F}^2(t) &= \sigma\{M(s) : s_2 \leq t_2\}.
\end{aligned}$$

Definition 2.1. We say that M satisfies condition $(F4)$ if for all $t \in \Re_+^2$, the σ-fields $\mathcal{F}^1(t)$ and $\mathcal{F}^2(t)$ are conditionally independent, given $\mathcal{F}(t)$.

Henceforth, it will be assumed that the single line process M on $\Re_+^2$ satisfies (F4). The label (F4) was given to the property of conditional independence in the seminal paper [2], and now has become commonplace in the literature. Intuitively, one can think of M as the initial points of infection in the spread of an air-born disease under prevailing winds from the southwest: since there are no points in $[0,t_1]\times[t_2,\infty)$ southwest of $[t_1,\infty)\times[0,t_2]$ and vice versa, the behaviour of M in either region will not affect the other. The most important consequence of the (F4) assumption is the following simplification of (4):

Lemma 2.2 ([10], Lemma 5.3)**.** *Let M be a single line process on $\Re^2_+$ satisfying (F4). Then the law (i.e. the finite dimensional distributions) of M is uniquely determined by the set of avoidance probabilities*

$$P_0(t) := P(M(t)=0) = P(t\in\zeta_1(M)^c),\ t\in\Re^2_+. \tag{5}$$

On $\Re_+$, the avoidance probabilities of a single jump process are simply defined by the survival probability of the jump τ: $P_0(t)=P(M(t)=0)=P(\tau>t)$. In both one and two dimensions, we refer to P_0 as the *avoidance function* of the single jump or the single line process, respectively.

Avoidance probabilities can be expressed in terms of an intensity, which in turn, has a martingale interpretation. For a single jump process on $\Re_+$, if τ has continuous distribution F with density f, then $P_0(t)=e^{-\Lambda(t)}$ where

$$\Lambda(t)=\int_0^t \frac{f(s)}{1-F(s)}ds.$$

Let $\lambda(s) := \frac{f(s)}{1-F(s)}$; λ is known as the intensity function (or hazard, in the terminology of survival analysis) and Λ is the integrated intensity (hazard). The process

$$M(t)-\int_0^t \lambda(s)I(\tau\geq s)ds$$

is a martingale with quadratic variation (cf. [11])

$$\langle M\rangle(t)=\int_0^t \lambda(s)I(\tau\geq s)ds.$$

This interpretation has a natural extension to any single line process M on $\Re^2_+$ that satisfies (F4). In analogy to the one dimensional case, for $t\in\Re^2_+$ define

$$\Lambda(t) := -\ln P_0(t).$$

It is an easy consequence of (F4) that Λ is a measure on $\Re^2_+$ and *it will always be assumed that Λ has a density λ*. We refer to λ and Λ as the intensity and integrated intensity, respectively, of M.

Just as is the case in one dimension, in two dimensions there is a planar martingale associated with M. We begin with some notation: let $M(t-) := M(A_t\setminus\{t\})$ and define the increment of a function $f:\Re^2\to\Re$ on a rectangle $(s,t]=(s_1,t_1]\times(s_2,t_2]$ by $f(s,t]=f(t_1,t_2)-f(s_1,t_2)-f(t_1,s_2)+f(s_1,s_2)$. Next, recall that

$$\mathcal{F}(t)=\sigma\{M(s):s\in A_t\}=\sigma\{M(s):s_1\leq t_1 \text{ and } s_2\leq t_2\}$$

and define

$$\mathcal{F}^*(t) = \sigma\{M(s) : s_1 \le t_1 \text{ or } s_2 \le t_2\}.$$

Any process X that is $\mathcal{F}$-adapted is

- a weak martingale if for every $s \le t \in \Re^2_+$,

$$E[X(s,t] \mid \mathcal{F}(s)] = 0;$$

- a strong martingale if for every $s \le t \in \Re^2_+$,

$$E[X(s,t] \mid \mathcal{F}^*(s)] = 0.$$

It is shown in Example 7.4 of [9] that if M is a single line process satisfying (F4) with intensity λ, then

$$\Gamma(t) := M(t) - \int_{A_t} I_{\{M(s-)=0\}} \lambda(s) d(s) \tag{6}$$

is a strong martingale.

Furthermore, according to [2], a quadratic variation of a strong martingale X is any process $\langle X \rangle$ that satisfies

$$E[(X(s,t])^2 | \mathcal{F}(s)] = E[X^2(s,t] | \mathcal{F}(s)] = E[\langle X \rangle (s,t] | \mathcal{F}(s)] \ \forall s \le t.$$

(In other words, $X^2 - \langle X \rangle$ is a weak martingale.) Just as in one dimension, a quadratic variation for Γ is given by

$$\langle \Gamma \rangle (t) = \int_{A_t} I_{\{M(s-)=0\}} \lambda(s) ds. \tag{7}$$

These martingale properties of the single jump and the single line processes will be used to construct the nonparametric estimators in §3 and §4.

2.3. Renewal processes

In one dimension, the renewal property is very simply expressed as follows: the times between successive renewals are i.i.d random variables. However, as mentioned in the introduction, we take a different approach to defining the renewal property for a point process N on $\Re^2_+$, and now we make use of single line processes and the random sets $\zeta_k(N)$ of (1). To motivate what follows, we show how to express the renewal property on $\Re_+$ in terms of these concepts. First, if N is a point process on $\Re_+$, let $\tau_1, \tau_2, \dots$ denote the successive interarrival times (i.e. the times between successive jumps of N). Next, for any set $B \subseteq \Re^d_+$ and $t \in \Re^d_+$ let

$$B \oplus t = \{b + t : b \in B\} \text{ and } B \ominus t = \{b - t : b \in B\}.$$

Now, returning to a point process N on $\Re_+$, the sets $\zeta_k(N)$ can be defined recursively as

$$\zeta_{k+1}(N) = [\tau_{k+1}, \infty) \oplus \min(\zeta_k(N))$$

where $\zeta_0(N) = \Re_+$; equivalently,

$$\zeta_{k+1}(N) = \zeta_1(M_{k+1}) \oplus \min(\zeta_k(N)) \tag{8}$$

where $M_1, M_2, \ldots$ are the single jump processes associated with $\tau_1, \tau_2, \ldots$, and $\min(\zeta_k(N)) = \sum_{j=1}^{k} \tau_j$. We know that N is a renewal process if and only if $\tau_1, \tau_2, \ldots$ are i.i.d. In terms of the sets $\zeta_k(N)$, N is renewal if and only if given $\zeta_k(N)$, the conditional distribution of the random set $\zeta_{k+1}(N)$ does not depend on k.

Although this characterization of the renewal property may appear somewhat unnatural in one dimension, it is the key to understanding how to make use of single line processes to extend the renewal property to two dimensions. Let N denote any point process on $\Re^2_+$ and as before, let $\Delta(N)$ denote the set of jump points of N. The *first line* of N is the single line process N_1 whose jump points are the minimal jump points of N: i.e.

$$\Delta(N_1) = \min(\Delta(N)) = \{\tau \in \Delta(N) : \tau' \not\leq \tau \ \forall \tau' \in \Delta(N) \text{ such that } \tau' \neq \tau\}.$$

In other words, N_1 is the single line process defined by $\zeta_1(N_1) = \zeta_1(N)$.

Given a single line process M on $\Re^2_+$ satisfying (F4), we begin with $N_1 =_{\mathcal{D}} M$ (i.e. N_1 and M have the same avoidance function). For $k \geq 1$, we recall that $\zeta_k(N) \setminus \zeta_k^+(N)$ is a collection of disjoint, incomparable rectangles determined by $\zeta_k(N)$. Therefore, given $\zeta_k(N)$, from (3) we see that the set $\zeta_{k+1}(N)$ is determined by single line processes, each of which consists of the set of minimal jump points of N within one of these rectangles (see also Figure 1). The point process N is renewal if (F4) is satisfied and the process is independently regenerated on each of these rectangles, in each case according to the same law as M, suitably shifted by the lower left corner of the rectangle.

Formally, we express these ideas as follows: for any strictly simple point process N on $\Re^2_+$, we have

$$(\zeta_k(N) \setminus \zeta_k^+(N))^o = \cup_{\epsilon \in \min \zeta_k(N)} C_\epsilon,$$

where the sets C_ϵ are incomparable open rectangles; ϵ is the lower left corner of C_ϵ. For $k = 0$, $\zeta_0(N) = \Re^2_+$ and there is only one such set: $C_0 = (\Re^2_+)^o$. Referring to (3), we see that $\zeta_{k+1}(N)$ is determined by $\cup_{\epsilon \in \min \zeta_k(N)} \min(\Delta(N) \cap C_\epsilon)$. Now, for $k \geq 0$ and $\epsilon \in \min \zeta_k(N)$ define the single line process M_ϵ

$$\Delta(M_\epsilon) = \min(\Delta(N) \cap C_\epsilon) \ominus \epsilon.$$

(Note: $\Delta(M_\epsilon)$ is empty if N has no jumps in C_ϵ.)

Definition 2.3. The point process N on $\Re^2_+$ is renewal if there exists a single line process M satisfying (F4) such that for every $k \geq 0$, given $\zeta_k(N)$ the single line processes M_ϵ are independent for all $\epsilon \in \min(\zeta_k(N))$ and

$$\Delta(M_\epsilon) =_{\mathcal{D}} (\Delta(M)) \cap (C_\epsilon \ominus \epsilon). \tag{9}$$

For a renewal process N on $\Re_+^2$, we now have in analogy to (8) (and noting that in $\Re_+$, $\zeta_k^+(N) = \emptyset \ \forall k$),

$$\zeta_{k+1}(N) = \zeta_k^+(N) \cup \cup_{\epsilon \in \min \zeta_k(N)} (\zeta_1(M_\epsilon) \oplus \epsilon). \tag{10}$$

As shown in [9], for any single line process M there exists a unique renewal point process N with first line $N_1 =_{\mathcal{D}} M$. In fact, just as is the case in one dimension, a renewal process N is a Poisson process if and only if the intensity of N_1 is constant (cf. [9]).

We now see that a spatial renewal process N is characterized by the law of its first line N_1, which in turn (under (F4)) is identified by its integrated intensity function Λ, or equivalently, by its avoidance function P_0. This fact is analogous to the case of the renewal process in one dimension: it is characterized by the law of its first jump, which is identified by its integrated hazard (the integrated intensity) or its survival function (avoidance function). Finally, while the integrated intensity refers to the single line process, we make the trivial observation that the avoidance function is the same for both N and N_1 since

$$P_0(t) = P(N_1(t) = 0) = P(N(t) = 0),$$

and so we refer to "the" avoidance function of N or N_1 without ambiguity.

3. Nonparametric estimation for i.i.d. copies of a single line process (synchronous data)

We now turn to the problem of estimating the avoidance probability function P_0 of a renewal process on $\Re_+$ or $\Re_+^2$. The easiest situation is that of synchronous data, where we fully observe a fixed number n of renewals - in other words, we observe $M_1, M_2, \ldots, M_n$, n i.i.d. copies of the single line process M. As always, we assume that (F4) is satisfied for processes on $\Re_+^2$.

Since $P_0(t) = P(M(t) = 0)$, it is clear that P_0 can be estimated by the corresponding empirical probabilities. We will discuss this further in subsection §3.2. Alternatively, we can approach the problem using the relationship

$$P_0(t) = e^{-\Lambda(t)};$$

we first find a martingale estimator of Λ in §3.1, and then use a product limit to find a second estimator of P_0 in §3.2. As is well known from survival analysis, the empirical and product limit estimators of P_0 are identical in $\Re_+$; however, this is not the case in $\Re_+^2$. The estimators of P_0 will be compared in §3.2.

3.1. Estimator of $\boldsymbol{\Lambda}$

Throughout this section, all statements are valid in both one and two dimensions.

To estimate the integrated intensity Λ we use the martingale properties of M outlined in §2.2. Recall that

$$\Gamma(t) = M(t) - \int_{A_t} I_{\{M(s-)=0\}}\lambda(s)ds$$

is a (strong) martingale with quadratic variation

$$\langle\Gamma\rangle(t) = \int_{A_t} I_{\{M(s-)=0\}}\lambda(s)ds.$$

Let $M_1, \dots, M_n$ be i.i.d. copies of M. Then

$$\Gamma_n(t) := \sum_{i=1}^{n}\left(M_i(t) - \int_{A_t} I_{\{M_i(s-)=0\}}\lambda(s)ds\right) \tag{11}$$

is a (strong) martingale with quadratic variation

$$\langle\Gamma_n\rangle(t) = \sum_{i=1}^{n}\int_{A_t} I_{\{M_i(s-)=0\}}\lambda(s)ds. \tag{12}$$

Therefore, defining

$$Z_n(s) = \sum_{i=1}^{n} I_{\{M_i(s-)=0\}}, \tag{13}$$

we have

$$\sum_{i=1}^{n} M_i(dt) - Z_n(t)\lambda(t)dt = \Gamma_n(dt).$$

Treating the martingale as noise and setting the right hand side of the preceding equation to 0, we are led to the estimator $d(\hat{\Lambda}_n(t))$, where

$$d(\hat{\Lambda}_n(t)) = \frac{\sum_{i=1}^{n} M_i(dt)}{Z_n(t)}.$$

Finally, we arrive at a Nelson-Aalen-type estimator for the integrated intensity:

$$\hat{\Lambda}_n(t) = \int_{A_t} \frac{\sum_{i=1}^{n} M_i(ds)}{Z_n(s)} = \sum_{i=1}^{n}\int_{A_t}\frac{M_i(ds)}{Z_n(s)}. \tag{14}$$

We observe that $\hat{\Lambda}_n$ is a discrete measure with support on the jump points of $M_1, \dots, M_n$; the mass assigned to any jump point τ is equal to $(Z_n(\tau))^{-1}$.

Computational notes:

- **Renewal processes on $\Re_+$:** In one dimension, (14) simplifies as follows. The point processes M_i each have exactly one jump τ_i; let $\tau_{(1)} < \tau_{(2)} < \cdots < \tau_{(n)}$ denote the corresponding order statistics. Then $Z_n(\tau_{(i)}) = 1/(n-i+1)$ and (14) becomes

$$\hat{\Lambda}_n(t) = \sum_{i=1}^{n}\frac{I_{\{\tau_{(i)}\le t\}}}{n-i+1}. \tag{15}$$

- **Renewal processes on $\Re_+^2$:** In two dimensions, the situation is more complicated since in principle each point process M_i can have infinitely many jumps. In practice, we observe n i.i.d. copies of M on some bounded set A_T. Number the $M_i(T)$ points of M_i lying in A_T from left to right: $\tau_{i,1}, \tau_{i,2}, \dots, \tau_{i,M_i(T)}$ (note that M_i may not have any jump points in A_T). For any $t \in A_T$ we have

$$\int_{A_t} \frac{M_i(ds)}{Z_n(s)} = \sum_{j=1}^{M_i(T)} \frac{I_{\{\tau_{i,j} \in A_t\}}}{Z_n(\tau_{i,j})},$$

and so

$$\hat{\Lambda}_n(t) = \sum_{i=1}^{n} \sum_{j=1}^{M_i(T)} I_{\{\tau_{i,j} \in A_t\}} (Z_n(\tau_{i,j}))^{-1}. \tag{16}$$

If $M_i(t) = 0$, then the i^{th} summand in (14) and (16) is 0. Next, consider $Z_n(\tau_{i,j})$. Since M_i is a single line process, $M_i(\tau_{i,j}-) = 0$ and so $Z_n(\tau_{i,j}) \geq 1$ and (16) is always well-defined. We have

$$Z_n(\tau_{i,j}) = 1 + \sum_{k \neq i} I_{\{M_k(\tau_{i,j}-)=0\}}.$$

We now turn to the asymptotic behaviour of Λ_n, but first we need to introduce some notation. Let $C(A_T)$ denote the continuous functions on A_T, and let $D(A_T)$ denote the space of cadlag functions for $T \in \Re_+$ and the Banach space of all functions $f : A_T \to \Re$ continuous from the upper right quadrant and with limits from the other quadrants for $T \in \Re_+^2$. Both $C(A_T)$ and $D(A_T)$ are equipped with the sup norm. Products of these spaces will always be equipped with a product norm. Note that the following theorem is valid in both one and two dimensions, and the method of proof is identical in both cases.

Theorem 3.1. *Consider n i.i.d. observations of a single jump process on $\Re_+$ (respectively, a single line process on $\Re_+^2$ satisfying (F4)). Assume that the intensity λ is uniformly bounded above on A_T. Then as $n \to \infty$,*

$$\sqrt{n}(\hat{\Lambda}_n(\cdot) - \Lambda(\cdot)) \to_{\mathcal{D}} G_\Lambda$$

in $D(A_T)$, where G_Λ is a continuous mean zero Gaussian process on A_T with covariance function

$$Cov(G_\Lambda(t), G_\Lambda(s)) = \int_{A_t \cap A_s} e^{\Lambda(u)} \lambda(u) du.$$

Proof. In what follows, all variables s, t, u are in either $\Re_+$ or $\Re_+^2$; the proof is identical in both cases.

Recalling the definition of Z_n, note that

$$\Lambda(t) = \int_{A_t} \frac{\sum_{i=1}^{n} \lambda(s) I_{\{M_i(s-)=0\}}}{Z_n(s)} ds$$

and so it is straightforward to see that

$$\hat{\Lambda}_n(t) - \Lambda(t) = \int_{A_t} \frac{\Gamma_n(ds)}{Z_n(s)}, \tag{17}$$

where Γ_n is the (strong) martingale defined in (11).

We use the functional delta method to prove convergence. First, note that both Γ_n and Z_n are sums of i.i.d. processes and

$$E(\Gamma_n(\cdot)) = 0,\ Var(\Gamma_n(\cdot)/\sqrt{n}) = E(\langle\Gamma_n\rangle(\cdot))/n = \int_{A.} e^{-\Lambda(u)}\lambda(u)du,$$

$$E(Z_n(\cdot)/n) = P_0(\cdot) = e^{-\Lambda(\cdot)},\ Var(Z_n(\cdot)/\sqrt{n}) = e^{-\Lambda(\cdot)}(1-e^{-\Lambda(\cdot)}).$$

By the Jain-Marcus Theorem ([12], Example 2.11.13),

$$\sqrt{n}\left(\frac{1}{n}\Gamma_n(\cdot), \frac{1}{n}Z_n(\cdot) - e^{-\Lambda(\cdot)}\right) \to_{\mathcal{D}} (G_\Gamma, G_Z) \tag{18}$$

in $D(A_T)\times D(A_T)$, where G_Γ, G_Z are tight, continuous mean zero Gaussian processes. Continuity of both processes is a consequence of the easily verified continuity of the corresponding variance functions (cf. [12], pg. 41).

Next, note that $\hat{\Lambda}_n - \Lambda$ depends on the pair $(\frac{1}{n}\Gamma_n, \frac{1}{n}Z_n)$ through the composition map

$$(a,b) \longrightarrow \left(a, \frac{1}{b}\right) \longrightarrow \int \frac{1}{b}da.$$

By Lemma 3.9.17 of [12] in the case of $\Re_+$ or Lemma 4.1 of [3] in the case of $\Re_+^2$, this map is Hadamard-differentiable tangentially to $C(A_T)\times D(A_T)$ on a domain of the type $\{(a,b), \int|da| \le K, b \ge \epsilon\}$ for given K and $\epsilon > 0$, at every (a,b) such that $1/b$ is of bounded variation. The pair $(\frac{1}{n}\Gamma_n, \frac{1}{n}Z_n)$ is contained in this domain with probability tending to 1 for $K > 2E(M(T))$ and $\epsilon < P_0(T) = e^{-\Lambda(T)}$. The derivative map is given by (cf. [12], Example 3.9.19 and [3], pg. 1508)

$$(\alpha,\beta) \longrightarrow \int \frac{1}{b}d\alpha - \int \frac{\beta}{b^2}da,$$

where the first integral is defined by integration by parts if α is not of bounded variation. Now apply the functional delta method with $a \equiv 0$, $\alpha = G_\Gamma$, $b(\cdot) = P_0(\cdot) = e^{-\Lambda(\cdot)}$ and $\beta = G_Z$ to conclude that $\sqrt{n}(\hat{\Lambda}_n(\cdot) - \Lambda(\cdot)) \to_{\mathcal{D}} G_\Lambda$, where

$$G_\Lambda(t) = \int_{A_t} \frac{dG_\Gamma(s)}{P_0(s)} = \int_{A_t} e^{\Lambda(s)} dG_\Gamma(s). \tag{19}$$

The covariance structure of G_Λ is found by observing that $G_\Lambda(\cdot)$ is also the limit in distribution of

$$\frac{1}{\sqrt{n}}\int_{A.} e^{\Lambda(u)}\Gamma_n(du), \tag{20}$$

a sum of i.i.d. processes. Therefore,

$$\begin{aligned} Cov(G_\Lambda(t), G_\Lambda(s)) &= Cov\left(\int_{A_t} e^{\Lambda(u)}\Gamma(du), \int_{A_s} e^{\Lambda(u)}\Gamma(du)\right) \\ &= E\left[\int_{A_T} I_{A_t}(u)e^{\Lambda(u)}\Gamma(du) \times \int_{A_T} I_{A_s}(u)e^{\Lambda(u)}\Gamma(du)\right] \\ &= E\left[\int_{A_T} I_{A_t}(u)e^{\Lambda(u)} I_{A_s}(u)e^{\Lambda(u)}\langle\Gamma\rangle(du)\right] \\ &= E\left[\int_{A_t\cap A_s} e^{2\Lambda(u)} I_{\{M(u-)=0\}}\lambda(u)du\right]. \end{aligned}$$

The second last equality above is standard for martingales on $\Re_+$; for strong martingales on $\Re_+^2$, see [2], Theorem 2.2 (c). Now taking expectations, it is easy to see that

$$Cov(G_\Lambda(t), G_\Lambda(s)) = \int_{A_t\cap A_s} e^{\Lambda(u)}\lambda(u)du.$$

□

3.2. Estimators of P_0

We now proceed with finding nonparametric estimators of P_0 in both $\Re_+$ and $\Re_+^2$: first we look at empirical estimators and next we use the results of §3.1 to find martingale estimators.

Method 1: Empirical probabilities

The easiest method is simply to use the empirical avoidance probability: denote the empirical estimator of P_0 by $\tilde{P}_0^{(n)}$ where for $t \in \Re_+$ (respectively, $t \in \Re_+^2$),

$$\tilde{P}_0^{(n)}(t) := \frac{1}{n}\sum_{i=1}^{n} I_{\{M_i(t)=0\}}. \tag{21}$$

Method 2: Product limit estimator (martingale approach)

Again, we use exactly the same approach in both $\Re_+$ and $\Re_+^2$. Defining the product limit integral π as usual in one dimension and as in [1], equation (10.3.11) (or [6], equation (28)) in two dimensions, by continuity of Λ we have

$$P_0(t) = e^{-\Lambda(t)} = \pi_{s\in A_t}\left(1 - d\Lambda(s)\right). \tag{22}$$

This leads us to a product limit estimator $\hat{P}_0^{(n)}$:

$$\hat{P}_0^{(n)}(t) := \pi_{s\in A_t}\left(1 - d\hat{\Lambda}_n(s)\right). \tag{23}$$

Computational notes:

- **Renewal processes on** $\Re_+$: In one dimension, it is a well known result in survival analysis when the data points are not subject to censoring, the product limit and empirical estimators of the avoidance (survival) function are identical: $\hat{P}_0^{(n)} = \tilde{P}_0^{(n)}$.
- **Renewal processes on** $\Re_+^2$: The situation in two dimensions is different. As before, we observe n i.i.d. copies of M on some bounded set A_T. Using the same notation as in (16), we have

$$\hat{P}_0^{(n)}(t) = \prod_{i=1}^{n} \prod_{j=1}^{M_i(T)} \left(1 - I_{\{\tau_{i,j} \in A_t\}}(Z_n(\tau_{i,j}))^{-1}\right). \tag{24}$$

  Unlike one dimension, this estimator is not equivalent to the empirical estimator. Consider the following simple example on $[0,1]^2$: Let $n = 2$, and suppose that M_1 has 2 jump points, at $(\frac{1}{4}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{4})$, respectively, while M_2 has one jump at $(\frac{3}{4}, \frac{3}{4})$. If $t = (t_1, t_2)$, then

$$\tilde{P}_0^{(n)}(t) = \begin{cases} 1 & \text{if } t \in \{[0,\frac{1}{4}) \times [0,1]\} \cup \{[0,1] \times [0,\frac{1}{4})\} \cup \{[0,\frac{1}{2}) \times [0,\frac{1}{2})\} \\ \frac{1}{2} & \text{if } t \in \{[\frac{1}{4},\frac{3}{4}) \times [\frac{1}{2},1]\} \cup \{[\frac{1}{2},1] \times [\frac{1}{4},\frac{3}{4})\} \\ 0 & \text{if } t \in [\frac{3}{4},1] \times [\frac{3}{4},1] \end{cases},$$

  while

$$\hat{P}_0^{(n)}(t) = \begin{cases} 1 & \text{if } t \in \{[0,\frac{1}{4}) \times [0,1]\} \cup \{[0,1] \times [0,\frac{1}{4})\} \cup \{[0,\frac{1}{2}) \times [0,\frac{1}{2})\} \\ \frac{1}{2} & \text{if } t \in \{[\frac{1}{4},\frac{1}{2}) \times [\frac{1}{2},1]\} \cup \{[\frac{1}{2},1] \times [\frac{1}{4},\frac{1}{2})\} \\ \frac{1}{4} & \text{if } t \in \{[\frac{1}{2},\frac{3}{4}) \times [\frac{1}{2},1]\} \cup \{[\frac{1}{2},1] \times [\frac{1}{2},\frac{3}{4})\} \\ 0 & \text{if } t \in [\frac{3}{4},1] \times [\frac{3}{4},1] \end{cases}.$$

  The difference in the estimators results from the fact that $\tilde{P}_0^{(n)}(t)$ reflects only whether or not any jump points of M_i fall into A_t whereas $\hat{P}_0^{(n)}$ takes into consideration the total number of jump points in M_i that fall into A_t. As a result, we will see below that $\hat{P}_0^{(n)}$ is to be preferred. In addition, the martingale approach used to construct $\hat{P}_0^{(n)}$ will be seen to adapt readily to asynchronous data, while the empirical estimator does not.

Both estimators satisfy a central limit theorem (CLT). In one dimension, it is the usual empirical central limit theorem applied to the survival function. In two dimensions, we have the following:

Theorem 3.2. *Let $M_1, \ldots, M_n$ be i.i.d. single line processes on $\Re_+^2$ satisfying (F4) and with common intensity λ uniformly bounded above on A_T. Let G_Λ be the Gaussian limit in Theorem 3.1. Then as $n \to \infty$,*

$$\sqrt{n}(\tilde{P}_0^{(n)}(\cdot) - P_0(\cdot)) \to_{\mathcal{D}} \tilde{G}_P(\cdot),$$

and

$$\sqrt{n}(\hat{P}_0^{(n)}(\cdot) - P_0(\cdot)) \to_{\mathcal{D}} G_P =_{\mathcal{D}} P_0(\cdot) G_\Lambda(\cdot) = e^{-\Lambda(\cdot)} G_\Lambda(\cdot)$$

in $D(A_T)$, where $\tilde{G}_P$ and G_P are both continuous mean zero Gaussian processes on A_T with the following covariance structures:

$$Cov(\tilde{G}_P(t), \tilde{G}_P(s)) = e^{-(\Lambda(t)+\Lambda(s))}(e^{\Lambda(A_t \cap A_s)} - 1) \tag{25}$$

and

$$Cov(G_P(t), G_P(s)) = e^{-(\Lambda(A_t)+\Lambda(A_s))} \int_{A_t \cap A_s} e^{\Lambda(u)} \lambda(u) du. \tag{26}$$

Proof. The CLT for $\tilde{P}_0^{(n)}$ follows from the Jain-Marcus Theorem ([12], Example 2.11.13) for sums of independent stochastic processes. The covariance structure follows easily by observing that

$$\begin{aligned} &Cov(I_{\{M(t)=0\}} I_{\{M(s)=0\}}) \\ &\quad = \quad P(M(A_t \cup A_s) = 0) - P(M(A_t) = 0)P(M(A_s) = 0) \\ &\quad = \quad e^{-\Lambda(A_t \cup A_s)} - e^{-(\Lambda(t)+\Lambda(s))} = e^{-(\Lambda(t)+\Lambda(s))}(e^{\Lambda(A_t \cap A_s)} - 1). \end{aligned}$$

Turning our attention to $\hat{P}_0^{(n)}$, we again make use of the functional delta method. Hadamard differentiability of the product integral map $\mathcal{P} : D(A_T) \to D(A_T)$ defined by

$$\mathcal{P}(f)(t) = \prod_{s \in A_t} (1 + df(s))$$

was established for two dimensions in [6], Theorem 3.2. An argument analogous to that used in the one-dimensional case in [12], pg. 392, shows that

$$\begin{aligned} \sqrt{n}(\hat{P}_0^{(n)}(\cdot) - P_0(\cdot)) \quad &= \quad \sqrt{n}\left(\mathcal{P}(-\hat{\Lambda}_n) - \mathcal{P}(-\Lambda)\right) \\ &\to_{\mathcal{D}} \quad \mathcal{P}'_{-\Lambda}(-G_\Lambda) \\ &= \quad \prod_{s \in A.} (1 - d\Lambda(s)) \int_{A.} \frac{-dG_\Lambda(u)}{(1 - \Delta\Lambda(u))} \qquad (27) \\ &= \quad e^{-\Lambda(\cdot)} \int_{A.} (-dG_\Lambda) \qquad (28) \\ &=_{\mathcal{D}} \quad P_0(\cdot) G_\Lambda(\cdot). \end{aligned}$$

The equality in (27) follows from equation (38) of [6] and so (28) follows by continuity of Λ ("$\Delta\Lambda(u)$" in (27) denotes the mass assigned to the singleton u by the measure Λ).

The covariance structure of G_P follows immediately from that of G_Λ. □

Comparison of $\tilde{P}_0^{(n)}$ and $\hat{P}_0^{(n)}$ for synchronous data on $\Re_+^2$

Asymptotically, both estimators are unbiased, but as noted previously, $\hat{P}_0^{(n)}$ uses more of the available information. We are also able to see that $\hat{P}_0^{(n)}(t)$ has a smaller asymptotic variance than $\tilde{P}_0^{(n)}(t)$

Comment 3.3. For completeness, we have applied all the martingale arguments to point processes on $\Re_+$ as well as on $\Re_+^2$. As observed above, for synchronous data on $\Re_+$, the product limit estimator of the survival probability $S = 1 - F$ coincides with the usual empirical survival function. However, we shall see in the next section that the martingale approach provides a new estimator for asynchronous data on $\Re_+$.

4. Nonparametric estimation for asynchronous data

In this section, we deal with estimation of the avoidance function given asynchronous data: instead of fixing the number of renewals, we fix the set A_T on which the renewals are observed. For data on $\Re_+$, a simple generalization of the empirical estimator is generally used, but given the more complex structure of the renewal process on $\Re_+^2$, the martingale approach is more appropriate. We shall see that asynchronous data is censored and the martingale methods proposed in §3 readily adapt to this framework, just as they do in multivariate survival analysis (cf. [8]). For clarity and a better understanding of the censoring issue, we will begin with the martingale approach to estimation of P_0 in one dimension in §4.1; this will motivate the approach taken in §4.2 for the estimator of P_0 in two dimensions.

4.1. Estimation of avoidance probabilities on $\Re_+$

We have a renewal process on $\Re_+$. The interarrival times are $\tau_1, \tau_2, \ldots$ where the τ_i's are i.i.d. F and the arrival times (renewals) are $X_1, X_2, \ldots$ where $X_n = \sum_{i=1}^n \tau_i$. We assume that F is continuous with density f, and $m := E[\tau] < \infty$. We observe the renewal process on $A_T = [0, T]$. Let $N(T)$ be the number of renewals before T. Setting $X_0 = 0$, define the backward recurrence time at T as $V_T = T - X_{N(T)}$. We want to estimate the avoidance probability $P_0(t) = P(\tau > t) = 1 - F(t)$. For any $t > 0$, define

$$I(t) := \sum_{i=1}^{N(T)} I_{\{\tau_i \leq t\}}.$$

Method 1: Classic estimator of P_0

According to Karr's modified estimator of the interarrival distribution F ([11], equation (8.13), pg. 313), an appropriate estimator is

$$\hat{P}_0^K(t) = \begin{cases} 1 - \frac{I(t)}{N(T)+1}, & t \leq V_T \\ 1 - \frac{I(t)}{N(T)}, & t > V_T \end{cases}, \tag{29}$$

where $\frac{0}{0}$ is interpreted as 1. The estimator satisfies a CLT:

Theorem 4.1 ([11], Theorem 8.9)**.** *As $T \to \infty$,*

$$\sqrt{T}(\hat{P}_0^K(\cdot) - P_0(\cdot)) \to_{\mathcal{D}} \hat{G}_P$$

in $D[0, \infty)$ where $\hat{G}_P$ is a continuous mean zero Gaussian process with covariance function $mP_0(t)(1 - P_0(s))$ for $0 < s \leq t < \infty$.

Method 2: Product limit estimator of P_0 (martingale approach)
The following estimator will be justified using martingale techniques. With the same notation as above, the martingale estimator of $P_0(\cdot)$ is

$$\hat{P}_0(t) = \begin{cases} 1 - \frac{I(t)}{N(T)+1}, & t \le V_T \\ \left[\frac{N(T)-I(V_T)+1}{N(T)+1}\right] \cdot \left[\frac{N(T)-I(t)}{N(T)-I(V_T)}\right], & t > V_T \end{cases}, \tag{30}$$

where $\frac{0}{0}$ is interpreted as 1.

The martingale estimator $\hat{P}_0$ is constructed as follows. Since we observe the renewal process on the finite interval $[0, T]$, our observations of $\tau_1, \tau_2, \dots$ are in fact *censored*. τ_1 is censored by T, τ_2 is censored by $T - \tau_1 = T - X_1$, τ_3 is censored by $T - X_2$ and so on. In general, τ_i is censored by $D_i = \max(T - X_{i-1}, 0) = (T - X_{i-1})^+$ (set $X_0 = 0$). We know that $I_{\{\tau_i \le t\}} - \int_0^t \lambda(s) I_{\{\tau_i \ge s\}} ds$ is a martingale. Since D_i is independent of τ_i, it is true that the stopped process $\gamma_i(t) := I_{\{\tau_i \le t \wedge D_i\}} - \int_0^t \lambda(s) I_{\{\tau_i \ge s\}} I_{\{D_i \ge s\}} ds$ is also a martingale. We have

$$\begin{aligned} \sum_{i=1}^{\infty} \gamma_i(t) &= \sum_{i=1}^{\infty} \left(I_{\{\tau_i \le t \wedge D_i\}} - \int_0^t \lambda(s) I_{\{\tau_i \ge s\}} I_{\{D_i \ge s\}} ds \right) \\ &= \sum_{i=1}^{\infty} I_{\{\tau_i \le t \wedge (T - X_{i-1})^+\}} - \int_0^t \lambda(s) \sum_{i=1}^{\infty} I_{\{s \le \tau_i \wedge (T - X_{i-1})^+\}} ds \\ &= \sum_{i=1}^{N(T)} I_{\{\tau_i \le t\}} - \int_0^t \lambda(s) Z(s) ds, \end{aligned} \tag{31}$$

where

$$Z(s) := \sum_{j=1}^{\infty} I_{\{s \le \tau_j \wedge (T - X_{j-1})^+\}} = \sum_{j=1}^{N(T)} I_{\{s \le \tau_j\}} + I_{\{s \le V_T\}}. \tag{32}$$

Equations (31) and (32) follow from the observation that $\tau_i \le (T - X_{i-1})^+ \Leftrightarrow i \le N(T)$, and so

$$\tau_j \wedge (T - X_{j-1})^+ = \begin{cases} \tau_j, & j \le N(T) \\ V_T, & j = N(T) + 1 \\ 0, & j > N(T) + 1 \end{cases}.$$

The sum $\sum_{i=1}^{\infty} \gamma_i$ is the sum of martingales, and for each fixed t has mean 0 (by dominated convergence, as both terms on the right hand side of (31) have expectations bounded above by $E[N(T)]$). Therefore, we will treat $\sum_{i=1}^{\infty} \gamma_i(t)$ as noise and set (31) equal to 0. As before, this leads to the following estimator of the integrated intensity:

$$\hat{\Lambda}(t) = \sum_{i=1}^{N(T)} \frac{I_{\{\tau_i \le t\}}}{Z(\tau_i)}.$$

This in turn yields the corresponding product limit estimator of P_0:

$$\hat{P}_0(t) = \prod_{i=1}^{N(T)} \left(1 - \frac{I_{\{\tau_i \le t\}}}{Z(\tau_i)}\right) = \prod_{i=1}^{N(T)} \left(1 - \frac{I_{\{\tau_{(i)} \le t\}}}{Z(\tau_{(i)})}\right), \tag{33}$$

where $\tau_{(1)} < \tau_{(2)} < \cdots < \tau_{(N(T))}$ are the order statistics associated with the $\tau_i^\prime s$. Letting $s = \tau_{(i)}$ in (32),

$$Z(\tau_{(i)}) = \sum_{j=1}^{N(T)} I_{\{\tau_{(i)} \le \tau_j\}} + I_{\{\tau_{(i)} \le V_T\}} = (N(T) - i + 1) + I_{\{\tau_{(i)} \le V_T\}}. \tag{34}$$

Substituting (34) in (33) and recalling the definition of $I(t)$, we obtain

$$\begin{aligned} \hat{P}_0(t) &= \prod_{i=1}^{N(T)} \left(1 - \frac{I_{\{\tau_{(i)} \le t\}}}{(N(T) - i + 1) + I_{\{\tau_{(i)} \le V_T\}}}\right) \\ &= \prod_{i=1}^{I(t)} \left(1 - \frac{1}{(N(T) - i + 1) + I_{\{\tau_{(i)} \le V_T\}}}\right). \end{aligned} \tag{35}$$

Now, if $t \le V_T$ then $\tau_{(i)} \le t \Rightarrow \tau_{(i)} \le V_T \Rightarrow (N(T) - i + 1) + I_{\{\tau_{(i)} \le V_T\}} = N(T) - i + 2$ for every $i, 1 \le i \le I(t)$. Therefore, by successive cancellation in (35), if $t \le V_T$,

$$\hat{P}_0(t) = 1 - \frac{I(t)}{N(T) + 1}.$$

On the other hand, if $t > V_T$, note that $I(V_T) \le I(t)$. If $i \le I(V_T)$, then $(N(T) - i + 1) + I_{\{\tau_{(i)} \le V_T\}} = N(T) - i + 2$, while for $i > I(V_T)$, $(N(T) - i + 1) + I_{\{\tau_{(i)} \le V_T\}} = N(T) - i + 1$. Again, by successive cancellation in (35) (and remembering that $0/0 = 1$), if $t > V_T$

$$\hat{P}_0(t) = \frac{N(T) - I(V_T) + 1}{N(T) + 1} \cdot \frac{N(T) - I(t)}{N(T) - I(V_T)}.$$

This yields (30).

Comparison of $\hat{P}_0^K$ and $\hat{P}_0$

Comparing the two estimators, we see that they agree if $t \le V_T$. For $t > V_T$ (in which case $I(V_T) \le I(t)$), if $I(V_T) < N(T)$, then

$$\hat{P}_0(t) = \frac{N(T)}{N(T) + 1} \cdot \frac{N(T) - I(V_T) + 1}{N(T) - I(V_T)} \hat{P}_0^K(t);$$

in this case, $\hat{P}_0(t) > \hat{P}_0^K(t)$ if and only if $1 \le I(V_T) \le I(t) < N(T)$. If $t > V_T$ and $I(V_T) = N(T)$, then $\hat{P}_0(t) = \frac{1}{N(T)+1}$, while $\hat{P}_0^K(t) = 0$ (since $I(V_T) = N(T) \Rightarrow I(t) = N(t)$ for $t > V_T$). To summarize, $\hat{P}_0(t) \ge \hat{P}_0^K(t)$, with strict inequality if $t > V_T$ and $1 \le I(V_T) \le I(t) < N(T)$ or $I(V_T) = I(t) = N(T)$. In fact, the estimators are asymptotically equivalent:

Lemma 4.2. *As $T \to \infty$,*

$$\sup_{t \in \Re_+} |\hat{P}_0(t) - \hat{P}_0^K(t)| \to_P 0.$$

Proof. We have seen that the estimators agree if $t \leq V_T$ and that for $t > V_T$,

$$\hat{P}_0(t) = \frac{N(T)}{N(T)+1} \cdot \left[1 + \frac{1}{N(T) - I(V_T)}\right] \hat{P}_0^K(t).$$

Since $N(T) \to \infty$, it suffices to show that $\frac{I(V_T)}{N(T)}$ converges in distribution to a nonnegative random variable W which satisfies $P(W < 1) = 1$. By strong uniform consistency of $\hat{P}_0^K$, (see [11], Proposition 8.8), it follows that with probability 1, $|(I(V_T)/N(T)) - F(V_T)| \to 0$. On the other hand, as the renewal process converges to stationarity, V_T converges in distribution to a random variable V with density $\frac{1}{m}(1-F)$. It follows that $F(V) < 1$ with probability 1. Letting $W = F(V)$ completes the proof. □

Since the censoring D_i depends on $\tau_1, \dots \tau_{i-1}$, the martingales γ_i are not independent, nor is Z the sum of i.i.d. processes. Consequently, we cannot apply the same methods as in §3 to analyze the asymptotic behaviour of $\hat{P}_0$. However, by Lemma 4.2, it follows that the asymptotic behaviour of $\hat{P}_0$ is identical to that of $\hat{P}_0^K$ and we have the following (cf. Theorem 4.1):

Corollary 4.3. *As $T \to \infty$,*

$$\sqrt{T}(\hat{P}_0(\cdot) - P_0(\cdot)) \to_{\mathcal{D}} \hat{G}_P$$

in $D[0,\infty)$ where $\hat{G}_P$ is a continuous mean zero Gaussian process with covariance function $mP_0(t)(1-P_0(s))$ for $0 < s \leq t < \infty$.

An empirical analysis of the estimators indicates that both $\hat{P}_0$ and $\hat{P}_0^K$ are slightly biased upwards (see Tables 2-7 of Appendix A.1). However, comparing the results in Tables 2-4, for example, it is clear that for a fixed t, the bias of both estimators decreases as T increases. This being said, since $\hat{P}_0^K \leq \hat{P}_0$, $\hat{P}_0^K$ is preferred. However, the martingale approach developed here and in the preceding section for two-dimensional synchronous data motivates the development of a nonparametric estimator of avoidance probabilities for asynchronous data on $\Re_+^2$. This is the topic of the next subsection.

4.2. Estimation of the integrated intensity and avoidance probabilities on $\Re_+^2$

For renewal processes on $\Re_+^2$, the asynchronous data model arises from observing a renewal process N on a rectangle A_T. Referring to (9) and (10), this is equivalent to observing a random number of i.i.d. realizations of the single line process $M =_{\mathcal{D}} N_1$ restricted to (or censored by) rectangles of the form

$$A_{D(\epsilon)} := (C_\epsilon \cap A_T) \ominus \epsilon,$$

where $D(\epsilon) = \min(T, \sup(C_\epsilon)) \ominus \epsilon$. In each case, the realization of M is independent of the random point $D(\epsilon) \in \Re^2_+$.

We use the same notation and martingale approach of §3: let M be a single line process satisfying (F4) with intensity λ. $M(t) - \int_{A_t} I_{\{M(s-)=0\}}\lambda(s)ds$ is a two-dimensional strong martingale. However, suppose that M is censored: we only observe M on some (random) rectangle A_D (i.e. for $t \in \Re^2_+$, we observe $M(t \wedge D) = M(A_t \cap A_D)$). If $D \in \Re^2_+$ is independent of M, then

$$\begin{aligned}\gamma(t) &:= M(t \wedge D) - \int_{A_t} I_{\{s \le D\}} I_{\{M(s-)=0\}}\lambda(s)ds \\ &= \int_{A_t} I_{\{s \le D\}} M(ds) - \int_{A_t} I_{\{s \le D\}} I_{\{M(s-)=0\}}\lambda(s)ds\end{aligned}$$

is also a strong martingale ([8], Theorem 3.12).

For the asynchronous data model, we number our (censored) observations of M first by k (i.e. the realization of M (suitably translated) defines jump points in a rectangle in $(\zeta_k \setminus \zeta_k^+)^o \cap A_T$) and then from left to right according to the positions of the rectangles. With this numbering, we have an array $(M_{k,i})$ of i.i.d. copies of M. Then, although $M_{k,i}$ is subject to censoring by a random point $D_{k,i}$ which can depend on $\{M_{h,j}, h < k\}$, we have that $D_{k,i}$ and $M_{k,i}$ are independent $\forall k, i$. As in §4.1, it follows that

$$\begin{aligned}\sum_{k=1}^\infty \sum_{i=1}^\infty \gamma_{k,i}(t) &= \sum_{k=1}^\infty \sum_{i=1}^\infty \left(M_{k,i}(t \wedge D_{k,i}) - \int_{A_t} I_{\{s \le D_{k,i}\}} I_{\{M_{k,i}(s-)=0\}}\lambda(s)ds \right) \\ &= \sum_{k=1}^\infty \sum_{i=1}^\infty \int_{A_t} I_{\{s \le D_{k,i}\}} M_{k,i}(ds) \\ &\quad - \int_{A_t} \lambda(s) \left(\sum_{k=1}^\infty \sum_{i=1}^\infty I_{\{s \le D_{k,i}\}} I_{\{M_{k,i}(s-)=0\}} \right) ds \qquad (36)\end{aligned}$$

is the sum of strong martingales. Of course, only finitely many terms in the sum will be non-zero. Therefore, defining

$$Z(s) = \sum_{k=1}^\infty \sum_{i=1}^\infty I_{\{s \le D_{k,i}\}} I_{\{M_{k,i}(s-)=0\}}, \qquad (37)$$

we have

$$\sum_{k=1}^\infty \sum_{i=1}^\infty I_{\{t \le D_{k,i}\}} M_{k,i}(dt) - Z(t)\lambda(t)dt = \sum_{k=1}^\infty \sum_{i=1}^\infty \gamma_{k,i}(dt).$$

Treating the sum of strong martingales as noise and setting the right hand side of the above equation to 0, we are led to the estimator $d(\hat{\Lambda}(t))$, where

$$d(\hat{\Lambda}(t)) = \frac{\sum_{k=1}^\infty \sum_{i=1}^\infty I_{\{t \le D_{k,i}\}} M_{k,i}(dt)}{Z(t)}.$$

Finally, as in §3.1, we arrive at a Nelson-Aalen-type estimator for the integrated intensity:

$$\hat{\Lambda}(t) = \int_{A_t} \frac{\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} I_{\{s\le D_{k,i}\}} M_{k,i}(ds)}{Z(s)} = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\int_{A_t} \frac{I_{\{s\le D_{k,i}\}} M_{k,i}(ds)}{Z(s)}. \tag{38}$$

This leads us to the corresponding product limit estimator of the avoidance probabilities:

$$\hat{P}_0(t) = \pi_{s\in A_t}\left(1 - d\hat{\Lambda}(s)\right). \tag{39}$$

Computational notes: Letting the set of *observed* jump points of $M_{k,i}$ be denoted by $\Delta_{k,i} := \Delta(M_{k,i}) \cap A_{D_{k,i}}$, if $\tau \in \Delta_{k,i}$, then

$$Z(\tau) = 1 + \sum_{(h,j)\neq(k,i)} I_{\{\tau\le D_{h,j}\}} I_{\{M_{h,j}(\tau-)=0\}}.$$

Finally, from (38) and (39) we have

$$\hat{\Lambda}(t) = \sum_k \sum_i \sum_{\tau\in\Delta_{k,i}} \frac{I_{\{\tau\le t\}}}{Z(\tau)} \tag{40}$$

and

$$\hat{P}_0(t) = \prod_k \prod_i \prod_{\tau\in\Delta_{k,i}} \left(1 - \frac{I_{\{\tau\le t\}}}{Z(\tau)}\right). \tag{41}$$

Note that if $\Delta_{k,i}$ is empty, then the corresponding term in the product is equal to 1.

If $T = (T_1, T_2)$, we conjecture that $\sqrt{T_1T_2}(\hat{\Lambda} - \Lambda)$ and $\sqrt{T_1T_2}(\hat{P}_0 - P_0)$ both satisfy functional CLTs as $\min(T_1, T_2) \to \infty$. The CLT for the avoidance probability function would follow from that for the integrated intensity via the functional delta method, as in Theorem 3.2. A CLT for $\hat{\Lambda}$ would likely follow from martingale techniques (CLTs are available for strong martingales), but also would likely require convergence of N to some form of stationarity; this problem has not yet been explored in the context of the spatial renewal process and is well beyond the scope of this article. To support our conjecture, in Appendix A.2 we present some empirical evidence of convergence to normality of $\hat{P}_0$ based on simulations. While Figure 4 exhibits some unusual behaviour (perhaps due to the quick convergence of the estimator to 0 as t goes to 0), these plots reveal that for a fixed T, as t decreases, the plots become smoother and the estimates seem to converge to normality. This is intuitive since as t decreases relative to T, the amount of censoring present decreases.

4.3. Parametric and nonparametric comparison

Here we include a brief simulation study wherein we compare parametric and nonparametric estimators of the integrated intensity Λ for asynchronous data

Table 1
Average and Standard Errors (SE) of Parametric ($\tilde{\Lambda}(t)$) and Nonparametric ($\hat{\Lambda}(t)$) Estimates of $\Lambda(t) = \lambda t_1^{\alpha} t_2^{\beta}$

					True Value	$\tilde{\Lambda}(t)$		$\hat{\Lambda}(t)$	
A_T	t	λ	α	β	$\Lambda(t)$	Average	SE	Average	SE
$[0,25]^2$	(1.00,1.00)	1.00	2.00	2.00	1.0000	1.0175	0.1045	0.8968	0.2948
$[0,25]^2$	(1.00,1.00)	0.75	1.50	1.50	0.7500	0.7585	0.0713	0.7005	0.2331
$[0,25]^2$	(1.00,1.00)	0.50	1.50	1.50	0.5000	0.5081	0.0470	0.4807	0.1278
$[0,25]^2$	(0.25,0.25)	1.00	1.50	2.00	0.0078	0.0078	0.0009	0.0077	0.0017
$[0,30]^2$	(0.25,0.25)	1.00	1.50	2.00	0.0078	0.0078	0.0007	0.0077	0.0014
$[0,30]^2$	(0.25,0.25)	2.00	2.00	2.00	0.0078	0.0078	0.0007	0.0077	0.0015
$[0,30]^2$	(0.10,0.10)	0.50	1.00	2.00	0.0005	0.0005	0.0001	0.0005	0.0003
$[0,30]^2$	(0.15,0.15)	1.00	1.00	2.00	0.0034	0.0034	0.0003	0.0033	0.0006
$[0,30]^2$	(0.75,0.75)	1.00	1.00	2.00	0.4219	0.4255	0.0339	0.4132	0.1076

for various A_T and t in terms of bias and standard error. Specifically, for each of 1000 simulated data, we compute the estimate found in (40) in the case of nonparametric, and in the case of parametric, we estimate the model parameters using numerical maximum likelihood as outlined in [4] and evaluate the function at the appropriate t. The model we consider is as found in [4], termed multiplicative generalized homogeneous Poisson process, with integrated intensity function as $\Lambda(t) = \lambda t_1^{\alpha} t_2^{\beta}$, $\lambda, \alpha, \beta > 0$. As a measure of the performance of both estimators, we find the average and empirical standard error of the 1000 estimates. We also include the true value of the integrated intensity function. Naturally, the nonparametric estimators do not perform as well as the parametric. This is more noticeable when t is larger relative to T, due to the greater number of censored renewals.

5. Conclusion

We have seen that martingale methods provide a unified approach to nonparametric estimation of avoidance probabilities for renewal processes in both one and two dimensions, regardless of the data structure (synchronous or asynchronous). Directions for further research include:

- convergence to stationarity of the spatial renewal process
- a CLT for $\hat{P}_0$ in the case of asynchronous spatial renewal data
- smoothed estimators of Λ and P_0
- semiparametric estimation of Λ and P_0.

Appendix A: Empirical results

A.1. Empirical comparisons of $\hat{P}_0^K$ and $\hat{P}_0$

The estimators $\hat{P}_0^K$ and $\hat{P}_0$ of §4.1 are compared in the following tables.

TABLE 2

Averages and Standard Errors (SE) of Nonparametric Estimators of $P_0(t)$ based on 1000 Simulations of Renewal Processes on [0,5] with $Exp(1)$ Interarrival Times

	$\hat{P}_0(t)$		$\hat{P}_0^K(t)$		True Value
t	Average	SE	Average	SE	$P_0(t)$
0.25	0.8112	0.1610	0.8078	0.1639	0.7788
0.50	0.6552	0.2109	0.6461	0.2149	0.6065
1.00	0.4328	0.2367	0.4132	0.2394	0.3679
2.00	0.1765	0.2278	0.1588	0.2115	0.1353
3.00	0.0682	0.1812	0.0620	0.1682	0.0498
4.00	0.0188	0.1163	0.0182	0.1136	0.0183

TABLE 3

Averages and Standard Errors (SE) of Nonparametric Estimators of $P_0(t)$ based on 1000 Simulations of Renewal Processes on [0,20] with $Exp(1)$ Interarrival Times

	$\hat{P}_0(t)$		$\hat{P}_0^K(t)$		True Value
t	Average	SE	Average	SE	$P_0(t)$
0.25	0.7884	0.0916	0.7875	0.0919	0.7788
0.50	0.6197	0.1102	0.6164	0.1106	0.6065
1.00	0.3845	0.1119	0.3778	0.1118	0.3679
2.00	0.1460	0.0814	0.1384	0.0791	0.1353
3.00	0.0577	0.0566	0.0527	0.0521	0.0498
4.00	0.0212	0.0366	0.0188	0.0318	0.0183

TABLE 4

Averages and Standard Errors (SE) of Nonparametric Estimators of $P_0(t)$ based on 1000 Simulations of Renewal Processes on [0,100] with $Exp(1)$ Interarrival Times

	$\hat{P}_0(t)$		$\hat{P}_0^K(t)$		True Value
t	Average	SE	Average	SE	$P_0(t)$
0.25	0.7813	0.0412	0.7811	0.0412	0.7788
0.50	0.6103	0.0510	0.6096	0.0511	0.6065
1.00	0.3734	0.0498	0.3720	0.0498	0.3679
2.00	0.1411	0.0350	0.1396	0.0347	0.1353
3.00	0.0529	0.0231	0.0519	0.0227	0.0498
4.00	0.0195	0.0141	0.0189	0.0138	0.0183

TABLE 5

Averages and Standard Errors (SE) of Nonparametric Estimators of $P_0(t)$ based on 1000 Simulations of Renewal Processes on [0,5] with $Exp(1.25)$ Interarrival Times

	$\hat{P}_0(t)$		$\hat{P}_0^K(t)$		True Value
t	Average	SE	Average	SE	$P_0(t)$
0.25	0.7596	0.1687	0.7557	0.1712	0.7316
0.50	0.5858	0.2005	0.5749	0.2029	0.5353
1.00	0.3438	0.2046	0.3211	0.2011	0.2865
2.00	0.1170	0.1732	0.1002	0.1483	0.0821
3.00	0.0300	0.1124	0.0264	0.0992	0.0235
4.00	0.0057	0.0585	0.0052	0.0534	0.0067

TABLE 6
Averages and Standard Errors (SE) of Nonparametric Estimators of $P_0(t)$ based on 1000 Simulations of Renewal Processes on [0,20] with $Exp(1.25)$ Interarrival Times

	$\hat{P}_0(t)$		$\hat{P}_0^K(t)$		True Value
t	Average	SE	Average	SE	$P_0(t)$
0.25	0.7440	0.0869	0.7428	0.0874	0.7316
0.50	0.5544	0.1030	0.5513	0.1030	0.5353
1.00	0.3058	0.0988	0.3000	0.0977	0.2865
2.00	0.0922	0.0667	0.0869	0.0637	0.0821
3.00	0.0284	0.0392	0.0255	0.0357	0.0235
4.00	0.0072	0.0198	0.0067	0.0179	0.0067

TABLE 7
Averages and Standard Errors (SE) of Nonparametric Estimators of $P_0(t)$ based on 1000 Simulations of Renewal Processes on [0,100] with $Exp(1.25)$ Interarrival Times

	$\hat{P}_0(t)$		$\hat{P}_0^K(t)$		True Value
t	Average	SE	Average	SE	$P_0(t)$
0.25	0.7345	0.0413	0.7342	0.0413	0.7316
0.50	0.5401	0.0442	0.5394	0.0442	0.5353
1.00	0.2900	0.0423	0.2888	0.0422	0.2865
2.00	0.0835	0.0248	0.0825	0.0246	0.0821
3.00	0.0248	0.0141	0.0243	0.0138	0.0235
4.00	0.0074	0.0078	0.0071	0.0076	0.0067

A.2. Behaviour of $\hat{P}_0$ for asynchronous data on $\Re_+^2$

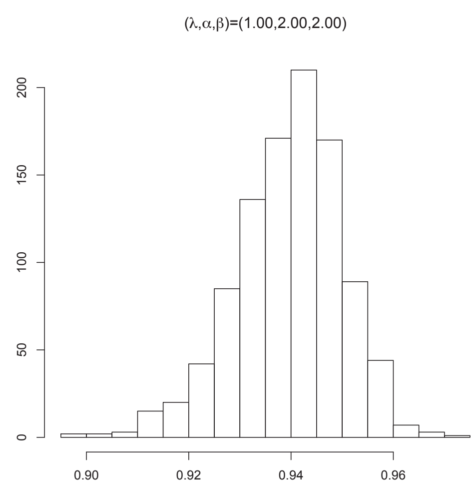

FIG 2. *Histogram of Estimates when $A_T = [0,30]^2$ and (λ,α,β)=(1.00,2.00,2.00), t=(0.5,0.5) (mean=0.9396, median=0.9405, sd=0.0105, true value=0.9394).*

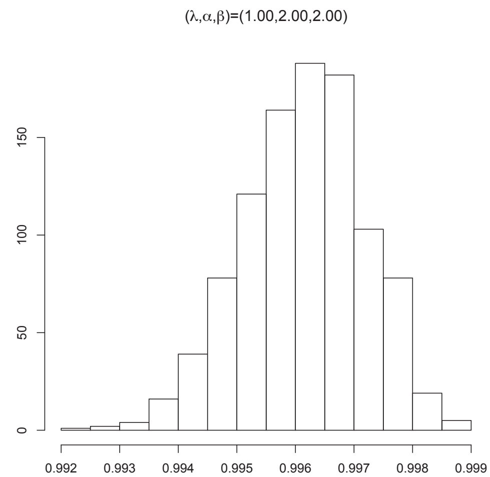

FIG 3. *Histogram of Estimates when* A_T=$[0, 30]^2$ *and* (λ, α, β)*=(1.00,2.00,2.00), t=(0.25,0.25) (mean=0.99615, median=0.99623, sd=0.00103, true value=0.99610).*

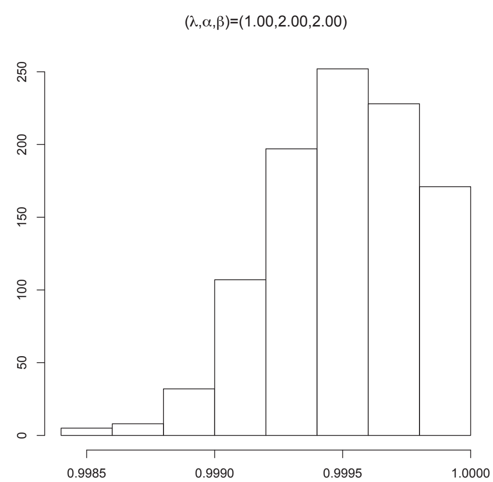

FIG 4. *Histogram of Estimates when* A_T=$[0, 30]^2$ *and* (λ, α, β)*=(1.00,2.00,2.00), t=(0.15,0.15) (mean=0.99950, median=0.99952, sd=0.00028, true value=0.99949).*

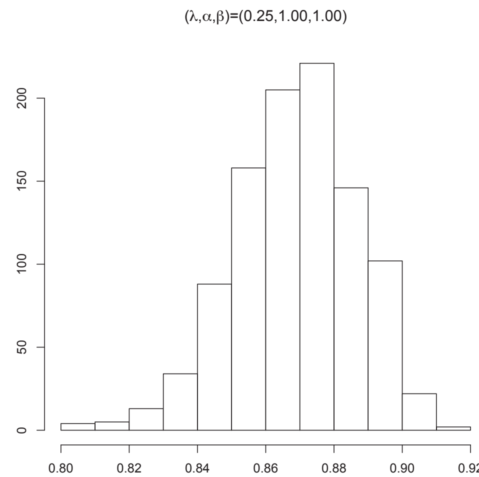

FIG 5. *Histogram of Estimates when* A_T*=*$[0,30]^2$ *and* (λ,α,β)*=(0.25,1.00,1.00), t=(0.75,0.75) (mean=0.86879, median=0.86963, sd=0.0178391, true value=0.868815).*

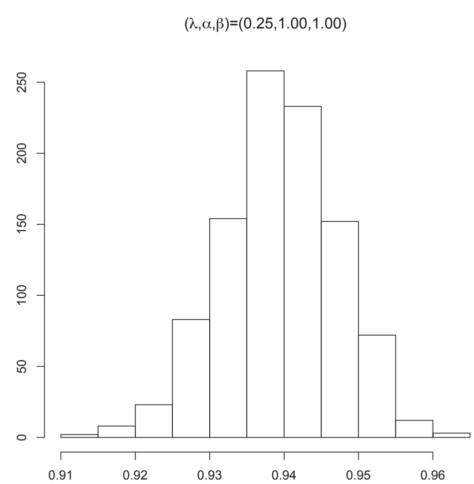

FIG 6. *Histogram of Estimates when* A_T*=*$[0,30]^2$ *and* (λ,α,β)*=(0.25,1.00,1.00), t=(0.5,0.5) (mean=0.939415, median=0.939534, sd=0.0078288, true value=0.939413).*

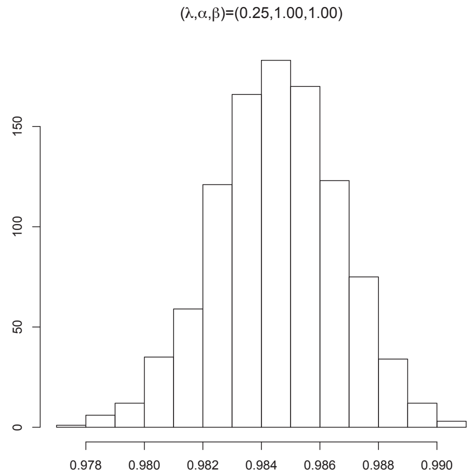

Fig 7. *Histogram of Estimates when* A_T*=*$[0,30]^2$ *and* (λ,α,β)*=(0.25,1.00,1.00), t=(0.25,0.25) (mean=0.98454, median=0.984573, sd=0.002117, true value=0.984496).*

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