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THE TOPOLOGY OF SCALING LIMITS OF POSITIVE GENUS RANDOM QUADRANGULATIONS[1]

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We discuss scaling limits of large bipartite quadrangulations of positive genus. For a given g, we consider, for every $n \geq 1$, a random quadrangulation $\mathfrak{q}_n$ uniformly distributed over the set of all rooted bipartite quadrangulations of genus g with n faces. We view it as a metric space by endowing its set of vertices with the graph metric. As n tends to infinity, this metric space, with distances rescaled by the factor $n^{-1/4}$, converges in distribution, at least along some subsequence, toward a limiting random metric space. This convergence holds in the sense of the Gromov–Hausdorff topology on compact metric spaces. We show that, regardless of the choice of the subsequence, the limiting space is almost surely homeomorphic to the genus g-torus.

1. Introduction.

1.1. *Motivation.* The present work is a sequel to a work by Bettinelli [5], whose aim is to investigate the topology of scaling limits for random maps of arbitrary genus. A map is a cellular embedding of a finite graph (possibly with multiple edges and loops) into a compact connected orientable surface without boundary, considered up to orientation-preserving homeomorphisms. By *cellular*, we mean that the faces of the map—the connected components of the complement of edges—are all homeomorphic to disks. The genus of the map is defined as the genus of the surface into which it is embedded. For technical reasons, it will be convenient to deal with rooted maps, meaning that one of the half-edges—or oriented edges—is distinguished.

We will particularly focus on bipartite quadrangulations: a map is a quadrangulation if all its faces have degree 4; it is bipartite if each vertex can be colored in black or white, in such a way that no edge links two vertices that have the same color. Although in genus $g = 0$, all quadrangulations are bipartite, this is no longer true in positive genus $g \geq 1$.

A natural way to generate a large random bipartite quadrangulation of genus g is to choose it uniformly at random from the set $\mathcal{Q}_n$ of all rooted bipartite quadrangulations of genus g with n faces, and then consider the limit as n goes to infinity. A natural setting for this problem is to consider quadrangulations as metric spaces

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endowed with their graph metric, properly rescaled by the factor $n^{-1/4}$ [21] and to study their limit in the Gromov–Hausdorff topology [15]. From this point of view, the planar case $g = 0$ has largely been studied during the last decade. Le Gall [18] showed the convergence of these metric spaces along some subsequence. It is believed that the convergence holds without the "along some subsequence" part in the last sentence, and Le Gall gave a conjecture for a limiting space to this sequence [18]. Although the whole convergence is yet to be proved, some information is available on the accumulation points of this sequence. Le Gall and Paulin [20] proved that every possible limiting metric space is almost surely homeomorphic to the two-dimensional sphere. Miermont [22] later gave a variant proof of this fact.

We showed in [5] that the convergence along some subsequence still holds in any fixed positive genus g. In this work, we show that the topology of every possible limiting space is that of the genus g-torus $\mathbb{T}_g$.

1.2. *Main results*. We will work in fixed genus g. On the whole, we will not let it figure in the notation, in order to lighten them. As the case $g = 0$ has already been studied, we suppose $g \geq 1$.

Recall that the Gromov–Hausdorff distance between two compact metric spaces $(\mathcal{X}, \delta)$ and $(\mathcal{X}', \delta')$ is defined by

$$d_{\mathrm{GH}}((\mathcal{X}, \delta), (\mathcal{X}', \delta')) := \inf\{\delta_{\mathcal{H}}(\varphi(\mathcal{X}), \varphi'(\mathcal{X}'))\},$$

where the infimum is taken over all isometric embeddings $\varphi : \mathcal{X} \to \mathcal{X}''$ and $\varphi' : \mathcal{X}' \to \mathcal{X}''$ of $\mathcal{X}$ and $\mathcal{X}'$ into the same metric space $(\mathcal{X}'', \delta'')$, and $\delta_{\mathcal{H}}$ stands for the usual Hausdorff distance between compact subsets of $\mathcal{X}''$. This defines a metric on the set $\mathbb{M}$ of isometry classes of compact metric spaces [8], Theorem 7.3.30, making it a Polish space.[2]

For any map $\mathfrak{m}$, we call $V(\mathfrak{m})$ its set of vertices. There exists on $V(\mathfrak{m})$ a natural graph metric $d_{\mathfrak{m}}$: for any vertices a and $b \in V(\mathfrak{m})$, the distance $d_{\mathfrak{m}}(a, b)$ is defined as the number of edges of any shortest path linking a to b. The main result of [5] is the following.

PROPOSITION 1. *Let $\mathfrak{q}_n$ be uniformly distributed over the set $\mathcal{Q}_n$ of all bipartite quadrangulations of genus g with n faces. Then, from any increasing sequence of integers, we may extract a subsequence $(n_k)_{k\geq 0}$ such that there exists a metric space $(\mathfrak{q}_\infty, d_\infty)$ satisfying*

$$\left(V(\mathfrak{q}_{n_k}), \frac{1}{\gamma n_k^{1/4}} d_{\mathfrak{q}_{n_k}}\right) \xrightarrow[k\to\infty]{(d)} (\mathfrak{q}_\infty, d_\infty)$$

in the sense of the Gromov–Hausdorff topology, where

$$\gamma := \left(\tfrac{8}{9}\right)^{1/4}.$$

[2]This is a simple consequence of Gromov's compactness theorem [8], Theorem 7.4.15.

Moreover, the Hausdorff dimension of the limit space $(\mathfrak{q}_\infty, d_\infty)$ *is almost surely equal to* 4, *regardless of the choice of the sequence of integers.*

Remark that the constant γ is not necessary in this statement (simply change d_∞ into γd_∞). We kept it for the sake of consistency with [5], and because of our definition of d_∞ later in the paper, although it is irrelevant for the moment. Note also that, a priori, the metric space $(\mathfrak{q}_\infty, d_\infty)$ depends on the subsequence $(n_k)_{k\geq 0}$. Similarly to the planar case, we believe that the extraction in Proposition 1 is not necessary, and we conjecture the space $(\mathfrak{q}_\infty, d_\infty^*)$ for the limit, where d_∞^* was defined at the end of Section 6.3 in [5]. We also believe that the space $(\mathfrak{q}_\infty, d_\infty^*)$ is somewhat universal, in the sense that we conjecture it as the scaling limit of more general classes of random maps. More precisely, we think that Proposition 1 still holds while replacing the class of quadrangulations with some other "reasonable" class of maps, as well as the constant γ, which is inherent to the class of quadrangulations, with the appropriate constant. In particular, our approach can be generalized to the case of $2p$-angulations, $p \geq 2$, by following the same lines as Le Gall in [18].

We may now state our main result, which identifies the topology of $(\mathfrak{q}_\infty, d_\infty)$, regardless of the subsequence $(n_k)_{k\geq 0}$.

THEOREM 2. *The metric space* $(\mathfrak{q}_\infty, d_\infty)$ *is a.s. homeomorphic to the* g-*torus* $\mathbb{T}_g$.

In the general picture, we rely on the same techniques as in the planar case. The starting point is to use a bijection due to Chapuy, Marcus and Schaeffer [10] between bipartite quadrangulations of genus g with n faces and so-called well-labeled g-trees with n edges. The study of the scaling limit as $n \to \infty$ of uniform random well-labeled g-trees with n edges was the major purpose of [5]. This study leads to the construction of a continuum random g-tree, which generalizes Aldous's CRT [1, 2]. The first step of our proof is to carry out the analysis of Le Gall [18] in the nonplanar case and see the space $(\mathfrak{q}_\infty, d_\infty)$ as a quotient of this continuum random g-tree via an equivalence relation defined in terms of Brownian labels on it. We then adapt Miermont's approach [22], and use the notion of 1-regularity introduced by Whyburn [26] and studied by Whyburn and Begle [3, 26] in order to see that the genus remains the same in the limit.

Finally, we deduce the technical estimates we need from the planar case thanks to a bijection due to Chapuy [9] between well-labeled g-trees and well-labeled plane trees with g distinguished triples of vertices.

We will use the background provided in [5]. We briefly recall it in Section 2. In Section 3, we define real g-trees and explain how we may see $(\mathfrak{q}_\infty, d_\infty)$ as a quotient of such objects. Theorem 8 in Section 4 gives a criteria telling which points are identified in this quotient, and Section 5 is dedicated to the proof of Theorem 2. Finally, we expose in Section 6 Chapuy's bijection, and use it to prove four technical lemmas stated during Section 4.

2. Preliminaries. In this section, we recall the notation, settings and results from [5] that we will need for this work. We refer the reader to [5] for more details.

We use the following formalism for maps. For any map $\mathfrak{m}$, we denote by $V(\mathfrak{m})$ and $E(\mathfrak{m})$, respectively, its sets of vertices and edges. We also call $\vec{E}(\mathfrak{m})$ its set of half-edges, and $\mathfrak{e}_* \in \vec{E}(\mathfrak{m})$ its root. For any half-edge $\mathfrak{e}$, we write $\bar{\mathfrak{e}}$ its reverse—so that $E(\mathfrak{m}) = \{\{\mathfrak{e}, \bar{\mathfrak{e}}\} : \mathfrak{e} \in \vec{E}(\mathfrak{m})\}$—as well as $\mathfrak{e}^-$ and $\mathfrak{e}^+$ its origin and end. Finally, we say that $\check{E}(\mathfrak{m}) \subset \vec{E}(\mathfrak{m})$ is an orientation of the half-edges if for every edge $\{\mathfrak{e}, \bar{\mathfrak{e}}\} \in E(\mathfrak{m})$ exactly one of $\mathfrak{e}$ or $\bar{\mathfrak{e}}$ belongs to $\check{E}(\mathfrak{m})$.

2.1. *The Chapuy–Marcus–Schaeffer bijection.* The first main tool we will need consists of the Chapuy–Marcus–Schaeffer bijection [10], Corollary 2 to Theorem 1, which allows us to code (rooted) quadrangulations by so-called well-labeled (rooted) g-trees.

A g-tree is a map of genus g with only one face. This notion naturally generalizes the notion of plane tree: in particular, 0-trees are plane trees. It may be convenient to represent a g-tree $\mathfrak{t}$ with n edges by a $2n$-gon whose edges are pairwise identified (see Figure 1). We note $\mathfrak{e}_1 := \mathfrak{e}_*, \mathfrak{e}_2, \dots, \mathfrak{e}_{2n}$ the half-edges of $\mathfrak{t}$ arranged according to the clockwise order around this $2n$-gon. The half-edges are said to be arranged according to the *facial order* of $\mathfrak{t}$. Informally, for $2 \le i \le 2n$, $\mathfrak{e}_i$ is the "first half-edge to the left after $\mathfrak{e}_{i-1}$." We call *facial sequence* of $\mathfrak{t}$ the sequence $\mathfrak{t}(0), \mathfrak{t}(1), \dots, \mathfrak{t}(2n)$ defined by $\mathfrak{t}(0) = \mathfrak{t}(2n) = \mathfrak{e}_1^- = \mathfrak{e}_{2n}^+$ and for $1 \le i \le 2n-1$, $\mathfrak{t}(i) = \mathfrak{e}_i^+ = \mathfrak{e}_{i+1}^-$. Imagine a fly flying along the boundary of the unique face of $\mathfrak{t}$. Let it start at time 0 by following the root $\mathfrak{e}_*$, and let it take one unit of time to follow each half-edge, then $\mathfrak{t}(i)$ is the vertex where the fly is at time i.

Let $\mathfrak{t}$ be a g-tree. Two vertices $u, v \in V(\mathfrak{t})$ are said to be *neighbors*, and we write $u \sim v$, if there is an edge linking them.

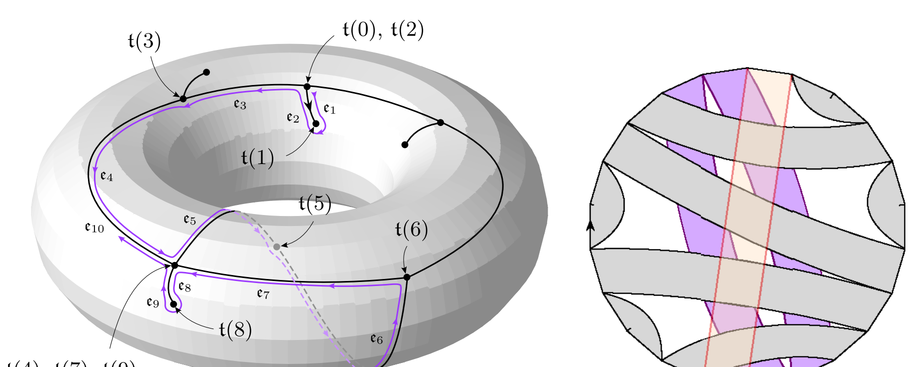

FIG. 1. Left. *The facial order and facial sequence of a g-tree*. Right. *Its representation as a polygon whose edges are pairwise identified.*

DEFINITION 1. A *well-labeled g-tree* is a pair $(\mathfrak{t}, \mathfrak{l})$ where $\mathfrak{t}$ is a g-tree and $\mathfrak{l} : V(\mathfrak{t}) \to \mathbb{Z}$ is a function (thereafter called *labeling function*) satisfying:

(i) $\mathfrak{l}(\mathfrak{e}_*^-) = 0$, where $\mathfrak{e}_*$ is the root of $\mathfrak{t}$;
(ii) if $u \sim v$, then $|\mathfrak{l}(u) - \mathfrak{l}(v)| \le 1$.

We call $\mathcal{T}_n$ the set of all well-labeled g-trees with n edges. A *pointed quadrangulation* is a pair $(\mathfrak{q}, v^\bullet)$ consisting in a quadrangulation $\mathfrak{q}$ together with a vertex $v^\bullet \in V(\mathfrak{q})$. We call $\mathcal{Q}_n^\bullet := \{(\mathfrak{q}, v^\bullet) : \mathfrak{q} \in \mathcal{Q}_n, v^\bullet \in V(\mathfrak{q})\}$ the set of all pointed bipartite quadrangulations of genus g with n faces.

The Chapuy–Marcus–Schaeffer bijection is a bijection between the sets $\mathcal{T}_n \times \{-1, +1\}$ and $\mathcal{Q}_n^\bullet$. We briefly describe here the mapping from $\mathcal{T}_n \times \{-1, +1\}$ onto $\mathcal{Q}_n^\bullet$, and we refer the reader to [10] for a more precise description. Let $(\mathfrak{t}, \mathfrak{l}) \in \mathcal{T}_n$ be a well-labeled g-tree with n edges and $\varepsilon_\pm \in \{-1, +1\}$. As above, we write $\mathfrak{t}(0), \mathfrak{t}(1), \dots, \mathfrak{t}(2n)$ its facial sequence. The pointed quadrangulation $(\mathfrak{q}, v^\bullet)$ corresponding to $((\mathfrak{t}, \mathfrak{l}), \varepsilon_\pm)$ is then constructed as follows. First, shift all the labels in such a way that the minimal label is equal to 1. Let us call $\tilde{\mathfrak{l}} := \mathfrak{l} - \min \mathfrak{l} + 1$ this shifted labeling function. Then, add an extra vertex $v^\bullet$ carrying the label $\tilde{\mathfrak{l}}(v^\bullet) := 0$ inside the only face of $\mathfrak{t}$. Finally, following the facial sequence, for every $0 \le i \le 2n - 1$, draw an arc—without crossing any edge of $\mathfrak{t}$ or arc already drawn—between $\mathfrak{t}(i)$ and $\mathfrak{t}(\mathrm{succ}(i))$, where $\mathrm{succ}(i)$ is the *successor* of i, defined by

$$\mathrm{succ}(i) := \begin{cases} \inf\{k \ge i : \tilde{\mathfrak{l}}(\mathfrak{t}(k)) = \ell\}, & \text{if } \{k \ge i : \tilde{\mathfrak{l}}(\mathfrak{t}(k)) = \ell\} \ne \varnothing, \\ \inf\{k \ge 1 : \tilde{\mathfrak{l}}(\mathfrak{t}(k)) = \ell\}, & \text{otherwise,} \end{cases} \tag{1}$$

where $\ell = \tilde{\mathfrak{l}}(\mathfrak{t}(i)) - 1$, and with the conventions $\inf \varnothing = \infty$, and $\mathfrak{t}(\infty) = v^\bullet$.

The quadrangulation $\mathfrak{q}$ is then defined as the map whose set of vertices is $V(\mathfrak{t}) \cup \{v^\bullet\}$, whose edges are the arcs we drew and whose root is the first arc drawn, oriented *from* $\mathfrak{t}(0)$ if $\varepsilon_\pm = -1$ or *toward* $\mathfrak{t}(0)$ if $\varepsilon_\pm = +1$; see Figure 2.

Because of the way we drew the arcs of $\mathfrak{q}$, we see that for any vertex $v \in V(\mathfrak{q})$, $\tilde{\mathfrak{l}}(v) = d_\mathfrak{q}(v^\bullet, v)$. When seen as a vertex in $V(\mathfrak{q})$, we write $\mathfrak{q}(i)$ instead of $\mathfrak{t}(i)$. In particular, $\{\mathfrak{q}(i), 0 \le i \le 2n\} = V(\mathfrak{q}) \setminus \{v^\bullet\}$.

We end this section by giving an upper bound for the distance between two vertices $\mathfrak{q}(i)$ and $\mathfrak{q}(j)$, in terms of the labeling function $\mathfrak{l}$:

$$\begin{aligned} &d_\mathfrak{q}(\mathfrak{q}(i), \mathfrak{q}(j)) \\ &\quad \le \mathfrak{l}(\mathfrak{t}(i)) + \mathfrak{l}(\mathfrak{t}(j)) - 2\max\Bigl(\min_{k \in \overrightarrow{[\![i,j]\!]}} \mathfrak{l}(\mathfrak{t}(k)), \min_{k \in \overrightarrow{[\![j,i]\!]}} \mathfrak{l}(\mathfrak{t}(k))\Bigr) + 2, \end{aligned} \tag{2}$$

where we note, for $i \le j$, $[\![i, j]\!] := [i, j] \cap \mathbb{Z} = \{i, i+1, \dots, j\}$, and

$$\overrightarrow{[\![i,j]\!]} := \begin{cases} [\![i, j]\!], & \text{if } i \le j, \\ [\![i, 2n]\!] \cup [\![0, j]\!], & \text{if } j < i. \end{cases} \tag{3}$$

We refer the reader to [23], Lemma 4, for a detailed proof of this bound.

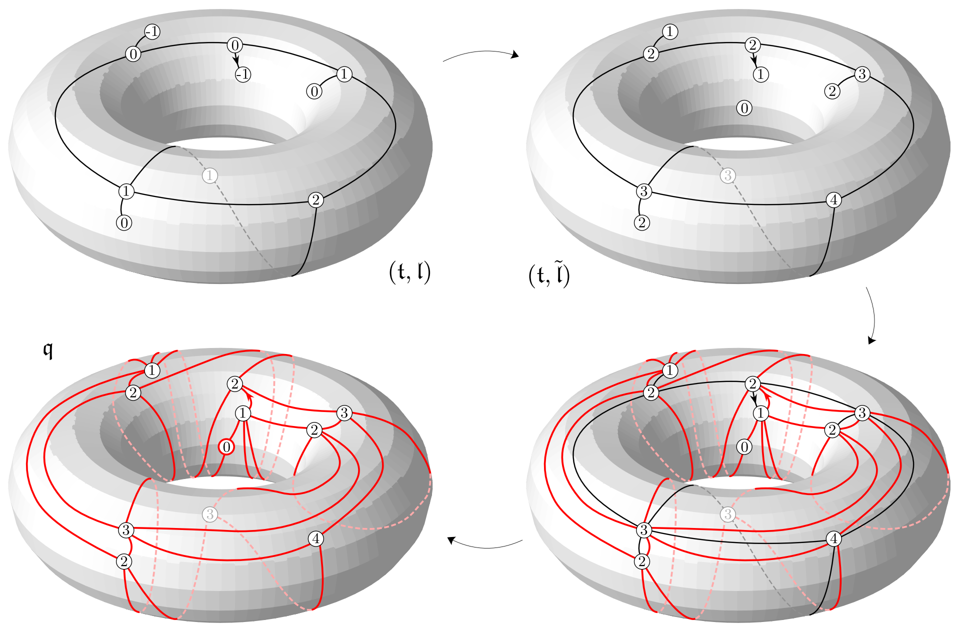

FIG. 2. *The Chapuy–Marcus–Schaeffer bijection. In this example*, $\varepsilon_\pm = +1$. *On the bottom–left picture, the vertex* $v^\bullet$ *has a thicker* (*red*) *borderline*.

2.2. *Decomposition of a* g*-tree*. We explained in [5] how to decompose a g-tree into simpler objects. Roughly speaking, a g-tree is a scheme (a g-tree whose all vertices have degree at least 3) in which every half-edge is replaced by a forest.

2.2.1. *Forests*. We adapt the standard formalism for plane trees—as found in [24] for instance—to forests. Let us call $\mathcal{U} := \bigcup_{n=1}^{\infty} \mathbb{N}^n$, where $\mathbb{N} := \{1, 2, \ldots\}$. If $u \in \mathbb{N}^n$, we write $|u| := n$. For $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_p) \in \mathcal{U}$, we let $uv := (u_1, \ldots, u_n, v_1, \ldots, v_p)$ be the concatenation of u and v. If $w = uv$ for some $u, v \in \mathcal{U}$, we say that u is a *ancestor* of w and that w is a *descendant* of u. In the case where $v \in \mathbb{N}$, we may also use the terms *parent* and *child* instead.

DEFINITION 2. A *forest* is a finite subset $\mathfrak{f} \subset \mathcal{U}$ satisfying the following:

(i) there is an integer $t(\mathfrak{f}) \geq 1$ such that $\mathfrak{f} \cap \mathbb{N} = [\![1, t(\mathfrak{f}) + 1]\!]$;
(ii) if $u \in \mathfrak{f}$, $|u| \geq 2$, then its parent belongs to $\mathfrak{f}$;
(iii) for every $u \in \mathfrak{f}$, there is an integer $c_u(\mathfrak{f}) \geq 0$ such that $ui \in \mathfrak{f}$ if and only if $1 \leq i \leq c_u(\mathfrak{f})$;
(iv) $c_{t(\mathfrak{f})+1}(\mathfrak{f}) = 0$.

The integer $t(\mathfrak{f})$ is called the *number of trees* of $\mathfrak{f}$.

For $u = (u_1, \ldots, u_p) \in \mathfrak{f}$, we call $\mathfrak{a}(u) := u_1$ its oldest ancestor. A *tree* of $\mathfrak{f}$ is a level set for $\mathfrak{a}$: for $1 \leq j \leq t(\mathfrak{f})$, the jth tree of $\mathfrak{f}$ is the set $\{u \in \mathfrak{f} : \mathfrak{a}(u) = j\}$. The

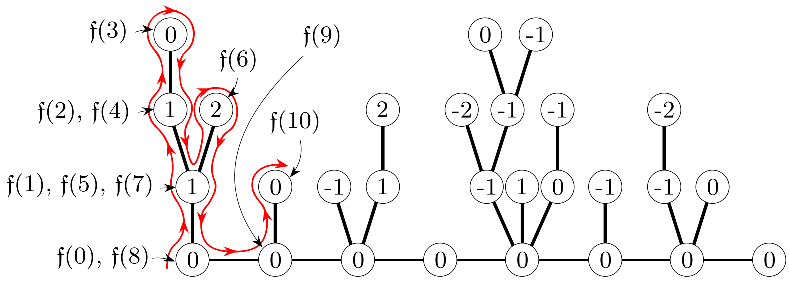

FIG. 3. *The facial sequence of a well-labeled forest from* $\mathfrak{F}_7^{20}$.

integer $\mathfrak{a}(u)$ hence records which tree u belongs to. We call $\mathfrak{f} \cap \mathbb{N} = \{\mathfrak{a}(u), u \in \mathfrak{f}\}$ the *floor* of the forest $\mathfrak{f}$.

For $u, v \in \mathfrak{f}$, we write $u \sim v$ if either u is a parent or child of v, or $u, v \in \mathbb{N}$ and $|u - v| = 1$. It is convenient, when representing a forest, to draw edges between u's and v's such that $u \sim v$; see Figure 3. We say that an edge drawn between a parent and its child is a *tree edge* whereas an edge drawn between two consecutive tree roots, that is, between some i and $i + 1$, will be called a *floor edge*. We call $\mathcal{F}_\sigma^m := \{\mathfrak{f} : t(\mathfrak{f}) = \sigma, |\mathfrak{f}| = m + \sigma + 1\}$ the set of all forests with σ trees and m tree edges.

DEFINITION 3. A *well-labeled forest* is a pair $(\mathfrak{f}, \mathfrak{l})$ where $\mathfrak{f}$ is a forest, and $\mathfrak{l} : \mathfrak{f} \to \mathbb{Z}$ is a function satisfying:

(i) for all $u \in \mathfrak{f} \cap \mathbb{N}$, $\mathfrak{l}(u) = 0$;
(ii) if $u \sim v$, $|\mathfrak{l}(u) - \mathfrak{l}(v)| \leq 1$.

Let $\mathfrak{F}_\sigma^m := \{(\mathfrak{f}, \mathfrak{l}) : \mathfrak{f} \in \mathcal{F}_\sigma^m\}$ be the set of well-labeled forests with σ trees and m tree edges.

Encoding by contour and spatial contour functions. There is a very convenient way to code forests and well-labeled forests. Let $\mathfrak{f} \in \mathcal{F}_\sigma^m$ be a forest. Let us begin by defining its *facial sequence* $\mathfrak{f}(0), \mathfrak{f}(1), \dots, \mathfrak{f}(2m + \sigma)$ as follows (see Figure 3): $\mathfrak{f}(0) := 1$, and for $0 \leq i \leq 2m + \sigma - 1$:

◇ if $\mathfrak{f}(i)$ has children that do not appear in the sequence $\mathfrak{f}(0), \mathfrak{f}(1), \dots, \mathfrak{f}(i)$, then $\mathfrak{f}(i + 1)$ is the first of these children, that is, $\mathfrak{f}(i + 1) := \mathfrak{f}(i) j_0$ where

$$j_0 = \min\{j \geq 1 : \mathfrak{f}(i) j \notin \{\mathfrak{f}(0), \mathfrak{f}(1), \dots, \mathfrak{f}(i)\}\};$$

◇ otherwise, if $\mathfrak{f}(i)$ has a parent [i.e., $|\mathfrak{f}(i)| \geq 2$], then $\mathfrak{f}(i + 1)$ is this parent;
◇ if neither of these cases occur, which implies that $|\mathfrak{f}(i)| = 1$, then $\mathfrak{f}(i + 1) := \mathfrak{f}(i) + 1$.

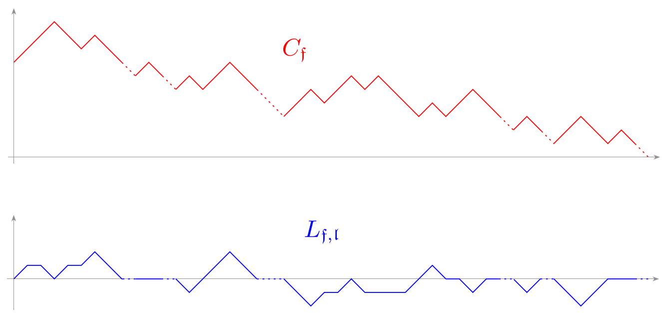

FIG. 4. *The contour pair of the well-labeled forest appearing in Figure 3. The paths are dashed on the intervals corresponding to floor edges.*

Each tree edge is visited exactly twice—once going from the parent to the child, once going the other way around—whereas each floor edge is visited only once—from some i to $i+1$. As a result, $\mathfrak{f}(2m+\sigma) = t(\mathfrak{f}) + 1$.

The *contour pair* $(C_{\mathfrak{f}}, L_{\mathfrak{f},\mathfrak{l}})$ of $(\mathfrak{f}, \mathfrak{l})$ consists in the *contour function* $C_{\mathfrak{f}} : [0, 2m + \sigma] \to \mathbb{R}_+$ of $\mathfrak{f}$ and the *spatial contour function* $L_{\mathfrak{f},\mathfrak{l}} : [0, 2m+\sigma] \to \mathbb{R}$ defined by

$$C_{\mathfrak{f}}(i) := |\mathfrak{f}(i)| + t(\mathfrak{f}) - \mathfrak{a}(\mathfrak{f}(i)) \quad \text{and} \quad L_{\mathfrak{f},\mathfrak{l}}(i) := \mathfrak{l}(\mathfrak{f}(i)), \qquad 0 \le i \le 2m+\sigma,$$

and linearly interpolated between integer values (see Figure 4). It entirely determines $(\mathfrak{f}, \mathfrak{l})$.

2.2.2. *Decomposition of a well-labeled g-tree into simpler objects.* We explain here how to decompose a well-labeled g-tree. See [5] for a more precise description.

DEFINITION 4. We call *scheme* of genus g a g-tree with no vertices of degree one or two. A scheme is said to be *dominant* when it only has vertices of degree exactly three.

We call $\mathfrak{S}$ the finite set of all schemes of genus g and $\mathfrak{S}^*$ the set of all dominant schemes of genus g.

Let us first explain how to decompose a g-tree (without labels) into a scheme, a family of forests and an integer. Let $\mathfrak{s}$ be a scheme. We suppose that we have forests $\mathfrak{f}^{\mathfrak{e}} \in \mathcal{F}_{\sigma^{\mathfrak{e}}}^{m^{\mathfrak{e}}}$, $\mathfrak{e} \in \vec{E}(\mathfrak{s})$, where for all $\mathfrak{e}$, $\sigma^{\bar{\mathfrak{e}}} = \sigma^{\mathfrak{e}}$, as well as an integer $u \in [\![0, 2m^{\mathfrak{e}_*} + \sigma^{\mathfrak{e}_*} - 1]\!]$, where $\mathfrak{e}_*$ denotes the root of $\mathfrak{s}$. We construct a g-tree as follows. First, we replace every edge $\{\mathfrak{e}, \bar{\mathfrak{e}}\}$ in $\mathfrak{s}$ with a chain of $\sigma^{\mathfrak{e}} = \sigma^{\bar{\mathfrak{e}}}$ edges. Then, for every half-edge $\mathfrak{e} \in \vec{E}(\mathfrak{s})$, we replace the chain of half-edges corresponding to it with the forest $\mathfrak{f}^{\mathfrak{e}}$, in such a way that its floor matches with the chain. In other words, we "graft" the forest $\mathfrak{f}^{\mathfrak{e}}$ to the left of $\mathfrak{e}$. Finally, the root of the g-tree is the half-edge linking $\mathfrak{f}^{\mathfrak{e}_*}(u)$ to $\mathfrak{f}^{\mathfrak{e}_*}(u+1)$ in the forest grafted to the left of $\mathfrak{e}_*$.

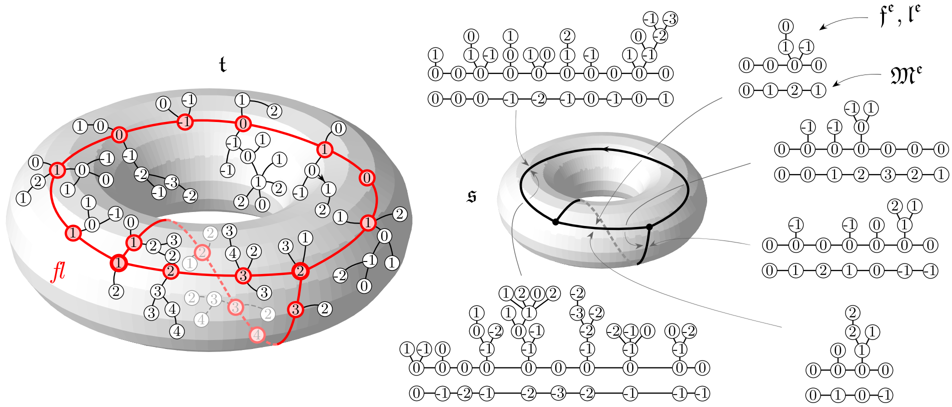

FIG. 5. *Decomposition of a well-labeled g-tree $\mathfrak{t}$ into its scheme $\mathfrak{s}$, the collection of its Motzkin paths $(\mathfrak{M}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$ and the collection of its well-labeled forests $(\mathfrak{f}^{\mathfrak{e}},\mathfrak{l}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$. In this example, the integer $u=10$. The floor of $\mathfrak{t}$ is more thickly outlined, and its two nodes are even more thickly outlined.*

PROPOSITION 3. *The above description provides a bijection between the set of all g-trees and the set of all triples $(\mathfrak{s},(\mathfrak{f}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})},u)$ where $\mathfrak{s}\in\mathfrak{S}$ is a scheme (of genus g), the forests $\mathfrak{f}^{\mathfrak{e}}\in\mathcal{F}_{\sigma^{\mathfrak{e}}}^{m^{\mathfrak{e}}}$ are such that $\sigma^{\bar{\mathfrak{e}}}=\sigma^{\mathfrak{e}}$ for all $\mathfrak{e}$ and $u\in[\![0,2m^{\mathfrak{e}_*}+\sigma^{\mathfrak{e}_*}-1]\!]$.*

Moreover, g-trees with n edges correspond to triples satisfying the condition $\sum_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}(m^{\mathfrak{e}}+\frac{1}{2}\sigma^{\mathfrak{e}})=n$.

Let $\mathfrak{t}$ be a g-tree and $(\mathfrak{s},(\mathfrak{f}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})},u)$ be the corresponding triple. We say that $\mathfrak{s}$ is the scheme of $\mathfrak{t}$ and that the forests $\mathfrak{f}^{\mathfrak{e}}$, $\mathfrak{e}\in\vec{E}(\mathfrak{s})$, are its forests. The set $V(\mathfrak{s})$ may be seen as a subset of $\mathfrak{t}$; we call *nodes* its elements. Finally, we call *floor* of $\mathfrak{t}$ the set *fl* of vertices we obtain after replacing the edges of $\mathfrak{s}$ by chains of edges (see Figure 5).

We now deal with well-labeled g-trees. We will need the following definition:

DEFINITION 5. We call *Motzkin path* a sequence $(M_n)_{0\le n\le\sigma}$ for some $\sigma\ge 0$ such that $M_0=0$ and for $0\le i\le\sigma-1$, $M_{i+1}-M_i\in\{-1,0,1\}$. We write $\sigma(M):=\sigma$ its lifetime.

Let $\mathfrak{s}$ be a scheme. We suppose that we have well-labeled forests $(\mathfrak{f}^{\mathfrak{e}},\mathfrak{l}^{\mathfrak{e}})\in\mathfrak{F}_{\sigma^{\mathfrak{e}}}^{m^{\mathfrak{e}}}$, $\mathfrak{e}\in\vec{E}(\mathfrak{s})$, where for all $\mathfrak{e}$, $\sigma^{\bar{\mathfrak{e}}}=\sigma^{\mathfrak{e}}$, as well as an integer $u\in[\![0,2m^{\mathfrak{e}_*}+\sigma^{\mathfrak{e}_*}-1]\!]$. Suppose moreover that we have a family of Motzkin paths $(\mathfrak{M}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$ such that $\mathfrak{M}^{\mathfrak{e}}$ is defined on $[\![0,\sigma^{\mathfrak{e}}]\!]$ and $\mathfrak{M}^{\mathfrak{e}}(\sigma^{\mathfrak{e}})=l^{\mathfrak{e}^+}-l^{\mathfrak{e}^-}$ for some family of integers $(l^v)_{v\in V(\mathfrak{s})}$ with $l^{\mathfrak{e}_*^-}=0$. We suppose that the Motzkin paths satisfy the following relation:

$$\mathfrak{M}^{\bar{\mathfrak{e}}}(i)=\mathfrak{M}^{\mathfrak{e}}(\sigma^{\mathfrak{e}}-i)-l^{\mathfrak{e}},\qquad 0\le i\le\sigma^{\mathfrak{e}}\qquad\text{where } l^{\mathfrak{e}}:=l^{\mathfrak{e}^+}-l^{\mathfrak{e}^-}.$$

We will say that a quadruple $(\mathfrak{s}, (\mathfrak{M}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}, (\mathfrak{f}^{\mathfrak{e}}, \mathfrak{l}^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}, u)$ satisfying these constraints is *compatible*.

We construct a well-labeled g-tree as follows. We begin by suitably relabeling the forests. For every half-edge $\mathfrak{e}$, first, we shift the labels of $\mathfrak{M}^{\mathfrak{e}}$ by $l^{\mathfrak{e}^-}$ so that it goes from $l^{\mathfrak{e}^-}$ to $l^{\mathfrak{e}^+}$. Then we shift all the labels of $(\mathfrak{f}^{\mathfrak{e}}, \mathfrak{l}^{\mathfrak{e}})$ tree by tree according to the Motzkin path: precisely, we change $\mathfrak{l}^{\mathfrak{e}}$ into $w \in \mathfrak{f}^{\mathfrak{e}} \mapsto l^{\mathfrak{e}^-} + \mathfrak{M}^{\mathfrak{e}}(\mathfrak{a}(w) - 1) + \mathfrak{l}^{\mathfrak{e}}(w)$. Then we replace the half-edge $\mathfrak{e}$ with this forest, as in the previous section. As before, we find the position of the root thanks to u. Finally, we shift all the labels for the root label to be equal to 0.

PROPOSITION 4. *The above description provides a bijection between the set of all well-labeled g-trees and the set of all compatible quadruples.*

Moreover, g-trees with n edges correspond to quadruples satisfying the condition $\sum_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}(m^{\mathfrak{e}} + \frac{1}{2}\sigma^{\mathfrak{e}}) = n$.

If we call $(C^{\mathfrak{e}}, L^{\mathfrak{e}})$ the contour pair of $(\mathfrak{f}^{\mathfrak{e}}, \mathfrak{l}^{\mathfrak{e}})$, then we may retrieve the oldest ancestor of $\mathfrak{f}^{\mathfrak{e}}(i)$ thanks to $C^{\mathfrak{e}}$ by the relation

$$\mathfrak{a}(\mathfrak{f}^{\mathfrak{e}}(i)) - 1 = \sigma^{\mathfrak{e}} - \underline{C}^{\mathfrak{e}}(i),$$

where we use the notation

$$\underline{X}_s := \inf_{[0,s]} X$$

for any process $(X_s)_{s\geq 0}$. The function

$$\mathfrak{L}^{\mathfrak{e}} := \bigl(L^{\mathfrak{e}}(t) + \mathfrak{M}^{\mathfrak{e}}(\sigma^{\mathfrak{e}} - \underline{C}^{\mathfrak{e}}(t))\bigr)_{0\leq t\leq 2m^{\mathfrak{e}}+\sigma^{\mathfrak{e}}}, \tag{4}$$

then records the labels of the forest $\mathfrak{f}^{\mathfrak{e}}$, once shifted tree by tree according to the Motzkin path $\mathfrak{M}^{\mathfrak{e}}$. This function will be used in Section 2.4.

Through the Chapuy–Marcus–Schaeffer bijection, a uniform random quadrangulation corresponds to a uniform random well-labeled g-tree. It can then be decomposed into a scheme, a collection of well-labeled forests, a collection of Motzkin paths and an integer, as explained above. The following section exposes the scaling limits of these objects.

2.3. *Scaling limits*. Let us define the space $\mathcal{K}$ of continuous real-valued functions on $\mathbb{R}_+$ killed at some time

$$\mathcal{K} := \bigcup_{x\in\mathbb{R}_+} \mathcal{C}([0,x], \mathbb{R}).$$

For an element $f \in \mathcal{K}$, we will define its lifetime $\sigma(f)$ as the only x such that $f \in \mathcal{C}([0,x], \mathbb{R})$. We endow this space with the following metric:

$$d_{\mathcal{K}}(f,g) := |\sigma(f) - \sigma(g)| + \sup_{y\geq 0}\bigl|f\bigl(y\wedge\sigma(f)\bigr) - g\bigl(y\wedge\sigma(g)\bigr)\bigr|.$$

Throughout this section, m and σ will denote positive real numbers and l will be any real number.

2.3.1. *Brownian bridges, first-passage Brownian bridges and Brownian snake.* We define here the Brownian bridge $B^{0\to l}_{[0,m]}$ on $[0,m]$ from 0 to l and the first-passage Brownian bridge $F^{0\to -\sigma}_{[0,m]}$ on $[0,m]$ from 0 to $-\sigma$. Informally, $B^{0\to l}_{[0,m]}$ and $F^{0\to -\sigma}_{[0,m]}$ are a standard Brownian motion β on $[0,m]$ conditioned, respectively, on the events $\{\beta_m = l\}$ and $\{\inf\{s \geq 0 : \beta_s = -\sigma\} = m\}$. Because both theses events occur with probability 0, we need to define these objects properly. There are several equivalent ways to do so; see for example [4, 6, 25]. We call p_a the density of a centered Gaussian variable with variance a, as well as p'_a its derivative

$$p_a(x) := \frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{x^2}{2a}\right) \quad \text{and} \quad p'_a(x) = -\frac{x}{a} p_a(x).$$

Let $(\beta_t)_{0\leq t\leq m}$ be a standard Brownian motion. As explained in [14], Proposition 1, the law of the Brownian bridge is characterized by the equation $B^{0\to l}_{[0,m]}(m) = l$ and the formula

$$\mathbb{E}\big[f\big((B^{0\to l}_{[0,m]}(t))_{0\leq t\leq m'}\big)\big] = \mathbb{E}\left[f((\beta_t)_{0\leq t\leq m'}) \frac{p_{m-m'}(l-\beta_{m'})}{p_m(l)}\right]$$

for all bounded measurable functions f on $\mathcal{K}$, for all $0 \leq m' < m$. We define the law of the first-passage Brownian bridge in a similar way, by letting

$$\begin{aligned} &\mathbb{E}\big[f\big((F^{0\to -\sigma}_{[0,m]}(t))_{0\leq t\leq m'}\big)\big] \\ &\quad = \mathbb{E}\left[f((\beta_t)_{0\leq t\leq m'}) \frac{p'_{m-m'}(-\sigma-\beta_{m'})}{p'_m(-\sigma)} \mathbb{1}_{\{\underline{\beta}_{m'} > -\sigma\}}\right] \end{aligned} \tag{5}$$

for all bounded measurable functions f on $\mathcal{K}$, for all $0 \leq m' < m$ and $F^{0\to -\sigma}_{[0,m]}(m) = -\sigma$.

For any real numbers l_1, l_2, $\sigma_1 > \sigma_2$, we define the bridge on $[0,m]$ from l_1 to l_2 and the first-passage bridge on $[0,m]$ from σ_1 to σ_2 by

$$B^{l_1\to l_2}_{[0,m]}(s) := l_1 + B^{0\to l_2-l_1}_{[0,m]} \quad \text{and} \quad F^{\sigma_1\to\sigma_2}_{[0,m]} := \sigma_1 + F^{0\to\sigma_2-\sigma_1}_{[0,m]}.$$

See [5], Section 5.1, for a more precise description of these objects. In particular, [5], Lemmas 10 and 14, show that these objects appear as the limits of their discrete analogs.

Conditionally given a first-passage Brownian bridge $F = F^{\sigma\to 0}_{[0,m]}$, we define a Gaussian process $(Z_{[0,m]}(s))_{0\leq s\leq m}$ with covariance function

$$\operatorname{cov}\big(Z_{[0,m]}(s), Z_{[0,m]}(s')\big) = \inf_{[s\wedge s', s\vee s']} (F - \underline{F}).$$

The process $(F^{\sigma\to 0}_{[0,m]}, Z_{[0,m]})$ has the law of the so-called Brownian snake's head; see [12, 16] for more details.

2.3.2. *Convergence results.* Recall that $\mathfrak{S}^*$ is the set of all dominant schemes of genus g, that is, schemes with only vertices of degree 3. For any $\mathfrak{s} \in \mathfrak{S}$, we identify an element $(m, \sigma, l, u) \in \mathbb{R}_+^{\vec{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}} \times (\mathbb{R}_+^*)^{\check{E}(\mathfrak{s})} \times \mathbb{R}^{V(\mathfrak{s})\setminus\{\mathfrak{e}_*^-\}} \times \mathbb{R}_+$ with an element of $\mathbb{R}_+^{\vec{E}(\mathfrak{s})} \times (\mathbb{R}_+^*)^{\vec{E}(\mathfrak{s})} \times \mathbb{R}^{V(\mathfrak{s})} \times \mathbb{R}_+$ by setting:

- ◇ $m^{\mathfrak{e}_*} := 1 - \sum_{\mathfrak{e}\in\vec{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}} m^{\mathfrak{e}}$,
- ◇ $\sigma^{\bar{\mathfrak{e}}} := \sigma^{\mathfrak{e}}$ for every $\mathfrak{e} \in \check{E}(\mathfrak{s})$,
- ◇ $l^{\mathfrak{e}_*^-} := 0$.

We write

$$\Delta_{\mathfrak{s}} := \left\{(x_{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} \in [0,1]^{\vec{E}(\mathfrak{s})}, \sum_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} x_{\mathfrak{e}} = 1\right\},$$

the simplex of dimension $|\vec{E}(\mathfrak{s})| - 1$. Note that m lies in $\Delta_{\mathfrak{s}}$ as long as $m^{\mathfrak{e}_*} \geq 0$. We define the probability μ by, for all measurable function φ on $\bigcup_{\mathfrak{s}\in\mathfrak{S}}\{\mathfrak{s}\} \times \Delta_{\mathfrak{s}} \times (\mathbb{R}_+^*)^{\vec{E}(\mathfrak{s})} \times \mathbb{R}^{V(\mathfrak{s})} \times [0,1]$,

$$\mu(\varphi) = \frac{1}{\Upsilon} \sum_{\mathfrak{s}\in\mathfrak{S}^*} \int_{\mathcal{S}^{\mathfrak{s}}} d\mathcal{L}^{\mathfrak{s}}\, \mathbb{1}_{\{m^{\mathfrak{e}_*}\geq 0, u<m^{\mathfrak{e}_*}\}} \varphi(\mathfrak{s}, m, \sigma, l, u) \times \prod_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} -p'_{m^{\mathfrak{e}}}(\sigma^{\mathfrak{e}}) \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} p_{\sigma^{\mathfrak{e}}}(l^{\mathfrak{e}}),$$

where $l^{\mathfrak{e}} := l^{\mathfrak{e}^+} - l^{\mathfrak{e}^-}$, the measure $d\mathcal{L}^{\mathfrak{s}} = d(m^{\mathfrak{e}})\, d(\sigma^{\mathfrak{e}})\, d(l^{v})\, du$ is the Lebesgue measure on the set

$$\mathcal{S}^{\mathfrak{s}} := [0,1]^{\vec{E}(\mathfrak{s})\setminus\{\mathfrak{e}_*\}} \times (\mathbb{R}_+^*)^{\check{E}(\mathfrak{s})} \times \mathbb{R}^{V(\mathfrak{s})\setminus\{\mathfrak{e}_*^-\}} \times [0,1]$$

and

$$\Upsilon = \sum_{\mathfrak{s}\in\mathfrak{S}^*} \int_{\mathcal{S}^{\mathfrak{s}}} d\mathcal{L}^{\mathfrak{s}}\, \mathbb{1}_{\{m^{\mathfrak{e}_*}\geq 0, u<m^{\mathfrak{e}_*}\}} \prod_{\mathfrak{e}\in\vec{E}(\mathfrak{s})} -p'_{m^{\mathfrak{e}}}(\sigma^{\mathfrak{e}}) \prod_{\mathfrak{e}\in\check{E}(\mathfrak{s})} p_{\sigma^{\mathfrak{e}}}(l^{\mathfrak{e}}) \tag{6}$$

is a normalization constant. We gave a nonintegral expression for this constant in [5].

Let $(\mathfrak{t}_n, \mathfrak{l}_n)$ be uniformly distributed over the set $\mathcal{T}_n$ of well-labeled g-trees with n vertices. We call $\mathfrak{s}_n$ its scheme and we define, as in Section 2.2, $(\mathfrak{f}_n^{\mathfrak{e}}, \mathfrak{l}_n^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}$ its well-labeled forests, $(m_n^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}$ and $(\sigma_n^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}$, respectively, their sizes and lengths, $(l_n^v)_{v\in V(\mathfrak{s}_n)}$ the shifted labels of its nodes, $(\mathfrak{M}_n^{\mathfrak{e}})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}$ its Motzkin paths and u_n the integer recording the position of the root in the first forest $\mathfrak{f}_n^{\mathfrak{e}_*}$. We call $(C_n^{\mathfrak{e}}, L_n^{\mathfrak{e}})$ the contour pair of the well-labeled forest $(\mathfrak{f}_n^{\mathfrak{e}}, \mathfrak{l}_n^{\mathfrak{e}})$, and we extend the definition of $\mathfrak{M}_n^{\mathfrak{e}}$ to $[0, \sigma_n^{\mathfrak{e}}]$ by linear interpolation. We then define the rescaled versions of these objects [recall that $\gamma := (8/9)^{1/4}$]

$$m^{\mathfrak{e}}_{(n)} := \frac{2m^{\mathfrak{e}}_n + \sigma^{\mathfrak{e}}_n}{2n}, \qquad \sigma^{\mathfrak{e}}_{(n)} := \frac{\sigma^{\mathfrak{e}}_n}{\sqrt{2n}}, \qquad l^v_{(n)} := \frac{l^v_n}{\gamma n^{1/4}}, \qquad u_{(n)} := \frac{u_n}{2n},$$

$$C^{\mathfrak{e}}_{(n)} := \left(\frac{C^{\mathfrak{e}}_n(2nt)}{\sqrt{2n}}\right)_{0\le t\le m^{\mathfrak{e}}_{(n)}}, \qquad L^{\mathfrak{e}}_{(n)} := \left(\frac{L^{\mathfrak{e}}_n(2nt)}{\gamma n^{1/4}}\right)_{0\le t\le m^{\mathfrak{e}}_{(n)}},$$

$$\mathfrak{M}^{\mathfrak{e}}_{(n)} := \left(\frac{\mathfrak{M}^{\mathfrak{e}}_n(\sqrt{2n}t)}{\gamma n^{1/4}}\right)_{0\le t\le \sigma^{\mathfrak{e}}_{(n)}}.$$

REMARK. Throughout this paper, the notation with a parenthesized n will always refer to suitably rescaled objects, as in the definitions above.

We described in [5] the limiting law of these objects:

PROPOSITION 5. *The random vector*

$$\big(\mathfrak{s}_n, (m^{\mathfrak{e}}_{(n)})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}, (\sigma^{\mathfrak{e}}_{(n)})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}, (l^v_{(n)})_{v\in V(\mathfrak{s}_n)}, u_{(n)},$$
$$(C^{\mathfrak{e}}_{(n)}, L^{\mathfrak{e}}_{(n)})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}, (\mathfrak{M}^{\mathfrak{e}}_{(n)})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}\big)$$

converges in law toward the random vector

$$\big(\mathfrak{s}_\infty, (m^{\mathfrak{e}}_{\infty})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)}, (\sigma^{\mathfrak{e}}_{\infty})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)}, (l^v_{\infty})_{v\in V(\mathfrak{s}_\infty)}, u_{\infty},$$
$$(C^{\mathfrak{e}}_{\infty}, L^{\mathfrak{e}}_{\infty})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)}, (\mathfrak{M}^{\mathfrak{e}}_{\infty})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)}\big),$$

whose law is defined as follows:

◇ *the law of the vector*

$$\mathfrak{I}_\infty := \big(\mathfrak{s}_\infty, (m^{\mathfrak{e}}_{\infty})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)}, (\sigma^{\mathfrak{e}}_{\infty})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)}, (l^v_{\infty})_{v\in V(\mathfrak{s}_\infty)}, u_{\infty}\big)$$

is the probability μ,

◇ *conditionally given* $\mathfrak{I}_\infty$:
- *the processes* $(C^{\mathfrak{e}}_{\infty}, L^{\mathfrak{e}}_{\infty})$, $\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)$, *and* $(\mathfrak{M}^{\mathfrak{e}}_{\infty})$, $\mathfrak{e}\in\check{E}(\mathfrak{s}_\infty)$, *are independent*;
- *the process* $(C^{\mathfrak{e}}_{\infty}, L^{\mathfrak{e}}_{\infty})$ *has the law of a Brownian snake's head on* $[0, m^{\mathfrak{e}}_{\infty}]$ *going from* $\sigma^{\mathfrak{e}}_{\infty}$ *to* 0

$$(C^{\mathfrak{e}}_{\infty}, L^{\mathfrak{e}}_{\infty}) \overset{(d)}{=} \big(F^{\sigma^{\mathfrak{e}}_{\infty}\to 0}_{[0,m^{\mathfrak{e}}_{\infty}]}, Z_{[0,m^{\mathfrak{e}}_{\infty}]}\big);$$

- *the process* $(\mathfrak{M}^{\mathfrak{e}}_{\infty})$ *has the law of a Brownian bridge on* $[0,\sigma^{\mathfrak{e}}_{\infty}]$ *from* 0 *to* $l^{\mathfrak{e}}_{\infty} := l^{\mathfrak{e}^+}_{\infty} - l^{\mathfrak{e}^-}_{\infty}$

$$(\mathfrak{M}^{\mathfrak{e}}_{\infty}) \overset{(d)}{=} B^{0\to l^{\mathfrak{e}}_{\infty}}_{[0,\sigma^{\mathfrak{e}}_{\infty}]};$$

- *the Motzkin paths are linked through the relation*

$$\mathfrak{M}^{\bar{\mathfrak{e}}}_{\infty}(s) = \mathfrak{M}^{\mathfrak{e}}_{\infty}(\sigma^{\mathfrak{e}}_{\infty} - s) - l^{\mathfrak{e}}_{\infty}.$$

Applying Skorokhod's representation theorem, we may and will assume that this convergence holds almost surely. As a result, note that for n large enough, $\mathfrak{s}_n = \mathfrak{s}_\infty$.

2.4. *Maps seen as quotients of* $[0,1]$. Let $\mathfrak{q}_n$ be uniformly distributed over the set $\mathcal{Q}_n$ of bipartite quadrangulations of genus g with n faces. Conditionally given $\mathfrak{q}_n$, we take $v_n^\bullet$ uniformly over $V(\mathfrak{q}_n)$ so that $(\mathfrak{q}_n, v_n^\bullet)$ is uniform over the set $\mathcal{Q}_n^\bullet$ of pointed bipartite quadrangulations of genus g with n faces. Recall that every element of $\mathcal{Q}_n$ has the same number of vertices, $n+2-2g$. Through the Chapuy–Marcus–Schaeffer bijection, $(\mathfrak{q}_n, v_n^\bullet)$ corresponds to a uniform well-labeled g-tree with n edges $(\mathfrak{t}_n, \mathfrak{l}_n)$. The parameter $\varepsilon_\pm \in \{-1, 1\}$ appearing in the bijection will be irrelevant to what follows.

Recall the notation $\mathfrak{t}_n(0), \mathfrak{t}_n(1), \dots, \mathfrak{t}_n(2n)$ and $\mathfrak{q}_n(0), \mathfrak{q}_n(1), \dots, \mathfrak{q}_n(2n)$ from Section 2.1. For technical reasons, it will be more convenient, when traveling along the g-tree, not to begin by its root but rather by the first edge of the first forest. Precisely, we define

$$\dot{\mathfrak{t}}_n(i) := \begin{cases} \mathfrak{t}_n(i - u_n + 2n), & \text{if } 0 \le i \le u_n, \\ \mathfrak{t}_n(i - u_n), & \text{if } u_n \le i \le 2n, \end{cases}$$

where u_n is the integer recording the position of the root in the first forest of $\mathfrak{t}_n$. We define $\dot{\mathfrak{q}}_n(i)$ in a similar way, and endow $[\![0, 2n]\!]$ with the pseudo-metric d_n defined by

$$d_n(i, j) := d_{\mathfrak{q}_n}(\dot{\mathfrak{q}}_n(i), \dot{\mathfrak{q}}_n(j)).$$

We define the equivalence relation $\sim_n$ on $[\![0, 2n]\!]$ by declaring that $i \sim_n j$ if $\dot{\mathfrak{q}}_n(i) = \dot{\mathfrak{q}}_n(j)$, that is, if $d_n(i, j) = 0$. We call π_n the canonical projection from $[\![0, 2n]\!]$ to $[\![0, 2n]\!]_{/\sim_n}$, and we slightly abuse notation by seeing d_n as a metric on $[\![0, 2n]\!]_{/\sim_n}$ defined by $d_n(\pi_n(i), \pi_n(j)) := d_n(i, j)$. In what follows, we will always make the same abuse with every pseudo-metric. The metric space $([\![0, 2n]\!]_{/\sim_n}, d_n)$ is then isometric to $(V(\mathfrak{q}_n) \setminus \{v_n^\bullet\}, d_{\mathfrak{q}_n})$, which is at d_{GH}-distance at most 1 from the space $(V(\mathfrak{q}_n), d_{\mathfrak{q}_n})$.

We extend the definition of d_n to noninteger values by linear interpolation and define its rescaled version: for $s, t \in [0, 1]$, we let

$$d_{(n)}(s, t) := \frac{1}{\gamma n^{1/4}} d_n(2ns, 2nt). \tag{7}$$

Spatial contour function of $(\mathfrak{t}_n, \mathfrak{l}_n)$. The spatial contour function of the pair $(\mathfrak{t}_n, \mathfrak{l}_n)$ is the function $\mathfrak{L}_n : [0, 2n] \to \mathbb{R}$, defined by

$$\mathfrak{L}_n(i) := \mathfrak{l}_n(\dot{\mathfrak{t}}_n(i)) - \mathfrak{l}_n(\dot{\mathfrak{t}}_n(0)), \qquad 0 \le i \le 2n,$$

and linearly interpolated between integer values. Its rescaled version is

$$\mathfrak{L}_{(n)} := \left(\frac{\mathfrak{L}_n(2nt)}{\gamma n^{1/4}} \right)_{0 \le t \le 1}.$$

Recall definition (4) of the process $\mathfrak{L}_n^{\mathfrak{e}}$. We define its rescaled version by

$$\mathfrak{L}_{(n)}^{\mathfrak{e}} := \left(\frac{\mathfrak{L}_n^{\mathfrak{e}}(2nt)}{\gamma n^{1/4}}\right)_{0\le t\le m_{(n)}^{\mathfrak{e}}} = \left(L_{(n)}^{\mathfrak{e}}(t) + \mathfrak{M}_{(n)}^{\mathfrak{e}}\left(\sigma_{(n)}^{\mathfrak{e}} - \underline{C}_{(n)}^{\mathfrak{e}}(t)\right)\right)_{0\le t\le m_{(n)}^{\mathfrak{e}}}.$$

Proposition 5 shows that $\mathfrak{L}_{(n)}^{\mathfrak{e}}$ converges in the space $(\mathcal{K}, d_{\mathcal{K}})$ toward

$$\mathfrak{L}_{\infty}^{\mathfrak{e}} := \left(L_{\infty}^{\mathfrak{e}}(t) + \mathfrak{M}_{\infty}^{\mathfrak{e}}\left(\sigma_{\infty}^{\mathfrak{e}} - \underline{C}_{\infty}^{\mathfrak{e}}(t)\right)\right)_{0\le t\le m_{\infty}^{\mathfrak{e}}}.$$

We can express $\mathfrak{L}_{(n)}$ in terms of the processes $\mathfrak{L}_{(n)}^{\mathfrak{e}}$'s by concatenating them. For $f, g \in \mathcal{K}_0$ two functions started at 0, we call $f \bullet g \in \mathcal{K}_0$ their concatenation defined by $\sigma(f \bullet g) := \sigma(f) + \sigma(g)$ and, for $0 \le t \le \sigma(f \bullet g)$,

$$f \bullet g(t) := \begin{cases} f(t), & \text{if } 0 \le t \le \sigma(f), \\ f(\sigma(f)) + g(t - \sigma(f)), & \text{if } \sigma(f) \le t \le \sigma(f) + \sigma(g). \end{cases}$$

We arrange the half-edges of $\mathfrak{s}_n$ according to its facial order, beginning with the root $\mathfrak{e}_1 = \mathfrak{e}_*, \dots, \mathfrak{e}_{2(6g-3)}$, so that $\mathfrak{L}_{(n)} = \mathfrak{L}_{(n)}^{\mathfrak{e}_1} \bullet \mathfrak{L}_{(n)}^{\mathfrak{e}_2} \bullet \cdots \bullet \mathfrak{L}_{(n)}^{\mathfrak{e}_{2(6g-3)}}$. By continuity of the concatenation, $\mathfrak{L}_{(n)}$ converges in $(\mathcal{K}, d_{\mathcal{K}})$ toward $\mathfrak{L}_{\infty} := \mathfrak{L}_{\infty}^{\mathfrak{e}_1} \bullet \mathfrak{L}_{\infty}^{\mathfrak{e}_2} \bullet \cdots \bullet \mathfrak{L}_{\infty}^{\mathfrak{e}_{2(6g-3)}}$, where the half-edges of $\mathfrak{s}_\infty$ are arranged in the same way.

Upper bound for $d_{(n)}$. Bound (2) provides us with an upper bound on $d_{(n)}$. We define

$$d_n^\circ(i,j) := \mathfrak{L}_n(i) + \mathfrak{L}_n(j) - 2\max\Big(\min_{k\in\overrightarrow{[\![i,j]\!]}} \mathfrak{L}_n(k), \min_{k\in\overrightarrow{[\![j,i]\!]}} \mathfrak{L}_n(k)\Big) + 2,$$

we extend it to $[0, 2n]$ by linear interpolation and define its rescaled version $d_{(n)}^\circ$ as we did for d_n by (7). We readily obtain that

$$d_{(n)}(s,t) \le d_{(n)}^\circ(s,t). \tag{8}$$

Moreover, the process $(d_{(n)}^\circ(s,t))_{0\le s,t\le 1}$ converges in $(\mathcal{C}([0,1]^2, \mathbb{R}), \|\cdot\|_\infty)$ toward the process $(d_\infty^\circ(s,t))_{0\le s,t\le 1}$ defined by

$$d_\infty^\circ(s,t) := \mathfrak{L}_\infty(s) + \mathfrak{L}_\infty(t) - 2\max\Big(\min_{x\in\overrightarrow{[s,t]}} \mathfrak{L}_\infty(x), \min_{x\in\overrightarrow{[t,s]}} \mathfrak{L}_\infty(x)\Big),$$

where

$$\overrightarrow{[s,t]} := \begin{cases} [s,t], & \text{if } s \le t, \\ [s,1] \cup [0,t], & \text{if } t < s. \end{cases} \tag{9}$$

Tightness of the processes $d_{(n)}$'s. In [5], Lemma 19, we showed the tightness of the processes $d_{(n)}$'s laws thanks to the inequality (8). As a result, from any increasing sequence of integers, we may extract a (deterministic) subsequence $(n_k)_{k\ge 0}$ such that there exists a function $d_\infty \in \mathcal{C}([0,1]^2, \mathbb{R})$ satisfying

$$(d_{(n_k)}(s,t))_{0\le s,t\le 1} \xrightarrow[k\to\infty]{(\mathrm{d})} (d_\infty(s,t))_{0\le s,t\le 1}. \tag{10}$$

By Skorokhod's representation theorem, we will assume that this convergence holds almost surely. We can check that the function d_∞ is actually a pseudometric. We define the equivalence relation associated with it by saying that $s \sim_\infty t$ if $d_\infty(s,t) = 0$, and we call $\mathfrak{q}_\infty := [0,1]_{/\sim_\infty}$. We proved in [5] that

$$\left(V(\mathfrak{q}_{n_k}), \frac{1}{\gamma n_k^{1/4}} d_{\mathfrak{q}_{n_k}}\right) \xrightarrow[k\to\infty]{(\mathrm{d})} (\mathfrak{q}_\infty, d_\infty)$$

in the sense of the Gromov–Hausdorff topology.

From now on, we fix such a subsequence $(n_k)_{k\geq 0}$. We will always focus on this particular subsequence in the following, and we will consider convergences when $n \to \infty$ to hold along this particular subsequence.

3. Real g-trees. In the discrete setting, it is sometimes convenient to work directly with the space $\mathfrak{t}_n$ instead of $[\![0, 2n]\!]$. In the continuous setting, we will see $\mathfrak{q}_\infty$ as a quotient of a continuous version of a g-tree, which we will call real g-tree. In other words, we will see the identifications $s \sim_\infty t$ as of two different kinds: some are inherited "from the g-tree structure," whereas the others come "from the map structure."

3.1. *Definitions.* As g-trees generalize plane trees in genus g, real g-trees are the objects that naturally generalize real trees. We will only use basic facts on real trees in this work. See, for example, [17] for more detail.

We consider a fixed dominant scheme $\mathfrak{s} \in \mathfrak{S}^*$. Let $(m^\mathfrak{e})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$ and $(\sigma^\mathfrak{e})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$ be two families of positive numbers satisfying $\sum_\mathfrak{e} m^\mathfrak{e} = 1$ and $\sigma^\mathfrak{e} = \sigma^{\bar{\mathfrak{e}}}$ for all $\mathfrak{e}$. As usual, we arrange the half-edges of $\mathfrak{s}$ according to its facial order, $\mathfrak{e}_1 = \mathfrak{e}_*, \dots, \mathfrak{e}_{2(6g-3)}$. For every $s \in [0,1)$, there exists a unique $1 \leq k \leq 2(6g-3)$ such that

$$\sum_{i=1}^{k-1} m^{\mathfrak{e}_i} \leq s < \sum_{i=1}^{k} m^{\mathfrak{e}_i}.$$

We let $\mathfrak{e}(s) := \mathfrak{e}_k$ and $\langle s\rangle := s - \sum_{i=1}^{k-1} m^{\mathfrak{e}_i} \in [0, m^{\mathfrak{e}(s)})$. By convention, we set $\mathfrak{e}(1) = \mathfrak{e}_1$ and $\langle 1\rangle = 0$. Beware that these notions depend on the family $(m^\mathfrak{e})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$. There should be no ambiguity in what follows.

Let us suppose we have a family $(h^\mathfrak{e})_{\mathfrak{e}\in\vec{E}(\mathfrak{s})}$ of continuous functions $h^\mathfrak{e} : [0, m^\mathfrak{e}] \to \mathbb{R}_+$ such that $h^\mathfrak{e}(0) = \sigma^\mathfrak{e}$ and $h^\mathfrak{e}(m^\mathfrak{e}) = 0$. It will be useful to consider their concatenation: we define the continuous function $h : [0,1] \to \mathbb{R}_+$ going from $\sum_\mathfrak{e} \sigma^\mathfrak{e}$ to 0 by

$$h := (h^{\mathfrak{e}_1} - \sigma^{\mathfrak{e}_1}) \bullet (h^{\mathfrak{e}_2} - \sigma^{\mathfrak{e}_2}) \bullet \cdots \bullet \left(h^{\mathfrak{e}_{2(6g-3)}} - \sigma^{\mathfrak{e}_{2(6g-3)}}\right) + \sum_{i=1}^{2(6g-3)} \sigma^{\mathfrak{e}_i}. \tag{11}$$

We define the relation $\simeq$ on $[0,1]$ as the coarsest equivalence relation for which $s \simeq t$ if one of the following occurs:

(12a) $$h(s) = h(t) = \inf_{[s\wedge t, s\vee t]} h;$$

(12b) $$h(s) = \underline{h}(s), \qquad h(t) = \underline{h}(t),$$
$$\mathfrak{e}(s) = \overline{\mathfrak{e}(t)} \quad \text{and} \quad h^{\mathfrak{e}(s)}(\langle s\rangle) = \sigma^{\mathfrak{e}(t)} - h^{\mathfrak{e}(t)}(\langle t\rangle);$$

(12c) $$\langle s\rangle = \langle t\rangle = 0 \quad \text{and} \quad \mathfrak{e}(s)^- = \mathfrak{e}(t)^-.$$

If we see the $h^{\mathfrak{e}}$'s as contour functions (in a continuous setting), the first item identifies numbers coding the same point in one of the forests. The second item identifies the floors of forests "facing each other": the numbers s and t should code floor points (two first equalities) of forests facing each other (third equality) and correspond to the same point (fourth equality). Finally, the third item identifies the nodes. We call *real g-tree* any space $\mathscr{T} := [0,1]_{/\simeq}$ obtained by such a construction.[3]

We now define the notions we will use throughout this work (see Figure 6). For $s \in [0,1]$, we write $\mathscr{T}(s)$ its equivalence class in the quotient $\mathscr{T} = [0,1]_{/\simeq}$. Similarly to the discrete case, the floor of $\mathscr{T}$ is defined as follows.

DEFINITION 6. We call *floor* of $\mathscr{T}$ the set $\mathit{fl} := \mathscr{T}(\{s : h(s) = \underline{h}(s)\})$.

For $a = \mathscr{T}(s) \in \mathscr{T} \setminus \mathit{fl}$, let $l := \inf\{t \le s : \underline{h}(t) = \underline{h}(s)\}$ and $r := \sup\{t \ge s : \underline{h}(t) = \underline{h}(s)\}$. The set $\tau_a := \mathscr{T}([l,r])$ is a real tree rooted at $\rho_a := \mathscr{T}(l) = \mathscr{T}(r) \in \mathit{fl}$.

DEFINITION 7. We call *tree* of $\mathscr{T}$ a set of the form τ_a for any $a \in \mathscr{T} \setminus \mathit{fl}$.

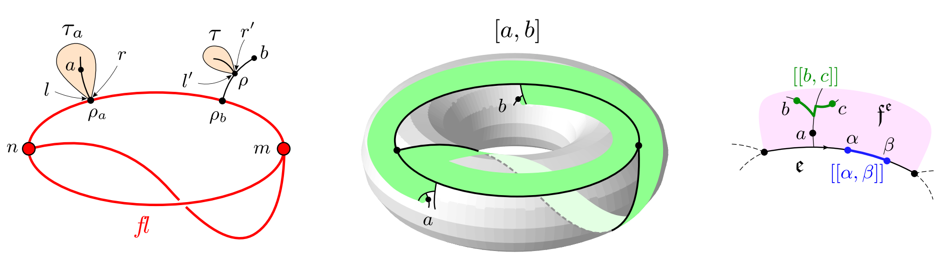

FIG. 6. Left. *On this picture, we can see the floor* fl, *the two nodes* n *and* m, *an example of tree* τ_a *and an example of tree* τ *to the left of* $[[\rho_b, b]]$ *rooted at* ρ. Middle. *The set* $[a,b]$. Right. *On this picture,* a *is an ancestor of* b *and* c, *and we can see the sets* $[[b,c]]$, $[[\alpha,\beta]]$ *and* $\mathfrak{f}^{\mathfrak{e}}$.

[3]There should be a more intrinsic definition for these spaces in terms of compact metric spaces that are locally real trees. As we will need to use this construction in what follows, we chose to define them as such for simplicity.

If $a \in \mathfrak{fl}$, we simply set $\rho_a := a$. Let τ be a tree of $\mathscr{T}$ rooted at ρ, and $a, b \in \tau$. We call $[[a, b]]$ the range of the unique injective path linking a to b. In particular, the set $[[\rho, a]]$ will be of interest. It represents the ancestral lineage of a in the tree τ. We say that a is an *ancestor* of b, and we write $a \preceq b$, if $a \in [[\rho, b]]$. We write $a \prec b$ if $a \preceq b$ and $a \neq b$.

DEFINITION 8. Let $b = \mathscr{T}(t) \in \mathscr{T} \setminus \mathfrak{fl}$ and $\rho \in [[\rho_b, b]] \setminus \{\rho_b, b\}$. Let $l' := \inf\{s \leq t : \mathscr{T}(s) = \rho\}$ and $r' := \sup\{s \leq t : \mathscr{T}(s) = \rho\}$. Then, provided $l' \neq r'$, we call the *tree to the left* of $[[\rho_b, b]]$ rooted at ρ the set $\mathscr{T}([l', r'])$.

We define the *tree to the right* of $[[\rho_b, b]]$ rooted at ρ in a similar way, by replacing "$\leq$" with "$\geq$" in the definitions of l' and r'.

DEFINITION 9. We call *subtree* of $\mathscr{T}$ any tree of $\mathscr{T}$, or any tree to the left or right of $[[\rho_b, b]]$ for some $b \in \mathscr{T} \setminus \mathfrak{fl}$.

Note that subtrees of $\mathscr{T}$ are real trees, and that trees of $\mathscr{T}$ are also subtrees of $\mathscr{T}$. For a subtree τ, the maximal interval $[s, t]$ such that $\tau = \mathscr{T}([s, t])$ is called the *interval coding* the subtree τ.

DEFINITION 10. For $\mathfrak{e} \in \vec{E}(\mathfrak{s})$, we call the *forest to the left of* $\mathfrak{e}$ the set $\mathfrak{f}^{\mathfrak{e}} := \mathscr{T}(\overline{\{s : \mathfrak{e}(s) = \mathfrak{e}\}})$.

The *nodes* of $\mathscr{T}$ are the elements of $\mathscr{T}(\{s : \langle s \rangle = 0\})$. In what follows, we will identify the nodes of $\mathscr{T}$ with the vertices of $\mathfrak{s}$. In particular, the two nodes $\mathfrak{e}^-$ and $\mathfrak{e}^+$ lie in $\mathfrak{f}^{\mathfrak{e}}$. We extend the definition of $[[a, b]]$ to the floor of $\mathfrak{f}^{\mathfrak{e}}$: for $a, b \in \mathfrak{f}^{\mathfrak{e}} \cap \mathfrak{fl}$, let $s, t \in \overline{\{r : \mathfrak{e}(r) = \mathfrak{e}\}}$ be such that $a = \mathscr{T}(s)$ and $b = \mathscr{T}(t)$. We define

$$[[a, b]] := \mathscr{T}([s \wedge t, s \vee t]) \cap \mathfrak{fl}$$

the range of the unique[4] injective path from a to b that stays inside $\mathfrak{f}^{\mathfrak{e}}$. For clarity, we write the set $[[\mathfrak{e}^-, \mathfrak{e}^+]]$ simply as $[[\mathfrak{e}]]$. Note that, in particular, $[[\mathfrak{e}]] = \mathfrak{f}^{\mathfrak{e}} \cap \mathfrak{f}^{\bar{\mathfrak{e}}} = \mathfrak{f}^{\mathfrak{e}} \cap \mathfrak{fl}$.

Let $a, b \in \mathscr{T}$. There is a natural way[5] to explore $\mathscr{T}$ from a to b. If $\inf \mathscr{T}^{-1}(a) \leq \sup \mathscr{T}^{-1}(b)$, then let $t := \inf\{r \geq \inf \mathscr{T}^{-1}(a) : b = \mathscr{T}(r)\}$ and $s := \sup\{r \leq t : a = \mathscr{T}(r)\}$. If $\sup \mathscr{T}^{-1}(b) < \inf \mathscr{T}^{-1}(a)$, then let $t := \inf \mathscr{T}^{-1}(b)$ and $s := \sup \mathscr{T}^{-1}(a)$. We define

$$[a, b] := \mathscr{T}(\overrightarrow{[s, t]}), \tag{13}$$

[4]Note that $\mathfrak{e}^+ \neq \mathfrak{e}^-$ because $\mathfrak{s}$ is a dominant scheme.

[5]Note that, if $a, b \in \mathfrak{fl}$, there are other possible ways to explore the g-tree between them. Indeed, a point of $\mathfrak{fl}$ is visited twice—or three times if it is a node—when we travel around $\mathfrak{fl}$. In particular, this definition depends on the position of the root in $\mathfrak{s}$ for such points. In what follows, we never use this definition for such points, so there will be no confusion.

where $\overrightarrow{[s,t]}$ is defined by (9).

We call $\mathscr{T}_n$ (resp., $\mathscr{T}_\infty$) the real g-tree obtained from the scheme $\mathfrak{s}_n$ (resp., $\mathfrak{s}_\infty$) and the family $(C^{\mathfrak{e}}_{(n)})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_n)}$ [resp., $(C^{\mathfrak{e}}_{\infty})_{\mathfrak{e}\in\vec{E}(\mathfrak{s}_\infty)}$]. For the sake of consistency with [5], we call $\mathfrak{C}_{(n)}$ and $\mathfrak{C}_\infty$ the functions obtained by (11) in this construction. We also call $\simeq_{(n)}$ and $\simeq_\infty$ the corresponding equivalence relations. When dealing with $\mathscr{T}_\infty$, we add an ∞ symbol to the notation defined above: for example, the floor of $\mathscr{T}_\infty$ will be noted fl_∞, and its forest to the left of $\mathfrak{e}$ will be noted $\mathfrak{f}^{\mathfrak{e}}_\infty$. It is more natural to use $\mathfrak{t}_n$ rather than $\mathscr{T}_n$ in the discrete setting. As $\mathfrak{t}_n$ may be viewed as a subset of $\mathscr{T}_n$, we will use for $\mathfrak{t}_n$ the formalism we defined above simply by restriction. Note that the notions of floor, forests, trees and nodes are consistent with the definitions we gave in Section 2.2 in that case.

Note that, because the functions $C^{\mathfrak{e}}_\infty$'s are first-passage Brownian bridges, the probability that there exists $\varepsilon > 0$ such that $C^{\mathfrak{e}}_\infty(s) > C^{\mathfrak{e}}_\infty(0)$ for all $s \in (0,\varepsilon)$ is equal to 0. As a result, there are almost surely no trees rooted at the nodes of $\mathscr{T}_\infty$. Moreover, the fact that the forests $\mathfrak{f}^{\mathfrak{e}}$ and $\mathfrak{f}^{\bar{\mathfrak{e}}}$ are independent yields that, almost surely, we cannot have a tree in $\mathfrak{f}^{\mathfrak{e}}$ and a tree in $\mathfrak{f}^{\bar{\mathfrak{e}}}$ rooted at the same point. As a consequence, we see that, almost surely, all the points of $\mathscr{T}_\infty$ are of order less than 3.

3.2. *Maps seen as quotients of real g-trees.* Consistently with the notation $\mathfrak{t}_n(i)$ and $\mathfrak{q}_n(i)$ in the discrete setting, we call $\mathscr{T}_\infty(s)$ [resp., $\mathfrak{q}_\infty(s)$] the equivalence class of $s \in [0,1]$ in $\mathscr{T}_\infty = [0,1]_{/\simeq_\infty}$ (resp., in $\mathfrak{q}_\infty = [0,1]_{/\sim_\infty}$).

LEMMA 6. *The equivalence relation $\simeq_\infty$ is coarser than $\sim_\infty$, so that we can see $\mathfrak{q}_\infty$ as the quotient of $\mathscr{T}_\infty$ by the equivalence relation on $\mathscr{T}_\infty$ induced from $\sim_\infty$.*

PROOF. By definition of $\simeq_\infty$, it suffices to show that if $s < t$ satisfy (12a), (12b) or (12c), then $s \sim_\infty t$. Let us first suppose that s and t satisfy (12a), that is,

$$\mathfrak{C}_\infty(s) = \mathfrak{C}_\infty(t) = \inf_{[s,t]} \mathfrak{C}_\infty.$$

In a first time, we moreover suppose that $\mathfrak{C}_\infty(r) > \mathfrak{C}_\infty(s)$ for all $r \in (s,t)$. Using Proposition 5, we can find integers $0 \le s_n < t_n \le 2n$ such that $(s_{(n)}, t_{(n)}) := (s_n/2n, t_n/2n) \to (s,t)$ and $\mathfrak{C}_{(n)}(s_{(n)}) = \mathfrak{C}_{(n)}(t_{(n)}) = \inf_{[s_{(n)},t_{(n)}]} \mathfrak{C}_{(n)}$. The latter condition imposes that $\dot{\mathfrak{t}}_n(s_n) = \dot{\mathfrak{t}}_n(t_n)$ so that $d_n(s_n,t_n) = 0$ and $s \sim_\infty t$ by (10).

Equation (5) shows that, for every $\mathfrak{e}$, the law of $C^{\mathfrak{e}}_\infty$ is absolutely continuous with respect to the Wiener measure on any interval $[0, m^{\mathfrak{e}}_\infty - \varepsilon]$, for $\varepsilon > 0$. Because local minimums of Brownian motion are pairwise distinct, this is also true for any $C^{\mathfrak{e}}_\infty$, and thus for the whole process $\mathfrak{C}_\infty$ by construction. If there exists $r \in (s,t)$ for which $\mathfrak{C}_\infty(r) = \mathfrak{C}_\infty(s)$, it is thus unique. We may then apply the previous reasoning to (s,r) and (r,t) and find that $s \sim_\infty r$ and $r \sim_\infty t$, so that $s \sim_\infty t$.

Let us now suppose that s and t satisfy (12b). If there is $0 \le r < s$ such that $\mathfrak{C}_\infty(r) = \mathfrak{C}_\infty(s)$, then $r \simeq_\infty s$ by (12a). The same holds with t instead of s. We may thus restrict our attention to s and t for which $\mathfrak{C}_\infty(r) > \mathfrak{C}_\infty(s)$ for all $r \in [0, s)$ and $\mathfrak{C}_\infty(r) > \mathfrak{C}_\infty(t)$ for all $r \in [0, t)$. Let us call $\mathfrak{e} = \mathfrak{e}(s) = \overline{\mathfrak{e}(t)}$. In order to avoid confusion, we use the notation $\langle \cdot \rangle_n$ and $\mathfrak{e}_n(\cdot)$ when dealing with the functions $\mathfrak{C}^{\mathfrak{e}}_{(n)}$'s. We know that for n large enough, we have $\mathfrak{s}_n = \mathfrak{s}_\infty$. We only consider such n's in the following. We first find $0 \le s_n \le 2n$ such that $s_{(n)} := s_n/2n \to s$, $\mathfrak{e}_n(s_{(n)}) = \mathfrak{e}$, and $\mathfrak{C}_{(n)}(s_{(n)}) = \underline{\mathfrak{C}}_{(n)}(s_{(n)})$. We define

$$t_{(n)} := \inf\left\{ r \in \frac{1}{2n}[\![0, 2n]\!] : \mathfrak{e}_n(r) = \bar{\mathfrak{e}}, \mathfrak{C}^{\bar{\mathfrak{e}}}_{(n)}(\langle r \rangle_n) = \sigma^{\mathfrak{e}}_{(n)} - \mathfrak{C}^{\mathfrak{e}}_{(n)}(\langle s_{(n)} \rangle_n) \right\},$$

so that $t_{(n)} \simeq_{(n)} s_{(n)}$, and then $d_{(n)}(s_{(n)}, t_{(n)}) = 0$. Taking an extraction if needed, we may suppose that $t_{(n)} \to t' \sim_\infty s$. By construction, $\mathfrak{e}(t') = \mathfrak{e}(t)$ and $\mathfrak{C}_\infty(t') = \underline{\mathfrak{C}}_\infty(t') = \mathfrak{C}_\infty(t)$. So t' and t fulfill requirement (12a) and $t' \sim_\infty t$ by the above argument. The case of (12c) is easier and may be treated in a similar way. □

This lemma allows us to define a pseudo-metric and an equivalence relation on $\mathscr{T}_\infty$, still denoted by d_∞ and $\sim_\infty$, by setting $d_\infty(\mathscr{T}_\infty(s), \mathscr{T}_\infty(t)) := d_\infty(s, t)$ and declaring $\mathscr{T}_\infty(s) \sim_\infty \mathscr{T}_\infty(t)$ if $s \sim_\infty t$. The metric space $(\mathfrak{q}_\infty, d_\infty)$ is then isometric to $(\mathscr{T}_{\infty/\sim_\infty}, d_\infty)$. We define d°_∞ on $\mathscr{T}_\infty$ by letting

$$d^\circ_\infty(a, b) := \inf\{d^\circ_\infty(s, t) : a = \mathscr{T}_\infty(s), b = \mathscr{T}_\infty(t)\}.$$

We will see in Lemma 9 that there is a.s. only one point where the function $\mathfrak{L}_\infty$ reaches its minimum. On this event, the following lemma holds.

LEMMA 7. *Let $s^\bullet$ be the unique point where $\mathfrak{L}_\infty$ reaches its minimum. Then*

$$d_\infty(s, s^\bullet) = \mathfrak{L}_\infty(s) - \mathfrak{L}_\infty(s^\bullet).$$

Moreover, $s \sim_\infty t$ implies $\mathfrak{L}_\infty(s) = \mathfrak{L}_\infty(t)$.

PROOF. This readily comes from the discrete setting. Let $0 \le s^\bullet_n \le 2n$ be an integer where $\mathfrak{L}_n$ reaches its minimum. By extracting if necessary, we may suppose that $s^\bullet_n/2n$ converges and its limit is necessarily $s^\bullet$. Let $0 \le s_n \le 2n$ be such that $s_n/2n \to s$. From the Chapuy–Marcus–Schaeffer bijection, $d_n(s_n, s^\bullet_n) = \mathfrak{L}_n(s_n) - \mathfrak{L}_n(s^\bullet_n) + 1$. Letting $n \to \infty$ after renormalizing yields the first assertion. The second one follows from the first one and the triangle inequality. □

As a result of Lemmas 6 and 7, we can define $\mathfrak{L}_\infty$ on $\mathscr{T}_\infty$ by $\mathfrak{L}_\infty(\mathscr{T}_\infty(s)) := \mathfrak{L}_\infty(s)$. When $(a, b) \notin (\mathit{fl}_\infty)^2$, we have

$$d^\circ_\infty(a, b) = \mathfrak{L}_\infty(a) + \mathfrak{L}_\infty(b) - 2\max\Big(\min_{x \in [a,b]} \mathfrak{L}_\infty(x), \min_{x \in [b,a]} \mathfrak{L}_\infty(x)\Big), \tag{14}$$

where $[a, b]$ was defined by (13).

4. Points identifications.

This section is dedicated to the proof of the following theorem:

THEOREM 8. *Almost surely, for every $a, b \in \mathscr{T}_\infty$, $a \sim_\infty b$ is equivalent to $d^\circ_\infty(a,b) = 0$.*

We already know that $d^\circ_\infty(a,b) = 0$ implies $a \sim_\infty b$ from the bound $d_\infty \le d^\circ_\infty$. We will show the converse through a series of lemmas. We adapt the approach of Le Gall [18] to our setting.

4.1. *Preliminary lemmas.*

Let us begin by giving some information on the process $(\mathfrak{C}_\infty, \mathfrak{L}_\infty)$.

LEMMA 9. *The set of points where $\mathfrak{L}_\infty$ reaches its minimum is a.s. a singleton.*

Let $f : [0,1] \to \mathbb{R}$ be a continuous function. We say that $s \in [0,1)$ is a *right-increase point* of f if there exists $t \in (s,1]$ such that $f(r) \ge f(s)$ for all $s \le r \le t$. A *left-increase point* is defined in a symmetric way. We call $\mathrm{IP}(f)$ the set of all (left or right) increase points of f.

LEMMA 10. *A.s.*, $\mathrm{IP}(\mathfrak{C}_\infty)$ *and* $\mathrm{IP}(\mathfrak{L}_\infty)$ *are disjoint sets.*

As the proofs of these lemmas are rather technical and unrelated to what follows, we postpone them to Section 6.

4.2. *Key lemma.*

REMARK. In what follows, every discrete path denoted by the letter "$\wp$" will always be a path in the *map*, never in the tree, that is, a path using the edges of the map.

Let τ be a subtree of $\mathfrak{t}_n$ and $\wp = (\wp(0), \wp(1), \dots, \wp(r))$ be a path in $\mathfrak{q}_n$ that avoids the base point $v_n^\bullet$. We say that the arc $(\wp(0), \wp(1))$ enters the subtree τ from the left (resp., from the right) if $\wp(0) \notin \tau$, $\wp(1) \in \tau$ and $\mathfrak{l}_n(\wp(1)) - \mathfrak{l}_n(\wp(0)) = -1$ [resp., $\mathfrak{l}_n(\wp(1)) - \mathfrak{l}_n(\wp(0)) = 1$]. We say that the path $\wp$ *passes through* the subtree τ between times i and j, where $0 < i \le j < r$, if:

- ◇ $\wp(i-1) \notin \tau$; $\wp([\![i,j]\!]) \subseteq \tau$; $\wp(j+1) \notin \tau$,
- ◇ $\mathfrak{l}_n(\wp(i)) - \mathfrak{l}_n(\wp(i-1)) = \mathfrak{l}_n(\wp(j+1)) - \mathfrak{l}_n(\wp(j))$.

The first condition states that $\wp$ "visits" τ, whereas the second one ensures that it really goes "through." It enters and exits τ going "in the same direction."

We say that a vertex $a_n \in \mathfrak{t}_n$ converges toward a point $a \in \mathscr{T}_\infty$ if there exists a sequence of integers $s_n \in [\![0, 2n]\!]$ coding a_n [i.e., $a_n = \dot{\mathfrak{t}}_n(s_n)$] such that $s_n/2n$

admits a limit s satisfying $a = \mathscr{T}_\infty(s)$. Let $[\![l_n, r_n]\!]$ be the intervals coding subtrees $\tau_n \subseteq \mathfrak{t}_n$. We say that the subtree τ_n converges toward a subtree $\tau \subseteq \mathscr{T}_\infty$ if the sequences $l_n/2n$ and $r_n/2n$ admit limits l and r such that the interval coding τ is $[l, r]$. The following lemma is adapted from Le Gall [18], end of Proposition 4.2.

LEMMA 11. *With full probability, the following occurs. Let $a, b \in \mathscr{T}_\infty$ be such that $\mathfrak{L}_\infty(a) = \mathfrak{L}_\infty(b)$. We suppose that there exists a subtree τ rooted at ρ such that* $\inf_\tau \mathfrak{L}_\infty < \mathfrak{L}_\infty(a) < \mathfrak{L}_\infty(\rho)$. *We further suppose that we can find vertices a_n, $b_n \in \mathfrak{t}_n$ and subtrees τ_n in $\mathfrak{t}_n$ converging, respectively, toward a, b, τ and satisfying the following property: for infinitely many n's, there exists a geodesic path $\wp_n$ in $\mathfrak{q}_n$ from a_n to b_n that avoids the base point $v_n^\bullet$ and passes through the subtree τ_n.*

Then, $a \not\sim_\infty b$.

PROOF. The idea is that if a and b were identified, then all the points in the discrete subtrees close (in a certain sense) to the geodesic path would be close to a in the limit. Fine estimates on the sizes of balls yield the result. We proceed to the rigorous proof.

We reason by contradiction and suppose that $a \sim_\infty b$. We only consider integers n for which the hypothesis holds. We call ρ_n the root of τ_n, and we set, for $\varepsilon > 0$,

$$\mathcal{U}_\infty^\varepsilon := \left\{ y \in \tau : \mathfrak{L}_\infty(y) < \mathfrak{L}_\infty(a) + \varepsilon; \forall x \in [[\rho, y]], \mathfrak{L}_\infty(x) > \mathfrak{L}_\infty(a) + \frac{\varepsilon}{8} \right\}.$$

We first show that $\mathcal{U}_\infty^\varepsilon \subseteq B_\infty(a, 2\varepsilon)$, where $B_\infty(a, 2\varepsilon)$ denotes the closed ball of radius 2ε centered at a in the metric space $(\mathfrak{q}_\infty, d_\infty)$. Let $y \in \mathcal{U}_\infty^\varepsilon$. We can find $y_n \in \tau_n \setminus \{\rho_n\}$ converging toward y. For n large enough, we have

$$d_{\mathfrak{q}_n}(a_n, b_n) \le \frac{\varepsilon}{32} n^{1/4}, \qquad \sup_{c \in \wp_n} |\mathfrak{l}_n(c) - \mathfrak{l}_n(a_n)| \le \frac{\varepsilon}{32} n^{1/4},$$

$$\mathfrak{l}_n(y_n) \le \mathfrak{l}_n(a_n) + \frac{3}{2}\varepsilon n^{1/4}, \qquad \forall x \in [[\rho_n, y_n]] \qquad \mathfrak{l}_n(x) \ge \mathfrak{l}_n(a_n) + \frac{\varepsilon}{16} n^{1/4}.$$

The first inequality comes from the fact that $a \sim_\infty b$. The second inequality is a consequence of the first one. The third inequality holds because $(\mathfrak{l}_n(y_n) - \mathfrak{l}_n(v_n^\bullet))/\gamma n^{1/4} \to \mathfrak{L}_\infty(y)$ and $(\mathfrak{l}_n(a_n) - \mathfrak{l}_n(v_n^\bullet))/\gamma n^{1/4} \to \mathfrak{L}_\infty(a)$. Finally, the fourth inequality follows by compactness of $[[\rho, y]]$.

From now on, we only consider such n's. We call $t_n := \sup\{t : y_n = \dot{\mathfrak{t}}_n(t)\}$ the last integer coding y_n, and $[\![l_n, r_n]\!]$ the interval coding τ_n. We also call $i \le j$ two integers such that $\wp_n$ passes through τ_n between times i and j. For the sake of simplicity, we suppose that $\wp_n$ enters τ_n from the left.[6] Notice that the path $\wp_n$ does not intersect $[[\rho_n, y_n]]$, because the labels on $[[\rho_n, y_n]]$ are strictly greater

[6]The case where $\wp_n$ enters τ_n from the right may be treated by considering the path $h \mapsto \wp_n(d_{\mathfrak{q}_n}(a_n, b_n) - h)$ instead of $\wp_n$.

than the labels on $\wp_n$. Let k be the largest integer in $[\![i-1, j]\!]$ such that $\wp_n(k)$ belongs to the set $\{\wp_n(i-1)\} \cup \dot{\mathfrak{t}}_n([\![l_n, t_n]\!])$. Then $\wp_n(k+1) \in \{\wp_n(j+1)\} \cup \dot{\mathfrak{t}}_n([\![t_n, r_n]\!])$. Moreover, $\mathfrak{l}_n(\wp_n(k+1)) = \mathfrak{l}_n(\wp_n(k)) - 1$: otherwise, all the vertices in $[\wp_n(k+1), \wp_n(k)]$ would have labels greater than $\mathfrak{l}_n(\wp_n(k))$, and it is easy to see that this would prohibit $\wp_n$ from exiting τ_n by going "to the right," in the sense that we would not have $\mathfrak{l}_n(\wp_n(j+1)) = \mathfrak{l}_n(\wp_n(j)) - 1$. As a result, when performing the Chapuy–Marcus–Schaeffer bijection for the arc linking $\wp_n(k)$ to $\wp_n(k+1)$, we have to visit y_n. Then, going through consecutive successors of t_n, we are bound to hit $\wp_n(k+1)$, so that $d_{\mathfrak{q}_n}(y_n, \wp_n) \le \mathfrak{l}_n(y_n) - \mathfrak{l}_n(\wp_n(k+1))$. This yields that $d_{\mathfrak{q}_n}(a_n, y_n) \le d_{\mathfrak{q}_n}(a_n, b_n) + d_{\mathfrak{q}_n}(y_n, \wp_n) \le 2\varepsilon\gamma n^{1/4}$, and, by taking the limit, $d_\infty(a, y) \le 2\varepsilon$.

We conclude thanks to two lemmas, whose proofs are postponed to Section 6. They are derived from similar results in the planar case: [18], Lemma 2.4, and [19], Corollary 6.2. We call λ the volume measure on $\mathfrak{q}_\infty$, that is, the image of the Lebesgue measure on [0, 1] by the canonical projection from [0, 1] to $\mathfrak{q}_\infty$.

LEMMA 12. *Almost surely, for every $\eta > 0$ and every subtree τ rooted at ρ, the condition* $\inf_\tau \mathfrak{L}_\infty < \mathfrak{L}_\infty(\rho) - \eta$ *implies that*

$$\liminf_{\varepsilon\to 0} \varepsilon^{-2}\lambda\Bigg(\bigg\{y \in \tau : \mathfrak{L}_\infty(y) < \mathfrak{L}_\infty(\rho) - \eta + \varepsilon;\ \forall x \in [[\rho, y]], \mathfrak{L}_\infty(x) > \mathfrak{L}_\infty(\rho) - \eta + \frac{\varepsilon}{8}\bigg\}\Bigg) > 0.$$

LEMMA 13. *Let $\delta \in (0, 1]$. For every $p \ge 1$,*

$$\mathbb{E}\bigg[\bigg(\sup_{\varepsilon>0}\bigg(\sup_{x\in\mathfrak{q}_\infty} \frac{\lambda(B_\infty(x,\varepsilon))}{\varepsilon^{4-\delta}}\bigg)\bigg)^p\bigg] < \infty.$$

We apply Lemma 12 to τ and $\eta = \mathfrak{L}_\infty(\rho) - \mathfrak{L}_\infty(a) > 0$, and we find that, for ε small enough,

$$\lambda(\mathcal{U}_\infty^\varepsilon) \ge \varepsilon^{5/2}.$$

The inclusion $\mathcal{U}_\infty^\varepsilon \subseteq B_\infty(a, 2\varepsilon)$ yields that

$$S := \sup_{\varepsilon>0}\bigg(\sup_{x\in\mathfrak{q}_\infty} \frac{\lambda(B_\infty(x,\varepsilon))}{\varepsilon^{7/2}}\bigg) = \infty.$$

Lemma 13 applied to $\delta = 1/2$ and $p = 1$ yields that S is integrable, so that $S < \infty$ a.s. This is a contradiction. $\square$

4.3. *Set overflown by a path.* We call fl_n the floor of $\mathfrak{t}_n$. Let $i \in [\![0, 2n]\!]$, and let $\operatorname{succ}(i)$ be its successor in $(\mathfrak{t}_n, \mathfrak{l}_n)$, defined by (1). We moreover suppose that

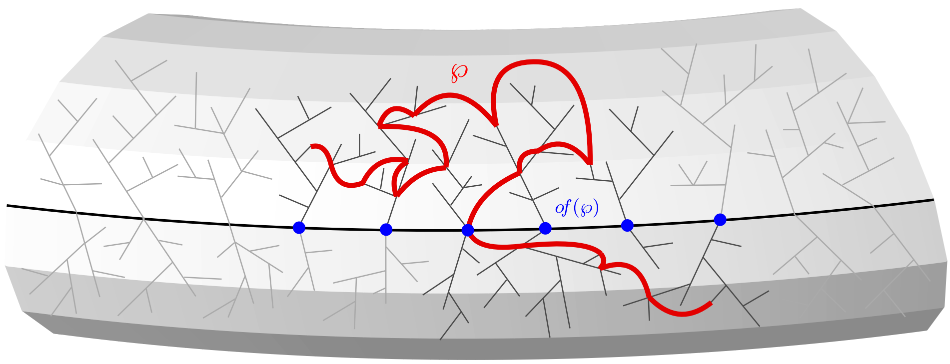

FIG. 7. *The set overflown by the path ℘ is the set of (blue) large dots.*

$\mathrm{succ}(i) \neq \infty$. We say that the arc linking $\mathfrak{t}_n(i)$ to $\mathfrak{t}_n(\mathrm{succ}(i))$ *overflies* the set

$$\mathfrak{t}_n(\overrightarrow{[\![i, \mathrm{succ}(i)]\!]}) \cap \mathit{fl}_n,$$

where $\overrightarrow{[\![i, \mathrm{succ}(i)]\!]}$ was defined by (3). We define the set overflown by a path $\wp$ in $\mathfrak{q}_n$ that avoids the base point $v_n^\bullet$ as the union of the sets its arcs overfly; see Figure 7. We denote it by $\mathit{of}(\wp) \subseteq \mathit{fl}_n$.

LEMMA 14. *Let $a \sim_\infty b \in \mathscr{T}_\infty$ and $\alpha, \beta \in \mathfrak{f}_\infty^{\mathfrak{e}} \cap \mathit{fl}_\infty$. We suppose that, for n sufficiently large, there exist vertices $\alpha_n, \beta_n \in \mathfrak{f}_n^{\mathfrak{e}} \cap \mathit{fl}_n$ and $a_n, b_n \in \mathfrak{t}_n$ converging, respectively, toward α, β, a and b. If, for infinitely many n's, there exists a geodesic path $\wp_n$ from a_n to b_n that overflies $[[\alpha_n, \beta_n]]$, then for all $c \in [[\alpha, \beta]]$,*

$$\mathfrak{L}_\infty(c) \geq \mathfrak{L}_\infty(a) = \mathfrak{L}_\infty(b).$$

Moreover, if there exists $c \in [[\alpha, \beta]]$ for which $\mathfrak{L}_\infty(c) = \mathfrak{L}_\infty(a)$, then $a \sim_\infty c$.

PROOF. Let $c \in [[\alpha, \beta]]$. We can find vertices $c_n \in [[\alpha_n, \beta_n]]$ converging to c. By definition, there is an arc of $\wp_n$ that overflies c_n. Say it links a vertex labeled l to a vertex v labeled $l-1$. From the Chapuy–Marcus–Schaeffer construction, we readily obtain that $\mathfrak{l}_n(c_n) \geq l$. Using the fact that $\mathfrak{l}_n(a_n) - l \leq d_{\mathfrak{q}_n}(a_n, b_n)$, we find

$$\mathfrak{l}_n(c_n) \geq \mathfrak{l}_n(a_n) - d_{\mathfrak{q}_n}(a_n, b_n).$$

Moreover, we can construct a path from c_n to v going through consecutive successors of c_n. As a result, $d_{\mathfrak{q}_n}(c_n, \wp_n) \leq \mathfrak{l}_n(c_n) - l + 1$, so that

$$d_{\mathfrak{q}_n}(c_n, a_n) \leq \mathfrak{l}_n(c_n) - \mathfrak{l}_n(a_n) + 2d_{\mathfrak{q}_n}(a_n, b_n) + 1.$$

Both claims follow by taking limits in these inequalities after renormalization, and by using the fact that $d_{\mathfrak{q}_n}(a_n, b_n) = o(n^{1/4})$. □

4.4. *Points identifications.* We proceed in three steps. We first show that points of fl_∞ are not identified with any other points, then that points cannot be identified with their strict ancestors, and finally Theorem 8.

4.4.1. *Floor points are not identified with any other points.*

LEMMA 15. *A.s., if $a \in \mathit{fl}_\infty$ and $b \in \mathscr{T}_\infty$ are such that $a \sim_\infty b$, then $a = b$.*

PROOF. Let $a \in \mathit{fl}_\infty$ and $b \in \mathscr{T}_\infty \setminus \{a\}$ be such that $a \sim_\infty b$. We first suppose that a is not a node. There exists $\mathfrak{e} \in \vec{E}(\mathfrak{s}_\infty)$ such that $a \in \mathfrak{f}_\infty^{\mathfrak{e}} \cap \mathfrak{f}_\infty^{\bar{\mathfrak{e}}}$, and we can find s, t satisfying $a = \mathscr{T}_\infty(s) = \mathscr{T}_\infty(t)$, $\mathfrak{e}(s) = \mathfrak{e}$ and $\mathfrak{e}(t) = \bar{\mathfrak{e}}$. Without loss of generality, we may suppose that $s < t$. Until further notice, we will moreover suppose that $\rho_b \notin [[\mathfrak{e}]]$.

We restrict ourselves to the case $\mathfrak{s}_n = \mathfrak{s}_\infty$, which happens for n sufficiently large. We can find $a_n \in \mathit{fl}_n$ and $b_n \in \mathfrak{t}_n$ converging toward a and b and satisfying $\rho_{b_n} \notin [[\mathfrak{e}]]$. Let $\wp_n$ be a geodesic path (in $\mathfrak{q}_n$, for $d_{\mathfrak{q}_n}$) from a_n to b_n. It has to overfly at least $[[a_n, \mathfrak{e}^-]]$ or $[[a_n, \mathfrak{e}^+]]$. Indeed, every pair $(x, y) \in [[a_n, \mathfrak{e}^-]] \times [[a_n, \mathfrak{e}^+]]$ breaks $\mathfrak{t}_n$ into connected components, and the points a_n and b_n do not belong to the same of these components. There has to be an arc of $\wp_n$ that links a point belonging to the component containing a_n to one of the other components. Such an arc overflies x or y.

Let us suppose that, for infinitely many n's, $\wp_n$ overflies $[[a_n, \mathfrak{e}^-]]$. Then, Lemma 14 ensures that $\mathfrak{L}_\infty(c) \geq \mathfrak{L}_\infty(a) = \mathfrak{L}_\infty(b)$ for all $c \in [[a, \mathfrak{e}^-]]$. Properties of Brownian snakes show that the labels on $[[a, \mathfrak{e}^-]]$ are Brownian. Precisely, we may code $[[\mathfrak{e}]]$ by the interval $[0, \sigma^{\mathfrak{e}}]$ as follows. For $x \in [0, \sigma^{\mathfrak{e}}]$, we define $T_x := \inf\{r \geq \langle s\rangle : \mathfrak{C}_\infty(r) = \mathfrak{C}_\infty(\langle s\rangle) - x\}$. Then $[[\mathfrak{e}]] = \mathscr{T}_\infty(\{T_x, 0 \leq x \leq \sigma_{\mathfrak{e}}\})$, and

$$\big(\mathfrak{L}_\infty(T_x) - \mathfrak{L}_\infty(\langle s\rangle)\big)_{0 \leq x \leq \sigma_{\mathfrak{e}}} = (\mathfrak{M}_\infty^{\mathfrak{e}}(x))_{0 \leq x \leq \sigma_{\mathfrak{e}}},$$

where, conditionally given $\mathfrak{I}_\infty$, the process $\mathfrak{M}_\infty^{\mathfrak{e}}$ (defined during Proposition 5) has the law of a certain Brownian bridge. Using the fact that local minimums of Brownian motion are distinct, we can find $d \in [[a, \mathfrak{e}^-]] \setminus \{a\}$ such that $\mathfrak{L}_\infty(c) > \mathfrak{L}_\infty(a)$ for all $c \in [[a, d]] \setminus \{a\}$.

Because $a \in \mathit{fl}_\infty$, s and t are both increase points of $\mathfrak{C}_\infty$ and thus are not increase points of $\mathfrak{L}_\infty$, by Lemma 10. As a result, there exist two trees $\tau^1 \subseteq \mathfrak{f}_\infty^{\mathfrak{e}}$ and $\tau^2 \subseteq \mathfrak{f}_\infty^{\bar{\mathfrak{e}}}$ rooted at $\rho^1, \rho^2 \in [[a, d]] \setminus \{a\}$ satisfying $\inf_{\tau^i} \mathfrak{L}_\infty < \mathfrak{L}_\infty(a) < \mathfrak{L}_\infty(\rho^i)$ (see Figure 8).

Similarly, if for infinitely many n's, $\wp_n$ overflies $[[a_n, \mathfrak{e}^+]]$, then we can find two trees $\tau^3 \subseteq \mathfrak{f}_\infty^{\mathfrak{e}}$ and $\tau^4 \subseteq \mathfrak{f}_\infty^{\bar{\mathfrak{e}}}$ rooted at $\rho^3, \rho^4 \in [[a, \mathfrak{e}^+]] \setminus \{a\}$ satisfying $\inf_{\tau^i} \mathfrak{L}_\infty < \mathfrak{L}_\infty(a) < \mathfrak{L}_\infty(\rho^i)$, and $\mathfrak{L}_\infty(c) > \mathfrak{L}_\infty(a)$ for all $c \in [[\rho^3, \rho^4]]$. Three cases may occur:

(i) for n large enough, $\wp_n$ does not overfly $[[a_n, \mathfrak{e}^+]]$ (and therefore overflies $[[a_n, \mathfrak{e}^-]]$);

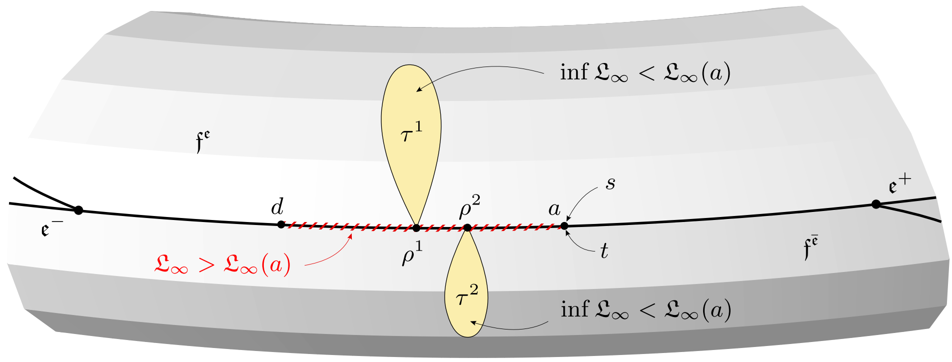

FIG. 8. *The trees τ^1 and τ^2.*

(ii) for n large enough, $\wp_n$ does not overfly $[[a_n, \mathfrak{e}^-]]$ (and therefore overflies $[[a_n, \mathfrak{e}^+]]$);

(iii) for infinitely many n's, $\wp_n$ overflies $[[a_n, \mathfrak{e}^+]]$, and for infinitely many n's, $\wp_n$ overflies $[[a_n, \mathfrak{e}^-]]$.

In case (i), the trees τ^1 and τ^2 are well defined. Let $\tau_n^1 \subseteq \mathfrak{f}_n^{\mathfrak{e}}$, $\tau_n^2 \subseteq \mathfrak{f}_n^{\bar{\mathfrak{e}}}$ be trees rooted at ρ_n^1, $\rho_n^2 \in [[a_n, \mathfrak{e}^-]]$ converging to τ^1 and τ^2. We claim that, for n sufficiently large, $\wp_n$ passes through τ_n^1 or τ_n^2. First, notice that, for n large enough, $\wp_n \cap [[\rho_n^1, \rho_n^2]] = \varnothing$. Otherwise, for infinitely many n's, we could find $\alpha_n \in \wp_n \cap [[\rho_n^1, \rho_n^2]]$, and, up to extraction, we would have $\alpha_n \to \alpha \in [[\rho^1, \rho^2]] \subseteq [[a, d]] \setminus \{a\}$. Furthermore, $d_{\mathfrak{q}_n}(a_n, \alpha_n) \le d_{\mathfrak{q}_n}(a_n, b_n)$ so that $a \sim_\infty \alpha$, and $\mathfrak{L}_\infty(a) = \mathfrak{L}_\infty(\alpha)$ by Lemma 7, which is impossible. For n even larger, it holds that $\inf_{\tau_n^i} \mathfrak{l}_n < \inf_{\wp_n} \mathfrak{l}_n$. Roughly speaking, $\wp_n$ cannot go from a tree located at the right of τ_n^1 (resp., at the left of τ_n^2) to a tree located at its left in $\mathfrak{f}_n^{\mathfrak{e}}$ (resp., to a tree located at its right in $\mathfrak{f}_n^{\bar{\mathfrak{e}}}$) without entering it. Then $\wp_n$ has to enter τ_n^1 from the right or τ_n^2 from the left and pass through one of these trees (see Figure 9).

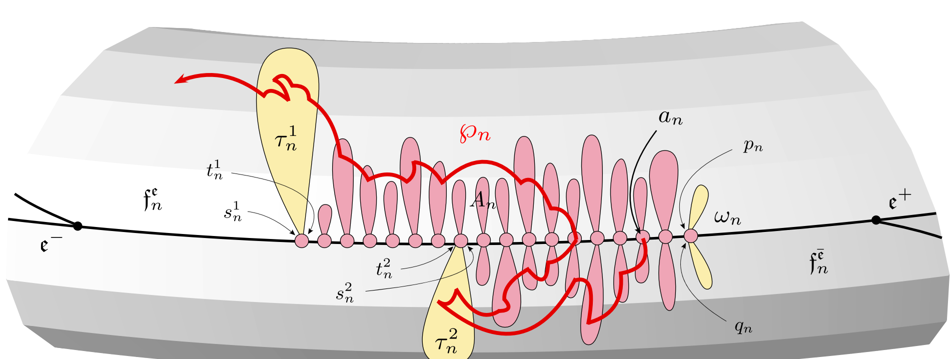

FIG. 9. *The path $\wp_n$ passing through the tree τ_n^1.*

More precisely, we call $[\![s_n^1, t_n^1]\!]$ and $[\![s_n^2, t_n^2]\!]$ the sets coding the subtrees τ_n^1 and τ_n^2. Let $\omega_n \in [[a_n, \mathfrak{e}^+]]$ be a point that is not overflown by $\wp_n$, $p_n := \inf\{t_n^1 \le r \le 2n : \omega_n = \mathfrak{t}_n(r)\}$ and $q_n := \sup\{0 \le r \le s_n^2 : \omega_n = \mathfrak{t}_n(r)\}$. Then, we let

$$A_n := \mathfrak{t}_n(\overrightarrow{[\![t_n^1, p_n]\!]} \cup \overrightarrow{[\![q_n, s_n^2]\!]}).$$

We call $\wp_n(i-1)$ the last point of $\wp_n$ belonging to A_n. Such a point exists because $a_n \in A_n$ and $b_n \notin A_n$. The remarks in the preceding paragraphs yield that neither $\wp_n(i-1)$ nor $\wp_n(i)$ belong to $[[\rho_n^1, \rho_n^2]]$, and, because of the way arcs are constructed in the Chapuy–Marcus–Schaeffer bijection, we see that $\wp_n(i) \in \tau_n^1 \cup \tau_n^2$. Without loss of generality, we may assume that $\wp_n(i) \in \tau_n^1$. Because $\wp_n$ does not overfly ω_n, it enters τ_n^1 from the right at time i, that is, $\mathfrak{l}_n(\wp_n(i)) = \mathfrak{l}_n(\wp_n(i-1)) + 1$. Let $\wp_n(j+1)$ be the first point after $\wp_n(i)$ not belonging to τ_n^1. It exists because $b_n \notin \tau_n^1$. Then, because $\wp_n(j+1) \notin A_n$ and $\wp_n$ does not overfly ω_n, we see that $\mathfrak{l}_n(\wp_n(j+1)) = \mathfrak{l}_n(\wp_n(j)) + 1$, so that $\wp_n$ passes through τ_n^1 between times i and j.

In case (ii), we apply the same reasoning with τ^3 and τ^4 instead of τ^1 and τ^2. In case (iii), the four trees τ^1, τ^2, τ^3 and τ^4 are well defined, and we obtain that $\wp_n$ has to pass through one of their discrete approximations. We then conclude by Lemma 11 that $a \not\sim_\infty b$, which contradicts our hypothesis.

We treat the case where $\rho_b \in [[\mathfrak{e}]] \setminus \{a\}$ in a similar way, simply by replacing $\mathfrak{e}^+$ (resp., $\mathfrak{e}^-$) by ρ_b if $\rho_b \in [[a, \mathfrak{e}^+]]$ (resp., $\rho_b \in [[a, \mathfrak{e}^-]]$). When a is a node, we apply the same arguments, finding up to six trees (one for each forest containing a). Finally, if $\rho_b = a$, then a is a strict ancestor of b. This will be a particular case of Lemma 16. □

4.4.2. *Points are not identified with their strict ancestors.*

LEMMA 16. *A.s., for every $a, b \in \mathscr{T}_\infty$ such that $\rho_a = \rho_b$ and $a \prec b$, we have $a \not\sim_\infty b$.*

The proof of this lemma uses the same kind of arguments we used in Section 4.4.1, is slightly easier than the proof of Lemma 15 and is very similar to Le Gall's proof for Proposition 4.2 in [18], so that we leave the details to the reader.

4.4.3. *Points a, b are only identified when $d_\infty^\circ(a, b) = 0$.*

LEMMA 17. *A.s., for every tree $\tau \subseteq \mathscr{T}_\infty$ rooted at $\rho \in \mathfrak{fl}_\infty$ and all $a, b \in \tau \setminus \{\rho\}$ satisfying $a \sim_\infty b$, we have $d_\infty^\circ(a, b) = 0$.*

PROOF. Let $\tau \subseteq \mathscr{T}_\infty$ be a tree rooted at $\rho \in \mathfrak{fl}_\infty$ and $a, b \in \tau \setminus \{\rho\}$ satisfying $a \neq b$ and $a \sim_\infty b$. By Lemma 16, we know that $a \not\prec b$ and $b \not\prec a$. As a consequence, we have either $s < t$ for all $(s, t) \in \mathscr{T}_\infty^{-1}(a) \times \mathscr{T}_\infty^{-1}(b)$ or $s > t$

for all $(s,t) \in \mathscr{T}_\infty^{-1}(a) \times \mathscr{T}_\infty^{-1}(b)$. Without loss of generality, we will assume that the first case occurs. Let us suppose that $d_\infty^\circ(a,b) > 0$. By Lemma 7, we know that $\mathfrak{L}_\infty(a) = \mathfrak{L}_\infty(b)$, and by (14), we have both $\inf_{[a,b]} \mathfrak{L}_\infty < \mathfrak{L}_\infty(a)$ and $\inf_{[b,a]} \mathfrak{L}_\infty < \mathfrak{L}_\infty(a)$. As a result, there are two subtrees $\tau^1 \subseteq [a,b]$ and $\tau^2 \subseteq [b,a]$ rooted at $\rho^1 \in [[a,b]] \setminus \{a,b\}$ and $\rho^2 \in ([[\rho,a]] \cup [[\rho,b]] \cup \mathit{fl}_\infty) \setminus \{a,b\}$ satisfying $\inf_{\tau^i} \mathfrak{L}_\infty < \mathfrak{L}_\infty(a)$.

Let $\tau_n \subseteq \mathfrak{t}_n$ be a tree rooted at ρ_n and $a_n, b_n \in \mathfrak{t}_n$ be points converging to τ, a, and b. Let $\tau_n^1 \subseteq [a_n, b_n]$ and $\tau_n^2 \subseteq [b_n, a_n]$ be subtrees rooted at $\rho_n^1 \in [[a_n, b_n]] \setminus \{a_n, b_n\}$ and $\rho_n^2 \in ([[\rho_n, a_n]] \cup [[\rho_n, b_n]] \cup \mathit{fl}_n) \setminus \{a_n, b_n\}$ converging toward τ^1 and τ^2. We consider a geodesic path $\wp_n$ from a_n to b_n. Recall that $a \sim_\infty b$ implies that $d_{\mathfrak{q}_n}(a_n, b_n) = o(n^{1/4})$.

Because every point in $[[\rho, \rho^1]]$ is a strict ancestor to a or b, for n large enough, $\wp_n$ does not intersect $[[\rho_n, \rho_n^1]]$. Otherwise, we could find an accumulation point α identified with a and b, such that $\alpha \prec a$ or $\alpha \prec b$ (possibly both), and this would contradict Lemma 16. If $\rho^2 \in \tau$, for n large, $\wp_n$ does not intersect $[[\rho_n, \rho_n^2]]$ either. The same reasoning yields that $\wp_n$ does not intersect fl_n for n sufficiently large, because of Lemma 15.

Let $[\![s_n^1, t_n^1]\!]$ and $[\![s_n^2, t_n^2]\!]$ be the sets coding the subtrees τ_n^1 and τ_n^2. We let

$$A_n := \dot{\mathfrak{t}}_n(\overrightarrow{[\![t_n^2, s_n^1]\!]}) \quad \text{and} \quad B_n := \dot{\mathfrak{t}}_n(\overrightarrow{[\![t_n^1, s_n^2]\!]}).$$

By convention, if $\rho_n^2 \notin \mathfrak{f}_n^{\mathfrak{e}}$, we set $[[\rho_n, \rho_n^2]] := \varnothing$. It is easy to see that $a_n \in A_n$, $b_n \in B_n$, $A_n \cap B_n \subseteq [[\rho_n, \rho_n^1]] \cup [[\rho_n, \rho_n^2]] \cup \mathit{fl}_n$ and $A_n \cup B_n \cup \tau_n^1 \cup \tau_n^2 = \mathfrak{t}_n$.

We conclude as in the proof of Lemma 15. We call $\wp_n(i-1)$ the last point of $\wp_n$ belonging to A_n. Such a point exists because $a_n \in A_n$ and $b_n \notin A_n$. The remarks in the preceding paragraphs yield that, for n large enough, neither $\wp_n(i-1)$ nor $\wp_n(i)$ belong to $A_n \cap B_n$. For n even larger, $\inf_{\tau_n^j} \mathfrak{l}_n < \inf_{\wp_n} \mathfrak{l}_n$, and because of the way arcs are constructed in the Chapuy–Marcus–Schaeffer bijection, we see that $\wp_n(i) \in \tau_n^1 \cup \tau_n^2$. The path $\wp_n$ either enters τ_n^1 from the left or enters τ_n^2 from the right. Without loss of generality, we may suppose that $\wp_n(i) \in \tau_n^1$. Let $\wp_n(i'+1)$ be the first point after $\wp_n(i)$ not belonging to τ_n^1. Then $\wp_n(i'+1) \in B_n \cup \tau_n^2$. If $\wp_n$ passes through τ_n^1 between times i and i', we are done. Otherwise, $\wp_n(i'+1) \in \tau_n^2$ because of the condition $\inf_{\tau_n^2} \mathfrak{l}_n < \inf_{\wp_n} \mathfrak{l}_n$ (informally, $\wp_n$ cannot pass over τ_n^2 without entering it). We consider the first point $\wp_n(i''+1)$ after $\wp_n(i')$ not belonging to τ_n^2, and reiterate the argument. Because $\wp_n$ is a finite path, we see that $\wp_n$ will eventually pass through τ_n^1 or τ_n^2; see Figure 10.

If $\wp_n$ passes through τ_n^1 (resp., τ_n^2) for infinitely many n's, a reasoning similar to the one we used in the proof of Lemma 14 yields that $\mathfrak{L}_\infty(\rho^1) > \mathfrak{L}_\infty(a)$ [resp., $\mathfrak{L}_\infty(\rho^2) > \mathfrak{L}_\infty(a)$]. We conclude by Lemma 11 that $a \sim_\infty b$. This is a contradiction. □

LEMMA 18. *A.s., for all $a, b \in \mathscr{T}_\infty \setminus \mathit{fl}_\infty$ such that $\rho_a \neq \rho_b$ and $a \sim_\infty b$, we have $d_\infty^\circ(a,b) = 0$.*

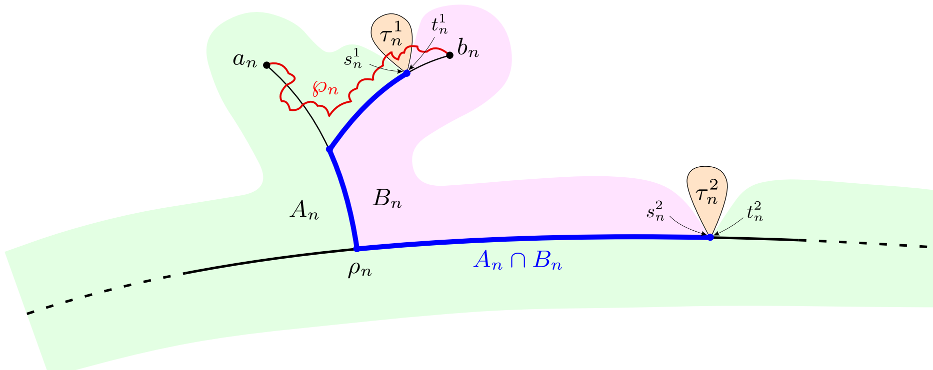

FIG. 10. *The path $\wp_n$ passing through the subtree τ_n^1.*

PROOF. The proof of this lemma is very similar to that of Lemma 17. Let $a, b \in \mathscr{T}_\infty \setminus \mathit{fl}_\infty$ be such that $\rho_a \neq \rho_b$ and $a \sim_\infty b$. Here again, we may suppose that $s < t$ for all $(s,t) \in \mathscr{T}_\infty^{-1}(a) \times \mathscr{T}_\infty^{-1}(b)$, and we can find two subtrees $\tau^1 \subseteq [a,b]$ and $\tau^2 \subseteq [b,a]$ rooted at $\rho^1, \rho^2 \in ([[\rho_a, a]] \cup [[\rho_b, b]] \cup \mathit{fl}_\infty) \setminus \{a, b\}$ satisfying $\inf_{\tau^i} \mathfrak{L}_\infty < \mathfrak{L}_\infty(a)$. As before, we consider the discrete approximations a_n, b_n, $\tau_n^1 = \mathfrak{i}_n([\![s_n^1, t_n^1]\!])$ and $\tau_n^2 = \mathfrak{i}_n([\![s_n^2, t_n^2]\!])$ of a, b, τ^1 and τ^2. Let $\wp_n$ be a geodesic path from a_n to b_n. We still define

$$A_n := \mathfrak{i}_n(\overrightarrow{[\![t_n^2, s_n^1]\!]}) \quad \text{and} \quad B_n := \mathfrak{i}_n(\overrightarrow{[\![t_n^1, s_n^2]\!]}),$$

and we see by the same arguments as in Lemma 17 that, for n sufficiently large, $\wp_n$ does not intersect $A_n \cap B_n$. We then conclude exactly as before. □

Theorem 8 follows from Lemmas 15, 16, 17 and 18. A straightforward consequence of Theorem 8 is that, if the equivalence class of $a = \mathscr{T}_\infty(s)$ for $\sim_\infty$ is not trivial, then s is an increase point of $\mathfrak{L}_\infty$. By Lemma 10, the equivalence class of a for $\simeq_\infty$ is then trivial. Such points may be called *leaves* by analogy with tree terminology.

5. 1-regularity of quadrangulations. The goal of this section is to prove Theorem 2. To that end, we use the notion of regular convergence, introduced by Whyburn [26].

5.1. 1-*regularity.* Recall that $(\mathbb{M}, d_{\mathrm{GH}})$ is the set of isometry classes of compact metric spaces, endowed with the Gromov–Hausdorff metric. We say that a metric space $(\mathcal{X}, \delta)$ is a *path metric space* if any two points $x, y \in \mathcal{X}$ may be joined by a path isometric to a real segment—necessarily of length $\delta(x,y)$. We call $\mathbb{PM}$ the set of isometry classes of path metric spaces. By [8], Theorem 7.5.1, $\mathbb{PM}$ is a closed subset of $\mathbb{M}$.

DEFINITION 11. We say that a sequence $(\mathcal{X}_n)_{n\geq 1}$ of path metric spaces is 1-*regular* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for n large enough, every loop of diameter less than δ in $\mathcal{X}_n$ is homotopic to 0 in its ε-neighborhood.

This definition is actually slightly stronger than Whyburn's original definition [26]. See the discussion in the second section of [22] for more details. We also chose here not to restrict the notion of 1-regularity only to converging sequences of path metric spaces, as it was done in [22, 26], because the notion of 1-regularity (as stated here) is not directly related to the convergence of the sequence of path metric spaces. Our main tool is the following theorem, which is a simple consequence of Begle [3], Theorem 7.

PROPOSITION 19. *Let $(\mathcal{X}_n)_{n\geq 1}$ be a sequence of path metric spaces all homeomorphic to the g-torus $\mathbb{T}_g$. Suppose that $\mathcal{X}_n$ converges toward $\mathcal{X}$ for the Gromov–Hausdorff topology, and that the sequence $(\mathcal{X}_n)_{n\geq 1}$ is* 1*-regular. Then $\mathcal{X}$ is homeomorphic to $\mathbb{T}_g$ as well.*

5.2. *Representation as metric surfaces.* In order to apply Proposition 19, we construct a path metric space $(\mathcal{S}_n, \delta_n)$ homeomorphic to $\mathbb{T}_g$, and an embedded graph that is a representative of the map $\mathfrak{q}_n$, such that the restriction of $(\mathcal{S}_n, \delta_n)$ to the embedded graph is isometric to $(V(\mathfrak{q}_n), d_{\mathfrak{q}_n})$. We use the method provided by Miermont in [22], Section 3.1.

We write $F(\mathfrak{q}_n)$ the set of faces of $\mathfrak{q}_n$. Let (X_f, D_f), $f \in F(\mathfrak{q}_n)$ be n copies of the hollow bottomless unit cube

$$X_f := [0,1]^3 \setminus \big((0,1)^2 \times [0,1)\big)$$

endowed with the intrinsic metric D_f inherited from the Euclidean metric. (The distance between two points x and y is the Euclidean length of a minimal path in X_f linking x to y.)

Let $f \in F(\mathfrak{q}_n)$, and let e_1, e_2, e_3 and e_4 be the four half-edges incident to f, ordered according to the counterclockwise order. For $0 \leq t \leq 1$, we define:

$$\begin{aligned} c_{e_1}(t) &= (t, 0, 0) \in X_f;\\ c_{e_2}(t) &= (1, t, 0) \in X_f;\\ c_{e_3}(t) &= (1-t, 1, 0) \in X_f;\\ c_{e_4}(t) &= (0, 1-t, 0) \in X_f. \end{aligned}$$

In this way, we associate with every half-edge $e \in \vec{E}(\mathfrak{q}_n)$ a path along one of the four edges of the square ∂X_f, where f is the face located to the left of e.

We then define the relation $\approx$ as the coarsest equivalence relation for which $c_e(t) \approx c_{\bar{e}}(1-t)$ for all $e \in \vec{E}(\mathfrak{q}_n)$ and $t \in [0,1]$. This corresponds to gluing the spaces X_f's along their boundaries according to the map structure of $\mathfrak{q}_n$. The topological quotient $\mathcal{S}_n := (\coprod_{f\in F(\mathfrak{q}_n)} X_f)_{/\approx}$ is a two-dimensional CW-complex satisfying the following. Its 1-skeleton $\mathcal{E}_n = (\coprod_{f\in F(\mathfrak{q}_n)} \partial X_f)_{/\approx}$ is an embedding of $\mathfrak{q}_n$ with faces $X_f \setminus \partial X_f$. To the edge $\{e, \bar{e}\} \in E(\mathfrak{q}_n)$ corresponds the edge of $\mathcal{S}_n$ made of the equivalence class of the points in $c_e([0,1])$. Its 0-skeleton $\mathcal{V}_n$ is in one-to-one correspondence with $V(\mathfrak{q}_n)$. Its vertices are the equivalence classes of the corners of the squares ∂X_f.

We endow the space $\coprod_{f\in F(\mathfrak{q}_n)} X_f$ with the largest pseudo-metric δ_n compatible with D_f, $f \in F(\mathfrak{q}_n)$ and $\approx$, in the sense that $\delta_n(x,y) \le D_f(x,y)$ for $x, y \in X_f$ and $\delta_n(x,y) = 0$ whenever $x \approx y$. Its quotient—still noted δ_n—then defines a pseudo-metric on $\mathcal{S}_n$ (which actually is a true metric, as we will see in Proposition 20). As usual, we define $\delta_{(n)} := \delta_n/\gamma n^{1/4}$ its rescaled version.

We rely on the following proposition. It was actually stated in [22] for the two-dimensional sphere but readily extends to the g-torus.

PROPOSITION 20 ([22], Proposition 1). *The space $(\mathcal{S}_n, \delta_n)$ is a path metric space homeomorphic to $\mathbb{T}_g$. Moreover, the restriction of $\mathcal{S}_n$ to $\mathcal{V}_n$ is isometric to $(V(\mathfrak{q}_n), d_{\mathfrak{q}_n})$, and any geodesic path in $\mathcal{S}_n$ between points in $\mathcal{V}_n$ is a concatenation of edges of $\mathcal{S}_n$. Finally, $d_{\mathrm{GH}}((V(\mathfrak{q}_n), d_{\mathfrak{q}_n}), (\mathcal{S}_n, \delta_n)) \le 3$, so that, by Proposition 1,*

$$(\mathcal{S}_{n_k}, \delta_{(n_k)}) \xrightarrow[k\to\infty]{(d)} (\mathfrak{q}_\infty, d_\infty)$$

in the sense of the Gromov–Hausdorff topology.

5.3. *Proof of Theorem 2.* We prove here that $(\mathfrak{q}_\infty, d_\infty)$ is a.s. homeomorphic to $\mathbb{T}_g$ by means of Propositions 19 and 20. To this end, we only need to show that the sequence $(\mathcal{S}_{n_k}, \delta_{(n_k)})_k$ is 1-regular. At first, we only consider simple loops made of edges. We proceed in two steps: Lemma 21 shows that there are no noncontractible "small" loops; then Lemma 22 states that the small loops are homotopic to 0 in their ε-neighborhood.

LEMMA 21. *A.s., there exists $\varepsilon_0 > 0$ such that for all k large enough, any noncontractible simple loop made of edges in $\mathcal{S}_{n_k}$ has diameter greater than ε_0.*

PROOF. The basic idea is that a noncontractible loop in $\mathcal{S}_n$ has to intersect $\mathfrak{fl}_n$ and to "jump" from a forest to another one. At the limit, the loop transits from a forest to another by visiting two points that $\sim_\infty$ identifies. If the loops vanish at the limit, then these two identified points become identified with a point in $\mathfrak{fl}_\infty$, creating an increase point for both $\mathfrak{L}_\infty$ et $\mathfrak{C}_\infty$. We proceed to the rigorous proof.

We argue by contradiction and assume that, with positive probability, along some (random) subsequence of the sequence $(n_k)_{k\ge0}$, there exist noncontractible simple loops $\wp_n$ made of edges in $\mathcal{S}_n$ with diameter tending to 0 (with respect to the rescaled metric $\delta_{(n)}$). We reason on this event.

Because $\wp_n$ is noncontractible, it has to intersect fl_n: if not, $\wp_n$ would entirely be drawn in the unique face of $\mathfrak{s}_n$, which is homeomorphic to a disk, by definition of a map. It would thus be contractible, by the Jordan curve theorem. Let $a_n \in \wp_n \cap \mathit{fl}_n$. Up to further extraction, we may suppose that $a_n \to a \in \mathit{fl}_\infty$. Notice that every time $\wp_n$ intersects fl_n, it has to be "close" to a_n. Precisely, if $b_n \in \wp_n \cap \mathit{fl}_n$ tends to b, then $\delta_{(n)}(a_n, b_n) \le \operatorname{diam}(\wp_n) \to 0$, which yields $a \sim_\infty b$, and $a = b$ by Lemma 15. Moreover, for n sufficiently large, the base point $v_n^\bullet \notin \wp_n$: otherwise, for infinitely many n's, $(\mathfrak{l}_n(a_n) - \min \mathfrak{l}_n + 1)/\gamma n^{1/4} \le \operatorname{diam}(\wp_n) \to 0$, so that $\mathfrak{L}_\infty$ would reach its minimum at a, and we know by Lemma 9 that this is not the case.

Let us first suppose that a is not a node of $\mathscr{T}_\infty$. There exists $\mathfrak{e} \in \vec{E}(\mathfrak{s}_\infty)$ such that $a \in \mathfrak{f}_\infty^{\mathfrak{e}} \cap \mathfrak{f}_\infty^{\bar{\mathfrak{e}}}$ and for n large enough, $a_n \in \mathfrak{f}_n^{\mathfrak{e}} \cap \mathfrak{f}_n^{\bar{\mathfrak{e}}}$. For n even larger, the whole loop $\wp_n$ "stays in $\mathfrak{f}_n^{\mathfrak{e}} \cup \mathfrak{f}_n^{\bar{\mathfrak{e}}}$." Precisely, for all $\mathfrak{e}' \in \vec{E}(\mathfrak{s}_\infty) \setminus \{\mathfrak{e}, \bar{\mathfrak{e}}\}$, we have $\wp_n \cap \mathfrak{f}_n^{\mathfrak{e}'} = \varnothing$. Otherwise, since $\vec{E}(\mathfrak{s}_\infty)$ is finite, there would exist $\mathfrak{e}' \notin \{\mathfrak{e}, \bar{\mathfrak{e}}\}$ such that for infinitely many n's, we can find $c_n \in \wp_n \cap \mathfrak{f}_n^{\mathfrak{e}'}$. Up to extraction, $c_n \to c \in \mathfrak{f}_\infty^{\mathfrak{e}'}$, so that $c \neq a$ (a is not a node) and $c \sim_\infty a$, which is impossible, by Lemma 15.

We claim that there exists an arc of $\wp_n$ linking a point $b_n \in \mathfrak{f}_n^{\mathfrak{e}}$ to some point in $\mathfrak{f}_n^{\bar{\mathfrak{e}}}$ that overflies either $[[\rho_{b_n}, \mathfrak{e}^+]]$ or $[[\mathfrak{e}^-, \rho_{b_n}]]$ (see Figure 11). Let us suppose for a moment that this does not hold. In particular, there is no arc linking a point in $\mathfrak{f}_n^{\mathfrak{e}} \setminus \mathit{fl}_n$ to a point in $\mathfrak{f}_n^{\bar{\mathfrak{e}}} \setminus \mathit{fl}_n$. It will be more convenient here to write $\wp_n$ as $(a_n = v_1, \alpha_1, v_2, \alpha_2, \dots, v_{r-1}, \alpha_{r-1}, v_r = a_n)$ where the v_i's are vertices, and the α_i's are arcs. Let $i := \inf\{j \in [\![2, r]\!] : v_j \in \mathit{fl}_n\}$ be the index of the first time $\wp_n$ returns to fl_n. Then $v_2, \dots, v_{i-1}$ belong to the same set $\mathfrak{f}_n^{\mathfrak{e}} \setminus \mathit{fl}_n$ or $\mathfrak{f}_n^{\bar{\mathfrak{e}}} \setminus \mathit{fl}_n$, and $(\alpha_1, v_2, \alpha_2, \dots, v_{i-1}, \alpha_{i-1})$ is thus drawn inside the face of $\mathfrak{s}_n$. As a result, the path $(v_1, \alpha_1, v_2, \dots, v_{i-1}, \alpha_{i-1}, v_i)$ is homotopic to the segment $[[v_1, v_i]]$. Repeating the argument for every "excursion" away from fl_n, we see that $\wp_n$ is homotopic to a finite concatenation of segments all included in the topological segment $[[2, \sigma_n^{\mathfrak{e}}]]$, where we used the notation of Section 2.2.1 for the forest $\mathfrak{f}_n^{\mathfrak{e}}$; see Figure 11. It follows that $\wp_n$ is contractible, which is a contradiction.

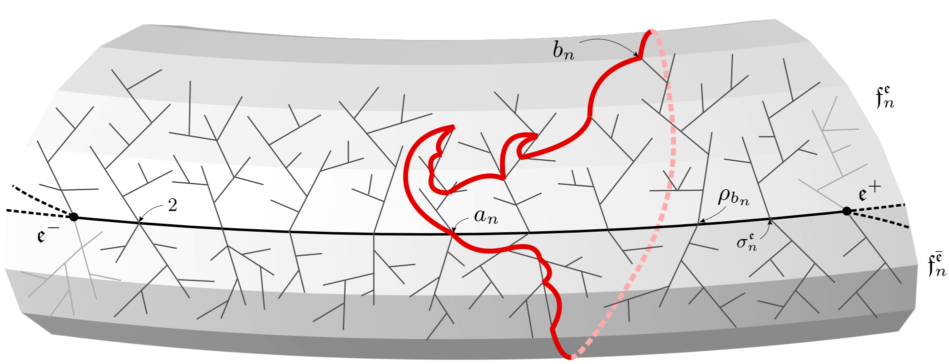

FIG. 11. *A noncontractible loop intersecting fl_n at a_n and "jumping" from $\mathfrak{f}_n^{\mathfrak{e}}$ to $\mathfrak{f}_n^{\bar{\mathfrak{e}}}$ at b_n.*

We consider the case where the arc from the previous paragraph overflies $[[\rho_{b_n}, \mathfrak{e}^+]]$. The other case is treated in a similar way. From the construction of the Chapuy–Marcus–Schaeffer bijection, we can find integers $s_n \leq t_n$ such that $b_n = \mathfrak{t}_n(s_n)$, $\mathfrak{e}^+ = \mathfrak{t}_n(t_n)$ and for all $s_n \leq r \leq t_n$, $\mathfrak{L}_n(r) \geq \mathfrak{L}_n(s_n)$. Up to further extraction, we may suppose that $s_n/2n \to s$ and $t_n/2n \to t$. Therefore, for all $s \leq r \leq t$, $\mathfrak{L}_\infty(r) \geq \mathfrak{L}_\infty(s)$. Moreover, the fact that $b_n \to a \neq \mathfrak{e}^+$ yields $s < t$, so that s is an increase point for $\mathfrak{L}_\infty$. But $\mathscr{T}_\infty(s) = a$ and s has to be an increase point for $\mathfrak{C}_\infty$. By Lemma 10, this cannot happen.

If a is a node, there are three half-edges $\mathfrak{e}_1$, $\mathfrak{e}_2$ and $\mathfrak{e}_3$ such that $a = \mathfrak{e}_1^+ = \mathfrak{e}_2^+ = \mathfrak{e}_3^+$. A reasoning similar to what precedes yields the existence of an arc of $\wp_n$ linking a point b_n in one of the three sets $\mathfrak{f}^{\mathfrak{e}_i} \cup \mathfrak{f}^{\bar{\mathfrak{e}}_{i+1}}$, $i = 1, 2, 3$ (where we use the convention $\mathfrak{e}_4 = \mathfrak{e}_1$) to a point lying in another one of these three sets that overflies either, if $b_n \in \mathfrak{f}_\infty^{\mathfrak{e}_i}$, $[[\rho_{b_n}, a]] \cup [[\mathfrak{e}_{i+1}]]$ or $[[\mathfrak{e}_i^-, \rho_{b_n}]]$, or, if $b_n \in \mathfrak{f}_\infty^{\mathfrak{e}_{i+1}}$, $[[\rho_{b_n}, \mathfrak{e}_{i+1}^+]]$ or $[[\mathfrak{e}_i]] \cup [[a, \rho_{b_n}]]$. We conclude by similar arguments. □

We now turn our attention to contractible loops. Let $\wp$ be a contractible simple loop in $\mathcal{S}_n$ made of edges. Then $\wp$ splits $\mathcal{S}_n$ into two domains. Only one of these is homeomorphic to a disk.[7] We call it the *inner domain* of $\wp$, and we call the other one the *outer domain* of $\wp$. In particular, these domains are well defined for loops whose diameter is smaller than ε_0, when n is large enough.

LEMMA 22. *A.s., for all $\varepsilon > 0$, there exists $0 < \delta < \varepsilon \wedge \varepsilon_0$ such that for all k sufficiently large, the inner domain of any simple loop made of edges in $\mathcal{S}_{n_k}$ with diameter less than δ has diameter less than ε.*

PROOF. We adapt the method used by Miermont in [22]. The idea is that a contractible loop separates a whole part of the map from the base point. Then the labels in one of the two domains it separates are larger than the labels on the loop. In the g-tree, this corresponds to having a part with labels larger than the labels on the "border." In the continuous limit, this creates an increase point for both $\mathfrak{C}_\infty$ and $\mathfrak{L}_\infty$.

Suppose that, with positive probability, there exists $0 < \varepsilon < \varepsilon_0$ for which, along some (random) subsequence of the sequence $(n_k)_{k \geq 0}$, there exist contractible simple loops $\wp_n$ made of edges in $\mathcal{S}_n$ with diameter tending to 0 (with respect to the rescaled metric $\delta_{(n)}$) and whose inner domains are of diameter larger than ε. Let us reason on this event. First, notice that, because $g \geq 1$, the outer domain of $\wp_n$

[7] This is a consequence of the Jordan–Schönflies theorem, applied in the universal cover of $\mathcal{S}_n$, which is either the plane when $g = 1$, or the unit disk when $g \geq 2$; see, for example, [13], Theorem 1.7.

contains at least one noncontractible loop, so that its diameter is larger than $\varepsilon_0 > \varepsilon$ by Lemma 21.

Let $s^\bullet$ be the unique point where $\mathfrak{L}_\infty$ reaches its minimum, and $s_n^\bullet$ be an integer where $\mathfrak{L}_n$ reaches its minimum. We call $w_n^\bullet := \dot{\mathfrak{t}}_n(s_n^\bullet)$ the corresponding point in the g-tree. This is a vertex at δ_n-distance 1 from $v_n^\bullet$. Let us take x_n in the domain that does not contain $w_n^\bullet$, such that the distance between x_n and $\wp_n$ is maximal. (If $w_n^\bullet \in \wp_n$, we take x_n in either of the two domains according to some convention.) Let $y_n \in \wp_n \cap ([[\rho_{w_n^\bullet}, w_n^\bullet]] \cup \mathit{fl}_n \cup [[\rho_{x_n}, x_n]])$ be such that there exists an injective path[8] $\mathfrak{p}_n$ in $\mathfrak{t}_n$ from x_n to y_n that intersects $\wp_n$ only at y_n. In other words, when going from x_n to $w_n^\bullet$ along some injective path, y_n is the first vertex belonging to $\wp_n$ we meet; see Figure 12. Such a point exists because x_n and $w_n^\bullet$ do not belong to the same of the two components delimited by $\wp_n$. Up to further extraction, we suppose that $s_n^\bullet/2n \to s^\bullet$, $x_n \to x$ and $y_n \to y$. We call $\mathfrak{p} \subseteq [[\rho_{w^\bullet}, w^\bullet]] \cup \mathit{fl}_\infty \cup [[\rho_x, x]]$ the injective path corresponding to $\mathfrak{p}_n$ in the limit, that is, the path defined as $\mathfrak{p}_n$ "without the subscripts n." Because the distance between two points in the same domain as x_n is smaller than $2\delta_{(n)}(x_n, \wp_n) + \operatorname{diam}(\wp_n)$, we obtain that $\delta_{(n)}(x_n, y_n) \geq \varepsilon/4$, as soon as $\operatorname{diam}(\wp_n) \leq \varepsilon/2$. In particular, we see that $x \neq y$, and that the path $\mathfrak{p}$ is not reduced to a single point.

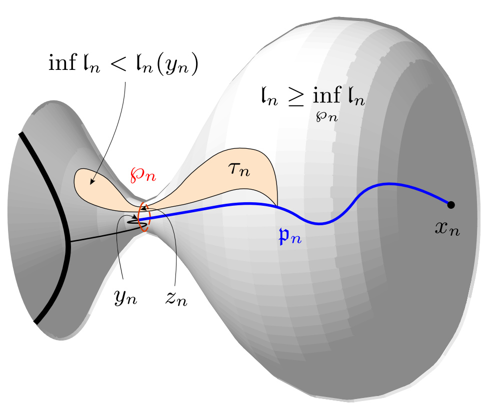

FIG. 12. *The path $\wp_n$ intersects τ_n. This figure represents the case where $y_n \in [[\rho_{x_n}, x_n]]$.*

[8]Depending on the case, the path $\mathfrak{p}_n$ will be of one of the following forms:

- ◇ $[[x_n, y_n]]$, with $y_n \in [[\rho_{x_n}, x_n]]$;
- ◇ $[[x_n, \rho_{x_n}]] \cup [[\rho_{x_n}, y_n]]$, with $y_n \in \mathit{fl}_n$;
- ◇ $[[x_n, \rho_{x_n}]] \cup [[\rho_{x_n}, e_1^+]] \cup [[e_2]] \cup \cdots \cup [[e_k]] \cup [[e_k^+, y_n]]$ for some half-edges $e_1, e_2, \ldots, e_k$ of $\mathfrak{s}_n$ satisfying $e_i^+ = e_{i+1}^-$, with $y_n \in \mathit{fl}_n$;
- ◇ $[[x_n, \rho_{x_n}]] \cup [[\rho_{x_n}, e_1^+]] \cup [[e_2]] \cup \cdots \cup [[e_k]] \cup [[e_k^+, \rho_{w_n^\bullet}]] \cup [[\rho_{w_n^\bullet}, y_n]]$ for some half-edges $e_1, e_2, \ldots, e_k$ of $\mathfrak{s}_n$ satisfying $e_i^+ = e_{i+1}^-$, with $y_n \in [[\rho_{w_n^\bullet}, w_n^\bullet]]$.

Let us first suppose that $y \neq w^\bullet := \mathscr{T}_\infty(s^\bullet)$. (In particular, $w_n^\bullet \notin \wp_n$ for n large, so that there is no ambiguity on which domain to chose x_n.) In that case, $y \in ([[\rho_{w^\bullet}, w^\bullet]] \cup \mathit{fl}_\infty \cup [[\rho_x, x]]) \setminus \{x, w^\bullet\}$, so that the points in $\mathscr{T}_\infty^{-1}(y)$ are increase points of $\mathfrak{C}_\infty$. By Lemma 10, we can find a subtree[9] τ, not containing y, satisfying $\inf_\tau \mathfrak{L}_\infty < \mathfrak{L}_\infty(y)$ and rooted on the path $\mathfrak{p}$.

We consider a discrete approximation τ_n rooted on $\mathfrak{p}_n$. Because the loop $\wp_n$ is contractible, all the labels of the points in the same domain as x_n are larger than $\inf_{\wp_n} \mathfrak{l}_n$. Indeed, the labels represent the distances (up to an additive constant) in $\mathfrak{q}_n$ to the base point, and every geodesic path from such a point to the base point has to intersect $\wp_n$. For n large enough, it holds that $\inf_{\tau_n} \mathfrak{l}_n < \inf_{\wp_n} \mathfrak{l}_n$. As a consequence, τ_n cannot entirely be included in the domain containing x_n. Therefore, the set $\wp_n \cap \tau_n$ is not empty, so that we can find $z_n \in \wp_n \cap \tau_n$. Up to extraction, we may suppose that $z_n \to z$.

On one hand, $\delta_{(n)}(y_n, z_n) \leq \operatorname{diam}(\wp_n)$, so that $y \sim_\infty z$. On the other hand, $z \in \tau$ and $y \notin \tau$, so that $y \neq z$. Because y is not a leaf, this contradicts Theorem 8.

When $y = w^\bullet$, we use a different argument. Let $a_n = \mathfrak{t}_n(\alpha_n)$ and $b_n = \mathfrak{t}_n(\beta_n)$ be, respectively, in the inner and outer domains of $\wp_n$, such that their distance to $\wp_n$ is maximal. Because a_n and b_n do not belong to the same domain, we can find

$$t_n^1 \in \overrightarrow{[\![\alpha_n, \beta_n]\!]} \quad \text{and} \quad t_n^2 \in \overrightarrow{[\![\beta_n, \alpha_n]\!]}$$

such that $\mathfrak{t}_n(t_n^1), \mathfrak{t}_n(t_n^2) \in \wp_n$. Up to extraction, we suppose that

$$\frac{\alpha_n}{2n} \to \alpha, \qquad \frac{\beta_n}{2n} \to \beta, \qquad \frac{t_n^1}{2n} \to t^1 \in \overrightarrow{[\alpha, \beta]} \quad \text{and} \quad \frac{t_n^2}{2n} \to t^2 \in \overrightarrow{[\beta, \alpha]}.$$

Because $\operatorname{diam}(\wp_n) \to 0$, we have $\mathscr{T}_\infty(t^1) = \mathscr{T}_\infty(t^2) = w^\bullet$. Moreover, the argument we used to prove that $x \neq y$ yields that $\mathscr{T}_\infty(\alpha) \neq w^\bullet$ and $\mathscr{T}_\infty(\beta) \neq w^\bullet$. As a result, we obtain that $t^1 \neq t^2$. This contradicts Lemma 9. $\square$

It remains to deal with general loops that are not necessarily made of edges. We reason on the set of full probability where Lemmas 21 and 22 hold. We fix $0 < \varepsilon < \operatorname{diam}(\mathfrak{q}_\infty)/4$. Let ε_0 be as in Lemma 21 and δ as in Lemma 22. For k sufficiently large, the conclusions of both lemmas hold, together with the inequality $\delta \gamma n_k^{1/4} \geq 12$. Now, take any loop $\mathscr{L}$ drawn in $\mathcal{S}_{n_k}$ with diameter less than $\delta/2$. Consider the union of the closed faces[10] visited by $\mathscr{L}$. The boundary of this union

[9]Here again, we need to distinguish between some cases:

- ◇ if $y \in [[\rho_x, x]]$, then $\mathfrak{p} = [[x, y]]$, and τ is a tree to the left or right of $[[\rho_x, x]]$ rooted at some point in $[[x, y]] \setminus \{x, y\}$;
- ◇ if $y \in \mathit{fl}_\infty \setminus \{\rho_x\}$, then τ is a tree of $\mathscr{T}_\infty$ rooted on $(\mathfrak{p} \cap \mathit{fl}_\infty) \setminus \{y\}$;
- ◇ if $y \in [[\rho_{w^\bullet}, w^\bullet]] \setminus \{\rho_{w^\bullet}\}$, then τ is a tree to the left or right of $[[\rho_{w^\bullet}, y]]$.

[10]We call *closed face* the closure of a face.

consists in simple loops made of edges in $\mathcal{S}_{n_k}$. Let us call Λ the set of these simple loops.

Because every face of $\mathcal{S}_{n_k}$ has a diameter smaller than $3/\gamma n_k^{1/4}$, we see that for all $\lambda \in \Lambda$, $\mathrm{diam}(\lambda) \le \mathrm{diam}(\mathscr{L}) + 6/\gamma n_k^{1/4} \le \delta$. Then, by Lemma 21, λ is contractible and, by Lemma 22, its inner domain is of diameter less than ε. By definition, for all $\lambda \in \Lambda$, $\mathscr{L}$ entirely lies either inside the inner domain of λ, or inside its outer domain. We claim that there exists one loop in Λ such that $\mathscr{L}$ lies in its inner domain. Then, it will be obvious that $\mathscr{L}$ is homotopic to 0 in its ε-neighborhood.

Let us suppose that $\mathscr{L}$ lies in the outer domain of every loop $\lambda \in \Lambda$. Then, every face of $\mathcal{S}_{n_k}$ is either visited by $\mathscr{L}$, or included in the inner domain of some loop $\lambda \in \Lambda$. As a result, we obtain that $\mathrm{diam}(\mathfrak{q}_\infty) \le \mathrm{diam}(\mathscr{L}) + 2\sup_{\lambda\in\Lambda} \mathrm{diam}(\lambda) + 6/\gamma n_k^{1/4} \le 3\delta$. This is in contradiction with our choice of δ.

6. Transfering results from the planar case through Chapuy's bijection. In order to prove Lemmas 9, 10, 12 and 13, we rely on similar results for the Brownian snake driven by a normalized excursion $(\mathbb{e}, Z)$. This means that $\mathbb{e}$ has the law of a normalized Brownian excursion, and, conditionally given $\mathbb{e}$, the process Z is a Gaussian process with covariance

$$\mathrm{cov}(Z_x, Z_y) = \inf_{[x\wedge y, x\vee y]} \mathbb{e}.$$

We first focus on the proofs of Lemmas 9 and 10. Lemmas 3.1 and 3.2 in [20] state that, a.s., Z reaches its minimum at a unique point, and that, a.s., $\mathrm{IP}(\mathbb{e})$ and $\mathrm{IP}(Z)$ are disjoint sets. We will use a bijection due to Chapuy [9] to transfer these results to our case.

6.1. *Chapuy's bijection.* Chapuy's bijection consists in "opening" g-trees into plane trees. We briefly describe it here. See [9] for more details. Let $\mathfrak{t}$ be a g-tree whose scheme $\mathfrak{s}$ is dominant. Such a g-tree will be called *dominant* in the following. As usual, we arrange the half-edges of $\mathfrak{s}$ according to its facial order: $\mathfrak{e}_1 = \mathfrak{e}_*, \dots, \mathfrak{e}_{2(6g-3)}$. Let v be one of the nodes of $\mathfrak{t}$. We can see it as a vertex of $\mathfrak{s}$. Let us call $\mathfrak{e}_{i_1}$, $\mathfrak{e}_{i_2}$ and $\mathfrak{e}_{i_3}$ the three half-edges starting from v (i.e., $v = \mathfrak{e}_{i_1}^- = \mathfrak{e}_{i_2}^- = \mathfrak{e}_{i_3}^-$), where $i_1 < i_2 < i_3$. We say that v is *intertwined* if the half-edges $\mathfrak{e}_{i_1}$, $\mathfrak{e}_{i_2}$, $\mathfrak{e}_{i_3}$ are arranged according to the counterclockwise order around v (see Figure 13). When v is intertwined, we may *slice* it: we define a new map, denoted by $\mathfrak{t} \setminus\!\!\setminus v$, by slicing the node v into three new vertices v^1, v^2 and v^3 (see Figure 13).

The map obtained by such an operation turns out to be a dominant $(g-1)$-tree. After repeating g times this operation, we are left with a plane tree. In that regard, we call *opening sequence* of $\mathfrak{t}$ a g-uple $(v_1, \dots, v_g)$ such that v_g is an intertwined node of $\mathfrak{t}$, and for all $1 \le i \le g-1$, the vertex v_i is an intertwined node of $\mathfrak{t} \setminus\!\!\setminus v_g \setminus\!\!\setminus \cdots \setminus\!\!\setminus v_{i+1}$. We can show that every g-tree has exactly $2g$ intertwined nodes, and thus $2^g g!$ opening sequences.

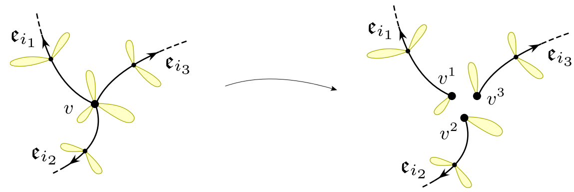

FIG. 13. *Slicing an intertwined node v.*

To reverse the slicing operation, we have to intertwine and glue back the three vertices together. We then need to record which vertices are to be glued together. This motivates the following definition: we call *tree with g triples* a pair $(\mathbf{t}, (c_1, \dots, c_g))$, where:

- ◇ $\mathbf{t}$ is a (rooted) plane tree;
- ◇ for $1 \le i \le g$, $c_i = \{v_i^1, v_i^2, v_i^3\} \subseteq V(\mathbf{t})$ is a set of three vertices of $\mathbf{t}$;
- ◇ the vertices v_i^j, $1 \le i \le g$, $1 \le j \le 3$, are pairwise distinct;
- ◇ the vertices of the tree

$$\bigcup_{i,i',j,j'} [[v_i^j, v_{i'}^{j'}]]$$

have degree at most 3, and the v_i^j's have degree exactly 1 in that tree. (As in the case of g-trees, the set $[[a, b]]$ represents the range of the unique path linking a and b in the tree.)

Let $\mathfrak{t}$ be a g-tree together with an opening sequence $(v_1, \dots, v_g)$. For all $1 \le i \le g$, let us call c_i the triple of vertices obtained from the slicing of v_i, as well as $\mathbf{t} := \mathfrak{t} \mathbin{\backslash\!\backslash} v_g \mathbin{\backslash\!\backslash} \cdots \mathbin{\backslash\!\backslash} v_1$ the plane tree. We define $\Phi(\mathfrak{t}, (v_1, \dots, v_g)) := (\mathbf{t}, (c_1, \dots, c_g))$. Then Φ is a bijection from the set of all dominant g-tree equipped with an opening sequence into the set of all trees with g triples.

Now, when the g-tree is well-labeled, we can do the same slicing operation, and the three vertices we obtain all have the same label. We call *well-labeled tree with g triples* a tree with g triples $(\mathbf{t}, (c_1, \dots, c_g))$ carrying a labeling function $\mathbf{l}: V(\mathbf{t}) \to \mathbb{Z}$ such that:

- ◇ $\mathbf{l}(e^-) = 0$, where e is the root of $\mathbf{t}$;
- ◇ for every pair of neighboring vertices $v \sim v'$, we have $\mathbf{l}(v) - \mathbf{l}(v') \in \{-1, 0, 1\}$;
- ◇ for all $1 \le i \le g$, we have $\mathbf{l}(v_i^1) = \mathbf{l}(v_i^2) = \mathbf{l}(v_i^3)$.

We call $\mathcal{W}_n$ the set of all well-labeled trees with g triples having n edges. The bijection Φ then extends to a bijection between dominant well-labeled g-trees equipped with an opening sequence and well-labeled trees with g triples.

6.2. *Contour pair of an opened g-tree.* The contour pair of an opened g-tree can be obtained from the contour pair of the g-tree itself (and vice versa). The labeling function is basically the same, but read in a different order. The contour function is slightly harder to recover, because half of the forests are to be read with the floor directed "upward" instead of "downward." Because we will deal at the same time with g-trees and plane trees in this section, we will use a Gothic font for objects related to g-trees, and a boldface font for objects related to plane trees. In the following, we use the notation of Section 2.2.

Let $(\mathfrak{t}, \mathfrak{l})$ be a well-labeled dominant g-tree with scheme $\mathfrak{s}$ and $(\mathbf{t}, \mathbf{l})$ be one of the $2^g g!$ corresponding opened well-labeled trees. The intertwined nodes of the g-tree correspond to intertwined nodes of its scheme, so that the opening sequence used to open $(\mathfrak{t}, \mathfrak{l})$ into $(\mathbf{t}, \mathbf{l})$ naturally corresponds to an opening sequence of $\mathfrak{s}$. Let $\mathbf{s}$ be the tree obtained by opening $\mathfrak{s}$ along this opening sequence. We identify the half-edges of $\mathfrak{s}$ with the half-edges of $\mathbf{s}$, and arrange them according to the facial order of $\mathbf{s}$: $\mathbf{e}_1 = \mathfrak{e}_*, \mathbf{e}_2, \dots, \mathbf{e}_{2(6g-3)}$. (Beware that this is not the usual arrangement according to the facial order of $\mathfrak{s}$.) Now, the plane tree $\mathbf{t}$ is obtained by replacing every half-edge $\mathbf{e}$ of $\mathbf{s}$ with the corresponding forest $\mathfrak{f}^{\mathbf{e}}$ of Proposition 4, as in Section 2.2.2.

We call $(C^{\mathfrak{e}}, L^{\mathfrak{e}})$ the contour pair of $(\mathfrak{f}^{\mathfrak{e}}, \mathfrak{l}^{\mathfrak{e}})$, we let $\mathfrak{C}^{\mathfrak{e}} := C^{\mathfrak{e}} - \sigma^{\mathfrak{e}}$ and we define $\mathfrak{L}^{\mathfrak{e}}$ by (4). For any edge $\{\mathbf{e}_i, \mathbf{e}_j\} \neq \{\mathfrak{e}_*, \bar{\mathfrak{e}}_*\}$ with $i < j$, we will visit the forest $\mathfrak{f}^{\mathbf{e}_i}$ while "going up" and the forest $\mathfrak{f}^{\mathbf{e}_j}$ while "coming down" when we follow the contour of $\mathbf{t}$. Precisely, we define

$$\mathbf{C}^{\mathbf{e}_i} := \mathfrak{C}^{\mathbf{e}_i} - 2\underline{\mathfrak{C}}^{\mathbf{e}_i} \quad \text{and} \quad \mathbf{C}^{\mathbf{e}_j} := \mathfrak{C}^{\mathbf{e}_j}. \tag{15}$$

The first function is the concatenation of the contour functions of the trees in $\mathfrak{f}^{\mathbf{e}_i}$ with an extra "up step" between every consecutive trees. The second one is the concatenation of the contour functions of the trees in $\mathfrak{f}^{\mathbf{e}_j}$ with an extra "down step" between every consecutive trees. It is merely the contour function of $\mathfrak{f}^{\mathbf{e}_j}$ shifted in order to start at 0. What happens to the forests $\mathfrak{f}^{\mathfrak{e}_*}$ and $\mathfrak{f}^{\bar{\mathfrak{e}}_*}$ is a little more intricate. Let us first call (see Figure 14)

$$\begin{aligned} x &:= \inf\{s : \mathfrak{C}^{\mathfrak{e}_*}(s) = \underline{\mathfrak{C}}^{\mathfrak{e}_*}(u)\}, \\ y &:= \inf\{s : \mathfrak{C}^{\bar{\mathfrak{e}}_*}(s) = -\sigma^{\mathfrak{e}_*} - \underline{\mathfrak{C}}^{\mathfrak{e}_*}(u)\}. \end{aligned} \tag{16}$$

When visiting the forest $\mathfrak{f}^{\bar{\mathfrak{e}}_*}$, the floor is directed downward up to time y and then upward:

$$\mathbf{C}^{\bar{\mathfrak{e}}_*} := (\mathfrak{C}^{\bar{\mathfrak{e}}_*}(s))_{0 \le s \le y} \bullet \Big(\mathfrak{C}^{\bar{\mathfrak{e}}_*}(y+s) - 2 \inf_{[y,y+s]} \mathfrak{C}^{\bar{\mathfrak{e}}_*} + \mathfrak{C}^{\bar{\mathfrak{e}}_*}(y)\Big)_{0 \le s \le m^{\bar{\mathfrak{e}}_*} - y}. \tag{17}$$

Finally, the forest $\mathfrak{f}^{\mathfrak{e}_*}$ is visited twice. The first time (when beginning the contour), it is visited between times u and $m^{\mathfrak{e}_*}$, and the floor is directed upward:

$$\mathbf{C}^{\mathfrak{e}_*,1} := \Big(\mathfrak{C}^{\mathfrak{e}_*}(u+s) - 2 \inf_{[u,u+s]} \mathfrak{C}^{\mathfrak{e}_*} + \mathfrak{C}^{\mathfrak{e}_*}(u)\Big)_{0 \le s \le m^{\mathfrak{e}_*} - u}. \tag{18}$$

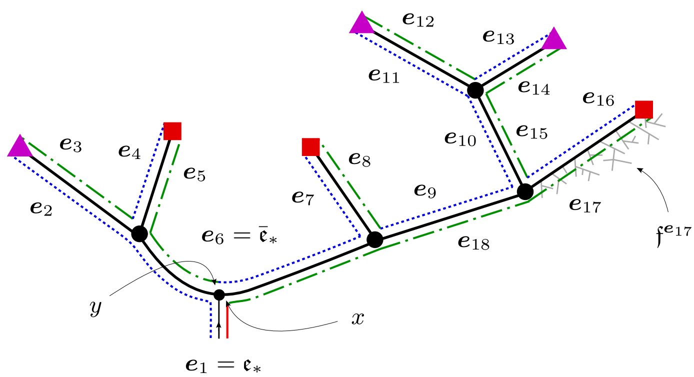

FIG. 14. *Opening of a 2-tree. The squares form one triple and the triangles the other one. The (blue) short dashes correspond to the upward-directed floors and the (green) long dashes to the downward-directed floors. The (red) solid line on the right of the root corresponds to the part of the tree containing the root that has to be visited at the end. The forest $\mathfrak{f}^{\mathbf{e}_{17}}$ is also represented on this figure.*

The second time (when finishing the contour), we visit it between times 0 and x with the floor directed downward, then we visit a part of the tree containing the root between times x and u:

$$\mathbf{C}^{\mathfrak{e}_*,2} := (\mathfrak{C}^{\mathfrak{e}_*}(s))_{0\le s\le x} \bullet \Big(\mathfrak{C}^{\mathfrak{e}_*}(x+s) - 2\inf_{[x+s,u]}\mathfrak{C}^{\mathfrak{e}_*} + \underline{\mathfrak{C}}^{\mathfrak{e}_*}(u)\Big)_{0\le s\le u-x}. \tag{19}$$

The contour pair of $(\mathbf{t},\mathbf{l})$ is then given by

$$\begin{cases} \mathbf{C} := \mathbf{C}^{\mathbf{e}_1,1} \bullet \mathbf{C}^{\mathbf{e}_2} \bullet \mathbf{C}^{\mathbf{e}_3} \bullet \cdots \bullet \mathbf{C}^{\mathbf{e}_{2(6g-3)}} \bullet \mathbf{C}^{\mathbf{e}_1,2}, \\ \mathbf{L} := \mathfrak{L}^{\mathbf{e}_1,1} \bullet \mathfrak{L}^{\mathbf{e}_2} \bullet \mathfrak{L}^{\mathbf{e}_3} \bullet \cdots \bullet \mathfrak{L}^{\mathbf{e}_{2(6g-3)}} \bullet \mathfrak{L}^{\mathbf{e}_1,2}, \end{cases} \tag{20}$$

where

$$\mathfrak{L}^{\mathbf{e}_1,1} := \big(\mathfrak{L}^{\mathbf{e}_1}(u+s) - \mathfrak{L}^{\mathbf{e}_1}(u)\big)_{0\le s\le m^{\mathbf{e}_1}-u} \quad \text{and} \quad \mathfrak{L}^{\mathbf{e}_1,2} := (\mathfrak{L}^{\mathbf{e}_1}(s))_{0\le s\le u}.$$

6.3. *Opened uniform well-labeled g-tree.* As in Section 2.3, we let $(\mathfrak{t}_n,\mathfrak{l}_n)$ be uniformly distributed over the set $\mathcal{T}_n$ of well-labeled g-trees with n edges, and, applying Skorokhod's representation theorem, we assume that the convergence of Proposition 5 holds almost surely. Let $(\mathfrak{i}_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables uniformly distributed over $[\![1, 2^g g!]\!]$ and independent of $(\mathfrak{t}_n,\mathfrak{l}_n)_{n\in\mathbb{N}}$. With any dominant scheme $\mathfrak{s}\in\mathfrak{S}_*$ and integer $\mathfrak{i}\in[\![1,2^g g!]\!]$, we associate a deterministic opening sequence. When $(\mathfrak{t}_n,\mathfrak{l}_n)$ is dominant, we may then define $(\mathbf{t}_n,\mathbf{l}_n)$ as the opened tree of $(\mathfrak{t}_n,\mathfrak{l}_n)$ according to the opening sequence determined by the integer $\mathfrak{i}_n$. In this case, we call $(\mathbf{C}_n,\mathbf{L}_n)$ the contour pair of $(\mathbf{t}_n,\mathbf{l}_n)$. When $(\mathfrak{t}_n,\mathfrak{l}_n)$ is not dominant, we simply set $(\mathbf{C}_n,\mathbf{L}_n) = (\mathbf{0}_{2n},\mathbf{0}_{2n})$, where we write $\mathbf{0}_\zeta : x\in[0,\zeta]\mapsto 0$. We also let

$$\mathbf{C}_{(n)} := \left(\frac{\mathbf{C}_n(2nt)}{\sqrt{2n}}\right)_{0\le t\le 1} \quad \text{and} \quad \mathbf{L}_{(n)} := \left(\frac{\mathbf{L}_n(2nt)}{\gamma n^{1/4}}\right)_{0\le t\le 1}$$

be the rescaled versions of $\mathbf{C}_n$ and $\mathbf{L}_n$.

We now work at fixed ω for which Proposition 5 holds, $\mathfrak{s}_\infty \in \mathfrak{S}_*$, and such that for all $\mathfrak{i} \in [\![1, 2^g g!]\!]$, $|\{n \in \mathbb{N} : \mathfrak{i}_n = \mathfrak{i}\}| = \infty$. Note that the set of such ω's is of full probability. For n large enough, $\mathfrak{s}_n = \mathfrak{s}_\infty \in \mathfrak{S}_*$, so that $(\mathbf{t}_n, \mathbf{l}_n)$ is well defined. For all n such that $\mathfrak{s}_n = \mathfrak{s}_\infty$ and $\mathfrak{i}_n = \mathfrak{i}$, we always open the g-tree $(\mathfrak{t}_n, \mathfrak{l}_n)$ according to the same opening sequence, so that the ordering $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{2(6g-3)}$ of the half-edges of $\mathfrak{s}_n$ is always the same. As a result, we obtain that

$$(\mathbf{C}_{(n)}, \mathbf{L}_{(n)}) \xrightarrow[n\to\infty]{n : \mathfrak{i}_n = \mathfrak{i}} (\mathbf{C}^{\mathfrak{i}}_\infty, \mathbf{L}^{\mathfrak{i}}_\infty),$$

where $(\mathbf{C}^{\mathfrak{i}}_\infty, \mathbf{L}^{\mathfrak{i}}_\infty)$ is defined by (15)–(19) and (20) when replacing every occurrence of $\mathfrak{C}^{\mathfrak{e}}$ by $\mathfrak{C}^{\mathfrak{e}}_\infty := C^{\mathfrak{e}}_\infty - \sigma^{\mathfrak{e}}_\infty$ and every occurrence of $\mathfrak{L}^{\mathfrak{e}}$ by $\mathfrak{L}^{\mathfrak{e}}_\infty$. Note that $(\mathbf{C}_{(n)}, \mathbf{L}_{(n)})$ has exactly $2^g g!$ a priori distinct accumulation points, each corresponding to one of the ways of opening the real g-tree $\mathscr{T}_\infty$.

Now, because every $\mathfrak{L}^{\mathfrak{e}}_\infty$ goes from 0 to 0, it is easy to see that for all $\mathfrak{i}$, the points where $\mathfrak{L}_\infty$ reaches its minimum are in one-to-one correspondence with the points where $\mathbf{L}^{\mathfrak{i}}_\infty$ reaches its minimum. Moreover, we can see that if $\mathfrak{C}_\infty$ and $\mathfrak{L}_\infty$ have a common increase point, then at least one of the pairs $(\mathbf{C}^{\mathfrak{i}}_\infty, \mathbf{L}^{\mathfrak{i}}_\infty)$ will also have a common increase point. Indeed, let us suppose that $\mathfrak{C}_\infty$ and $\mathfrak{L}_\infty$ have a common increase point. Then, there exists $\mathfrak{e} \in \vec{E}(\mathfrak{s}_\infty)$ such that $\mathfrak{C}^{\mathfrak{e}}_\infty$ and $\mathfrak{L}^{\mathfrak{e}}_\infty$ have a common increase point $s \in [0, m^{\mathfrak{e}}_\infty]$. We use the following lemma:

LEMMA 23. *Let $f : [0, m] \to \mathbb{R}$ be a function.*

- ◇ *If $s \in [0, m)$ is an increase point of f, then s is an increase point of $f - 2\underline{f}$ as well.*
- ◇ *If $s \in (0, m]$ is an increase point of f, then s is an increase point of $r \mapsto f(r) - 2\inf_{[r,m]} f$.*

We postpone the proof of this lemma and finish our argument. If $s < m^{\mathfrak{e}}_\infty$, then s is a common increase point of $\mathbf{C}^{\mathfrak{e}}_\infty$ and $\mathbf{L}^{\mathfrak{e}}_\infty$ thanks to Lemma 23. When $\mathfrak{e} = \mathfrak{e}_*$, this fact remains true if we define $\mathbf{C}^{\mathfrak{e}}_\infty := \mathbf{C}^{\mathfrak{e},2}_\infty \bullet \mathbf{C}^{\mathfrak{e},1}_\infty$. Note that x is an increase point of $\mathbf{C}^{\mathfrak{e}}_\infty$, even if 0 is not an increase point of the second function defining $\mathbf{C}^{\mathfrak{e},2}_\infty$ in (19). In this case, for all $\mathfrak{i}$, $\mathbf{C}^{\mathfrak{i}}_\infty$ and $\mathbf{L}^{\mathfrak{i}}_\infty$ have a common increase point.

Let us now suppose that $s = m^{\mathfrak{e}}_\infty$, and let us fix $\mathfrak{i} \in [\![1, 2^g g!]\!]$. We consider the opening corresponding to $\mathfrak{i}$. If $\mathbf{e}_i = \mathfrak{e}$ is visited while coming down in the contour of the opened tree, then we conclude as above. If both $\mathbf{e}_i$ and $\mathbf{e}_{i+1}$ are visited while going up, then 0 will be an increase point of $\mathbf{C}^{\mathbf{e}_{i+1}}_\infty$, so that $\mathbf{C}^{\mathfrak{i}}_\infty$ and $\mathbf{L}^{\mathfrak{i}}_\infty$ will still have a common increase point. In the remaining case where $\mathbf{e}_i$ is visited while going up and $\mathbf{e}_{i+1}$ is visited while coming down (i.e., $\mathbf{e}_{i+1} = \bar{\mathbf{e}_i}$), we cannot conclude that $\mathbf{C}^{\mathfrak{i}}_\infty$ and $\mathbf{L}^{\mathfrak{i}}_\infty$ have a common increase point. This, however, only happens when the node $\mathfrak{e}^+$ belongs to the opening sequence. But when we pick an opening sequence, we can always choose not to pick a given node, because at each stage of the process, we have at least 2 intertwined nodes. This implies that at least one of the opening sequences will not contain $\mathfrak{e}^+$, and the corresponding pair $(\mathbf{C}^{\mathrm{i}}_\infty, \mathbf{L}^{\mathrm{i}}_\infty)$ will have a common increase point.

PROOF OF LEMMA 23. Let $s \in [0, m)$ be an increase point of f. If s is a right-increase point of f, then $f(r) \geq f(s)$ when $s \leq r \leq t$ for some $t > s$. For such r's, $\underline{f}(r) = \underline{f}(s)$, so that $f(r) - 2\underline{f}(r) \geq f(s) - 2\underline{f}(s)$, and s is a right-increase point of $f - 2\underline{f}$.

If s is a left-increase point of f, then $f(r) \geq f(s)$ when $t \leq r \leq s$ for some $t < s$. If $f(s) > \underline{f}(s)$, then, using the fact that $\underline{f}(s) = \underline{f}(r) \wedge \inf_{[r,s]} f$, we obtain that $\underline{f}(r) = \underline{f}(s)$ when $t \leq r \leq s$ and conclude as above that s is a left-increase point of $f - 2\underline{f}$. Finally, if $f(s) = \underline{f}(s)$, then for all $r \geq s$, we have $f(r) - 2\underline{f}(r) = (f(r) - \underline{f}(r)) - \underline{f}(r) \geq 0 - \underline{f}(s) = f(s) - 2\underline{f}(s)$, and because $s < m$, we conclude that s is a right-increase point of $f - 2\underline{f}$.

We obtain the second assertion of the lemma by applying the first one to $m - s$ and the function $x \mapsto f(m - x)$. $\square$

6.4. *Uniform well-labeled tree with g triples.* Conditionally on the event $D_n := \{(\mathbf{C}_n, \mathbf{L}_n) \neq (\mathbf{0}_{2n}, \mathbf{0}_{2n})\}$, the distribution of $(\mathbf{C}_n, \mathbf{L}_n)$ is that of the contour pair of a uniform well-labeled tree with g triples. We use this fact to see that the law of $(\mathbf{C}_{(n)}, \mathbf{L}_{(n)})$ converges weakly toward a law absolutely continuous with respect to the law of $(\mathbb{e}, Z)$. Let (τ_n, λ_n) be uniformly distributed over the set $\mathcal{T}_n^0$ of all well-labeled plane trees with n edges. We call (Γ_n, Λ_n) the contour pair of (τ_n, λ_n) and define as usual the rescaled versions of both functions,

$$\Gamma_{(n)} := \left(\frac{\Gamma_n(2nt)}{\sqrt{2n}} \right)_{0 \leq t \leq 1} \quad \text{and} \quad \Lambda_{(n)} := \left(\frac{\Lambda_n(2nt)}{\gamma n^{1/4}} \right)_{0 \leq t \leq 1}. \tag{21}$$

For all $n \geq 1$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}$, we define

$$X_n(k) := |\{v \in \tau_n : \lambda_n(v) = k\}| \quad \text{and} \quad X_{(n)}(x) := \frac{1}{n} \gamma n^{1/4} X_n(\lfloor \gamma n^{1/4} x \rfloor),$$

respectively, the profile and rescaled profile of (τ_n, λ_n). We let $\mathcal{I}$ be the one-dimensional ISE (random) measure defined by

$$\langle \mathcal{I}, h \rangle := \int_0^1 dt\, h(Z_t)$$

for every nonnegative measurable function h. By [7], Theorem 2.1, it is known that $\mathcal{I}$ a.s. has a continuous density f_{ISE} with compact support. In other words, $\langle \mathcal{I}, h \rangle = \int_{\mathbb{R}} dx\, h(x) f_{\mathrm{ISE}}(x)$ for every nonnegative measurable function h.

PROPOSITION 24. *The triple $(\Gamma_{(n)}, \Lambda_{(n)}, X_{(n)})$ converges weakly toward the triple $(\mathbb{e}, Z, f_{\mathrm{ISE}})$ in the space $\mathcal{C}([0,1], \mathbb{R})^2 \times \mathcal{C}_c(\mathbb{R})$ endowed with the product topology.*

PROOF. It is known that the pair $(\Gamma_{(n)}, \Lambda_{(n)})$ converges weakly to $(\mathbb{e}, Z)$: in [11], Theorem 5, Chassaing and Schaeffer proved this fact with $\lfloor 2nt \rfloor$ instead of $2nt$ in the definition (21). The claim as stated here easily follows by using the uniform continuity of $(\mathbb{e}, Z)$. Using [7], Theorem 3.6, and the fact that f_{ISE} is a.s. uniformly continuous [7], Theorem 2.1, we also obtain that the sequence $X_{(n)}$ converges weakly to f_{ISE}. As a result, the sequences of the laws of the processes $\Gamma_{(n)}$, $\Lambda_{(n)}$ and $X_{(n)}$ are tight. The sequence (ν_n) of the laws of $(\Gamma_{(n)}, \Lambda_{(n)}, X_{(n)})$ is then tight as well, and, by Prokhorov's lemma, the set $\{\nu_n, n \geq 0\}$ is relatively compact. Let ν be an accumulation point of the sequence (ν_n). There exists a subsequence along which $(\Gamma_{(n)}, \Lambda_{(n)}, X_{(n)})$ converges weakly toward a random variable $(\mathbb{e}', Z', f')$ with law ν. Thanks to Skorokhod's theorem, we may and will assume that this convergence holds almost surely along this subsequence. We know that

$$(\mathbb{e}', Z') \overset{(\mathrm{d})}{=} (\mathbb{e}, Z) \quad \text{and} \quad f' \overset{(\mathrm{d})}{=} f_{\mathrm{ISE}}.$$

It remains to see that f' is the density of the occupation measure of Z', that is,

$$\int_0^1 dt\, h(Z'_t) = \int_{\mathbb{R}} dx\, h(x) f'(x) \tag{22}$$

for all h continuous with compact support. First, notice that

$$\begin{aligned} \frac{1}{n} \sum_{k \in \mathbb{Z}} X_n(k) h(\gamma^{-1} n^{-1/4} k) &= \frac{1}{n} \int_{\mathbb{R}} dx\, X_n(\lfloor x \rfloor) h(\gamma^{-1} n^{-1/4} \lfloor x \rfloor) \\ &= \int_{\mathbb{R}} dx\, X_{(n)}(x) h(\gamma^{-1} n^{-1/4} \lfloor \gamma n^{1/4} x \rfloor) \\ &\to \int_{\mathbb{R}} dx\, f'(x) h(x) \end{aligned}$$

by dominated convergence, a.s. as $n \to \infty$ along the subsequence we consider. It is convenient to introduce now the notation $\langle\langle s \rangle\rangle_n$ defined as follows: for $s \in [0, 2n)$, we set

$$\langle\langle s \rangle\rangle_n := \begin{cases} \lceil s \rceil, & \text{if } \Gamma_n(\lceil s \rceil) - \Gamma_n(\lfloor s \rfloor) = 1, \\ \lfloor s \rfloor, & \text{if } \Gamma_n(\lceil s \rceil) - \Gamma_n(\lfloor s \rfloor) = -1. \end{cases}$$

Then, if we denote by $\tau_n(i)$ the ith vertex of the facial sequence of τ_n, and by ρ_n the root of τ_n, we obtain that the time the process $(\tau_n(\langle\langle s \rangle\rangle_n))_{s \in [0, 2n)}$ spends at each vertex $v \in \tau_n \setminus \{\rho_n\}$ is exactly 2. So we have

$$\begin{aligned} &\frac{1}{n} \sum_{k \in \mathbb{Z}} X_n(k) h(\gamma^{-1} n^{-1/4} k) \\ &\quad = \frac{1}{n} \sum_{v \in \tau_n \setminus \{\rho_n\}} h(\gamma^{-1} n^{-1/4} \lambda_n(v)) + \frac{1}{n} h(0) \end{aligned}$$

$$= \frac{1}{2n} \int_0^{2n} ds\, h(\gamma^{-1} n^{-1/4} \Lambda_n(\langle\langle s \rangle\rangle_n)) + \frac{1}{n} h(0)$$
$$= \int_0^1 ds\, h(\gamma^{-1} n^{-1/4} \Lambda_n(\langle\langle 2ns \rangle\rangle_n)) + \frac{1}{n} h(0)$$
$$\to \int_0^1 dt\, h(Z'_t)$$

a.s. along the subsequence considered. We used the fact that

$$\gamma^{-1} n^{-1/4} \Lambda_n(\langle\langle 2ns \rangle\rangle_n) \to Z'_s,$$

which is obtained by using the uniform continuity of Z'.

This proves that $(\mathbb{e}', Z', f')$ has the same law as $(\mathbb{e}, Z, f_{\mathrm{ISE}})$. Thus the only accumulation point ν of the sequence (ν_n) is the the law of the process $(\mathbb{e}, Z, f_{\mathrm{ISE}})$. By relative compactness of the set $\{\nu_n, n \geq 0\}$, we obtain the weak convergence of the sequence (ν_n) toward ν. $\square$

We define

$$W := \frac{(\int f_{\mathrm{ISE}}^3)^g}{\mathbb{E}[(\int f_{\mathrm{ISE}}^3)^g]}.$$

This quantity is well defined [9], Lemma 10. We also define the law of the pair $(\mathbf{C}_\infty, \mathbf{L}_\infty)$ by the following formula: for every bounded Borel function φ on $\mathcal{C}([0,1], \mathbb{R})^2$,

$$\mathbb{E}[\varphi(\mathbf{C}_\infty, \mathbf{L}_\infty)] = \mathbb{E}[W \varphi(\mathbb{e}, Z)]. \tag{23}$$

PROPOSITION 25. *The pair $(\mathbf{C}_{(n)}, \mathbf{L}_{(n)})$ converges weakly toward the pair $(\mathbf{C}_\infty, \mathbf{L}_\infty)$ in the space $(\mathcal{C}([0,1], \mathbb{R})^2, \|\cdot\|_\infty)$ of pair of continuous real-valued functions on* $[0, 1]$ *endowed with the uniform topology.*

PROOF. Let f be a bounded continuous function on $\mathcal{C}([0,1], \mathbb{R})^2$. We have

$$\mathbb{E}\big[f\big(\mathbf{C}_{(n)}, \mathbf{L}_{(n)}\big)\big] = \mathbb{P}(D_n) \sum_{\substack{(\tau,\lambda) \in \mathcal{T}_n^0 \\ (\tau,\lambda) \leftrightarrow (\mathbf{C},\mathbf{L})}} f(\mathbf{C}, \mathbf{L}) \mathbb{P}\big((\tau_n, \lambda_n) = (\tau, \lambda) | D_n\big)$$
$$+ \mathbb{P}(\overline{D}_n) f(\mathbf{0}_{2n}, \mathbf{0}_{2n}),$$

where we used the notation $(\tau, \lambda) \leftrightarrow (\mathbf{C}, \mathbf{L})$ to mean that the well-labeled tree (τ, λ) is coded by the contour pair $(\mathbf{C}, \mathbf{L})$. It was shown in [9], Lemma 8, that the number of well-labeled trees with g triples having n edges is equivalent to the number of well-labeled plane trees having n edges, together with g triples of vertices (not necessarily distinct and not arranged) such that all the vertices of the same triple have the same label. More precisely, we have

$$\mathbb{P}\big((\tau_n, \lambda_n) = (\tau, \lambda) | D_n\big) = \frac{1}{|\mathcal{W}_n|}\bigg(\sum_{k\in\mathbb{Z}} |\{v \in \tau : \lambda(v) = k\}|^3\bigg)^g + O(n^{-1/4}).$$

And, because f is bounded and $\mathbb{P}(D_n) \to 1$, we obtain that

$$\mathbb{E}\big[f\big(\mathbf{C}_{(n)}, \mathbf{L}_{(n)}\big)\big] \sim \frac{|\mathcal{T}_n^0|}{|\mathcal{W}_n|}\mathbb{E}\bigg[\bigg(\sum_{k\in\mathbb{Z}} X_n(k)^3\bigg)^g f\big(\Gamma_{(n)}, \Lambda_{(n)}\big)\bigg].$$

Using the asymptotic formulas $|\mathcal{T}_n^0| \sim \sqrt{\pi}\,12^n n^{-3/2}$, as well as $|\mathcal{W}_n| \sim c_g 12^n \times n^{(5g-3)/2}$ for some positive constant c_g only depending on g ([9], Lemma 8), as well as the computation

$$n^{-5/2}\sum_{k\in\mathbb{Z}} X_n(k)^3 = n^{-5/2}\int_{\mathbb{R}} dx\, X_n(\lfloor x\rfloor)^3 = \gamma^{-2}\int_{\mathbb{R}} dx\, X_{(n)}(x)^3,$$

we see that there exists a positive constant c such that

$$\mathbb{E}\big[f\big(\mathbf{C}_{(n)}, \mathbf{L}_{(n)}\big)\big] \sim c\mathbb{E}\bigg[\bigg(\int_{\mathbb{R}} dx\, X_{(n)}(x)^3\bigg)^g f\big(\Gamma_{(n)}, \Lambda_{(n)}\big)\bigg].$$

Now, let $\varepsilon > 0$. Thanks to [9], Lemma 10, we see that both quantities $\mathbb{E}[(\int f_{\mathrm{ISE}}^3)^g]$ and $\sup_n \mathbb{E}[(\int X_{(n)}^3)^{g+1}]$ are finite. Then, using the fact that

$$\mathbb{E}\bigg[\bigg(\int X_{(n)}^3\bigg)^g \mathbb{1}_{\{\int X_{(n)}^3 > L\}}\bigg] \le \frac{1}{L}\mathbb{E}\bigg[\bigg(\int X_{(n)}^3\bigg)^{g+1}\bigg],$$

we obtain that, for L sufficiently large,

$$\sup_n \mathbb{E}\bigg[\bigg(\int_{\mathbb{R}} dx\, X_{(n)}(x)^3\bigg)^g f\big(\Gamma_{(n)}, \Lambda_{(n)}\big)\mathbb{1}_{\{\int X_{(n)}^3 > L\}}\bigg] < \varepsilon$$

and

$$\mathbb{E}\bigg[\bigg(\int f_{\mathrm{ISE}}^3\bigg)^g f(\mathbb{e}, Z)\mathbb{1}_{\{\int f_{\mathrm{ISE}}^3 > L\}}\bigg] < \varepsilon.$$

Thanks to the Proposition 24, for n sufficiently large,

$$\bigg|\mathbb{E}\bigg[\bigg(\int_{\mathbb{R}} dx X_{(n)}(x)^3\bigg)^g f\big(\Gamma_{(n)}, \Lambda_{(n)}\big)\mathbb{1}_{\{\int X_{(n)}^3 \le L\}}\bigg] - \mathbb{E}\bigg[\bigg(\int f_{\mathrm{ISE}}^3\bigg)^g f(\mathbb{e}, Z)\mathbb{1}_{\{\int f_{\mathrm{ISE}}^3 \le L\}}\bigg]\bigg| < \varepsilon.$$

This yields the existence of a constant C such that

$$\mathbb{E}\big[f\big(\mathbf{C}_{(n)}, \mathbf{L}_{(n)}\big)\big] \underset{n\to\infty}{\longrightarrow} C\mathbb{E}\bigg[\bigg(\int f_{\mathrm{ISE}}^3\bigg)^g f(\mathbb{e}, Z)\bigg],$$

and we compute the value of C by taking $f \equiv 1$. $\square$

Thanks to (23), we see that the properties that hold almost surely for the pair $(\mathbb{e}, Z)$ also hold almost surely for $(\mathbf{C}_\infty, \mathbf{L}_\infty)$. We may now conclude thanks to [20], Lemma 3.1, that

$$\begin{aligned}
&\mathbb{P}(\exists s \neq t : \mathfrak{L}_\infty(s) = \mathfrak{L}_\infty(t) = \min \mathfrak{L}_\infty) \\
&\quad \leq \frac{1}{2^g g!} \sum_{\mathfrak{i}=1}^{2^g g!} \mathbb{P}(\exists s \neq t : \mathbf{L}_\infty^{\mathfrak{i}}(s) = \mathbf{L}_\infty^{\mathfrak{i}}(t) = \min \mathbf{L}_\infty^{\mathfrak{i}}) \\
&\quad = \mathbb{P}(\exists s \neq t : \mathbf{L}_\infty(s) = \mathbf{L}_\infty(t) = \min \mathbf{L}_\infty) = 0,
\end{aligned}$$

and, by [20], Lemma 3.2,

$$\begin{aligned}
\mathbb{P}(\mathrm{IP}(\mathfrak{C}_\infty) \cap \mathrm{IP}(\mathfrak{L}_\infty) \neq \varnothing) &\leq \sum_{\mathfrak{i}=1}^{2^g g!} \mathbb{P}(\mathrm{IP}(\mathbf{C}_\infty^{\mathfrak{i}}) \cap \mathrm{IP}(\mathbf{L}_\infty^{\mathfrak{i}}) \neq \varnothing) \\
&= 2^g g! \mathbb{P}(\mathrm{IP}(\mathbf{C}_\infty) \cap \mathrm{IP}(\mathbf{L}_\infty) \neq \varnothing) \\
&= 0.
\end{aligned}$$

This concludes the proof of Lemmas 9 and 10.

6.5. *Remaining proofs*.

6.5.1. *Proof of Lemma* 12. Chapuy's bijection may naturally be transposed in the continuous setting. Let $\mathfrak{i} \in [\![1, 2^g g!]\!]$ be an integer corresponding to an opening sequence, and $\mathbf{T}_\infty^{\mathfrak{i}}$ the real tree coded by $\mathbf{C}_\infty^{\mathfrak{i}}$. The interval $[0, 1]$ may be split into $2g + 1$ intervals coding the two halves of $\mathfrak{f}_\infty^{\mathfrak{e}_*}$ and the other forests of $\mathscr{T}_\infty$. Through the continuous analog of Chapuy's bijection, these intervals are reordered into an order corresponding to the opening sequence. We call $\varphi^{\mathfrak{i}} : [0, 1] \to [0, 1]$ the bijection accounting for this reordering. It is a cadlag function with derivative 1 satisfying $\mathfrak{L}_\infty(s) = \mathbf{L}_\infty^{\mathfrak{i}}(\varphi^{\mathfrak{i}}(s))$ for all $s \in [0, 1]$.

In order to see that Lemma 12 is a consequence of [18], Lemma 2.4, let us first see what happens to subtrees of $\mathscr{T}_\infty$ through the continuous analog of Chapuy's bijection. It is natural to call root of $\mathscr{T}_\infty$ the point $\partial := \mathscr{T}_\infty(u_\infty)$, where the real number u_∞ was defined in Proposition 5 as the limit of the integer coding the root in $\mathfrak{t}_n$, properly rescaled. Using classical properties of the Brownian motion together with Proposition 5, it is easy to see that, almost surely, ∂ is a leaf of $\mathscr{T}_\infty$, so that τ_∂ is well defined. Any subtree of $\mathscr{T}_\infty$ not included in τ_∂ (these subtrees require extra care, we will treat them separately) is transformed through Chapuy's bijection into some subtree of the opened tree $\mathbf{T}_\infty^{\mathfrak{i}}$ (i.e., into some tree to the left or right of some branch of $\mathbf{T}_\infty^{\mathfrak{i}}$). This is easy to see when the subtree is not rooted at a node of $\mathscr{T}_\infty$, and we saw at the end of Section 3.1 that, almost surely, all the subtrees are rooted outside the set of nodes of $\mathscr{T}_\infty$.

We reason by contradiction to rule out these subtrees. We call $\mathscr{L}$ the Lebesgue measure on [0, 1]. Let us suppose that there exist $\eta > 0$, and some subtree τ, coded by $[l, r]$, not included in τ_∂, such that $\inf_{[l,r]} \mathfrak{L}_\infty < \mathfrak{L}_\infty(l) - \eta$, and

$$\liminf_{\varepsilon\to 0} \varepsilon^{-2}\mathscr{L}\biggl(\biggl\{s \in [l, r] : \mathfrak{L}_\infty(s) < \mathfrak{L}_\infty(l) - \eta + \varepsilon; \forall x \in [\mathfrak{C}_\infty(l), \mathfrak{C}_\infty(s)], \mathfrak{L}_\infty\bigl(\sup\{t \le s : \mathfrak{C}_\infty(t) = x\}\bigr) > \mathfrak{L}_\infty(l) - \eta + \frac{\varepsilon}{8}\biggr\}\biggr) = 0. \tag{24}$$

Note that, by definition of $\mathbf{C}^{\mathrm{i}}_\infty$, the function $s \mapsto \mathfrak{C}_\infty(s) - \mathbf{C}^{\mathrm{i}}_\infty(\varphi^{\mathrm{i}}(s))$ is constant on $[l, r]$. Let us call $l' := \varphi^{\mathrm{i}}(l)$ and $r' := \varphi^{\mathrm{i}}(r)$. It is easy to see that (24) remains true when replacing, respectively, l, r, $\mathfrak{C}_\infty$ and $\mathfrak{L}_\infty$ with l', r', $\mathbf{C}^{\mathrm{i}}_\infty$ and $\mathbf{L}^{\mathrm{i}}_\infty$. Thanks to Proposition 25, the conclusion of [18], Lemma 2.4, is also true for the opened tree $\mathbf{T}^{\mathrm{i}}_\infty$, and the fact that $[l', r']$ codes a subtree of the opened tree yields a contradiction.

We then use a re-rooting argument to conclude. With positive probability, τ_∂ is no longer the tree containing the root in the uniformly re-rooted g-tree. Let us suppose that, with positive probability, there exists a subtree of $\mathscr{T}_\infty$ included in τ_∂, satisfying the hypotheses but not the conclusion of Lemma 12. Then, with positive probability, there will exist a subtree not included in the tree containing the root of the uniformly re-rooted g-tree, satisfying the hypotheses but not the conclusion of Lemma 12. The fact that the uniformly re-rooted g-tree has the same law as $\mathscr{T}_\infty$ yields a contradiction.

6.5.2. *Proof of Lemma* 13. Using the same arguments as in [19], we can see that Lemma 13 is a consequence of the following lemma (see [19], Corollary 6.2):

LEMMA 26. *For every* $p \ge 1$ *and every* $\delta \in (0, 1]$, *there exists a constant* $c_{p,\delta} < \infty$ *such that, for every* $\varepsilon > 0$,

$$\mathbb{E}\biggl[\biggl(\int_0^1 \mathbb{1}_{\{\mathfrak{L}_\infty(s) \le \min \mathfrak{L}_\infty + \varepsilon\}}\, ds\biggr)^p\biggr] \le c_{p,\delta}\,\varepsilon^{4p-\delta}.$$

PROOF. This readily comes from [19], Lemma 6.1, stating that for every $p \ge 1$ and every $\delta \in (0, 1]$, there exists a constant $c'_{p,\delta} < \infty$ such that, for every $\varepsilon > 0$,

$$\mathbb{E}\biggl[\biggl(\int_0^1 \mathbb{1}_{\{Z_s \le \min Z + \varepsilon\}}\, ds\biggr)^p\biggr] \le c'_{p,\delta}\,\varepsilon^{4p-\delta}.$$

Obviously, this still holds for $\delta \in (1, 2]$. Using the link between $\mathfrak{L}_\infty$ and $\mathbf{L}_\infty$, as well as Proposition 25, we see that, for $p \geq 1$ and $\delta \in (0, 1]$,

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^1 \mathbb{1}_{\{\mathfrak{L}_\infty(s)\leq \min \mathfrak{L}_\infty+\varepsilon\}}\,ds\right)^p\right] &= \mathbb{E}\left[\left(\int_0^1 \mathbb{1}_{\{\mathbf{L}_\infty(s)\leq \min \mathbf{L}_\infty+\varepsilon\}}\,ds\right)^p\right]\\ &= \mathbb{E}\left[W\left(\int_0^1 \mathbb{1}_{\{Z_s\leq \min Z+\varepsilon\}}\,ds\right)^p\right]\\ &\leq (\mathbb{E}[W^2]c'_{2p,2\delta})^{1/2}\varepsilon^{4p-\delta}\\ &= c_{p,\delta}\varepsilon^{4p-\delta},\end{aligned}$$

where $c_{p,\delta} := (\mathbb{E}[W^2]c'_{2p,2\delta})^{1/2} < \infty$, by [9], Lemma 10. $\square$

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