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DIFFUSION PROCESSES IN THIN TUBES AND THEIR LIMITS ON GRAPHS

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The present paper is concerned with diffusion processes running on tubular domains with conditions on nonreaching the boundary, respectively, reflecting at the boundary, and corresponding processes in the limit where the thin tubular domains are shrinking to graphs. The methods we use are probabilistic ones. For shrinking, we use big potentials, respectively, reflection on the boundary of tubes. We show that there exists a unique limit process, and we characterize the limit process by a second-order differential generator acting on functions defined on the limit graph, with Kirchhoff boundary conditions at the vertices.

1. Introduction. The present paper is concerned with diffusion processes running on tubular domains with Dirichlet (i.e., absorbing-like) (resp., Neumann, i.e., reflecting) boundary conditions, and the respective processes obtained in the limit where the thin tubular domains shrink to graphs. Problems of this type have been intensively studied before in the case of Neumann boundary conditions, both by probabilistic tools [21, 22] and analytic tools [2, 8–10, 12, 13, 15, 38, 41]. The case of Dirichlet boundary conditions was known to present special difficulties, which explains why there have been, up to now, fewer works concerned with this case, and, in fact, these are only concerned with either special graphs or special shrinking procedures, leading mainly (with the exception of [2, 9, 10, 12]) to limiting processes which "decouple at vertices" [7, 11, 15].

Before explaining these difficulties and entering into details let us motivate the reasons to undertake such studies, pointing out also some connections with other problems and giving some historical remarks.

In many problems of analysis and probability one encounters differential operators defined on structures which have small dimensions in one or more directions. Let us mention as examples the modeling of fluid motion in narrow tubes, or in nearly two-dimensional domains (see, e.g., [42]), the propagation of electric signals along nearly one-dimensional neurons (see, e.g., [3, 7, 11]), the propagation of

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electromagnetic waves in wave guides [31], the propagation of quantum mechanical effects in thin wires (in the context of nanotechnology); see, for example, [2, 9, 10, 12, 13, 15, 17, 24, 32, 33, 35, 41, 48]. Such geometrical structures tend in a certain limit (mathematically well described in general through a Gromov topology) to a graph. Modeling dynamical systems or processes on such structures by corresponding ones on a graph might present certain advantages (e.g., PDEs becoming ODEs on graphs; more dimensional spectral problems reduced to one-dimensional ones). In any case the study of dynamics and processes on graphs can be considered as an idealization or a "first approximation" for the study of the corresponding objects in more realistic situations.

There is a rich literature on differential operators on graphs. Diffusion operators and evolution equations were considered originally in work by Lumer [37], and subsequently by many authors; see, for example, [5, 40, 49, 50]. Elliptic and parabolic nonlinear equations on graphs have been discussed, for example, in relations to applications in biology, for example, in [11]; see also, for example, [3, 7] for nonlinear diffusions on graphs in connection with neurobiology. Heat kernels on graphs have been studied in particular in [39]. Hyperbolic nonlinear equations on graphs have been studied, for example, in [31].

In quantum mechanics, Schrödinger equations on graphs are considered as models of nanostructures; see, for example, [6, 17, 32, 33]. Work has been particularly intense in the study of spectral properties of Schödinger-type operators on graphs; see, for example, [24, 32, 33, 35]. Such models of quantum mechanics on graphs also play an important role in the study of the relation between classical chaos and quantum chaos; see, for example, [16, 24, 35, 43, 44].

For the study of the limit of differential operators on thin domains of $\mathbb{R}^n$ (and corresponding PDEs) degenerating into geometric graphs (and corresponding ODEs) we refer to [30, 42, 50] and especially to the surveys by Raugel [42] (which discuss topics like spectral properties, asymptotics and attractors). For the study of parabolic equations and associated semi-groups and diffusion processes we also refer to [42]. Corresponding hyperbolic problems in connection with the modeling of ferroelectric materials have been discussed, for example, in [1].

Probabilistic methods for the study of processes on thin domains of $\mathbb{R}^n$ have been developed by Freidlin and Wentzell in the case of Neumann boundary conditions. They exploit the consideration of slow, respectively fast, components going back to [20], applied to the thin tubes problem [21]. In these studies the basic probabilistic observation is that for a Brownian motion in a thin tube along a line, the component in the transverse direction is fast, and the one in the longitudinal direction is slow. The control in the limit exploits the assumption on the reflecting properties of the fast component, together with a projection technique onto the longitudinal direction. In [21] it is shown that the diffusion coefficient for this limit process is obtained by averaging the diffusion coefficient for the process in tubular domains with respect to the invariant measure of the fast component with suitable changed space and time scales.

Analytically the Laplacian in the transverse direction has a constant eigenvalue 0 (ground state in the transverse direction), which then yields a natural identification of the subspace of L^2—over the thin tube corresponding to the eigenvalue 0 for the Laplacian in the transverse direction with the L^2—space along an edge. Results about this approximation concern convergence of eigenvalues, eigenfunctions, resolvents and semigroups [13, 15, 25, 38]. Besides, operatorial and variational methods also methods of Dirichlet form theory have been used [8].

The identification stressed above is no longer possible in the case of Dirichlet boundary conditions on the boundary of the thin tube, since the lowest eigenvalue of the Laplacian in the transverse direction diverges like $1/\varepsilon^2$, where $\varepsilon > 0$ is the width of the narrow tube. (For a probabilistic study of the first-order asymptotics of the lowest eigenvalue of the Dirichlet Laplacian in tubular neighborhoods of submanifolds of Riemannian manifolds, see [28].) This has been pointed out clearly and posed as an open problem by Exner (see [4]). In order to nevertheless manage analytically the limit to a graph, one has to perform a renormalization procedure, first introduced in [2], and extended in [9, 10], for the case of a V-graph (waveguide). More general cases with Dirichlet boundary conditions have been managed in the case where the shrinking at vertices is quicker than the one at the edges; however, then one has "no communication between the different edges" (i.e., "decoupling") on the graphs; see [25, 38, 41]. The interest in discussing the case of Dirichlet-boundary conditions is particularly clear in the physics of conductors, where such boundary conditions arise most naturally, both in classical and quantum mechanical problems. However, in the other type of applications we have mentioned there is also an interest in studying boundary conditions that are different from the Neumann ones, since boundary conditions influence the limit behavior, and one is interested to obtain on the graphs the most general possible boundary conditions at the vertices (even in the case of an "N-spider graph" there are N^2-different possible self-adjoint realizations of a Laplacian on the spider; see, for example, [17, 29]).

The present paper mainly discusses the case of shrinking by potentials, and the goal is to determine the limit process on a given graph. This shrinking by potentials corresponds to confining the process in thin tubes around the graph, not reaching the boundary almost surely, and in this sense is related with Dirichlet boundary conditions (the latter property corresponding however to a completely absorbing boundary). In Sections 2 and 3 we consider special cases, because the consideration of these cases illustrate better the methods we use.

In Section 2 the case of a thin tube Ω^ε in $\mathbb{R}^n$ shrinking to a curve γ in $\mathbb{R}^n$ is discussed. The tube Ω^ε has a uniform width $\varepsilon > 0$. In the tube we have a nondegenerate diffusion process X^ε with a drift consisting of two parts, one continuous and bounded, the other of gradient type, pushing away from the boundary, so that the first hitting time of X^ε at the boundary $\partial\Omega^\varepsilon$ is infinite almost surely. We also construct a diffusion process X on γ and show (Theorem 2.2) that if $X^\varepsilon(0)$ converges weakly to $X(0)$, then also X^ε converges weakly to X. If pathwise uniqueness holds both for X^ε and X, then X^ε also converges to X almost surely as $\varepsilon \downarrow 0$. We also state corresponding results for a process in Ω^ε with a reflecting boundary condition on the boundary $\partial\Omega^\varepsilon$ (Theorem 2.3). These results are obtained in a similar way as those obtained by our shrinking with potentials in the first part of Section 2.

In Section 3 we discuss the case of shrinking N thin tubes in $\mathbb{R}^n$ to an N-spider graph in $\mathbb{R}^n$. In this section, we often use the methods discovered by Freidlin and Wentzell [21], extend their method to the case of diffusion processes instead of Brownian motions and apply it to the case of shrinking by potentials. The process X^ε in the domain Ω^ε consisting of N tubes is defined in a similar way as in Section 2, $\varepsilon > 0$ being the parameter of shrinking to the N-spider graph Γ for $\varepsilon \downarrow 0$. We prove again that the first hitting time of X^ε at the boundary $\partial\Omega^\varepsilon$ is infinite and that the laws of $\{X^\varepsilon : \varepsilon > 0\}$ are tight in the topology of probability measures on $C([0, +\infty))$, if their initial distributions are tight. We then show that any limit process is strong Markov and study the transition probabilities from the vertex O to any edge of the spider graph Γ. This requires quite detailed estimates of the behavior of the process X^ε in a neighborhood of O in Ω^ε. These results imply that the boundary condition at O should be a weighted Kirchhoff boundary condition for the functions in the domain of the generator of the limit processes X. (This is one of the types of boundary conditions known from the general discussions on boundary conditions for processes on graphs; see, for example, [12, 17, 29, 32–34].) The weights are determined explicitly from the construction, as transition probabilities to the edges (Lemma 3.7). This is crucial to determine the generator of the unique limit process X (Theorem 3.8). Similar considerations lead to corresponding results for the case where X^ε is a diffusion in Ω^ε with reflecting boundary conditions on $\partial\Omega^\varepsilon$ (Theorem 3.9).

In Section 4 we state the results in the case of thin tubes around general graphs, which are obtained immediately from the results in Sections 2 and 3. These are systems consisting of thin tubes around finitely ramified graphs in $\mathbb{R}^n$ with edges which consist of C^3-curves. Theorem 4.1 presents a result similar to the one for an N-spider graph, showing, in particular, convergence of the diffusion process X^ε not leaving the system Ω^ε of tubes around the general graph to a diffusion process X on the graph. Again its generator is determined and an extension is given to the case of a diffusion with reflecting boundary conditions on $\partial\Omega^\varepsilon$. Since the latter result is not only for a Brownian motion in the thin tubes, but also for reflecting diffusion processes in the thin tubes, it is also an extension of previous results of Freidlin and Wentzell [21].

All random variables discussed in the present paper are defined on a probability space with probability measure P, and $E[\cdot]$ denotes their expectation with respect to P. For a locally compact topological subspace A of $\mathbb{R}^n$, let $C_0(A) := \{f \in C(A) : \lim_{|x|\to+\infty} f(x) = 0\}$.

2. The case of curves. In this section, we consider shrinking of thin tubes to curves. Let n be an integer larger than or equal to 2. Let $\gamma \in C^3(\mathbb{R};\mathbb{R}^n)$ such that $|\dot{\gamma}| = 1$ [with $\dot{\gamma}$ the derivatives of $t \to \gamma(t)$, and $|\cdot|$ the norm in $\mathbb{R}^n$], and assume that γ has no self-crossing point, and $\ddot{\gamma}$ is a bounded function with a compact support. Let $\varepsilon > 0$, $\langle \cdot, \cdot \rangle$ be the inner product on $\mathbb{R}^n$, and $d(x,\gamma)$ be the distance between x and γ. Note that $d(x,\gamma)$ is Lipschitz continuous in x. Define domains $\{\Omega^\varepsilon\}$ by

$$\Omega^\varepsilon := \{x \in \mathbb{R}^n : d(x,\gamma) < \varepsilon\}.$$

Consider a differentiable function u on $[0,1)$ such that

$$u(0) = 0, \qquad u' \geq 0 \qquad \lim_{R \uparrow 1} u'(R) = +\infty \quad \text{and} \quad -\lim_{R \uparrow 1} \frac{u(R)}{\log(1-R)} = +\infty.$$

For example, if we define $u(r) := r^\alpha/(1-r^\alpha)$ for $r \in [0,1)$ where $\alpha > 0$, then u satisfies the conditions above. Let

$$U^\varepsilon(x) = u(\varepsilon^{-1} d(x,\gamma)), \qquad x \in \Omega^\varepsilon.$$

For $\varepsilon > 0$, consider a diffusion process X^ε given by the following equation:

$$\begin{aligned} X^\varepsilon(t) = X^\varepsilon(0) &+ \int_0^{t\wedge\zeta^\varepsilon} \sigma(X^\varepsilon(s))\,dW(s) + \int_0^{t\wedge\zeta^\varepsilon} b(X^\varepsilon(s))\,ds \\ &- \int_0^{t\wedge\zeta^\varepsilon} (\nabla U^\varepsilon)(X^\varepsilon(s))\,ds, \end{aligned} \tag{2.1}$$

where $X^\varepsilon(0)$ is an Ω^ε-valued random variable, W is an n-dimensional Wiener process, $\sigma \in C_b(\mathbb{R}^n;\mathbb{R}^n \otimes \mathbb{R}^n)$, $b \in C_b(\mathbb{R}^n;\mathbb{R}^n)$ and ζ^ε is the first hitting time of X^ε at the boundary $\partial\Omega^\varepsilon$ of Ω^ε. Let $a := \sigma\sigma^T$ (with σ^T the transpose of σ), and assume that a is a uniformly positive definite matrix. Then, the solution X^ε of (2.1) exists uniquely; see, for example, [47].

LEMMA 2.1. *$\zeta^\varepsilon = +\infty$ almost surely for small $\varepsilon > 0$.*

PROOF. Assume $n \geq 3$. Note that X^ε does not hit γ almost surely in this case. Let X_x^ε be the solution of (2.1) replacing $X^\varepsilon(0)$ and ζ^ε by x and ζ_x^ε, respectively, where ζ_x^ε is the first hitting time of X_x^ε at $\partial\Omega^\varepsilon$. It is sufficient to show that $\zeta_x^\varepsilon = +\infty$ almost surely for x near to $\partial\Omega^\varepsilon$. By the tubular neighborhood theorem and Theorem 1 in [18], there exists a C^2-diffeomorphism $\phi = (\phi_1, \phi_2)$ from $\Omega^\varepsilon \setminus \gamma$ to $\{y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 < |y_2| < \varepsilon\}$ which satisfies, for small ε,

$$\phi_1(x) = \gamma^{-1} \circ \pi(x) \quad \text{and} \quad \phi_2(x) = d(x,\gamma)\nabla d(x,\gamma), \qquad x \in \Omega^\varepsilon \setminus \gamma,$$

where $\pi(x)$ is the nearest point in γ from x. Note that ϕ is a C^2-function on Ω^ε and $\langle \nabla\pi, \nabla U^\varepsilon \rangle = 0$ for small ε. Hence, $\langle \nabla\phi_1, \nabla U^\varepsilon \rangle = 0$ and $\nabla\phi_2 \nabla U^\varepsilon =$

$\varepsilon^{-1}u'(\varepsilon^{-1}d(\cdot,\gamma))\nabla d(\cdot,\gamma)$. By Itô's formula, we have

$$\begin{aligned}\phi_1(X_x^\varepsilon(t)) = \phi_1(x) &+ \int_0^{t\wedge\zeta_x^\varepsilon} \nabla\phi_1(X_x^\varepsilon(s))\sigma(X_x^\varepsilon(s))\,dW(s)\\ &+ \int_0^{t\wedge\zeta_x^\varepsilon} \nabla\phi_1(X_x^\varepsilon(s))b(X_x^\varepsilon(s))\,ds\\ &+ \frac{1}{2}\sum_{i,j=1}^n \int_0^{t\wedge\zeta_x^\varepsilon} a_{ij}(X_x^\varepsilon(s))\,\partial_i\partial_j\phi_1(X_x^\varepsilon(s))\,ds,\end{aligned}\tag{2.2}$$

$$\begin{aligned}\phi_2(X_x^\varepsilon(t)) = \phi_2(x) &+ \int_0^{t\wedge\zeta_x^\varepsilon} \nabla\phi_2(X_x^\varepsilon(s))\sigma(X_x^\varepsilon(s))\,dW(s)\\ &+ \int_0^{t\wedge\zeta_x^\varepsilon} \nabla\phi_2(X_x^\varepsilon(s))b(X_x^\varepsilon(s))\,ds\\ &+ \frac{1}{2}\sum_{i,j=1}^n \int_0^{t\wedge\zeta_x^\varepsilon} a_{ij}(X_x^\varepsilon(s))\,\partial_i\partial_j\phi_2(X_x^\varepsilon(s))\,ds\\ &- \varepsilon^{-1}\int_0^{t\wedge\zeta_x^\varepsilon} u'(\varepsilon^{-1}d(X_x^\varepsilon(s),\gamma))\nabla d(\cdot,\gamma)|_{X_x^\varepsilon(s)}\,ds.\end{aligned}\tag{2.3}$$

Moreover, again by Itô's formula,

$$\begin{aligned}&|\phi_2(X_x^\varepsilon(t))|^2\\ &\quad= |\phi_2(x)|^2 + 2\int_0^{t\wedge\zeta_x^\varepsilon} \langle\phi_2(X_x^\varepsilon(s)), \nabla\phi_2(X_x^\varepsilon(s))\sigma(X_x^\varepsilon(s))\,dW(s)\rangle\\ &\qquad+ 2\int_0^{t\wedge\zeta_x^\varepsilon} \langle\phi_2(X_x^\varepsilon(s)), \nabla\phi_2(X_x^\varepsilon(s))b(X_x^\varepsilon(s))\rangle\,ds\\ &\qquad+ \int_0^{t\wedge\zeta_x^\varepsilon} \left\langle\phi_2(X_x^\varepsilon(s)), \sum_{i,j=1}^n a_{ij}(X_x^\varepsilon(s))\,\partial_i\partial_j\phi_2(X_x^\varepsilon(s))\right\rangle ds\\ &\qquad- 2\varepsilon^{-1}\int_0^{t\wedge\zeta_x^\varepsilon} |\phi_2(X_x^\varepsilon(s))|u'(\varepsilon^{-1}d(X_x^\varepsilon(s),\gamma))\,ds\\ &\qquad+ \int_0^{t\wedge\zeta_x^\varepsilon} \operatorname{trace}[\nabla\phi_2(X_x^\varepsilon(s))\sigma(X_x^\varepsilon(s))(\nabla\phi_2(X_x^\varepsilon(s))\sigma(X_x^\varepsilon(s)))^T]\,ds.\end{aligned}$$

Let

$$\begin{aligned}\bar a &:= \sup\{|(\nabla\phi_2(x)\sigma(x))^T\xi|^2 : x\in\Omega^\varepsilon, \xi\in\{y\in\mathbb{R}^n : |y|=1\}\},\\ \bar b &:= \sup_{x\in\Omega^\varepsilon}\Bigg(2\langle\phi_2(x),\nabla\phi_2(x)b(x)\rangle + \left\langle\phi_2(x), \sum_{i,j=1}^n a_{ij}(x)\,\partial_i\partial_j\phi_2(x)\right\rangle\\ &\qquad\qquad + \operatorname{trace}[\nabla\phi_2(x)\sigma(x)(\nabla\phi_2(x)\sigma(x))^T]\Bigg).\end{aligned}$$

Take $c_0 \in (0,1)$ such that $\sup_{x\in[c_0,1)}(\bar{b}-2xu'(x)) \le 0$ and

$$f(x) := \int_{c_0^2\varepsilon^2}^{x} \exp\biggl(-2\int_{c_0^2\varepsilon^2}^{y} \frac{\bar{b}-2\varepsilon^{-1}\sqrt{z}u'(\varepsilon^{-1}\sqrt{z})}{\bar{a}z}\,dz\biggr)\,dy, \qquad x \in [0,\varepsilon^2).$$

Then, by Itô's formula, for δ such that $0<\delta<1-c_0$ and for x such that $c_0\varepsilon \le d(x,\gamma) \le \varepsilon(1-\delta)$, we have that

$$E\bigl[f\bigl(|\phi_2(X_x^\varepsilon(T^{c_0\varepsilon}\wedge T^{\varepsilon(1-\delta)}))|^2\bigr)\bigr] \le f(d(x,\gamma)^2),$$

where $T^c := \inf\{t>0 : d(X_x^\varepsilon,\gamma)=c\}$ for $c>0$. Since

$$\begin{aligned} &E\bigl[f\bigl(|\phi_2(X_x^\varepsilon(T^{c_0\varepsilon}\wedge T^{\varepsilon(1-\delta)}))|^2\bigr)\bigr]\\ &\qquad = f(c_0^2\varepsilon^2)P\bigl(T^{c_0\varepsilon}<T^{\varepsilon(1-\delta)}\bigr) + f\bigl(\varepsilon^2(1-\delta)^2\bigr)P\bigl(T^{c_0\varepsilon}>T^{\varepsilon(1-\delta)}\bigr) \end{aligned}$$

and

$$P\bigl(T^{c_0\varepsilon}<T^{\varepsilon(1-\delta)}\bigr) + P\bigl(T^{c_0\varepsilon}>T^{\varepsilon(1-\delta)}\bigr) = 1,$$

we have

$$P\bigl(T^{c_0\varepsilon}>T^{\varepsilon(1-\delta)}\bigr) \le \frac{f(d(x,\gamma)^2)-f(c_0^2\varepsilon^2)}{f(\varepsilon^2(1-\delta)^2)-f(c_0^2\varepsilon^2)}.$$

The assumptions on u imply that $f(\varepsilon^2(1-\delta)^2)$ diverges to $+\infty$ as $\delta\to 0$. Hence, the proof is achieved from the fact that $T^{\varepsilon(1-\delta)}$ converges to ζ_x^ε as $\delta\to 0$.

In the case where $n=2$, since X^ε can hit γ, we need a little arrangement. Let Ω_+^ε and Ω_-^ε be the two domains consisting of $\Omega^\varepsilon\setminus\gamma$, and $\theta^\varepsilon(x)$ be 1 if $x\in\Omega_+^\varepsilon$, -1 if $x\in\Omega_-^\varepsilon$ and 0 if $x\in\gamma$. By the tubular neighborhood theorem and Theorem 1 in [18] again, there exists a C^2-diffeomorphism $\phi=(\phi_1,\phi_2)$ from Ω^ε to $\{y=(y_1,y_2)\in\mathbb{R}\times(-\varepsilon,\varepsilon)\}$ which satisfies, for small ε,

$$\phi_1(x)=\gamma^{-1}\circ\pi(x) \quad\text{and}\quad \phi_2(x)=\theta^\varepsilon(x)d(x,\gamma), \qquad x\in\Omega^\varepsilon,$$

such that (2.2) and (2.3) hold. Thus, we can discuss this case in a similar way as the case where $n\ge 3$. $\square$

THEOREM 2.2. *Define a diffusion process X by the solution of the following equation*:

$$\begin{aligned} X(t) = X(0) &+ \int_0^t \dot{\gamma}\circ\gamma^{-1}(X(s))\langle\dot{\gamma}\circ\gamma^{-1}(X(s)),\sigma(X(s))\,dW(s)\rangle\\ &+ \int_0^t \dot{\gamma}\circ\gamma^{-1}(X(s))\langle\dot{\gamma}\circ\gamma^{-1}(X(s)),b(X(s))\rangle\,ds\\ &+ \frac{1}{2}\int_0^t \ddot{\gamma}\circ\gamma^{-1}(X(s))|\sigma(X(s))^T\dot{\gamma}\circ\gamma^{-1}(X(s))|^2\,ds\\ &+ \int_0^t \dot{\gamma}\circ\gamma^{-1}(X(s))\langle\sigma(X(s))^T\ddot{\gamma}\circ\gamma^{-1}(X(s)),\\ &\qquad\qquad \sigma(X(s))^T\dot{\gamma}\circ\gamma^{-1}(X(s))\rangle\,ds. \end{aligned} \tag{2.4}$$

Note that X is uniquely determined as a process on γ.

If $X^\varepsilon(0)$ converges to a γ-valued random variable $X(0)$ weakly, then the process X^ε converges weakly to X in the sense of their laws on $C([0,+\infty);\mathbb{R}^n)$ as $\varepsilon \downarrow 0$.

Moreover, if pathwise uniqueness holds for (2.4) and (2.1) for all $\varepsilon > 0$, and $X^\varepsilon(0)$ converges to a γ-valued random variable $X(0)$ almost surely, then X^ε converges to X almost surely, as $\varepsilon \downarrow 0$.

PROOF. Note that equation (2.2) holds even if we replace X_x^ε, x and ζ_x^ε by X^ε, $X^\varepsilon(0)$ and ζ^ε, respectively. Lemma 2.1 implies

$$\sup_{t\in[0,+\infty)} d(X^\varepsilon(t),\gamma) \to 0, \qquad \varepsilon \downarrow 0, \tag{2.5}$$

almost surely. Hence, the boundedness of the coefficients implies the tightness of the process $\phi_1(X^\varepsilon)$. Let X be any limit process of subsequence of X^ε. Then, we have $X \in C([0,+\infty);\gamma)$ almost surely by (2.5). Hence, taking $\varepsilon \downarrow 0$ in (2.2) with replacing X_x^ε, x and ζ_x^ε by X^ε, $X^\varepsilon(0)$ and ζ^ε, respectively,

$$\begin{aligned}\phi_1(X(t)) = \phi_1(X(0)) &+ \int_0^t \nabla\phi_1(X(s))\sigma(X(s))\,d\tilde{W}(s)\\ &+ \int_0^t \nabla\phi_1(X(s))b(X(s))\,ds\\ &+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t a_{ij}(X(s))\,\partial_i\partial_j\phi_1(X(s))\,ds,\end{aligned}$$

where $\tilde{W}$ is an Wiener process.

Noting that $\phi_1(X(\cdot))$ is a stochastic process on $\mathbb{R}$ and $|\nabla\phi_1(x)\sigma(x)| > 0$ for $x \in \gamma$, the law of $\phi_1(X(\cdot))$ is uniquely determined by this equation; see Theorem 3.3 of Chapter IV in [27]. Applying Itô's formula to $\gamma(\phi_1(X(t)))$ and noting that $\gamma(\phi_1(X(\cdot))) = X(\cdot)$, $\partial_i\phi_1 = \dot\gamma_i \circ \gamma^{-1}$ on γ for $i = 1,2,\ldots,N$ and $\partial_i\partial_j\phi_1 = (\dot\gamma_i \circ \gamma^{-1})(\ddot\gamma_j \circ \gamma^{-1}) + (\dot\gamma_j \circ \gamma^{-1})(\ddot\gamma_i \circ \gamma^{-1})$ on γ for $i,j = 1,2,\ldots,N$, we have that X satisfies (2.4); therefore, the first assertion holds. The second assertion is obtained in a similar way. □

The argument above is also available in the case where the boundary $\partial\Omega^\varepsilon$ carries a Neumann boundary condition, for the generator of the process, in the following sense. Consider a diffusion process $\widehat{X}^\varepsilon$ which is associated with

$$\frac{1}{2}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \sum_{j=1}^n b_j(x)\frac{\partial}{\partial x_j}$$

in Ω^ε and reflecting on $\partial\Omega^\varepsilon$. Then, $\widehat{X}^\varepsilon$ can be expressed by the following equation:

$$\widehat{X}^\varepsilon(t) = \widehat{X}^\varepsilon(0) + \int_0^t \sigma(\widehat{X}^\varepsilon(s))\,dW(s) + \int_0^t b(\widehat{X}^\varepsilon(s))\,ds + \Phi^\varepsilon(\widehat{X}^\varepsilon)(t), \tag{2.6}$$

where Φ^ε is a singular drift which forces the reflecting boundary condition on $\partial\Omega^\varepsilon$; see [46]. Discussing this case in a similar way as above, we obtain the following theorem.

THEOREM 2.3. *Define a diffusion process $\widehat{X}$ by the solution of the following equation*:

$$\begin{aligned}\widehat{X}(t) = \widehat{X}(0) &+ \int_0^t \dot{\gamma}\circ\gamma^{-1}(\widehat{X}(s))\langle\dot{\gamma}\circ\gamma^{-1}(\widehat{X}(s)), \sigma(\widehat{X}(s))\,dW(s)\rangle \\ &+ \int_0^t \dot{\gamma}\circ\gamma^{-1}(\widehat{X}(s))\langle\dot{\gamma}\circ\gamma^{-1}(\widehat{X}(s)), b(\widehat{X}(s))\rangle\,ds \\ &+ \frac{1}{2}\int_0^t \ddot{\gamma}\circ\gamma^{-1}(\widehat{X}(s))|\sigma(\widehat{X}(s))^T\dot{\gamma}\circ\gamma^{-1}(\widehat{X}(s))|^2\,ds \\ &+ \int_0^t \dot{\gamma}\circ\gamma^{-1}(\widehat{X}(s))\langle\sigma(\widehat{X}(s))^T\ddot{\gamma}\circ\gamma^{-1}(\widehat{X}(s)), \\ &\qquad\qquad \sigma(\widehat{X}(s))^T\dot{\gamma}\circ\gamma^{-1}(\widehat{X}(s))\rangle\,ds.\end{aligned} \tag{2.7}$$

If $\widehat{X}^\varepsilon(0)$ converges to a γ-valued random variable $\widehat{X}(0)$ weakly, then the process $\widehat{X}^\varepsilon$ converges weakly to $\widehat{X}$ in the sense of their laws on $C([0,+\infty);\mathbb{R}^n)$ as $\varepsilon\downarrow 0$. Moreover, if pathwise uniqueness holds for (2.7) *and* (2.6) *for all $\varepsilon > 0$, and $\widehat{X}^\varepsilon(0)$ converges to a γ-valued random variable $\widehat{X}(0)$ almost surely, then $\widehat{X}^\varepsilon$ converges to $\widehat{X}$ almost surely, as $\varepsilon\downarrow 0$.*

REMARK 2.4. In this section, the shape of tubes was taken to be cylindrical and the "confining" potential U^ε has been defined by the scaling of a fixed function U. However, neither the shape of the tubes nor the scaling property are essential. If U^ε is "along γ" (in the sense that the gradient of U^ε is normal to the tangent of γ), the same results hold. In the case where U^ε is not along γ, some effect of U^ε remains in the limit process; see [19, 45].

3. The case of N-spiders. In this section, we consider the shrinking of thin tubes to N-spider graphs. The argument in this section is the main part of this article. Consider an n-dimensional Euclidean space $\mathbb{R}^n$, let $d(\cdot,\cdot)$ be the distance function in $\mathbb{R}^n$ and let O be the origin. Let $\{e_i\}_{i=1}^N$ be N different unit vectors in $\mathbb{R}^n$ and $I_i := \{se_i : s\in[0,\infty)\}$. Consider an N-spider graph Γ defined by $\Gamma := \bigcup_{i=1}^N I_i$. Γ is also called an N-star graph. Let A be the set in $\mathbb{R}^n$ given by

$$A := \bigcup_{i,j:i\neq j}\{x\in\mathbb{R}^n : \langle x, e_i\rangle = \langle x, e_j\rangle\}.$$

For $x\in\mathbb{R}^n\setminus A$, let $\pi(x)$ be the nearest point in Γ from x. Note that $\pi(x)$ is uniquely determined for all $x\in\mathbb{R}^n\setminus A$.

Let u_i be given similarly to u in Section 2 for $i = 1, 2, \dots, N$ (so that u_i determines the potential acting in the thin tube around I_i). Let c_i be a positive number for $i = 1, 2, \dots, N$,

$$\kappa := \max\{2\sqrt{2}c_i/\sqrt{1 - \langle e_i, e_j\rangle} : i, j = 1, 2, \dots, N, i \neq j\}$$

and $\kappa_0 \in (0, \kappa)$. c_i has the interpretation of width of the tube around I_i. Let U be a function on $\mathbb{R}^n$ with values in $[0, \infty]$, and assume

$$U(x) = u_i(c_i^{-1} d(x, \Gamma)), \qquad x \in \{x \in \mathbb{R}^n : \pi(x) \in I_i, d(x, I_i) < c_i, |x| \geq \kappa\},$$

$$U(x) = +\infty, \qquad x \in \{x \in \mathbb{R}^n : \pi(x) \in I_i, d(x, I_i) \geq c_i, |x| \geq \kappa\},$$

$$U(x) < +\infty, \qquad x \in \{x \in \mathbb{R}^n : |x| \leq \kappa_0\},$$

$\Omega := \{x : U(x) < \infty\}$ is a simply connected and unbounded domain, $\partial\Omega$ is a C^2-manifold and $U|_\Omega$ is a C^1-function in Ω. This structure Ω is sometimes called a "fattened" N-spider. In addition, we assume

$$\lim_{m\to\infty} \langle -\nabla U(x_m), \nabla d(x_m, \partial\Omega)\rangle = +\infty \quad \text{and} \quad -\lim_{m\to\infty} \frac{U(x_m)}{\log(d(x_m, \partial\Omega))} = +\infty$$

for any sequence $\{x_m\}$ which converges to a point $x \in \partial\Omega$. Define domains $\{\Omega_i : i = 1, 2, \dots, N\}$ in $\mathbb{R}^n$ by

$$\Omega_i := \{x \in \Omega \setminus A : \pi(x) \in I_i, |x| \geq \kappa\}$$

for $i = 1, 2, \dots, N$. Let $\Omega^\varepsilon := \varepsilon\Omega$, $\Omega_i^\varepsilon := \varepsilon\Omega_i$, and $U^\varepsilon(x) = U(\varepsilon^{-1}x)$ for $x \in \mathbb{R}^n$ for all $\varepsilon > 0$. Note that $U^\varepsilon(x) \in [0, +\infty)$ for $x \in \Omega^\varepsilon$, ∂U^ε is a C^2-manifold, and $U^\varepsilon|_{\Omega^\varepsilon}$ is a C^1-function on Ω^ε. Consider a diffusion process X^ε given by the following equation:

$$\begin{aligned} X^\varepsilon(t) = X^\varepsilon(0) &+ \int_0^{t\wedge\zeta^\varepsilon} \sigma(X^\varepsilon(s))\, dW(s) + \int_0^{t\wedge\zeta^\varepsilon} b(X^\varepsilon(s))\, ds \\ &- \int_0^{t\wedge\zeta^\varepsilon} (\nabla U^\varepsilon)(X^\varepsilon(s))\, ds, \end{aligned} \tag{3.1}$$

where $X^\varepsilon(0)$ is an Ω^ε-valued random variable, ζ^ε is the first hitting time of X^ε at $\partial\Omega^\varepsilon$, W is an n-dimensional Wiener process, $\sigma \in C_b(\mathbb{R}^n; \mathbb{R}^n \otimes \mathbb{R}^n)$ and $b \in C_b(\mathbb{R}^n; \mathbb{R}^n)$. Define a stochastic process X_x^ε by the solution of (3.1) with replacing $X^\varepsilon(0)$ by x, and P_x^ε by the law of X_x^ε on $C([0, \infty); \mathbb{R}^n)$. Let $a(x) := \sigma(x)\sigma^T(x)$, and assume that a is a uniformly positive definite matrix. Define a second-order elliptic differential operator L on Ω^ε by

$$L := \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i};$$

then the generator of X^ε is a closed extension of $(L - \nabla U^\varepsilon \cdot \nabla)$ in $L^2(\Omega^\varepsilon, dx)$ for any $\varepsilon > 0$. Since a is a uniformly positive definite matrix, the process X^ε exists uniquely for all $\varepsilon > 0$.

The following lemma implies that X^ε does not exit from Ω^ε almost surely.

LEMMA 3.1. *$\zeta^\varepsilon = +\infty$ almost surely for all $\varepsilon > 0$.*

PROOF. Locally, the discussion in the proof of Lemma 2.1 is available. Hence, by using the strong Markov property of X^ε, we have the assertion. □

Next we shall study the tightness of $\{X^\varepsilon : \varepsilon > 0\}$.

LEMMA 3.2. *If the laws of $\{X^\varepsilon(0) : \varepsilon > 0\}$ are tight, then the laws of $\{X^\varepsilon : \varepsilon > 0\}$ are also tight in the sense of laws on $C([0,\infty);\mathbb{R}^n)$.*

PROOF. In view of Theorem 2.1 in [21] it is sufficient to show that for any $\rho > 0$ there exists a positive constant C_ρ such that for all $y \in \mathbb{R}^n$ there exists a function f_ρ^y on $\mathbb{R}^n$ which satisfies the following:

(i) $f_\rho^y(y) = 1$, $f_\rho^y(x) = 0$ for $|x-y| \geq \rho$ and $0 \leq f_\rho^y \leq 1$.
(ii) $(f_\rho^y(X^\varepsilon(t)) + C_\rho t; t \geq 0)$ is a submartingale for sufficiently small ε.

Now we choose f_ρ^y and C_ρ satisfying the conditions above. Fix $\rho > 0$, and take $\varepsilon_0 > 0$ such that $\varepsilon_0 < \rho/(16\kappa)$. When $y \in \overline{\Omega^{\varepsilon_0}}$ (where $\overline{\Omega^{\varepsilon_0}}$ denotes the closure of Ω^{ε_0} in $\mathbb{R}^n$) and $|y| > \rho/2$, choose $f_\rho^y \in C^\infty(\mathbb{R}^n)$ such that:

- $f_\rho^y(x) = f_\rho^y(\pi(x))$ for $x \in \Omega^{\varepsilon_0} \setminus A$ and $f_\rho^y(x) = 0$ for $|x-y| \geq \rho/4$;
- $f_\rho^y(y) = 1$, $0 \leq f_\rho^y \leq 1$, $\|\nabla f\|_\infty \leq 8/\rho$ and $\|\nabla^2 f\|_\infty \leq 64/\rho^2$.

Since $f_\rho^y(x) = 0$ for $|x| \leq 2\kappa\varepsilon_0$ and $\nabla\pi(x)\nabla U^\varepsilon(x) = 0$ for $|x| \geq 2\kappa\varepsilon_0$, it follows by Itô's formula that

$$f_\rho^y(X^\varepsilon(t)) - \int_0^t L f_\rho^y(X^\varepsilon(s))\,ds$$

is a martingale for all $\varepsilon < \varepsilon_0$. Hence, choosing C_ρ larger than $(8/\rho + 64/\rho^2) \times (\|\sigma\|_\infty^2/2 + \|b\|_\infty)$, conditions (i) and (ii) are satisfied for $\varepsilon < \varepsilon_0$.

When $y \in \overline{\Omega^{\varepsilon_0}}$ and $|y| \leq \rho/2$, choose $f_\rho^y \in C^\infty(\mathbb{R}^n)$ such that:

- $f_\rho^y(x) = f_\rho^y(\pi(x))$ for $x \in \Omega^{\varepsilon_0} \setminus A$, $f_\rho^y(x) = 1$ for $|x| \leq \rho/4$, and $f_\rho^y(x) = 0$ for $|x-y| \geq \rho$;
- $f_\rho^y(y) = 1$, $0 \leq f_\rho^y \leq 1$, $\|\nabla f\|_\infty \leq 8/\rho$ and $\|\nabla^2 f\|_\infty \leq 64/\rho^2$.

Here, note that $2\kappa\varepsilon \leq \rho/4$ for $\varepsilon < \varepsilon_0$. Similarly to the case where $y \in \overline{\Omega^{\varepsilon_0}}$ and $|y| > \rho/2$, one proves that conditions (i) and (ii) are satisfied for $\varepsilon < \varepsilon_0$ with the same C_ρ as above.

When $y \notin \overline{\Omega^{\varepsilon_0}}$, choose $f_\rho^y \in C^\infty(\mathbb{R}^n)$ such that $f_\rho^y(y) = 1$, $f_\rho^y(x) = 0$ for $x \in \overline{\Omega^{\varepsilon_0}}$, and f_ρ^y satisfies condition (i) above. Since X^ε moves in Ω^ε, $f_\rho^y(X^\varepsilon(t)) = 0$ for all t and $\varepsilon < \varepsilon_0$.

Thus, for all $\rho > 0$, $\{f_\rho^y : y \in \mathbb{R}^n\}$ and C_p are chosen in such a way that conditions (i) and (ii) are satisfied. □

Now, we assume the tightness of $\{X^\varepsilon(0) : \varepsilon > 0\}$. By Lemma 3.2 we can choose a subsequence $\{X^{\varepsilon'} : \varepsilon' > 0\}$ of $\{X^\varepsilon : \varepsilon > 0\}$ such that the laws of its members converge weakly in the sense of laws on $C([0,\infty);\mathbb{R}^n)$. Define X as the limit process of this subsequence, and to simplify the notation denote the subsequence ε' by ε again. From now on we fix X as the limit process of X^ε.

For $w \in C([0,+\infty);\mathbb{R}^n)$, let $\tilde{T}^c(w) := \inf\{t > 0 : |w(t)| = c\}$ and $T^c(w) := \inf\{t > 0 : w(t) \notin A, |\pi(w(t))| = c\}$ for $c > 0$.

Theorem 2.2 determines the behavior of X on $\Gamma \setminus O$. Hence, to characterize X, we need to determine the boundary condition for X at O. Now we give some lemmas. The following lemma implies that the edge which X goes to, starting from O, is independent of the edge which X comes from. Therefore, we obtain in particular that X is a strong Markov process on Γ.

LEMMA 3.3. *Let $\{\delta(\varepsilon) : \varepsilon > 0\}$ be positive numbers satisfying the condition that* $\lim_{\varepsilon \downarrow 0} \varepsilon^{-1}\delta(\varepsilon) = +\infty$. *For* $B \in \mathscr{B}(\mathbb{R}^n)$ [$\mathscr{B}(\mathbb{R}^n)$ *denoting the Borel subsets of* $\mathbb{R}^n$],

$$\sup\{|P_x^\varepsilon(w(T^{\delta(\varepsilon)}) \in B) - P_O^\varepsilon(w(T^{\delta(\varepsilon)}) \in B)| : x \in \Omega^\varepsilon, |x| \le 3\kappa\varepsilon\}$$

converges to 0 *as* $\varepsilon \downarrow 0$.

PROOF. Define a process $\widehat{X}_x^\varepsilon$ by the solution of the equation

$$\begin{aligned}\widehat{X}_x^\varepsilon(t) &= x + \int_0^t \sigma(\varepsilon \widehat{X}_x^\varepsilon(s))\, d\widehat{W}(s) + \varepsilon \int_0^t b(\varepsilon \widehat{X}_x^\varepsilon(s))\, ds \\ &\quad - \int_0^t (\nabla U)(\widehat{X}_x^\varepsilon(s))\, ds\end{aligned} \tag{3.2}$$

for $x \in \Omega$ and $\varepsilon > 0$, where $\widehat{W}$ is an n-dimensional Wiener process defined by $\widehat{W}(t) = \varepsilon^{-1}W(\varepsilon^2 t)$ for $t \in [0,\infty)$. It is easy to see that the law of $(\widehat{X}_x^\varepsilon(t) : t \ge 0)$ is equal to $(\varepsilon^{-1}X_{\varepsilon x}^\varepsilon(\varepsilon^2 t) : t \ge 0)$ for $x \in \Omega$. Letting $\widehat{P}_x^\varepsilon$ be the law of $\widehat{X}_x^\varepsilon$ on $C([0,\infty);\mathbb{R}^n)$, we have

$$\widehat{P}_x^\varepsilon(w(t) \in dx) = P_{\varepsilon x}^\varepsilon(\varepsilon^{-1}w(\varepsilon^2 t) \in dx) \tag{3.3}$$

for $t \in [0,\infty)$, $x \in \Omega$ and $\varepsilon > 0$. By (3.3), it is sufficient to show that

$$|\widehat{P}_x^\varepsilon(w(T^{\delta(\varepsilon)/\varepsilon}) \in \varepsilon^{-1}B) - \widehat{P}_O^\varepsilon(w(T^{\delta(\varepsilon)/\varepsilon}) \in \varepsilon^{-1}B)| \to 0 \tag{3.4}$$

as ε tends to 0, uniformly in $x \in \{y \in \Omega : |y| \le 3\kappa\}$. Define stopping times

$$\tau_0(w) := \inf\{t > 0 : w(t) \notin A, |\pi(w(t))| > 3\kappa\},$$

$$\tilde{\tau}_k(w) := \inf\{t > \tau_{k-1} : w(t) \notin A, |\pi(w(t))| > 4\kappa\}, \qquad k \in \mathbb{N},$$

$$\tau_k(w) := \inf\{t > \tilde{\tau}_k : w(t) \notin A, |\pi(w(t))| < 3\kappa\}, \qquad k \in \mathbb{N},$$

for $w \in C([0,\infty);\mathbb{R}^n)$. Note that $|w(\tau_k)| = 3\kappa$ for $k = 0, 1, 2, \dots$, and $|w(\tilde{\tau}_k)| = 4\kappa$ for $k = 1, 2, 3, \dots$ almost surely under $\widehat{P}_x^{\varepsilon}$ for $x \in \Omega$ and $|x| \le 2\kappa\varepsilon$. Since $\triangle\pi(x) = 0$ and $\nabla\pi(x)\nabla U(x) = 0$ for $|x| \ge 2\kappa$, Itô's formula implies

$$\begin{aligned}\pi(\widehat{X}_x^{\varepsilon}(t)) &= \pi(\tilde{\tau}_k(\widehat{X}_x^{\varepsilon})) + \int_{\tilde{\tau}_k(\widehat{X}_x^{\varepsilon})}^{t} \nabla\pi(\widehat{X}_x^{\varepsilon}(s))\sigma(\varepsilon\widehat{X}^{\varepsilon}(s))\,d\widehat{W}(s)\\ &\quad + \varepsilon\int_{\tilde{\tau}_k(\widehat{X}_x^{\varepsilon})}^{t} \nabla\pi(\widehat{X}_x^{\varepsilon}(s))b(\varepsilon\widehat{X}^{\varepsilon}(s))\,ds\end{aligned} \tag{3.5}$$

for $t \in [\tilde{\tau}_k(\widehat{X}_x^{\varepsilon}), \tau_k(\widehat{X}_x^{\varepsilon})]$, $x \in \Omega$ and $|x| \le 3\kappa\varepsilon$. Since the diffusion coefficient of the one-dimensional process $|\pi(\widehat{X}_x^{\varepsilon}(t))|$ is uniformly elliptic, and $T^{\delta(\varepsilon)/\varepsilon}$ diverges to infinity as $\varepsilon \downarrow 0$ almost surely under $\widehat{P}_x^{\varepsilon}$, there exists a sequence $\{\eta(\varepsilon)\}$ converging to 0 as $\varepsilon \downarrow 0$ such that

$$\sup_{|x|=4\kappa} \widehat{P}_x^{\varepsilon}\big(T^{\delta(\varepsilon)/\varepsilon} < T^{3\kappa}\big) \le \eta(\varepsilon).$$

On the other hand, since $\sigma\sigma^T$ is uniformly positive definite, $\widehat{X}_x^{\varepsilon}$ hits $\{x \in \Omega : |x| < \delta'\}$ with positive probability for all $x \in \Omega$, $\varepsilon > 0$, $\delta' > 0$. Hence, letting $\alpha(\varepsilon)$ be a sequence of positive numbers such that $\alpha(\varepsilon) \le 2\kappa$, and $\alpha(\varepsilon)$ converges to 0 as $\varepsilon \downarrow 0$, we obtain that

$$p(\varepsilon) := \inf_{|x|=3\kappa} \widehat{P}_x^{\varepsilon}\big(\tilde{T}^{\alpha(\varepsilon)} < T^{4\kappa}\big) > 0$$

for all $\varepsilon > 0$, and that $p(\varepsilon)$ converges to 0 as $\varepsilon \downarrow 0$. Moreover, we have

$$\begin{aligned}&\widehat{P}_x^{\varepsilon}\big(T^{\delta(\varepsilon)/\varepsilon} < \tilde{T}^{\alpha(\varepsilon)}\big)\\
&\quad = \sum_{k=1}^{\infty} \widehat{P}_x^{\varepsilon}\big(T^{\delta(\varepsilon)/\varepsilon} < \tau_k, \tilde{\tau}_k < \tilde{T}^{\alpha(\varepsilon)}\big)\\
&\quad = \sum_{k=1}^{\infty} \int_{\{x_1 \in \Omega : |\pi(x_1)|=3\kappa\}} \int_{\{y_1 \in \Omega : |\pi(y_1)|=4\kappa\}}\\
&\qquad \cdots \int_{\{x_k \in \Omega : |\pi(x_k)|=3\kappa\}} \int_{\{y_k \in \Omega : |\pi(y_k)|=4\kappa\}} \widehat{P}_{y_k}^{\varepsilon}\big(T^{\delta(\varepsilon)/\varepsilon} < T^{3\kappa}\big)\\
&\qquad\qquad \times \widehat{P}_{x_k}^{\varepsilon}\big(w(T^{4\kappa}) \in dy_k, T^{4\kappa} < \tilde{T}^{\alpha(\varepsilon)}\big)\\
&\qquad\qquad \times \widehat{P}_{y_{k-1}}^{\varepsilon}\big(w(T^{3\kappa}) \in dx_k, T^{\delta(\varepsilon)/\varepsilon} > T^{3\kappa}\big)\\
&\qquad\qquad \times \cdots \times \widehat{P}_{x_1}^{\varepsilon}\big(w(T^{4\kappa}) \in dy_1, T^{4\kappa} < \tilde{T}^{\alpha(\varepsilon)}\big)\\
&\qquad\qquad \times \widehat{P}_{x}^{\varepsilon}\big(w(T^{3\kappa}) \in dx_1, T^{\delta(\varepsilon)/\varepsilon} > T^{3\kappa}\big)\\
&\quad \le \eta(\varepsilon)\sum_{k=1}^{\infty}\big(1 - p(\varepsilon)\big)^k\\
&\quad = \frac{\eta(\varepsilon)(1 - p(\varepsilon))}{p(\varepsilon)}.\end{aligned}$$

Hence, if $\eta(\varepsilon)/p(\varepsilon)$ converges to 0 as $\varepsilon \downarrow 0$, $\widehat{P}_x^\varepsilon(T^{\delta(\varepsilon)/\varepsilon} < \tilde{T}^{\alpha(\varepsilon)})$ converges to 0 as $\varepsilon \downarrow 0$. Now we choose $\alpha(\varepsilon)$ so that $\eta(\varepsilon)/p(\varepsilon)$ converges to 0 as $\varepsilon \downarrow 0$. Then $\widehat{P}_x^\varepsilon(T^{\delta(\varepsilon)/\varepsilon} < \tilde{T}^{\alpha(\varepsilon)})$ converges to 0 as $\varepsilon \downarrow 0$. Thus, for (3.4), it is sufficient to prove that

$$\sup_{|x|\le\alpha(\varepsilon)} \left|\widehat{P}_x^\varepsilon\big(w\big(T^{\delta(\varepsilon)/\varepsilon}\big) \in \varepsilon^{-1}B\big) - \widehat{P}_O^\varepsilon\big(w\big(T^{\delta(\varepsilon)/\varepsilon}\big) \in \varepsilon^{-1}B\big)\right| \to 0 \tag{3.6}$$

as $\varepsilon \downarrow 0$. To show this convergence, we use the coupling method. Let $\sigma^l \in C_b^\infty(\mathbb{R}^n; \mathbb{R}^n \otimes \mathbb{R}^n)$ and $b^l \in C_b^\infty(\mathbb{R}^n; \mathbb{R}^n)$ for $l = 1, 2, \dots,$ such that

$$\lim_{l\to\infty} \sup_{|x|\le M} |\sigma^l(x) - \sigma(x)| = 0 \quad \text{and} \quad \lim_{l\to\infty} \sup_{|x|\le M} |b^l(x) - b(x)| = 0 \qquad \text{for } M > 0.$$

Let x be fixed, and consider a pair of stochastic processes $(\widetilde{X}_x^{\varepsilon,l}, \widetilde{X}_O^{\varepsilon,l})$ defined by

$$\begin{aligned}
\widetilde{X}_x^{\varepsilon,l}(t) &= x + \int_0^t \sigma^l(\varepsilon\widetilde{X}_x^{\varepsilon,l}(s))\, d\widehat{W}(s) \\
&\qquad + \varepsilon \int_0^t b^l(\varepsilon\widetilde{X}_x^{\varepsilon,l}(s))\, ds - \int_0^t (\nabla U)(\widetilde{X}_x^{\varepsilon,l}(s))\, ds, \\
\widetilde{X}_O^{\varepsilon,l}(t) &= \int_0^t \sigma^l(\varepsilon\widetilde{X}_O^{\varepsilon,l}(s)) H^{\varepsilon,l}(\widetilde{X}_x^{\varepsilon,l}(s), \widetilde{X}_O^{\varepsilon,l}(s))\, d\widehat{W}(s) \\
&\qquad + \varepsilon \int_0^t b^l(\varepsilon\widetilde{X}_O^{\varepsilon,l}(s))\, ds - \int_0^t (\nabla U)(\widetilde{X}_O^{\varepsilon,l}(s))\, ds,
\end{aligned}$$

where

$$H^{\varepsilon,l}(x_1, x_2) := I_n - \frac{2\sigma^l(\varepsilon x_2)^{-1}(x_1 - x_2)(x_1 - x_2)^T(\sigma^l(\varepsilon x_2)^{-1})^T}{|\sigma^l(\varepsilon x_2)^{-1}(x_1 - x_2)|^2}$$

for $x_1, x_2 \in \mathbb{R}^n$, and I_n is the unit matrix. Note that $(\widetilde{X}_x^{\varepsilon,l}, \widetilde{X}_O^{\varepsilon,l})$ is uniquely determined because of the smoothness of σ^l and b^l. We define

$$\begin{gathered}
V(y) := |y|^{-1}y, \qquad \Phi^{\varepsilon,l}(x_1, x_2) := \sigma^l(\varepsilon x_1) - \sigma^l(\varepsilon x_2)H^{\varepsilon,l}(x_1, x_2), \\
\Psi^{\varepsilon,l}(x_1, x_2) := \varepsilon b^l(\varepsilon x_1) - (\nabla U)(x_1) - \varepsilon b^l(\varepsilon x_2) + (\nabla U)(x_2)
\end{gathered}$$

for $y \in \{z \in \mathbb{R}^n : |z| \le 2\kappa_0\}$ and $x_1, x_2 \in \{z \in \mathbb{R}^n : |z| \le \kappa_0\}$. Similarly to the argument in Section 3 in [36], there exists a positive constant K such that $\inf_l\{|\Phi^{\varepsilon,l}(x_1, x_2)^T V(x_1 - x_2)|^2\} \ge K$ for $x_1, x_2 \in \{z \in \mathbb{R}^n : |z| \le \kappa_0\}$ for small ε. By the equi-continuity of $\{\sigma^l\}$, we can choose $\rho \in (0, 2\kappa_0)$ satisfying

$$\begin{aligned}
&2\langle x_1 - x_2, \Psi^{\varepsilon,l}(x_1, x_2)\rangle + \mathrm{trace}(\Phi^{\varepsilon,l}(x_1, x_2)\Phi^{\varepsilon,l}(x_1, x_2)^T) \\
&\quad - |\Phi^{\varepsilon,l}(x_1, x_2)^T V(x_1 - x_2)|^2 \le K/12
\end{aligned}$$

for $|x_1 - x_2| < \rho, |x_1| \le \kappa_0, |x_2| \le \kappa_0, l = 1, 2, 3, \dots$ (see [36]). For $\varepsilon' \in [0, \rho)$, define a stopping time $\mathscr{T}^{\varepsilon'}$ on $C([0,\infty); \mathbb{R}^n) \times C([0,\infty); \mathbb{R}^n)$ by

$$\begin{aligned}
\mathscr{T}^{\varepsilon'}(w, w') := \inf\{t > 0 : |w(t) - w'(t)| \notin (\varepsilon, \rho),& \\
|w(t)| \ge \kappa_0,\ \text{or } |w'(t)| \ge \kappa_0\}&
\end{aligned}$$

for $w, w' \in C([0,\infty);\mathbb{R}^n)$. By Itô's formula and the choice of ρ, we have

$$\begin{aligned}
&\rho^{2/3} P\big(|\widetilde{X}_x^{\varepsilon,l}(\mathscr{T}^{\varepsilon'}(\widetilde{X}_x^{\varepsilon,l},\widetilde{X}_O^{\varepsilon,l})) - \widetilde{X}_O^{\varepsilon,l}(\mathscr{T}^{\varepsilon'}(\widetilde{X}_x^{\varepsilon,l},\widetilde{X}_O^{\varepsilon,l}))| = \rho\big)\\
&\quad\le E[|\widetilde{X}_x^{\varepsilon,l}(\mathscr{T}^{\varepsilon'}(\widetilde{X}_x^{\varepsilon,l},\widetilde{X}_O^{\varepsilon,l})) - \widetilde{X}_O^{\varepsilon,l}(\mathscr{T}^{\varepsilon'}(\widetilde{X}_x^{\varepsilon,l},\widetilde{X}_O^{\varepsilon,l}))|^{2/3}]\\
&\quad= |x|^{2/3} - \frac{1}{9}E\bigg[\int_0^{\mathscr{T}^{\varepsilon'}(\widetilde{X}_x^{\varepsilon,l},\widetilde{X}_O^{\varepsilon,l})} |\widetilde{X}_x^{\varepsilon,l}(s) - \widetilde{X}_O^{\varepsilon,l}(s)|^{-4/3}\\
&\qquad\qquad\times \big|\Phi^{\varepsilon,l}(\widetilde{X}_x^{\varepsilon,l}(s),\widetilde{X}_O^{\varepsilon,l}(s))^T\\
&\qquad\qquad\times V\big(\widetilde{X}_x^{\varepsilon,l}(s) - \widetilde{X}_O^{\varepsilon,l}(s)\big)\big|^2\,ds\bigg]\\
&\qquad+ \frac{2}{3}E\bigg[\int_0^{\mathscr{T}^{\varepsilon'}(\widetilde{X}_x^{\varepsilon,l},\widetilde{X}_O^{\varepsilon,l})} |\widetilde{X}_x^{\varepsilon,l}(s) - \widetilde{X}_O^{\varepsilon,l}(s)|^{-4/3}\\
&\qquad\qquad\times \{2\langle \widetilde{X}_x^{\varepsilon,l}(s) - \widetilde{X}_O^{\varepsilon,l}(s), \Psi^{\varepsilon,l}(\widetilde{X}_x^{\varepsilon,l}(s),\widetilde{X}_O^{\varepsilon,l}(s))\rangle\\
&\qquad\qquad\quad+ \operatorname{trace}\big(\Phi^{\varepsilon,l}(\widetilde{X}_x^{\varepsilon,l}(s),\widetilde{X}_O^{\varepsilon,l}(s))\\
&\qquad\qquad\qquad\times \Phi^{\varepsilon,l}(\widetilde{X}_x^{\varepsilon,l}(s),\widetilde{X}_O^{\varepsilon,l}(s))^T\big)\\
&\qquad\qquad\quad- \big|\Phi^{\varepsilon,l}(\widetilde{X}_x^{\varepsilon,l}(s),\widetilde{X}_O^{\varepsilon,l}(s))^T\\
&\qquad\qquad\qquad\times V\big(\widetilde{X}_x^{\varepsilon,l}(s) - \widetilde{X}_O^{\varepsilon,l}(s)\big)\big|^2\}\,ds\bigg]\\
&\quad\le |x|^{2/3} - \frac{K}{18}E\bigg[\int_0^{\mathscr{T}^{\varepsilon'}(\widetilde{X}_x^{\varepsilon,l}(s),\widetilde{X}_O^{\varepsilon,l}(s))} |\widetilde{X}_x^{\varepsilon,l}(s) - \widetilde{X}_O^{\varepsilon,l}(s)|^{-4/3}\,ds\bigg]\\
&\quad\le |x|^{2/3} - \frac{K}{18\rho^{4/3}}E[\mathscr{T}^{\varepsilon'}(\widetilde{X}_x^{\varepsilon,l},\widetilde{X}_O^{\varepsilon,l})].
\end{aligned}$$

Hence, letting $\varepsilon' \downarrow 0$, we have the following two estimates:

$$\begin{aligned}
&P\big(|\widetilde{X}_x^{\varepsilon,l}(\mathscr{T}^0(\widetilde{X}_x^{\varepsilon,l},\widetilde{X}_O^{\varepsilon,l})) - \widetilde{X}_O^{\varepsilon,l}(\mathscr{T}^0(\widetilde{X}_x^{\varepsilon,l},\widetilde{X}_O^{\varepsilon,l}))| = \rho\big),\\
&\qquad\le \rho^{-2/3}|x|^{2/3},
\end{aligned}\tag{3.7}$$

$$E[\mathscr{T}^0(\widetilde{X}_x^{\varepsilon,l},\widetilde{X}_O^{\varepsilon,l})] \le \frac{18\rho^{4/3}}{K}|x|^{2/3}.\tag{3.8}$$

On the other hand, by Itô's formula,

$$\begin{aligned}
&E\bigg[\bigg\{\bigg(|\widetilde{X}_x^{\varepsilon,l}(\mathscr{T}^0(\widetilde{X}_x^{\varepsilon,l},\widetilde{X}_O^{\varepsilon,l})) - x| - \frac{\kappa_0}{2}\bigg)_+\bigg\}^2\bigg]\\
&\quad= E\bigg[\int_0^{\mathscr{T}^0(\widetilde{X}_x^{\varepsilon,l},\widetilde{X}_O^{\varepsilon,l})}\bigg(|\widetilde{X}_x^{\varepsilon,l}(s) - x| - \frac{\kappa_0}{2}\bigg)_+ |\widetilde{X}_x^{\varepsilon,l}(s) - x|^{-1}\mathbb{I}_{\{|\widetilde{X}_x^{\varepsilon,l}(s)-x|\ge\kappa_0/2\}}
\end{aligned}$$

$$
\begin{aligned}
&\times \Big\{ 2\langle \widetilde{X}_x^{\varepsilon,l}(s) - x, \varepsilon b^l(\varepsilon \widetilde{X}_x^{\varepsilon,l}(s)) - \nabla U(\widetilde{X}_x^{\varepsilon,l}(s))\rangle \\
&\qquad + \operatorname{trace}[\sigma^l(\varepsilon \widetilde{X}_x^{\varepsilon,l}(s))\sigma^l(\varepsilon \widetilde{X}_x^{\varepsilon,l}(s))^T] \\
&\qquad\qquad - \left| \sigma^l(\varepsilon \widetilde{X}_x^{\varepsilon,l}(s))^T \frac{\widetilde{X}_x^{\varepsilon,l}(s) - x}{|\widetilde{X}_x^{\varepsilon,l}(s) - x|}\right|^2 \Big\} \\
&+ \mathbb{I}_{\{|\widetilde{X}_x^{\varepsilon,l}(s) - x| \ge \kappa_0/2\}} \left| \sigma^l(\varepsilon \widetilde{X}_x^{\varepsilon,l}(s))^T \frac{\widetilde{X}_x^{\varepsilon,l}(s) - x}{|\widetilde{X}_x^{\varepsilon,l}(s) - x|}\right|^2 ds \Big] \\
\le{}& C E[\mathscr{T}^0(\widetilde{X}_x^{\varepsilon,l}, \widetilde{X}_O^{\varepsilon,l})],
\end{aligned}
$$

where $z_+ := \max\{0, z\}$ for $z \in \mathbb{R}$ and C is a positive constant independent of l and x. This inequality together with (3.8) implies

$$
P\big(|\widetilde{X}_x^{\varepsilon,l}(\mathscr{T}^0(\widetilde{X}_x^{\varepsilon,l}, \widetilde{X}_O^{\varepsilon,l})) - x| = \kappa_0\big) \le \frac{72C\rho^{4/3}}{\kappa_0^2 K}|x|^{2/3}. \tag{3.9}
$$

Similarly, we have

$$
P\big(|\widetilde{X}_O^{\varepsilon,l}(\mathscr{T}^0(\widetilde{X}_x^{\varepsilon,l}, \widetilde{X}_O^{\varepsilon,l}))| = \kappa_0\big) \le \frac{72C'\rho^{4/3}}{\kappa_0^2 K}|x|^{2/3}, \tag{3.10}
$$

where C' is a positive constant. Noting that $\widetilde{X}_x^{\varepsilon,l}$ and $\widetilde{X}_O^{\varepsilon,l}$ converge to $\widehat{X}_x^{\varepsilon}$ and $\widehat{X}_O^{\varepsilon}$ in law as $l \to +\infty$, respectively, for each ε, by the coupling inequality (see [36]) we have

$$
\begin{aligned}
&\sup_{|x|\le\alpha(\varepsilon)} \big|\widehat{P}_x^{\varepsilon}\big(w\big(T^{\delta(\varepsilon)/\varepsilon}\big) \in \varepsilon^{-1}B\big) - \widehat{P}_O^{\varepsilon}\big(w\big(T^{\delta(\varepsilon)/\varepsilon}\big) \in \varepsilon^{-1}B\big)\big| \\
&\quad \le \sup_{|x|\le\alpha(\varepsilon)} \sup_l \big| P\big(\widetilde{X}_x^{\varepsilon,l}\big(T^{\delta(\varepsilon)/\varepsilon}\big) \in \varepsilon^{-1}B\big) - P\big(\widetilde{X}_O^{\varepsilon,l}\big(T^{\delta(\varepsilon)/\varepsilon}\big) \in \varepsilon^{-1}B\big)\big| \\
&\quad \le \sup_{|x|\le\alpha(\varepsilon)} \sup_l P\big(|\widetilde{X}_x^{\varepsilon,l}(\mathscr{T}^0(\widetilde{X}_x^{\varepsilon,l}, \widetilde{X}_O^{\varepsilon,l})) - \widetilde{X}_O^{\varepsilon,l}(\mathscr{T}^0(\widetilde{X}_x^{\varepsilon,l}, \widetilde{X}_O^{\varepsilon,l}))| \ne 0\big) \\
&\quad \le \sup_{|x|\le\alpha(\varepsilon)} \sup_l P\big(|\widetilde{X}_x^{\varepsilon,l}(\mathscr{T}^0(\widetilde{X}_x^{\varepsilon,l}, \widetilde{X}_O^{\varepsilon,l})) - \widetilde{X}_O^{\varepsilon,l}(\mathscr{T}^0(\widetilde{X}_x^{\varepsilon,l}, \widetilde{X}_O^{\varepsilon,l}))| = \rho\big) \\
&\qquad + \sup_{|x|\le\alpha(\varepsilon)} \sup_l P\big(|\widetilde{X}_x^{\varepsilon,l}(\mathscr{T}^0(\widetilde{X}_x^{\varepsilon,l}, \widetilde{X}_O^{\varepsilon,l})) - x| = \kappa_0\big) \\
&\qquad + \sup_{|x|\le\alpha(\varepsilon)} \sup_l P\big(|\widetilde{X}_O^{\varepsilon,l}(\mathscr{T}^0(\widetilde{X}_x^{\varepsilon,l}, \widetilde{X}_O^{\varepsilon,l}))| = \kappa_0\big).
\end{aligned}
$$

This inequality, together with (3.7), (3.9) and (3.10) yields (3.6). □

The next lemma implies that O is not absorbing for X.

LEMMA 3.4.

$$\int_0^t E\big[\mathbb{I}_{\{x\,:\,|x|\le\delta'\}}(X(s))\big]\,ds = O(\delta')$$

as $\delta' \downarrow 0$, *for all* $t \ge 0$.

PROOF. To simplify the notation, let $X^\varepsilon(0) = x^\varepsilon \in \Omega^\varepsilon$. It is sufficient to show that

$$\int_0^t E\big[\mathbb{I}_{\{x\,:\,|\pi(x)|\le\delta'\}}(X(s))\big]\,ds = O(\delta')$$

as $\delta' \downarrow 0$. By Fatou's lemma, we have

$$\begin{aligned}
&\int_0^t E\big[\mathbb{I}_{\{x\,:\,|\pi(x)|\le\delta'\}}(X(s))\big]\,ds\\
&\qquad\le \liminf_{\varepsilon\downarrow 0}\int_0^t E\big[\mathbb{I}_{\{x\,:\,|\pi(x)|\le 3\kappa\varepsilon\}}(X^\varepsilon(s))\big]\,ds \qquad (3.11)\\
&\qquad\quad + \liminf_{\varepsilon\downarrow 0}\int_0^t E\big[\mathbb{I}_{\{x\,:\,3\kappa\varepsilon\le|\pi(x)|\le\delta'\}}(X^\varepsilon(s))\big]\,ds.
\end{aligned}$$

To show that the second term is $O(\delta')$ as $\delta' \downarrow 0$, let f be a continuous function on $\mathbb{R}$ such that $\mathbb{I}_{\{x\in\mathbb{R}\,:\,3\kappa\varepsilon\le x\le\delta'\}} \le f \le \mathbb{I}_{\{x\in\mathbb{R}\,:\,2\kappa\varepsilon\le x\le 2\delta'\}}$ and $F(x) := \int_0^x\int_0^y f(z)\,dz\,dy$. Noting that $\pi(x) = \langle e_i, x\rangle e_i$ for $x \in \Omega_i$ and $i = 1, 2, \dots, N$, we have $\nabla\pi(x)\pi(x) = \pi(x)$ for $x \in \Omega$ such that $|x| \ge 2\kappa$. Since $\nabla\pi(x)\nabla U^\varepsilon(x) = 0$ and $\triangle\pi(x) = 0$ for $x \in \Omega^\varepsilon$ such that $|x| \ge 2\kappa\varepsilon$, we have

$$\begin{aligned}
&E[F(|\pi(X^\varepsilon(t))|)] - F(|\pi(x^\varepsilon)|)\\
&\quad = \frac{1}{2}\int_0^t E\left[f(|\pi(X^\varepsilon(s))|)\left|\sigma(X^\varepsilon(s))^T\frac{\pi(X^\varepsilon(s))}{|\pi(X^\varepsilon(s))|}\right|^2\right]ds\\
&\quad\quad + \int_0^t E\left[F'(|\pi(X^\varepsilon(s))|)\left\langle\frac{\pi(X^\varepsilon(s))}{|\pi(X^\varepsilon(s))|}, b(X^\varepsilon(s))\right\rangle\right]ds.
\end{aligned}$$

It is easy to see that $E[|X^\varepsilon(t)|^2]$ is dominated uniformly in $\varepsilon > 0$. Moreover, it holds that $0 \le F' \le 2\delta'$ and $0 \le F(x) \le 2\delta' x$ for $x \in \mathbb{R}_+$. Thus, by uniform ellipticity of $a = \sigma\sigma^T$, we have the following estimate:

$$\int_0^t E\big[\mathbb{I}_{\{x\in\mathbb{R}\,:\,3\kappa\varepsilon\le x\le\delta'\}}(|\pi(X^\varepsilon(s))|)\big]\,ds \le C\delta'$$

for some constant C. Hence,

$$\liminf_{\varepsilon\downarrow 0}\int_0^t E\big[\mathbb{I}_{\{x\,:\,3\kappa\varepsilon\le|\pi(x)|\le\delta'\}}(X^\varepsilon(s))\big]\,ds = O(\delta')$$

as $\delta' \downarrow 0$. This yields that the second term of (3.11) is equal to $O(\delta')$ as $\delta' \downarrow 0$.

The proof is finished by showing that

$$\int_0^t E\big[\mathbb{I}_{\{x\,:\,|\pi(x)|\le 3\kappa\varepsilon\}}(X^\varepsilon(s))\big]\,ds = O(\varepsilon) \tag{3.12}$$

as $\varepsilon \downarrow 0$. Define stopping times $\{\tau_k^\varepsilon, \tilde{\tau}_k^\varepsilon\}$ by

$$\begin{aligned}
&\tau_0^\varepsilon(w) := 0,\\
&\tilde{\tau}_k^\varepsilon(w) := \inf\{u > \tau_{k-1}^\varepsilon(w) : |\pi(w(u))| > 4\kappa\varepsilon\}, \qquad k \in \mathbb{N},\\
&\tau_k^\varepsilon(w) := \inf\{u > \tilde{\tau}_k^\varepsilon(w) : |\pi(w(u))| < 3\kappa\varepsilon\}, \qquad k \in \mathbb{N},
\end{aligned}$$

for $w \in C([0,\infty);\mathbb{R}^n)$. Then,

$$\begin{aligned}
&\int_0^t E\big[\mathbb{I}_{\{x\,:\,|\pi(x)|\le 3\kappa\varepsilon\}}(X^\varepsilon(s))\big]\,ds\\
&\qquad \le \sum_{k=1}^\infty \int \left(\int T^{4\kappa\varepsilon}(w) P_x^\varepsilon(dw)\right) P_{x^\varepsilon}^\varepsilon\big(w(\tau_k^\varepsilon)\in dx, \tau_k^\varepsilon \le t\big)\\
&\qquad \le \sup_{x\in\{y\in\Omega\,:\,|\pi(y)|=3\kappa\varepsilon\}} \left(\int T^{4\kappa\varepsilon}(w) P_x^\varepsilon(dw)\right) \sum_{k=1}^\infty P_{x^\varepsilon}^\varepsilon(\tau_k^\varepsilon \le t).
\end{aligned}$$

By using the notation in the proof of Lemma 3.3, we have

$$\sup_{x\in\{y\in\Omega\,:\,|\pi(y)|=3\kappa\varepsilon\}} \int T^{4\kappa\varepsilon}(w) P_x^\varepsilon(dw) = \varepsilon^2 \sup_{x\in\{y\in\Omega\,:\,|\pi(y)|=3\kappa\}} \int T^{4\kappa}(w) \widehat{P}_x^\varepsilon(dw).$$

It is easy to see that

$$\sup_{\varepsilon>0} \sup_{x\in\{y\in\Omega\,:\,|\pi(y)|=3\kappa\}} \int T^{4\kappa}(w) \widehat{P}_x^\varepsilon(dw) < +\infty.$$

Hence, for (3.12), it is sufficient to show that

$$\sum_{k=1}^\infty P_{x^\varepsilon}^\varepsilon(\tilde{\tau}_k^\varepsilon \le t) \le C\varepsilon^{-1} \tag{3.13}$$

for some constant C. For $w \in C([0,\infty);\Omega)$, let $\mathcal{N}_t(w)$ be the number of transitions of w from the set $\{x \in \Omega^\varepsilon : |\pi(x)| = 3\kappa\}$ to the set $\{x \in \Omega^\varepsilon : |\pi(x)| = 4\kappa\}$ during the time interval $[0,t]$. Then,

$$\sum_{k=1}^\infty P_{x^\varepsilon}^\varepsilon(\tilde{\tau}_k^\varepsilon \le t) = \int \mathcal{N}_{\varepsilon^{-2}t}(w) \widehat{P}_{\varepsilon^{-1}x^\varepsilon}^\varepsilon(dw). \tag{3.14}$$

Take $f \in C^\infty([0,\infty))$ such that $f \ge 0$, $0 \le f' \le 1$, $f'' \ge 0$, $\operatorname{supp} f'' \subset [2\kappa, 3\kappa]$, $f(x) = 0$ for $x \le 2\kappa$ and $f(3\kappa) < f(4\kappa)$. Define $\widehat{Y}_x^{\varepsilon,i}$ by

$$\widehat{Y}_x^{\varepsilon,i}(t) := f(\langle e_i, \widehat{X}_x^\varepsilon(t)\rangle \mathbb{I}_{\Omega_i}(\widehat{X}_x^\varepsilon(t)))$$

for $x \in \Omega$ and $i = 1, 2, \dots, N$. Since $\langle e_i, \nabla U(x)\rangle = 0$ for $x \in \{\Omega_i : |x| \geq 2\kappa\varepsilon\}$, by Itô's formula we have

$$\begin{aligned}\widehat{Y}^{\varepsilon,i}_x(t) = f(\langle e_i, x\rangle \mathbb{I}_{\Omega_i}(x)) &\\ &+ \int_0^t f'(\langle e_i, \widehat{X}^\varepsilon_x(s)\rangle \mathbb{I}_{\Omega_i}(\widehat{X}^\varepsilon_x(s)))\langle e_i, \sigma(\varepsilon \widehat{X}^\varepsilon_x(s))\, d\widehat{W}(s)\rangle \\ &+ \varepsilon \int_0^t f'(\langle e_i, \widehat{X}^\varepsilon_x(s)\rangle \mathbb{I}_{\Omega_i}(\widehat{X}^\varepsilon_x(s)))\langle e_i, b(\varepsilon \widehat{X}^\varepsilon_x(s))\rangle\, ds \\ &+ \frac{1}{2}\int_0^t f''(\langle e_i, \widehat{X}^\varepsilon_x(s)\rangle \mathbb{I}_{\Omega_i}(\widehat{X}^\varepsilon_x(s)))|\sigma(\varepsilon \widehat{X}^\varepsilon_x(s))^T e_i|^2\, ds.\end{aligned}$$

It is clear that

$$E[\mathcal{N}_{\varepsilon^{-2}t}(\widehat{X}^\varepsilon_{\varepsilon^{-1}x^\varepsilon})] \leq \sum_{i=1}^N \sup_{x:|\pi(x)|\leq 4\kappa} E[\widetilde{\mathcal{N}}_{\varepsilon^{-2}t}(\widehat{Y}^{\varepsilon,i}_x)],$$

where $\widetilde{\mathcal{N}}_t(w)$ is the number of up-crossing of w for the interval $[f(3\kappa), f(4\kappa)]$ during the time interval $[0, t]$. Hence, by (3.13) and (3.14), it is sufficient to show that

$$\sup_{x:|\pi(x)|\leq 4\kappa} E[\widetilde{\mathcal{N}}_{\varepsilon^{-2}t}(\widehat{Y}^{\varepsilon,i}_x)] \leq C\varepsilon^{-1} \tag{3.15}$$

with a constant C for all $i = 1, 2, \dots, N$. Let i be fixed and $m \in \mathbb{N}$. Define τ_k and $\tilde{\tau}_k$ by

$$\begin{aligned}\tilde{\tau}_0 &:= 0,\\ \tau_0 &:= \inf\{u > 0 : \widehat{Y}^{\varepsilon,i}_x(u) \leq f(3\kappa)\},\\ \tilde{\tau}_k &:= \inf\{u > \tau_{k-1} : \widehat{Y}^{\varepsilon,i}_x(u) \geq f(4\kappa)\}, \qquad k \in \mathbb{N},\\ \tau_k &:= \inf\{u > \tilde{\tau}_k : \widehat{Y}^{\varepsilon,i}_x(u) \leq f(3\kappa)\}, \qquad k \in \mathbb{N}.\end{aligned}$$

Then,

$$\begin{aligned}E[\widehat{Y}^{\varepsilon,i}_x(t \wedge \tilde{\tau}_m)] - E[\widehat{Y}^{\varepsilon,i}_x(t \wedge \tau_0)] &\\ &= \sum_{k=1}^m E[\widehat{Y}^{\varepsilon,i}_x(\tilde{\tau}_k \wedge t) - \widehat{Y}^{\varepsilon,i}_x(\tau_{k-1} \wedge t)] \\ &\quad + \sum_{k=1}^{m-1} E[\widehat{Y}^{\varepsilon,i}_x(\tau_k \wedge t) - \widehat{Y}^{\varepsilon,i}_x(\tilde{\tau}_k \wedge t)] \\ &= \sum_{k=1}^m E[\widehat{Y}^{\varepsilon,i}_x(\tilde{\tau}_k \wedge t) - \widehat{Y}^{\varepsilon,i}_x(\tau_{k-1} \wedge t)]\end{aligned}$$

$$+\varepsilon\sum_{k=1}^{m-1}E\biggl[\int_{\tilde{\tau}_k\wedge t}^{\tau_k\wedge t}f'(\langle e_i,\widehat{X}_x^\varepsilon(s)\rangle\mathbb{I}_{\Omega_i}(\widehat{X}_x^\varepsilon(s)))\langle e_i,b(\varepsilon\widehat{X}_x^\varepsilon(s))\rangle\,ds\biggr]$$

$$+\frac{1}{2}\sum_{k=1}^{m-1}E\biggl[\int_{\tilde{\tau}_k\wedge t}^{\tau_k\wedge t}f''(\langle e_i,\widehat{X}_x^\varepsilon(s)\rangle\mathbb{I}_{\Omega_i}(\widehat{X}_x^\varepsilon(s)))|\sigma(\varepsilon\widehat{X}_x^\varepsilon(s))^T e_i|^2\,ds\biggr].$$

Since $f''\geq 0$, we have

$$\begin{aligned}&\left|\sum_{k=1}^{m}E[\widehat{Y}_x^{\varepsilon,i}(\tilde{\tau}_k\wedge t)-\widehat{Y}_x^{\varepsilon,i}(\tau_{k-1}\wedge t)]\right|\\&\qquad\leq E[\widehat{Y}_x^{\varepsilon,i}(t\wedge\tilde{\tau}_m)]+E[\widehat{Y}_x^{\varepsilon,i}(t\wedge\tau_0)]+C_1\varepsilon t\end{aligned}\tag{3.16}$$

with a positive constant C_1. Let

$$\tilde{\tau}_*:=\max\{\tilde{\tau}_k:\tilde{\tau}_k\leq t,k=1,2,3,\ldots\},$$

$$\tau_*:=\max\{\tau_k:\tau_k\leq t,k=1,2,3,\ldots\},$$

$$M(t):=\int_0^t f'(\langle e_i,\widehat{X}_x^\varepsilon(s)\rangle\mathbb{I}_{\Omega_i}(\widehat{X}_x^\varepsilon(s)))\langle e_i,\sigma(\varepsilon\widehat{X}_x^\varepsilon(s))\,d\widehat{W}(s)\rangle.$$

When $\tilde{\tau}_*\leq\tau_*$, $\widehat{Y}_x^{\varepsilon,i}(t)\leq f(4\kappa)$. When $\tau_*\leq\tilde{\tau}_*$, $f''(\langle e_i,\widehat{X}_x^\varepsilon(s)\rangle\mathbb{I}_{\Omega_i}(\widehat{X}_x^\varepsilon(s)))=0$ for $s\in[\tilde{\tau}_*,t]$. Thus, we have

$$\begin{aligned}\widehat{Y}_x^{\varepsilon,i}(t)&=\widehat{Y}_x^{\varepsilon,i}(\tilde{\tau}_*)+M(t)-M(\tilde{\tau}_*)\\&\quad+\varepsilon\int_{\tilde{\tau}_*}^t f'(\langle e_i,\widehat{X}_x^\varepsilon(s)\rangle\mathbb{I}_{\Omega_i}(\widehat{X}_x^\varepsilon(s)))\langle e_i,b(\varepsilon\widehat{X}_x^\varepsilon(s))\rangle\,ds.\end{aligned}$$

Hence,

$$\widehat{Y}_x^{\varepsilon,i}(t)\leq f(4\kappa)+2\sup_{0\leq s\leq t}|M(s)|+C_2\varepsilon t$$

for $|x|\leq 4\kappa$ with a constant C_2. By the Burkholder–Davis–Gundy inequality we have

$$E\Bigl[\sup_{0\leq s\leq t}\widehat{Y}_x^{\varepsilon,i}(s)\Bigr]\leq f(4\kappa)+2C_3\sqrt{t}+C_2\varepsilon t$$

for $|x|\leq 4\kappa$ with a constant C_3. Thus, letting $m\to+\infty$ on (3.16), we have for $|x|\leq 4\kappa$

$$\bigl(f(4\kappa)-f(3\kappa)\bigr)E\Bigl[\widetilde{\mathcal{N}}_t(\widehat{Y}_x^{\varepsilon,i})\Bigr]\leq 2f(4\kappa)+4C_3\sqrt{t}+(C_1+2C_2)\varepsilon t.$$

Therefore, replacing t by $\varepsilon^{-2}t$, (3.15) is obtained. □

The lemmas above yield that the boundary condition at O is a weighted Kirchhoff boundary condition. Hence, the next step is to determine the weights associated with the edges. Let Y_x^ε be a diffusion process defined by the solution of the following stochastic differential equation:

$$Y_x^\varepsilon(t) = x + \sigma(O)W(t) - \int_0^t (\nabla U^\varepsilon)(Y_x^\varepsilon(s))\,ds. \tag{3.17}$$

Note that Y_x^ε is a special case of X^ε with the condition $X^\varepsilon(0) = x$, and Y_x^ε does not hit Ω^ε almost surely. Denote the law of Y_x^ε on $C([0,\infty);\mathbb{R}^n)$ by Q_x^ε. It is easy to see that the law of Y_x^ε is the same as that of $\varepsilon Y^1_{\varepsilon^{-1}x}(\varepsilon^{-2})$. By (3.2) one has that the law of $\widehat{X}_x^\varepsilon$ converges to that of Y_x^1 as $\varepsilon \downarrow 0$, and therefore, the law of X_x^ε and that of Y_x^ε are getting closer as $\varepsilon \downarrow 0$. In particular, we have

$$\lim_{\varepsilon\downarrow 0}\left|P_O^\varepsilon\big(w(T^{c\varepsilon}) \in \Omega_i^\varepsilon\big) - Q_O^\varepsilon\big(w(T^{c\varepsilon}) \in \Omega_i^\varepsilon\big)\right| = 0$$

for all $c > 0$ and $i = 1, 2, \dots, N$. Since this holds for all $c > 0$, it is possible to choose a subsequence of ε (denote the subsequence by ε again) and positive numbers $\beta(\varepsilon)$ which satisfy $\lim_{\varepsilon\downarrow 0}\beta(\varepsilon) = +\infty$, and

$$\lim_{\varepsilon\downarrow 0}\left|P_O^\varepsilon\big(w(T^{\beta(\varepsilon)\varepsilon}) \in \Omega_i^\varepsilon\big) - Q_O^\varepsilon\big(w(T^{\beta(\varepsilon)\varepsilon}) \in \Omega_i^\varepsilon\big)\right| = 0 \tag{3.18}$$

for $i = 1, 2, \dots, N$. Let $\delta(\varepsilon) := \varepsilon\beta(\varepsilon)$. Then, $\delta(\varepsilon)$ satisfies the conditions in Lemma 3.3.

Now we assume that $\sigma(O) = I_n$ where I_n means the unit matrix. This assumption enables us to determine the weights of the edges explicitly. Let

$$p_i := \frac{c_i^{n-1}\int_0^1 r^{n-2}e^{-u_i(r)}\,dr}{\sum_{i=1}^N c_i^{n-1}\int_0^1 r^{n-2}e^{-u_i(r)}\,dr}.$$

We remark that when u_i is independent of i, then we have $p_i := c_i^{n-1}/(\sum_{i=1}^N c_i^{n-1})$; hence the weights $\{p_i\}$ are determined by the ratio of the area of the cross-section around the edge I_i. Then, the following lemma holds.

LEMMA 3.5. *If $\sigma(O) = I_n$, then*

$$\lim_{\varepsilon\downarrow 0}\sup_{|x|\le 3\kappa\varepsilon}\left|P_x^\varepsilon\big(w(T^{\delta(\varepsilon)}) \in \Omega_i^\varepsilon\big) - p_i\right| = 0$$

for $i = 1, \dots, N$.

PROOF. Applying Lemma 3.3 to both $X_\cdot^\varepsilon$ and $Y_\cdot^\varepsilon$, and using (3.18), it is sufficient to show that

$$\lim_{\varepsilon\downarrow 0}\left|Q_O^\varepsilon\big(w(T^{\delta(\varepsilon)}) \in \Omega_i^\varepsilon\big) - p_i\right| = 0 \tag{3.19}$$

for $i = 1, \dots, N$.

We make a similar discussion as in the proof of Theorem 6.1 in [21]. Let ν^ε be the invariant measure of the Markov chain $\{Y^\varepsilon(\tau_k^\varepsilon)\}$, where τ_k^ε are stopping times defined by

$$\begin{aligned}
&\tau_0^\varepsilon(w) := 0,\\
&\tilde{\tau}_k^\varepsilon(w) := \inf\{u > \tau_{k-1}^\varepsilon(w) : |\pi(w(u))| > \delta(\varepsilon)\}, \qquad k \in \mathbb{N},\\
&\tau_k^\varepsilon(w) := \inf\{u > \tilde{\tau}_k^\varepsilon(w) : |\pi(w(u))| < 3\kappa\varepsilon\}, \qquad k \in \mathbb{N}.
\end{aligned}$$

Define a measure μ^ε on Ω^ε by

$$\mu^\varepsilon(dx) := \exp(-U^\varepsilon(x))\,dx, \qquad x \in \Omega^\varepsilon,$$

a function space $\mathscr{D}(\mathscr{E}^\varepsilon)$ by $\{f \in C^2(\Omega^\varepsilon) : \lim_{x : d(x,\partial\Omega^\varepsilon)\to 0} f(x) = 0\}$ and a bilinear form $\mathscr{E}^\varepsilon$ by

$$\mathscr{E}^\varepsilon(f, g) := \int_{\Omega^\varepsilon} \langle \nabla f(x), \nabla g(x)\rangle \mu^\varepsilon(dx), \qquad f, g \in \mathscr{D}(\mathscr{E}^\varepsilon).$$

Then, the pre-Dirichlet form $(\mathscr{E}^\varepsilon, \mathscr{D}(\mathscr{E}^\varepsilon))$ on $L^2(\Omega^\varepsilon, \mu^\varepsilon)$ is closable, and Y^ε is associated to the Dirichlet form obtained by closing $(\mathscr{E}^\varepsilon, \mathscr{D}(\mathscr{E}^\varepsilon))$. Note that μ^ε is an invariant measure of Y^ε; see [23]. By Theorem 2.1 in [26] we have

$$\mu^\varepsilon(B) = \int_{\{x\in\Omega^\varepsilon : |\pi(x)|=3\kappa\varepsilon\}} \nu^\varepsilon(dx) \int \left[\int_0^{\tau_1^\varepsilon(w)} \mathbb{I}_B(w(t))\,dt\right] Q_x^\varepsilon(dw)$$

for $B \in \mathscr{B}(\mathbb{R}^n)$. Let $B_i^\varepsilon := \{x \in \Omega_i^\varepsilon : \delta(\varepsilon) \le |\pi(x)| \le 2\delta(\varepsilon)\}$. Then,

$$\begin{aligned}
\mu^\varepsilon(B_i^\varepsilon)&\\
&= \int_{\{x\in\Omega^\varepsilon : |\pi(x)|=3\kappa\varepsilon\}} \nu^\varepsilon(dx) \int \left[\mathbb{I}_{\Omega_i}(w(\tilde{\tau}_1^\varepsilon)) \int_{\tilde{\tau}_1^\varepsilon(w)}^{\tau_1^\varepsilon(w)} \mathbb{I}_{B_i^\varepsilon}(w(t))\,dt\right] Q_x^\varepsilon(dw)\\
&= \int_{\{x\in\Omega^\varepsilon : |\pi(x)|=3\kappa\varepsilon\}} \nu^\varepsilon(dx) \int \mathbb{I}_{\Omega_i}\big(w\big(T^{\delta(\varepsilon)}\big)\big) Q_x^\varepsilon(dw)\\
&\qquad\qquad \times \int \left[\int_0^{T^{3\kappa\varepsilon}} \mathbb{I}_{B_i^\varepsilon}(\tilde{w}(t))\,dt\right] Q^\varepsilon_{w(T^{\delta(\varepsilon)})}(d\tilde{w}).
\end{aligned} \tag{3.20}$$

On the other hand, let

$$Z(t) := -\delta(\varepsilon) + \check{W}(t), \qquad \check{T} := \inf\{t > 0 : |Z(t)| > 2\delta(\varepsilon) - 3\kappa\varepsilon\},$$

where $\check{W}$ is a one-dimensional Wiener process starting from 0, and

$$F(x) := \int_{-2\delta(\varepsilon)}^{x} \int_{-2\delta(\varepsilon)}^{y} \mathbb{I}_{[-\delta(\varepsilon),\delta(\varepsilon)]}(z)\,dz\,dy, \qquad x \in \mathbb{R}.$$

Then, by Itô's formula we have

$$E[F(Z(\check{T}))] - F(-\delta(\varepsilon)) = \frac{1}{2} E\left[\int_0^{\check{T}} \mathbb{I}_{[-\delta(\varepsilon),\delta(\varepsilon)]}(Z_t)\,dt\right].$$

Since F can be computed explicitly, we see that $F(-\delta(\varepsilon)) = 0$ and

$$\begin{aligned} E[F(Z(\check{T}))] &= F\big(2\delta(\varepsilon) - 3\kappa\varepsilon\big)P\big(Z(\check{T}) = 2\delta(\varepsilon) - 3\kappa\varepsilon\big) \\ &= \frac{\delta(\varepsilon) - 3\kappa\varepsilon}{4\delta(\varepsilon) - 6\kappa\varepsilon}\big[2\delta(\varepsilon)^2 + 2\delta(\varepsilon)\big(\delta(\varepsilon) - 3\kappa\varepsilon\big)\big]. \end{aligned}$$

Thus, it follows that

$$E\bigg[\int_0^{\check{T}} \mathbb{I}_{[-\delta(\varepsilon),\delta(\varepsilon)]}(Z(t))\,dt\bigg] = 2\delta(\varepsilon)^2 + o(\delta(\varepsilon)^2).$$

On the other hand, the strong Markov property and the reflection principle imply that

$$\int\bigg(\int_0^{T^{3\kappa\varepsilon}} \mathbb{I}_{B_i^\varepsilon}(w(t))\,dt\bigg)Q_y^\varepsilon(dw) = E\bigg[\int_0^{\check{T}} \mathbb{I}_{[-\delta(\varepsilon),\delta(\varepsilon)]}(Z(t))\,dt\bigg]$$

for all $y \in \{x \in \Omega^\varepsilon : |\pi(x)| = \delta(\varepsilon)\}$, because the left-hand side is independent of the behavior of w moving in $\{x \in \Omega^\varepsilon : |\pi(x)| \geq \delta(\varepsilon)\}$ under Q_y^ε. Hence, it holds that

$$\int\bigg(\int_0^{T^{3\kappa\varepsilon}} \mathbb{I}_{B_i^\varepsilon}(w(t))\,dt\bigg)Q_y^\varepsilon(dw) = 2\delta(\varepsilon)^2 + o(\delta(\varepsilon)^2) \tag{3.21}$$

for all $y \in \{x \in \Omega^\varepsilon : |\pi(x)| = \delta(\varepsilon)\}$. By Lemma 3.3, (3.20) and (3.21), we have

$$\begin{aligned} \mu^\varepsilon(B_i^\varepsilon) = &\big(2\delta(\varepsilon)^2 + o(\delta(\varepsilon)^2)\big)\nu^\varepsilon\big(\{x \in \Omega^\varepsilon : |\pi(x)| = 3\kappa\varepsilon\}\big) \\ &\times \big(Q_O^\varepsilon\big(w\big(T^{\delta(\varepsilon)}\big) \in \Omega_i^\varepsilon\big) + o_\varepsilon(1)\big). \end{aligned} \tag{3.22}$$

Since $\sum_{i=1}^N Q_O^\varepsilon(w(T^{\delta(\varepsilon)}) \in \Omega_i^\varepsilon) = 1$, we have, as $\varepsilon \downarrow 0$

$$\nu^\varepsilon\big(\{x \in \Omega^\varepsilon : |\pi(x)| = 3\kappa\varepsilon\}\big) = \frac{1}{2}\delta(\varepsilon)^{-2}\sum_{i=1}^N \mu^\varepsilon(B_i^\varepsilon) + o_\varepsilon(1). \tag{3.23}$$

Dividing both sides of (3.22) by those of (3.23), we obtain that

$$Q_O^\varepsilon\big(w\big(T^{\delta(\varepsilon)}\big) \in \Omega_i^\varepsilon\big) = \frac{\mu^\varepsilon(B_i^\varepsilon)}{\sum_{i=1}^N \mu^\varepsilon(B_i^\varepsilon)} + o_\varepsilon(1).$$

By the definition of μ^ε, the continuity of σ and b, and $\sigma(O) = I_n$, $\mu^\varepsilon(B_i^\varepsilon)$ can be expressed explicitly as

$$\mu^\varepsilon(B_i^\varepsilon) = \omega_{n-2}\delta(\varepsilon)c_i^{n-1}\varepsilon^{n-1}\int_0^1 r^{n-2}e^{-u_i(r)}\,dr,$$

where ω_{n-2} is the area of the $(n-2)$-dimensional unit sphere. Therefore, (3.19) is proved. □

The statement in Lemma 3.5 can be improved as follows.

LEMMA 3.6.

$$\lim_{\delta'\downarrow 0}\lim_{\varepsilon\downarrow 0}\sup_{|x|\le 3\kappa\varepsilon}\left|P_x^\varepsilon\left(w(T^{\delta'})\in\Omega_i^\varepsilon\right)-p_i\right|=0$$

for $i=1,\dots,N$.

PROOF. In view of Lemma 3.3, it is sufficient to show

$$\lim_{\delta'\downarrow 0}\lim_{\varepsilon\downarrow 0}\left|P_O^\varepsilon\left(w(T^{\delta'})\in\Omega_i^\varepsilon\right)-p_i\right|=0$$

for $i=1,2,\dots,N$. Define stopping times $\{\tau_k^\varepsilon,\tilde\tau_k^\varepsilon\}$ by

$$\begin{aligned}
&\tau_0^\varepsilon(w):=0,\\
&\tilde\tau_k^\varepsilon(w):=\inf\{u>\tau_{k-1}^\varepsilon(w):|\pi(w(u))|>\delta(\varepsilon)\},\qquad k\in\mathbb{N},\\
&\tau_k^\varepsilon(w):=\inf\{u>\tilde\tau_k^\varepsilon(w):|\pi(w(u))|<3\kappa\varepsilon\},\qquad k\in\mathbb{N}.
\end{aligned}$$

By the strong Markov property, we have

$$\begin{aligned}
&P_O^\varepsilon\left(w(T^{\delta'})\in\Omega_i^\varepsilon\right)\\
&\quad=\sum_{k=1}^\infty\int\mathbb{I}_{\{\tau_{k-1}^\varepsilon<T^{\delta'}\}}(w)P_O^\varepsilon(dw)\\
&\qquad\times\int P_y^\varepsilon(T^{\delta'}<T^{3\kappa\varepsilon})\mathbb{I}_{\Omega_i^\varepsilon}(y)P_{w(\tau_{k-1}^\varepsilon)}\left(w\left(T^{\delta(\varepsilon)}\right)\in dy\right)
\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}
p_i&=p_i\sum_{k=1}^\infty P_O^\varepsilon(\tau_{k-1}^\varepsilon<T^{\delta'}<\tau_k^\varepsilon)\\
&=\sum_{k=1}^\infty\int\mathbb{I}_{\{\tau_{k-1}^\varepsilon<T^{\delta'}\}}(w)P_O^\varepsilon(dw)\\
&\qquad\times\int p_iP_y^\varepsilon(T^{\delta'}<T^{3\kappa\varepsilon})P_{w(\tau_{k-1}^\varepsilon)}\left(w\left(T^{\delta(\varepsilon)}\right)\in dy\right)
\end{aligned}\tag{3.25}$$

for $i=1,2,\dots,N$. Let h_-^ε and h_+^ε be functions on $[0,\infty)$ given by

$$h_-^\varepsilon(z):=\max_i\sup_{x\in\Omega_i^\varepsilon:|\pi(x)|=\max\{z,3\kappa\varepsilon\}}\frac{2\langle e_i,b(x)\rangle}{|\sigma(x)^Te_i|^2},$$

$$h_+^\varepsilon(z):=\min_i\inf_{x\in\Omega_i^\varepsilon:|\pi(x)|=\max\{z,3\kappa\varepsilon\}}\frac{2\langle e_i,b(x)\rangle}{|\sigma(x)^Te_i|^2},$$

respectively. Define functions s_-^ε and s_+^ε on $[0,\infty)$ by

$$s_-^\varepsilon(z):=\int_0^z\exp\left(-\int_0^{z'}h_-^\varepsilon(z'')\,dz''\right)dz',$$

$$s_+^\varepsilon(z) := \int_0^z \exp\biggl(-\int_0^{z'} h_+^\varepsilon(z'')\,dz''\biggr)\,dz',$$

respectively. Then, for $y \in \{x \in \Omega_i^\varepsilon : |\pi(x)| = \delta(\varepsilon)\}$ we have

$$\begin{aligned}
&\int s_-^\varepsilon\bigl(\bigl|\pi\bigl(w(T^{\delta'} \wedge T^{3\kappa\varepsilon})\bigr)\bigr|\bigr)P_y^\varepsilon(dw) - s_-^\varepsilon(\delta(\varepsilon))\\
&\quad = \int s_-^\varepsilon\bigl(\langle e_i, w(T^{\delta'} \wedge T^{3\kappa\varepsilon})\rangle\bigr)P_y^\varepsilon(dw) - s_-^\varepsilon(\langle e_i, y\rangle)\\
&\quad = -\frac{1}{2}\int\biggl[\int_0^{T^{\delta'}\wedge T^{3\kappa\varepsilon}} h_-^\varepsilon(w(s))|\sigma(w(s))^T e_i|^2\\
&\qquad\qquad\qquad \times \exp\biggl(-\int_0^{w(s)} h_-^\varepsilon(z')\,dz'\biggr)\,ds\biggr]P_y^\varepsilon(dw)\\
&\qquad + \int\biggl[\int_0^{T^{\delta'}\wedge T^{3\kappa\varepsilon}} \langle e_i, b(w(s))\rangle \exp\biggl(-\int_0^{w(s)} h_-^\varepsilon(z')\,dz'\biggr)\,ds\biggr]P_y^\varepsilon(dw)\\
&\quad \le 0.
\end{aligned}$$

Hence, it holds that

$$s_-^\varepsilon(\delta')P_y^\varepsilon(T^{\delta'} < T^{3\kappa\varepsilon}) + s_-^\varepsilon(3\kappa\varepsilon)P_y^\varepsilon(T^{\delta'} > T^{3\kappa\varepsilon}) \le s_-^\varepsilon(\delta(\varepsilon))$$

for $y \in \{x \in \Omega^\varepsilon : |\pi(x)| = \delta(\varepsilon)\}$. Since

$$P_y^\varepsilon(T^{\delta'} < T^{3\kappa\varepsilon}) + P_y^\varepsilon(T^{\delta'} > T^{3\kappa\varepsilon}) = 1,$$

we have

$$P_y^\varepsilon(T^{\delta'} < T^{3\kappa\varepsilon}) \le \frac{s_-^\varepsilon(\delta(\varepsilon)) - s_-^\varepsilon(3\kappa\varepsilon)}{s_-^\varepsilon(\delta') - s_-^\varepsilon(3\kappa\varepsilon)} \tag{3.26}$$

for $y \in \{x \in \Omega^\varepsilon : |\pi(x)| = \delta(\varepsilon)\}$. Similarly we have

$$P_y^\varepsilon(T^{\delta'} < T^{3\kappa\varepsilon}) \ge \frac{s_+^\varepsilon(\delta(\varepsilon)) - s_+^\varepsilon(3\kappa\varepsilon)}{s_+^\varepsilon(\delta') - s_+^\varepsilon(3\kappa\varepsilon)} \tag{3.27}$$

for $y \in \{x \in \Omega^\varepsilon : |\pi(x)| = \delta(\varepsilon)\}$. Let $\mathcal{N}_{T^{\delta'}}(X_O^\varepsilon)$ be the number of transitions of X_O^ε from the set $\{x \in \Omega^\varepsilon : |\pi(x)| = 3\kappa\varepsilon\}$ to the set $\{x \in \Omega^\varepsilon : |\pi(x)| = \delta(\varepsilon)\}$ during the time interval $[0, T^{\delta'}(X_O^\varepsilon)]$. By Lemma 3.5, (3.24), (3.25), (3.26) and (3.27), we have

$$\begin{aligned}
&P_O^\varepsilon\bigl(w(T^{\delta'}) \in \Omega_i^\varepsilon\bigr) - p_i\\
&\quad \le \frac{s_-^\varepsilon(\delta(\varepsilon)) - s_-^\varepsilon(3\kappa\varepsilon)}{s_-^\varepsilon(\delta') - s_-^\varepsilon(3\kappa\varepsilon)}\\
&\qquad \times \sum_{k=1}^\infty \int P_{w(\tau_{k-1}^\varepsilon)}\bigl(w\bigl(T^{\delta(\varepsilon)}\bigr) \in \Omega_i^\varepsilon\bigr)\mathbb{I}_{\{\tau_{k-1}^\varepsilon < T^{\delta'}\}}(w)P_O^\varepsilon(dw)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{s_+^\varepsilon(\delta(\varepsilon)) - s_+^\varepsilon(3\kappa\varepsilon)}{s_+^\varepsilon(\delta') - s_+^\varepsilon(3\kappa\varepsilon)} p_i \sum_{k=1}^{\infty} \int \mathbb{I}_{\{\tau_{k-1}^\varepsilon < T^{\delta'}\}}(w) P_O^\varepsilon(dw) \\
&\le \left(\frac{s_-^\varepsilon(\delta(\varepsilon)) - s_-^\varepsilon(3\kappa\varepsilon)}{s_-^\varepsilon(\delta') - s_-^\varepsilon(3\kappa\varepsilon)} - \frac{s_+^\varepsilon(\delta(\varepsilon)) - s_+^\varepsilon(3\kappa\varepsilon)}{s_+^\varepsilon(\delta') - s_+^\varepsilon(3\kappa\varepsilon)} \right) p_i E[\mathcal{N}_{T^{\delta'}}(X_O^\varepsilon)] \\
&\quad + o_\varepsilon(1) \frac{s_-^\varepsilon(\delta(\varepsilon)) - s_-^\varepsilon(3\kappa\varepsilon)}{s_-^\varepsilon(\delta') - s_-^\varepsilon(3\kappa\varepsilon)} E[\mathcal{N}_{T^{\delta'}}(X_O^\varepsilon)].
\end{aligned}
$$

By the definitions of s_-^ε and s_+^ε, we obtain

$$
\limsup_{\varepsilon\downarrow 0} \delta(\varepsilon)^{-1} \left(\frac{s_-^\varepsilon(\delta(\varepsilon)) - s_-^\varepsilon(3\kappa\varepsilon)}{s_-^\varepsilon(\delta') - s_-^\varepsilon(3\kappa\varepsilon)} - \frac{s_+^\varepsilon(\delta(\varepsilon)) - s_+^\varepsilon(3\kappa\varepsilon)}{s_+^\varepsilon(\delta') - s_+^\varepsilon(3\kappa\varepsilon)} \right) \le C,
$$

where C is a constant independent of δ', and for each $\delta' > 0$

$$
\frac{s_-^\varepsilon(\delta(\varepsilon)) - s_-^\varepsilon(3\kappa\varepsilon)}{s_-^\varepsilon(\delta') - s_-^\varepsilon(3\kappa\varepsilon)} = O(\delta(\varepsilon)).
$$

On the other hand, a similar discussion as in the proof of Lemma 3.4 implies

$$
E[\mathcal{N}_{T^{\delta'}}(X_O^\varepsilon)] = \delta(\varepsilon)^{-1} o_{\delta'}(1).
$$

Therefore, we have

$$
\limsup_{\delta'\downarrow 0} \limsup_{\varepsilon\downarrow 0} \big(P_O^\varepsilon\big(w(T^{\delta'}) \in \Omega_i^\varepsilon\big) - p_i \big) \le 0.
$$

Similarly we obtain

$$
\limsup_{\delta'\downarrow 0} \limsup_{\varepsilon\downarrow 0} \big(p_i - P_O^\varepsilon\big(w(T^{\delta'}) \in \Omega_i^\varepsilon\big) \big) \le 0.
$$

These inequalities yield the conclusion. $\square$

We need a little more improvement of Lemma 3.6 as follows.

LEMMA 3.7.

$$
\lim_{\delta'\downarrow 0} \lim_{\delta\downarrow 0} \lim_{\varepsilon\downarrow 0} \sup_{|x|\le\delta} \big| P_x^\varepsilon\big(w(T^{\delta'}) \in \Omega_i^\varepsilon\big) - p_i \big| = 0
$$

for $i = 1, \dots, N$.

PROOF. In view of Lemma 3.6 it is sufficient to show

$$
\lim_{\delta'\downarrow 0} \lim_{\delta\downarrow 0} \lim_{\varepsilon\downarrow 0} \sup_{3\kappa\varepsilon\le|x|\le\delta} \big| P_x^\varepsilon\big(w(T^{\delta'}) \in \Omega_i^\varepsilon\big) - p_i \big| = 0.
$$

By Lemma 3.6 again,

$$
\begin{aligned}
&\bigl|P_x^\varepsilon\bigl(w(T^{\delta'}) \in \Omega_i^\varepsilon\bigr) - p_i\bigr| \\
&\quad = \biggl|\int_{\{y\in\Omega^\varepsilon : |\pi(y)|=3\kappa\varepsilon\}} P_y^\varepsilon\bigl(\tilde{w}(T^{\delta'}) \in \Omega_i^\varepsilon\bigr) P_x^\varepsilon\bigl(w(T^{3\kappa\varepsilon}) \in dy, T^{3\kappa\varepsilon} < T^{\delta'}\bigr) \\
&\qquad\qquad + P_x^\varepsilon(T^{3\kappa\varepsilon} > T^{\delta'})\mathbb{I}_{\Omega_i^\varepsilon}(x) - p_i\biggr| \\
&\quad = \bigl|\bigl(p_i + o_{\varepsilon,\delta'}(1)\bigr)P_x^\varepsilon(T^{3\kappa\varepsilon} < T^{\delta'}) + P_x^\varepsilon(T^{3\kappa\varepsilon} > T^{\delta'})\mathbb{I}_{\Omega_i^\varepsilon}(x) - p_i\bigr| \\
&\quad \le p_i\bigl|P_x^\varepsilon(T^{3\kappa\varepsilon} < T^{\delta'}) - 1\bigr| + P_x^\varepsilon(T^{3\kappa\varepsilon} > T^{\delta'})\mathbb{I}_{\Omega_i^\varepsilon}(x) + o_{\varepsilon,\delta'}(1).
\end{aligned}
$$

Here, $o_{\varepsilon,\delta'}(1)$ means a term which converges to 0 as $\delta' \downarrow 0$ after letting $\varepsilon \downarrow 0$. Hence, it is sufficient to show for $\delta' > 0$ and $i = 1, 2, \dots, N$

$$
\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \inf_{x\in\Omega_i^\varepsilon : 3\kappa\varepsilon\le|x|\le\delta} P_x^\varepsilon(T^{3\kappa\varepsilon} < T^{\delta'}) = 1. \tag{3.28}
$$

Let $T^O(w) := \inf\{t \ge 0 : w(t) = O\}$ and fix i. By Theorem 2.2 the law of $(T^{3\kappa\varepsilon}(X_{x^\varepsilon}^\varepsilon), T^{\delta'}(X_{x^\varepsilon}^\varepsilon))$ converges to that of $(T^O(X_x), T^{\delta'}(X_x))$ as $\varepsilon \downarrow 0$ for $x^\varepsilon \in \{y \in \Omega^\varepsilon : 3\kappa\varepsilon \le |y| \le \delta\}$ such that x^ε converges to $x \in I_i$, where the process X_x is determined by the following stochastic differential equation:

$$
\begin{aligned}
X_x(t) = x + \int_0^t \langle e_i, \sigma(X_x(s))\,dW(s)\rangle + \int_0^t \langle e_i, b(X_x(s))\rangle\,ds,& \\
t \in [0, T^O(X_x) \wedge T^{\delta'}(X_x)].&
\end{aligned}
$$

By using $I_i = \bigcap_{\varepsilon'>0}\bigcup_{\varepsilon<\varepsilon'} \Omega_i^\varepsilon$ and compactness of $\{y \in \mathbb{R}^n : |y| \le \delta\}$, we have

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \inf_{x\in\Omega_i^\varepsilon : 3\kappa\varepsilon\le|x|\le\delta} P_x^\varepsilon(T^{3\kappa\varepsilon} < T^{\delta'}) \\
&\qquad = \inf_{x\in I_i : 0\le|x|\le\delta} P\bigl(T^O(X_x) < T^{\delta'}(X_x)\bigr).
\end{aligned}
$$

Since $\sigma\sigma^T$ is uniformly positive definite, we have

$$
\lim_{\delta\downarrow 0} \inf_{x\in I_i : 0\le|x|\le\delta} P\bigl(T^O(X_x) < T^{\delta'}(X_x)\bigr) = 1.
$$

This proves (3.28). $\square$

The lemmas above determine the boundary condition for X at O. Now let us characterize X by a generator of a process on Γ. Let

$$
\partial_{e_i} f(x) := \lim_{s\to 0} \frac{1}{s}\bigl(f(x + se_i) - f(x)\bigr)
$$

for any differentiable function f on I_i and $i = 1, 2, \dots, N$. Define a second-order differential operator $\mathcal{L}_i$ on I_i by

$$\mathcal{L}_i := \frac{1}{2}|\sigma^T(x)e_i|^2 \, \partial_{e_i}^2 + \langle b(x), e_i\rangle \, \partial_{e_i} \tag{3.29}$$

for $i = 1, 2, \dots, N$. Define the second-order differential operator $\mathcal{L}$ on $C_0(\Gamma)$

$$\begin{aligned}\mathscr{D}(\mathcal{L}) := \Bigg\{ & f \in C_0(\Gamma) : f|_{I_i \setminus O} \in C_b^2(I_i \setminus O) \text{ for all } i = 1, 2, \dots, N, \\ & \lim_{s\downarrow 0} \mathcal{L}_i f(se_i) \text{ has a common value for } i = 1, 2, \dots, N, \\ & \qquad \sum_{i=1}^N p_i \Big(\lim_{s \downarrow 0} (\partial_{e_i} f)(se_i) \Big) = 0 \Bigg\},\end{aligned}$$

$$\begin{aligned}\mathcal{L}f(x) &:= \mathcal{L}_i f(x), \qquad x \in I_i \setminus O, \\ \mathcal{L}f(O) &:= \lim_{s\downarrow 0} \mathcal{L}_i f(se_i).\end{aligned}$$

Note that $\mathcal{L}f(O)$ does not depend on the selection of $i = 1, 2, \dots, N$. We call $\{p_i\}$ the weights of the Kirchhoff boundary condition at O, and call $\sum_{i=1}^N p_i(\lim_{s\downarrow 0}(\partial_{e_i} f)(se_i)) = 0$ the weighted Kirchhoff boundary condition at O.

THEOREM 3.8. *Consider diffusion processes X^ε defined by* (3.1). *Assume that $\sigma(O) = I_n$ and the law of $X^\varepsilon(0)$ converges to a probability measure μ_0 on Γ. Then, X^ε converges weakly on $C([0, +\infty); \mathbb{R}^n)$ to the diffusion process X as $\varepsilon \downarrow 0$, where X is determined by the conditions that the law of $X(0)$ is equal to μ_0 and*

$$E\bigg[f(X(t)) - f(X(s)) - \int_s^t \mathcal{L}f(X(u))\, du \bigg| \mathscr{F}_s \bigg] = 0 \tag{3.30}$$

for $t \geq s \geq 0$ and $f \in \mathscr{D}(\mathcal{L})$, where $(\mathscr{F}_t)$ is the filtration generated by X. Therefore, $\mathcal{L}$ is the generator of X.

PROOF. From Lemma 3.2 we have that $\{X^\varepsilon\}$ is tight. We are going to show that there is a unique limit point in this family. Let X be any limit point of $\{X^\varepsilon\}$, and denote the sequence converging to X by $\{X^\varepsilon\}$ again. Since this martingale problem is well-posed (see [21]; [14] for the relationship between martingale problems and partial differential equations, and [37] for the uniqueness of the semigroup generated by $\mathcal{L}$), it is sufficient to prove that X satisfies (3.30). Fix $s \geq 0$. Let δ' be a positive number. Define the following stopping times:

$$\begin{aligned}\tilde{\tau}_0 &:= s, \\ \tau_0 &:= \inf\{u \geq s : X(u) = O\}, \\ \tilde{\tau}_k &:= \inf\{u > \tau_{k-1} : |X(u)| > \delta'\}, \qquad k \in \mathbb{N}, \\ \tau_k &:= \inf\{u > \tilde{\tau}_k : X(u) = O\}, \qquad k \in \mathbb{N}.\end{aligned}$$

Then, for $f \in \mathscr{D}(\mathcal{L})$, $s \le t$

$$
\begin{aligned}
E\Big[f(X(t)) - f(X(s)) - \int_s^t \mathcal{L} f(X(u))\, du \Big| \mathscr{F}_s \Big]
&= E\Bigg[\sum_{k=1}^{\infty} \Big(f\big(X(t \wedge \tilde{\tau}_k)\big) - f\big(X(t \wedge \tau_{k-1})\big) - \int_{t \wedge \tau_{k-1}}^{t \wedge \tilde{\tau}_k} \mathcal{L} f(X(u))\, du \Big) \Big| \mathscr{F}_s \Bigg] \\
&\quad + \sum_{k=0}^{\infty} E\Big[f\big(X(t \wedge \tau_k)\big) - f\big(X(t \wedge \tilde{\tau}_k)\big) - \int_{t \wedge \tilde{\tau}_k}^{t \wedge \tau_k} \mathcal{L} f(X(u))\, du \Big| \mathscr{F}_s \Big].
\end{aligned}
$$

Because of Theorem 2.2 the second sum vanishes. We estimate the first sum as follows:

$$
\begin{aligned}
&\Bigg| E\Bigg[\sum_{k=1}^{\infty} \Big(f\big(X(t \wedge \tilde{\tau}_k)\big) - f\big(X(t \wedge \tau_{k-1})\big) - \int_{t \wedge \tau_{k-1}}^{t \wedge \tilde{\tau}_k} \mathcal{L} f(X(u))\, du \Big) \Bigg] \Big| \mathscr{F}_s \Bigg| \\
&\quad \le \Bigg| E\Bigg[\sum_{k : \tilde{\tau}_k < t} \big(f(X(\tilde{\tau}_k)) - f(X(\tau_{k-1})) \big) \Big| \mathscr{F}_s \Bigg] \Bigg| \\
&\qquad + \|\mathcal{L} f\|_\infty E\Big[\int_s^t \mathbb{I}_{\{x : |x| \le \delta'\}}(X(u))\, du \Big| \mathscr{F}_s \Big] + \sup_{|x| \le \delta'} |f(x) - f(O)|.
\end{aligned}
$$

Clearly, the third term on the right-hand side converges to 0 as $\delta' \downarrow 0$. By Lemma 3.4 the second term on the right-hand side converges to 0 as $\delta' \downarrow 0$. The first sum on the right-hand side is equal to

$$
\begin{aligned}
&\Bigg| \sum_{k=1}^{\infty} \sum_{i=1}^{N} \big(f(\delta' e_i) - f(O) \big) P\big(X(\tilde{\tau}_k) \in I_i, \tilde{\tau}_k < t | \mathscr{F}_s \big) \Bigg| \\
&\quad = \Bigg| \sum_{k=1}^{\infty} \sum_{i=1}^{N} \Big(\delta' \lim_{s \downarrow 0} f'(s e_i) + o(\delta') \Big) P\big(X(\tilde{\tau}_k) \in I_i, \tilde{\tau}_k < t | \mathscr{F}_s \big) \Bigg|
\end{aligned}
\tag{3.31}
$$

Let $\delta \in (0, \delta')$ and let, for any $\varepsilon > 0$:

$$
\begin{aligned}
\tau_0^{\varepsilon,\delta} &:= \inf\{u > s : |\pi(X^\varepsilon(u))| < \delta\}, \\
\tilde{\tau}_k^{\varepsilon,\delta} &:= \inf\{u > \tau_{k-1}^{\varepsilon,\delta} : |\pi(X^\varepsilon(u))| > \delta'\}, \qquad k \in \mathbb{N}, \\
\tau_k^{\varepsilon,\delta} &:= \inf\{u > \tilde{\tau}_k^{\varepsilon,\delta} : |\pi(X^\varepsilon(u))| < \delta\}, \qquad k \in \mathbb{N}.
\end{aligned}
$$

The distributions of the pairs $(X^\varepsilon, \tilde{\tau}_k^{\varepsilon,\delta}, \tau_k^{\varepsilon,\delta})$ converge weakly to those of $(X, \tilde{\tau}_k, \tau_k)$ as $\delta \downarrow 0$ after $\varepsilon \downarrow 0$. Hence, by Lemma 3.7 we have

$$
\begin{aligned}
&P\big(X(\tilde{\tau}_k) \in I_i, \tilde{\tau}_k < t | \mathscr{F}_s \big) \\
&\quad = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} P\big(X^\varepsilon(\tilde{\tau}_k^{\varepsilon,\delta}) \in \Omega_i^\varepsilon, \tilde{\tau}_k^{\varepsilon,\delta} < t | \mathscr{F}_s \big)
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\int P_y^{\varepsilon}\big(w(T^{\delta'})\in\Omega_i^{\varepsilon}\big)P\big(X^{\varepsilon}(\tau_{k-1}^{\varepsilon,\delta})\in dy,\tilde{\tau}_k^{\varepsilon,\delta}<t|\mathscr{F}_s\big)\\
&= \big(p_i+o_{\delta'}(1)\big)P(\tilde{\tau}_k<t|\mathscr{F}_s).
\end{aligned}
$$

Note that $\sum_{k=1}^{\infty}P(\tilde{\tau}_k<t|\mathscr{F}_s)$ is equal to the expectation of the number of transitions of X from the point O to the set $\{x\in\Gamma:|x|=\delta'\}$ during the time interval $[s,t]$ [with respect to a general initial condition $X(0)$]. Approximating that by the expectation of the number of transitions of X^{ε} from the set $\{x\in\Omega^{\varepsilon}:|\pi(x)|=\delta\}$ to the set $\{x\in\Omega^{\varepsilon}:|\pi(x)|=\delta'\}$ during the time interval $[s,t]$, similarly as in the proof of Lemma 3.4 we obtain the estimate

$$
\sum_{k=1}^{\infty}P(\tilde{\tau}_k<t|\mathscr{F}_s)\le\frac{C_t}{\delta'}
$$

with a positive constant C_t depending only on t. Hence, by (3.31) we have

$$
\begin{aligned}
&\left|E\left[\sum_{k:\tilde{\tau}_k<t}\big(f\big(X(t\wedge\tilde{\tau}_k)\big)-f\big(X(t\wedge\tau_{k-1})\big)\big)\Big|\mathscr{F}_s\right]\right|\\
&\qquad\le\frac{C_t}{\delta'}\left|\sum_{i=1}^{N}\delta'\lim_{s\downarrow 0}f'(se_i)p_i+o(\delta')\right|.
\end{aligned}
$$

Since $f\in\mathscr{D}(\mathcal{L})$, the right-hand side converges to 0 as $\delta'\downarrow 0$. $\square$

Similarly as in Section 2, the argument above is also available in the case where the boundary of Ω^{ε} carries a Neumann boundary condition. Consider a diffusion process X^{ε} which is associated to L in Ω^{ε} and satisfies the reflecting boundary condition on $\partial\Omega^{\varepsilon}$. Then, X^{ε} can be expressed by the following equation:

$$
\widehat{X}^{\varepsilon}(t)=\widehat{X}^{\varepsilon}(0)+\int_0^t\sigma(\widehat{X}^{\varepsilon}(s))\,dW(s)+\int_0^t b(\widehat{X}^{\varepsilon}(s))\,ds+\Phi^{\varepsilon}(\widehat{X}^{\varepsilon})(t), \tag{3.32}
$$

where Φ^{ε} is a singular drift which forces the process to be reflecting on $\partial\Omega^{\varepsilon}$; see [46]. Note that $\widehat{X}^{\varepsilon}$ depends on Ω^{ε} but is independent of U^{ε}. Discussing this case in a similar way as we did in the case of Dirichlet boundary condition we obtain the following theorem. Let

$$
\begin{aligned}
\widehat{p}_i &:= \frac{c_i^{n-1}}{\sum_{i=1}^{N}c_i^{n-1}},\\
\mathscr{D}(\widehat{\mathcal{L}}) &:= \Big\{f\in C_0(\Gamma):f|_{I_i\setminus O}\in C_b^2(I_i\setminus O)\text{ for all }i=1,2,\dots,N,\\
&\qquad \lim_{s\downarrow 0}\mathcal{L}_i f(se_i)\text{ has a common value for }i=1,2,\dots,N,\\
&\qquad\qquad \sum_{i=1}^{N}\widehat{p}_i\Big(\lim_{s\downarrow 0}(\partial_{e_i}f)(se_i)\Big)=0\Big\},
\end{aligned}
$$

$$\widehat{\mathcal{L}}f(x) := \mathcal{L}_i f(x), \qquad x \in I_i \setminus O,$$
$$\widehat{\mathcal{L}}f(O) := \lim_{s\downarrow 0} \mathcal{L}_i f(se_i),$$

where $\mathcal{L}_i$ is given by (3.29). Note that $\widehat{\mathcal{L}}f(O)$ does not depend on the selection of $i = 1, 2, \dots, N$.

THEOREM 3.9. *Consider the diffusion processes $\widehat{X}^\varepsilon$ defined by (3.32). Assume that $\sigma(O) = I_n$ and the law of $\widehat{X}^\varepsilon(0)$ converges to a probability measure μ_0 on Γ. Then, $\{\widehat{X}^\varepsilon\}$ converge weakly on $C([0, +\infty); \mathbb{R}^n)$ to the diffusion process $\widehat{X}$ as $\varepsilon \downarrow 0$, where $\widehat{X}$ is determined by the conditions that the law of $\widehat{X}(0)$ is equal to μ_0 and*

$$E\left[f(\widehat{X}(t)) - f(\widehat{X}(s)) - \int_s^t \widehat{\mathcal{L}}f(\widehat{X}(u))\,du \,\middle|\, \mathscr{F}_s\right] = 0$$

for $t \geq s \geq 0$ and $f \in \mathscr{D}(\widehat{\mathcal{L}})$, where $(\mathscr{F}_t)$ is the filtration generated by $\widehat{X}$. Therefore, $\widehat{\mathcal{L}}$ is the generator of $\widehat{X}$.

REMARK 3.10. The weights $\{\widehat{p}_i\}$ of the case of Neumann boundary condition can be obtained from the wights $\{p_i\}$ discussed in Theorem 3.8 in the heuristic limit where the potential u_i around each edge takes only the value 0 on $[0, 1)$ and $+\infty$ on $[1, +\infty)$.

REMARK 3.11. As mentioned in Remark 2.4, we can discuss similarly the case where the shapes of the tubes $\{\Omega_i^\varepsilon\}$ are not cylindrical. However, if U^ε is not defined by a scaling of a fixed function U, the weights of the weighted Kirchhoff boundary condition cannot be determined uniquely. To handle this more general case, we have to assume that U^ε satisfies some uniform bound.

4. The case of general graphs. In this section we present results obtained by combining the results of Sections 2 and 3, and, in this way, we cover more general graphs. Let Λ be a finite or countable set, Ξ be a subset of $\Lambda \times \Lambda$, $\{V_\lambda : \lambda \in \Lambda\}$ be vertices in $\mathbb{R}^n$, $\{E_{\lambda,\lambda'} : (\lambda, \lambda') \in \Xi\}$ be C^3-curves with ends $\{V_\lambda, V_{\lambda'}\}$ and $G := \bigcup_{(\lambda,\lambda')\in\Xi} E_{\lambda,\lambda'}$. Denote $\lambda \sim \lambda'$ if $(\lambda, \lambda') \in \Xi$.

Let us denote the length of $E_{\lambda,\lambda'}$ by $|E_{\lambda,\lambda'}|$. Define $(\gamma_{\lambda,\lambda'}(s) : s \in [0, |E_{\lambda,\lambda'}|])$ as the arc-length parameterization of $E_{\lambda,\lambda'}$ with $\gamma_{\lambda,\lambda'}(0) = V_\lambda$. Assume that the number of $\{V_\lambda : \lambda \in \Lambda\} \cap \{x \in \mathbb{R}^n : |x| \leq M\}$ is finite for all $M > 0$, $|E_{\lambda,\lambda'}|$ is finite for all $(\lambda, \lambda') \in \Xi$ and

$$\lim_{s\downarrow 0} \langle \dot{\gamma}_{\lambda,\lambda_1}(s), \dot{\gamma}_{\lambda,\lambda_2}(s)\rangle < 1$$

for all $\lambda \sim \lambda_1$ and $\lambda \sim \lambda_2$ such that $\lambda_1 \neq \lambda_2$. Let $c_{\lambda,\lambda'}$ be a positive number for $(\lambda, \lambda') \in \Xi$, and let

$$\kappa_\lambda := \max\Big\{2\sqrt{2}c_{\lambda,\lambda_1}\Big/\sqrt{1 - \lim_{s\downarrow 0}\langle\dot{\gamma}_{\lambda,\lambda_1}(s), \dot{\gamma}_{\lambda,\lambda_2}(s)\rangle} : \lambda_1, \lambda_2 \in \Lambda \text{ such that } \lambda \sim \lambda_1, \lambda \sim \lambda_2, \lambda_1 \neq \lambda_2\Big\}$$

for $\lambda \in \Lambda$. Let $\pi(x)$ be a point in G which is nearest to $x \in \mathbb{R}^n$. Assume that there exists a small $\varepsilon_0 > 0$ and positive numbers $\{\kappa_\lambda\}$ such that $\pi(x)$ is uniquely determined for all $x \in \bigcup_{\lambda\sim\lambda'}\{x \in \mathbb{R}^n : d(x, E_{\lambda,\lambda'}) < c_{\lambda,\lambda'}\varepsilon, d(x, V_\lambda) \geq \kappa_\lambda\varepsilon$ and $d(x, V_{\lambda'}) \geq \kappa_{\lambda'}\varepsilon\}$ and for all $\varepsilon \in (0, \varepsilon_0]$, and that $\ddot{\gamma}_{\lambda,\lambda'}(s) = 0$ for sufficiently small s for each $(\lambda, \lambda') \in \Xi$.

Let $u_{\lambda,\lambda'}$ be given similarly to u in Section 2 for $(\lambda, \lambda') \in \Xi$. For $\varepsilon \in (0, \varepsilon_0]$, let U^ε be a function on $\mathbb{R}^n$ with values in $[0, +\infty]$, and assume

$$U^\varepsilon(x) = u_{\lambda,\lambda'}(c_{\lambda,\lambda'}^{-1}\varepsilon^{-1}d(x, E_{\lambda,\lambda'})),$$

$$x \in \{x \in \mathbb{R}^n : \pi(x) \in E_{\lambda,\lambda'}, d(x, E_{\lambda,\lambda'}) < c_{\lambda,\lambda'}\varepsilon, d(x, V_\lambda) \geq \kappa_\lambda\varepsilon, d(x, V_{\lambda'}) \geq \kappa_{\lambda'}\varepsilon\},$$

$$U^\varepsilon(x) = +\infty,$$

$$x \in \{x \in \mathbb{R}^n : \pi(x) \in E_{\lambda,\lambda'}, d(x, E_{\lambda,\lambda'}) \geq c_{\lambda,\lambda'}\varepsilon, d(x, V_\lambda) \geq \kappa_\lambda\varepsilon, d(x, V_{\lambda'}) \geq \kappa_{\lambda'}\varepsilon\},$$

$\Omega^\varepsilon := \{x : U^\varepsilon(x) < \infty\}$ is a simply connected domain, $\partial\Omega^\varepsilon$ is an $(n-1)$-dimensional C^2-manifold embedded in $\mathbb{R}^n$ and $U^\varepsilon|_{\Omega^\varepsilon}$ is a C^1-function on Ω^ε. In addition, we assume

$$\lim_{m\to\infty}\langle -\nabla U(x_m), \nabla d(x_m, \partial\Omega^\varepsilon)\rangle = +\infty \quad \text{and} \quad -\lim_{m\to\infty}\frac{U^\varepsilon(x_m)}{\log(d(x_m, \partial\Omega^\varepsilon))} = +\infty$$

for any sequence $\{x_m\}$ which converges to a point $x \in \partial\Omega^\varepsilon$.

Consider a diffusion process X^ε given by the following equation:

$$\begin{aligned} X^\varepsilon(t) = X^\varepsilon(0) &+ \int_0^t \sigma(X^\varepsilon(s))\,dW(s) + \int_0^t b(X^\varepsilon(s))\,ds \\ &- \int_0^t (\nabla U^\varepsilon)(X^\varepsilon(s))\,ds, \end{aligned} \tag{4.1}$$

where $X^\varepsilon(0)$ is an Ω^ε-valued random variable, W is an n-dimensional Wiener process, $\sigma \in C_b(\mathbb{R}^n; \mathbb{R}^n \otimes \mathbb{R}^n)$ and $b \in C_b(\mathbb{R}^n; \mathbb{R}^n)$. Let $a := \sigma\sigma^T$, and assume that a is uniformly positive definite. Define a second-order elliptic differential operator L on Ω^ε by

$$L := \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i}.$$

Then X^ε is associated with $(L - \langle\nabla U^\varepsilon, \nabla\rangle)$. Similarly to Section 3, it holds that X^ε does not exit from Ω^ε almost surely. Assume that $\sigma(V_\lambda) = \sigma_\lambda I_n$ for all $\lambda \in \Lambda$ where $\sigma_\lambda > 0$.

For $(\lambda, \lambda') \in \Xi$, define a second-order differential operator $\mathcal{L}_{\lambda,\lambda'}$ on $E_{\lambda,\lambda'}$ by

$$\begin{aligned}
&\mathcal{L}_{\lambda,\lambda'} f(x) \\
&\quad := \frac{1}{2}|\sigma(x)^T \dot{\gamma}_{\lambda,\lambda'} \circ \gamma_{\lambda,\lambda'}^{-1}(x)|^2 \frac{d^2}{ds^2}(f \circ \gamma_{\lambda,\lambda'})(\gamma_{\lambda,\lambda'}^{-1}(x)) \\
&\qquad + [\langle b(x), \dot{\gamma}_{\lambda,\lambda'} \circ \gamma_{\lambda,\lambda'}^{-1}(x)\rangle \\
&\qquad\quad + \langle \sigma(x)^T \ddot{\gamma}_{\lambda,\lambda'} \circ \gamma_{\lambda,\lambda'}^{-1}(x), \\
&\qquad\qquad \sigma(x)^T \dot{\gamma}_{\lambda,\lambda'} \circ \gamma_{\lambda,\lambda'}^{-1}(x)\rangle] \frac{d}{ds}(f \circ \gamma_{\lambda,\lambda'})(\gamma_{\lambda,\lambda'}^{-1}(x)),
\end{aligned}$$

for $x \in E_{\lambda,\lambda'}$ and $f \in C_b^2(E_{\lambda,\lambda'})$ where s is the parameter for the arc-length parametrization $\gamma_{\lambda,\lambda'}$. Let

$$p_{\lambda,\lambda'} := \frac{c_{\lambda,\lambda'}^{n-1} \int_0^1 r^{n-2} \exp(-u_{\lambda,\tilde{\lambda}}(r))\, dr}{\sum_{\tilde{\lambda} : \tilde{\lambda} \sim \lambda} c_{\lambda,\tilde{\lambda}}^{n-2} \int_0^1 r^{n-1} \exp(-u_{\lambda,\tilde{\lambda}}(r))\, dr}.$$

By using these notations, define the second-order differential operator $\mathcal{L}$ on $C_0(G)$ by

$$\begin{aligned}
\mathscr{D}(\mathcal{L}) := \Bigg\{ & f \in C_0(G) : f|_{E_{\lambda,\lambda'} \setminus \{V_\lambda, V_{\lambda'}\}} \in C_b^2(E_{\lambda,\lambda'} \setminus \{V_\lambda, V_{\lambda'}\}) \text{ for } \lambda \sim \lambda', \\
& \text{for } \lambda \in \Lambda, \lim_{s \downarrow 0} \mathcal{L}_{\lambda,\lambda'} f(\gamma_{\lambda,\lambda'}(s)) \text{ has a common value for } \lambda' : \lambda \sim \lambda', \\
& \sum_{\lambda' : \lambda' \sim \lambda} p_{\lambda,\lambda'} \lim_{s \downarrow 0} \left(\frac{d}{ds} (f \circ \gamma_{\lambda,\lambda'}(s)) \right) = 0 \text{ for } \lambda \in \Lambda \Bigg\},
\end{aligned}$$

$$\mathcal{L} f(x) := \mathcal{L}_{\lambda,\lambda'} f(x), \qquad x \in E_{\lambda,\lambda'}, (\lambda, \lambda') \in \Xi,$$

$$\mathcal{L} f(V_\lambda) := \lim_{x \to V_\lambda} \mathcal{L}_{\lambda,\lambda'} f(x), \qquad \lambda \in \Lambda,$$

where the limit $x \to V_\lambda$ is along $E_{\lambda,\lambda'}$. Note that $\mathcal{L} f(V_\lambda)$ does not depend on the selection of λ'.

Since by locality the behavior of diffusion processes associated with differential operators is determined in a given point by the behavior in its neighborhoods, we have the following theorem by Theorem 2.2 and 3.8.

THEOREM 4.1. *Consider the diffusion process X^ε defined by* (4.1). *Assume that the law of $X^\varepsilon(0)$ converges to a probability measure μ_0 on G. Then, $\{X^\varepsilon\}$ converge weakly on $C([0, +\infty); \mathbb{R}^n)$ to the diffusion process X as $\varepsilon \downarrow 0$, where X determined by the conditions that the law of $X(0)$ is equal to μ_0 and*

$$E\left[f(X(t)) - f(X(s)) - \int_s^t \mathcal{L} f(X(u))\, du \,\middle|\, \mathscr{F}_s \right] = 0$$

for $t \geq s \geq 0$ and all $f \in \mathscr{D}(\mathcal{L})$, where $(\mathscr{F}_t)$ is the filtration generated by X. The operator $\mathcal{L}$ as defined above is thus the generator of X.

Similarly as in Sections 2 and 3, our discussion is also available for the case where the boundary Ω^ε carries a Neumann boundary condition for the process. Consider a diffusion process $\widehat{X}^\varepsilon$ which is associated with L in Ω^ε and reflecting on $\partial\Omega^\varepsilon$ [defined similarly as the process described by (3.32)].

Let

$$\widehat{p}_{\lambda,\lambda'} := \frac{c_{\lambda,\lambda'}^{n-1}}{\sum_{\tilde{\lambda}:\tilde{\lambda}\sim\lambda} c_{\lambda,\tilde{\lambda}}^{n-1}},$$

$$\begin{aligned}\mathscr{D}(\widehat{\mathcal{L}}) := \Bigg\{ & f \in C_0(G) : f|_{E_{\lambda,\lambda'}\setminus\{V_\lambda,V_{\lambda'}\}} \in C_b^2(E_{\lambda,\lambda'} \setminus \{V_\lambda, V_{\lambda'}\}) \text{ for } \lambda \sim \lambda', \\ & \text{for } \lambda \in \Lambda, \lim_{s\downarrow 0} \mathcal{L}_{\lambda,\lambda'} f(\gamma_{\lambda,\lambda'}(s)) \text{ has a common value for } \lambda' : \lambda \sim \lambda', \\ & \sum_{\lambda':\lambda'\sim\lambda} \widehat{p}_{\lambda,\lambda'} \lim_{s\downarrow 0}\left(\frac{d}{ds}(f \circ \gamma_{\lambda,\lambda'}(s))\right) = 0 \text{ for } \lambda \in \Lambda \Bigg\},\end{aligned}$$

$$\widehat{\mathcal{L}} f(x) := \mathcal{L}_{\lambda,\lambda'} f(x), \qquad x \in E_{\lambda,\lambda'}, (\lambda,\lambda') \in \Xi,$$

$$\widehat{\mathcal{L}} f(V_\lambda) := \lim_{x\to V_\lambda} \mathcal{L}_{\lambda,\lambda'} f(x), \qquad \lambda \in \Lambda,$$

where the limit $x \to V_\lambda$ is along $E_{\lambda,\lambda'}$. Then, we obtain the following theorem.

THEOREM 4.2. *Consider the diffusion process $\widehat{X}^\varepsilon$ defined above. Assume that the law of $\widehat{X}^\varepsilon(0)$ converges to μ_0. Then, $\{\widehat{X}^\varepsilon\}$ converge weakly on $C([0,+\infty);\mathbb{R}^n)$ to the diffusion process $\widehat{X}$ as $\varepsilon \downarrow 0$, where $\widehat{X}$ is determined by the conditions that the law of $\widehat{X}(0)$ is equal to μ_0 and*

$$E\left[f(\widehat{X}(t)) - f(\widehat{X}(s)) - \int_s^t \widehat{\mathcal{L}} f(\widehat{X}(u))\,du \,\Big|\, \mathscr{F}_s \right] = 0$$

for $t \geq s \geq 0$ and $f \in \mathscr{D}(\widehat{\mathcal{L}})$ where $(\mathscr{F}_t)$ is the filtration generated by $\widehat{X}$. The operator $\widehat{\mathcal{L}}$ as defined above is thus the generator of $\widehat{X}$.

REMARK 4.3. As mentioned in Remarks 2.4 and 3.11, similar discussions can be given for the case where the shapes of the tubes are not cylindrical. In the case where $\sigma = I_n$, $b = 0$, and $E_{\lambda,\lambda'}$ are straight, the result of Theorem 4.2 coincides with Theorem 6.1 in [21].

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