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MULTI-POINT GREEN'S FUNCTIONS FOR SLE AND AN ESTIMATE OF BEFFARA

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In this paper we define and prove of the existence of the multi-point Green's function for SLE—a normalized limit of the probability that an SLE_κ curve passes near to a pair of marked points in the interior of a domain. When $\kappa < 8$ this probability is nontrivial, and an expression can be written in terms two-sided radial SLE. One of the main components to our proof is a refinement of a bound first provided by Beffara [*Ann. Probab.* **36** (2008) 1421–1452]. This work contains a proof of this bound independent from the original.

1. Introduction. The Schramm–Loewner evolution (SLE) is a random process first introduced by Oded Schramm in [12] as a candidate for scaling limits of models from statistical physics which are believed to be conformally invariant. Since its introduction, SLE has been rigorously established as the scaling limit for a number of these processes, including the loop-erased random walk [10], the percolation exploration process [14] and the interface of the Gaussian free field [13]. For a general introduction to SLE see, for example, [5, 9, 15].

Chordal SLE_κ for $\kappa > 0$ in the upper half-plane ($\mathbb{H}$) is a one-parameter family of noncrossing random curves $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(\infty^-) = \infty$. Depending on κ, the geometry of the curve has several different phases. When $0 < \kappa \leq 4$, the curves are simple (no self intersections). When $\kappa > 4$, the curves are no longer simple, but they remain noncrossing. When $\kappa \geq 8$, the curve is space filling, passing through every point in $\overline{\mathbb{H}}$.

When examining geometric questions about the SLE curves, such as the almost sure Hausdorff dimension in [3], it is often useful to be able to provide estimates on the probability that the process $\gamma(t)$ passes near a series of marked points in $\mathbb{H}$. However, the non-Markovian nature of this process makes estimating such probabilities difficult.

When trying to understand the probability that SLE_κ gets near to some point $z \in \mathbb{H}$ it is convenient to consider the conformal radius of z in $H_t := \mathbb{H} \setminus \gamma(0, t]$, which we denote by $\Upsilon_t(z)$, instead of the Euclidean distance from z to $\gamma(0, t]$; see Section 2.1 for the definition. This change does little to the geometry of the

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problem being considered since the conformal radius differs from the Euclidean distance by at most a universal multiplicative constant.

The Green's function for SLE_κ from 0 to ∞ in $\mathbb{H}$ for $\kappa < 8$ is a form of the normalized probability of passing near to a point in $\mathbb{H}$. It is defined by

$$\lim_{\varepsilon \to 0} \varepsilon^{d-2} \mathbb{P}\{\Upsilon_\infty(z) < \varepsilon\} = c_* G_{\mathbb{H}}(z; 0, \infty),$$

where $d := 1 + \kappa/8$ is the Hausdorff dimension of the SLE_κ, and c_* is some known constant depending on κ. The Green's function was first computed in [11] (although they neither used this name nor definition), and the exact formula found there is given in Section 2.1. The existence of the limit requires some argument, and a form of it is proven in Lemma 2.10.

We wish to show analogously that

$$\lim_{\varepsilon, \delta \to 0} \varepsilon^{d-2} \delta^{d-2} \mathbb{P}\{\Upsilon_\infty(z) < \varepsilon; \Upsilon_\infty(w) < \delta\}$$

exists and can be written as

$$c_*^2 G_{\mathbb{H}}(z; 0, \infty) \mathbb{E}_z^*[G_{H_{T_z}}(w; z, \infty)] + c_*^2 G_{\mathbb{H}}(w; 0, \infty) \mathbb{E}_w^*[G_{H_{T_w}}(z; w, \infty)],$$

where $\mathbb{E}_z^*$ is the expectation of a particular form of SLE called *two-sided radial SLE*, which can be understood as chordal SLE conditioned to pass though the point z, and $G_{H_{T_z}}$ is the Green's function for SLE in the domain remaining at the time it does so. The form of the limit as the sum of two similar terms comes from the two possible orders that the curve can pass through z and w, and each term individually can be thought of as an ordered Green's function.

To prove this result, we will use techniques similar to those used in [3], where Beffara (in slightly different notation) established the estimate that there exists some $c > 0$ such that for any two points $z, w \in \mathbb{H}$ with $\mathrm{Im}(z), \mathrm{Im}(w) \geq 1$

$$\mathbb{P}\{\Upsilon_\infty(z) < \varepsilon; \Upsilon_\infty(w) < \varepsilon\} < c\varepsilon^{2(2-d)} |z - w|^{d-2}.$$

Similar techniques arise since both proofs need to make rigorous the heuristic that an SLE curve conditioned to pass through z and then w will do so directly—without first approaching very near w before passing through z. Figure 1 demonstrates some of the issues which can occur which make this a tricky statement to make rigorous.

In the process of proving the existence of the multi-point Green's function for SLE, we also obtain an independent proof of a mild generalization of Beffara's estimate—that there exists a $c > 0$ such that for any $z, w \in \mathbb{H}$ with $\mathrm{Im}(z)$, $\mathrm{Im}(w) \geq 1$

$$\mathbb{P}\{\Upsilon_\infty(z) < \varepsilon; \Upsilon_\infty(w) < \delta\} < c\varepsilon^{2-d} \delta^{2-d} |z - w|^{d-2}.$$

While it may be possible to derive some of the lemmas we require directly from the proof in [3], we include a complete proof of them, along with Beffara's original estimate, so that the proof of our main result is completely self-contained.

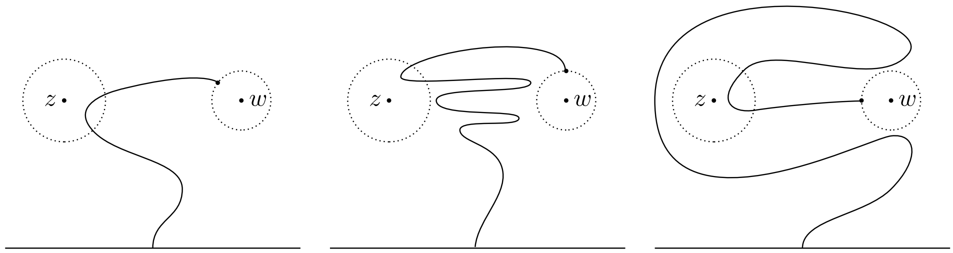

FIG. 1. *We wish to show that curves that get near z then near w concentrate on curves like those in the left image. Estimating the probability of such curves is easy by repeated application of the Green's function. However, such simple estimation gives the same order of magnitude to curves like those in the center image. This issue can be overcome as long as getting near to w before z decreases the probability that the SLE gets even closer to w later on. This is often the case; however, the right image shows an example where it is not. In this case, once the curve gets near to z, it is essentially guaranteed to pass near w. Controlling for these issues forms the bulk of this work.*

It is worth noting that Beffara's estimate itself immediately yields an upper bound on the multi-point Green's function. For a lower bound, and an application of the multi-point Green's function to the proof of the existence of the "natural parametrization" of SLE, a parametrization of SLE by what can be thought of as a form d-dimensional arc length; see [8].

The paper is structured as follows. Section 2.1 begins by establishing the notation used throughout the paper, and to provide a few simple deterministic and random bounds required in the proofs that follow. Section 2.4 then gives a brief introduction to two-sided radial SLE and collects the facts about this process that we require to show the existence of the multi-point Green's function. Section 3 provides a proof of the existence of the multi-point Green's function assuming an estimate derived from our proof of Beffara's estimate. The rest of the paper is dedicated to our independent proof of Beffara's estimate. To aid in the presentation of this proof, we have separated the bounds required by the type of argument required: topological lemmas, probabilistic lemmas and combinatorial lemmas. The proof of one of the topological lemmas is left to the Appendix as the result is intuitive and the formal proof of it does little to aid the understanding of our main results.

In this paper we fix $\kappa < 8$ and constants implicitly depend on κ.

2. Preliminaries.

2.1. *Notation.* We set

$$a = \frac{2}{\kappa}, \qquad d = 1 + \frac{\kappa}{8} = 1 + \frac{1}{4a},$$
$$\beta = \frac{8}{\kappa} - 1 = 4a - 1 > 0.$$

The Green's function for chordal SLE$_\kappa$ (from 0 to ∞ in $\mathbb{H}$) is

$$G(x+iy) = G(re^{i\theta}) = r^{d-2}\sin^{4a+1/(4a)-2}\theta = y^{d-2}\sin^{\beta}\theta.$$

The Green's function can be defined for other simply connected domains as we now demonstrate. If D is a simply connected domain, z_1, z_2 are distinct boundary points, let $\Phi_D : D \to \mathbb{H}$ be a conformal transformation with $\Phi_D(z_1) = 0$, $\Phi_D(z_2) = \infty$. This is unique up to a final dilation. If $w \in D$, we define

$$S_D(w; z_1, z_2) = \sin\arg\Phi_D(w),$$

which is independent of the choice of Φ_D and gives a conformal invariant. We let $\Upsilon_D(w)$ be (twice the) conformal radius of w in D; that is, if $f : \mathbb{D} \to D$ is a conformal transformation with $f(0) = w$, then $\Upsilon_D(w) = 2|f'(0)|$. This satisfies the scaling rule

$$\Upsilon_{f(D)}(f(w)) = |f'(w)|\Upsilon_D(w).$$

It is easy to check that $\Upsilon_{\mathbb{H}}(x+iy) = y$, and, more generally,

$$\Upsilon_D(w) = \frac{\operatorname{Im}(\Phi_D(w))}{|\Phi_D'(w)|}.$$

The Green's function for SLE$_\kappa$ from z_1 to z_2 in D is defined by

$$G_D(w; z_1, z_2) = \Upsilon_D(w)^{d-2} S(w; z_1, z_2)^{\beta}.$$

It satisfies the scaling rule

$$G_D(w; z_1, z_2) = |f'(w)|^{2-d} G_{f(D)}(f(w); f(z_1), f(z_2)).$$

For a proof that the Green's function so defined satisfies the limit claimed in the Introduction; see Lemma 2.10.

Let $\operatorname{inrad}_D(w) = \operatorname{dist}(w, \partial D)$ denote the inradius. Using the Koebe (1/4)-theorem, we know that

$$\frac{1}{2}\operatorname{inrad}_D(w) \le \Upsilon_D(w) \le 2\operatorname{inrad}_D(w). \tag{1}$$

Therefore,

$$G_D(w; z_1, z_2) \asymp \operatorname{inrad}_D(w)^{d-2} S_D(w; z_1, z_2)^{\beta},$$

where we write $f_1 \asymp f_2$ if there exists some constant c such that $f_1 \le c f_2$ and $f_2 \le c f_1$. We write

$$\partial D = \partial_1 D \cup \partial_2 D \cup \{z_1, z_2\},$$

where $\partial_1 D, \partial_2 D$ denote the two open arcs of ∂D with endpoints z_1, z_2. Let $\hat{S}_D(w; z_1, z_2)$ be the minimum of the harmonic measures of $\partial_1 D, \partial_2 D$ from w. This is a conformal invariant, and a simple computation in $\mathbb{H}$ shows that

$$\hat{S}_D(w; z_1, z_2) = \frac{1}{\pi}\min\{\arg\Phi_D(w), \pi - \arg\Phi_D(w)\},$$

and hence

$$\hat{S}_D(w; z_1, z_2) \asymp S_D(w; z_1, z_2)$$

and

$$G_D(w; z_1, z_2) \asymp \mathrm{inrad}_D(w)^{d-2} \hat{S}_D(w; z_1, z_2)^{\beta}.$$

To bound the harmonic measure, it is often useful to use the Beurling estimate. We recall it here; for a proof see, for example, [2], Chapter V. Let B_t be a standard Brownian motion and τ_D denote the first exit time of some domain D for this Brownian motion.

PROPOSITION 2.1 (Beurling estimate). *There is a constant $c > 0$ such that if $z \in \mathbb{D}$, and K is a connected compact subset of $\overline{\mathbb{D}}$ with $0 \in K$ and $K \cap \partial\mathbb{D} \neq \varnothing$, then*

$$\mathbb{P}^z\{B[0, \tau_{\mathbb{D}}] \cap K = \varnothing\} \le c|z|^{1/2}.$$

We may derive from this the following consequence.

PROPOSITION 2.2. *There is a constant $c > 0$ such that if K is a connected compact subset of $\overline{\mathbb{H}}$ with $K \cap \mathbb{R} \neq \varnothing$, and $z_0 \in \mathbb{H}$, $\varepsilon > 0$ are such that $B_\varepsilon(z_0) \cap K \neq \varnothing$ then for $w \in \mathbb{H}$,*

$$\mathbb{P}^w\{B[0, \tau_{\mathbb{H}\setminus K}] \cap B_\varepsilon(z_0) \neq \varnothing\} \le c\left[\frac{\varepsilon}{|z_0 - w|}\right]^{1/2}.$$

PROOF. Consider the map

$$g(z) := \frac{\varepsilon}{z - z_0}, \qquad g : \mathbb{C} \setminus B_\varepsilon(z_0) \to \mathbb{D}.$$

Let $K' = g([\mathbb{C} \setminus \mathbb{H}] \cup [K \setminus B_\varepsilon(z_0)])$, and note that K' is a connected compact subset of $\mathbb{D}$ with $0 \in K'$ and $K' \cap \partial\mathbb{D} \neq \varnothing$. Thus by Proposition 2.1 we know

$$\mathbb{P}^{g(w)}\{B[0, \tau_{\mathbb{D}}] \cap K' = \varnothing\} \le c|g(w)|^{1/2},$$

which, by the conformal invariance of Brownian motion and the definition of g, is the desired statement. $\square$

If $j = 1, 2$, let $\Delta_{D,j}(w; z_1, z_2)$ be the infimum of all s such that there exists a curve $\eta : [0, 1) \to D$ contained in the disk of radius s about w with $\eta(0) = w$, $\eta(1^-) \in \partial_j D$. Note that

$$\mathrm{inrad}_D(w) = \min\{\Delta_{D,1}(w; z_1, z_2), \Delta_{D,2}(w; z_1, z_2)\}.$$

We let

$$\Delta_D^*(w; z_1, z_2) = \max\{\Delta_{D,1}(w; z_1, z_2), \Delta_{D,2}(w; z_1, z_2)\}.$$

The Beurling estimate implies that there is a $c < \infty$ such that the probability a Brownian motion starting at w reaches distance $\Delta^*_D(w; z_1, z_2)$ before leaving D is bounded above by

$$c\left[\frac{\mathrm{inrad}_D(w)}{\Delta^*_D(w; z_1, z_2)}\right]^{1/2}.$$

Therefore,

$$S_D(w; z_1, z_2) \asymp \hat{S}_D(w; z_1, z_2) \le c\left[\frac{\mathrm{inrad}_D(w)}{\Delta^*_D(w; z_1, z_2)}\right]^{1/2}, \tag{2}$$

which gives us the upper bound

$$G_D(w; z_1, z_2) \le c\, \mathrm{inrad}_D(w)^{d-2+\beta/2} \Delta^*_D(w; z_1, z_2)^{-\beta/2}.$$

We will also need a fact which is a form of continuity of the Green's function under a small perturbation of the domain. First consider the following two lemmas on the conformal radius.

LEMMA 2.3. *Let $\mathcal{B}_r$ denote the closed disk of radius e^{-r} about the origin. Suppose D is a simply connected subdomain of $\mathbb{D}$ containing the origin and $e^{-r} <$* $\mathrm{inrad}_D(0)$. *Suppose B_t is a Brownian motion starting at the origin, and let*

$$\tau_D = \inf\{t : B_t \notin D\}, \qquad \tau_{\mathbb{D}} = \inf\{t : B_t \notin \mathbb{D}\}, \qquad \sigma_{r,D} = \inf\{t \ge \tau_D : B_t \in \mathcal{B}_r\}.$$

Then

$$\mathbb{P}\{\tau_D < \sigma_{r,D} < \tau_{\mathbb{D}}\} = -\frac{1}{r} \log[\Upsilon_D(0)/2].$$

PROOF. Let $f : D \to \mathbb{D}$ be the conformal transformation with $f'(0) > 0$. Since $\Upsilon_D(0)$ is twice the usual conformal radius, $-\log[\Upsilon_D(0)/2] = \log f'(0)$. Let $g(z) = \log[|f(z)|/|z|]$ which is a bounded harmonic function on D, and hence

$$\log f'(0) = g(0) = \mathbb{E}[g(B_\tau)] = -\mathbb{E}[\log|B_\tau|].$$

For $e^{-r} \le |w| < 1$, $-\log|w|/r$ is the probability that a Brownian motion starting at w hits $\mathcal{B}$ about the origin before leaving the $\mathbb{D}$. Therefore,

$$\log f'(0) = r\mathbb{P}\{\tau_D < \sigma_{r,D} < \tau_{\mathbb{D}}\}. \qquad \square$$

LEMMA 2.4. *There exists a $c > 0$ such that for any two simply connected domains $D_1 \subseteq D_2$ and a point $w \in D_1 \cap D_2$, then*

$$0 \le \Upsilon_{D_2}(w) - \Upsilon_{D_1}(w) \le c\, \mathrm{diam}(D_2 \setminus D_1).$$

PROOF. Without loss of generality, we may assume $\mathrm{inrad}(D_2) = 1$. If $\mathrm{inrad}(D_1) \leq 7/8$, then $\mathrm{diam}(D_2 \setminus D_1) \geq 1/8$, and we can use the estimate $\mathrm{inrad}(D) \asymp \Upsilon(D)$. If $\mathrm{inrad}(D_1) \geq 7/8$, then we can use the previous lemma, conformal invariance, and the Koebe (1/4)-theorem to see $\Upsilon_{D_2}(w) - \Upsilon_{D_1}(w)$ is comparable to the probability that a Brownian motion starting at w hits $D_2 \setminus D_1$ and returns to $\mathcal{B} = B_{1/16}(w)$, the disk of radius $1/16$ about w without leaving D_2. Using the Beurling estimate, we see the probability of hitting $D_2 \setminus D_1$ is bounded above by $c\,\mathrm{diam}(D_2 \setminus D_1)^{1/2}$ and using it again the probability of getting back to $\mathcal{B}$ before leaving D_2 is bounded by $c\,\mathrm{diam}(D_2 \setminus D_1)^{1/2}$. □

We will need some notion of closeness of two nested domains before we can state our lemma. Although the following definitions are very general, we will use them only in the case where the domains are the complements of a single curve considered up to two different times.

DEFINITION. Given two nested simply connected domains $D_1 \subseteq D_2 \subseteq \mathbb{H}$ with marked boundary points $z_1 \in \partial D_1$ and $z_2 \in \partial D_2$, we say (D_1, z_1) and (D_2, z_2) are *R-close near z* if the following holds. Let $B_R^{(i)}(z)$ denote the connected component of $B_R(z) \cap D_i$ which contains z. Then:

- $z_1 \in \partial B_R^{(1)}(z)$,
- $z_2 \in \partial B_R^{(2)}(z)$ and
- $D_2 \setminus D_1 \subseteq B_R(z)$.

LEMMA 2.5. *There exists $c > 0$ such that the following holds. Suppose $z, w \in \mathbb{H}$, $D_1 \subseteq D_2 \subseteq \mathbb{H}$ are simply connected domains, and $z_1 \in \partial D_1$, $z_2 \in \partial D_2$. If:*

- $z, w \in D_1 \cap D_2$,
- (D_1, z_1) *and* (D_2, z_2) *are R-close near z for* $R \leq \mathrm{inrad}_{D_1}(w) \wedge \frac{1}{2}|z - w|$,
- $\infty \in \partial D_1 \cap \partial D_2$,

then

$$|G_{D_1}(w; z_1, \infty) - G_{D_2}(w; z_2, \infty)| \leq c\,\mathrm{inrad}_{D_1}(w)^{d-2-(\beta\wedge 1)/2} R^{(\beta\wedge 1)/2}.$$

One need not fix the point z in the beginning of this lemma by simply making the second bullet point of this lemma say that there exists some z so that the domains are R-close near z; however we write it in this form since we will always use this lemma with a fixed z and w already in mind.

PROOF OF LEMMA 2.5. Recall that

$$G_D(w; z_1, z_2) = \Upsilon_D(w)^{d-2} S_D(w; z_1, z_2)^{\beta},$$

where $S(w; z_1, z_2)$ is the sine of the argument of w after applying the unique (up to scaling) conformal map, Φ_D, that sends D to $\mathbb{H}$ while sending z_1 to 0 and z_2 to ∞. Writing, as before,

$$\partial D = \partial_1 D \cup \{z_1\} \cup \partial_2 D \cup \{z_2\},$$

where the union is written in counter-clockwise order, this argument is conformally invariant and can be computed by

$$\arg \Phi_D(w) = \pi \cdot \mathbb{P}^w\{B_\tau \in \partial_2 D\} \qquad \text{where } \tau = \inf\{t : B_t \in \partial D\},$$

where $\mathbb{P}^w$ is the probability for a standard Brownian motion started at w.

Consider our case. Write

$$\partial D_1 = \partial_1 D_1 \cup \{z_1\} \cup \partial_2 D_1 \cup \{\infty\} \quad \text{and} \quad \partial D_2 = \partial_1 D_2 \cup \{z_2\} \cup \partial_2 D_2 \cup \{\infty\}$$

again with the union written in counter-clockwise order. Note that the condition that (D_1, z_1) and (D_2, z_2) are R-close near z implies that

$$\partial_1 D_1 \setminus B_R(z) = \partial_1 D_2 \setminus B_R(z) \quad \text{and} \quad \partial_2 D_1 \setminus B_R(z) = \partial_2 D_2 \setminus B_R(z). \tag{3}$$

Define

$$\tau_1 = \inf\{t : B_t \in \partial D_1\} \quad \text{and} \quad \tau_2 = \inf\{t : B_t \in \partial D_2\}$$

and note that, since $B_0 = w$, $\tau_1 \le \tau_2$.

We may write that

$$\begin{aligned} |\arg \Phi_{D_1}(w) - \arg \Phi_{D_2}(w)| &= |\pi \cdot \mathbb{P}^w\{B_{\tau_1} \in \partial_2 D_1\} - \pi \cdot \mathbb{P}^w\{B_{\tau_2} \in \partial_2 D_2\}| \\ &\le 2\pi \cdot \mathbb{P}^w\{B_t \in B_R(z) \text{ for some } t \le \tau_2\}, \end{aligned}$$

where the last line follows since, if considered path-wise, the Brownian motion must enter $B_R(z)$ if it is to hit a different side of the boundary in D_1 versus D_2 by (3). By the Beurling estimate (Proposition 2.2),

$$|\arg \Phi_{D_1}(w) - \arg \Phi_{D_2}(w)| \le c\left(\frac{R}{|z-w|}\right)^{1/2}.$$

By noting that $\operatorname{inrad}_{D_1}(w) \le c|z-w|$ by the choice of R and the definition of R-close, and splitting into the cases when $\beta \ge 1$ versus $\beta < 1$ we see

$$|S_{D_1}(w; z_1, \infty)^\beta - S_{D_2}(w; z_2, \infty)^\beta| \le c\left(\frac{R}{\operatorname{inrad}_{D_1}(w)}\right)^{(\beta\wedge 1)/2}.$$

Consider the term involving the conformal radius. By using Lemma 2.4 and recalling that $d - 2 < 0$ and $\Upsilon_{D_1}(w) \le \Upsilon_{D_2}(w)$, we see

$$\begin{aligned} |\Upsilon_{D_2}(w)^{d-2} - \Upsilon_{D_1}(w)^{d-2}| &\le (2-d)\Upsilon_{D_1}(w)^{d-3}|\Upsilon_{D_2}(w) - \Upsilon_{D_1}(w)| \\ &\le c\Upsilon_{D_1}(w)^{d-2}\left(\frac{R}{\operatorname{inrad}_{D_1}(w)}\right). \end{aligned}$$

Combining these, noting that $R < \operatorname{inrad}_{D_1}(w)$, gives

$$\begin{aligned}
&|G_{D_1}(w; z_1, \infty) - G_{D_2}(w; z_2, \infty)| \\
&\quad \le |\Upsilon_{D_1}(w)^{d-2} S_{D_1}(w; z_1, \infty)^\beta - \Upsilon_{D_1}(w)^{d-2} S_{D_2}(w; z_2, \infty)^\beta| \\
&\qquad + |\Upsilon_{D_1}(w)^{d-2} S_{D_2}(w; z_2, \infty)^\beta - \Upsilon_{D_2}(w)^{d-2} S_{D_2}(w; z_2, \infty)^\beta| \\
&\quad \le c\Upsilon_{D_1}(w)^{d-2}\left(\frac{R}{\operatorname{inrad}_{D_1}(w)}\right)^{(\beta\wedge 1)/2} + c\Upsilon_{D_1}(w)^{d-2}\left(\frac{R}{\operatorname{inrad}_{D_1}(w)}\right) \\
&\quad \le c \operatorname{inrad}_{D_1}(w)^{d-2-(\beta\wedge 1)/2} R^{(\beta\wedge 1)/2}
\end{aligned}$$

as desired. $\square$

2.2. *Schramm–Loewner evolution*. The chordal Schramm–Loewner evolution with parameter κ (from 0 to ∞ in $\mathbb{H}$ parametrized so that the half-plane capacity grows at rate $a = 2/\kappa$) is the random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ satisfying the following. Let H_t denote the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$, and let g_t be the unique conformal transformation of H_t onto $\mathbb{H}$ with $g_t(z) - z \to 0$ as $z \to \infty$. Then g_t satisfies the Loewner differential equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \qquad g_0(z) = z, \tag{4}$$

where $U_t = -B_t$ is a standard Brownian motion. For $z \in \overline{\mathbb{H}} \setminus \{0\}$, the solution of this initial value problem exists up to time $T_z \in (0, \infty]$.

Suppose $z \in \mathbb{H}$, and let

$$Z_t = Z_t(z) = X_t + iY_t = g_t(z) - U_t.$$

Then the Loewner differential equation becomes the SDE

$$dZ_t = \frac{a}{Z_t}\, dt + dB_t. \tag{5}$$

Let

$$\begin{aligned}
S_t = S_t(z) &= S_{H_t}(z; \gamma(t), \infty) = \sin\arg Z_t, \\
\Upsilon_t = \Upsilon_t(z) &= \Upsilon_{H_t}(z; \gamma(t), \infty) = \frac{Y_t}{|g_t'(z)|}, \\
M_t = M_t(z) &= G_{H_t}(z; \gamma(t), \infty) = \Upsilon_t^{d-2} S_t^\beta.
\end{aligned}$$

Either by direct computation or by using the Schwarz lemma, we can see that Υ_t decreases in t, and hence we can define $\Upsilon = \Upsilon_{T_z-}$. If $0 < \kappa \le 4$, the SLE paths are simple and with probability one $T_z = \infty$. If $4 < \kappa < 8$, $T_z < \infty$ and by (1) we know

$$\Upsilon \asymp \operatorname{dist}[z, \gamma(0, T_z] \cup \mathbb{R}] = \operatorname{dist}[z, \gamma(0, \infty) \cup \mathbb{R}]. \tag{6}$$

Using Itô's formula, we can see that M_t is a local martingale satisfying

$$dM_t = \frac{aX_t}{X_t^2 + Y_t^2} M_t \, dB_t.$$

We will need the following estimate for SLE; see [1] for a proof. By a *crosscut* in D we will mean a simple curve $\eta : (0, 1) \to D$ with $\eta(0^+), \eta(1^-) \in \partial D$. We call $\eta(0^+), \eta(1^-)$ the endpoints of the crosscut.

PROPOSITION 2.6. *There exists $c < \infty$ such that if η is a crosscut in $\mathbb{H}$ with $-\infty < \eta(1^-) \le \eta(0^+) = -1$, then the probability that an SLE$_\kappa$ curve from 0 to ∞ intersects η is bounded above by $c\,\mathrm{diam}(\eta)^\beta$ where $\beta = 4a - 1$ is as defined in Section 2.1.*

2.3. *Radial parametrization.* In order to prove the existence of multi-point Green's functions, we will need to study the behavior of the SLE curve from the perspective of $z \in \mathbb{H}$. To do so, it is useful to parametrize the curve so that the conformal radius seen from z decays deterministically. We fix $z \in \mathbb{H}$ and let

$$\sigma(t) = \inf\{s : \Upsilon_s = e^{-2at}\}.$$

Under this parametrization, the "starting time" is $-\log(\Upsilon_0)/2a$, and the total lifetime of the curve is $\log(\Upsilon_0/\Upsilon)/2a$. Let $\Theta_t = \arg Z_{\sigma(t)}(z)$, $\hat{S}_t = S_{\sigma(t)}(z) = \sin \Theta_t$. Using Itô's formula one can see that Θ_t satisfies

$$d\Theta_t = (1 - 2a) \cot \Theta_t \, dt + d\hat{W}_t,$$

where $\hat{W}_t$ is a standard Brownian motion. Since $a > 1/4$, comparison to a Bessel process shows that solutions to this process leave $(0, \pi)$ in finite time. This reflects that fact that chordal SLE$_\kappa$ does not reach z for $\kappa < 8$ and hence $\Upsilon > 0$. Let

$$\hat{M}_t = M_{\sigma(t)}(z) = e^{-2at(d-2)} \hat{S}_t^\beta = e^{-(2a-1/2)t} \hat{S}_t^\beta.$$

This is a time change of a local martingale and hence is a local martingale; indeed, Itô's formula gives

$$d\hat{M}_t = (4a - 1) \cot \Theta_t \, d\hat{W}_t.$$

Using Girsanov's theorem (see, e.g., [4]), we can define a new probability measure $\mathbb{P}^*$ which corresponds to paths "weighted locally by the local martingale $\hat{M}_t$." For the time being, we treat this as an arbitrary change of measure; however, in Section 2.4 we will see that is precisely the change of measure which gives two-sided radial SLE. Intuitively, $\hat{M}_t$ weights more heavily those paths whose continuations are likely to get much closer to z. For more examples of the application of Girsanov's theorem to the study of SLE, and a general outline of the way Girsanov's theorem is used below, see [6].

In this weighting,

$$d\hat{W}_t = (4a - 1) \cot \Theta_t \, dt + dW_t,$$

where W_t is a standard Brownian motion with respect to $\mathbb{P}^*$. In particular,

$$d\Theta_t = 2a \cot \Theta_t \, dt + dW_t. \tag{7}$$

Since $2a > 1/2$, we can see by comparison with a Bessel process that with respect to $\mathbb{P}^*$, the process stays in $(0, \pi)$ for all times. Using this we can show that $\hat{M}_t$ is actually a martingale, and the measure $\mathbb{P}^*$ can be defined by

$$\mathbb{P}^*[V] = \hat{M}_0^{-1} \mathbb{E}[\hat{M}_t 1_V] \qquad \text{for } V \in \mathcal{F}_t,$$

where $\mathcal{F}_t$ denotes the σ-algebra generated by $\{\hat{W}_s : 0 \le s \le t\}$. Much of the analysis of SLE$_\kappa$ as it gets close to z uses properties of the simple SDE (7). Recall that we assume that $a > 1/4$ and all constants can depend on a.

LEMMA 2.7. *There exists $c < \infty$ such that if Θ_t satisfies (7) with $\Theta_0 = x \in (0, \pi/2)$, then if $0 < y < 1$ and*

$$\tau = \inf\{t : \Theta_t \in \{y, \pi/2\}\},$$

then

$$\mathbb{P}^*\{\Theta_\tau = y\} \le c(y/x)^{1-4a}.$$

PROOF. Itô's formula shows that $F(\Theta_{t \wedge \tau})$ is a $\mathbb{P}^*$-martingale where

$$F(s) = \int_s^{\pi/2} (\sin u)^{-4a} \, du, \qquad \frac{F''(s)}{F'(s)} = -4a \cot s.$$

Note that $F(\pi/2) = 0$ and

$$F(s) \sim \frac{s^{1-4a}}{1-4a}, \qquad s \to 0^+.$$

The optional sampling theorem implies that

$$F(x) = \mathbb{P}^*\{\Theta_\tau = y\} F(y). \qquad \square$$

LEMMA 2.8. *The invariant density for the SDE (7) is*

$$f(x) = C_{4a} \sin^{4a} x, \qquad 0 < x < \pi, \qquad C_{4a} := \left[\int_0^\pi \sin^{4a} x\right]^{-1}. \tag{8}$$

PROOF. This can be quickly verified and is left to the reader. $\square$

One can use standard techniques for one-dimensional diffusions to discuss the rate of convergence to the equilibrium distribution. We will state the one result that we need; see [8] for more details. If F is a nonnegative function on $(0, \pi)$, let

$$I_F := C_{4a} \int_0^\pi F(x) \sin^{4a} x \, dx.$$

LEMMA 2.9. *There exists $u < \infty$ such that for every $t_0 > 0$ there exists $c < \infty$ such that if F is a nonnegative function with $I_F < \infty$ and $t \geq t_0$,*

$$|\mathbb{E}[F(\Theta_t)] - I_F| \leq ce^{-ut} I_F.$$

Note that this estimate applies uniformly over all starting points x.

An important case for us is $F(x) = [\sin x]^{-\beta} = \sin^{1-4a} x$. Let

$$c_* = I_F = \frac{C_{4a}}{C_1} = \frac{2}{\int_0^\pi \sin^{4a} x\, dx}. \tag{9}$$

We will take advantage of this uniform bound to give a concrete estimate on how well the Green's function approximates the probability of getting near a point.

LEMMA 2.10. *There exists $u > 0$ such that if D is a simply connected domain, and z_1, z_2 are points in its boundary, $r \leq 3/4$, γ is an SLE$_\kappa$ curve from z_1 to z_2, $w \in D$, and D_∞ denotes the connected component of $D \setminus \gamma(0, \infty)$ containing w, then*

$$\begin{aligned}\mathbb{P}\{\Upsilon_{D_\infty}(w) \leq r \cdot \Upsilon_D(w)\} &= c_* r^{2-d} S_D(w; z_1, z_2)^\beta [1 + O(r^u)] \\ &= c_* r^{2-d} \Upsilon_D(w)^{2-d} G_D(w; z_1, z_2)[1 + O(r^u)],\end{aligned}$$

where c_* is as defined in (9). In particular, there exists $c < \infty$ such that for all $r \leq 3/4$,

$$\mathbb{P}\{\Upsilon_{D_\infty}(w) \leq r \cdot \Upsilon_D(w)\} \leq cr^{2-d} S_D(w; z_1, z_2)^\beta.$$

PROOF. By conformal invariance we may assume $\Upsilon_D(w) = 1$ and define t by $r = e^{-2at}$. Let $\sigma = \inf\{s : \Upsilon_s = r\}$. Then,

$$\begin{aligned}\mathbb{P}\{\sigma < \infty\} &= \mathbb{E}[1\{\sigma < \infty\}] \\ &= r^{2-d} \mathbb{E}[\hat{M}_t \hat{S}_t^{-\beta}] \\ &= r^{2-d} S_D(w; z_1, z_2)^\beta \mathbb{E}^*[\hat{S}_t^{-\beta}] \\ &= c_* r^{2-d} S_D(w; z_1, z_2)^\beta [1 + O(e^{-ut})] \\ &= c_* r^{2-d} \Upsilon_D(w)^{2-d} G_D(w; z_1, z_2)[1 + O(e^{-ut})]. \qquad \square\end{aligned}$$

Using (1) and (2), we immediately get the following lemma which is in the form that we will use.

LEMMA 2.11. *There exists $C < \infty$, such that if D is a simply connected domain, and z_1, z_2 are points in its boundary, $r \leq 3/4$, and γ is an SLE$_\kappa$ curve from z_1 to z_2, then*

$$\mathbb{P}\{\mathrm{dist}[w, \gamma[0, \infty)] \leq r \cdot \mathrm{inrad}_D(w)\} \leq Cr^{2-d} \left[\frac{\mathrm{inrad}_D(w)}{\Delta_D^*(w; z_1, z_2)}\right]^{\beta/2}.$$

2.4. *Two-sided radial SLE*. We call SLE_κ under the measure $\mathbb{P}^*$ in the previous subsection *two-sided radial* SLE_κ (from 0 to ∞ through z in $\mathbb{H}$ stopped when it reaches z). Roughly speaking it is chordal SLE_κ conditioned to go through z (stopped when it reaches z). Of course this is an event of probability zero, so we cannot define the process exactly this way. We may provide a direct definition by driving the Loewner equation by the process defined in (7) rather than a standard Brownian motion. This definition uses the radial parametrization. We could also just as well use the capacity parametrization, in which case with probability one $T_z < \infty$.

One may justify the definition above examining its relationship to SLE_κ conditioned to get close to z. This next proposition is just a restatement of the definition of the measure $\mathbb{P}^*$ when restricted to curves stopped at a particular stopping time.

PROPOSITION 2.12. *Suppose γ is a chordal SLE_κ path from 0 to ∞ and $z \in \mathbb{H}$. For $\varepsilon \le \mathrm{Im}(z)$, let $\rho_\varepsilon = \inf\{t : \Upsilon_t(z) = \varepsilon\}$. Let μ, μ^* be the two measures on $\{\gamma(t) : 0 \le t \le \rho_\varepsilon\}$ corresponding to chordal SLE_κ restricted to the event $\{\rho_\varepsilon < \infty\}$ and two-sided radial SLE_κ through z, respectively. Then μ, μ^* are mutually absolutely continuous with respect to each other with the Radon–Nikodym derivative*

$$\frac{d\mu^*}{d\mu} = \frac{G_{H_{\rho_\varepsilon}}(z; \gamma(\rho_\varepsilon), \infty)}{G_{\mathbb{H}}(z; 0, \infty)} = \frac{\varepsilon^{d-2} S_{\rho_\varepsilon}(z)^\beta}{G_{\mathbb{H}}(z; 0, \infty)}.$$

Note that as $\varepsilon \to 0$ the Radon–Nikodym derivative tends to infinity. This reflects the fact that μ^* is a probability measure and that the total mass of μ is of order ε^{2-d} (see Lemma 2.10).

This proposition seems to indicate that there is a still a significant difference between two-sided radial SLE_κ going though z and SLE_κ conditioned to get within a specific distance. However, by using the methods of Lemma 2.9 we get the following improvement.

PROPOSITION 2.13. *There exists $u > 0, c < \infty$ such that the following is true. Suppose γ is a chordal SLE_κ path from 0 to ∞ and $z \in \mathbb{H}$. For $\varepsilon \le \mathrm{Im}(z)$, let $\rho_\varepsilon = \inf\{t : \Upsilon_t(z) = \varepsilon\}$. Suppose $\varepsilon' < 3\varepsilon/4$. Let μ', μ^* be the two probability measures on $\{\gamma(t) : 0 \le t \le \rho_\varepsilon\}$ corresponding to chordal SLE_κ conditioned on the event $\{\rho_{\varepsilon'} < \infty\}$ and two-sided radial SLE_κ through z, respectively. Then μ', μ^* are mutually absolutely continuous with respect to each other and the Radon–Nikodym derivative satisfies*

$$\left|\frac{d\mu^*}{d\mu'} - 1\right| \le c(\varepsilon'/\varepsilon)^u.$$

From the definition, it is easy to show that there is a subsequence $t_n \uparrow T_z$ with $\gamma(t_n) \to z$. In fact, in [7], a stronger fact is proven: for $0 < k < 8$, with probability one, the two-sided radial measure produces a curve, by which we mean that with probability one $\gamma(T_z^-) = z$.

LEMMA 2.14. *Let $\rho_\varepsilon = \inf\{t : \Upsilon_t(z) = \varepsilon\}$. There exists $\alpha > 0$ so that for any $z \in \mathbb{H}$ there exists $c_z < \infty$, so that for any ε and R with $\varepsilon \le R \le \mathrm{Im}(z)$ we have*

$$\mathbb{P}^*\{\gamma[\rho_\varepsilon, T_z] \not\subseteq B_R(z)\} \le c_z\left(\frac{\varepsilon}{R}\right)^\alpha.$$

PROOF. This result was shown for a two-sided radial through 0 from 1 to -1 in $\mathbb{D}$ in [7], Theorem 3. Since c_z is allowed to depend on z, the form in this lemma can be obtained by conformal invariance. $\square$

We will also need this bound in a chordal form, rather than two-sided radial form. In order to prove the chordal form, we need the following lemma.

LEMMA 2.15. *Let $\rho_\varepsilon = \inf\{t : \Upsilon_t(z) = \varepsilon\}$. There exists $c < \infty$, such that if $z \in \mathbb{H}$ and $\varepsilon \le \mathrm{Im}(z)/2$, $0 < \theta_0 \le \pi/2$,*

$$\mathbb{P}\{S_{\rho_\varepsilon}(z) < \sin(\theta_0) | \rho_\varepsilon < \infty\} \le c\theta_0^2.$$

PROOF. First note that by Proposition 2.12 and Lemma 2.10 we have that

$$\mathbb{P}\{S_{\rho_\varepsilon}(z) < \sin(\theta_0) | \rho_\varepsilon < \infty\} \le c\mathbb{E}^*[S_{\rho_\varepsilon}^{-\beta}(z) 1\{S_{\rho_\varepsilon}(z) < \sin(\theta_0)\}].$$

By applying the techniques from Lemma 2.9 with the function

$$F(\theta) = \sin(\theta)^{-\beta} 1\{\sin(\theta) < \sin(\theta_0)\},$$

and noting that the integral is

$$\int_0^\pi \sin(\theta)^{-\beta} 1\{\sin(\theta) < \sin(\theta_0)\} \sin^{4a} d\theta = 2\int_0^{\theta_0} \sin(\theta)\, d\theta = O(\theta_0^2),$$

we get the result. $\square$

LEMMA 2.16. *Let $\rho_\varepsilon = \inf\{t : \Upsilon_t(z) = \varepsilon\}$. Fix $\varepsilon < \eta < R < 1$ and $z \in \mathbb{H}$, then there exists some c depending only on z and $\alpha > 0$ such that*

$$\mathbb{P}\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z) | \rho_\varepsilon < \infty\} \le c\left(\frac{\eta}{R}\right)^\alpha.$$

PROOF. Let $0 < \theta < \pi/2$ be arbitrary; we will fix its precise value later. We apply Lemmas 2.15 and 2.14 with the above to see that

$$\begin{aligned}
&\mathbb{P}\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z) | \rho_\varepsilon < \infty\} \\
&\quad = \mathbb{P}\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z); S_{\rho_\varepsilon}(z) \ge \sin(\theta) | \rho_\varepsilon < \infty\} \\
&\qquad + \mathbb{P}\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z); S_{\rho_\varepsilon}(z) < \sin(\theta) | \rho_\varepsilon < \infty\} \\
&\quad \le c\mathbb{E}^*[S_{\rho_\varepsilon}^{-\beta}(z) 1\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z); S_{\rho_\varepsilon}(z) \ge \sin(\theta)\}] + c\theta^2
\end{aligned}$$

$$\leq c\theta^{-\beta}\mathbb{P}^*\{\gamma[\rho_\eta, \rho_\varepsilon] \not\subseteq B_R(z)\} + c\theta^2$$
$$\leq c\theta^{-\beta}\mathbb{P}^*\{\gamma[\rho_\eta, T_z] \not\subseteq B_R(z)\} + c\theta^2$$
$$\leq c\theta^{-\beta}(\eta/R)^\alpha + c\theta^2,$$

where c is being used generically. Thus by an appropriate choice of θ, for example,

$$\theta = (\eta/R)^{\alpha/(2+\beta)},$$

we get the desired bound. $\square$

3. Multi-point Green's function. In this section we consider two distinct points $z, w \in \mathbb{H}$. To simplify notation, we write

$$\xi = \xi_\varepsilon = \xi_{z,\varepsilon} = \inf\{t : \Upsilon_t(z) \leq \varepsilon\},$$
$$\chi = \chi_\delta = \chi_{w,\delta} = \inf\{t : \Upsilon_t(w) \leq \delta\}.$$

Although we will write ξ, χ, it is important to remember that these quantities depend on $z, \varepsilon, w, \delta$. We let $\mathbb{P}, \mathbb{E}$ denote probabilities and expectations for SLE_κ from 0 to ∞ in $\mathbb{H}$ and $\mathbb{P}^*, \mathbb{E}^*$ for the corresponding quantities for a two-sided radial through z. The multi-point Green's function, which we write

$$G(z, w) = G_{\mathbb{H}}(z, w; 0, \infty),$$

roughly corresponds to the probability that SLE in $\mathbb{H}$ from 0 to ∞ goes through z and then through w. This quantity is not symmetric. Although we do not have a closed from for this quantity, we can define it precisely.

DEFINITION. The multi-point Green's function $G(z, w)$ is defined by

$$G(z, w) = G(z)\mathbb{E}^*[G_H(w; z, \infty)],$$

where H is the unbounded component of $\mathbb{H} \setminus \gamma(0, T_z]$.

It is worth noting that if w is swallowed by the two-sided radial SLE curve before reaching z, this Green's function gives that event weight zero since the curve w is unreachable no matter how close the curve was to w before reaching z.

In [11], the exact formula for $G_{\mathbb{H}}(z; 0, \infty)$ was found by considering the martingale $G_{H_t}(z, \gamma(t), \infty)$ and then using Itô's formula and scaling to find the ODE that it satisfies, which could then be explicitly solved. When attempting the same technique here, a three real variable PDE result, which does not immediately seem to admit a closed form solution. A derivation of this PDE may be found in Appendix B.

The justification for this definition comes from the following theorem. Implicit in the statement is that the limit can be taken along any sequence of ε, δ going to zero.

THEOREM 1. *If* $z, w \in \mathbb{H}$, *then*

$$\lim_{\varepsilon,\delta\to 0^+} \varepsilon^{d-2}\delta^{d-2}\mathbb{P}\{\xi < \chi < \infty\} = c_*^2 G(z, w),$$

where c_* *is as defined in* (9).

When

$$d = \left(1 + \frac{\kappa}{8}\right) \wedge 2$$

is the dimension rather than simply $d = 1 + \kappa/8$, this theorem still defines an interesting quantity for $\kappa \geq 8$. Since the curve is space filling for $\kappa \geq 8$, the limit is trivial and

$$\lim_{\varepsilon,\delta\to 0^+} \varepsilon^{d-2}\delta^{d-2}\mathbb{P}\{\xi < \chi < \infty\} = \mathbb{P}\{\xi_0 < \chi_0\} = c_* G(z, w).$$

This agrees with the above definition of $G(z, w)$ since we may take two-sided radial through z for $\kappa \geq 8$ to be the measure on γ stopped at the time the curve passes through z and

$$G_D(w; z_1, z_2) = 1\{w \in D\}.$$

Since this case requires no further work, we will continue to assume that $\kappa < 8$.

We will need one lemma that will follow from our work on Beffara's estimate, which we will prove in Section 4.

LEMMA 3.1. *There exists* $\alpha > 0$, *such that if* $z, w \in \mathbb{H}$, *then there exists* $c = c_{z,w} < \infty$, *such that for all* $\varepsilon, \delta, r > 0$,

$$\mathbb{P}\{\xi < \chi < \infty; \operatorname{inrad}_\xi(w) \leq r\} \leq c\varepsilon^{2-d}\delta^{2-d}r^\alpha.$$

More precise results than this are obtained in this paper, but this is all that is required in this section.

Before going through the details of the proof, we briefly sketch the argument. To estimate

$$\mathbb{P}\{\xi < \chi < \infty\},$$

we wish to show that this probability is carried mostly on curves which get within ε of z in conformal radius before decreasing the conformal radius of w much at all. To show that the curves which do not do this are negligible, we use Lemma 3.1.

On the event that the curve stays bounded away from w, we know the Green's function for getting to w stays uniformly bounded, allowing us to use convergence of the conditioned measures $\mathbb{E}[\cdot \,|\xi < \infty]$ to $\mathbb{E}^*[\cdot]$, the two-sided radial measure, as measures on the SLE curve up until some fixed conformal radius $\eta \gg \varepsilon$.

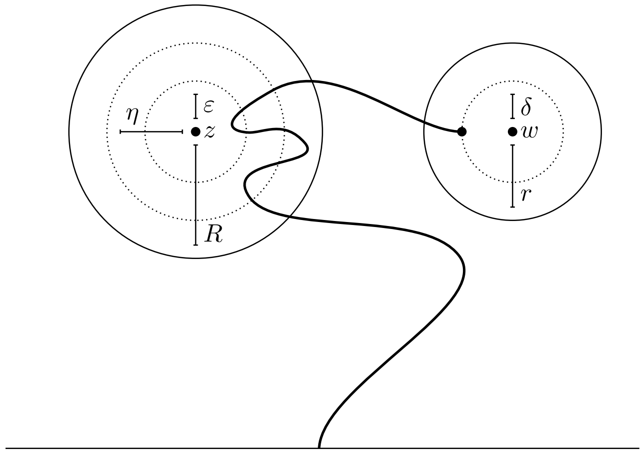

FIG. 2. *A diagram of the proof of Theorem 1. Dotted circles represent conformal radii and solid circles refer to geometric radii. The bold curve gives an example of the approximate shape of a curve contributing to the leading order event.*

This would be everything if it were not for the fact that the tip of the curves (the portion very near z) under the conditioned measure versus the two-sided radial measure have very different distribution. To handle this, we use Lemmas 2.14 and 2.16 to show that under both measures the tip stays close to z most of the time in Euclidean distance, and then Lemma 2.5 tells us that the Green's function for getting to w is insensitive to these changes.

To aid in the understanding of the proof, Figure 2 shows diagrammatically the various distances considered and the approximate shape of a curve in the main term.

PROOF OF THEOREM 1 GIVEN LEMMA 3.1. We first split according to how close we get to w before getting close to z. Fixing some $r < |z - w|/2$, by Lemma 3.1 we see that for some $\alpha > 0$

$$\begin{aligned}\mathbb{P}\{\xi < \chi < \infty\} &= \mathbb{P}\{\xi < \chi < \infty; \operatorname{inrad}_\xi(w) > r\} \\ &\quad + \mathbb{P}\{\xi < \chi < \infty; \operatorname{inrad}_\xi(w) < r\} \\ &= \mathbb{P}\{\xi < \chi < \infty; \operatorname{inrad}_\xi(w) > r\} + O(\varepsilon^{2-d}\delta^{2-d}r^\alpha).\end{aligned}$$

Let $\mathcal{F}_\xi$ denote the σ-algebra generated by the stopping time ξ. By applying Lemma 2.10 to w in the domain H_ξ, we see if $\delta \le r/2$,

$$\begin{aligned}&\mathbb{P}\{\xi < \chi < \infty; \operatorname{inrad}_\xi(w) > r | \mathcal{F}_\xi\} \\ &\quad = 1\{\xi < \infty; \operatorname{inrad}_\xi(w) > r\} c_* \delta^{2-d} G_{H_\xi}(w; \gamma(\xi), \infty)\big[1 + O\big((\delta/r)^u\big)\big].\end{aligned}$$

Applying Lemma 2.10 to z in $\mathbb{H}$ combined with the previous equation implies

$$\begin{aligned}&c_*^{-2}\varepsilon^{d-2}\delta^{d-2}G_{\mathbb{H}}(z;0,\infty)^{-1}\mathbb{P}\{\xi<\chi<\infty;\operatorname{inrad}_\xi(w)>r\}\\&\quad=\bigl[1+O\bigl(\varepsilon^u+(\delta/r)^u\bigr)\bigr]\mathbb{E}[G_{H_\xi}(w;\gamma(\xi),\infty)1\{\operatorname{inrad}_\xi(w)>r\}|\xi<\infty].\end{aligned}$$

For simplicity of notation, given a stopping time τ, we let

$$\mathbb{E}_\tau[\cdot]=\mathbb{E}[\cdot\,|\tau<\infty]\quad\text{and}\quad G_{\tau,r}=G_{H_\tau}(w;\gamma(\tau),\infty)1\{\operatorname{inrad}_\tau(w)>r\},$$

and hence we may rewrite this as

$$\begin{aligned}&\mathbb{P}\{\xi<\chi<\infty;\operatorname{inrad}_\xi(w)>r\}\\&\quad=c_*^2\varepsilon^{2-d}\delta^{2-d}G_{\mathbb{H}}(z;0,\infty)\bigl[1+O\bigl(\varepsilon^u+(\delta/r)^u\bigr)\bigr]\mathbb{E}_\xi[G_{\xi,r}].\end{aligned}$$

We wish to transform this expression from the conditioned measure to the two-sided radial measure, and from considering the situation at time ξ (the time it first gets within conformal radius ε) to T_z (the time under the two-sided radial measure that z is first contained in the boundary of H_{T_z}). To do so we will pass through a series of steps.

Fix some η, R so that $\varepsilon<\eta<R<|z-w|/2$. We wish to control the difference

$$\begin{aligned}|\mathbb{E}_\xi[G_{\xi,r}]-\mathbb{E}_\xi[G_{\xi_\eta,r}]|&\le\mathbb{E}_\xi\bigl[|G_{\xi,r}-G_{\xi_\eta,r}|1\{\gamma[\xi_\eta,\xi]\subseteq B_R(z)\}\bigr]\\&\quad+\mathbb{E}_\xi\bigl[|G_{\xi,r}-G_{\xi_\eta,r}|1\{\gamma[\xi_\eta,\xi]\not\subseteq B_R(z)\}\bigr].\end{aligned}$$

By Lemma 2.5 and the fact that the inradius about w cannot decrease between ξ_η and ξ if $\gamma[\xi_\eta,\xi]\subseteq B_R(z)$, we see that

$$\mathbb{E}_\xi\bigl[|G_{\xi,r}-G_{\xi_\eta,r}|1\{\gamma[\xi_\eta,\xi]\subseteq B_R(z)\}\bigr]=O\bigl(r^{d-2-(\beta\wedge1)/2}R^{(\beta\wedge1)/2}\bigr).$$

On the second term, the difference is no bigger than $O(r^{d-2})$ on an event, which by Lemma 2.16 is $O((\eta/R)^{\alpha'})$ for some $\alpha'>0$. Putting it all together yields

$$|\mathbb{E}_\xi[G_{\xi,r}]-\mathbb{E}_\xi[G_{\xi_\eta,r}]|=O\bigl(r^{d-2-(\beta\wedge1)/2}R^{(\beta\wedge1)/2}+r^{d-2}(\eta/R)^{\alpha'}\bigr).$$

By Lemma 2.13, we know for events in $\mathcal{F}_{\xi_\eta}$ we have

$$\left|\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}_\xi}-1\right|=O\bigl((\varepsilon/\eta)^u\bigr),$$

and hence we have

$$|\mathbb{E}_\xi[G_{\xi_\eta,r}]-\mathbb{E}^*[G_{\xi_\eta,r}]|=O\bigl(r^{d-2}(\varepsilon/\eta)^u\bigr).$$

Analogously to before, consider splitting the difference

$$\begin{aligned}|\mathbb{E}^*[G_{\xi_\eta,r}]-\mathbb{E}^*[G_{T_z,r}]|&\le\mathbb{E}^*\bigl[|G_{\xi_\eta,r}-G_{T_z,r}|1\{\gamma[\xi_\eta,T_z]\subseteq B_R(z)\}\bigr]\\&\quad+\mathbb{E}^*\bigl[|G_{\xi_\eta,r}-G_{T_z,r}|1\{\gamma[\xi_\eta,T_z]\not\subseteq B_R(z)\}\bigr].\end{aligned}$$

By Lemma 2.5 and the fact that the inradius about w cannot decrease between ξ_η and T_z if $\gamma[\xi_\eta, T_z] \subseteq B_R(z)$, we again see

$$\mathbb{E}^*\big[|G_{\xi_\eta,r} - G_{T_z,r}| 1\{\gamma[\xi_\eta, T_z] \subseteq B_R(z)\}\big] = O\big(r^{d-2-(\beta\wedge 1)/2} R^{(\beta\wedge 1)/2}\big).$$

The second term is on an event which by Lemma 2.14 is $O((\eta/R)^{\alpha'})$, and hence

$$|\mathbb{E}^*[G_{\xi_\eta,r}] - \mathbb{E}^*[G_{T_z,r}]| = O\big(r^{d-2-(\beta\wedge 1)/2} R^{(\beta\wedge 1)/2} + r^{d-2}(\eta/R)^{\alpha'}\big).$$

We may easily see that

$$\mathbb{P}^*\{\mathrm{inrad}_{T_z}(w) = 0\} \le \sum_{k\ge 1} \mathbb{P}^*\{\mathrm{inrad}_{\xi_{1/k}}(w) = 0\} = 0$$

by the fact that $\mathbb{P}^*$ is absolutely continuous with respect to $\mathbb{P}$ until the stopping time $\xi_{1/k}$ combined with that fact that two-sided radial SLE generates a curve with probability one. Hence, since $G_{T_z}(w; z, \infty) \ge 0$, we have that

$$\mathbb{E}^*[G_{T_z}(w; z, \infty) 1\{\mathrm{inrad}_{T_z}(w) > r\}] \to \mathbb{E}^*[G_{T_z}(w; z, \infty)] \qquad \text{as } r \to 0.$$

Combining all these terms and by combining exponents, we see there exists some $b > 0$ such that

$$\begin{aligned} &\varepsilon^{d-2}\delta^{d-2}\mathbb{P}\{\xi < \chi < \infty\} \\ &\quad = c_*^2 G_{\mathbb{H}}(z; 0, \infty)\big[1 + O\big(\varepsilon^b + (\delta/r)^b\big)\big]\mathbb{E}^*[G_{T_z,r}] \\ &\qquad + O\big(r^b + (R/r)^b + (R/r)^b(\eta/R)^b + (\varepsilon/r)^b(\varepsilon/\eta)^b\big). \end{aligned}$$

Thus by choosing r, η and R so that as $\varepsilon, \delta \to 0$ we also have

$$\begin{aligned} &r \to 0, \qquad \delta/r \to 0, \qquad \varepsilon/r \to 0, \\ &R/r \to 0, \qquad \eta/R \to 0, \qquad \varepsilon/\eta \to 0, \end{aligned}$$

we see that

$$\varepsilon^{d-2}\delta^{d-2}\mathbb{P}\{\xi < \chi < \infty\} \to c_*^2 G_{\mathbb{H}}(z; 0, \infty)\mathbb{E}^*[G_{T_z}(w; z, \infty)]$$

as desired. □

This same argument generalizes to show that we can define higher-order Green's functions of SLE as well (those that give normalized probabilities for passing through k marked points in the interior), and that the resulting multi-point Green's functions can be written in terms of expectations under the two-sided radial measure of lower-order Green's functions, for instance,

$$\begin{aligned} &\lim_{\varepsilon_1,\varepsilon_2,\varepsilon_3\to 0} \varepsilon_1^{d-2}\varepsilon_2^{d-2}\varepsilon_3^{d-2}\mathbb{P}\{\xi_{\varepsilon_1,z_1} < \xi_{\varepsilon_2,z_2} < \xi_{\varepsilon_3,z_3}\} \\ &\qquad = c_*^3 G_{\mathbb{H}}(z_1; 0, \infty)\mathbb{E}^*[G_{H_{T_{z_1}}}(z_2, z_3; z_1, \infty)], \end{aligned}$$

where $\mathbb{E}^*$ is the two-sided radial measure passing through z_1.

Note that we may obtain the multi-point Green's function as defined in the Introduction by summing this over the case where it gets near to z then w and the case where it gets near to w then z.

The remainder of this paper is dedicated to providing a proof of Lemma 3.1 and a sharpened version of Beffara's estimate.

4. Proof of Beffara's estimate and Lemma 3.1. To complete our proof of the existence of multi-point Green's functions we require a proof of Lemma 3.1. We also wish to prove Befarra's estimate which is the following theorem.

THEOREM 2 (Beffara's estimate). *There exists a* $c > 0$ *such that for all* z, $w \in \mathbb{H}$ *with* $\mathrm{Im}(z), \mathrm{Im}(w) \geq 1$ *we have that*

$$\mathbb{P}\{\Upsilon_\infty(z) < \varepsilon, \Upsilon_\infty(w) < \delta\} \leq c\varepsilon^{2-d}\delta^{2-d}|z-w|^{d-2}. \tag{10}$$

The hard work will be establishing the result when z, w are far apart. We use the notation introduced in Section 2.1. For later convenience, we write this proposition in terms of the usual radius rather than the conformal radius, but it is easy to convert to conformal radius using the Koebe $(1/4)$-theorem. We use the notation

$$\Delta_t(z) = \mathrm{inrad}_{H_t}(z).$$

PROPOSITION 4.1. *For every* $0 < \theta < \infty$, *there exists* $c < \infty$, *such that if* $z, w \in \mathbb{H}$ *with*

$$\mathrm{Im}(z), \mathrm{Im}(w) \geq \theta \quad \textit{and} \quad |z-w| \geq \theta/9,$$

then

$$\mathbb{P}\{\Delta_\infty(z) \leq \varepsilon, \Delta_\infty(w) \leq \delta\} \leq c\varepsilon^{2-d}\delta^{2-d}.$$

PROOF OF THEOREM 2 GIVEN PROPOSITION 4.1. Without loss of generality we assume that $1 \leq \mathrm{Im}(z) \leq \mathrm{Im}(w)$. We first claim that is suffices to prove (10) when $1 = \mathrm{Im}(z) \leq \mathrm{Im}(w)$. Indeed, if this is true and $r > 1$, scaling implies that

$$\begin{aligned}\mathbb{P}\{\Upsilon_\infty(rz) < \varepsilon, \Upsilon_\infty(rw) < \delta\} &= \mathbb{P}\{\Upsilon_\infty(z) < \varepsilon/r, \Upsilon_\infty(w) < \delta/r\} \\ &\leq c(\varepsilon/r)^{2-d}(\delta/r)^{2-d}|z-w|^{d-2} \\ &< c\varepsilon^{2-d}\delta^{2-d}|rz-rw|^{d-2}.\end{aligned}$$

Suppose $\varepsilon > |z-w|/10$. Then, using the one-point estimate Lemma 2.10, we get

$$\begin{aligned}\mathbb{P}\{\Upsilon_\infty(z) < \varepsilon, \Upsilon_\infty(w) < \delta\} &\leq \mathbb{P}\{\Upsilon_\infty(w) < \delta\} \\ &\leq c\delta^{2-d} \\ &\leq c10^{2-d}\varepsilon^{2-d}\delta^{2-d}|w-z|^{d-2}.\end{aligned}$$

A similar argument with δ shows that it suffices to prove (10) with $\mathrm{Im}(z) = 1$ and $\varepsilon, \delta < |z - w|/10$. If $|z - w| \geq 1/9$, we can apply Proposition 4.1 directly. So for the remainder of the proof, we let $u = |z - w|$ and assume

$$1 = \mathrm{Im}(z) \leq \mathrm{Im}(w), \qquad u \leq \frac{1}{9}, \qquad \varepsilon, \delta \leq \frac{u}{10}.$$

We will use the growth and distortion theorems which we now recall (see, e.g., [9], Section 3.2). Suppose $f : \mathbb{D} \to \mathbb{C}$ is a univalent function with $f(0) = 0$, $|f'(0)| = 1$. Then if $|\zeta| < 1$,

$$\frac{|\zeta|}{(1 + |\zeta|)^2} \leq |f(\zeta)| \leq \frac{|\zeta|}{(1 - |\zeta|)^2}, \tag{11}$$

$$\frac{1 - |\zeta|}{(1 + |\zeta|)^3} \leq |f'(\zeta)| \leq \frac{1 + |\zeta|}{(1 - |\zeta|)^3}. \tag{12}$$

Let $\tau = \inf\{t : |\gamma(t) - z| = 8u\} = \inf\{t : \Delta_t(z) = 8u\}$. The triangle inequality implies that $7u \leq \Delta_\tau(w) \leq 9u$. Lemma 2.10 implies that

$$\mathbb{P}\{\tau < \infty\} \leq cu^{2-d}. \tag{13}$$

Let g_τ be the usual conformal map, and let $h = sg_\tau$ where $s > 0$ is chosen so that $\mathrm{Im}(h(z)) = 1$. By the Schwarz lemma and the Koebe (1/4)-theorem, $4u \leq \Upsilon_{H_\tau}(z) \leq 16u$, and since $\Upsilon_{\mathbb{H}}(h(z)) = 1$,

$$\frac{1}{16u} \leq |h'(z)| \leq \frac{1}{4u}.$$

Since h is a conformal transformation of the disk of radius $8u$ about z, (11) implies

$$\tfrac{4}{81} \leq (8/9)^2 u|h'(z)| \leq |h(w) - h(z)| \leq (8/7)^2 u|h'(z)| \leq \tfrac{16}{49}.$$

Since $\varepsilon \leq u/10 = (8u)/80$, if $|z - \zeta| \leq \varepsilon$, (11) implies

$$|h(\zeta) - h(z)| \leq \left(\frac{80}{79}\right)^2 \varepsilon |h'(z)| \leq \left(\frac{80}{79}\right)^2 \frac{\varepsilon}{4u} \leq \frac{2\varepsilon}{7u}.$$

Applying (12), we can see that

$$|h'(w)| \leq \frac{(10/9)}{(8/9)^3} |h'(z)| \leq \frac{(10/9)}{(8/9)^3} \frac{1}{4u}.$$

Applying (11) to the disk of radius $7u$ about w and using $\delta \leq u/10 = (7u)/70$, we see that for $|\zeta - w| \leq \delta$,

$$|h(\zeta) - h(w)| \leq \left(\frac{70}{69}\right)^2 \delta |h'(w)| \leq \left(\frac{70}{69}\right)^2 \delta \frac{(10/9)}{(8/9)^3} \frac{1}{4u} \leq \frac{3\delta}{7u}.$$

Using the estimates of the previous paragraph, we can see by conformal invariance and the Markov property, that

$$\mathbb{P}\{\Delta_\infty(z) \leq \varepsilon, \Delta_\infty(w) \leq \delta | \tau < \infty\}$$

is bounded above by the supremum of

$$\mathbb{P}\{\Delta_\infty(z') \le \varepsilon', \Delta_\infty(w') \le \delta'\},$$

where the supremum is over

$$\operatorname{Im}(z') = 1, \qquad \frac{4}{81} \le |z' - w'| \le \frac{16}{49}, \qquad \varepsilon' \le \frac{2\varepsilon}{7u}, \qquad \delta' \le \frac{3\delta}{7u}.$$

Proposition 4.1 implies that there exists c' such that this supremum is bounded by

$$c'(\varepsilon')^{2-d}(\delta')^{2-d} \le c\varepsilon^{2-d}\delta^{2-d}u^{2(d-2)}.$$

If we combine this with (13), we get

$$\mathbb{P}\{\Delta_\infty(z') \le \varepsilon', \Delta_\infty(w') \le \delta'\} \le c\varepsilon^{2-d}\delta^{2-d}u^{d-2},$$

which is what we needed to prove. □

By an analogous argument to how we obtained Theorem 2 from Proposition 4.1, we may obtain Lemma 3.1 from Proposition 4.2.

PROPOSITION 4.2. *For every* $0 < \theta < \infty$, *there exists* $c < \infty$ *and* $\alpha > 0$ *such that if* $z, w \in \mathbb{H}$ *with*

$$\operatorname{Im}(z), \operatorname{Im}(w) \ge \theta \quad \textit{and} \quad |z - w| \ge \theta/9,$$

then for $\rho > \delta$

$$\mathbb{P}\{\Delta_\infty(z) \le \varepsilon, \Delta_\infty(w) \le \delta, \Delta_\sigma(w) \le \rho\} \le c\varepsilon^{2-d}\delta^{2-d}\rho^\alpha, \tag{14}$$

where $\sigma = \inf\{t : \Delta_t(z) \le \varepsilon$ *or* $\Delta_t(w) \le \delta\}$.

This proposition will follow immediately from the work required to show Proposition 4.1.

To prove the proposition, we will show that there exists a $c < \infty$ such that (14) holds if $|z - w| \ge 2\sqrt{2}$ and $\operatorname{Im}(z), (w) \ge 1$. By scaling one can easily deduce the result for all $\theta > 0$ with a θ-dependent constant. We fix z, w with $|z - w| \ge 2\sqrt{2}$ and $\operatorname{Im}(z), \operatorname{Im}(w) \ge 1$, and denote by $\mathcal{I}$ some fixed vertical or diagonal line such that

$$\operatorname{dist}(z, \mathcal{I}), \operatorname{dist}(w, \mathcal{I}) \ge 1, \tag{15}$$

and z, w lie in different components of $\mathbb{H} \setminus \mathcal{I}$. We will further assume, without loss of generality, that z is in the component of $\mathbb{H} \setminus \mathcal{I}$ which contains arbitrarily large negative real numbers in it's boundary (more informally that z is in the left component).

4.1. *An excursion measure estimate.* Our main result will require an estimate of the "distance" between two boundary arcs in a simply connected domain. We will use excursion measure to gauge the distance; we could also use extremal distance, but we find excursion measure more convenient.

Suppose η is a crosscut in $\mathbb{H}$ with $-\infty < \eta(1^-) \le \eta(0^+) \le -1$. Let H_η denote the unbounded component of $\mathbb{H} \setminus \eta$. Let $\mathcal{E}(\eta) = \mathcal{E}_{\mathbb{H}_\eta}(\eta, [0, \infty))$ denote the excursion measure between η and $[0, \infty)$ in H_η, the definition of which we now recall (see [9], Section 5.2, for more details). If $z \in H_\eta$, let $h_\eta(z)$ be the probability that a Brownian motion starting at z exits H_η at η. For $x \ge 0$, let $\partial_y h_\eta(x)$ denote the partial derivative. Then

$$\mathcal{E}(\eta) = \int_0^\infty \partial_y h_\eta(x)\, dx.$$

The excursion measure $\mathcal{E}_D(V_1, V_2)$ is defined for any domain and boundary arcs V_1, V_2 in a similar way and is a conformal invariant. If V_2 is smooth, then we can compute $\mathcal{E}_D(V_1, V_2)$ by a similar integral

$$\mathcal{E}_D(V_1, V_2) = \int_{V_2} \partial_{\mathbf{n}} h_{V_1}(z) |dz|,$$

where $\mathbf{n}$ denotes the inward normal. We need the following easy estimate.

LEMMA 4.3. *There exist c_1, c_2 such that if η is a crosscut in $\mathbb{H}$ with $-\infty < \eta(1^-) \le \eta(0^+) = -1$ and* $\operatorname{diam}(\eta) \le 1/2$, *then*

$$c_1 \operatorname{diam}(\eta) \le \mathcal{E}(\eta) \le c_2 \operatorname{diam}(\eta).$$

SKETCH OF PROOF. In fact, we get an estimate

$$\partial_y h_\eta(x) \asymp \frac{\operatorname{diam}(\eta)}{(x+1)^2}.$$

The key estimate used here is the fact that that if $\operatorname{Re}(z) \ge 0$,

$$h_\eta(z) \asymp \frac{\operatorname{Im}(z) \operatorname{diam}(\eta)}{(|z|+1)^2}. \qquad \square$$

LEMMA 4.4. *There exists a $C < \infty$ such that the following is true. Suppose $H \subset \mathbb{C}$ is a half-plane bounded by the line $L = \partial H$, D is a simply connected subdomain of $\mathbb{H}$ and $z \in \partial D$ with $d(z, L) > \frac{1}{2}$. Suppose I is a subinterval of $L \cap \partial D$. Then for every $\varepsilon < \frac{1}{2}$, the excursion measure between I and $V := \partial D \cap \{w : |w - z| \le \varepsilon\}$ is bounded above by $C\varepsilon^{1/2}$.*

PROOF. Without loss of generality we assume that $H = \mathbb{H}$, $z = i/2$. Let $h(w)$ denote the probability that a Brownian motion starting at w exits D at V. Then the excursion measure is exactly

$$\int_I \partial_y h(x)\, dx.$$

Hence it suffices to give an estimate

$$\partial_y h(x) \le c\varepsilon^{1/2}[1 \wedge x^{-2}]. \tag{16}$$

For $|x| \le 4$, this follows from the Beurling estimate. For $|x| \ge 4$, we first consider the excursion "probability" to reach $\mathrm{Re}(w) = x/2$. By the gambler's ruin estimate, this is bounded by $O(|x|^{-1})$. Then we need to consider the probability that a Brownian motion starting at z' with $\mathrm{Re}(z') = x/2$ reaches the disk of radius 1 about z without leaving D. By comparison with the same probability in the domain $\mathbb{H}$, we see that this is bounded above by $O(|x|^{-1})$. Finally from there we need to hit V which contributes a factor of $O(\varepsilon^{1/2})$ by the Beurling estimate. Combining these estimates gives (16). $\square$

LEMMA 4.5. *There exists $c > 0$ such that the following holds. Let D be a simply connected domain, and let γ be a chordal SLE$_\kappa$ path from z_1 to z_2 in D. Let $\eta : (0, 1) \to D$ be a crosscut in D. Let $\xi : (0, 1) \to D$ be another crosscut with $\xi(0^+) = z_1$, and let D_1, D_2 denote the components of $D \setminus \xi$. Suppose $\eta \subset D_1$ and $z_2 \in \partial D_2$. Then,*

$$\mathbb{P}\{\gamma(0, \infty) \cap \eta(0, 1) \ne \varnothing\} \le c\mathcal{E}_D(\eta, \xi)^\beta.$$

See Figure 3 for a diagram of the setup of this lemma.

PROOF OF LEMMA 4.5. By conformal invariance, we may assume that $D = \mathbb{H}$, $z_1 = 0$, $z_2 = \infty$, and it suffices to prove the result when $\mathcal{E}_D(\eta, \xi) \le 1$ in which case the endpoints of η are nonzero. Without loss of generality we assume that they lie on the negative real axis, and by scale invariance we may assume $\eta(1^-) \le \eta(0^+) = -1$. Then monotonicity of the excursion measure implies that

$$\mathcal{E}_D(\eta, \xi) \ge \mathcal{E}_D(\eta).$$

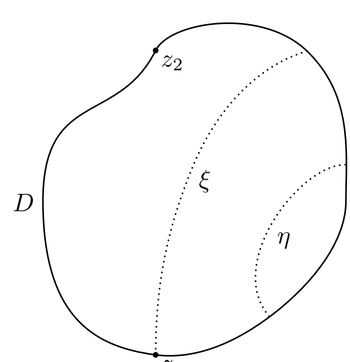

FIG. 3. *The setup for Lemma 4.5.*

Lemma 4.3 implies that if $\operatorname{diam}(\eta) < 1/2$, then $\mathcal{E}_D(\eta) \asymp \operatorname{diam}(\eta)$. Since $\mathcal{E}_D(\eta) \leq 1$ one can see there is a c_0 so that $\operatorname{diam}(\eta) \leq c_0$. The result then follows from Proposition 2.6. $\square$

Given the proof, the form of this lemma may seem odd as the curve ξ is discarded half way through; indeed, the result could be stated with $\mathcal{E}_D(\eta)$ rather than $\mathcal{E}_D(\eta, \xi)$ in the inequality. However, $\mathcal{E}_D(\eta)$ is hard to estimate directly and, in every case in this paper, the method of estimation is to find a curve ξ and proceed as above.

4.2. *Topological lemmas*. The most challenging portion of this proof is gaining simultaneous control of the distances to the near and far edges of the curve. Luckily, we may eliminate a number of hard cases of the computations that follow by purely topological means. For clarity of presentation, we have isolated these topological lemmas here in a separate section. Let $z, w, \mathcal{I}$ be as described in the paragraph around equation (15). We call γ a noncrossing curve (from 0 to ∞ in $\mathbb{H}$) if is generated by the Loewner equation (4) with some driving function U_t, and, as before, we let H_t be the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$ and $\partial_1 H_t, \partial_2 H_t$ be the preimages (considered as prime ends) under g_t of $(-\infty, U_t)$ and (U_t, ∞). We call a simple curve $\omega : (0, \infty) \to H_t$ with $\omega(0^+) = \gamma(t)$ and $\omega(\infty) = \infty$ an *infinite crosscut of* H_t. Such curves can be obtained as preimages under g_t of simple curves from U_t to ∞ in $\mathbb{H}$. An important observation is that infinite crosscuts of H_t separate $\partial_1 H_t$ from $\partial_2 H_t$.

We now define a particular crosscut of H_t contained in $\mathcal{I}$ that separates z from w.

DEFINITION. Let γ be a noncrossing curve, and let $\mathcal{I}_t = \mathcal{I} \setminus \gamma(0, t]$. We denote by $I_t = I_t(\mathcal{I}, z, w, \gamma)$ the unique open interval contained in $\mathcal{I}$ such that the following four properties hold. For any $t \leq t'$ we have:

- I_t is a connected component of $\mathcal{I}_t$,
- $I_{t'} \subseteq I_t$,
- $H_t \setminus I_t$ has exactly two connected components, one containing z and one containing w and
- $I_t = I_{t'}$ whenever $\gamma(t, t'] \cap \mathcal{I} = \varnothing$.

We let H_t^z, H_t^w denote the components of $H_t \setminus I_t$ that contain z and w, respectively.

Seeing that this notion is well defined is nontrivial, despite the intuitive nature what it should be (see Figure 4). To avoid breaking the flow of the document, the proof that it is well defined has been deferred to Appendix A.

LEMMA 4.6. *Suppose γ is a noncrossing curve with $z, w \notin \gamma(0, \infty)$ and $I_t = I_t(\mathcal{I}, z, w, \gamma)$ as above. Suppose $\gamma(t) \in \overline{I}_t$. If I_t is not bounded, then*

$$\Delta^*_{H_t}(z, \gamma(t), \infty) \geq 1, \qquad \Delta^*_{H_t}(w, \gamma(t), \infty) \geq 1.$$

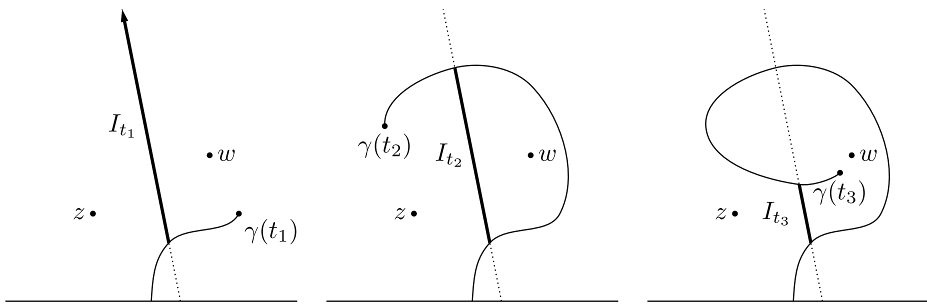

FIG. 4. *A few steps showing the behavior of I_t for some times $0 < t_1 < t_2 < t_3$.*

PROOF. Suppose I_t is not bounded. Then I_t is an infinite crosscut of H_t. Suppose that $\Delta^*_{H_t}(z, \gamma(t), \infty) < 1$. Then there is a crosscut η contained in a disc of radius strictly less than one centered on z which has one end point in $\partial_1 H_t$ and one end point in $\partial_2 H_t$. Hence η must intersect I_t. However, $\operatorname{dist}(z, I_t) \geq \operatorname{dist}(z, \mathcal{I}) \geq 1$ which is a contradiction. Therefore, $\Delta^*_{H_t}(z, \gamma(t), \infty) \geq 1$. $\square$

LEMMA 4.7. *Suppose γ is a noncrossing curve with $z, w \notin \gamma(0, \infty)$ and $I_t = I_t(\mathcal{I}, z, w, \gamma)$ as above. Suppose $\gamma(t) \in \overline{I}_t$. If I_t is bounded, and H^z_t is bounded, then*

$$\Delta^*_{H_t}(z, \gamma(t), \infty) \geq 1.$$

PROOF. Suppose I_t is bounded, H^z_t is bounded, and $\Delta^*_{H_t}(z, \gamma(t), \infty) < 1$. Then there is a crosscut η of H^z_t which has one end point in $\partial_1 H_t$ and one end point in $\partial_2 H_t$. Since H^z_t is bounded and $\gamma(t) \in \overline{I}_t$, we may find an infinite crosscut ω of H_t that never enters H^z_t [take a simple curve from ∞ in H_t until it first hits I_t and then continue the curve along I_t to reach $\gamma(t)$]. Since η and ω do not intersect, we get a contradiction. $\square$

Given these simple observations, we can restrict the manner in which the various distances to the curve can be decreased.

LEMMA 4.8. *Suppose γ is a noncrossing curve with $z, w \notin \gamma(0, \infty)$ and $I_t = I_t(\mathcal{I}, z, w, \gamma)$ as above. Suppose t_0 is a time so that $\gamma(t_0) \in \overline{I}_{t_0}$. Let $\zeta = \inf\{t > t_0 | \gamma(t) \in I_{t-}\}$. Then at most one of the following holds*:

- $\Delta_{H_\zeta,1}(z, \gamma(\zeta), \infty) < \Delta_{H_{t_0},1}(z, \gamma(t_0), \infty) \wedge 1$, *or*
- $\Delta_{H_\zeta,2}(z, \gamma(\zeta), \infty) < \Delta_{H_{t_0},2}(z, \gamma(t_0), \infty) \wedge 1$.

PROOF. If $\zeta = t_0$, the above statement follows immediately, so we may assume $\zeta > t_0$. Consider the noncrossing loop $\ell = \gamma[t_0, \zeta] \cup L$ where L is the line connecting $\gamma(\zeta)$ and $\gamma(t_0)$. Partition $\mathbb{H}$ into two sets, the infinite component of

$\mathbb{H} \setminus \ell$, which we will denote by A_∞, and the union of the finite components of $\mathbb{H} \setminus \ell$, which we will denote by A_0. The point z is either in A_∞ or A_0. As the cases are similar, assume $z \in A_\infty$. Since ℓ is a noncrossing loop, we either have a curve $\eta : [0, 1) \to A_\infty$ with $\eta(0) = z$ and $\eta(1^-) \in \partial_1 H_\zeta$ or $\eta(1^-) \in \partial_2 H_\zeta$, but not both. This yields that only one of the $\Delta_{H_\zeta, j}(z, \gamma(\zeta), \infty)$ could have decreased past the minimum of 1 and its previous value. $\square$

LEMMA 4.9. *Suppose γ is a noncrossing curve with $z, w \notin \gamma(0, \infty)$ and $I_t = I_t(\mathcal{I}, z, w, \gamma)$ as above. Suppose t_0 is a time so that $\gamma(t_0) \in \overline{I}_{t_0}$, and let $\zeta = \inf\{t > t_0 | \gamma(t) \in I_{t-}\}$. Suppose for some $s < 1$,*

$$\Delta^*_\zeta(z) \le s < \Delta^*_{t_0}(z).$$

Then $\Delta_{t_0}(z) \le s$, and $H^w_{t_0}$ and H^w_ζ are bounded.

PROOF. By the previous lemma, we have that either $\Delta^1_\zeta(z) \ge \Delta^1_{t_0}(z) \wedge 1$ or $\Delta^2_\zeta(z) \ge \Delta^2_{t_0}(z) \wedge 1$. This implies that $\Delta^*_\zeta(z) \ge \Delta_{t_0}(z) \wedge 1$, and hence $\Delta_{t_0}(z) \wedge 1 \le s$ which is the first assertion.

We now prove that H^w_ζ is bounded. Assume first that both H^w_ζ and H^z_ζ are unbounded. Then I_ζ is unbounded, and by Lemma 4.6 we have that

$$\Delta^*_{H_\zeta}(z, \gamma(t), \infty) \ge 1,$$

which is a contradiction. Thus one of H^w_ζ or H^z_ζ is bounded. If H^z_ζ is bounded, then by Lemma 4.7 we have

$$\Delta^*_{H_\zeta}(z, \gamma(t), \infty) \ge 1,$$

which is again a contradiction. Thus H^w_ζ is bounded, as desired.

By the definition of ζ and I_t, we know $\gamma(t_0, \zeta)$ is contained in precisely one of $H^z_{t_0}$ or $H^w_{t_0}$. Since

$$\Delta^*_\zeta(z) < 1 \le \Delta^*_{t_0}(z)$$

by assumption, we know $\gamma(t_0, \zeta) \subseteq H^z_{t_0}$. Assume that $H^w_{t_0}$ were unbounded. Then there is a curve η from w to ∞ contained in $H^w_{t_0}$. Since H^w_ζ is bounded $\eta \cap \partial H^w_\zeta$ is nonempty. By definition,

$$\partial H^w_\zeta \subseteq \gamma(0, t_0] \cup \gamma(t_0, \zeta] \cup I_\zeta.$$

We now show η cannot intersect any of the three sets on the right. Since η is in $H^w_{t_0}$, we know $\eta \cap (\gamma(0, t_0] \cup I_{t_0}) = \varnothing$ and moreover, since $I_\zeta \subseteq I_{t_0}$, that $\eta \cap I_\zeta = \varnothing$. Since $\gamma(t_0, \zeta) \subseteq H^z_{t_0}$, we know $\eta \cap \gamma(t_0, \zeta) = \varnothing$. Thus we have a contradiction, and $H^w_{t_0}$ must be bounded, as desired. $\square$

4.3. *Main SLE estimates.* We now use the above topological restrictions to help us establish the needed SLE estimates. Let T_z (resp., T_w) denote the first time that z (resp., w) is not in H_t, and let $T = T_z \wedge T_w$ denote the first time that one of z, w is not in H_t. Note that if the curve is to approach z and w to within ε and δ as desired, it must do so before $T_z \vee T_w$.

We also define the following recursive set of stopping times. Let $\tau_0 = 0$. Given $\tau_j < T$, define $\hat{\tau}_j$ as the infimum over times $t > \tau_j$ such that

$$\Delta_t(z) \le \tfrac{1}{2}\Delta_{\tau_j}(z) \quad \text{or} \quad \Delta_t(w) \le \tfrac{1}{2}\Delta_{\tau_j}(w).$$

Given this, let τ_{j+1} be the infimum over times $t > \hat{\tau}_j$ such that $\gamma(t) \in \overline{I}_{\hat{\tau}_j}$. These times are understood to be infinite when past T, and hence at least one of the points can no longer be approached by the curve. The sequence of stopping times $\{\tau_k\}_{k\ge 0}$ are called renewal times. We let $R_{k+1} = 0$ if $\tau_{k+1} < \infty$ and $\Delta_{\tau_{k+1}}(z) \le \frac{1}{2}\Delta_{\tau_k}(z)$; in this case, we can see that $\Delta_{\tau_{k+1}}(w) > \frac{1}{2}\Delta_{\tau_k}(w)$. If $\tau_{k+1} < \infty$ and $\Delta_{\tau_{k+1}}(w) \le \frac{1}{2}\Delta_{\tau_k}(w)$, we set $R_{k+1} = 1$. We set $R_{k+1} = \infty$ if $\tau_{k+1} = \infty$. Less formally, the renewal times encode when our curve halved its distance to either z or w and then returned to I_t, while R_k specifies which point we halved the distance to. Let $\mathcal{F}_k = \mathcal{F}_{\tau_k}$.

LEMMA 4.10. *There exist $c < \infty, \alpha > 0$ such that for all $k \ge 0$, $r \le 1/2$,*

$$\mathbb{P}\{R_{k+1} = 0; \Delta_{\tau_{k+1}}(z) \le r\Delta_{\tau_k}(z) | \mathcal{F}_k\} \le c1\{\tau_k < T\}\Delta_{\tau_k}(z)^{\alpha} r^{2-d}.$$

PROOF. We assume $\tau_k < T$, and we write $\tau = \tau_k$, $\xi = \xi(z; r\Delta_\tau(z))$. First, consider the event that either I_τ is not bounded, or both I_τ and H_τ^z are bounded. By Lemmas 4.6 and 4.7, we have $\Delta_\tau^*(z) \ge 1$. Thus by Lemma 2.11, we get

$$\mathbb{P}\{\xi < \infty | \mathcal{F}_k\} \le cr^{2-d}\Delta_\tau(z)^{\beta/2}.$$

Suppose that I_τ is bounded, and H_τ^w is bounded. We split into the following two cases: $\Delta_\tau^*(z) \le \sqrt{\Delta_\tau(z)}$ and $\Delta_\tau^*(z) > \sqrt{\Delta_\tau(z)}$. If $\Delta_\tau^*(z) > \sqrt{\Delta_\tau(z)}$, then Lemma 2.11 implies

$$\mathbb{P}\{\xi < \infty | \mathcal{F}_k\} \le cr^{2-d}\Delta_\tau(z)^{\beta/4}.$$

Suppose $\Delta_\tau^*(z) \le \sqrt{\Delta_\tau(z)}$. Then there exist simple curves $\eta_1, \eta_2 : [0, 1) \to H_\tau^z$ contained in the disk of radius $2\Delta_\tau^*(z)$ about z with $\eta^j(0) = z$ and $\eta^j(1+) \in \partial_j H_\tau$. At the time ξ we can consider the line segment L from $\gamma(\xi)$ to z. There exists a crosscut of H_ξ, $\hat{\eta}$, contained in $L \cup \eta_1$ or in $L \cup \eta_2$, one of whose endpoints is $\gamma(\xi)$, that disconnects I_ξ from infinity. Using Lemma 4.4, we see that

$$\mathcal{E}_{H_\xi}(\hat{\eta}, I_\xi) \le c\Delta_\tau^*(z)^{1/2} \le c\Delta_\tau(z)^{1/4}.$$

Thus, using Lemma 4.5 we see that

$$\mathbb{P}\{\xi < \tau_{k+1} < \infty | \mathcal{F}_k\} \le c\Delta_\tau(z)^{\beta/4}\mathbb{P}\{\xi < \infty | \mathcal{F}_k\} \le cr^{2-d}\Delta_\tau(z)^{\beta/4}. \qquad \square$$

REMARK. The proof of the last lemma was not difficult given the estimates we have derived. However, it is useful to summarize the basic idea. If $\Delta^*_\tau(z)$ is not too small, then it suffices to estimate

$$\mathbb{P}\{R_{k+1}=0;\,\Delta_{\tau_{k+1}}(z)\le r\Delta_{\tau_k}(z)|\mathcal{F}_k\}$$

by

$$\mathbb{P}\{\xi<\infty|\mathcal{F}_k\}.$$

However, if $\Delta^*_{\tau_k}(z)$ is not much bigger than $\Delta_{\tau_k}(z)$ this estimate is not sufficient. In this case, we need to use

$$\mathbb{P}\{\xi<\infty|\mathcal{F}_k\}\mathbb{P}\{\tau_{k+1}<\infty|\mathcal{F}_k,\xi<\infty\}.$$

The above argument provides a good bound on the probability that the near side gets even closer. To complete our argument, we must also provide a bound limiting the probability that the far side can get closer as well.

LEMMA 4.11. *There exists $c<\infty$ such that for all $k\ge 0$, $s\le 1/4$, if*

$$\xi^*=\inf\{t>\tau_k|\Delta^*_t(z)\le s\}\quad \textit{and}\quad \eta^*=\inf\{t>\xi^*|\gamma(t)\in I_{t^-}\},$$

then

$$\mathbb{P}\{\eta^*<\infty,\,\Delta^*_{\eta^*}(z)\le s|\Delta^*_{\tau_k}(z)>s,\mathcal{F}_{\tau_k}\}\le cs^{\beta/2}.$$

PROOF. Assume $\Delta^*_{\tau_k}(z)>s$. If $\eta^*<\infty$ we may define

$$\varpi=\sup\{t<\eta^*|\gamma(t)\in I_{t^-}\}$$

to be the previous time that γ crossed I_{t^-} before η^*. Note that $\tau_k\le\varpi<\xi^*<\eta^*$ and $\Delta^*_\varpi(z)>s$. By considering the two times ϖ and η^* in Lemma 4.9, we see that H^w_ϖ is bounded.

Consider the situation at time ξ^*. By the definition of the stopping times, there must be a curve $\nu:(0,1)\to H_{\xi^*}$ which contains z, is never more than distance $2s$ from z, has $\nu(0^+)\in\partial_1 H_{\xi^*}$ and $\nu(1^-)\in\partial_2 H_{\xi^*}$ such that ν separates I_{ξ^*}, and hence w, from infinity. Since ν is at least distance $1/2$ from I_{ξ^*} we know from Lemma 4.4 that the excursion measure between ν and I_{ξ^*} in H_{ξ^*} is bounded above by $Cs^{1/2}$. Then an application of Lemma 4.5 tells us that the probability of γ returning to I_{ξ^*} is bounded above by $Cs^{\beta/2}$ which gives the lemma. □

The following two lemmas combine the methods of the above two bounds.

LEMMA 4.12. *There exist $c<\infty,\alpha>0$ such that for all $k\ge 0$, $r\le 1/2$, $s\le 1/4$,*

$$\begin{aligned}\mathbb{P}\{R_{\tau_{k+1}}=0;\,\Delta_{\tau_{k+1}}(z)\le r\Delta_{\tau_k}(z);\,\Delta^*_{\tau_{k+1}}(w)\le s|\mathcal{F}_k\}\\ \le c1\{\tau_k<T\}\Delta_{\tau_k}(z)^\alpha[s^\alpha+1\{\Delta^*_{\tau_k}(w)\le s\}]r^{2-d}.\end{aligned}$$

PROOF. If $\Delta^*_{\tau_k}(w) \le s$, then the desired statement reduces to Lemma 4.10. Thus, we may assume that $\Delta^*_{\tau_k}(w) > s$.

Let $\zeta^* = \zeta^*_k$ be the infimum over times $t > \tau_k$ so that $\Delta^*_t(w) \le s$ and $\gamma(t) \in I_{t^-}$. Let $\sigma = \sigma_k = \inf\{t > \tau_k | \Delta_t(z) \le r\Delta_{\tau_k}(z)\}$. If $\Delta^*_{\tau_k}(w) > s$, $\Delta^*_{\tau_{k+1}}(w) \le s$, and $\sigma < \infty$, then $\zeta^* < \sigma$ since the curve γ would need to intersect I_σ before approaching w and hence would force the renewal time τ_{k+1} before ζ_k.

By the same argument as in Lemma 4.11, we know if $\Delta^*_{\tau_k}(w) > s$ and $\zeta^* < \infty$, there is a time ω, $\tau_k \le \omega < \zeta^*$ for which there is a curve connecting $\partial_1 H_\omega$ to $\partial_2 H_\omega$ passing through $\gamma(\omega)$ contained in a disk of radius $2s$ about w separating I_κ from infinity. Then, by Lemma 4.5, we have that

$$\mathbb{P}\{\zeta^* < \infty | \Delta^*_{\tau_k}(z) > s, \mathcal{F}_{\tau_k}\} \le cs^\alpha.$$

By Lemma 4.9 we know $H^z_{\zeta^*}$ is bounded. Lemma 4.7 implies that $\Delta^*_{\zeta^*}(z) = 1$, and hence by Lemma 4.4

$$\mathbb{P}\{R_{\tau_{k+1}} = 0; \Delta_{\tau_{k+1}} \le r\Delta_{\tau_k}(z) | \mathcal{F}_{\zeta^*}, \zeta^* < \infty\} \le c1\{\tau_k < T\}\Delta_{\zeta^*}(z)^\alpha r^{2-d}.$$

Combining the above two bounds gives the desired result. $\square$

LEMMA 4.13. *There exist $c < \infty, \alpha > 0$ such that for all $k \ge 0$, $r \le 1/2$, $s > 0$,*

$$\begin{aligned}\mathbb{P}\{R_{\tau_{k+1}} = 0; \Delta_{\tau_{k+1}}(z) \le r\Delta_{\tau_k}(z); \Delta^*_{\tau_{k+1}}(z) \le s | \mathcal{F}_k\} \\ \le c1\{\tau_k < T\}\Delta_{\tau_k}(z)^\alpha [s^\alpha + 1\{\Delta^*_{\tau_k}(z) \le s\}] r^{2-d}.\end{aligned}$$

PROOF. If $\Delta^*_{\tau_k}(z) \le s$ or $s \ge 1/4$, the conclusion reduces to Lemma 4.10. Thus we may assume that $\Delta^*_{\tau_k}(z) > s$, $s \le 1/4$. Let E denote the event

$$E = \{R_{\tau_{k+1}} = 0; \Delta_{\tau_{k+1}}(z) \le r\Delta_{\tau_k}(z); \Delta^*_{\tau_{k+1}}(z) \le s; \Delta^*_{\tau_k}(z) > s\}.$$

Let

$$\sigma = \inf\{t | \Delta_t(z) \le r\Delta_{\tau_k}(z)\},$$

and note that on the event E,

$$\tau_{k+1} = \inf\{t > \sigma | \gamma(t) \in I_{t^-}\}.$$

Define ξ to be the infimum over times $t \ge \sigma$ such that there is a curve $\eta : (0,1) \to H_t$ with $\eta(0^+) = \gamma(t)$ and $\eta(1^-) \in \partial H_t$ with η contained entirely in the ball of radius $2s$ about z, and η separating I_t from ∞.

We now claim that given $\mathcal{F}_\sigma$ either $\xi < \tau_{k+1}$ or $\Delta^*_{\tau_{k+1}}(z) > s$. To see this, suppose neither holds. Since $\Delta^*_{\tau_{k+1}}(z) \le s$, for every $s < s' \le 2s \le 1/2$, there is a crosscut η of $H_{\tau_{k+1}}$ going through z whose endpoints are in $\partial_1 H_{\tau_{k+1}}$, $\partial_2 H_{\tau_{k+1}}$, respectively, and which is contained in the disk of radius s' about z. By Lemma 4.9 we know η must disconnect $I_{\tau_{k+1}}$ from ∞ since $H^w_{\tau_{k+1}}$ must be bounded. We can choose such an η such that at least one endpoint of η is not in $\gamma[0, \tau_k]$, for otherwise all such η would be a crosscuts of H_{τ_k} separating w from infinity which would imply that $\Delta^*_{\tau_k}(z) \leq s$.

Let $\zeta = \sup\{t \leq \tau_{k+1} | \gamma(t) \in \overline{\eta}\} > \tau_k$ and note that $\tau_k < \zeta < \tau_{k+1}$. If $\zeta \geq \sigma$ we are done since this η demonstrates that $\xi < \tau_{k+1}$.

Thus assume $\zeta < \sigma$. In this case, we will construct a curve in H_σ satisfying the conditions in the definition of ξ. Since $\zeta < \sigma$ we know the curve η defined above disconnects I_σ from infinity in H_σ. By the definition of σ as the first time that $\Delta_\sigma(z) \leq r\Delta_{\tau_k}(z)$, the straight open line segment, L, from $\gamma(\sigma)$ to z is contained in H_σ. Additionally, since $\Delta_\sigma(z) \leq \Delta^*_\sigma(z) \leq s$, we know $\eta(0,1) \cup L$ is contained entirely in the ball of radius $2s$ about z. Thus we may find a curve $\hat{\eta}$ contained in $\eta(0,1) \cup L$ which separates I_σ from infinity in H_σ with $\eta(0^+) = \gamma(t)$ and $\eta(1^-) \in \partial H_t$ and with η contained entirely in the ball of radius $2s$ about z, proving that $\xi = \sigma < \tau_{k+1}$. Thus we have reached a contradiction.

On the event E we know $\Delta^*_{\tau_{k+1}}(z) \leq s$, and thus the above argument tells us $\xi < \tau_{k+1}$. We have therefore shown that if $\Delta^*_{\tau_k}(z) > s$, $s \leq 1/4$, then

$$\mathbb{P}(E|\mathcal{F}_k) \leq \mathbb{P}\{\sigma \leq \xi < \tau_{k+1} < \infty | \mathcal{F}_k\}.$$

We may now argue as in the second part of the proof of Lemma 4.10 to obtain $\mathbb{P}\{\sigma < \infty | \mathcal{F}_k\} \leq c\Delta_{\tau_k}(z)^\alpha$ and $\mathbb{P}\{\tau_{k+1} < \infty | \mathcal{F}_\xi\} \leq cs^\alpha$. $\square$

4.4. *Combinatorial estimates*. We have now completed the bulk of the probabilistic estimates. Most of what remains is a combinatorial argument to sum up the bounds proven above across all possible ways that the SLE curve may approach z and w in turn.

Without loss of generality, assume that $\delta = 2^{-m}$ and $\varepsilon = 2^{-n}$, and let

$$\begin{aligned}
\xi_z &= \xi_{z,\varepsilon} = \inf\{t : \Delta_t(z) \leq 2^{-n}\}, \\
\xi_w &= \xi_{w,\delta} = \inf\{t : \Delta_t(w) \leq 2^{-m}\}, \\
\xi &= \xi_z \vee \xi_w = \inf\{t : \Delta_t(z) \leq 2^{-n}, \Delta_t(w) \leq 2^{-m}\}.
\end{aligned}$$

These are similar to χ and ξ from the previous sections; however now the times denote the first time that the curve gets within a small Euclidean distance of the point, rather than a small conformal radius of the point. Let σ be the minimal τ_k such that $\Delta_{\tau_k}(z) < 2^{-n+1}$ or $\Delta_{\tau_k}(w) < 2^{-m+1}$. Let k_σ be the index so that $\sigma = \tau_{k_\sigma}$. If such a renewal time does not exist, let $k_\sigma = \infty$ and $\sigma = \infty$. Note that if ξ is finite, then so is σ.

Let $V_{z,k}$, V_z denote the events (and their indicator functions)

$$V_{z,k} = \{k_\sigma = k, R_\sigma = 0\}, \qquad V_z = \bigcup_{k=1}^{\infty} V_{z,k}.$$

We define V_w analogously. By the definition of σ, on the event the event V_z,

$$\Delta_{\tau_{k_\sigma - 1}}(z) \geq 2^{-n+1}, \qquad \Delta_{\tau_{k_\sigma - 1}}(w) \geq 2^{-m+1}, \qquad \Delta_\sigma(z) < 2^{-n+1}.$$

Also,

$$\Delta_\sigma(w) > 2^{-m},$$

for if $\Delta_\sigma(w) \leq 2^{-m}$, there would have been a renewal time after $\tau_{k_\sigma - 1}$ but before $\tau_k = \sigma$. Note that

$$\{\xi < \infty\} \subset [V_z \cap \{\xi_w < \infty\}] \cup [V_w \cap \{\xi_z < \infty\}].$$

We will concentrate on the event $V_z \cap \{\xi_w < \infty\}$; similar arguments handle the event $V_w \cap \{\xi_z < \infty\}$.

Define the integers (i_l, j_l) by stating that at the renewal time τ_l,

$$2^{-i_l} < \Delta_{\tau_l}(z) \leq 2^{-i_l+1}, \qquad 2^{-j_l} < \Delta_{\tau_l}(w) \leq 2^{-j_l+1}.$$

If $\sigma < \infty$, we write $(i_\sigma, j_\sigma) = (i_{k_\sigma}, j_{k_\sigma})$. On the event $k_\sigma = k$, $R_\sigma = 0$, there is a finite sequence of ordered triples

$$\pi = [(i_0, j_0, 0), (i_1, j_1, R_1), \ldots, (i_{k-1}, j_{k-1}, R_{k-1}), (i_k, j_k, R_k) = (i_\sigma, j_\sigma, 0)],$$
$$i_l, j_l \in \{1, 2, 3, \ldots\}, R_l \in \{0, 1\}.$$

We have the following properties for $0 \leq l \leq k - 1$:

- If $R_{l+1} = 0$, then $i_{l+1} \geq i_l + 1$ and $j_l \leq j_{l+1} \leq j_l + 1$.
- If $R_{l+1} = 1$, then $i_l \leq i_{l+1} \leq i_l + 1$ and $j_{l+1} \geq j_l + 1$.

We call any sequence of triples satisfying these two properties a *legal* sequence of length k. For any i, j, k, let $\mathcal{S}_k(i, j, 0)$ denote the collection of legal finite sequences of length k whose final triple is

$$(i_k, j_k, R_k) = (i, j, 0).$$

If π is a legal finite sequence of length k, let $V_{z,\pi}$ be the event that $k_\sigma = k$, $R_\sigma = 0$ and the renewal times up to and including σ give the sequence π. Figure 5 illustrates this definition.

Define K_l for $1 \leq l \leq k$ by

$$K_l = \begin{cases} i_{l-1}, & \text{if } R_l = 0, \\ j_{l-1}, & \text{if } R_l = 1. \end{cases}$$

The next proposition gives the fundamental estimate.

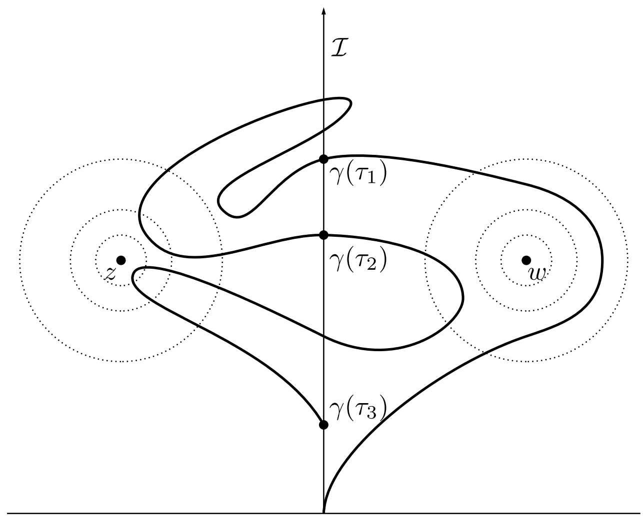

FIG. 5. *A curve γ (shown in bold) in $V_{z,\pi}$ with $\pi = [(0,0,0),(0,1,1),(2,1,0),(3,1,0)]$.*

PROPOSITION 4.14. *There exist c and an $\alpha > 0$ such that the following holds. Let i, j, k be integers, and let $\pi \in \mathcal{S}_k(i, j, 0)$. Then*

$$\mathbb{P}[V_{z,\pi} \cap \{\xi_w < \infty\}] \le c^k 2^{(m+n)(d-2)} e^{-\alpha(i+j-n)} \prod_{l=1}^{k} e^{-\alpha K_l}.$$

PROOF. Note that on the event V_z we may say by Lemma 2.11 that

$$\mathbb{P}\{\xi_w < \infty | \mathcal{F}_k\} \le c\left[\frac{2^{-j}}{\Delta^*_{\tau_k}(w)}\right]^{\beta/2} 2^{(m-j)(d-2)}.$$

We will proceed by splitting the event $V_{z,\pi}$ into the case where $\Delta^*_{\tau_k}(w) \ge 2^{-j}$ and the case where it is not.

Before doing so, we will discuss how to estimate $\mathbb{P}[V_{z,\pi}]$ without any further conditions, as it is important to our bounds below. By the definition of $\mathcal{S}_k(i, j, 0)$ we have that

$$\pi = [(i_0, j_0, 0), (i_1, j_1, R_1), \dots, (i_k, j_k, 0)],$$

where the sequence of triples is a legal sequence as described above.

We will estimate this probability by applying Lemma 4.10 to approximate the probability of each step; which is to say the probability that given that the SLE at time τ_l yields the triple (i_l, j_l, R_l) we get the triple $(i_{l+1}, j_{l+1}, R_{l+1})$

at the time τ_{l+1}. As the two cases are similar, we assume that $R_{l+1} = 0$. Since $K_{l+1} = i_l$, we know the distance to z at time τ_l is less than $2^{-K_{l+1}}$. We wish the distance from z to the SLE curve to decrease by at least a factor of $2^{i_{l+1}-i_l-1}$. The probability of this is shown by Lemma 4.10 to be of the order $c2^{-\alpha K_{l+1}}2^{(2-d)(i_{l+1}-i_l)}$ by absorbing factors into α and c we may rewrite this bound as $ce^{-\alpha K_{l+1}}2^{(d-2)(i_{l+1}-i_l+j_{l+1}-j_l)}$ as j_{l+1} can be at most one greater than j_l.

To get the probability of $V_{z,\pi}$, we need only multiply through each of these k individual probabilities to get that

$$\begin{aligned}
\mathbb{P}[V_{z,\pi}] &\leq \prod_{l=1}^{k} ce^{-\alpha K_l}2^{(d-2)(i_l-i_{l-1}+j_l-j_{l-1})} \\
&= c^k \exp\left\{\log(2)(d-2)\sum_{l=1}^{k}(i_l - i_{l-1} + j_l - j_{l-1})\right\}\prod_{l=1}^{k} e^{-\alpha K_l} \\
&= c^k 2^{(d-2)(i+j)}\prod_{l=1}^{k} e^{-\alpha K_l},
\end{aligned}$$

where we have absorbed the $2^{(d-2)(i_0+j_0)}$ (bounded above by a constant given the restrictions of z and w) into the c^k term in the last line by redefining c.

We now return to our main estimate. Note that, when $\Delta^*_{\tau_k}(w) \geq 2^{-j/2}$, we have

$$\begin{aligned}
&\mathbb{P}[V_{z,\pi} \cap \{\Delta^*_{\tau_k}(w) \geq 2^{-j/2}\} \cap \{\xi_w < \infty\}] \\
&\quad \leq c\mathbb{P}[V_{z,\pi} \cap \{\Delta^*_{\tau_k}(w) \geq 2^{-j/2}\}]2^{-\beta j/4}2^{(m-j)(d-2)} \\
&\quad \leq c\mathbb{P}[V_{z,\pi}]2^{-\beta j/4}2^{(m-j)(d-2)} \\
&\quad \leq c^k 2^{-\beta j/4}2^{(m-j)(d-2)}2^{(i+j)(d-2)}\prod_{l=1}^{k} e^{-\alpha K_l} \\
&\quad = c^k 2^{-\beta j/4}2^{(m+n)(d-2)}2^{(i-n)(d-2)}\prod_{l=1}^{k} e^{-\alpha K_l} \\
&\quad \leq c^k 2^{(m+n)(d-2)}2^{-\mu(i+j-n)}\prod_{l=1}^{k} e^{-\alpha K_l},
\end{aligned}$$

where the third line follows from the above discussion, and the last line holds for some choice of $\mu > 0$.

Thus we need only understand the event

$$\begin{aligned}
&\mathbb{P}[V_{z,\pi} \cap \{\Delta^*_{\tau_k}(w) < 2^{-j/2}\} \cap \{\xi_w < \infty\}] \\
&\quad \leq c\mathbb{P}[V_{z,\pi} \cap \{\Delta^*_{\tau_k}(w) < 2^{-j/2}\}]2^{(m-j)(d-2)}.
\end{aligned}$$

For the event $\{\Delta^*_{\tau_k}(w) < 2^{-j/2}\}$ there must be at least one l such that $\Delta^*_{\tau_l}(w) \geq 2^{-j/2}$ and $\Delta^*_{\tau_{l+1}}(w) < 2^{-j/2}$. By using Lemma 4.12 for that single step if $R_l = 1$ or Lemma 4.13 if $R_l = 0$ and 4.10 for all other steps, we have that

$$\begin{aligned}
&\mathbb{P}[V_{z,\pi} \cap \{\Delta^*_{\tau_k}(w) < 2^{-j/2}\}] \\
&\quad \leq \sum_{l=0}^{k-1} \mathbb{P}[V_{z,\pi} \cap \{\Delta^*_{\tau_l}(w) \geq 2^{-j/2}; \Delta^*_{\tau_{l+1}}(w) < 2^{-j/2}\}] \\
&\quad \leq kc^k 2^{-\alpha j/2} 2^{(i+j)(d-2)} \prod_{l=1}^{k} e^{-\alpha K_l}.
\end{aligned}$$

By combining this with the above event we see that

$$\begin{aligned}
&\mathbb{P}[V_{z,\pi} \cap \{\Delta^*_{\tau_k}(w) < 2^{-j/2}\} \cap \{\xi_w < \infty\}] \\
&\quad \leq kc^k 2^{(m-j)(d-2)} 2^{-\alpha j/2} 2^{(i+j)(d-2)} \prod_{l=1}^{k} e^{-\alpha K_l} \\
&\quad \leq c^k 2^{(m+n)(d-2)} 2^{-\mu(j+i-n)} \prod_{l=1}^{k} e^{-\alpha K_l}
\end{aligned}$$

for some choice of μ and where c is being used generically to absorb the leading k. Thus by choosing μ and α to be the same (which we can do by taking the minimum for both) we get the desired result. $\square$

We will now show how this proposition implies the main theorem. The proof rests upon the following combinatorial lemma.

LEMMA 4.15. *For every $\alpha > 0$, there exist c and a $u > 0$ such that for all k*

$$\sum_{\pi \in \mathcal{S}_k(i,j,0)} \prod_{l=1}^{k} e^{-\alpha K_l} \leq ce^{-uk^2}.$$

PROOF. We fix α and allow all constants to depend on α. Let

$$\Sigma_k = \sum_{[m]_k} \prod_{l=1}^{k} e^{-\alpha m_l},$$

where the sum is over all strictly increasing finite sequences of positive integers, written as $[m]_k := [m_1, m_2, \ldots, m_k]$. We first claim that

$$\Sigma_k \leq c_1 e^{-\alpha k^2/4}.$$

Consider the following recursive relation:

$$\begin{aligned}\Sigma_k &= \sum_{[m]_k} \prod_{l=1}^{k} e^{-\alpha m_l} \\ &\le \sum_{[m]_{k-1}} \sum_{m_k=k}^{\infty} e^{-\alpha m_k} \prod_{l=1}^{k-1} e^{-\alpha m_l} \\ &= \Sigma_{k-1} \sum_{j=k}^{\infty} e^{-\alpha j} \\ &\le c_2 \Sigma_{k-1} e^{-\alpha k}.\end{aligned}$$

Applying this bound inductively to Σ_k yields

$$\Sigma_k \le c_2^k \exp\left\{-\alpha \sum_{i=1}^{k} i\right\} \le c_1 e^{-\alpha k^2/4}$$

as desired.

To choose a legal sequence in $\mathcal{S}_k(i, j, 0)$, there are 2^{k-1} ways to choose the values $R_1, \dots, R_{k-1}$. Given the values of $R_1, \dots, R_{k-1}$ we choose the increases of the integers. If $R_l = 0$, then $i_l > i_{l-1}$ and $j_l = j_{l-1}$ or $j_l = j_{l-1} + 1$. The analogous inequalities hold if $R_1 = 1$. There are 2^k ways to choose whether $j_l = j_{l-1}$ or $j_l = j_{l-1} + 1$ (or the corresponding jump for i_l if $R_1 = 1$). In the other components we have to increase by an integer. We therefore get that the sum is bounded above by

$$\begin{aligned}2^{k-1} \max_{0 \le l \le k-1} 2^l \Sigma_l \cdot 2^{k-l-1} \Sigma_{k-l-1} &\le c^k \max_{0 \le l \le k-1} e^{-\alpha l^2/4} e^{-\alpha(k-l-1)^2/4} \\ &\le c e^{-uk^2}.\end{aligned}$$

□

By combining Proposition 4.14 and Lemma 4.15, there exist c such that

$$\sum_{k=1}^{\infty} \sum_{\pi \in \mathcal{S}_k(i,j,0)} \mathbb{P}[V_{z,\pi} \cap \{\xi_w < \infty\}] \le c 2^{(m+n)(d-2)} e^{-\alpha(j+i-n)},$$

and hence by summing over $i \ge n-1$, $j \ge 0$ we get

$$\begin{aligned}\mathbb{P}[V_z \cap \{\xi < \infty\}] &\le \mathbb{P}[V_z \cap \{\xi_w < \infty\}] \\ &= \sum_{i=n-1}^{\infty} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \sum_{\pi \in \mathcal{S}_k(i,j,0)} \mathbb{P}[V_{z,\pi} \cap \{\xi_w < \infty\}] \\ &\le c 2^{(m+n)(d-2)} = c \varepsilon^{2-d} \delta^{2-d}.\end{aligned}$$

By the symmetry of z, w we have the bound

$$\mathbb{P}[V_w \cap \{\xi < \infty\}] \leq c\varepsilon^{2-d}\delta^{2-d}$$

and hence

$$\mathbb{P}\{\Delta_\infty(z) \leq \varepsilon, \Delta_\infty(w) \leq \delta\} = \mathbb{P}\{\xi < \infty\} \leq c\varepsilon^{2-d}\delta^{2-d}$$

as required to complete the proof of Proposition 4.1, and hence the proof of Beffara's estimate.

With the proof set up in this way, we may now rapidly complete our proof of the existence of the multi-point Green's function. By mirroring the proof above, we may conclude that for $\rho = e^{-\ell}$ (and hence for all ρ) that

$$\begin{aligned}\mathbb{P}[V_z \cap \{\xi < \infty, \Delta_\sigma(w) \leq \rho\}] &\leq \mathbb{P}[V_z \cap \{\xi_w < \infty, \Delta_\sigma(w) \leq \rho\}] \\ &= \sum_{i=n-1}^{\infty} \sum_{j=\ell}^{\infty} \sum_{k=1}^{\infty} \sum_{\pi \in \mathcal{S}_k(i,j,0)} \mathbb{P}[V_{z,\pi} \cap \{\xi_w < \infty\}] \\ &\leq c2^{(m+n)(d-2)} e^{-\alpha\ell} = c\varepsilon^{2-d}\delta^{2-d}\rho^\alpha.\end{aligned}$$

This proves Proposition 4.2 and hence completes the proof of the existence of the multi-point Green's function.

APPENDIX A: THE EXISTENCE OF THE I_t

The aim of this Appendix is to prove the existence of the separating set I_t desired above.

DEFINITION. Let γ be a curve in the upper half-plane, and let $z, w, \mathcal{I}$ be a pair or distinct points in $\mathbb{H}$ separated by the line $\mathcal{I}$. Let $\mathcal{I}_t = \mathcal{I} \setminus \gamma(0, t]$. We will denote by I_t the unique open interval contained in $\mathcal{I}$ such that the following four properties hold. For any $t \leq t'$ we have:

- I_t is a connected component of $\mathcal{I}_t$,
- the I_t are decreasing, which is to say $I_{t'} \subseteq I_t$,
- $H_t \setminus I_t$ has exactly two connected components, one containing z and one containing w and
- $I_t = I_{t'}$ whenever $\gamma(t, t'] \cap \mathcal{I} = \varnothing$.

It may, at first glance, seem simple to define such sets inductively. However, in general, the set of times that a curve γ crosses $\mathcal{I}$ may be uncountable and have no well-defined notion of "the previous crossing." To avoid this issue and show this notion is well defined, we require a few topological lemmas.

LEMMA A.1. *Let U be a connected open set in $\mathbb{C}$ separated by a smooth simple curve $\eta : [0, 1] \to \overline{U}$. Let $V \subset U$ be a connected open subset. Then for any points $z, w \in V$, there exits a curve $\xi : [0, 1] \to V$ from z to w which intersects η a finite number of times.*

PROOF. This proof mirrors the classic proof that a connected open set is path connected. Define an equivalence relation on V where points $z, w \in V$ are equivalent, if z can be connected to w by a curve ξ which intersects η a finite number of times. This can readily be shown to satisfy the requirements of an equivalence relation.

Let V_α denote the open connected components of $V \setminus \eta$. If z, w are both in the same V_α, then they may be connected by a curve which does not intersect η; hence each V_α is contained entirely in a single equivalence class.

Consider a disc, D, contained in V centered on a point $\eta(t_0)$ for some $t_0 \in (0, 1)$ with components V_α and V_β on either side of η near this point. Since η is smooth and simple, by choosing D sufficiently small we may find a diffeomorphism ϕ so that $\phi(D) = \mathbb{D}$ and $\phi(\eta \cap D) = \{it : t \in (-1, 1)\}$. Connect $-1/2$ to $1/2$ by the straight line between them, which only intersects the image of η once. Taking the image of this line under ϕ^{-1} gives a curve ξ satisfying the conditions of the equivalence relation connecting two points, one in V_α and one in V_β. Thus components of $V \setminus \eta$ which are directly separated by η are in the same equivalence class. Since V is connected, the only equivalence class is V itself. $\square$

Suppose U is a connected open set in $\mathbb{C}$ separated by a curve $\eta : (0, 1) \to U$ into two components U_1, U_2 with points $z \in U_1$ and $w \in U_2$. Let V be a connected subset of U. Define $\mathcal{D}_V(z, w; \eta)$ to be the the set of connected components of $V \cap \eta$ which disconnects z from w in V.

COROLLARY A.2. *Let U be a connected open set in $\mathbb{C}$ separated by a smooth simple curve $\eta : [0, 1] \to \overline{U}$ into two components U_1, U_2 with $z \in U_1$ and $w \in U_2$. Let $V \subset U$ be a connected open subset containing z and w. Then $|\mathcal{D}_V(z, w, \eta)|$ is finite and odd.*

PROOF. To see that the number is finite, take the curve ξ between z and w as in the above lemma, and note that any η_i which separates z from w must intersect ξ.

To see that it is odd, consider the connected components of $V' := V \setminus \bigcup_{\gamma \in \mathcal{D}_V(z,w;\eta)} \gamma$. There are exactly $|\mathcal{D}_V(z, w; \eta)| + 1$ such components. η separates U into two components, and hence the components of V' are alternately contained in U_1 and U_2. Since the component containing z is in U_1, and the component containing w is in U_2, there must be an even number of components of V', which makes $|\mathcal{D}_V(z, w; \eta)|$ odd. $\square$

This general topological lemma has the following consequence in our setting. To simplify notation, we will define $\mathcal{D}_t = \mathcal{D}_{H_t}(z, w, \mathcal{I})$.

COROLLARY A.3. *Fix $0 \le t' \le t < \infty$. Then a connected component I of $\mathcal{I}_{t'}$ separates z from w in $H_{t'}$ if and only if the number of elements of $\mathcal{D}_t$ contained in I is odd.*

PROOF. The "only if" direction is precisely Corollary A.2. Thus we wish to show that if the number of elements of $\mathcal{D}_t$ contained in I is odd, then I separates z from w.

Assume not, so the number of elements of $\mathcal{D}_t$ contained in I is odd but I does not separate z from w. $H_{t'} \setminus I$ has two components, one of which contains both z and w. Consider any curve η connecting z to w. Without loss of generality assume that η crosses each element of $\mathcal{D}_t$ exactly once by simply removing any portion of the curve between the first and last times that it crosses each element of $\mathcal{D}_t$. Since η crosses each element of $\mathcal{D}_t$ contained in I precisely once, we know η crosses I an odd number of times, and hence it must start and end in different components of $H_{t'} \setminus I$ which contradicts the fact that it connects z to w. $\square$

We may now use this to prove that I_t is well defined.

PROOF OF WELL-DEFINEDNESS OF I_t. For a component I of $\mathcal{I}_t$ and $t' < t$, let $C_{t'}(I)$ denote the component of $\mathcal{I}_{t'}$ which contains I. We claim there exists a unique component of $\mathcal{I}_t$, which we will denote I_t, such that for all $0 \le t' \le t$, we have $C_{t'}(I_t) \in \mathcal{D}_{t'}$. Note that such an I_t clearly satisfies all the conditions of the definition.

First we prove existence. Let $\{J_i\}_{i=1}^{\infty}$ be the connected components of $\mathcal{I}_t$. Assume that none satisfy the above condition, which is to say that for each i there exists a $t_i \le t$ so that $C_{t_i}(J_i)$ does not separate z from w in H_{t_i}. Now $\{C_{t_i}(J_i)\}_{i=1}^{\infty}$ covers $\mathcal{I}_t$ since the J_i did as well, and moreover since by construction the $C_{t_i}(J_i)$ are either contained in each other or disjoint, we may find a subcollection $\{C_{t_{i_k}}(J_{i_k})\}_{k=1}^{\infty}$ which covers $\mathcal{I}_t$ with all elements pairwise disjoint. By Corollary A.3 there are an even number of elements of $\mathcal{D}_t$ contained in $C_{t_{i_k}}(J_{i_k})$ for each k. However, since they cover disjointly, this implies that $|\mathcal{D}_t|$ is even, which contradicts Corollary A.2 completing the proof of existence.

Now we establish uniqueness. Let $I_t^{(1)}, I_t^{(2)}, \dots, I_t^{(\ell)}$ denote the components of $\mathcal{I}_t$ such that for all $0 \le t' \le t$ we have $C_{t'}(I_t^{(i)}) \in \mathcal{D}_{t'}$, and assume that $\ell > 1$. Define

$$t_0 = \sup\{t' : \exists_{i \neq j} \text{ s.t. } C_{t'}(I_t^{(i)}) = C_{t'}(I_t^{(j)})\}.$$

By this definition, it is clear that $\gamma(t_0) \in \mathcal{I}$. Moreover, there exists a $t_1 < t_0$ such that $\gamma[t_1, t_0) \cap \mathcal{I} = \varnothing$ since if there did not then $\gamma(t_0)$ is a limit point of $\gamma(0, t_0) \cap \mathcal{I}$ which implies that an earlier time would have separated all the $I_t^{(i)}$ from each other contradicting the choice of t_0. The components of $\mathcal{I}_{t_0}$ are precisely those of $\mathcal{I}_{t_1}$ except for a single component, call it J, which is split into J_1, J_2 in $\mathcal{I}_{t_0}$ by $\gamma(t_0)$. By the choice of t_0, J is $C_{t_1}(I_t^{(i)})$ for some i and both of J_1, J_2 are $C_{t_0}(I_t^{(i)})$ for some i. This is a contradiction since by Corollary A.3 each of J, J_1, J_2 must contain an odd number of elements of $\mathcal{D}_t$. $\square$

APPENDIX B: THE PDE FOR THE GREEN'S FUNCTION

We outline here the derivation of the PDE which governs the ordered version of the multi-point Green's function. From Theorem 1, we know that for $z, w \in \mathbb{H}$ with

$$\xi = \xi_\varepsilon = \xi_{z,\varepsilon} = \inf\{t : \Upsilon_t(z) \leq \varepsilon\},$$
$$\chi = \chi_\delta = \chi_{w,\delta} = \inf\{t : \Upsilon_t(w) \leq \delta\},$$

we have that

$$G_{\mathbb{H}}(z, w; 0, \infty) = \frac{1}{c_*^2} \lim_{\varepsilon,\delta\to 0^+} \varepsilon^{d-2}\delta^{d-2}\mathbb{P}\{\xi < \chi < \infty\}.$$

By the domain Markov property, and conformal invariance of SLE, one can deduce that

$$\begin{aligned} M_t &:= \mathbb{E}[G_{\mathbb{H}}(z, w; 0, \infty)|\mathcal{F}_t] \\ &= G_{H_t}(z, w; 0, \infty) \\ &= |Z_t'(z)|^{2-d}|Z_t'(w)|^{2-d} \cdot G_{\mathbb{H}}(Z_t(z), Z_t(w); 0, \infty) \end{aligned}$$

is a local martingale, where Z_t is the unique conformal map defined by (5) which maps H_t to $\mathbb{H}$, sending $\gamma(t)$ to 0. We will find the SDE which M_t satisfies and use that the drift must zero to find the differential equation that $G(x_1, y_1, x_1, y_2) := G_{\mathbb{H}}(x_1 + iy_1, x_2 + iy_2; 0, \infty)$ must satisfy.

From (5), we know that

$$dZ_t(z) = \frac{a}{Z_t(z)}\, dt + dB_t,$$

and hence, letting $Z_t(z) = X_t(z) + iY_t(z)$, we see that

$$dX_t(z) = \frac{aX_t(z)}{X_t(z)^2 + Y_t(z)^2}\, dt + dB_t,$$
$$dY_T(z) = -\frac{aY_t(z)}{X_t(z)^2 + Y_t(z)^2}\, dt.$$

To compute the SDE for $|Z_t'(z)|$, we must use the logarithm. First note that

$$dZ_t'(z) = -\frac{aZ_t'(z)}{Z_t(z)^2}\, dt$$

and hence that

$$d[\log Z_t'(z)] = \frac{dZ_t'(z)}{Z_t'(z)} = -\frac{a}{Z_t(z)^2}\, dt.$$

We may thus recover the norm of the absolute value by considering the real part, yielding

$$d|Z_t'(z)|^{2-d} = a(d-2)|Z_t'(z)|^{2-d} \frac{X_t(z)^2 - Y_t(z)^2}{(X_t(z)^2 + Y_t(z)^2)^2}\, dt.$$

From these, we may compute the equation for M_t. Note that only $X_t(z)$ and $X_t(w)$ have nonzero diffusion coefficients. Suppressing the arguments of G in the notation, we obtain the following:

$$\begin{aligned} dM_t = M_t\bigg[& a(d-2)\frac{X_t(z)^2 - Y_t(z)^2}{(X_t(z)^2+Y_t(z)^2)^2} + a(d-2)\frac{X_t(w)^2 - Y_t(w)^2}{(X_t(w)^2+Y_t(w)^2)^2} \\ & + \frac{aX_t(z)}{X_t(z)^2+Y_t(z)^2}\frac{\partial_{x_1}G}{G} + \frac{aX_t(w)}{X_t(w)^2+Y_t(w)^2}\frac{\partial_{x_2}G}{G} \\ & - \frac{aY_t(z)}{X_t(z)^2+Y_t(z)^2}\frac{\partial_{y_1}G}{G} - \frac{aY_t(w)}{X_t(w)^2+Y_t(w)^2}\frac{\partial_{y_2}G}{G} \\ & + \frac{1}{2}\frac{\partial_{x_1x_1}G}{G} + \frac{1}{2}\frac{\partial_{x_2x_2}G}{G} + \frac{\partial_{x_1x_2}G}{G}\bigg] dt \\ & + M_t\bigg[\frac{\partial_{x_1}G}{G} + \frac{\partial_{x_2}G}{G}\bigg] dB_t. \end{aligned}$$

Collecting together the drift terms and specializing to $t=0$ yields

$$\begin{aligned} 0 = & a(d-2)\frac{x_1^2-y_1^2}{(x_1^2+y_1^2)^2}G + a(d-2)\frac{x_2^2-y_2^2}{(x_2^2+y_2^2)^2}G \\ & + a\frac{x_1\,\partial_{x_1}G - y_1\,\partial_{y_1}G}{x_1^2+y_1^2} + a\frac{x_2\,\partial_{x_2}G - y_2\,\partial_{y_2}G}{x_2^2+y_2^2} \\ & + \frac{1}{2}\partial_{x_1x_1}G + \frac{1}{2}\partial_{x_2x_2}G + \partial_{x_1x_2}G. \end{aligned}$$

This PDE has a particularly nice structure. Let

$$L_i = a(d-2)\frac{x_i^2-y_i^2}{(x_i^2+y_i^2)^2} + a\frac{x_i\,\partial_{x_i} - y_i\,\partial_{y_i}}{x_i^2+y_i^2} + \frac{1}{2}\partial_{x_ix_i}.$$

This can be seen to be precisely the differential operator which arises in the computation of the single point Green's function, but now we have a copy for both z and w. With this we can rewrite the equation for the multi-point Green's function as

$$(L_1 + L_2 + \partial_{x_1x_2})G = 0.$$

Given this simple form it may be reasonable to look for solutions which are, in some sense, asymptotically $G(z)G(w)$. Additionally, it is worth noting that this extends to arbitrary n-point Green's functions by

$$\Bigg[\sum_{i=1}^{n} L_i + \sum_{1\le i<j\le n}\partial_{x_ix_j}\Bigg]G = 0$$

as one might expect.

The boundary conditions of this equation are not clear, and their determination may provide bounds of intrinsic interest.

The above equation shows the PDE in its most symmetric form; however, in order to find an explicit solution, it may be useful to exploit the scaling rule for the Green's function to reduce this to an equation for a function of three real variables. There is no unique way to do so, and no such reductions have lead to a particularly simple equation. A reasonable example would be to scale the above equation so that $y_1 = 1$ in which case we can find a three real variable function $\hat{G}$ so that

$$G(x_1, y_1, x_2, y_2) = y_1^{2(d-2)} \hat{G}\left(\frac{x_1}{y_1}, \frac{x_2}{y_1}, \frac{y_2}{y_1}\right).$$

From this the PDE can be derived; however, the result is not illuminating.

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