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STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS DRIVEN BY LÉVY PROCESSES AND QUASI-LINEAR PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS[1]

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In this article we study a class of stochastic functional differential equations driven by Lévy processes (in particular, α-stable processes), and obtain the existence and uniqueness of Markov solutions in small time intervals. This corresponds to the local solvability to a class of quasi-linear partial integro-differential equations. Moreover, in the constant diffusion coefficient case, without any assumptions on the Lévy generator, we also show the existence of a unique maximal weak solution for a class of semi-linear partial integro-differential equation systems under bounded Lipschitz assumptions on the coefficients. Meanwhile, in the nondegenerate case (corresponding to $\Delta^{\alpha/2}$ with $\alpha \in (1, 2]$), based upon some gradient estimates, the existence of global solutions is established too. In particular, this provides a probabilistic treatment for the nonlinear partial integro-differential equations, such as the multi-dimensional fractal Burgers equations and the fractal scalar conservation law equations.

1. Introduction. Consider the following multi-dimensional fractal Burgers equation in $\mathbb{R}^d$:

$$\partial_t u = \nu \Delta^{\alpha/2} u - (u \cdot \nabla u), \qquad t \geq 0,\ u_0 = \varphi, \tag{1}$$

where $u = (u^1, \dots, u^d)$ and $\nu > 0$ is a viscosity constant, and $\Delta^{\alpha/2}$ with $\alpha \in (0, 2)$ is the usual fractional Laplacian defined by

$$\Delta^{\alpha/2} u(x) := \lim_{\varepsilon \downarrow 0} \int_{|z| \geq \varepsilon} \frac{u(x+z) - u(x)}{|z|^{d+\alpha}} \, \mathrm{d}z.$$

This is a typical nonlinear partial integro-differential equation and is regarded as a simplified model for the classical Navier–Stokes equation when $\alpha = 2$. Recently, there has been great interest in studying the multi-dimensional Burgers turbulence (cf. [2, 17]), the fractal Burgers equation (cf. [3, 6, 11]) and the fractal conservation law equation (cf. [7]), etc. All these works are based on the analytic approaches, especially the energy method, Duhamel's formulation and the maximum principle.

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The purpose of the present paper is to give a probabilistic treatment for a large class of quasi-linear partial integro-differential equations. Let us first introduce the main idea. By reversing the time variable, one can write Burgers' equation (1) as the following equivalent backward form:

$$\partial_t u + \nu \Delta^{\alpha/2} u - (u \cdot \nabla u) = 0, \qquad t \le 0,\ u_0 = \varphi. \tag{2}$$

Now, consider the case of $\alpha = 2$, and for a given smooth solution $u_t(x) \in C_b^\infty(\mathbb{R}^d; \mathbb{R}^d)$ to the above equation, let $X_{t,s}(x)$ solve the following stochastic differential equation (abbreviated as SDE):

$$\mathrm{d}X_{t,s}(x) = -u_s(X_{t,s}(x))\,\mathrm{d}s + \sqrt{2\nu}\,\mathrm{d}W_s, \qquad s \in [t,0],\ X_{t,t}(x) = x, \tag{3}$$

where $(W_s)_{s\le 0}$ is a d-dimensional standard Brownian motion on $\mathbb{R}_- := (-\infty, 0]$. By Itô's formula and the Markov property of the solution, it is well known that

$$u_t(x) = \mathbb{E}\varphi(X_{t,0}(x)). \tag{4}$$

Conversely, assume that (u, X) solves the implicit system (3) and (4); then u also solves the backward Burgers' equation (2). This type of implicit stochastic differential equation has been systematically studied by Freidlin [8], Chapter 5; see also [4, 16].

Let us now substitute (4) for (3); then

$$\begin{aligned} \mathrm{d}X_{t,s}(x) &= -[\mathbb{E}\varphi(X_{s,0}(y))]_{y=X_{t,s}(x)}\,\mathrm{d}s + \sqrt{2\nu}\,\mathrm{d}W_s,\\ &\qquad\qquad s \in [t,0],\ X_{t,t}(x) = x. \end{aligned} \tag{5}$$

As the Markov property holds, one can write the above equation as a closed form,

$$\mathrm{d}X_{t,s}(x) = -\mathbb{E}^{\mathscr{F}_{t,s}}\varphi(X_{t,0}(x))\,\mathrm{d}s + \sqrt{2\nu}\,\mathrm{d}W_s, \qquad s \in [t,0],\ X_{t,t}(x) = x, \tag{6}$$

where $\mathscr{F}_{t,s} := \sigma\{W_r - W_t : r \in [t,s]\}$, and $\mathbb{E}^{\mathscr{F}_{t,s}}$ denotes the conditional expectation with respect to $\mathscr{F}_{t,s}$. The question is this: Suppose that the stochastic equation (6) admits a unique solution family $\{X_{t,s}(x) : t \le s \le 0, x \in \mathbb{R}^d\}$. Does $u_t(x)$ defined by (4) solve Burgers' equation (2)? To answer this question, the key point is to establish the following Markov property: for all $t_1 \le t_2 \le t_3 \le 0$ and $x \in \mathbb{R}^d$,

$$\mathbb{E}^{\mathscr{F}_{t_1,t_2}}(\varphi(X_{t_1,t_3}(x))) = \mathbb{E}(\varphi(X_{t_2,t_3}(y)))|_{y=X_{t_1,t_2}(x)} \qquad \text{a.s.} \tag{7}$$

so that equation (6) can be written back to (5). This is not obvious since SDE (6) involves a conditional expectation operator. On the other hand, one can replace the Brownian motion in equation (6) by an α-stable process, as is done in [18], so that we can give a probabilistic explanation for the Burgers equation (2).

Basing on this simple observation, in this paper we are mainly concerned about the following general stochastic functional differential equation (abbreviated as SFDE) driven by a Lévy process $(L_t)_{t\le 0}$:

$$\begin{aligned} \mathrm{d}X_{t,s}(x) &= G_s(X_{t,s-}(x), \mathbb{E}^{\mathscr{F}_{s-}}(\phi_s(X_{t,\cdot}(x))))\,\mathrm{d}L_s,\\ &\qquad\qquad s \in [t,0],\ X_{t,t}(x) = x, \end{aligned} \tag{8}$$

where $\mathscr{F}_s := \sigma\{L_{s'} - L_{s''} : s'' < s' \le s\}$, G and ϕ are some Lipschitz functionals (see below). In Section 2, we are devoted to proving the existence and uniqueness of a short time solution as well as the Markov property (7) for equation (8) under Lipschitz assumptions on G and ϕ. Moreover, a locally maximal solution is also achieved. Since Lévy processes usually have poor integrability, we have to carefully treat the big jump part of Lévy processes. Compared with the classical argument in Freidlin [9], it seems that SFDE (8) is easier to handle since it is a closed equation.

Next, in Section 3 we apply our results to a class of quasi-linear partial integro-differential equations (abbreviated as PIDE) and obtain the existence of short time solutions. Here, we discuss two cases: G and ϕ satisfy linear growth conditions, but Lévy processes have finite moments of arbitrary orders; G and ϕ are bounded, but equation (8) has a constant coefficient in the big jump part. This is natural since only the big jump is related to the moment of Lévy processes.

In Section 4, we turn to the investigation of the following system of semi-linear PIDEs (nonlinear transport equation):

$$
\begin{cases} \partial_t u_t + \mathcal{L}_0 u_t + (G_t(x, u_t) \cdot \nabla) u_t + F_t(x, u_t) = 0, \\ (t, x) \in \mathbb{R}_- \times \mathbb{R}^d, \qquad u_0(x) = \varphi(x) \in \mathbb{R}^m, \end{cases} \tag{9}
$$

where $\mathcal{L}_0$ is the generator of the Lévy process given by (15) below. It is observed that the following scalar conservation law equation can be written as the above form:

$$
\begin{cases} \partial_t u_t + \mathcal{L}_0 u_t + \operatorname{div}(g_t(x, u_t)) + f_t(x, u_t) = 0, \\ (t, x) \in \mathbb{R}_- \times \mathbb{R}^d, \qquad u_0(x) = \varphi(x) \in \mathbb{R}. \end{cases} \tag{10}
$$

In particular, the one-dimensional fractal Burgers equation (2) takes the above form. In equation (9), since there are not any analytic properties to be imposed on $\mathcal{L}_0$, one can not appeal to the Duhamel formula or the energy method to give an analytic treatment. In this situation, the probabilistic approach seems to be quite suitable. In fact, by using purely probabilistic argument, we shall prove in Theorem 4.2 below that PIDE (9) admits a unique maximal weak solution in the class of bounded Lipschitz functions. In the nondegenerate case (corresponding to the subcritical case for $\mathcal{L}_0 = \Delta^{\alpha/2}$ with $\alpha \in (1, 2]$), the existence of global solutions is also obtained by applying some gradient estimates. We mention that for the one-dimensional Burgers equation (1), it has been proved in [11] that the global analytic solution does exist for $\alpha \in [1, 2]$, and the finite time blow up solution also exists for $\alpha \in (0, 1)$. However, in the critical case of $\alpha = 1$, the existence of global solutions for the general equation (9) is left open.

We conclude this introduction by introducing the following conventions: The letter C with or without subscripts will denote a positive constant, whose value may change in different places. If we write $T = T(K_1, K_2, \ldots)$, this means that T depends only on these indicated arguments.

2. A stochastic functional differential equation: Short time existence.

2.1. *General facts about Lévy processes.* Let $(L_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^m$-valued Lévy process on the real line and defined on some complete probability space $(\Omega, \mathscr{F}, P)$, which means that:

- $(L_t)_{t\in\mathbb{R}}$ has independent and stationary increments, that is, for all $-\infty < t_1 < t_2 < \cdots < t_n < +\infty$, the random variables $(L_{t_2} - L_{t_1}, \dots, L_{t_n} - L_{t_{n-1}})$ are independent, and the distribution of $L_{t+s} - L_s$ does not depend on s.
- For P-almost all $\omega \in \Omega$, the mapping $t \mapsto L_t(\omega)$ is right-continuous and has left-limit (also called càdlàg in French).

Let $\mathscr{N}$ be the total of all P-null sets. For $-\infty \le t < s < +\infty$, define

$$\mathscr{F}_{t,s} := \sigma\{L_r - L_{r'}; r, r' \in (t, s]\} \vee \mathscr{N}.$$

By the independence of increments of the Lévy process, it is easy to see that for $-\infty \le t_1 < t_2 < t_3 < +\infty$, $\mathscr{F}_{t_1,t_2}$ and $\mathscr{F}_{t_2,t_3}$ are independent. For simplicity of notation, we write

$$\mathscr{F}_s = \mathscr{F}_{-\infty,s}, \qquad \mathscr{F}_{s-} := \bigvee_{t<s} \mathscr{F}_t.$$

It is clear that $\mathscr{F}_t \subset \mathscr{F}_s$ if $t < s$, and $s \mapsto \mathscr{F}_{s-}$ is left-continuous. Moreover, L_{s-} is $\mathscr{F}_{s-}$-measurable. Throughout this paper, we shall work on the negative time axes $\mathbb{R}_- := (-\infty, 0]$.

REMARK 2.1. For any measurable process $\eta_s \in L^1(\Omega, \mathscr{F}_0, P)$, $s \le 0$, by the predictable projection theorem (cf. [14], page 173, Theorem 5.3), there always exists a predictable version of $s \to \mathbb{E}(\eta_s|\mathscr{F}_{s-})$, which will be denoted by $\mathbb{E}^{\mathscr{F}_{s-}}(\eta_s)$. Moreover, for any $\xi \in L^1(\Omega, \mathscr{F}_0, P)$, by the regularization theorem of martingales (cf. [14], page 64, Proposition 2.7 and page 65, Theorem 2.9), we have

$$\lim_{s\uparrow t} \mathbb{E}^{\mathscr{F}_{s-}}(\xi) = \mathbb{E}^{\mathscr{F}_{t-}}(\xi) = \mathbb{E}^{\mathscr{F}_t}(\xi) \qquad \text{a.s.},$$

where the second equality follows by $P\{L_t = L_{t-}\} = 1$ and a monotone class argument.

By the Lévy–Khintchine formula (cf. [1], page 109, Corollary 2.4.20), the characteristic function of L_t is given by

$$\begin{aligned}\mathbb{E}(e^{i\xi\cdot L_t}) &= \exp\left\{t\left[ib\cdot\xi - \xi^{\mathrm{t}}A\xi + \int_{\mathbb{R}^m}[e^{i\xi\cdot z} - 1 - i\xi\cdot z1_{|z|\le1}]\nu(\mathrm{d}z)\right]\right\} \\ &=: e^{t\Psi(\xi)},\end{aligned} \tag{11}$$

where $\Psi(\xi)$ is a complex-valued function called the symbol of $(L_t)_{t\le0}$, and $b \in \mathbb{R}^m$, $A \in \mathbb{R}^m \times \mathbb{R}^m$ is a positive definite and symmetric matrix, ν is a Lévy measure on $\mathbb{R}^m$, that is, $\nu\{0\} = 0$ and

$$\int_{\mathbb{R}^m} 1 \wedge |z|^2 \nu(\mathrm{d}z) < +\infty. \tag{12}$$

We call

$$\mathscr{A} := (b, A, \nu) \tag{13}$$

the characteristic triple of L_t. If $b = 0$, $A = 0$ and $\nu(\mathrm{d}z) = \frac{\mathrm{d}z}{|z|^{m+\alpha}}$, where $\alpha \in (0, 2)$, then L_t is the α-stable process with the Lévy exponent $c_{m,\alpha}|\xi|^\alpha$, and its generator is the fractional Laplacian $\Delta^{\alpha/2}$ by multiplying a constant $c'_{m,\alpha}$.

By the Lévy–Itô decomposition (cf. [1], page 108, Theorem 2.4.16), L_t can be written as

$$L_t = bt + W_t^A + \int_{|z|\le 1} z\tilde{N}(t, \mathrm{d}z) + \int_{|z|>1} zN(t, \mathrm{d}z), \tag{14}$$

where W_t^A is a Brownian motion with covariance matrix $A = (a_{ij})$, $N(t, \mathrm{d}z)$ is the Poisson random point measure associated with $(L_t)_{t\le 0}$ given by

$$N(t, \Gamma) := \sum_{t<s\le 0} 1_\Gamma(L_s - L_{s-}), \qquad \Gamma \in \mathcal{B}(\mathbb{R}^m)$$

and $\tilde{N}(t, \mathrm{d}z) := N(t, \mathrm{d}z) - t\nu(\mathrm{d}z)$ is the compensated random martingale measure. Here, $(W_t^A)_{t\le 0}$ and $(N(t, \mathrm{d}z))_{t\le 0}$ are independent. The generator of L_t is given by

$$\begin{aligned} \mathcal{L}_0 u(x) = {} & \frac{1}{2} a_{ij}\partial_i\partial_j u + b_i\partial_i u \\ & + \int_{\mathbb{R}^m} \big[u(x+z) - u(x) - 1_{|z|<1}\partial_i u(x) z_i\big]\nu(\mathrm{d}z). \end{aligned} \tag{15}$$

Here and after, we use the usual convention for summation: the same index in a product will be summed automatically.

In the following, we denote by $\mathbb{D}$ the space of all càdlàg functions from $\mathbb{R}_-$ to $\mathbb{R}^d$, which is endowed with the locally uniform metric ρ. Notice that this metric is complete, but not separable. For given $t < 0$ and càdlàg function $f : [t, 0] \to \mathbb{R}^d$, we extend f to $\mathbb{R}_-$ in a natural manner by putting $f(s) = f(t)$ for $s < t$ so that $f \in \mathbb{D}$.

2.2. *A general case*. In this subsection, we consider the following general SFDE in $\mathbb{R}^d$ driven by the Lévy process $(L_s)_{s\le 0}$:

$$X_{t,s} = \xi + \int_{(t,s]} G_r(X_{t,r-}, \mathbb{E}^{\mathscr{F}_{r-}}(\phi_r(X_{t,\cdot})))\,\mathrm{d}L_r, \qquad t \le s \le 0, \tag{16}$$

where $\xi \in \mathscr{F}_t$, $G : \mathbb{R}_- \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^d \times \mathbb{R}^m$ is a measurable function, and $\phi : \mathbb{R}_- \times \mathbb{D} \to \mathbb{R}^k$ is a uniformly Lipschitz continuous functional in the sense that

$$\|\phi\|_{\mathrm{Lip}} := \sup_{s\in\mathbb{R}_-} \sup_{\omega\ne\omega'\in\mathbb{D}} \frac{|\phi_s(\omega) - \phi_s(\omega')|}{\rho(\omega, \omega')} < +\infty, \tag{17}$$

where $\rho(\omega, \omega') := \sum_n 2^{-n}(1 \wedge \sup_{s\in[-n,0]} |\omega(s) - \omega'(s)|)$ is the locally uniform metric on $\mathbb{D}$.

The definition about the solutions to equation (16) is given as follows:

DEFINITION 2.2. For fixed $t < 0$ and $\xi \in \mathscr{F}_t$, an $(\mathscr{F}_s)$-adapted càdlàg stochastic process $X_s =: X_{t,s}(\xi)$ is called a solution of equation (16) if for all $s \in [t, 0]$,

$$X_s = \xi + \int_{(t,s]} G_r(X_{r-}, \mathbb{E}^{\mathscr{F}_{r-}}(\phi_r(X_\cdot)))\, \mathrm{d}L_r \qquad \text{a.s.}$$

For $T < 0$, we say that equation (16) is (uniquely) solvable on $(T, 0]$ (or $[T, 0]$) if for all $t \in (T, 0]$ (or $t \in [T, 0]$) and $\xi \in \mathscr{F}_t$, equation (16) has a (unique) solution starting from ξ at time t.

REMARK 2.3. In this definition, it has been assumed that $\phi_r(X_\cdot) \in L^1(\Omega, \mathscr{F}_0, P)$ so that $\mathbb{E}^{\mathscr{F}_{r-}}(\phi_r(X_\cdot))$ makes sense by Remark 2.1, and further the stochastic integral with respect to the Lévy process in the definition makes sense.

Below, we make the following assumptions on the coefficients and the Lévy measure ν:

$(\mathbf{H}_G)$ For some $K_0, K_1 > 0$ and all $s \leq 0$, $x, x' \in \mathbb{R}^d$, $u, u' \in \mathbb{R}^k$,

$$|G_s(0,0)| \leq K_0, \qquad |G_s(x,u) - G_s(x',u')| \leq K_1(|x - x'| + |u - u'|).$$

$(\mathbf{H}_\nu^\beta)$ For some $\beta > 0$,

$$\int_{|z|\geq 1} |z|^\beta \nu(\mathrm{d}z) < +\infty.$$

REMARK 2.4. Condition $(\mathbf{H}_\nu^\beta)$, which is a restriction on the big jump of the Lévy process, is equivalent to saying that the β-order moment of the Lévy process is finite; cf. [15], Theorem 25.3. It should be noticed that for α-stable process, condition $(\mathbf{H}_\nu^\beta)$ is satisfied only for any $\beta < \alpha$.

Now we prove the following result about the existence and uniqueness of solutions for equation (16) in a short time.

THEOREM 2.5. *Assume that* $(\mathbf{H}_G)$ *and* $(\mathbf{H}_\nu^\beta)$ *hold for some* $\beta > 1$, *and* ϕ *is a Lipschitz continuous functional on* $\mathbb{D}$; *see* (17). *Then there exists a time* $T = T(K_1, \mathscr{A}, \beta, \|\phi\|_{\mathrm{Lip}}) < 0$ *such that equation* (16) *is uniquely solvable on* $[T, 0]$ *for any* L^β*-integrable initial value* $\xi \in \mathscr{F}_t$ *in the sense of Definition* 2.2, *and for some* $C = C(T, K_0)$ *and any* $t \in [T, 0]$,

$$\mathbb{E}\Big(\sup_{s\in[t,0]} |X_{t,s}(\xi)|^\beta\Big) \leq C\mathbb{E}|\xi|^\beta. \tag{18}$$

Moreover, if $\xi = x \in \mathbb{R}^d$ *is nonrandom, then for any* $t \in [T, 0)$, *the unique solution* $X_{t,s}$ *is* $\mathscr{F}_{t,s}$*-measurable for all* $s \in [t, 0]$.

PROOF. We prove the theorem for $\beta \in (1, 2)$. For $\beta \geq 2$, the proof is similar and simpler. Fix $t < 0$, which will be determined below. For $\xi \in L^\beta(\Omega, \mathscr{F}_t, P)$, set $X_{t,s}^{(0)} \equiv \xi$, and let $X_{t,s}^{(n)}$ be the Picard iteration sequence defined as follows: for $n \in \mathbb{N}$,

$$X_{t,s}^{(n)} = \xi + \int_{(t,s]} G_r\big(X_{t,r-}^{(n-1)}, \mathbb{E}^{\mathscr{F}_{r-}}\big(\phi_r\big(X_{t,\cdot}^{(n-1)}\big)\big)\big)\,\mathrm{d}L_r. \tag{19}$$

Set

$$Z_{t,s}^{(n)} := X_{t,s}^{(n+1)} - X_{t,s}^{(n)}.$$

Using the Lévy–Itô decomposition (14), one can write

$$\begin{aligned} Z_{t,s}^{(n)} &= \int_{(t,s]} \int_{|z|<1} \mathcal{G}_r^{(n)} \cdot z\tilde{N}(\mathrm{d}r, \mathrm{d}z) \\ &\quad + \int_{(t,s]} \int_{|z|\geq 1} \mathcal{G}_r^{(n)} \cdot zN(\mathrm{d}r, \mathrm{d}z) \\ &\quad + \int_{(t,s]} \mathcal{G}_r^{(n)} \cdot b\,\mathrm{d}r + \int_{(t,s]} \mathcal{G}_r^{(n)}\,\mathrm{d}W_r^A \\ &=: I_1^{(n)}(s) + I_2^{(n)}(s) + I_3^{(n)}(s) + I_4^{(n)}(s), \end{aligned}$$

where

$$\mathcal{G}_r^{(n)} := G_r\big(X_{t,r-}^{(n)}, \mathbb{E}^{\mathscr{F}_{r-}}\big(\phi_r\big(X_{t,\cdot}^{(n)}\big)\big)\big) - G_r\big(X_{t,r-}^{(n-1)}, \mathbb{E}^{\mathscr{F}_{r-}}\big(\phi_r\big(X_{t,\cdot}^{(n-1)}\big)\big)\big).$$

By Burkholder's inequality (cf. [10], Theorem 23.12) and Young's inequality, thanks to $\beta \in (1, 2)$, we have that for any $\varepsilon \in (0, 1)$,

$$\begin{aligned} &\mathbb{E}\Big(\sup_{r\in[t,0]} \big|I_1^{(n)}(r)\big|^\beta\Big) \\ &\quad \leq C\mathbb{E}\bigg(\int_{(t,0]} \int_{|z|<1} \big|\mathcal{G}_r^{(n)} \cdot z\big|^2 N(\mathrm{d}r, \mathrm{d}z)\bigg)^{\beta/2} \\ &\quad \leq C\mathbb{E}\bigg(\sup_{r\in[t,0]} \big|\mathcal{G}_r^{(n)}\big|^{2-\beta} \int_{(t,0]} \int_{|z|<1} \big|\mathcal{G}_r^{(n)}\big|^\beta \cdot |z|^2 N(\mathrm{d}r, \mathrm{d}z)\bigg)^{\beta/2} \\ &\quad \leq \varepsilon\mathbb{E}\Big(\sup_{r\in[t,0]} \big|\mathcal{G}_r^{(n)}\big|^\beta\Big) + C_\varepsilon\mathbb{E}\bigg(\int_{(t,0]} \int_{|z|<1} \big|\mathcal{G}_r^{(n)}\big|^\beta \cdot |z|^2 N(\mathrm{d}r, \mathrm{d}z)\bigg) \\ &\quad = \varepsilon\mathbb{E}\Big(\sup_{r\in[t,0]} \big|\mathcal{G}_r^{(n)}\big|^\beta\Big) + C_\varepsilon\mathbb{E}\bigg(\int_{[t,0]} \int_{|z|<1} \big|\mathcal{G}_r^{(n)}\big|^\beta \cdot |z|^2 \nu(\mathrm{d}z)\,\mathrm{d}r\bigg) \\ &\quad \leq \bigg(\varepsilon + C_\varepsilon|t| \int_{|z|<1} |z|^2 \nu(\mathrm{d}z)\bigg)\mathbb{E}\Big(\sup_{r\in[t,0]} \big|\mathcal{G}_r^{(n)}\big|^\beta\Big). \end{aligned}$$

Here and below, the constant C or C_ε is independent of t and n. For $I_2^{(n)}(s)$, by Itô's formula, we have

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{r\in[t,s]}\big|I_2^{(n)}(r)\big|^\beta\Big)\\
&\quad\le \mathbb{E}\bigg(\int_{(t,s]}\int_{|z|\ge 1}\big|\big|I_2^{(n)}(r-)+\mathcal{G}_r^{(n)}\cdot z\big|^\beta-\big|I_2^{(n)}(r-)\big|^\beta\big|N(\mathrm{d}r,\mathrm{d}z)\bigg)\\
&\quad=\mathbb{E}\bigg(\int_{(t,s]}\int_{|z|\ge 1}\big|\big|I_2^{(n)}(r-)+\mathcal{G}_r^{(n)}\cdot z\big|^\beta-\big|I_2^{(n)}(r-)\big|^\beta\big|\nu(\mathrm{d}z)\,\mathrm{d}r\bigg)\\
&\quad\le C\mathbb{E}\bigg(\int_{(t,s]}\big|I_2^{(n)}(r)\big|^\beta\,\mathrm{d}r\bigg)+C\bigg(\int_{|z|\ge 1}|z|^\beta\nu(\mathrm{d}z)\bigg)\mathbb{E}\bigg(\int_{(t,0]}\big|\mathcal{G}_r^{(n)}\big|^\beta\,\mathrm{d}r\bigg),
\end{aligned}
$$

which then implies that by $(\mathbf{H}_\nu^\beta)$ and Gronwall's inequality,

$$
\mathbb{E}\Big(\sup_{r\in[t,0]}\big|I_2^{(n)}(r)\big|^\beta\Big)\le C|t|\mathbb{E}\Big(\sup_{r\in[t,0]}\big|\mathcal{G}_r^{(n)}\big|^\beta\Big).
$$

Similarly, we have

$$
\mathbb{E}\Big(\sup_{r\in[t,0]}\big|I_3^{(n)}(r)\big|^\beta\Big)\le(|t|\cdot|b|)^\beta\mathbb{E}\Big(\sup_{r\in[t,0]}\big|\mathcal{G}_r^{(n)}\big|^\beta\Big),
$$

and for any $\varepsilon\in(0,1)$,

$$
\mathbb{E}\Big(\sup_{r\in[t,0]}\big|I_4^{(n)}(r)\big|^\beta\Big)\le(\varepsilon+C_\varepsilon|t|)\mathbb{E}\Big(\sup_{r\in[t,0]}\big|\mathcal{G}_r^{(n)}\big|^\beta\Big).
$$

Combining the above calculations, we obtain that for any $\varepsilon\in(0,1)$,

$$
\mathbb{E}\Big(\sup_{r\in[t,0]}\big|Z_{t,r}^{(n)}\big|^\beta\Big)\le(\varepsilon+C_\varepsilon|t|)\cdot\mathbb{E}\Big(\sup_{r\in[t,0]}\big|\mathcal{G}_r^{(n)}\big|^\beta\Big).\tag{20}
$$

Noticing that by $(\mathbf{H}_G)$,

$$
\big|\mathcal{G}_r^{(n)}\big|\le K_1\Big(\big|Z_{t,r-}^{(n-1)}\big|+\|\phi\|_{\mathrm{Lip}}\mathbb{E}^{\mathscr{F}_{r-}}\Big(\sup_{s\in[t,0]}\big|Z_{t,s}^{(n-1)}\big|\Big)\Big),
$$

and in view of $\beta>1$, we further have by Doob's maximal inequality,

$$
\mathbb{E}\Big(\sup_{s\in[t,0]}\big|Z_{t,s}^{(n)}\big|^\beta\Big)\le(\varepsilon+C_\varepsilon|t|)C_0\mathbb{E}\Big(\sup_{s\in[t,0]}\big|Z_{t,s}^{(n-1)}\big|^\beta\Big).
$$

Now, let us choose

$$
\varepsilon=\frac{1}{4C_0}\quad\text{and}\quad T:=-\frac{1}{4C_\varepsilon C_0},
$$

and then for all $t\in[T,0]$,

$$
\mathbb{E}\Big(\sup_{s\in[t,0]}\big|Z_{t,s}^{(n)}\big|^\beta\Big)\le\frac{1}{2}\mathbb{E}\Big(\sup_{s\in[t,0]}\big|Z_{t,s}^{(n-1)}\big|^\beta\Big)\le\cdots\le\frac{1}{2^n}\mathbb{E}\Big(\sup_{s\in[t,0]}\big|Z_{t,s}^{(0)}\big|^\beta\Big).\tag{21}
$$

On the other hand, notice that

$$Z^{(0)}_{t,s} = X^{(1)}_{t,s} - \xi = \int_{(t,s]} G_r\big(X^{(1)}_{t,r-}, \mathbb{E}^{\mathscr{F}_{r-}}(\phi_r(\xi))\big)\,\mathrm{d}L_r.$$

As above, and using Gronwall's inequality, it is easy to derive that

$$\mathbb{E}\Big(\sup_{s\in[t,0]} \big|Z^{(0)}_{t,s}\big|^\beta\Big) \le C\mathbb{E}|\xi|^\beta. \tag{22}$$

Hence, there exists an $(\mathscr{F}_s)$-adapted and càdlàg stochastic process $X_{t,s}$ such that

$$\lim_{n\to\infty} \mathbb{E}\Big(\sup_{s\in[t,0]} \big|X^{(n)}_{t,s} - X_{t,s}\big|^\beta\Big) = 0. \tag{23}$$

By taking limits for equation (19), it is easy to see that $X_{t,s}$ solves SFDE (16). Moreover, estimate (18) follows from (21), (22) and (23). The uniqueness is clear from the above proof.

Suppose now that $\xi = x$ is nonrandom. From the Picard iteration (19), one sees that for each $n \in \mathbb{N}$ and $s \in (t, 0]$, $X^{(n)}_{t,s}$ is $\mathscr{F}_{t,s}$-measurable. Indeed, suppose that $X^{(n-1)}_{t,s}$ is $\mathscr{F}_{t,s}$-measurable for each $s \in (t, 0]$, and then it is clear that $\phi_r(X^{(n-1)}_{t,\cdot})$ is independent of $\mathscr{F}_t$. Noticing that for $r > t$, $\mathscr{F}_{r-} = \mathscr{F}_{t,r-} \vee \mathscr{F}_t$ and $\mathscr{F}_{t,r-}$ is independent of $\mathscr{F}_t$, we have

$$\mathbb{E}^{\mathscr{F}_{r-}}\big(\phi_r\big(X^{(n-1)}_{t,\cdot}\big)\big) = \mathbb{E}^{\mathscr{F}_{t,r-}}\big(\phi_r\big(X^{(n-1)}_{t,\cdot}\big)\big).$$

By induction method, starting from equation (19) with $\xi = x$, one finds that $X^{(n)}_{t,s}$ is also $\mathscr{F}_{t,s}$-measurable for each $s \in (t, 0]$. So, the limit $X_{t,s}$ is also $\mathscr{F}_{t,s}$-measurable. □

REMARK 2.6. In this theorem, if $G_s(x, u) = G_s(x)$ does not depend on u, then the short time solution can be extended to any large time by the usual time shift technique.

2.3. *A special case.* In Theorem 2.5, since we require $\beta > 1$, the result rules out the α-stable process with $\alpha \in (0, 1]$. In this subsection, we drop assumption $(\mathbf{H}^\beta_\nu)$ in Theorem 2.5, and consider the following special form:

$$\begin{aligned}
X_{t,s} = \xi &+ \int_{(t,s]}\int_{|z|<1} G_r(X_{t,r-}, \mathbb{E}^{\mathscr{F}_{r-}}(\phi_r(X_{t,\cdot}))) \cdot z\tilde{N}(\mathrm{d}r, \mathrm{d}z) \\
&+ \int_{(t,s]}\int_{|z|\ge 1} zN(\mathrm{d}r, \mathrm{d}z) \\
&+ \int_{(t,s]} G_r(X_{t,r-}, \mathbb{E}^{\mathscr{F}_{r-}}(\phi_r(X_{t,\cdot}))) \cdot b\,\mathrm{d}r \\
&+ \int_{(t,s]} G_r(X_{t,r-}, \mathbb{E}^{\mathscr{F}_{r-}}(\phi_r(X_{t,\cdot})))\,\mathrm{d}W^A_r,
\end{aligned} \tag{24}$$

where $\xi \in \mathscr{F}_t$. In this equation, the big jump part has a constant coefficient. In order to make sense for the integrals, we need to assume that G and ϕ are bounded. We have:

THEOREM 2.7. *In addition to* $(\mathbf{H}_G)$, *we assume that* G *is bounded, and* ϕ *is a bounded Lipschitz continuous functional on* $\mathbb{D}$. *Then there exists a time* $T = T(K_1, \mathscr{A}, \|\phi\|_{\mathrm{Lip}}) < 0$ *such that SFDE* (24) *is uniquely solvable on* $[T, 0]$. *Moreover, if* $\xi = x \in \mathbb{R}^d$ *is nonrandom, then for any* $t \in [T, 0)$, *the unique solution* $X_{t,s}$ *is* $\mathscr{F}_{t,s}$*-measurable for all* $s \in [t, 0]$.

PROOF. For $t < 0$ and $\xi \in \mathscr{F}_t$, set $X^{(0)}_{t,s} \equiv \xi$, and let $X^{(n)}_{t,s}$ be the Picard iteration sequence defined as follows:

$$
\begin{aligned}
X^{(n)}_{t,s} = \xi &+ \int_{(t,s]}\int_{|z|\ge 1} zN(\mathrm{d}r, \mathrm{d}z) \\
&+ \int_{(t,s]}\int_{|z|<1} G_r\big(X^{(n-1)}_{t,r-}, \mathbb{E}^{\mathscr{F}_{r-}}\big(\phi_r\big(X^{(n-1)}_{t,\cdot}\big)\big)\big)\cdot z\tilde{N}(\mathrm{d}r, \mathrm{d}z) \\
&+ \int_{(t,s]} G_r\big(X^{(n-1)}_{t,r-}, \mathbb{E}^{\mathscr{F}_{r-}}\big(\phi_r\big(X^{(n-1)}_{t,\cdot}\big)\big)\big)\cdot b\,\mathrm{d}r \\
&+ \int_{(t,s]} G_r\big(X^{(n-1)}_{t,r-}, \mathbb{E}^{\mathscr{F}_{r-}}\big(\phi_r\big(X^{(n-1)}_{t,\cdot}\big)\big)\big)\,\mathrm{d}W^A_r.
\end{aligned}
\tag{25}
$$

Set

$$
Z^{(n)}_{t,s} := X^{(n+1)}_{t,s} - X^{(n)}_{t,s}.
$$

Then

$$
Z^{(n)}_{t,s} = \int_{(t,s]}\int_{|z|<1} \mathcal{G}^{(n)}_r \cdot z\tilde{N}(\mathrm{d}r, \mathrm{d}z) + \int_{(t,s]} \mathcal{G}^{(n)}_r \cdot b\,\mathrm{d}r + \int_{(t,s]} \mathcal{G}^{(n)}_r\,\mathrm{d}W^A_r,
$$

where

$$
\mathcal{G}^{(n)}_r := G_r\big(X^{(n)}_{t,r-}, \mathbb{E}^{\mathscr{F}_{r-}}\big(\phi_r\big(X^{(n)}_{t,\cdot}\big)\big)\big) - G_r\big(X^{(n-1)}_{t,r-}, \mathbb{E}^{\mathscr{F}_{r-}}\big(\phi_r\big(X^{(n-1)}_{t,\cdot}\big)\big)\big).
$$

Notice that $|\mathcal{G}^{(n)}_r| \le 2\|G\|_\infty$, and by $(\mathbf{H}_G)$,

$$
|\mathcal{G}^{(n)}_r|^2 \le 2K_1^2\Big(|Z^{(n-1)}_{t,r-}|^2 + \|\phi\|^2_{\mathrm{Lip}}\Big(\mathbb{E}^{\mathscr{F}_{r-}}\Big(\sup_{s\in[t,0]}|Z^{(n-1)}_{t,s}|\Big)\Big)^2\Big) =: \Phi^{(n)}_r.
$$

By Burkholder's inequality and (12), we have

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{s\in[t,0]}\Big|\int_{(t,s]}\int_{|z|<1} \mathcal{G}^{(n)}_r \cdot z\tilde{N}(\mathrm{d}r, \mathrm{d}z)\Big|^2\Big) &\le C\mathbb{E}\Big(\int_{(t,0]}\int_{|z|<1} |\mathcal{G}^{(n)}_r \cdot z|^2 N(\mathrm{d}r, \mathrm{d}z)\Big) \\
&\le C|t|\mathbb{E}\Big(\sup_{r\in[t,0]}\Phi^{(n)}_r\Big).
\end{aligned}
$$

Here and below, the constant C is independent of t and n. Similarly, we have

$$\mathbb{E}\Big(\sup_{s\in[t,0]}\Big|\int_{(t,s]}\mathcal{G}_r^{(n)}\cdot b\,\mathrm{d}r\Big|^2\Big)\le C|t|^2\mathbb{E}\Big(\sup_{r\in[t,0]}\Phi_r^{(n)}\Big)$$

and

$$\mathbb{E}\Big(\sup_{s\in[t,0]}\Big|\int_{(t,s]}\mathcal{G}_r^{(n)}\,\mathrm{d}W_r^A\Big|^2\Big)\le C|t|\mathbb{E}\Big(\sup_{r\in[t,0]}\Phi_r^{(n)}\Big).$$

Combining the above calculations and by Doob's maximal inequality, we obtain

$$\mathbb{E}\Big(\sup_{s\in[t,0]}|Z_{t,s}^{(n)}|^2\Big)\le C_0|t|\mathbb{E}\Big(\sup_{r\in[t,0]}\Phi_r^{(n)}\Big)\le C_1|t|\mathbb{E}\Big(\sup_{s\in[t,0]}|Z_{t,s}^{(n-1)}|^2\Big).$$

Now, let us choose

$$T:=-\frac{1}{2C_1},$$

and then for all $t\in[T,0]$,

$$\mathbb{E}\Big(\sup_{r\in[t,0]}|Z_{t,r}^{(n)}|^2\Big)\le\frac{1}{2}\mathbb{E}\Big(\sup_{r\in[t,0]}|Z_{t,r}^{(n-1)}|^2\Big)\le\cdots\le\frac{1}{2^n}\mathbb{E}\Big(\sup_{r\in[t,0]}|Z_{t,r}^{(1)}|^2\Big)\le\frac{C}{2^n}.$$

Hence, there exists an $(\mathscr{F}_s)$-adapted and càdlàg stochastic process $X_{t,s}$ such that

$$\lim_{n\to\infty}\mathbb{E}\Big(\sup_{s\in[t,0]}|X_{t,s}^{(n)}-X_{t,s}|^2\Big)=0.$$

By taking limits for equation (25), it is easy to see that $X_{t,s}$ solves SFDE (24). The remaining proof is the same as in Theorem 2.5. □

2.4. *Markov property.* In this subsection, we prove the Markov property for the solutions of equations (16) and (24), which is crucial for the development of the next section.

We first show the continuous dependence of the solutions with respect to the initial values.

PROPOSITION 2.8. *In the situation of Theorem 2.5, for $t\in[T,0]$, let $\xi^{(n)},\xi\in L^\beta(\Omega,\mathscr{F}_t,P)$. If $\xi^{(n)}$ converges to ξ in probability as $n\to\infty$, then $X_{t,s}^{(n)}$ converges to $X_{t,s}$ uniformly with respect to $s\in[t,0]$ in probability as $n\to\infty$, where $\{X_{t,s}^{(n)};t\le s\le 0\}$ and $\{X_{t,s};t\le s\le 0\}$ are the solutions of SFDE (16) corresponding to the initial values $\xi^{(n)}$ and ξ.*

PROOF. Define

$$A_n:=\{|\xi^{(n)}-\xi|\le 1\}\in\mathscr{F}_t.$$

Then we can write

$$1_{A_n}(X^{(n)}_{t,s} - X_{t,s}) = 1_{A_n}(\xi^{(n)} - \xi) + \int_{(t,s]}\int_{|z|<1} 1_{A_n}\mathcal{G}^{(n)}_r \cdot z\tilde{N}(\mathrm{d}r, \mathrm{d}z)$$
$$+ \int_{(t,s]}\int_{|z|\geq 1} 1_{A_n}\mathcal{G}^{(n)}_r \cdot zN(\mathrm{d}r, \mathrm{d}z)$$
$$+ \int_{(t,s]} 1_{A_n}\mathcal{G}^{(n)}_r \cdot b\,\mathrm{d}r + \int_{(t,s]} 1_{A_n}\mathcal{G}^{(n)}_r\,\mathrm{d}W^A_r,$$

where

$$\mathcal{G}^{(n)}_r := G_r(X^{(n)}_{t,r-}, \mathbb{E}^{\mathscr{F}_{r-}}(\phi_r(X^{(n)}_{t,\cdot}))) - G_r(X_{t,r-}, \mathbb{E}^{\mathscr{F}_{r-}}(\phi_r(X_{t,\cdot}))).$$

As in (21), we can prove that for all $t \in [T, 0]$,

$$\mathbb{E}\Big(1_{A_n} \sup_{s\in[t,0]} |X^{(n)}_{t,s} - X_{t,s}|^\beta\Big) \leq C\mathbb{E}(1_{A_n}|\xi^{(n)} - \xi|^\beta), \tag{26}$$

where C is independent of n.

Now, for any $\varepsilon > 0$, we have

$$P\Big\{\sup_{s\in[t,0]} |X^{(n)}_{t,s} - X_{t,s}| \geq \varepsilon\Big\} \leq P\Big\{1_{A_n} \sup_{s\in[t,0]} |X^{(n)}_{t,s} - X_{t,s}| \geq \varepsilon\Big\} + P(A^c_n)$$
$$\leq \frac{1}{\varepsilon^\beta}\mathbb{E}\Big(1_{A_n} \sup_{s\in[t,0]} |X^{(n)}_{t,s} - X_{t,s}|^\beta\Big) + P(A^c_n)$$
$$\leq \frac{C}{\varepsilon^\beta}\mathbb{E}(1_{A_n}|\xi^{(n)} - \xi|^\beta) + P(A^c_n).$$

The proof is then complete by letting $n \to \infty$. $\square$

REMARK 2.9. In the situation of Theorem 2.7, the conclusion of this proposition still holds, which can be proven by the same procedure.

The following lemma is a direct consequence of the uniqueness of solutions.

LEMMA 2.10. *Suppose that SFDE* (16) *is uniquely solvable on the time interval* $(T, 0]$. *Then for all* $T < t_1 < t_2 < t_3 \leq 0$ *and* $\xi \in \mathscr{F}_{t_1}$, *we have*

$$X_{t_2,t_3}(X_{t_1,t_2}(\xi)) = X_{t_1,t_3}(\xi) \qquad a.s. \tag{27}$$

Moreover, for any $T < t < s \leq 0$, $x_i \in \mathbb{R}^d$, $i = 1, \ldots, n$ *and disjoint* $\Lambda_i \in \mathscr{F}_t$, $i = 1, \ldots, n$ *with* $\bigcup_i \Lambda_i = \Omega$,

$$X_{t,s}\Big(\sum_i 1_{\Lambda_i} x_i\Big) = \sum_i 1_{\Lambda_i} X_{t,s}(x_i) \qquad a.s. \tag{28}$$

PROOF. For $T < t_1 < t_2 < s \le 0$, we can write

$$X_{t_1,s}(\xi) = X_{t_1,t_2}(\xi) + \int_{(t_2,s]} G_r(X_{t_1,r-}(\xi), \mathbb{E}^{\mathscr{F}_{r-}}(\phi(X_{t_1,\cdot}(\xi)))) \, \mathrm{d}L_r \qquad \text{a.s.}$$

On the other hand, if we set

$$Y_s := X_{t_2,s}(X_{t_1,t_2}(\xi)) \qquad \forall s \in [t_2, 0],$$

then Y_s satisfies

$$Y_s = X_{t_1,t_2}(\xi) + \int_{(t_2,s]} G_r(Y_{r-}, \mathbb{E}^{\mathscr{F}_{r-}}(\phi(Y_\cdot))) \, \mathrm{d}L_r \qquad \text{a.s.}$$

Equality (27) follows by the uniqueness.

As for (28), noticing that for all $r \in (t, 0]$,

$$\begin{aligned}
&\sum_i 1_{\Lambda_i} G_r(X_{t,r-}(x_i), \mathbb{E}^{\mathscr{F}_{r-}}(\phi_r(X_{t,\cdot}(x_i)))) \\
&\quad = \sum_i G_r(1_{\Lambda_i} X_{t,r-}(x_i), 1_{\Lambda_i} \mathbb{E}^{\mathscr{F}_{r-}}(\phi_r(X_{t,\cdot}(x_i)))) \\
&\quad = G_r\biggl(\sum_i 1_{\Lambda_i} X_{t,r-}(x_i), \mathbb{E}^{\mathscr{F}_{r-}}\biggl(\phi_r\biggl(\sum_i 1_{\Lambda_i} X_{t,\cdot}(x_i)\biggr)\biggr)\biggr),
\end{aligned}$$

it follows by the uniqueness as above. □

Now we can prove the following Markov property.

PROPOSITION 2.11. *In the situation of Theorem 2.5 or Theorem 2.7, let $\{X_{t,s}(x); T \le t < s \le 0\}$ be the solution family of SFDE (16) or (24). Then for any $T \le t_1 < t_2 < t_3 \le 0$, $x \in \mathbb{R}^d$ and bounded continuous function φ, we have*

$$\mathbb{E}^{\mathscr{F}_{t_2}}(\varphi(X_{t_1,t_3}(x))) = \mathbb{E}(\varphi(X_{t_2,t_3}(y)))|_{y=X_{t_1,t_2}(x)} \qquad a.s. \tag{29}$$

PROOF. We only prove (29) in the case of Theorem 2.5. By Proposition 2.8, the mapping $y \mapsto \mathbb{E}(\varphi(X_{t_2,t_3}(y))) := \Phi(y)$ is continuous. So, $\Phi(X_{t_1,t_2}(x))$ is $\mathscr{F}_{t_2}$-measurable. Thus, for proving (29), it suffices to prove that for any $\Lambda \in \mathscr{F}_{t_2}$,

$$\mathbb{E}(1_\Lambda \varphi(X_{t_1,t_3}(x))) = \mathbb{E}(1_\Lambda \Phi(X_{t_1,t_2}(x))).$$

Let $\xi^{(n)} = \sum_{i=1}^{m_n} x_i 1_{\Lambda_i}$ be a sequence of simple functions, where $x_i \in \mathbb{R}^d$, $\Lambda_i \in \mathscr{F}_{t_2}$ disjoint and $\bigcup_i \Lambda_i = \Omega$, and such that

$$\xi^{(n)} \to X_{t_1,t_2}(x) \qquad \text{in } L^\beta \text{ as } n \to \infty.$$

By Proposition 2.8 again, we have

$$\begin{aligned}\mathbb{E}(1_\Lambda\varphi(X_{t_1,t_3}(x))) &\overset{(27)}{=} \mathbb{E}(1_\Lambda\varphi(X_{t_2,t_3}(X_{t_1,t_2}(x))))\\ &= \lim_{n\to\infty}\mathbb{E}\big(1_\Lambda\varphi\big(X_{t_2,t_3}(\xi^{(n)})\big)\big)\\ &\overset{(28)}{=} \lim_{n\to\infty}\sum_{i=1}^{m_n}\mathbb{E}(1_\Lambda 1_{\Lambda_i}\varphi(X_{t_2,t_3}(x_i))).\end{aligned}$$

Since $X_{t_2,t_3}(x_i)$ is $\mathscr{F}_{t_2,t_3}$-measurable and independent of $\mathscr{F}_{t_2}$, we further have

$$\begin{aligned}\mathbb{E}(1_\Lambda\varphi(X_{t_1,t_3}(x))) &= \lim_{n\to\infty}\sum_{i=1}^{m_n}\mathbb{E}(1_\Lambda 1_{\Lambda_i}\Phi(x_i))\\ &= \lim_{n\to\infty}\mathbb{E}\big(1_\Lambda\Phi\big(\xi^{(n)}\big)\big) = \mathbb{E}(1_\Lambda\Phi(X_{t_1,t_2}(x))).\end{aligned}$$

The proof is complete. $\square$

2.5. *Locally maximal solutions*. Now, suppose that ϕ takes the following form:

$$\phi_s(\omega) = \varphi(\omega(0)) + \int_s^0 f_r(\omega(r))\,\mathrm{d}r, \qquad \omega\in\mathbb{D}, \tag{30}$$

where $\varphi:\mathbb{R}^d\to\mathbb{R}^k$ and $f:\mathbb{R}_-\times\mathbb{R}^d\to\mathbb{R}^k$ satisfy that for some $K_2>0$ and all $s\in\mathbb{R}_-$ and $x,x'\in\mathbb{R}^d$,

$$|\varphi(x)-\varphi(x')| + |f_s(x)-f_s(x')| \le K_2|x-x'|. \tag{31}$$

In this case, we have the following existence result of a unique maximal solution.

THEOREM 2.12. *Assume that* (31), $(\mathbf{H}_G)$ *and* $(\mathbf{H}_\nu^\beta)$ *hold for some* $\beta>1$. *Then there exists a time* $T=T(K_1,K_2,\mathscr{A},\beta)\in[-\infty,0)$ *such that SFDE* (16) *is solvable on* $(T,0]$ *for any initial value* $x\in\mathbb{R}^d$, *and if* T *is finite*, *then*

$$\lim_{t\downarrow T}\|u_t\|_{\mathrm{Lip}} := \sup_{x\neq x'\in\mathbb{R}^d}\frac{|u_t(x)-u_t(x')|}{|x-x'|} = +\infty, \tag{32}$$

where

$$u_t(x) := \mathbb{E}\left(\varphi(X_{t,0}(x)) + \int_t^0 f_s(X_{t,s}(x))\,\mathrm{d}s\right). \tag{33}$$

Moreover, the family of solutions $\{X_{t,s}(x), T<t<s\le 0, x\in\mathbb{R}^d\}$ *is unique in the class that for all* $T<t_1<t_2<t_3\le 0$ *and* $x\in\mathbb{R}^d$,

$$X_{t_1,t_2}(x)\in L^\beta(\Omega,\mathscr{F}_{t_1,t_2},P), \qquad X_{t_1,t_3}(x) = X_{t_2,t_3}(X_{t_1,t_2}(x)) \qquad a.s.$$

We also have the following uniform estimate: *for any* $T'\in(T,0)$ *and* $x\in\mathbb{R}^d$,

$$\sup_{t\in[T',0]}\mathbb{E}\Big(\sup_{s\in[t,0]}|X_{t,s}(x)|^\beta\Big) \le C_{T',x}. \tag{34}$$

PROOF. First of all, let T_1 be the existence time in Theorem 2.5. By (26), there exists a constant $C = C(K_1, K_2, \mathscr{A}, \beta) > 0$ such that for all $x, x' \in \mathbb{R}^d$ and $t \in [T_1, 0]$,

$$\mathbb{E}\Big(\sup_{s\in[t,0]} |X_{t,s}(x) - X_{t,s}(x')|^\beta\Big) \le C|x - x'|^\beta.$$

Using this estimate and (31), it is easy to check that

$$\|u_{T_1}\|_{\mathrm{Lip}} < +\infty.$$

Next, we consider the following SFDE on $[t, T_1]$:

$$X_{t,s}(x) = x + \int_{(t,s]} G_r\bigg(X_{t,r-}(x), \mathbb{E}^{\mathscr{F}_{r-}}\bigg(u_{T_1}(X_{t,T_1}(x)) + \int_r^{T_1} f_{r'}(X_{t,r'}(x))\,\mathrm{d}r'\bigg)\bigg)\,\mathrm{d}L_r.$$

Repeating the proof of Theorem 2.5, one can find another $T_2 < T_1$ so that this SFDE is uniquely solvable on $[T_2, T_1]$. Meanwhile, one can patch up the solution by setting

$$X_{t,s}(x) := X_{T_1,s}(X_{t,T_1}(x)) \qquad \forall s \in [T_1, 0],\ t \in [T_2, T_1].$$

It is easy to verify that $\{X_{t,s}(x), T_2 \le t < s \le 0, x \in \mathbb{R}^d\}$ solves SFDE (16) on $[T_2, 0]$. Proceeding this construction, we obtain a sequence of times

$$0 > T_1 > T_2 > \cdots > T_n \downarrow T,$$

and a family of solutions

$$\{X_{t,s}(x), T < t < s \le 0, x \in \mathbb{R}^d\}.$$

From the construction of T, one knows that (32) holds. As for the uniqueness, it can be proved piecewisely on each $[T_n, T_{n-1}]$. Estimate (34) follows from (18) and induction. □

REMARK 2.13. By this theorem, for obtaining the global solution, it suffices to give an a priori estimate for $\|u_T\|_{\mathrm{Lip}} = \|\nabla u_T\|_\infty$.

The following result can be proved similarly. We omit the details.

THEOREM 2.14. *In addition to* (31) *and* $(\mathbf{H}_G)$, *we assume that* G, φ *and* f *are uniformly bounded. Then there exists a time* $T = T(K_1, K_2, \mathscr{A}) \in [-\infty, 0)$ *such that SFDE* (24) *is solvable on* $(T, 0]$ *and estimate* (32) *holds provided* $T > -\infty$. *Moreover, the family of solutions* $\{X_{t,s}(x), T < t < s \le 0, x \in \mathbb{R}^d\}$ *is unique in the class that for all* $T < t_1 < t_2 < t_3 \le 0$ *and* $x \in \mathbb{R}^d$,

$$X_{t_1,t_2}(x) \in \mathscr{F}_{t_1,t_2}, \qquad X_{t_1,t_3}(x) = X_{t_2,t_3}(X_{t_1,t_2}(x)) \qquad a.s.$$

3. Application to quasi-linear partial integro-differential equations. In this section, we establish the connection between stochastic functional differential equations and a class of quasi-linear partial integro-differential equations. For this aim, we consider ϕ taking the form of (30) and assume that for some $k \in \mathbb{N}$, $(\mathbf{H}_k)$, G, f and φ are continuous functions in s, x, u, and for any $j = 1, \dots, k$, $\nabla^j G_s(x, u)$, $\nabla^j f_s(x)$, $\nabla^j \varphi(x)$ are uniformly bounded continuous functions with respect to $s \in \mathbb{R}_-$, where ∇^j denotes the jth order gradient with respect to x, u. We also denote

$$\mathscr{K} := \sup_{s \in \mathbb{R}_-} (\|\nabla G_s\|_\infty + \|\nabla f_s\|_\infty) + \|\nabla \varphi\|_\infty. \tag{35}$$

Under this assumption, it is clear that (31) and $(\mathbf{H}_G)$ hold. Let $u_t(x)$ be defined by (33). By Theorem 2.12, the mapping $x \mapsto u_t(x)$ is Lipschitz continuous. However, it is in general not C^2-differentiable since we have poor integrabilities for $\nabla X_{t,s}(x)$. We shall divide two cases to discuss this problem.

3.1. *Unbounded data and ν has finite moments of arbitrary orders.* In this subsection, we consider equation (16), and assume that $(\mathbf{H}_k)$ holds for some $k \geq 3$, and $(\mathbf{H}_\nu^\beta)$ holds for all $\beta \geq 2$. In this case, we can write

$$L_t = \hat{b}t + W_t^A + \int_{\mathbb{R}^m} z \tilde{N}(t, \mathrm{d}z),$$

where $\hat{b} := b + \int_{|z|>1} z\nu(\mathrm{d}z) \in \mathbb{R}^m$.

Let $T < 0$ be the maximal time given in Theorem 2.12 and $\{X_{t,s}(x), T < t < s \leq 0, x \in \mathbb{R}^d\}$ the solution family of equation (16). For simplicity of notation, below we shall write

$$\begin{aligned} \mathcal{G}_{t,r} &:= \mathcal{G}_{t,r}(x) \\ &:= G_r\left(X_{t,r-}(x), \mathbb{E}^{\mathscr{F}_{r-}}\left(\varphi(X_{t,0}(x)) + \int_r^0 f_{r'}(X_{t,r'}(x))\,\mathrm{d}r'\right)\right). \end{aligned} \tag{36}$$

Let $g : \mathbb{R}^d \to \mathbb{R}^k$ be a C^2-function with bounded first and second order partial derivatives. By Itô's formula (cf. [1], page 226, Theorem 4.4.7), we have

$$\begin{aligned} g(X_{t,s}) = g(x) &+ \int_{(t,s]} \int_{\mathbb{R}^m} [g(X_{t,r-} + \mathcal{G}_{t,r} \cdot z) - g(X_{t,r-}) \\ &\qquad - \partial_i g(X_{t,r-}) \mathcal{G}_{t,r}^{ij} z_j] \nu(\mathrm{d}z)\,\mathrm{d}r \\ &+ \int_{(t,s]} \partial_i g(X_{t,r-}) \mathcal{G}_{t,r}^{ij} \hat{b}_j\,\mathrm{d}r \\ &+ \frac{1}{2} \int_{(t,s]} \partial_i \partial_j g(X_{t,r-}) (\mathcal{G}_{t,r}^{\mathrm{t}} A \mathcal{G}_{t,r})^{ij}\,\mathrm{d}r + M_{t,s}^g, \end{aligned} \tag{37}$$

where

$$M_{t,s}^g := \int_{(t,s]} \int_{\mathbb{R}^m} [g(X_{t,r-} + \mathcal{G}_{t,r} \cdot z) - g(X_{t,r-})] \tilde{N}(\mathrm{d}r, \mathrm{d}z) + \int_{(t,s]} \partial_i g(X_{t,r-}) \mathcal{G}_{t,r}^{ij} \, \mathrm{d}(W_r^A)^j$$

is a square integrable $(\mathscr{F}_{t,s})$-martingale by (34). Here and below, the superscript "t" denotes the transpose of a matrix.

Fix $t \in (T, 0]$ and $h > 0$ so that $t - h \in (T, 0]$. By taking expectations for both sides of (37), we have

$$\frac{1}{h}[\mathbb{E}g(X_{t-h,t}) - g(x)] = I_1^g(h) + I_2^g(h) + I_3^g(h),$$

where

$$I_1^g(h) := \frac{1}{h}\mathbb{E}\biggl(\int_{t-h}^t \int_{\mathbb{R}^m} [g(X_{t-h,r} + \mathcal{G}_{t-h,r} \cdot z) - g(X_{t-h,r}) - \partial_i g(X_{t-h,r}) \mathcal{G}_{t-h,r}^{ij} z_j] \nu(\mathrm{d}z) \, \mathrm{d}r\biggr),$$

$$I_2^g(h) := \frac{1}{h}\mathbb{E}\biggl(\int_{t-h}^t \partial_i g(X_{t,r}) \mathcal{G}_{t-h,r}^{ij} \hat{b}_j \, \mathrm{d}r\biggr),$$

$$I_3^g(h) := \frac{1}{2h}\mathbb{E}\biggl(\int_{t-h}^t \partial_i \partial_j g(X_{t,r}) (\mathcal{G}_{t-h,r}^{\mathrm{t}} A \mathcal{G}_{t-h,r})^{ij} \, \mathrm{d}r\biggr).$$

We have:

LEMMA 3.1. *As $h \downarrow 0$, it holds that*

$$I_1^g(h) \to \int_{\mathbb{R}^m} \bigl[g\bigl(x + \mathscr{G}_t(x) \cdot z\bigr) - g(x) - \partial_i g(x) \mathscr{G}_t^{ij}(x) \cdot z_j\bigr] \nu(\mathrm{d}z),$$

$$I_2^g(h) \to \partial_i g(x) \mathscr{G}_t^{ij}(x) \hat{b}_j, \qquad I_3^g(h) \to \tfrac{1}{2} \partial_i \partial_j g(x) (\mathscr{G}_t^{\mathrm{t}}(x) A \mathscr{G}_t(x))^{ij},$$

where

$$\mathscr{G}_t(x) := G_t\biggl(x, \mathbb{E}\biggl(\varphi(X_{t,0}(x)) + \int_t^0 f_s(X_{t,s}(x)) \, \mathrm{d}s\biggr)\biggr).$$

PROOF. We only prove the first limit, the others are analogous. By the change of variables, we can write

$$I_1^g(h) = \mathbb{E}\biggl(\int_0^1 \int_{\mathbb{R}^m} [g(X_{t-h,t-hs} + \mathcal{G}_{t-h,t-hs} \cdot z) - g(X_{t-h,t-hs}) - \partial_i g(X_{t-h,t-hs}) \mathcal{G}_{t-h,t-hs}^{ij} z_j] \nu(\mathrm{d}z) \, \mathrm{d}s\biggr).$$

Notice that

$$
\begin{aligned}
X_{t-h,t-hs}(x) - x
&= \int_{(t-h,t-hs]} \int_{\mathbb{R}^m} \mathcal{G}_{t-h,r}(x) \cdot z \tilde{N}(\mathrm{d}r, \mathrm{d}z) + \int_{(t-h,t-hs]} \mathcal{G}_{t-h,r}(x) \cdot \hat{b} \, \mathrm{d}r \\
&\quad + \int_{(t-h,t-hs]} \mathcal{G}_{t-h,r}(x) \, \mathrm{d}W_r^A \\
&=: J_1(h) + J_2(h) + J_3(h).
\end{aligned}
$$

By the isometric property of stochastic integrals, we have

$$
\begin{aligned}
\mathbb{E}|J_1(h)|^2 &= \mathbb{E}\left(\int_{t-h}^{t-hs} \int_{\mathbb{R}^m} |\mathcal{G}_{t-h,r}(x) \cdot z|^2 \nu(\mathrm{d}z) \, \mathrm{d}r \right) \\
&\leq |h| \mathbb{E}\left(\sup_{r \in [t-h,t-hs]} |\mathcal{G}_{t-h,r}(x)|^2 \int_{\mathbb{R}^m} |z|^2 \nu(\mathrm{d}z) \right) \\
&\leq C|h| \mathbb{E}\left(1 + \sup_{r \in [t-h,0]} |X_{t-h,r}(x)|^2 \right) \overset{(34)}{\to} 0 \qquad \text{as } h \downarrow 0,
\end{aligned}
$$

where in the last inequality, we used Doob's maximal inequality and that G, ϕ and f in definition (36) of $\mathcal{G}_{t,r}$ are linear growth in x and u, respectively. Similarly,

$$
\mathbb{E}|J_2(h)|^2 + \mathbb{E}|J_3(h)|^2 \to 0 \qquad \text{as } h \downarrow 0.
$$

Hence, for fixed t, s, x,

$$
\lim_{h \downarrow 0} \mathbb{E}|X_{t-h,t-hs}(x) - x|^2 = 0. \tag{38}
$$

Noticing that

$$
g(x+y) - g(x) = y \cdot \int_0^1 \nabla g(x + \theta y) \, \mathrm{d}\theta, \tag{39}
$$

we have

$$
\begin{aligned}
&\mathbb{E}|g(X_{t-h,t-hs} + \mathcal{G}_{t-h,t-hs} \cdot z) - g(X_{t-h,t-hs}) - \partial_i g(X_{t-h,t-hs}) \mathcal{G}^{ij}_{t-h,t-hs} z_j| \\
&\quad = \mathbb{E}\left| \left(\int_0^1 [\partial_i g(X_{t-h,t-hs} + \theta \mathcal{G}_{t-h,t-hs} \cdot z) - \partial_i g(X_{t-h,t-hs})] \, \mathrm{d}\theta \right) \right. \\
&\qquad\qquad \left. \times \mathcal{G}^{ij}_{t-h,t-hs} z_j \right| \\
&\quad \leq C \mathbb{E}|\mathcal{G}_{t-h,t-hs}|^2 |z|^2 \leq C \mathbb{E}\left(1 + \sup_{r \in [t-h,0]} |X_{t-h,r}(x)|^2 \right) |z|^2 \overset{(34)}{\leq} C|z|^2,
\end{aligned}
$$

where the second-to-last inequality is the same as above, and the constant C is independent of h, s, z. Thus, for proving the first limit, by the dominated convergence theorem, it suffices to prove that for fixed $s \in [0, 1]$ and $z \in \mathbb{R}^m$,

$$\mathbb{E}\biggl(\biggl(\int_0^1 [\partial_i g(X_{t-h,t-hs} + \theta \mathcal{G}_{t-h,t-hs} \cdot z) - \partial_i g(X_{t-h,t-hs})] \, d\theta\biggr) \mathcal{G}^{ij}_{t-h,t-hs} z_j\biggr)$$
$$\to \biggl(\int_0^1 \bigl[\partial_i g\bigl(x + \theta \mathscr{G}_t(x) \cdot z\bigr) - \partial_i g(x)\bigr] \, d\theta\biggr) \mathscr{G}^{ij}_t(x) z_j \qquad \text{as } h \downarrow 0.$$

By (38) and Remark 2.1, this limit is easily obtained. □

We also need the following differentiability of the solution $X_{t,s}(x)$ with respect to x in the L^p-sense.

LEMMA 3.2. *For any $p \geq 2$, there exists a time $T_* = T_*(p, k, \mathscr{A}, \mathscr{K}) \in (T, 0)$, where $\mathscr{A}$ is defined by (13), and $\mathscr{K}$ is defined by (35), such that for any $T_* \leq t \leq s \leq 0$, the mapping $x \mapsto X_{t,s}(x)$ is C^{k-1}-differentiable in the L^p-sense and for any $j = 1, \dots, k-1$,*

$$\sup_{x \in \mathbb{R}^d} \sup_{s \in [t,0]} \mathbb{E}|\nabla^j X_{t,s}(x)|^p < +\infty.$$

PROOF. Since the proof is standard (cf. [13], Theorem 39 or [12], Section 4.6), we sketch it. Let $\{e_i, i = 1, \dots, d\}$ be the canonical basis of $\mathbb{R}^d$. For $\delta > 0$ and $i = 1, \dots, d$, define

$$X^{\delta,i}_{t,s} := X^{\delta,i}_{t,s}(x) = \frac{X_{t,s}(x + \delta e_i) - X_{t,s}(x)}{\delta}$$

and

$$\mathcal{G}^{\delta,i}_{t,s} := \mathcal{G}^{\delta,i}_{t,s}(x) = \frac{\mathcal{G}_{t,s}(x + \delta e_i) - \mathcal{G}_{t,s}(x)}{\delta},$$

where $\mathcal{G}_{t,s}(x)$ is defined by (36). Then,

$$X^{\delta,i}_{t,s} = e_i + \int_{(t,s]} \int_{\mathbb{R}^m} \mathcal{G}^{\delta,i}_{t,r} \cdot z \tilde{N}(dr, dz) + \int_{(t,s]} \mathcal{G}^{\delta,i}_{t,r} \cdot \hat{b} \, dr + \int_{(t,s]} \mathcal{G}^{\delta,i}_{t,r} \, dW^A_r. \tag{40}$$

As in (20), by Burkholder's inequality, we have that for any $p \geq 2$,

$$\mathbb{E}\Bigl(\sup_{r \in [t,0]} |X^{\delta,i}_{t,r}|^p\Bigr) \leq C_{p,\mathscr{A}} |t| \mathbb{E}\Bigl(\sup_{r \in [t,0]} |\mathcal{G}^{\delta,i}_{t,r}|^p\Bigr). \tag{41}$$

Moreover, by $(\mathbf{H}_k)$ and Doob's maximal inequality, we easily derive that

$$\mathbb{E}\Bigl(\sup_{r \in [t,0]} |\mathcal{G}^{\delta,i}_{t,r}|^p\Bigr) \leq C_{p,\mathscr{K}} \mathbb{E}\Bigl(\sup_{r \in [t,0]} |X^{\delta,i}_{t,r}|^p\Bigr).$$

Substituting this into (41), we find that for some $C_{p,\mathscr{A},\mathscr{K}} > 0$ independent of x, t and δ,

$$\mathbb{E}\Big(\sup_{r\in[t,0]} |X^{\delta,i}_{t,r}|^p\Big) \le C_{p,\mathscr{A},\mathscr{K}} |t| \mathbb{E}\Big(\sup_{r\in[t,0]} |X^{\delta,i}_{t,r}|^p\Big).$$

From this, we deduce that there exists a time $T_* = T_*(p, \mathscr{A}, \mathscr{K}) \in (T, 0)$ such that for all $t \in [T_*, 0]$,

$$\sup_{\delta\in(0,1)} \sup_{x\in\mathbb{R}^d} \sup_{t\in[T_*,0]} \mathbb{E}\Big(\sup_{r\in[t,0]} |X^{\delta,i}_{t,r}(x)|^p\Big) < +\infty. \tag{42}$$

On the other hand, let $Y^i_{t,s} = Y^i_{t,s}(x)$ satisfy the following SFDE:

$$\begin{aligned}
Y^i_{t,s} = e_i &+ \int_t^s \nabla_x G_r\Big(X_{t,r-}(x), \mathbb{E}^{\mathscr{F}_{r-}}\Big(\varphi(X_{t,0}(x)) + \int_r^0 f_{r'}(X_{t,r'}(x))\,\mathrm{d}r'\Big)\Big) \\
&\qquad\quad \times Y^i_{t,r}\,\mathrm{d}L_r \\
&+ \int_t^s \nabla_y G_r(X_{t,r}, \mathbb{E}^{\mathscr{F}_{r-}}(\phi_r(X_{t,\cdot}))) \\
&\qquad\quad \times \mathbb{E}^{\mathscr{F}_{r-}}\Big(\nabla\varphi(X_{t,0})Y^i_{t,0} + \int_r^0 \nabla f'_r(X_{t,r'})Y^i_{t,r'}\,\mathrm{d}r'\Big)\,\mathrm{d}L_r,
\end{aligned} \tag{43}$$

which can be solved on $[T_*, 0]$ as in Theorem 2.5. Using the uniform estimate (42) and formula (39), it is not hard to deduce that

$$\lim_{\delta\to 0} \mathbb{E}\Big(\sup_{r\in[t,0]} |X^{\delta,i}_{t,r}(x) - Y^i_{t,r}(x)|^p\Big) = 0.$$

In particular,

$$\sup_{x\in\mathbb{R}^d} \sup_{t\in[T_*,0]} \mathbb{E}\Big(\sup_{r\in[t,0]} |Y^i_{t,r}(x)|^p\Big) < +\infty.$$

The higher derivatives can be estimated similarly from (43). $\square$

Now we can prove the following result, which was originally due to [4, 8, 16].

THEOREM 3.3. *Assume that* $(\mathbf{H}_k)$ *holds for some* $k \ge 3$, *and* $(\mathbf{H}^\beta_\nu)$ *holds for all* $\beta \ge 2$. *Let* $\{X_{t,s}(x), T < t \le s \le 0, x \in \mathbb{R}^d\}$ *be the maximal solution of SFDE* (24) *in Theorem* 2.12, *and* $u_t(x)$ *be defined by*

$$u_t(x) := \mathbb{E}\varphi(X_{t,0}(x)) + \mathbb{E}\Big(\int_t^0 f_s(X_{t,s}(x))\,\mathrm{d}s\Big). \tag{44}$$

Then there exists a time $T_* = T_*(k, \mathscr{A}, \mathscr{K}) \in (T, 0)$ *such that for each* $t \in [T_*, 0]$, $x \mapsto u_t(x)$ *has bounded derivatives up to* $(k-1)$*-order, and solves the following quasi-linear partial integro-differential equation*:

$$u_t(x) = \varphi(x) + \int_t^0 [\mathcal{L}^{\mathrm{c}} u_s(x) + \mathcal{L}^{\mathrm{d}} u_s(x) + f_s(x)]\,\mathrm{d}s \qquad \forall (t, x) \in [T_*, 0] \times \mathbb{R}^d,$$

where

$$\mathcal{L}^{\mathrm{c}} u_t(x) := \partial_i u_t(x) G_t^{ij}(x, u_t(x))\hat{b}_j + \tfrac{1}{2}\partial_i\partial_j u_t(x)(G_t^{\mathrm{t}}(x, u_t(x)) A G_t(x, u_t(x)))^{ij}$$

and

$$\mathcal{L}^{\mathrm{d}} u_t(x) := \int_{\mathbb{R}^m} [u_t(x + G_t(x, u_t(x)) \cdot z) - u_t(x) - \partial_i u_t(x) G_t^{ij}(x, u_t(x)) \cdot z_j] \nu(\mathrm{d}z).$$

PROOF. We follow the argument of Friedman [9]. By Proposition 2.11, for $T < t - h < t \le 0$, we have

$$\begin{aligned} u_{t-h}(x) &= \mathbb{E}[(\mathbb{E}\varphi(X_{t,0}(y)))|_{y=X_{t-h,t}(x)}] + \mathbb{E}\biggl[\mathbb{E}\biggl(\int_t^0 f_r(X_{t,r}(y))\,\mathrm{d}r\biggr)\biggr|_{y=X_{t-h,t}(x)}\biggr] \\ &\quad + \mathbb{E}\biggl(\int_{t-h}^t f_r(X_{t-h,r}(x))\,\mathrm{d}r\biggr) \\ &= \mathbb{E}u_t(X_{t-h,t}(x)) + \mathbb{E}\biggl(\int_{t-h}^t f_r(X_{t-h,r}(x))\,\mathrm{d}r\biggr). \end{aligned}$$

By Lemma 3.2, it is easy to see that there exists a time $T_* = T_*(k, \mathscr{A}, \mathscr{K}) < 0$ such that for each $t \in [T_*, 0]$, $u_t(x)$ has bounded derivatives up to $(k-1)$-order. Thus, we can invoke Lemma 3.1 to derive that

$$\begin{aligned} &\frac{1}{h}(u_{t-h}(x) - u_t(x)) \\ &\quad = \frac{1}{h}(\mathbb{E}u_t(X_{t-h,t}(x)) - u_t(x)) + \frac{1}{h}\mathbb{E}\biggl(\int_{t-h}^t f_r(X_{t-h,r}(x))\,\mathrm{d}r\biggr) \\ &\quad \to \mathcal{L}^{\mathrm{c}} u_t(x) + \mathcal{L}^{\mathrm{d}} u_t(x) + f_t(x) \qquad \text{as } h \downarrow 0. \end{aligned}$$

On the other hand, from the above proof, it is also easy to see that for fixed $x \in \mathbb{R}^d$, $t \mapsto u_t(x)$ is Lipschitz continuous. Hence,

$$u_t(x) - \varphi(x) = -\int_t^0 \partial_s u_s(x)\,\mathrm{d}s = \int_t^0 [\mathcal{L}^{\mathrm{c}} u_s(x) + \mathcal{L}^{\mathrm{d}} u_s(x) + f_s(x)]\,\mathrm{d}s.$$

The proof is thus complete. □

3.2. *Bounded data and constant big jump.* In this subsection we assume that $(\mathbf{H}_k)$ holds for some $k \ge 3$, and G, φ and f are uniformly bounded and continuous functions. Consider the following SFDE:

$$\begin{aligned} X_{t,s}(x) = x &+ \int_{(t,s]}\int_{|z|<1} \mathcal{G}_{t,r}(x) \cdot z\tilde{N}(\mathrm{d}r, \mathrm{d}z) + \int_{(t,s]}\int_{|z|\ge 1} zN(\mathrm{d}r, \mathrm{d}z) \\ &+ \int_{(t,s]} \mathcal{G}_{t,r}(x) \cdot b\,\mathrm{d}r + \int_{(t,s]} \mathcal{G}_{t,r}(x)\,\mathrm{d}W_r^A, \end{aligned}$$

where $\mathcal{G}_{t,r}(x)$ is defined by (36). In this case, Lemmas 3.1 and 3.2 still hold. We just want to mention that (38) should be replaced by

$$X_{t-h,t-hs}(x) \to x \qquad \text{in probability as } h \downarrow 0,$$

and (40) becomes

$$X_{t,s}^{\delta,i} = e_i + \int_{(t,s]} \int_{|z|\le 1} \mathcal{G}_{t,r}^{\delta,i} \cdot z\tilde{N}(\mathrm{d}r, \mathrm{d}z) + \int_{(t,s]} \mathcal{G}_{t,r}^{\delta,i} \cdot b \,\mathrm{d}r + \int_{(t,s]} \mathcal{G}_{t,r}^{\delta,i} \,\mathrm{d}W_r^A.$$

Thus, the following result can be proved along the same lines as in Theorem 3.3. We omit the details.

THEOREM 3.4. *Assume that* $(\mathbf{H}_k)$ *holds for some* $k \ge 3$, *and* G, φ *and* f *are uniformly bounded and continuous functions. Let* $\{X_{t,s}(x), T < t \le s \le 0, x \in \mathbb{R}^d\}$ *be the short time solution of SFDE* (24) *in Theorem* 2.14, *and* $u_t(x)$ *be defined by* (44). *Then there exists a time* $T_* = T_*(k, \mathscr{A}, \mathscr{K}) \in (T, 0)$ *such that for each* $t \in [T_*, 0]$, $x \mapsto u_t(x)$ *has bounded derivatives up to* $(k-1)$*-order, and solves the following quasi-linear partial integro-differential equation*:

$$u_t(x) = \varphi(x) + \int_t^0 [\mathcal{L}^{\mathrm{c}} u_s(x) + \mathcal{L}^{\mathrm{d}} u_s(x) + f_s(x)] \,\mathrm{d}s \qquad \forall (t, x) \in [T_*, 0] \times \mathbb{R}^d,$$

where

$$\mathcal{L}^{\mathrm{c}} u_t(x) := \partial_i u_t(x) G_t^{ij}(x, u_t(x)) b_j + \tfrac{1}{2} \partial_i \partial_j u_t(x) (G_t^{\mathrm{t}}(x, u_t(x)) A G_t(x, u_t(x)))^{ij}$$

and

$$\begin{aligned}\mathcal{L}^{\mathrm{d}} u_t(x) := \int_{|z|<1} &\big[u_t\big(x + G_t(x, u_t(x)) \cdot z\big) - u_t(x) - \partial_i u_t(x) G_t^{ij}(x, u_t(x)) \cdot z_j\big] \\ &\times \nu(\mathrm{d}z) \\ &+ \int_{|z|\ge 1} [u_t(x+z) - u_t(x)] \nu(\mathrm{d}z).\end{aligned}$$

4. Semi-linear partial integro-differential equation: Existence and uniqueness of weak solutions. In this section we consider the following semi-linear partial integro-differential equation:

$$\partial_t u_t + \mathcal{L}_0 u_t + G_t^i(x, u_t) \partial_i u_t + F_t(x, u_t) = 0, \qquad u_0 = \varphi, t \le 0, \tag{45}$$

where $\mathcal{L}_0$ is the generator of the Lévy process L_t given by (15), and

$$\begin{aligned} &G \in \mathcal{B}\big(\mathbb{R}_-; \mathbb{W}^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^d)\big), \qquad F \in \mathcal{B}\big(\mathbb{R}_-; \mathbb{W}^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^k)\big), \\ &\varphi \in \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k). \end{aligned} \tag{46}$$

Here and below, $\mathbb{W}^{1,\infty}$ denotes the space of bounded and Lipschitz continuous functions, $\mathcal{B}$ or $\mathcal{B}_{\mathrm{loc}}$ denotes the space of uniformly or locally bounded measurable functions.

Let us first give the following definition about the maximal weak solution for equation (45).

DEFINITION 4.1. For $T \in [-\infty, 0)$, we call $u \in \mathcal{B}_{\mathrm{loc}}((T,0]; \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k))$ a maximal weak solution of equation (45) if

$$\lim_{t \downarrow T} \|\nabla u_t(x)\|_\infty = +\infty \qquad \text{when } T > -\infty, \tag{47}$$

and for all $\psi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^k)$ and $t \in (T, 0]$,

$$\langle u_t, \psi\rangle = \langle \varphi, \psi\rangle + \int_t^0 \langle u_s, \mathcal{L}_0^* \psi\rangle \, \mathrm{d}r + \int_t^0 \langle G_s^i(u_s)\partial_i u_s + F_s(u_s), \psi\rangle \, \mathrm{d}s, \tag{48}$$

where $\langle \varphi, \psi\rangle := \int_{\mathbb{R}^d} \varphi(x) \cdot \psi(x)\, \mathrm{d}x$, and $\mathcal{L}_0^*$ is the adjoint operator of $\mathcal{L}_0$ and given by

$$\mathcal{L}_0^* \psi(x) := \frac{1}{2} a_{ij} \partial_i \partial_j \psi - b_i \partial_i \psi + \int_{\mathbb{R}^d} [\psi(x-z) - \psi(x) + 1_{|z|<1} \partial_i \psi(x) z_i] \nu(\mathrm{d}z).$$

The main aim of this section is to prove the following existence and uniqueness of a maximal weak solution as well as the global solution for equation (45).

THEOREM 4.2. (i) (*Locally maximal weak solution*) *Under* (46), *there exists a unique maximal weak solution* $u_t(x)$ *for equation* (45) *in the sense of Definition* 4.1. *Moreover*, *let* T *be the maximal existence time*, *then for any* $t \in (T, 0]$,

$$\|u_t\|_\infty \le \|\varphi\|_\infty + |t| \sup_{s \in [t,0]} \|F_s\|_\infty. \tag{49}$$

(ii) (*Nonnegative solution*) *If for some* $j = 1, \dots, k$, *the components* φ^j *and* F^j *are nonnegative*, *then the corresponding component* u^j *of weak solution in* (i) *are also nonnegative*.

(iii) (*Global solution*) *Let* $\Psi(\xi)$ *be the Lévy symbol defined in* (11) *with* $b = A = 0$. *If for some* $\alpha \in (1, 2)$,

$$\mathrm{Re}(\Psi(\xi)) \asymp |\xi|^\alpha \qquad \text{as } |\xi| \to \infty, \tag{50}$$

where $a \asymp b$ *means that for some* $c_1, c_2 > 0$, $c_1 b \le a \le c_2 b$, *then the maximal existence time* T *in* (i) *equals to* $-\infty$. *In the case that* $b = \nu = 0$ *and* A *is strictly positive*, *then* T *also equals to* $-\infty$.

REMARK 4.3. Since we have estimate (49), it is easy to see that the assumption on G in (46) can be replaced by

$$G \in \mathcal{B}(\mathbb{R}_-; \mathbb{W}^{1,\infty}(\mathbb{R}^d \times \mathbb{B}_R; \mathbb{R}^d)) \qquad \forall R > 0,$$

where $\mathbb{B}_R := \{x \in \mathbb{R}^k : |x| \le R\}$.

For proving this theorem, let us begin with studying:

4.1. *Linear partial integro-differential equation*. In this subsection, we firstly study the existence and uniqueness of weak solutions for the following linear PIDE:

$$\partial_t u_t + \mathcal{L}_0 u_t + G_t^i(x)\partial_i u_t + H_t(x)u_t + f_t(x) = 0, \qquad u_0 = \varphi,\ t \le 0, \tag{51}$$

where $G:\mathbb{R}_- \times \mathbb{R}^d \to \mathbb{R}^d$, $H:\mathbb{R}_- \times \mathbb{R}^d \to \mathbb{R}^k \times \mathbb{R}^k$, $f:\mathbb{R}_- \times \mathbb{R}^d \to \mathbb{R}^k$ and $\varphi:\mathbb{R}^d \to \mathbb{R}^k$ are bounded measurable functions.

Let us start with the following case of smooth coefficients, which is the classical Feynman–Kac formula. Here, the main point is to prove the uniqueness.

THEOREM 4.4 (Feynman–Kac formula). *Assume that*

$$G \in \mathcal{B}(\mathbb{R}_-; C_b^\infty(\mathbb{R}^d; \mathbb{R}^d)), \qquad H \in \mathcal{B}\big(\mathbb{R}_-; C_b^\infty(\mathbb{R}^d; \mathbb{R}^k \times \mathbb{R}^k)\big),$$
$$f \in \mathcal{B}(\mathbb{R}_-; C_b^\infty(\mathbb{R}^d; \mathbb{R}^k)), \qquad \varphi \in C_b^\infty(\mathbb{R}^d; \mathbb{R}^k),$$

where C_b^∞ denotes the space of bounded smooth functions with bounded derivatives of all orders. Let $\{X_{t,s}(x), t \le s \le 0, x \in \mathbb{R}^d\}$ solve the following SDE:

$$X_{t,s}(x) = x + \int_t^s G_r(X_{t,r}(x))\,\mathrm{d}r + \int_t^s \mathrm{d}L_r,$$

and $\{Z_{t,s}(x), t \le s \le 0, x \in \mathbb{R}^d\}$ solve the following ODE:

$$Z_{t,s}(x) = \mathbb{I}_{m\times m} + \int_t^s H_r(X_{t,r}(x)) \cdot Z_{t,r}(x)\,\mathrm{d}x.$$

Define

$$u_t(x) := \mathbb{E}[Z_{t,0}(x)\varphi(X_{t,0}(x))] + \mathbb{E}\left[\int_t^0 Z_{t,r}(x) f_r(X_{t,r}(x))\,\mathrm{d}r\right]. \tag{52}$$

Then $u \in C(\mathbb{R}_-; C_b^\infty(\mathbb{R}^d; \mathbb{R}^k))$ uniquely solves the following linear PIDE:

$$u_t(x) = \varphi(x) + \int_t^0 [\mathcal{L}_0 u_s(x) + G_s^i(x)\partial_i u_s(x) + H_s(x)u_s(x) + f_s(x)]\,\mathrm{d}s \qquad \forall (t,x) \in \mathbb{R}_- \times \mathbb{R}^d. \tag{53}$$

PROOF. By smoothing the time variable and then taking limits, as in Section 3, by careful calculations, one can find that u defined by (52) belongs to $C(\mathbb{R}_-; C_b^\infty(\mathbb{R}^d; \mathbb{R}^k))$ and satisfies (53); see [9], page 148, Theorem 5.3 or [5, 18].

We now prove the uniqueness by the duality argument. Let $\hat{X}_{t,s}(x)$ solve the following SDE:

$$\hat{X}_{t,s}(x) = x - \int_t^s G_r(\hat{X}_{t,r}(x))\,\mathrm{d}r - \int_t^s \mathrm{d}L_r,$$

and $\hat{Z}_{t,s}(x)$ solve the following ODE:

$$\hat{Z}_{t,s}(x) = \mathbb{I}_{m\times m} + \int_t^s [H_r(\hat{X}_{t,r}(x))^{\mathrm{t}} + \operatorname{div} G_r(\hat{X}_{t,r}(x))\mathbb{I}_{m\times m}] \cdot \hat{Z}_{t,r}(x)\,\mathrm{d}x.$$

Fix $T < 0$ and $\psi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^k)$ define

$$\hat{u}_t(x) := \mathbb{E}[\hat{Z}_{T-t,0}(x)\psi(\hat{X}_{T-t,0}(x))].$$

As above, one can check that

$$\hat{u}_t(x) = \psi(x) + \int_T^t [\mathcal{L}_0^* \hat{u}_s(x) - G_s^i(x)\partial_i \hat{u}_s(x) + H_s(x)^{\mathrm{t}}\hat{u}_s(x) + \operatorname{div} G_s(x)\hat{u}_s(x)]\,\mathrm{d}s.$$

Let $u \in C(\mathbb{R}_-; C_b^\infty(\mathbb{R}^d; \mathbb{R}^k))$ satisfy equation (53) with $\varphi = f = 0$. Then by the integration by parts formula, we have for almost all $t \in [T, 0]$,

$$\begin{aligned}\partial_t \langle u_t, \hat{u}_t\rangle &= -\langle \mathcal{L}_0 u_t + G_t^i \partial_i u_t + H_t u_t, \hat{u}_t\rangle \\ &\quad + \langle u_t, \mathcal{L}_0^* \hat{u}_t - G_t \partial_i \hat{u}_t + H_t^{\mathrm{t}} \hat{u} + \operatorname{div} G_t \hat{u}_t\rangle \\ &= 0.\end{aligned}$$

From this, we get

$$\langle u_T, \psi\rangle = \langle u_T, \hat{u}_T\rangle = \langle u_0, \hat{u}_0\rangle = 0,$$

which leads to $u_T(x) = 0$ by the arbitrariness of ψ. □

For $\ell \in \mathbb{N}$, we introduce a family of mollifiers in $\mathbb{R}^\ell$. Let $\rho : \mathbb{R}^\ell \to [0, 1]$ be a smooth function satisfying that

$$\rho(x) = 0 \qquad \forall |x| > 1, \qquad \int_{\mathbb{R}^\ell} \rho(x)\,\mathrm{d}x = 1.$$

We shall call $\{\rho_\varepsilon(x) := \varepsilon^{-\ell}\rho(x/\varepsilon), \varepsilon \in (0, 1)\}$ a family of mollifiers in $\mathbb{R}^\ell$.

Next, we relax the regularity assumptions on G, H, f and φ, and prove the following:

THEOREM 4.5. *Assume that*

$$\begin{aligned} &G \in \mathcal{B}(\mathbb{R}_-; \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)), \qquad H \in \mathcal{B}(\mathbb{R}_- \times \mathbb{R}^d; \mathbb{R}^k \times \mathbb{R}^k), \\ &f \in \mathcal{B}(\mathbb{R}_-; \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k)), \qquad \varphi \in \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k).\end{aligned}$$

Let $u_t(x)$ be defined as in (52). *Then $u_t(x) \in \mathcal{B}_{\mathrm{loc}}(\mathbb{R}_-; \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k))$ is a unique weak solution of equation* (51) *in the sense of Definition* 4.1.

PROOF. We only prove the uniqueness. As for the existence, it follows by smoothing the coefficients and then taking limits as done in Theorem 4.8 below.

Suppose that $u \in \mathcal{B}_{\mathrm{loc}}(\mathbb{R}_-; \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k))$ is a weak solution of equation (51) with $\varphi = f = 0$ in the sense of Definition 4.1. We want to prove that $u \equiv 0$. Let ρ_ε be a family of mollifiers in $\mathbb{R}^d$. Define

$$u_t^\varepsilon(x) := u_t * \rho_\varepsilon(x), \qquad G_t^\varepsilon(x) := G_t * \rho_\varepsilon(x), \qquad H_t^\varepsilon(x) := H_t * \rho_\varepsilon(x).$$

Taking $\psi(\cdot) = \rho_\varepsilon(x - \cdot)$ in (48), one finds that $u_t^\varepsilon(x)$ satisfies

$$u_t^\varepsilon(x) = \int_t^0 [\mathcal{L}_0 u_s^\varepsilon(x) + G_s^{\varepsilon,i}(x)\partial_i u_s^\varepsilon(x) + H_s^\varepsilon(x) u_s^\varepsilon(x) + f_s^\varepsilon(x)]\,\mathrm{d}s,$$

where

$$f_s^\varepsilon(x) = (G_s^i \partial_i u) * \rho_\varepsilon(x) - G_s^{\varepsilon,i}(x)\partial_i u_s^\varepsilon(x) + (H_s u_s) * \rho_\varepsilon(x) - H_s^\varepsilon(x) u_s^\varepsilon(x).$$

By the property of convolutions, we have

$$\|f_s^\varepsilon\|_\infty \leq 2\|G_s\|_\infty \|\nabla u\|_\infty + 2\|H_s\|_\infty \|u\|_\infty, \tag{54}$$

and for fixed s and Lebesgue almost all $x \in \mathbb{R}^d$,

$$f_s^\varepsilon(x) \to 0, \qquad \varepsilon \to 0. \tag{55}$$

Let $X_{t,s}^\varepsilon(x)$ solve the following SDE:

$$X_{t,s}^\varepsilon(x) = x + \int_t^s G_r^\varepsilon(X_{t,r}^\varepsilon(x))\,\mathrm{d}r + \int_t^s \mathrm{d}L_r, \tag{56}$$

and $Z_{t,s}^\varepsilon(x)$ solve the following ODE:

$$Z_{t,s}^\varepsilon(x) = \mathbb{I}_{m\times m} + \int_t^s H_r^\varepsilon(X_{t,r}^\varepsilon(x)) \cdot Z_{t,r}^\varepsilon(x)\,\mathrm{d}x. \tag{57}$$

By Theorem 4.4, $u_t^\varepsilon(x)$ can be uniquely represented by

$$u_t^\varepsilon(x) := \mathbb{E}\left[\int_t^0 Z_{t,s}^\varepsilon(x) f_s^\varepsilon(X_{t,s}^\varepsilon(x))\,\mathrm{d}s\right].$$

For completing the proof, it suffices to prove that for each $(t, x) \in \mathbb{R}_- \times \mathbb{R}^d$,

$$u_t^\varepsilon(x) \to 0, \qquad \varepsilon \to 0.$$

Since by (57), $Z_{t,s}^\varepsilon(x)$ is uniformly bounded with respect to x, ε and $s \in [t, 0]$, we need only to show that for any nonnegative $\psi \in C_0^\infty(\mathbb{R}^d)$,

$$I_t^\varepsilon := \mathbb{E}\left[\int_t^0 \int_{\mathbb{R}^d} |f_s^\varepsilon|(X_{t,s}^\varepsilon(x))\psi(x)\,\mathrm{d}x\,\mathrm{d}s\right] \to 0, \qquad \varepsilon \to 0.$$

For any $R > 0$, by (54) and the change of variables, we have

$$
\begin{aligned}
I_t^\varepsilon &\le \mathbb{E}\bigg[\int_t^0 \int_{|X_{t,s}^\varepsilon(x)|\le R} |f_s^\varepsilon|(X_{t,s}^\varepsilon(x))\psi(x)\,\mathrm{d}x\,\mathrm{d}s\bigg] \\
&\quad + C_t\mathbb{E}\bigg[\int_t^0 \int_{|X_{t,s}^\varepsilon(x)|> R} \psi(x)\,\mathrm{d}x\,\mathrm{d}s\bigg] \\
&\le \mathbb{E}\bigg[\int_t^0 \int_{B_R} |f_s^\varepsilon|(x)\psi(X_{t,s}^{\varepsilon,-1}(x))\det(\nabla X_{t,s}^{\varepsilon,-1}(x))\,\mathrm{d}x\,\mathrm{d}s\bigg] \\
&\quad + C_t\int_t^0 \int_{\mathbb{R}^d} P\{|X_{t,s}^\varepsilon(x)| > R\}\psi(x)\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned} \tag{58}
$$

where $X_{t,s}^{\varepsilon,-1}(x)$ denotes the inverse of $x \mapsto X_{t,s}^\varepsilon(x)$. From equation (56), it is by now standard to prove that (e.g., see Kunita [12], Lemma 4.3.1)

$$
\det(\nabla X_{t,s}^{\varepsilon,-1}(x)) = \exp\bigg\{-\int_t^s (\operatorname{div} G_r^\varepsilon)(X_{t,r}^\varepsilon(X_{t,s}^{\varepsilon,-1}(x)))\,\mathrm{d}r\bigg\},
$$

which then yields

$$
C_0 := \sup_{\varepsilon\in(0,1)} \sup_{x\in\mathbb{R}^d} \sup_{s\in[t,0]} |\det(\nabla X_{t,s}^{\varepsilon,-1}(x))| < +\infty.
$$

Thus, for fixed $R > 0$, the first term in (58) is less than

$$
C_0\|\psi\|_\infty \int_t^0 \int_{B_R} |f_s^\varepsilon|(x)\,\mathrm{d}x\,\mathrm{d}s \overset{(55)}{\to} 0, \qquad \varepsilon \to 0.
$$

Moreover, by equation (56), we also have

$$
\lim_{R\to\infty} \sup_\varepsilon P\{|X_{t,s}^\varepsilon(x)| > R\} \le \lim_{R\to\infty} P\bigg\{|x| + \int_t^0 \|G_r\|_\infty\,\mathrm{d}r + |L_s - L_t| > R\bigg\} = 0.
$$

The proof is complete by first letting $\varepsilon \to 0$ and then $R \to \infty$ in (58). $\square$

As an easy corollary of this theorem, we first establish the uniqueness for equation (45).

THEOREM 4.6. *Under* (46), *there exists*, *at most*, *one weak solution for equation* (45).

PROOF. Let $u^{(i)} \in \mathcal{B}_{\mathrm{loc}}((T,0];\mathbb{W}^{1,\infty}(\mathbb{R}^d;\mathbb{R}^k))$, $i = 1,2$ be two weak solutions of equation (45) in the sense of Definition 4.1. Define $u_t(x) := u_t^{(1)}(x) - u_t^{(2)}(x)$. Then $u_t(x)$ satisfies that for all $\psi \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^k)$,

$$
\langle u_t, \psi\rangle = \int_t^0 \langle u_s, \mathcal{L}_0^*\psi\rangle\,\mathrm{d}r + \int_t^0 \big\langle G_s^i(u_s^{(1)})\partial_i u_s, \psi\big\rangle\,\mathrm{d}s + \int_t^0 \langle H_s u_s, \psi\rangle\,\mathrm{d}s,
$$

where

$$H_s(x) := \left(\int_0^1 \nabla_u G_s^i\big(x, u_s^{(1)}(x) + \theta u_s(x)\big)\,\mathrm{d}\theta\right)\partial_i u_s^{(2)}(x) \\ + \int_0^1 \nabla_u F_s\big(x, u_s^{(1)}(x) + \theta u_s(x)\big)\,\mathrm{d}\theta.$$

By (46), it is easy to verify that

$$(s,x) \mapsto G_s\big(x, u_s^{(1)}(x)\big) \in \mathcal{B}_{\mathrm{loc}}((T,0]; \mathbb{W}^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$$

and

$$(s,x) \mapsto H_s(x) \in \mathcal{B}_{\mathrm{loc}}((T,0]; \mathcal{B}(\mathbb{R}^d;\mathbb{R}^k)).$$

Thus, by Theorem 4.5, we conclude that $u_t(x) = 0$. □

4.2. *A special form*: $F_t(x,u) = f_t(x)$ *independent of* u. Consider the following SFDE:

$$X_{t,s}(x) = x + \int_t^s G_r\left(X_{t,r}(x), \mathbb{E}^{\mathscr{F}_r}\left(\varphi(X_{t,0}(x)) - \int_r^0 f_{r'}(X_{t,r'}(x))\,\mathrm{d}r'\right)\right)\mathrm{d}r \\ + \int_t^s \mathrm{d}L_r, \tag{59}$$

where $G : \mathbb{R}_- \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^d$, $f : \mathbb{R}_- \times \mathbb{R}^d \to \mathbb{R}^k$ and $\varphi : \mathbb{R}^d \to \mathbb{R}^k$ are bounded measurable functions.

We need the following continuous dependence of the solutions with respect to the coefficients.

PROPOSITION 4.7. *Suppose that* $(G^{(i)}, f^{(i)}, \varphi^{(i)})$, $i = 1,2$ *are two groups of bounded measurable functions, and for some* $K > 0$ *and all* $t \in \mathbb{R}_-$, $x, x' \in \mathbb{R}^d$ *and* $u, u' \in \mathbb{R}^k$,

$$\big|G_t^{(i)}(x,u) - G_t^{(i)}(x',u')\big| + \big|f_t^{(i)}(x) - f_t^{(i)}(x')\big| + \big|\varphi^{(i)}(x) - \varphi^{(i)}(x')\big| \\ \le K(|x-x'| + |u-u'|).$$

Then there exists a time $T < 0$ *depending only on* K *such that for all* $t \in [T,0]$ *and* $x, y \in \mathbb{R}^d$,

$$\sup_{s\in[t,0]} \mathbb{E}\big|X_{t,s}^{(1)}(x) - X_{t,s}^{(2)}(y)\big| \\ \le 2|x-y| + 2\int_t^0 \|G_r^{(1)} - G_r^{(2)}\|_\infty\,\mathrm{d}r \\ + 2K|T|\left(\|\varphi^{(1)} - \varphi^{(2)}\|_\infty + \int_t^0 \|f_r^{(1)} - f_r^{(2)}\|_\infty\,\mathrm{d}r\right), \tag{60}$$

where $X_{t,s}^{(i)}(x)$ *is the solution of* (59) *corresponding to* $(G^{(i)}, f^{(i)}, \varphi^{(i)})$.

PROOF. Set

$$Z_{t,s} := X^{(1)}_{t,s}(x) - X^{(2)}_{t,s}(y).$$

By (59) and the assumptions, we have

$$\begin{aligned}
\mathbb{E}|Z_{t,s}| \le |x-y| &+ \int_t^s \|G^{(1)}_r - G^{(2)}_r\|_\infty \,\mathrm{d}r \\
&+ K\int_t^s \Big(\mathbb{E}|Z_{t,r}| + \|\varphi^{(1)} - \varphi^{(2)}\|_\infty + K\mathbb{E}|Z_{t,0}| \\
&\qquad + \int_r^0 \|f^{(1)}_{r'} - f^{(2)}_{r'}\|_\infty \,\mathrm{d}r' + K\int_r^0 \mathbb{E}|Z_{t,r'}|\,\mathrm{d}r'\Big)\,\mathrm{d}r \\
\le |x-y| &+ \int_t^0 \|G^{(1)}_r - G^{(2)}_r\|_\infty \,\mathrm{d}r \\
&+ K|t|\Big(\|\varphi^{(1)} - \varphi^{(2)}\|_\infty + \int_t^0 \|f^{(1)}_r - f^{(2)}_r\|_\infty \,\mathrm{d}r\Big) \\
&+ K\int_t^s \mathbb{E}|Z_{t,r}|\,\mathrm{d}r + K^2|t|\Big(\mathbb{E}|Z_{t,0}| + \int_t^0 \mathbb{E}|Z_{t,r}|\,\mathrm{d}r\Big) \\
\le |x-y| &+ \int_t^0 \|G^{(1)}_r - G^{(2)}_r\|_\infty \,\mathrm{d}r \\
&+ K|t|\Big(\|\varphi^{(1)} - \varphi^{(2)}\|_\infty + \int_t^0 \|f^{(1)}_r - f^{(2)}_r\|_\infty \,\mathrm{d}r\Big) \\
&+ (K|t| + K^2|t| + K^2|t|^2) \sup_{r\in[t,0]} \mathbb{E}|Z_{t,r}|.
\end{aligned}$$

From this, we immediately conclude the proof. □

THEOREM 4.8. *Assume that (G, f, φ) are bounded measurable functions and satisfy for some $K > 0$ and all $t \in \mathbb{R}_-$, $x, x' \in \mathbb{R}^d$ and $u, u' \in \mathbb{R}^k$,*

$$|G_t(x,u) - G_t(x',u')| + |f_t(x) - f_t(x')| + |\varphi(x) - \varphi(x')| \le K(|x-x'| + |u-u'|).$$

Then there exists a time $T = T(K) < 0$ such that

$$u_t(x) := \mathbb{E}\varphi(X_{t,0}(x)) + \mathbb{E}\Big(\int_t^0 f_r(X_{t,r}(x))\,\mathrm{d}r\Big) \tag{61}$$

is a unique weak solution of equation (45) on $[T, 0]$ in the sense of Definition 4.1.

PROOF. Let $(G^\varepsilon, f^\varepsilon, \varphi^\varepsilon)$ be the smooth approximation of (G, f, φ) defined by

$$G^\varepsilon_t(x,u) := G * \rho^{(1)}_\varepsilon(t,x,u), \qquad f^\varepsilon_t(x) := f * \rho^{(1)}_\varepsilon(t,x),$$
$$\varphi^\varepsilon(x) = \varphi * \rho^{(1)}_\varepsilon(x),$$

where $\rho_\varepsilon^{(1)}$ [resp., $\rho_\varepsilon^{(2)}$ and $\rho_\varepsilon^{(3)}$] are the mollifiers in $\mathbb{R}^{d+k+1}$ (resp., $\mathbb{R}^{d+1}$ and $\mathbb{R}^d$). It is clear that

$$\|\nabla G_t^\varepsilon\|_\infty + \|\nabla f_t^\varepsilon\|_\infty + \|\nabla \varphi^\varepsilon\|_\infty \le K.$$

By Theorem 2.7 and Proposition 4.7, there exists a time $T = T(K)$ such that for all $T \le t \le s \le 0$ and $x \in \mathbb{R}^d$,

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}|X_{t,s}^\varepsilon(x) - X_{t,s}(x)| = 0,$$

where $X_{t,s}^\varepsilon$ (resp., $X_{t,s}$) is the solution family of SFDE (59) corresponding to the coefficients $(G^\varepsilon, f^\varepsilon, \varphi^\varepsilon)$ [resp., (G, f, φ)]. Using this limit, and by the dominated convergence theorem, it is easy to verify that for each $(t,x) \in [T,0] \times \mathbb{R}^d$,

$$u_t^\varepsilon(x) \to u_t(x), \tag{62}$$

where $u_t^\varepsilon(x)$ is defined through φ^ε, f^ε and $X_{t,s}^\varepsilon(x)$ as in (61). Moreover, by Proposition 4.7, we also have

$$\sup_{\varepsilon \in (0,1)} \sup_{t \in [T,0]} \|\nabla u_t^\varepsilon\|_\infty + \sup_{t \in [T,0]} \|\nabla u_t\|_\infty \le C_{T,K} < +\infty. \tag{63}$$

On the other hand, thanks to (63), by Theorem 3.4, there exists another time $T' = T'(K) \in [T, 0)$ independent of ε such that

$$u_t^\varepsilon(x) = \varphi^\varepsilon(x) + \int_t^0 \mathcal{L}_0 u_r^\varepsilon(x)\,\mathrm{d}r + \int_t^0 G_r^i(x, u_r^\varepsilon(x))\partial_i u_r^\varepsilon(x)\,\mathrm{d}r + \int_t^0 f_r^\varepsilon(x)\,\mathrm{d}r.$$

In particular, for all $\psi \in C_0^\infty(\mathbb{R}^d)$ and all $t \in [T', 0]$,

$$\begin{aligned}\langle u_t^\varepsilon, \psi\rangle &= \langle \varphi^\varepsilon, \psi\rangle + \int_t^0 \langle u_r^\varepsilon, \mathcal{L}_0^* \psi\rangle\,\mathrm{d}r + \int_t^0 \langle G_r^i(u_r^\varepsilon)\partial_i u_r^\varepsilon, \psi\rangle\,\mathrm{d}r \\ &\quad + \int_t^0 \langle f_r^\varepsilon, \psi\rangle\,\mathrm{d}r.\end{aligned} \tag{64}$$

We want to take limits for both sides of the above identity by (62). The key point is to prove

$$\int_t^0 \langle G_r^{\varepsilon,i}(u_r^\varepsilon)\partial_i u_r^\varepsilon, \psi\rangle\,\mathrm{d}r \to \int_t^0 \langle G_r^i(u_r)\partial_i u_r, \psi\rangle\,\mathrm{d}r,$$

which will be obtained by proving the following two limits:

$$\int_t^0 \langle \big(G_r^{\varepsilon,i}(u_r^\varepsilon) - G_r^i(u_r)\big)\partial_i u_r^\varepsilon, \psi\rangle\,\mathrm{d}r \to 0, \qquad \varepsilon \to 0,$$

$$\int_t^0 \langle G_r^i(u_r)\partial_i(u_r^\varepsilon - u_r), \psi\rangle\,\mathrm{d}r \to 0, \qquad \varepsilon \to 0.$$

The first limit is clear by (62), (63) and the dominated convergence theorem. The second limit follows by (62), (63) and the integration by parts formula. $\square$

Now we are in a position to give:

4.3. *Proof of Theorem* 4.2. We divide the proof into three steps.
(*Step* 1). For $h \in \mathcal{B}(\mathbb{R}_-; \mathbb{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^k))$, define

$$f_r^h(x) := F_r(x, h_r(x))$$

and

$$\mathscr{K} := \sup_{s \in \mathbb{R}_-} (\|\nabla G_s\|_\infty + \|\nabla F_s\|_\infty) + \|\nabla \varphi\|_\infty.$$

In this step, we prove the following claim:

CLAIM. *For given* $U \geq 4\|\nabla\varphi\|_\infty$, *there exists a time* $T = T(\mathscr{K}, U) < 0$ *such that for any bounded measurable function* $h : \mathbb{R}_- \times \mathbb{R}^d \to \mathbb{R}^m$ *satisfying* $\sup_{t \in [T,0]} \|\nabla h_t\|_\infty \leq U$, *it holds that*

$$\|u_t^h\|_\infty \leq \|\varphi\|_\infty + |t| \sup_{s \in [t,0]} \|f_s^h\|_\infty, \tag{65}$$

and

$$\sup_{t \in [T,0]} \|\nabla u_t^h\|_\infty \leq U, \tag{66}$$

where $u_t^h(x)$ *is defined by* (61) *in terms of* φ, f^h *and* $X_{t,s}^h(x)$, *and* $\{X_{t,s}^h(x), T \leq t \leq s \leq 0, x \in \mathbb{R}^d\}$ *is the unique solution family of SFDE* (59) *corresponding to* (G, f^h, φ).

PROOF OF THE CLAIM. By Proposition 4.7, there exists a time $T_1 := T_1(\mathscr{K}, U) < 0$ such that for all $x, y \in \mathbb{R}^d$,

$$\sup_{T_1 \leq t \leq s \leq 0} \mathbb{E}|X_{t,s}(x) - X_{t,s}(y)| \leq 2|x - y|.$$

Using this and by the definition of $u_t^h(x)$ [see (61)], we have

$$|u_t^h(x) - u_t^h(y)| \leq 2\|\nabla\varphi\|_\infty |x - y| + 2\int_t^0 (\|\nabla_x F_r\|_\infty + \|\nabla_u F_r\|_\infty U)|x - y|\, \mathrm{d}r.$$

So,

$$\begin{aligned} \sup_{s \in [t,0]} \|\nabla u_s^h\|_\infty &\leq 2\|\nabla\varphi\|_\infty + 2|t| \sup_{s \in [t,0]} (\|\nabla_x F_s\|_\infty + \|\nabla_u F_s\|_\infty U) \\ &\leq 2\|\nabla\varphi\|_\infty + 2|t|\mathscr{K}(U + 1). \end{aligned}$$

Since $U \geq 4\|\nabla\varphi\|_\infty$, choosing $T = \frac{U - 2\|\nabla\varphi\|_\infty}{2\mathscr{K}(U+1)} \wedge T_1$, we obtain (66). Estimate (65) follows from definition (61). $\square$

(*Step* 2). Set $u_t^0(x) := \varphi(x)$. We construct the following iteration approximation sequence: for $n \in \mathbb{N}$,

$$X_{t,s}^n(x) := X_{t,s}^{u^{n-1}}(x), \qquad u_t^n(x) := u_t^{u^{n-1}}(x),$$
$$f_t^n(x) := f_t^{u^{n-1}}(x) := F_t(x, u_t^{n-1}(x)).$$

By the above claim, there exists a time $T_1 = T_1(\mathscr{K}) < 0$ such that for all $n \in \mathbb{N}$,

$$\|u_t^n\|_\infty \le \|\varphi\|_\infty + |t| \sup_{s\in[t,0]} \|F_s\|_\infty, \qquad \sup_{t\in[T_1,0]} \|\nabla u_t^n\|_\infty \le 4\|\nabla\varphi\|_\infty. \tag{67}$$

Hence,

$$\|\nabla f_t^n\|_\infty \le \|\nabla_x F_t\|_\infty + \|\nabla_u F_t\|_\infty \|\nabla u_t^{n-1}\|_\infty \le \|\nabla_x F_t\|_\infty + 4\|\nabla_u F_t\|_\infty \|\nabla\varphi\|_\infty.$$

Thus, by the definition of $u_t^n(x)$ [see (61)] and Proposition 4.7 again, there exists another time $T = T(\mathscr{K}) \in [T_1, 0)$ such that for all $n, m \in \mathbb{N}$ and $t \in [T, 0)$,

$$\begin{aligned}\|u_t^n - u_t^m\|_\infty &\le \|\nabla\varphi\|_\infty \sup_{x\in\mathbb{R}^d} \mathbb{E}|X_{t,0}^n(x) - X_{t,0}^m(x)| + \int_t^0 \|f_r^n - f_r^m\|_\infty \,\mathrm{d}r \\ &\quad + \int_t^0 \|\nabla f_r^n\|_\infty \sup_{x\in\mathbb{R}^d} \mathbb{E}|X_{t,r}^n(x) - X_{t,r}^m(x)| \,\mathrm{d}r \\ &\le C\int_t^0 \|f_r^n - f_r^m\|_\infty \,\mathrm{d}r \le C\int_t^0 \|u_r^{n-1} - u_r^{m-1}\|_\infty \,\mathrm{d}r,\end{aligned}$$

where C is independent of n, m. By Gronwall's inequality, we obtain that

$$\lim_{n,m\to\infty} \sup_{t\in[T,0]} \|u_t^n - u_t^m\|_\infty = 0.$$

Hence, there exists a $u_t \in \mathcal{B}([T,0] \times \mathbb{R}^d; \mathbb{R}^k)$ such that

$$\lim_{n\to\infty} \sup_{t\in[T,0]} \|u_t^n - u_t\|_\infty = 0, \tag{68}$$

and by (67),

$$\|u_t\|_\infty \le \|\varphi\|_\infty + |t| \sup_{s\in[t,0]} \|F_s\|_\infty, \qquad \sup_{t\in[T,0]} \|\nabla u_t\|_\infty \le 4\|\nabla\varphi\|_\infty.$$

On the other hand, by Theorem 4.8, $u_t^n(x)$ satisfies that for all $\psi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^k)$,

$$\langle u_t^n, \psi\rangle = \langle\varphi, \psi\rangle + \int_t^0 \langle u_s^n, \mathcal{L}_0^*\psi\rangle \,\mathrm{d}r + \int_t^0 \langle G_s^i(u_s^n)\partial_i u_s^n + F_s(u_s^{n-1}), \psi\rangle \,\mathrm{d}s.$$

Thus, one can take limits as in Theorem 4.8 to obtain the existence of a short time weak solution for equation (45). Moreover, (ii) follows from (61). The existence of a maximal weak solution can be obtained as in the proof of Theorem 2.12 by shifting the time and the induction. Thus, we conclude the proof of (i). As for (ii), it follows by (68) and the definition of $u_t^n(x)$.

(*Step* 3). Let $u \in \mathcal{B}_{\mathrm{loc}}((T,0]; \mathbb{W}^{1,\infty}(\mathbb{R}^d;\mathbb{R}^k))$ be a maximal weak solution of equation (45). Define for $(t,x) \in (T,0] \times \mathbb{R}^d$,

$$b_t(x) := G_t(x, u_t(x)), \qquad f_t(x) := F_t(x, u_t(x)).$$

Then it is clear that

$$b \in \mathcal{B}_{\mathrm{loc}}((T,0]; \mathbb{W}^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)), \qquad f \in \mathcal{B}_{\mathrm{loc}}((T,0]; \mathbb{W}^{1,\infty}(\mathbb{R}^d;\mathbb{R}^k)).$$

For $t \in (T,0]$, let $\{X_{t,s}(x), t \le s \le 0, x \in \mathbb{R}^d\}$ solve the following SDE:

$$X_{t,s}(x) = x + \int_t^s b_r(X_{t,r}(x))\,\mathrm{d}r + \int_t^s \mathrm{d}L_r, \qquad s \in [t,0].$$

Define

$$\tilde{u}_t(x) := \mathbb{E}(\varphi(X_{t,0}(x))) + \int_t^0 \mathbb{E}(f_s(X_{t,s}(x)))\,\mathrm{d}s. \tag{69}$$

By Theorem 4.5, we have

$$\tilde{u}_t(x) = u_t(x) \qquad \forall (t,x) \in (T,0] \times \mathbb{R}^d.$$

Suppose now that $T > -\infty$. For completing the proof, by (47) it is enough to show that

$$\lim_{t \downarrow T} \|\nabla \tilde{u}_t(x)\|_\infty < +\infty.$$

It immediately follows from (69) and the following claim proved in [18], Theorem 4.5, which is stated in a slight variant.

CLAIM. *Under* (50) *or A nondegenerate*, *for any bounded continuous function φ and $T < t < s \le 0$*,

$$\|\nabla \mathbb{E}\varphi(X_{t,s}(\cdot))\|_\infty \le C_1(|t-s| \wedge 1)^{-1/\alpha}\|\varphi\|_\infty,$$

where C_1 only depends on d, α, T and the bound of b.

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