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A NATURAL PARAMETRIZATION FOR THE SCHRAMM–LOEWNER EVOLUTION

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The Schramm–Loewner evolution (SLE_κ) is a candidate for the scaling limit of random curves arising in two-dimensional critical phenomena. When $\kappa < 8$, an instance of SLE_κ is a random planar curve with almost sure Hausdorff dimension $d = 1 + \kappa/8 < 2$. This curve is conventionally parametrized by its half plane capacity, rather than by any measure of its d-dimensional volume.

For $\kappa < 8$, we use a Doob–Meyer decomposition to construct the unique (under mild assumptions) Markovian parametrization of SLE_κ that transforms like a d-dimensional volume measure under conformal maps. We prove that this parametrization is nontrivial (i.e., the curve is not entirely traversed in zero time) for $\kappa < 4(7 - \sqrt{33}) = 5.021\ldots$.

1. Introduction.

1.1. *Overview.* A number of measures on paths or clusters on planar lattices arising from critical statistical mechanical models are believed to exhibit some kind of conformal invariance in the scaling limit. The Schramm–Loewner evolution [*SLE* (see Section 2.1 for a definition)] was created by Schramm [13] as a candidate for the scaling limit of these measures.

For each fixed $\kappa \in (0, 8)$, an instance γ of SLE_κ is a random planar curve with almost sure Hausdorff dimension $d = 1 + \kappa/8 \in (1, 2)$ [3]. This curve is conventionally parametrized by its half plane capacity (see Section 2.1), rather than by any measure of its d-dimensional volume. Modulo time parametrization, it has been shown that several discrete random paths on grids (e.g., loop-erased random walk [9], Ising interfaces [17], harmonic explorer [14], percolation interfaces [16], uniform spanning trees [9]) have *SLE* as a scaling limit. In these cases, one would expect the natural discrete parametrization (in which each edge is traversed in the same amount of time) of the lattice paths to scale to a continuum parametrization of *SLE*. The goal of this paper is to construct a candidate for this parametrization, a candidate which is (like *SLE* itself) completely characterized by its conformal invariance symmetries, continuity and Markov properties. We call this a *natural*

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parameterization because this term is very often used for arc length parametrization of smooth curves, and we believe our candidate is the *SLE* analogue of this.

When $\kappa \geq 8$ (and SLE_κ is almost surely space-filling) the natural candidate is the area parameter $\Theta_t := \text{Area}\,\gamma([0,t])$. One could use something similar for $\kappa < 8$ if one could show that some conventional measure of the d-dimensional volume of $\gamma([0,t])$ [e.g., the d-dimensional Minkowski content or some sort of Hausdorff content (see Section 2.2)] was well defined and nontrivial. In that case, one could replace $\text{Area}\,\gamma([0,t])$ with the d-dimensional volume of $\gamma([0,t])$. We will take a slightly different route. Instead of directly constructing a d-dimensional volume measure (using one of the classical definitions), we will simply assume that there exists a locally finite measure on γ that *transforms* like a d-dimensional volume measure under conformal maps and then use this assumption (together with an argument based on the Doob–Meyer decomposition) to deduce what the measure must be. We conjecture that the measure we construct is equivalent to the d-dimensional Minkowski content, but we will not prove this. Most of the really hard work in this paper takes place in Section 5, where certain second moment bounds are used to prove that the measure one obtains from the Doob–Meyer decomposition is nontrivial (in particular, that it is almost surely not identically zero). At present, we are only able to prove this for $\kappa < 4(7 - \sqrt{33}) = 5.021\ldots$.

We mention that a variant of our approach, due to Alberts and the second coauthor of this work, appears in [1], which gives, for $\kappa \in (4,8)$, a natural *local* time parameter for the intersection of an SLE_κ curve with the boundary of the domain it is defined on. The proofs in [1] cite and utilize the Doob–Meyer-based techniques first developed for this paper; however, the second moment arguments in [1] are very different from the ones appearing in Section 5 of this work. It is possible that our techniques will have other applications. In particular, it would be interesting to see whether natural d-dimensional volume measures for other random d-dimensional sets with conformal invariance properties (such as conformal gaskets [15] or the intersection of an $SLE_{\kappa,\rho}$ with its boundary) can be constructed using similar tools. In each of these cases, we expect that obtaining precise second moment bounds will be the most difficult step.

A precise statement of our main results will appear in Section 3. In the meantime, we present some additional motivation and definitions.

Although discrete models, in particular, the self-avoiding walk, motivate our construction, we do not prove any results about discrete parametrizations in this paper. If a discrete model on a compact domain converges to *SLE* on that domain, then the discrete natural measure on the discrete curves (normalized to have median length one, say) will converge at least subsequentially to a random measure on the *SLE* curve. However, it would take some work to prove that this limiting measure is nontrivial (and not almost surely either an infinite measure or a zero measure, say) and that it satisfies the scaling axioms required by our main result. This is an interesting topic for future research.

1.2. *Self-avoiding walks*: *Heuristics and motivation.* In order to further explain and motivate our main results, we include a heuristic discussion of a single concrete example: the self-avoiding walk (SAW). We will not be very precise here; in fact, what we say here about SAWs is still only conjectural. All of the conjectural statements in this section can be viewed as consequences of the "conformal invariance Ansatz" that is generally accepted (often without a precise formulation) in the physics literature on conformal field theory. Let $D \subset \mathbb{C}$ be a simply connected bounded domain, and let z, w be distinct points on ∂D. Suppose that a lattice $\varepsilon\mathbb{Z}^2$ is placed on D, and let $\tilde{z}, \tilde{w} \in D$ be lattice points in $\varepsilon\mathbb{Z}^2$ "closest" to z, w. A SAW ω from $\tilde{z}$ to $\tilde{w}$ is a sequence of distinct points

$$\tilde{z} = \omega_0, \omega_1, \dots, \omega_k = \tilde{w},$$

with $\omega_j \in \varepsilon\mathbb{Z}^2 \cap D$ and $|\omega_j - \omega_{j-1}| = \varepsilon$ for $1 \le j \le k$. We write $|\omega| = k$. For each $\beta > 0$, we can consider the measure on SAWs from $\tilde{z}$ to $\tilde{w}$ in D that gives measure $e^{-\beta|\omega|}$, to each such SAW. There is a critical β_0, such that the partition function,

$$\sum_{\omega : \tilde{z} \to \tilde{w}, \omega \subset D} e^{-\beta_0 |\omega|},$$

neither grows nor decays exponentially as a function of ε as $\varepsilon \to 0$. It is believed that if we choose this β_0, and normalize so that this is a probability measure, then there is a limiting measure on paths that is the scaling limit.

It is further believed that the typical number of steps of a SAW in the measure above is of order ε^{-d} where the exponent $d = 4/3$ can be considered the fractal dimension of the paths. For fixed ε, let us define the scaled function

$$\hat{\omega}(j\varepsilon^d) = \omega_j, \qquad j = 0, 1, \dots, |\omega|.$$

We use linear interpolation to make this a continuous path $\hat{\omega} : [0, \varepsilon^d |\omega|] \to \mathbb{C}$. Then one expects that the following is true:

- As $\varepsilon \to 0$, the above probability measure on paths converges to a probability measure $\mu_D^\#(z, w)$ supported on continuous curves $\gamma : [0, t_\gamma] \to \mathbb{C}$ with $\gamma(0) = z, \gamma(t_\gamma) = w, \gamma(0, t_\gamma) \subset D$.
- The probability measures $\mu_D^\#(z, w)$ are conformally invariant. To be more precise, suppose $F : D \to D'$ is a conformal transformation that extends to ∂D at least in neighborhoods of z and w. For each γ in D connecting z and w, we will define a conformally transformed path $F \circ \gamma$ (with a parametrization described below) on D'. We then denote by $F \circ \mu_D^\#(z, w)$ the push-forward of the measure $\mu_D^\#(z, w)$ via the map $\gamma \to F \circ \gamma$. The conformal invariance assumption is

$$F \circ \mu_D^\#(z, w) = \mu_{D'}^\#(F(z), F(w)). \tag{1.1}$$

Let us now define $F \circ \gamma$. The path $F \circ \gamma$ will traverse the points $F(\gamma(t))$ in order; the only question is how "quickly" does the curve traverse these points. If we look at how the scaling limit is defined, we can see that if $F(z) = rz$ for some

$r > 0$, then the lattice spacing ε on D corresponds to lattice space $r\varepsilon$ on $F(D)$, and hence we would expect the time to traverse $r\gamma$ should be r^d times the time to traverse γ. Using this as a guide locally, we say that the amount of time needed to traverse $F(\gamma[t_1, t_2])$ is

$$\int_{t_1}^{t_2} |F'(\gamma(s))|^d \, ds. \tag{1.2}$$

This tells us how to parametrize $F \circ \gamma$, and we include this as part of the definition of $F \circ \gamma$. This is analogous to the known conformal invariance of Brownian motion in $\mathbb{C}$ where the time parametrization must be defined as in (1.2) with $d = 2$.

If there is to be a family of probability measures $\mu^\#_D(z, w)$ satisfying (1.1) for simply connected D, then we only need to define $\mu^\#_{\mathbb{H}}(0, \infty)$, where $\mathbb{H}$ is the upper half plane. To restrict the set of possible definitions, we introduce another property that one would expect the scaling limit of SAW to satisfy. The *domain Markov property* states that if t is a stopping time for the random path γ, then given $\gamma([0, t])$, the conditional law of the remaining path $\gamma'(s) := \gamma(t + s)$ (defined for $s \in [0, \infty)$) is

$$\mu^\#_{\mathbb{H}\setminus\gamma([0,t])}(\gamma(t), \infty),$$

independent of the parametrization of $\gamma([0, t])$.

If we consider γ and $F \circ \gamma$ as being defined only up to reparametrization, then Schramm's theorem states that (1.1) (here being considered as a statement about measures on paths defined up to reparametrization) and the domain Markov property (again interpreted up to reparametrization) characterize the path as being a chordal SLE_κ for some $\kappa > 0$. (In the case of the self-avoiding walk, another property called the "restriction property" tells us that we must have $\kappa = 8/3$ [8, 10].) Recall that if $\kappa \in (0, 8)$, Beffara's theorem (partially proved in [12] and completed in [3]) states that the Hausdorff dimension of SLE_κ is almost surely $d = 1 + \kappa/8$.

The main purpose of this paper is to remove the "up to reparametrization" from the above characterization. Roughly speaking, we will show that the conformal invariance assumption (1.1) and the domain Markov property uniquely characterize the law of the random parametrized path as being an SLE_κ with a *particular* parametrization that we will construct in this paper. We may interpret this parametrization as giving a d-dimensional volume measure on γ, which is uniquely defined up to a multiplicative constant. As mentioned in Section 1.1, one major caveat is that, due to limitations of certain second moment estimates we need, we are currently only able to prove that this measure is nontrivial (i.e., not identically zero) for $\kappa < 4(7 - \sqrt{33}) = 5.021\ldots$, although we expect this to be the case for all $\kappa < 8$.

2. *SLE* definition and limit constructions.

2.1. *Schramm–Loewner evolution* (*SLE*). We now provide a quick review of the definition of the Schramm–Loewner evolution (see [7], especially Chapters 6 and 7, for more details). We will discuss only chordal *SLE* in this paper, and we will call it just *SLE*.

Suppose that $\gamma:(0,\infty)\to\mathbb{H}=\{x+iy:y>0\}$ is a noncrossing curve with $\gamma(0+)\in\mathbb{R}$ and $\gamma(t)\to\infty$ as $t\to\infty$. Let H_t be the unbounded component of $\mathbb{H}\setminus\gamma(0,t]$. Using the Riemann mapping theorem, one can see that there is a unique conformal transformation

$$g_t: H_t \longrightarrow \mathbb{H}$$

satisfying $g_t(z)-z\to 0$ as $z\to\infty$. It has an expansion at infinity

$$g_t(z)=z+\frac{a(t)}{z}+O(|z|^{-2}).$$

The coefficient $a(t)$ equals $\operatorname{hcap}(\gamma(0,t])$ where $\operatorname{hcap}(A)$ denotes the half plane capacity from infinity of a bounded set A. There are a number of ways of defining hcap, for example,

$$\operatorname{hcap}(A)=\lim_{y\to\infty} y\mathbb{E}^{iy}[\operatorname{Im}(B_\tau)],$$

where B is a complex Brownian motion and $\tau=\inf\{t:B_t\in\mathbb{R}\cup A\}$.

DEFINITION. The *Schramm–Loewner evolution*, *SLE*$_\kappa$, (from 0 to infinity in $\mathbb{H}$) is the random curve $\gamma(t)$ such that g_t satisfies

$$\dot g_t(z)=\frac{a}{g_t(z)-V_t},\qquad g_0(z)=z, \tag{2.1}$$

where $a=2/\kappa$ and $V_t=-B_t$ is a standard Brownian motion.

Showing that the conformal maps g_t are well defined is easy. In fact, for given $z\in\mathbb{H}$, $g_t(z)$ is defined up to time $T_z=\sup\{t:\operatorname{Im} g_t(z)>0\}$. Also, g_t is the unique conformal transformation of $H_t=\{z\in\mathbb{H}:T_z>t\}$ onto $\mathbb{H}$ satisfying $g_t(z)-z\to 0$ as $z\to\infty$. It is not as easy to show that H_t is given by the unbounded component of $\mathbb{H}\setminus\gamma(0,t]$ for a curve γ. However, this was shown for $\kappa\neq 8$ by Rohde and Schramm [12]. If $\kappa\le 4$, the curve is simple and $\gamma(0,\infty)\subset\mathbb{H}$. If $\kappa>4$, the curve has double points and $\gamma(0,\infty)\cap\mathbb{R}\neq\varnothing$. For $\kappa\ge 8$, $\gamma(0,\infty)$ is plane filling; we will restrict our consideration to $\kappa<8$.

REMARK. We have defined chordal *SLE*$_\kappa$ so that it is *parametrized by capacity* with

$$\operatorname{hcap}(\gamma(0,t])=at.$$

It is more often defined with the capacity parametrization chosen so that $\operatorname{hcap}(\gamma[0,t])=2t$. In this case we need to choose $U_t=-\sqrt{\kappa}B_t$. We will choose the parametrization in (2.1), but this is only for our convenience. Under our parametrization, if $z\in\overline{\mathbb{H}}\setminus\{0\}$, then $Z_t=Z_t(z):=g_t(z)-U_t$ satisfies the Bessel equation,

$$dZ_t=\frac{a}{Z_t}\,dt+dB_t.$$

In this paper we will use both κ and a as notation; throughout, $a = 2/\kappa$.

We let

$$f_t = g_t^{-1}, \qquad \hat{f}_t(z) = f_t(z + U_t).$$

We recall the following scaling relation [7], Proposition 6.5.

LEMMA 2.1 (Scaling). *If $r > 0$, then the distribution of $g_{tr^2}(rz)/r$ is the same as that of $g_t(z)$; in particular, $g'_{tr^2}(rz)$ has the same distribution as $g'_t(z)$.*

For $\kappa < 8$, we let

$$d = 1 + \frac{\kappa}{8} = 1 + \frac{1}{4a}. \tag{2.2}$$

If $z \in \mathbb{C}$ we will write x_z, y_z for the real and imaginary parts of $z = x_z + iy_z$ and θ_z for the argument of z. Let

$$G(z) := y_z^{d-2}[(x_z/y_z)^2 + 1]^{1/2-2a} = |z|^{d-2} \sin^{\kappa/8+8/\kappa-2} \theta_z, \tag{2.3}$$

denote the "Green's function" for SLE_κ in $\mathbb{H}$. The value of d and the function G were first found in [12] and are characterized by the scaling rule $G(rz) = r^{d-2}G(z)$ and the fact that

$$M_t(z) := |g'_t(z)|^{2-d} G(Z_t(z)) \tag{2.4}$$

is a local martingale. In fact, for a given κ, the scaling rule $G(rz) = r^{d-2}G(z)$ and the requirement that (2.4) is a local martingale uniquely determines d and (up to a multiplicative constant) G. Note that if $K < \infty$,

$$\int_{|z| \le K} G(z)\, dA(z) = K^d \int_{|z| \le 1} G(z)\, dA(z) < \infty. \tag{2.5}$$

Here, and throughout this paper, we use dA to denote integration with respect to area. The Green's function will turn out to describe the expectation of the measure we intend to construct in later sections, as suggested by the following proposition.

PROPOSITION 2.2. *Suppose that there exists a parametrization for SLE_κ in $\mathbb{H}$ satisfying the domain Markov property and the conformal invariance assumption (1.1). For a fixed Lebesgue measurable subset $S \subset \mathbb{H}$, let $\Theta_t(S)$ denote the process that gives the amount of time in this parametrization spent in S before time t (in the half-plane capacity parametrization given above), and suppose further that $\Theta_t(S)$ is $\mathcal{F}_t$ adapted for all such S. If $\mathbb{E}\Theta_\infty(D)$ is finite for all bounded domains D, then it must be the case that (up to multiplicative constant)*

$$\mathbb{E}\Theta_\infty(D) = \int_D G(z)\, dA(z),$$

and more generally,

$$\mathbb{E}[\Theta_\infty(D) - \Theta_t(D) | \mathcal{F}_t] = \int_D M_t(z)\, dA(z).$$

PROOF. It is immediate from the conformal invariance assumption (which in particular implies scale invariance) that the measure ν defined by $\nu(\cdot) = \mathbb{E}\Theta_\infty(\cdot)$ satisfies $\nu(r\cdot) = r^d \nu(\cdot)$ for each fixed $r > 0$. To prove the proposition, it is enough to show that $\nu(\cdot) = \int_\cdot G(z)\, dA(z)$ (up to a constant factor), since the conditional statement at the end of the proposition then follows from the domain Markov property and conformal invariance assumptions.

The first observation to make is that ν is absolutely continuous with respect to Lebesgue measure, with a smooth Radon–Nikodym derivative. To see this, suppose that S is bounded away from the real axis, so that there exists a $t > 0$ such that almost surely no point in S is swallowed before time t. Then the conformal invariance assumption (1.1) and the domain Markov property imply that

$$\nu(S) = \mathbb{E} \int_{g_t(S)} |(g_t^{-1})'(z)|^{-d}\, d\nu(z).$$

The desired smoothness can be then deduced from the fact that the law of the pair $g_t(z), g_t'(z)$ has a smooth Radon–Nikodym derivative that varies smoothly with z (which follows from the Loewner equation and properties of Brownian motion). Recalling the scale invariance, we conclude that ν has the form

$$|z|^{d-2} F(\theta_z)\, dA(z)$$

for some smooth function F. Standard Itô calculus and the fact that M_t is a local martingale determine F up to a constant factor, implying that $F(z)|z|^{d-2} = G(z)$ (up to a constant factor). $\square$

REMARK. It is not clear whether it is necessary to assume in the statement of Proposition 2.2 that $\mathbb{E}\Theta_\infty(D) < \infty$ for bounded domains D. It is possible that if one had $\mathbb{E}\Theta_\infty(D) = \infty$ for some bounded D, then one could use some scaling arguments and the law of large numbers to show that in fact $\Theta_\infty(D) = \infty$ almost surely for domains D intersected by the path γ. If this is the case, then the assumption $\mathbb{E}\Theta_\infty(D) < \infty$ can be replaced by the weaker assumption that $\Theta_\infty(D) < \infty$ almost surely.

2.2. *Attempting to construct the parametrization as a limit.* As we mentioned in the Introduction, the parametrization we will construct in Section 3 is uniquely determined by certain conformal invariance assumptions and the domain Markov property. Leaving this fact aside, one could also motivate our definition by noting its similarity and close relationship to some of the other obvious candidates for a d-dimensional volume measure on an arc of an SLE_κ curve.

In this section, we will describe two of the most natural candidates: Minkowski measure and d-variation. While we are not able to prove that either of these candidates is well defined, we will point out that both of these candidates have variants that are more or less equivalent to the measure we will construct in Section 3. In each case we will define approximate parametrizations $\tau_n(t)$ and propose that a natural parametrization τ could be given by

$$\tau(t) = \lim_{n\to\infty} \tau_n(t),$$

if one could show that this limit (in some sense) exists and is nontrivial.

To motivate these constructions, we begin by assuming that any candidate for the natural parametrization of a d-dimensional object induces a "d-dimensional" measure on the path and hence satisfies an appropriate scaling relationship. In particular if $\gamma(t)$ is an SLE_κ curve that is parametrized so that $\mathrm{hcap}[\gamma(0,t]] = at$, then $\tilde{\gamma}(t) = r\gamma(t)$ is an SLE_κ curve parametrized so that $\mathrm{hcap}[\gamma(0,t]] = r^2 at$. If it takes time $\tau(t)$ to traverse $\gamma(0,t]$ in the natural parametrization, then it should take time $r^d\tau(t)$ to traverse $\tilde{\gamma}(0,t]$ in the natural parametrization. In particular, it should take roughly time $O(R^d)$ in the natural parametrization for the path to travel distance R.

2.2.1. *Minkowski content.* Let

$$\mathcal{N}_{t,\varepsilon} = \{z \in \mathbb{H} : \mathrm{dist}(z, \gamma(0,t]) \le \varepsilon\},$$
$$\tau_n(t) = n^{2-d}\,\mathrm{area}(\mathcal{N}_{t,1/n}).$$

We call the limit $\tau(t) = \lim_{n\to\infty} \tau_n(t)$, if it exists, the Minkowski content of $\gamma(0,t]$. Using the local martingale (2.4) one can show that as $\varepsilon \to 0+$,

$$\mathbf{P}\{z \in \mathcal{N}_{\infty,\varepsilon}\} \asymp G(z)\varepsilon^{2-d}. \tag{2.6}$$

We remark that a commonly employed alternative to Minkowski content is the d-dimensional Hausdorff content; the Hausdorff content of a set $X \subset D$ is defined to be the limit as $\varepsilon \to 0$ of the infimum—over all coverings of X by balls with some radii $\varepsilon_1, \varepsilon_2, \dots < \varepsilon$—of $\sum \Phi(\varepsilon_i)$ where $\Phi(x) = x^d$. We have at least some intuition, however, to suggest that the Hausdorff content of $\gamma([0,t])$ will be almost surely zero for all t. Even if this is the case, it may be that the Hausdorff content is nontrivial when Φ is replaced by another function [e.g., $\Phi(x) = x^d \log\log x$], in which case we would expect it to be equivalent, up to constant, to the d-dimensional Minkowski measure.

2.2.2. *Conformal Minkowski content.* Here is a variant of the Minkowski content that could be called the *conformal Minkowski content*. Let g_t be the conformal maps as above. If $t < T_z$, let

$$\Upsilon_t(z) = \frac{\mathrm{Im}[g_t(z)]}{|g_t'(z)|}.$$

We will call $\Upsilon_t(z)$ the *conformal radius* (*of H_t about z*) although it is $1/2$ times the usual definition. In other words, if $F : \mathbb{D} \to H_t$ is a conformal transformation with $F(0) = z$, then $|F'(0)| = 2\Upsilon_t(z)$. Using the Schwarz lemma or by doing a simple calculation, we can see that $\Upsilon_t(z)$ decreases in t and hence we can define

$$\Upsilon_t(z) = \Upsilon_{T_z -}(z), \qquad t \geq T_z.$$

Similarly, $\Upsilon(z) = \Upsilon_\infty(z)$ is well defined; this is the conformal radius with respect to z of the domain $D(\infty, z)$. The Koebe 1/4-theorem implies that $\Upsilon_t(z) \asymp \mathrm{dist}[z, \gamma(0,t] \cup \mathbb{R}]$; in fact, each side is bounded above by four times the other side. To prove (2.6) one can show that there is a c_* such that

$$\mathbf{P}\{\Upsilon(z) \leq \varepsilon\} \sim c_* G(z) \varepsilon^{2-d}, \qquad \varepsilon \to 0+.$$

This was first established in [7] building on the argument in [12]. The conformal Minkowski content is defined as in the previous paragraph replacing $\mathcal{N}_{t,\varepsilon}$ with

$$\mathcal{N}^*_{t,\varepsilon} = \{z \in \mathbb{H} : \Upsilon_t(z) \leq \varepsilon\}.$$

It is possible that this limit will be easier to establish. Assuming the limit exists, we can see that the expected amount of time (using the natural parametrization) that $\gamma(0, \infty)$ spends in a bounded domain D should be given (up to multiplicative constant) by

$$\int_D G(z)\, dA(z), \tag{2.7}$$

where A denotes area. This formula agrees with Proposition 2.2 and will be the starting point for our construction of the natural parametrization in Section 3.

2.2.3. *d-variation.* The idea that it should take roughly time R^d for the path to move distance R—and thus $\tau(t_2) - \tau(t_1)$ should be approximately $|\gamma(t_2) - \gamma(t_1)|^d$—motivates the following definition. Let

$$\tau_n(t) = \sum_{k=1}^{\lfloor tn \rfloor} \left| \gamma\left(\frac{k}{n}\right) - \gamma\left(\frac{k-1}{n}\right)\right|^d.$$

More generally, we can consider

$$\tau_n(t) = \sum_{t_{j-1,n} < t} |\gamma(t_{j,n}), -\gamma(t_{j-1,n})|^d,$$

where $t_{0,n} < t_{1,n} < t_{2,n} < \infty$ is a partition, depending on n, whose mesh goes to zero as $n \to \infty$, and as usual $d = 1 + \kappa/8$. It is natural to expect that for a wide class of partitions this limit exists and is independent of the choice of partitions. In the case $\kappa = 8/3$, a version of this was studied numerically by Kennedy [5].

2.2.4. *A variant of d-variation.* We next propose a variant of the d-variation in which an expression involving derivatives of $\hat{f}'$ (as defined in Section 2.1) takes the place of $|\gamma(t_2) - \gamma(t_1)|^d$. Suppose $\tau(t)$ were the natural parametrization. Since $\tau(1) < \infty$, we would expect that the average value of

$$\Delta_n \tau(j) := \tau\left(\frac{j+1}{n}\right) - \tau\left(\frac{j}{n}\right)$$

would be of order $1/n$ for typical $j \in \{1, 2, \dots, n\}$. Consider

$$\gamma^{(j/n)}\left[0, \frac{1}{n}\right] := g_{j/n}\left(\gamma\left[\frac{j}{n}, \frac{j+1}{n}\right]\right).$$

Since the hcap of this set is a/n, we expect that the diameter of the set is of order $1/\sqrt{n}$. Using the scaling properties, we guess that the time needed to traverse $\gamma^{(j/n)}[0, \frac{1}{n}]$ in the natural parametrization is of order $n^{-d/2}$. Using the scaling properties again, we guess that

$$\Delta_n \tau(j) \approx n^{-d/2} \left|\hat{f}'_{j/n}(i/\sqrt{n})\right|^d.$$

This leads us to define

$$\tau_n(t) = \sum_{k=1}^{\lfloor tn \rfloor} n^{-d/2} \left|\hat{f}'_{k/n}(i/\sqrt{n})\right|^d. \tag{2.8}$$

More generally, we could let

$$\tau_n(t) = \sum_{k=1}^{\lfloor tn \rfloor} n^{-d/2} \int_{\mathbb{H}} \left|\hat{f}'_{k/n}(z/\sqrt{n})\right|^d \nu(dz), \tag{2.9}$$

where ν is a finite measure on $\mathbb{H}$. It will turn out that the parametrization we construct in Section 3 can be realized as a limit of this form with a particular choice of ν. We expect that (up to a constant factor) this limit is independent of ν, but we will not prove this.

3. Natural parametrization.

3.1. *Notation.* We now summarize some of the key notation we will use throughout the paper. For $z \in \mathbb{H}$, we write

$$Z_t(z) = X_t(z) + iY_t(z) = g_t(z) - V_t,$$

$$R_t(z) = \frac{X_t(z)}{Y_t(z)}, \qquad \Upsilon_t(z) = \frac{Y_t(z)}{|g_t'(z)|},$$

$$M_t(z) = \Upsilon_t(z)^{d-2}\left(R_t(z)^2 + 1\right)^{1/2-2a} = |g_t'(z)|^{2-d} G(Z_t(z)).$$

At times we will write just $Z_t, X_t, Y_t, R_t, \Upsilon_t, M_t$, but it is important to remember that these quantities depend on z.

3.2. *Definition*. We will now give a precise definition of the natural time parametrization. It will be easier to restrict our attention to the time spent in a fixed domain bounded away from the real line. Let $\mathcal{D}$ denote the set of bounded domains $D \subset \mathbb{H}$ with $\operatorname{dist}(\mathbb{R}, D) > 0$. We write

$$\mathcal{D} = \bigcup_{m=1}^{\infty} \mathcal{D}_m,$$

where $\mathcal{D}_m$ denotes the set of domains D with

$$D \subset \{x + iy : |x| < m, 1/m < y < m\}.$$

Suppose for the moment that $\Theta_t(D)$ denotes the amount of time in the natural parametrization that the curve spends in the domain D. This is *not* defined at the moment so we are being heuristic. Using (2.7) or Proposition 2.2 we expect (up to a multiplicative constant that we set equal to one)

$$\mathbb{E}[\Theta_\infty(D)] = \int_D G(z)\, dA(z).$$

In particular, this expectation is finite for bounded D.

Let $\mathcal{F}_t$ denote the σ-algebra generated by $\{V_s : s \le t\}$. For any process Θ_t with finite expectations, we would expect that

$$\mathbb{E}[\Theta_\infty(D)|\mathcal{F}_t] = \Theta_t(D) + \mathbb{E}[\Theta_\infty(D) - \Theta_t(D)|\mathcal{F}_t].$$

If $z \in D$, with $t < T_z$, then the Markov property for *SLE* can be used to see that the conditional distribution of $\Upsilon(z)$ given $\mathcal{F}_t$ is the same as the distribution of $|g_t'(z)|^{-1}\Upsilon^*$ where Υ^* is independent of $\mathcal{F}_t$ with the distribution of $\Upsilon(Z_t(z))$. This gives us another heuristic way of deriving the formula in Proposition 2.2

$$\begin{aligned}\lim_{\delta\to 0+} \delta^{d-2}\mathbf{P}\{\Upsilon(z) < \delta|\mathcal{F}_t\} &= \lim_{\delta\to 0+} \delta^{d-2}\mathbf{P}\{\Upsilon^* \le \delta|g_t'(z)|\}\\ &= c_*|g_t'(z)|^{2-d}G(Z_t(z)) = c_* M_t(z).\end{aligned}$$

We therefore see that

$$\mathbb{E}[\Theta_\infty(D) - \Theta_t(D)|\mathcal{F}_t] = \Psi_t(D),$$

where

$$\Psi_t(D) = \int_D M_t(z) 1\{T_z > t\}\, dA(z).$$

We now use the conclusion of Proposition 2.2 to give a precise definition for $\Theta_t(D)$. The expectation formula from this proposition is

$$\Psi_t(D) = \mathbb{E}[\Theta_\infty(D)|\mathcal{F}_t] - \Theta_t(D). \tag{3.1}$$

The left-hand side is clearly supermartingale in t [since it is a weighted average of the $M_t(x)$, which are nonnegative local martingales and hence supermartingales].

It is reasonable to expect (though we have not proved this) that $\Psi_t(D)$ is in fact continuous as a function of D. Assuming the conclusion of Proposition 2.2, the first term on the right-hand side is a martingale, and the map $t \mapsto \Theta_t(D)$ is increasing. The reader may recall the continuous case of the standard Doob–Meyer theorem [4]: any continuous supermartingale can be written uniquely as the sum of a continuous adapted decreasing process with initial value zero and a continuous local martingale. If $\Psi_t(D)$ is a continuous supermartingale, it then follows that (3.1) is its Doob–Meyer decomposition. Since we have a formula for $\Psi_t(D)$, we could [if we knew $\Psi_t(D)$ was continuous] simply *define* $\Theta_t(D)$ to be the unique continuous, increasing, adapted process such that

$$\Theta_t(D) + \Psi_t(D)$$

is a local martingale.

Even when it is not known that $\Psi_t(D)$ is continuous, there is a canonical Doob–Meyer decomposition that we could use to define $\Theta_t(D)$, although the details are more complicated (see [4]). Rather than focus on these issues, what we will aim to prove in this paper is that there exists an adapted continuous decreasing $\Theta_t(D)$ for which $\Theta_t(D) + \Psi_t(D)$ is a *martingale*. If such a process exists, it is obviously unique, since if there were another such process $\tilde{\Theta}_t(D)$, then $\Theta_t(D) - \tilde{\Theta}_t(D)$ would be a continuous martingale with paths of bounded variation and hence identically zero. One consequence of having $\Theta_t(D) + \Psi_t(D)$ be a martingale (as opposed to merely a local martingale) is that $\Theta_t(D)$ is not identically zero; this is because $\Psi_t(D)$ is a strict supermartingale (i.e., not a martingale), since it is an average of processes $M_t(x)$ which are strict supermartingales (i.e., not martingales). Another reason for wanting $\Theta_t(D) + \Psi_t(D)$ to be a martingale is that this will imply that Θ_t (defined below) actually satisfies the hypotheses Proposition 2.2, and (by Proposition 2.2) is the unique process that does so. Showing the existence of an adapted continuous increasing $\Theta_t(D)$ that makes $\Theta_t(D) + \Psi_t(D)$ a martingale takes work. We conjecture that this is true for all $\kappa < 8$; in this paper we prove it for

$$\kappa < \kappa_0 := 4(7 - \sqrt{33}) = 5.021\dots. \tag{3.2}$$

DEFINITION.

- If $D \in \mathcal{D}$, then the natural parametrization $\Theta_t(D)$ is the unique continuous, increasing process such that

  $$\Psi_t(D) + \Theta_t(D)$$

  is a martingale (assuming such a process exists).
- If $\Theta_t(D)$ exists for each $D \in \mathcal{D}$, we define

  $$\Theta_t = \lim_{m \to \infty} \Theta_t(D_m),$$

  where $D_m = \{x + iy : |x| < m, 1/m < y < m\}$.

The statement of the main theorem includes a function ϕ related to the Loewner flow that is defined later in (3.14). Roughly speaking, we think of ϕ as

$$\phi(z) = \mathbf{P}\{z \in \gamma(0,1]|z \in \gamma(0,\infty)\}.$$

This equation as written does not make sense because we are conditioning on an event of probability zero. To be precise it is defined by

$$\mathbb{E}[M_1(z)] = M_0(z)[1-\phi(z)]. \tag{3.3}$$

Note that the conclusion of Proposition 2.2 and our definition of Θ imply that

$$\mathbb{E}\Theta_1(D) = \int_D \phi(z)G(z)\,dA(z).$$

This a point worth highlighting: the hypotheses of Proposition 2.2 determine not only the form of $\mathbb{E}[\Theta_\infty(D)]$ (up to multiplicative constant) but also $\mathbb{E}[\Theta_1(D)]$ and (by scaling) $\mathbb{E}[\Theta_t(D)]$ for general t.

In the theorem below, note that (3.5) is of the form (2.9) where $\nu(dz) = \phi(z)G(z)\,dA(z)$. Let κ_0 be as in (3.2), and let $a_0 = 2/\kappa_0$. Note that

$$\frac{16}{\kappa} + \frac{\kappa}{16} > \frac{7}{2}, \qquad 0 < \kappa < \kappa_0. \tag{3.4}$$

We will need this estimate later which puts the restriction on κ.

THEOREM 3.1.

- *For $\kappa < 8$ that are good in the sense of (3.23) and all $D \in \mathcal{D}$, there is an adapted, increasing, continuous process $\Theta_t(D)$ with $\Theta_0(D) = 0$ such that*

$$\Psi_t(D) + \Theta_t(D)$$

  is a martingale. Moreover, with probability one for all t

$$\begin{aligned}\Theta_t(D) = \lim_{n\to\infty} \sum_{j \le t2^n} \int_{\mathbb{H}} &|\hat{f}'_{(j-1)/2^n}(z)|^d \phi(z2^{n/2}) G(z) \\ &\times 1\{\hat{f}_{(j-1)/2^n}(z) \in D\}\,dA(z),\end{aligned} \tag{3.5}$$

  where ϕ is defined in (3.3).
- *If $\kappa < \kappa_0$, then κ is good.*

REMARK. The hypotheses and conclusion of Proposition 2.2 would imply that the summands in (3.5) are equal to the conditional expectations

$$\mathbb{E}[\Theta_{j2^{-n}}(D) - \Theta_{(j-1)2^{-n}}(D)|\mathcal{F}_{(j-1)2^{-n}}].$$

THEOREM 3.2. *For all $\kappa < 8$ and all $t < \infty$,*

$$\lim_{m\to\infty} \mathbb{E}[\Theta_t(D_m)] < \infty. \tag{3.6}$$

In particular, if $\kappa < 8$ is good, then Θ_t is a continuous process.

SKETCH OF PROOFS. The remainder of this paper is dedicated to proving these theorems. For Theorem 3.1, we start by discretizing time and finding an approximation for $\Theta_t(D)$. This is done in Sections 3.3 and 3.4 and leads to the sum in (3.5). This time discretization is the first step in proving the Doob–Meyer decomposition for any supermartingale. The difficult step comes in taking the limit. For general supermartingales, this is subtle and one can only take a weak limit (see [11]). However, if there are uniform second moment estimates for the approximations, one can take a limit both in L^2 and with probability one. We state the estimate that we will use in (3.23), and we call κ good if such an estimate exists. For completeness, we give a proof of the convergence in Section 4 assuming this bound; this section is similar to a proof of the Doob–Meyer decomposition for L^2 martingales in [2]. Hölder continuity of the paths follows. The hardest part is proving (3.23) and this is done in Section 5. Two arguments are given: one easier proof that works for $\kappa < 4$ and a more complicated argument that works for $\kappa < \kappa_0$. We conjecture that all $\kappa < 8$ are good. In this section we also establish (3.6) for all $\kappa < 8$ (see Theorem 5.1). Since $t \mapsto \Theta_t - \Theta_t(D_m)$ is increasing in t, and $\Theta_t(D_m)$ is continuous in t for good κ, the final assertion in Theorem 3.2 follows immediately.

Before proceeding, let us derive some simple scaling relations. It is well known that if g_t are the conformal maps for SLE_κ and $r > 0$, then $\tilde{g}_t(z) := r^{-1} g_{tr^2}(rz)$ has the same distribution as g_t. In fact, it is the solution of the Loewner equation with driving function $\tilde{V}_t = r^{-1} V_{r^2 t}$. The corresponding local martingale is

$$\begin{aligned}\tilde{M}_t(z) &= |\tilde{g}_t'(z)|^{2-d} G(\tilde{g}_t(z) - \tilde{V}_t) = |g'_{tr^2}(rz)|^{2-d} G(r^{-1} Z_t(z)) \\ &= r^{2-d} M_{r^2 t}(rz), \\ \tilde{\Psi}_t(D) &=: \int_D \tilde{M}_t(z)\, dA(z) = r^{2-d} \int_D M_{r^2 t}(rz)\, dA(z) = r^{-d} \Psi_{r^2 t}(rD).\end{aligned}$$

Hence, if $\Psi_t(rD) + \Theta_t(rD)$ is a local martingale, then so is $\tilde{\Psi}_t(D) + \tilde{\Theta}_t(D)$, where

$$\tilde{\Theta}_t(D) = r^{-d} \Theta_{r^2 t}(rD).$$

This scaling rule implies that it suffices to prove that $\Theta_t(D)$ exists for $0 \le t \le 1$.

3.3. *The forward-time local martingale.* The process $\Psi_t(D)$ is defined in terms of the family of local martingales $M_t(z)$ indexed by starting points $z \in \mathbb{H}$. If $z \notin \gamma(0, t]$, then $M_t(z)$ has a heuristic interpretation as the (appropriately normalized limit of the) probability that $z \in \gamma[t, \infty)$ given $\gamma(0, t]$.

Let $Z_t, X_t, Y_t, R_t, \Upsilon_t, M_t$ be as defined in Section 3.1, recalling that these quantities implicitly depend on the starting point $z \in \mathbb{H}$. The Loewner equation can be written as

$$dX_t = \frac{aX_t}{X_t^2 + Y_t^2}\, dt + dB_t, \qquad \partial_t Y_t = -\frac{aY_t}{X_t^2 + Y_t^2}, \tag{3.7}$$

and using Itô's formula and the chain rule we see that if $t < T_z$,

$$\partial_t \Upsilon_t = -\Upsilon_t \frac{2aY_t^2}{(X_t^2+Y_t^2)^2}, \qquad dM_t = M_t \frac{(1-4a)X_t}{X_t^2+Y_t^2}\, dB_t. \tag{3.8}$$

It is straightforward to check that with probability one

$$\sup_{0\le s<T_z\wedge t} M_s(z) \begin{cases} =\infty, & \text{if } z\in\gamma(0,t],\\ <\infty, & \text{otherwise.}\end{cases} \tag{3.9}$$

Moreover, if $4<\kappa<8$ and $z\notin\gamma(0,\infty)$, then $T_z<\infty$ and

$$M_{T_z-}(z)=0.$$

In other words, if we extend $M_t(z)$ to $t\ge T_z$ by $M_t(z)=M_{T_z-}(z)$, then for $z\notin \gamma(0,\infty)$, $M_t(z)$ is continuous in t and equals zero if $t\ge T_z$. Since $\gamma(0,\infty)$ has zero area, we can write

$$\Psi_t(D)=\int_D M_t(z)\,dA(z)=\int_D M_t(z)1\{T_z>t\}\,dA(z). \tag{3.10}$$

PROPOSITION 3.3. *If $z\in\mathbb{H}$, $M_t=M_t(z)$ is a local martingale but not a martingale. In fact,*

$$\mathbb{E}[M_t]=\mathbb{E}[M_0][1-\phi(z;t)]=G(z)[1-\phi(z;t)]. \tag{3.11}$$

Here $\phi(z;t)=\mathbf{P}\{T_z^*\le t\}$ is the distribution function of

$$T_z^*=\inf\{t:Y_t=0\},$$

where X_t+iY_t satisfies

$$\begin{aligned} dX_t &= \frac{(1-3a)X_t}{X_t^2+Y_t^2}\,dt+dW_t,\\ \partial_t Y_t &= -\frac{aY_t}{X_t^2+Y_t^2}, \qquad X_0+iY_0=z \end{aligned} \tag{3.12}$$

and W_t is a standard Brownian motion.

PROOF. The fact that M_t is a local martingale follows immediately from

$$dM_t=\frac{(1-4a)X_t}{X_t^2+Y_t^2}M_t\,dB_t.$$

To show that M_t is not a martingale, we will consider $\mathbb{E}[M_t]$. For every n, let $\tau_n=\inf\{t:M_t\ge n\}$. Then

$$\mathbb{E}[M_t]=\lim_{n\to\infty}\mathbb{E}[M_t;\tau_n>t]=\mathbb{E}[M_0]-\lim_{n\to\infty}\mathbb{E}[M_{\tau_n};\tau_n\le t].$$

If $z\notin\gamma(0,t]$, then $M_t(z)<\infty$. Therefore

$$\lim_{n\to\infty}\mathbb{E}[M_{\tau_n};\tau_n\le t]$$

denotes the probability that the process Z_t weighted (in the sense of the Girsanov theorem) by M_t reaches zero before time t. We claim that for t sufficiently large,

$$\lim_{n\to\infty} \mathbb{E}[M_{\tau_n}; \tau_n \le t] > 0. \tag{3.13}$$

We verify this by using the Girsanov theorem. For fixed n, $M_{t,n} := M_{t\wedge\tau_n}$ is a nonnegative martingale satisfying

$$dM_{t,n} = \frac{(1-4a)X_t}{X_t^2+Y_t^2} M_{t,n} 1\{\tau_n > t\}\, dB_t.$$

The Girsanov transformation considers the paths under the new measure $Q = Q^{(n)}$ defined by

$$Q(E) = M_0^{-1}\mathbb{E}[M_{t,n}1_E],$$

if E is $\mathcal{F}_t$-measurable. The Girsanov theorem tells us that in the new measure, X_t satisfies (3.12) where W_t is a standard Brownian motion in the new measure. It is fairly straightforward to show that if (X_t, Y_t) satisfy (3.12) and $a > 1/4$, then Y_t reaches zero in finite time. $\square$

The process satisfying (3.12) is called *two-sided radial* $SLE_{2/a}$ *from* 0 *and* ∞ *to* z *in* $\mathbb{H}$. Actually, it is the distribution only of one of the two arms, the arm from 0 to z. Heuristically, we think of this as $SLE_{2/a}$ from 0 to ∞ conditioned so that z is on the path. Let $T_z^* = \inf\{t : Z_t = 0\}$ where $Z_t = X_t + iY_t$ satisfies (3.12) with $Z_0 = z$. We have noted that $\mathbf{P}\{\tau < \infty\} = 1$. The function $\phi(z;t)$ will be important. We define

$$\phi(z) = \phi(z; 1), \tag{3.14}$$

$$\phi_t(z) = \mathbf{P}\{t \le \tau_z \le t+1\} = \phi(z; t+1) - \phi(z;t). \tag{3.15}$$

In particular, $\phi_0(z) = \phi(z)$. The scaling properties of *SLE* imply

$$\phi(z;t) = \phi\big(z/\sqrt{t}\big).$$

Let $Q = Q_z$ be the probability measure obtained by weighting by the local martingale $M_t(z)$. Then $\phi(z;t)$ denotes the distribution function of $T = T_z$ in the measure Q. If $t, s > 0$, then

$$Q[t < T < t+s \mid \mathcal{F}_t] = \phi(Z_t(z); s)1\{T > t\}.$$

Taking expectations, we get

$$\mathbb{E}[M_t(z)\phi(Z_t(z); s)] = G(z)[\phi(z; t+s) - \phi(z;t)]. \tag{3.16}$$

The next lemma describes the distribution of T under Q in terms of a functional of a simple one-dimensional diffusion.

LEMMA 3.4. *Suppose $a > 1/4$ and $X_t + iY_t$ satisfies*

$$dX_t = \frac{(1-3a)X_t}{X_t^2 + Y_t^2}\,dt + dW_t,$$

$$\partial Y_t = -\frac{aY_t}{X_t^2 + Y_t^2}\,dt, \qquad X_0 = x, \qquad Y_0 = 1,$$

where W_t is a standard Brownian motion. Let

$$T = \sup\{t : Y_t > 0\}.$$

Then

$$T = \int_0^\infty e^{-2as}\cosh^2 J_s\,ds = \frac{1}{4a} + \frac{1}{2}\int_0^\infty e^{-2as}\cosh(2J_s)\,ds,$$

where J_t satisfies

$$dJ_t = \left(\frac{1}{2} - 2a\right)\tanh J_t\,dt + dW_t, \qquad \sinh J_0 = x. \tag{3.17}$$

PROOF. Define the time change

$$\sigma(s) = \inf\{t : Y_t = e^{-as}\}.$$

Let $\hat{X}_s = X_{\sigma(s)}$, $\hat{Y}_s = Y_{\sigma(s)} = e^{-as}$. Since

$$-a\hat{Y}_t = \partial_t \hat{Y}_t = -\dot{\sigma}(t)\frac{a\hat{Y}_t^2}{\hat{X}_t^2 + \hat{Y}_t^2},$$

we have

$$\dot{\sigma}(s) = \hat{X}_t^2 + \hat{Y}_t^2 = e^{-2as}[K_s^2 + 1],$$

where $K_s = e^{as}\hat{X}_s$. Note that

$$\begin{aligned} d\hat{X}_s &= \left(\frac{1}{2} - 3a\right)\hat{X}_s\,ds + e^{-as}\sqrt{K_s^2 + 1}\,dW_s, \\ dK_s &= (1-2a)K_s\,ds + \sqrt{K_s^2 + 1}\,dW_s, \qquad K_0 = x. \end{aligned} \tag{3.18}$$

Using Itô's formula we see that if J_s satisfies (3.17) and $K_s = \sinh(J_s)$, then K_s satisfies (3.18). Also,

$$\sigma(\infty) = \int_0^\infty \dot{\sigma}(s)\,ds = \int_0^\infty e^{-2as}[K_s^2 + 1]\,ds = \int_0^\infty e^{-2as}\cosh^2 J_s\,ds. \qquad \square$$

Using the lemma one can readily see that there exist c, β such that

$$\phi\big(s(x+iy); s^2\big) = \phi(x+iy) \le c1\{y \le 2a\}e^{-\beta x^2}. \tag{3.19}$$

3.4. *Approximating* $\Theta_t(D)$. If $D \in \mathcal{D}$, the change of variables $z = Z_t(w)$ in (3.10) gives

$$\begin{aligned}\Psi_t(D) &= \int_{\mathbb{H}} |g_t'(w)|^{2-d} G(Z_t(w)) 1\{w \in D\}\, dA(w)\\ &= \int_{\mathbb{H}} |\hat{f}_t'(z)|^d G(z) 1\{\hat{f}_t(z) \in D\}\, dA(z), \qquad (3.20)\\ \mathbb{E}[\Psi_t(D)] &= \int_{\mathbb{H}} \mathbb{E}[|\hat{f}_t'(z)|^d;\ \hat{f}_t(z) \in D] G(z)\, dA(z).\end{aligned}$$

LEMMA 3.5. *If* $D \in \mathcal{D}$, $s, t \geq 0$,

$$\begin{aligned}&\mathbb{E}[\Psi_{s+t}(D)|\mathcal{F}_s]\\ &\quad = \Psi_s(D) - \int_{\mathbb{H}} |\hat{f}_s'(z)|^d G(z) \phi(z/\sqrt{t}) 1\{\hat{f}_s(z) \in D\}\, dA(z),\end{aligned} \qquad (3.21)$$

where $\phi(w)$ *is as defined in* (3.11) *and* (3.15).

PROOF. Recalling the definition of $\phi(w; t)$ in (3.11), we get

$$\begin{aligned}&\mathbb{E}[\Psi_{s+t}(D)|\mathcal{F}_s]\\ &\quad = \mathbb{E}\biggl[\int_D M_{s+t}(w)\, dA(z)\Big|\mathcal{F}_s\biggr]\\ &\quad = \int_D \mathbb{E}[M_{s+t}(w)|\mathcal{F}_s]\, dA(w)\\ &\quad = \int_D M_s(w)[1 - \phi(Z_s(w); t)]\, dA(w)\\ &\quad = \Psi_s(D) - \int_D M_s(w) \phi(Z_s(w); t)\, dA(w)\\ &\quad = \Psi_s(D) - \int_{\mathbb{H}} |g_s'(w)|^{2-d} G(Z_s(w)) \phi(Z_s(w); t) 1\{w \in D\}\, dA(w).\end{aligned}$$

If we use the change of variables $z = Z_s(w)$ and the scaling rule for ϕ, we get (3.21). $\square$

Using the last lemma, we see that a natural candidate for the process $\Theta_t(D)$ is given by

$$\Theta_t(D) = \lim_{n \to \infty} \Theta_{t,n}(D),$$

where

$$\begin{aligned}\Theta_{t,n}(D) &= \sum_{j \leq t2^n} \mathbb{E}[\Psi_{j2^{-n}}(D) - \Psi_{(j-1)2^{-n}}(D)|\mathcal{F}_{(j-1)2^{-n}}]\\ &= \lim_{n \to \infty} \sum_{j \leq t2^n} I_{j,n}(D),\end{aligned}$$

and

$$I_{j,n}(D)=\int_{\mathbb{H}}\bigl|\hat f'_{(j-1)2^{-n}}(z)\bigr|^d\phi(z2^{n/2})G(z)1\bigl\{\hat f_{(j-1)2^{-n}}(z)\in D\bigr\}\,dA(z). \tag{3.22}$$

Indeed, it is immediate that for fixed n

$$\Psi_{t,n}(D)+\Theta_{t,n}(D)$$

restricted to $t=\{k2^{-n}:k=0,1,\dots\}$ is a martingale.

To take the limit, one needs further conditions. One sufficient condition (see Section 4) is a second moment condition.

DEFINITION. We say that κ (or $a=2/\kappa$) is *good* with exponent $\alpha>0$ if $\kappa<8$ and the following holds. For every $D\in\mathcal{D}$ there exist $c<\infty$ such that for all n, and all $s,t\in\mathcal{Q}_n$ with $0<s<t\le 1$,

$$\mathbb{E}\bigl[[\Theta_{t,n}(D)-\Theta_{s,n}(D)]^2\bigr]\le c(t-s)^{1+\alpha}. \tag{3.23}$$

We say κ is *good* if this holds for some $\alpha>0$.

By scaling we see that $I_{j,n}(D)$ has the same distribution as

$$\int_{\mathbb{H}}|\hat f'_{j-1}(z2^{n/2})|^d\phi(z2^{n/2})G(z)1\{\hat f_{j-1}(z2^{n/2})\in 2^{n/2}D\}\,dA(z),$$

and the change of variables $w=z2^{n/2}$ converts this integral to

$$2^{-nd/2}\int_{\mathbb{H}}|\hat f'_{j-1}(w)|^d1\{\hat f_{j-1}(w)\in 2^{n/2}D\}\,d\mu(w),$$

where $d\mu(w)=\phi(w)G(w)\,dA(w)$. Therefore,

$$\Theta_{t,n}(D):=\sum_{j\le t2^n}I_{j,n}(D)$$

has the same distribution as

$$2^{-nd/2}\sum_{j\le t2^n}\int_{\mathbb{H}}|\hat f'_{j-1}(w)|^d1\{\hat f_{j-1}(w)\in 2^{n/2}D\}\,d\mu(w). \tag{3.24}$$

4. Doob–Meyer decomposition. In this section, we give a proof of the Doob–Meyer decomposition for submartingales satisfying a second moment bound. Although we will apply the results in this section to $\Theta_t(D)$ with $D\in\mathcal{D}$, it will be easier to abstract the argument and write just $L_t=-\Theta_t$. Suppose L_t is a submartingale with respect to $\mathcal{F}_t$ with $L_0=0$. We call a finite subset of $[0,1]$, $\mathcal{Q}$, which contains $\{0,1\}$ a *partition*. We can write the elements of a partition as

$$0=r_0<r_1<r_2<r_k=1.$$

We define

$$\|\mathcal{Q}\|=\max\{r_j-r_{j-1}\},\qquad c_*(\mathcal{Q})=\|\mathcal{Q}\|^{-1}\min\{r_j-r_{j-1}\}.$$

DEFINITION. Suppose $\mathcal{Q}_n$ is a sequence of partitions.

- If $\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \cdots$, we call the sequence *increasing* and let $\mathcal{Q} = \bigcup \mathcal{Q}_n$.
- If there exist $0 < u, c < \infty$ such that for each n, $\|\mathcal{Q}_n\| \le ce^{-un}$ we call the sequence *geometric*.
- If there exists $c > 0$ such that for all n, $c_*(\mathcal{Q}_n) \ge c$, we call the sequence *regular*.

The prototypical example of an increasing, regular, geometric sequence of partitions is the dyadic rationals

$$\mathcal{Q}_n = \{k2^{-n} : k = 0, 1, \ldots, 2^n\}.$$

Suppose $L_t, 0 \le t \le 1$, is a submartingale with respect to $\mathcal{F}_t$. Given a sequence of partitions $\mathcal{Q}_n$ there exist increasing processes $\Theta_{r,n}, r \in \mathcal{Q}_n$, such that

$$L_r - \Theta_{r,n}, \qquad r \in \mathcal{Q}_n,$$

is a martingale. Indeed if $\mathcal{Q}_n$ is given by $0 \le r_0 < r_1 < \cdots < r_{k_n} = 1$, then we can define the increasing process by $\Theta_{0,n} = 0$ and recursively

$$\Theta_{r_j,n} = \Theta_{r_{j-1},n} + \mathbb{E}[L_{r_j} - L_{r_{j-1}} | \mathcal{F}_{r_{j-1}}].$$

Note $\Theta_{r_j,n}$ is $\mathcal{F}_{r_{j-1}}$-measurable and if $s, t \in \mathcal{Q}_n$ with $s < t$,

$$\mathbb{E}[\Theta_{t,n} | \mathcal{F}_s] = \Theta_{s,n} + \mathbb{E}[L_t - L_s | \mathcal{F}_s]. \tag{4.1}$$

The proof of the next proposition follows the proof of the Doob–Meyer theorem in [2].

PROPOSITION 4.1. *Suppose $\mathcal{Q}_n = \{r(j,n);\ j = 0, 1, \ldots, k_n\}$ is an increasing sequence of partitions with*

$$\delta_n := \|\mathcal{Q}_n\| \to 0.$$

Suppose there exist $\beta > 0$ and $c < \infty$ such that for all n and all $s, t \in \mathcal{Q}_n$

$$\mathbb{E}[(\Theta_{t,n} - \Theta_{s,n})^2] \le c(t-s)^{\beta+1}. \tag{4.2}$$

Then there exists an increasing, continuous process Θ_t such that $L_t - \Theta_t$ is a martingale. Moreover, for each $t \in \mathcal{Q}$,

$$\Theta_t = \lim_{n\to\infty} \Theta_{t,n}, \tag{4.3}$$

where the limit is in L^2. In particular, for all $s < t$,

$$\mathbb{E}[(\Theta_t - \Theta_s)^2] \le c(t-s)^{\beta+1}.$$

If $u < \beta/2$, then with probability one, Θ_t is Hölder continuous of order u. If the sequence is geometric (i.e., if $\delta_n \to 0$ exponentially in n), then the limit in (4.3) exists with probability one.

PROOF. We first consider the limit for $t = 1$. For $m \geq n$, let

$$\Delta(n, m) = \max\{\Theta_{r(j,n),m} - \Theta_{r(j-1,n),m} : j = 1, \dots, k_n\}.$$

Using (4.2) and writing $k = k_n$, we can see that

$$\begin{aligned}
\mathbb{E}[\Delta(n, m)^2] &\leq \sum_{j=1}^{k} \mathbb{E}[(\Theta_{r(j,n),m} - \Theta_{r(j-1,n),m})^2] \\
&\leq c \sum_{j=1}^{k} [r(j, n) - r(j-1, n)]^{\beta+1} \\
&\leq c\delta_n^{\beta} \sum_{j=1}^{k} [r(j, n) - r(j-1, n)] \\
&= c\delta_n^{\beta}.
\end{aligned}$$

If $m \geq n$, then (4.1) shows that $Y_t := \Theta_{t,m} - \Theta_{t,n}$, $t \in \mathcal{Q}_n$, is a martingale, and (4.2) shows that it is square-integrable. Hence, with $k = k_n$,

$$\begin{aligned}
\mathbb{E}[(\Theta_{1,m} - \Theta_{1,n})^2] &= \sum_{j=1}^{k} \mathbb{E}[(Y_{r(j,n)} - Y_{r(j-1,n)})^2] \\
&\leq \mathbb{E}\left[(\Delta(n, n) + \Delta(n, m)) \sum_{j=1}^{k} |Y_{r(j,n)} - Y_{r(j-1,n)}|\right].
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{j=1}^{k} |Y_{r(j,n)} - Y_{r(j-1,n)}| \\
&\quad \leq \sum_{j=1}^{k} ([\Theta_{r(j,n),n} - \Theta_{r(j-1,n),n}] + [\Theta_{r(j,n),m} - \Theta_{r(j-1,n),m}]) \\
&\quad = \Theta_{1,n} + \Theta_{1,m}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}[(\Theta_{1,m} - \Theta_{1,n})^2] \\
&\quad \leq \mathbb{E}[(\Delta(n, n) + \Delta(n, m))(\Theta_{1,n} + \Theta_{1,m})] \\
&\quad \leq \mathbb{E}[(\Delta(n, n) + \Delta(n, m))^2]^{1/2} \mathbb{E}[(\Theta_{1,n} + \Theta_{1,m})^2]^{1/2} \\
&\quad \leq c\delta_n^{\beta}.
\end{aligned} \tag{4.4}$$

This shows that $\{\Theta_{1,m}\}$ is a Cauchy sequence, and completeness of L^2 implies that there is a limit. Similarly, for every $t \in \mathcal{Q}$, we can see that there exists Θ_t such that

$$\lim_{n\to\infty} \mathbb{E}[|\Theta_{t,n} - \Theta_t|^2] = 0, \qquad t \in \mathcal{Q}.$$

Moreover, we get the estimate

$$\mathbb{E}[(\Theta_t - \Theta_s)^2] = \lim_{n\to\infty} \mathbb{E}[(\Theta_{t,n} - \Theta_{s,n})^2] \le c(t-s)^{\beta+1}$$

for

$$0 \le s \le t \le 1, \qquad s, t \in \mathcal{Q}.$$

The L^2-maximal inequality implies then that

$$\mathbb{E}\Big[\sup_{s\le r\le t} (\Theta_r - \Theta_s)^2\Big] \le c(t-s)^{\beta+1}, \qquad 0 \le s \le t \le 1, s, t \in \mathcal{Q},$$

where the supremum is also restricted to $r \in \mathcal{Q}$. Let

$$M(j,n) = \sup\{(\Theta_t - \Theta_s)^2 : (j-1)2^{-n} \le s, t \le j2^{-n}, s, t \in \mathcal{Q}\},$$
$$M_n = \max\{M(j,n) : j = 1, \dots, n\}.$$

Since $\mathcal{Q}$ is dense, we can then conclude

$$\mathbb{E}[M(j,n)] \le c2^{-n(\beta+1)},$$
$$\mathbb{E}[M_n] \le \sum_{j=1}^{2^n} \mathbb{E}[M(j,n)] \le c2^{-n\beta}.$$

An application of the triangle inequality shows that if

$$Z_n = \sup\{(\Theta_t - \Theta_s)^2 : 0 \le s, t \le 1, s, t \in \mathcal{Q}, |s-t| \le 2^{-n}\},$$

then

$$\mathbb{E}[Z_n] \le c2^{-n\beta}.$$

The Chebyshev inequality and the Borel–Cantelli lemma show that if $u < \beta/2$, with probability one

$$\sup\left\{\frac{|\Theta_t - \Theta_s|}{(t-s)^u} : 0 \le s, t \le 1, s, t \in \mathcal{Q}\right\} < \infty.$$

In particular, we can choose a continuous version of the process Θ_t whose paths are Hölder continuous of order u for every $u < \beta/2$.

If the sequence is geometric, then (4.4) implies that there exist c, v such that

$$\mathbb{E}[(\Theta_{1,n+1} - \Theta_{1,n})^2] \le ce^{-nv},$$

which implies

$$\mathbf{P}\{|\Theta_{1,n+1} - \Theta_{1,n}| \geq e^{-nv/4}\} \leq ce^{-nv/2}.$$

Hence by the Borel–Cantelli lemma we can write

$$\Theta_1 = \Theta_{1,1} + \sum_{n=1}^{\infty}[\Theta_{n+1,1} - \Theta_{n,1}],$$

where the sum converges absolutely with probability one. $\square$

5. Moment bounds. In this section we show that κ is good for $\kappa < 4$. Much of what we do applies to other values of κ, so for now we let $\kappa < 8$. Let

$$d\mu(z) = G(z)\phi(z)\, dA(z), \qquad d\mu_t(z) = G(z)\phi(z;t)\, dA(z).$$

We note the scaling rule

$$d\mu_t(z) = t^{d/2}\, d\mu(z/\sqrt{t}).$$

From (3.19) we can see that

$$d\mu_{t^2}(x+iy) \leq cy^{d-2}[(x/y)^2+1]^{1/2-2a}e^{-\beta(x/t)^2}1\{y \leq 2at\}\, dx\, dy. \tag{5.1}$$

Note that this implies (with a different c)

$$d\mu_{t^2}(z) \leq c[\sin\theta_z]^{\kappa/8+8/\kappa-2}|z|^{\kappa/8-1}e^{-\beta|z|^2/t^2}\, dA(z). \tag{5.2}$$

We have shown that $\Theta_{t,n}(D)$ has the same distribution as $\tilde{\Theta}_{t,2^{n/2}}(D)$ where

$$\tilde{\Theta}_{t,n}(D) = n^{-d}\sum_{j \leq tn^2} I_{j-1,nD}$$

and

$$I_{s,D} = \int_{\mathbb{H}} |\hat{f}_s'(w)|^d 1\{f_s(w) \in D\}\, d\mu(w).$$

In this section we establish the following theorems which are the main estimate.

THEOREM 5.1. *If $\kappa < 8$, there exists c such that for all s,*

$$\mathbb{E}[I_{s,\mathbb{H}}] \leq cs^{d-2}. \tag{5.3}$$

THEOREM 5.2. *If $\kappa < 4$, then for every $m < \infty$ there exists $c = c_m$ such that if $D \in \mathcal{D}_m$ and $1 \leq s, t \leq n$, then*

$$\sum_{j=0}^{s^2-1}\mathbb{E}[I_{j+t^2,nD}I_{t^2,nD}] \leq c(s/t)^{\zeta}s^{2(d-1)}, \qquad s \leq t, \tag{5.4}$$

where $\zeta = 2 - \frac{3\kappa}{4}$. *In particular,*

$$\mathbb{E}\big[[\Theta_{1,n}(D) - \Theta_{\delta,n}(D)]^2\big] = 2^{-nd} \sum_{(1-\delta)2^n \le j,k \le 2^n} \mathbb{E}[I_{j,2^{n/2}D} I_{k,2^{n/2}D}]$$
$$\le c(1-\delta)^{d+\zeta/2} = c(1-\delta)^{2-\kappa/4}.$$

THEOREM 5.3. *If* $\kappa < \kappa_0$, *then for every* $m < \infty$ *there exists* $c = c_m$ *such that if* $D \in \mathcal{D}_m$ *and* $1 \le s, t \le n$, *then*

$$\sum_{j=0}^{s^2-1} \mathbb{E}[I_{j+t^2,nD} I_{t^2,nD}] \le c(s/t)^{\zeta} s^{2(d-1)}, \qquad s \le t, \tag{5.5}$$

where $\zeta = \frac{4}{\kappa} - \frac{3\kappa}{16} - 1$. *In particular,*

$$\mathbb{E}\big[[\Theta_{1,n}(D) - \Theta_{\delta,n}(D)]^2\big] = 2^{-nd} \sum_{(1-\delta)2^n \le j,k \le 2^n} \mathbb{E}[I_{j,2^{n/2}D} I_{k,2^{n/2}D}]$$
$$\le c(1-\delta)^{d+\zeta/2} = c(1-\delta)^{1/2+2/\kappa+\kappa/32}.$$

This section is devoted to proving Theorems 5.1–5.3. Note that

$$I_{s+t,D} I_{t,D} = \int_{\mathbb{H}} \int_{\mathbb{H}} |\hat{f}'_{s+t}(z)|^d |\hat{f}'_t(w)|^d 1\{\hat{f}_{t+s}(z), \hat{f}_t(w) \in D\}\, d\mu(z)\, d\mu(w).$$

In particular,

$$\mathbb{E}[I_{s,D}] = \int_{\mathbb{H}} \mathbb{E}[|\hat{f}'_s(w)|^d;\ \hat{f}_s(w) \in D]\, d\mu(w), \tag{5.6}$$

$$\begin{aligned}\mathbb{E}[I_{s+t,D} I_{t,D}] = \int_{\mathbb{H}} \int_{\mathbb{H}} \mathbb{E}[|\hat{f}'_{s+t}(z)|^d |\hat{f}'_t(w)|^d;& \\ \hat{f}_{t+s}(z), \hat{f}_t(w) \in D]\, d\mu(z)\, d\mu(w).&\end{aligned} \tag{5.7}$$

5.1. *Reverse-time flow.* In this subsection we define the reverse flow and set up some notation that will be useful. Suppose $s, t \ge 0$ are given. The expectations we need to estimate are of the form

$$\mathbb{E}[|\hat{f}'_t(z)|^d], \tag{5.8}$$

$$\mathbb{E}[|\hat{f}'_{s+t}(z)|^d |\hat{f}'_t(w)|^d;\ \hat{f}_{s+t}(z) \in D,\ \hat{f}_t(w) \in D]. \tag{5.9}$$

We fix $s, t \ge 0$ and allow quantities in this subsection to depend implicitly on s, t.

Let $\tilde{U}_r = V_{t+s-r} - V_{s+t}$. Then $\tilde{B}_r := -\tilde{U}_r$, $0 \le r \le s+t$, is a standard Brownian motion starting at the origin. Let $U_r = V_{t-r} - V_t = \tilde{U}_{s+r} - \tilde{U}_s$, $0 \le r \le t$. Then $B_r = -U_r$ is also a standard Brownian motion and $\{\tilde{U}_r : 0 \le r \le s\}$ is independent of $\{U_r : 0 \le r \le t\}$.

Let $\tilde{h}_r$, $0 \le r \le s+t$, be the solution to the reverse-time Loewner equation

$$\partial_r \tilde{h}_r(z) = \frac{a}{\tilde{U}_r - \tilde{h}_r(z)}, \qquad \tilde{h}_0(z) = z. \tag{5.10}$$

Let h_r, $0 \le r \le t$, be the solution to

$$\partial_r h_r(z) = \frac{a}{U_r - h_r(z)} = \frac{a}{\tilde{U}_{s+r} - [h_r(z) + \tilde{U}_s]}, \qquad h_0(z) = z.$$

Let $\tilde{h} = \tilde{h}_{s+t}$, $h = h_t$. Using only the Loewner equation, we can see that

$$\begin{aligned} &\hat{f}_{s+t}(z) = \tilde{h}_{s+t}(z) - \tilde{U}_{s+t}, \qquad \hat{f}_t(w) = h_t(w) - U_t, \\ &\tilde{h}_{s+t}(z) = h_t\big(\tilde{h}_s(z) - \tilde{U}_s\big) + \tilde{U}_s. \end{aligned} \tag{5.11}$$

Therefore the expectations in (5.8) and (5.9) equal

$$\mathbb{E}[|h'(z)|^d], \tag{5.12}$$

$$\mathbb{E}[|\tilde{h}'(z)|^d |h'(w)|^d; \tilde{h}(z) - \tilde{U}_{s+t} \in D, h(w) - U_t \in D], \tag{5.13}$$

respectively. Let

$$\mathcal{I}_t(z; D) = 1\{h_t(z) - U_t \in D\}, \qquad \mathcal{I}_t(z, w; D) = \mathcal{I}_t(z; D)\mathcal{I}_t(w; D).$$

Using (5.11), we can write (5.13) as

$$\mathbb{E}\big[|\tilde{h}_s'(z)|^d \big|h_t'\big(h_s(z) - \tilde{U}_s\big)\big|^d |h_t'(w)|^d; \mathcal{I}_t\big(\tilde{h}_s(z) - \tilde{U}_s, w; D\big)\big]. \tag{5.14}$$

We will derive estimates for $h, \tilde{h}$. Let $F_D(z, w; s+t, t)$ denote the expectation in (5.13) and let

$$F_D(z, w, t) = F_D(z, w; t, t) = \mathbb{E}[|h_t'(z)|^d |h_t'(w)|^d \mathcal{I}_t(z, w; D)].$$

We note the scaling relation: if $r > 0$,

$$F_D(z, w, s+t, t) = F_{rD}\big(rz, rw; r^2(s+t), r^2 t\big).$$

Since $\tilde{h}_s$ and h_t are independent, we can see by conditioning on the σ-algebra generated by $\{\tilde{U}_r : 0 \le r \le s\}$, we see that (5.14) yields

$$F(z, w; s+t, t) = \mathbb{E}\big[|h_s'(z)|^d F_D\big(h_s(z) - U_s, w, t\big)\big] \tag{5.15}$$

(since $\tilde{h}_s$ and h_s have the same distribution, we replaced $\tilde{h}_s, \tilde{U}_s$ with h_s, U_s).

We rewrite (5.6) and (5.7) as

$$\begin{aligned} \mathbb{E}[I_{t,D}] &= \int_{\mathbb{H}} \mathbb{E}[|h_t'(w)|^d \mathcal{I}_t(w; D)]\, d\mu(w), \\ \mathbb{E}[I_{s+t,D} I_{t,D}] &= \int_{\mathbb{H}} \int_{\mathbb{H}} F_D(z, w, s+t, t)\, d\mu(w)\, d\mu(z). \end{aligned} \tag{5.16}$$

The expressions on the left-hand side of (5.4) and (5.5) involve expectations at two different times. The next lemma shows that we can write these sums in terms of "two-point" estimates at a single time. Recall the definition of μ_s from Section 3.3.

LEMMA 5.4. *For all $D \in \mathcal{D}$ and $s \geq 0$,*

$$\int_{\mathbb{H}} \int_{\mathbb{H}} F_D(z, w, s+t, t)\, d\mu(w)\, d\mu(z) \\ = \int_{\mathbb{H}} \int_{\mathbb{H}} F_D(z, w, t)\, d\mu(w) [d\mu_{s+1} - d\mu_s](z).$$

In particular, if s is an integer,

$$\sum_{j=0}^{s^2-1} \mathbb{E}[I_{j+t,D} I_{t,D}] = \int_{\mathbb{H}} \int_{\mathbb{H}} F_D(z, w, t)\, d\mu(w)\, d\mu_{s^2}(z).$$

PROOF. Using (5.15), we write

$$\int_{\mathbb{H}} \int_{\mathbb{H}} F_D(z, w, s+t, t)\, d\mu(w)\, d\mu(z) = \int_{\mathbb{H}} \mathbb{E}[\Phi]\, d\mu(w),$$

where

$$\Phi = \Phi_D(w, s, t) = \int_{\mathbb{H}} |h_s'(z)|^d F_D\bigl(h_s(z) - U_s, w, t\bigr) \phi(z) G(z)\, dA(z).$$

We will change variables,

$$z' = h_s(z) - U_s = \hat{f}_s(z).$$

Here $h_s(z) = \hat{f}_s(z) + U_s = g_s^{-1}(z + U_s) + U_s$ for a conformal map g_s with the distribution of the forward-time flow with driving function $\hat{U}_r = U_{s-r} - U_s$. Then

$$z = \hat{f}_s^{-1}(z') = g_s(z') - \hat{U}_s = \hat{Z}_s(z'),$$

where $\hat{Z}_s(\cdot) = g_s(\cdot) - \hat{U}_s$. Then

$$\Phi = \int_{\mathbb{H}} |g_s'(z')|^{2-d} F(z', w, t) G(\hat{Z}_s(z')) \phi(\hat{Z}_s(z'))\, dA(z') \\ = \int_{\mathbb{H}} \hat{M}_s(z') F(z', w, t) \phi(\hat{Z}_s(z'))\, dA(z'),$$

where $\hat{M}_s$ denotes the forward direction local martingale as in Section 3.3. Taking expectation using (3.16), we get

$$\mathbb{E}[\Phi] = \int_{\mathbb{H}} F(z', w, t) G(z') [\phi(z'; s+1) - \phi(z'; s)]\, dA(z') \\ = \int_{\mathbb{H}} F(z', w, t)\, d[\mu_{s+1} - \mu_s](z').$$

This gives the first assertion. The second assertion follows from (5.16). $\square$

5.2. *The reverse-time martingale.* In this section we will collect facts about the reverse-time martingale; this is analyzed in more detail in [6]. Suppose that B_t is a standard Brownian motion and $h_t(z)$ is the solution to the reverse-time Loewner equation

$$\partial_t h_t(z) = \frac{a}{U_t - h_t(z)}, \qquad h_0(z) = z, \tag{5.17}$$

where $U_t = -B_t$ and $a = 2/\kappa$. Here $z \in \mathbb{C} \setminus \{0\}$. The solution exists for all times t if $z \notin \mathbb{R}$ and

$$\overline{h_t(z)} = h_t(\overline{z}).$$

For fixed t, h_t is a conformal transformation of $\mathbb{H}$ onto a subdomain of $\mathbb{H}$. Let

$$Z_t(z) = X_t(z) + iY_t(z) = h_t(z) - U_t,$$

and note that

$$dZ_t(z) = -\frac{a}{Z_t(z)}\,dt + dB_t,$$
$$dX_t(z) = -\frac{X_t(z)}{|Z_t(z)|^2}\,dt + dB_t, \qquad \partial_t Y_t(z) = \frac{a}{|Z_t(z)|^2}.$$

We use d for stochastic differentials and ∂_t for actual derivatives. Differentiation of (5.17) yields

$$\partial_t |h_t'(z)| = |h_t'(z)| \frac{a[X_t(z)^2 - Y_t(z)^2]}{|Z_t(z)|^4},$$
$$\partial_t \left[\frac{|h_t'(z)|}{Y_t(z)}\right] = -\left[\frac{|h_t'(z)|}{Y_t(z)}\right] \frac{2aY_t(z)^2}{|Z_t(z)|^4}.$$

In particular,

$$|h_t'(x+i)| \le Y_t(x+i) \le \sqrt{2at+1}. \tag{5.18}$$

Let

$$N_t(z) = |h_t'(z)|^d Y_t(z)^{1-d} |Z_t(z)| = |h_t'(z)|^d Y_t(z)^{-\kappa/8} |Z_t(z)|.$$

An Itô's formula calculation shows that $N_t(z)$ is a martingale satisfying

$$dN_t(z) = \frac{X_t(z)}{|Z_t(z)|^2} N_t(z)\,dB_t.$$

More generally, if $r > 0$ and

$$\lambda = r\left[1 + \frac{\kappa}{4}\right] - \frac{\kappa r^2}{8},$$

then

$$N_t = N_{t,r}(z) = |h_t'(z)|^{\lambda} Y_t^{-\kappa r^2/8} |Z_t(z)|^r \tag{5.19}$$

is a martingale satisfying

$$dN_t(z) = \frac{r X_t(z)}{|Z_t(z)|^2} N_t(z)\, dB_t. \tag{5.20}$$

Note that

$$\begin{aligned}
h_t(z) - h_t(w) &= Z_t(z) - Z_t(w), \\
\partial_t[Z_t(z) - Z_t(w)] &= [Z_t(z) - Z_t(w)] \frac{a}{Z_t(z) Z_t(w)}, \\
\partial_t |Z_t(z) - Z_t(w)| &= |Z_t(z) - Z_t(w)| \operatorname{Re}\left[\frac{a}{Z_t(z) Z_t(w)}\right] \\
&= |Z_t(z) - Z_t(w)| \\
&\quad \times \frac{a[X_t(z) X_t(w) - Y_t(z) Y_t(w)]}{|Z_t(z)|^2 |Z_t(w)|^2}, \\
\partial_t[|Z_t(z) - Z_t(w)||Z_t(z) - Z_t(\overline{w})|] &= [|Z_t(z) - Z_t(w)||Z_t(z) - Z_t(\overline{w})|] \\
&\quad \times \frac{2a X_t(z) X_t(w)}{|Z_t(z)|^2 |Z_t(w)^2|}.
\end{aligned}$$

Combining this with (5.20) and the stochastic product rule yields the following. We will only use this lemma with $m = 2$.

LEMMA 5.5. *Suppose $r \in \mathbb{R}$, $z_1, \ldots, z_m \in \mathbb{H}$, and $N_t(z_j)$ denotes the martingale in* (5.19). *Let*

$$\begin{aligned}
N_t &= N_t(z_1, \ldots, z_m) \\
&= \left[\prod_{j=1}^{m} N_t(z_j)\right] \left[\prod_{j \neq k} |Z_t(z_j) - Z_t(z_k)||Z_t(z_j) - Z_t(\overline{z}_k)|\right]^{-r^2\kappa/4}.
\end{aligned} \tag{5.21}$$

Then N_t is a martingale satisfying

$$dN_t = r N_t \left[\sum_{j=1}^{m} \frac{X_j(z_j)}{|Z_j(z_j)|^2}\right] dB_t.$$

5.3. *First moment*. The proof of Theorem 5.1 relies on the following estimate that can be found in [6], Theorem 9.1. Since it will not require much extra work here, we will also give a proof in this paper. Unlike the second moment estimates, there is no need to restrict this to $D \in \mathcal{D}$.

LEMMA 5.6. *Suppose $\kappa < 8$ and $\hat{u} > 2 - \frac{\kappa}{8}$. Then there exists c such that for all x, y and $s \ge y$,*

$$\mathbb{E}[|h'_{s^2}(z)|^d] \le cs^{d-2}|z|^{2-d}[\sin\theta_z]^{2-d-\hat{u}}. \tag{5.22}$$

PROOF. See Lemma 5.13. Note that scaling implies that it suffices to prove the result for $y_z = 1$. □

PROOF OF THEOREM 5.1 GIVEN (5.22). If $\kappa < 8$, we can find $\hat{u}$ satisfying

$$2 - \frac{\kappa}{8} < \hat{u} < \frac{8}{\kappa}. \tag{5.23}$$

Then,

$$\begin{aligned}\mathbb{E}[I_{s^2}] &\le \int_{\mathbb{H}} \mathbb{E}[|h'_{s^2}(z)|^d]\, d\mu(z)\\ &\le cs^{d-2}\int_{\mathbb{H}} |z|^{2-d}[\sin\theta_z]^{2-d-\hat{u}}|z|^{d-2}[\sin\theta_z]^{\kappa/8+8/\kappa-2}e^{-\beta|z|^2}\, dA(z)\\ &\le cs^{d-2}.\end{aligned}$$

The last inequality uses $\hat{u} < 8/\kappa$. □

5.4. *Proof of Theorem* 5.2. The martingale in (5.21) yields a simple two-point estimate for the derivatives. This bound is not always sharp, but it suffices for proving Theorem 5.2.

PROPOSITION 5.7. *For every $m < \infty$, there exists $c = c_m$ such that if $D \in \mathcal{D}_m$, $z, w \in \mathbb{H}$, $s, t > 0$,*

$$F_{sD}(z, w; ts^2) \le cs^{3\kappa/4-2}y_z^{-\kappa/8}|z|y_w^{-\kappa/8}|w||z-w|^{-\kappa/4}|z-\overline{w}|^{-\kappa/4}. \tag{5.24}$$

PROOF. By scaling we may assume $s = 1$. All constants in this proof depend on m but not otherwise on D. Let N_t be the martingale from Lemma 5.5 with $r = 1$. Then

$$\mathbb{E}[N_t] = N_0 = y_z^{-\kappa/8}|z|y_w^{-\kappa/8}|w||z-w|^{-\kappa/4}|z-\overline{w}|^{-\kappa/4}.$$

If $\mathcal{I}_t(z, w; D) = 1$, we have

$$\begin{aligned}Y_t(z)^{-\kappa/8}|Z_t(z)|Y_t(w)^{-\kappa/8}|Z_t(w)| &\ge c_1 > 0,\\ |h_t(z) - h_t(w)||h_t(z) - h_t(\overline{w})| &\le c_2 < \infty.\end{aligned}$$

Therefore,

$$|h'_t(z)|^d|h'_t(w)|^d\mathcal{I}_t(z, w; D) \le c_3N_t \tag{5.25}$$

and

$$\mathbb{E}[|h'_t(z)|^d|h'_t(w)|^d\mathcal{I}_t(z, w; D)] \le c\mathbb{E}[N_t]. \qquad \square$$

REMARK. Estimate (5.24) is not sharp if z and w are close in the hyperbolic metric. For example, suppose that $z = 2w = \varepsilon i$. Then using distortion estimates we see that

$$|h_t(z) - h_t(w)| \asymp |h_t'(z)||z - w| \asymp \varepsilon |h_t'(z)|.$$

On the event $\mathcal{I}_t(z, w; D) = 1$,

$$N_t \asymp \varepsilon^{-\kappa/4} |h_t'(z)|^{2d-\kappa/4}.$$

Typically, $|h_t'(z)| \ll \varepsilon^{-1}$, so estimate (5.25) in the proof is not sharp.

PROPOSITION 5.8. *If $\kappa < 4$, then for every positive integer m there exists $c = c_m$ such that if $D \in \mathcal{D}_m$ and $s \geq 1$, $t \geq 1$, $r > 0$*

$$\int_{\mathbb{H}} \int_{\mathbb{H}} F_{tD}(z, w, rt^2)\, d\mu(w)\, d\mu_{s^2}(z) \leq c(s/t)^{2-3\kappa/4} s^{\kappa/4}. \tag{5.26}$$

PROOF. As in the previous proof, constants in this proof may depend on m but not otherwise on D. By (5.24) the left-hand side of (5.26) is bounded above by a constant times

$$t^{3\kappa/4-2} \int_{\mathbb{H}} \int_{\mathbb{H}} y_z^{-\kappa/8} |z| y_w^{-\kappa/8} |w| |z - w|^{-\kappa/4} |z - \overline{w}|^{-\kappa/4}\, d\mu(w)\, d\mu_{s^2}(z).$$

Hence it suffices to show that there exists c such that for all s,

$$\begin{aligned} &\int_{\mathbb{H}} \int_{\mathbb{H}} y_z^{-\kappa/8} |z| y_w^{-\kappa/8} |w| |z - w|^{-\kappa/4} |z - \overline{w}|^{-\kappa/4}\, d\mu(w)\, d\mu_{s^2}(z) \\ &\quad < c s^{2-\kappa/2}. \end{aligned} \tag{5.27}$$

Recall from (5.2) that

$$d\mu_{s^2}(z) \leq c e^{-\beta(|z|/s)^2} y_z^{\kappa/8+8/\kappa-2} |z|^{1-8/\kappa}\, dA(z). \tag{5.28}$$

We write the integral in (5.27) as

$$\int_{\mathbb{H}} \Phi(z) y_z^{-\kappa/8} |z|\, d\mu_{s^2}(z),$$

where

$$\Phi(z) = \int_{\mathbb{H}} y_w^{-\kappa/8} |w| |z - w|^{-\kappa/4} |z - \overline{w}|^{-\kappa/4}\, d\mu(w).$$

We will show that

$$\Phi(z) \leq c|z|^{-\kappa/2}. \tag{5.29}$$

Using this and (5.28), the integral in (5.27) is bounded above by a constant times

$$\begin{aligned} \int_{\mathbb{H}} |z|^{2-\kappa/2-8/\kappa} e^{-\beta(|z|/s)^2} y_z^{8/\kappa-2}\, dA(z) &\leq \int_{\mathbb{H}} |z|^{-\kappa/2} e^{-\beta(|z|/s)^2}\, dA(z) \\ &= s^{2-\kappa/2} \int_0^\infty r^{1-\kappa/2} e^{-\beta r^2}\, dr. \end{aligned}$$

Hence it suffices to prove (5.29).

Using (5.1), we see that $\Phi(z)$ is bounded by a constant times

$$\Phi^*(z) := \int_{\mathbb{H}} K(z,w)\,dA(w),$$

where

$$K(z,w) = 1\{0 < y_w < 2a\} y_w^{8/\kappa - 2} |w|^{2-8/\kappa} |z-w|^{-\kappa/4} |z-\overline{w}|^{-\kappa/4} e^{-\beta x_w^2}.$$

We write $\Phi^*(z) = \Phi_1(z) + \Phi_2(z) + \Phi_3(z)$ where

$$\Phi_1(z) = \int_{|z-w|\le y_z/2} K(z,w)\,dA(w),$$

$$\Phi_2(z) = \int_{y_z/2<|z-w|<|z|/2} K(z,w)\,dA(w),$$

$$\Phi_3(z) = \int_{|z-w|\ge |z|/2} K(z,w)\,dA(w).$$

If $|z-w| \le y_z/2$ and $y_w \le 2a$, then

$$y_w \le 2a,\ |w| \asymp |z|,\ y_w \asymp y_z,\ |z-w|^{-\kappa/4}|z-\overline{w}|^{-\kappa/4} \asymp |z-w|^{-\kappa/4} y_z^{-\kappa/4}.$$

Also,

$$x_w^2 \ge |w|^2 - (2a)^2 \ge \frac{|z|^2}{4} - (2a)^2.$$

Hence

$$\Phi_1(z) \le c e^{-\beta |z|^2/4} |z|^{2-8/\kappa} y_z^{8/\kappa - \kappa/4 - 2} \int_{|w-z|<y_z/2} |z-w|^{-\kappa/4}\,dA(z).$$

Therefore,

$$\begin{aligned}\Phi_1(z) &\le c e^{-\beta|z|^2/2} |z|^{2-8/\kappa} y_z^{8/\kappa-\kappa/2} 1\{y_z \le 4a\}\\ &\le c|z|^{2-\kappa/2} e^{-\beta|z|^2/2} \le c|z|^{-\kappa/2}.\end{aligned}$$

Suppose $|z-w| \ge |z|/2$. Then $|z-w| \asymp |z|$ for $|w| \le 2|z|$ and $|z-w| \asymp |w|$ for $|w| \ge 2|z|$. Using $\kappa < 8$,

$$\begin{aligned}&\int_{y_w<2a,|w-z|\ge|z|/2,|w|\le 2|z|} y_w^{8/\kappa-2}|w|^{2-8/\kappa}|z-w|^{-\kappa/2} e^{-\beta x_w^2}\,dA(w)\\ &\qquad \le c|z|^{-\kappa/2} \int_{y_w<2a,|w|\le 2|z|} [\sin\theta_w]^{8/\kappa-2} e^{-\beta x_w^2}\,dA(w)\\ &\qquad \le c|z|^{-\kappa/2}[|z|^2 \wedge 1],\\ &\int_{y_w<2a,|w|>2|z|} y_w^{8/\kappa-2}|w|^{2-8/\kappa}|z-w|^{-\kappa/2} e^{-\beta x_w^2}\,dA(w)\\ &\qquad \le c e^{-\beta|z|^2} \int_{y_w\le 2a,|w|>2|z|} |w|^{-\kappa/2} [\sin\theta_w]^{8/\kappa-2} e^{-\beta x_w^2/2}\,dA(w) \le c|z|^{-\kappa/2}.\end{aligned}$$

Therefore, $\Phi_3(z) \le c|z|^{-\kappa/2}$.

We now consider $\Phi_2(z)$ which is bounded above by a constant times

$$\int_{y_w \le 2a, y_z/2 < |z-w| < |z|/2} y_w^{8/\kappa-2} |w|^{2-8/\kappa} |z-w|^{-\kappa/2} e^{-\beta x_w^2}\, dA(w).$$

Note that for w in this range, $x_w^2 \ge |w|^2 - (2a)^2 \ge (|z|/2)^2 - (2a)^2$ and $|w| \asymp |z|$ and hence we can bound this by

$$ce^{-\beta|z|^2/4} |z|^{2-8/\kappa} \int_{y_w \le 2a, y_z/2 < |z-w| < |z|/2} y_w^{8/\kappa-2} |z-w|^{-\kappa/2}\, dA(w).$$

The change of variables $w \mapsto w - x_z$ changes the integral to

$$\int_{y_w \le 2a, y_z/2 < |w-iy_z| < |z|/2} y_w^{8/\kappa-2} |w - iy_z|^{-\kappa/2}\, dA(w).$$

We split this integral into the integral over $|w| \le 2y_z$ and $|w| > 2y_z$. The integral over $|w| \le 2y_z$ is bounded by a constant times

$$y_z^{-\kappa/2} \int_{|w| \le 2y_z} y_w^{8/\kappa-2} \le c y_z^{8/\kappa-\kappa/2} \le c|z|^{8/\kappa-\kappa/2}.$$

The integral over $|w| > 2y_z$ is bounded by a constant times

$$\int_{y_w \le 2a, 2y_z \le |w| \le |z|/2} y_w^{8/\kappa-2} |w|^{-\kappa/2}\, dA(w) \le c|z|^{8/\kappa-\kappa/2}.$$

We therefore get

$$\Phi_2(z) \le ce^{-\beta|z|^2/4} |z|^{2-8/\kappa} |z|^{8/\kappa-\kappa/2} \le c|z|^{2-\kappa/2} e^{-\beta|z|^2/4} \le c|z|^{-\kappa/2}. \qquad \square$$

5.5. *Second moment.*

LEMMA 5.9. *If $\kappa < 8$ and $m < \infty$, there exists $c = c_m$ such that if $D \in \mathcal{D}_m$, $z, w \in \mathbb{H}$ and $2at \ge y_z, y_w$,*

$$F_{tD}(z, w, t^2) \le ct^{-\zeta} |z|^{\zeta/2} |w|^{\zeta/2} [\sin\theta_z]^{\zeta/2-1/4-2/\kappa} [\sin\theta_w]^{\zeta/2-1/4-2/\kappa}, \tag{5.30}$$

where

$$\zeta = \frac{4}{\kappa} - \frac{3\kappa}{16} - 1. \tag{5.31}$$

PROOF. By the Cauchy–Schwarz inequality, it suffices to prove the result for $z = w$, and by scaling we may assume $y_z = 1$. Therefore, it suffices to prove

$$F_D(x+i, x+i, t^2) \le ct^{-\zeta} (x^2+1)^{1/4+2/\kappa}, \qquad t \ge 1/2a.$$

We let $z = x + i$ and write $Z_t = X_t + iY_t = h_t(z) - U_t$. Consider the martingale $N_t = N_t(z)$ as in (5.19) with

$$r = \frac{4}{\kappa} + \frac{1}{2}, \qquad \lambda = \frac{2}{\kappa} + \frac{3\kappa}{32} + 1, \qquad r - \frac{\kappa r^2}{8} = \lambda - \frac{\kappa r}{4} = \frac{2}{\kappa} - \frac{\kappa}{32}. \tag{5.32}$$

Since $\mathbb{E}[N_{t^2}] = M_0$ and $Y_{t^2}, |Z_{t^2}| \asymp t$ when $\mathcal{I}_{t^2}(z; tD) = 1$,

$$\mathbb{E}[|h'_{t^2}(z)|^{2/\kappa+3\kappa/32+1}\mathcal{I}_{t^2}(z; tD)] \le ct^{\kappa/32-2/\kappa}(x^2+1)^{2/\kappa+1/4}.$$

From (5.18), we know that

$$|h'_{t^2}(z)| \le \sqrt{2at^2+1} \le ct.$$

Note that

$$2d - \left(\frac{2}{\kappa} + \frac{3\kappa}{32} + 1\right) = 1 - \frac{2}{\kappa} + \frac{5\kappa}{32}.$$

Hence

$$\begin{aligned}\mathbb{E}[|h'_{t^2}(z)|^{2d}\mathcal{I}_{t^2}(z; tD)] &\le ct^{1-2/\kappa+5\kappa/32}\mathbb{E}[|h'_{t^2}(z)|^{2/\kappa+3\kappa/32+1}\mathcal{I}_{t^2}(z; tD)] \\ &\le ct^{-\zeta}(x^2+1)^{2/\kappa+1/4}. \qquad \square\end{aligned}$$

REMARK. We have not given the motivation for the choice (5.32). See [6] for a discussion of this.

REMARK. The estimate (5.30) for $z \ne w$ makes use of the Cauchy–Schwarz inequality,

$$\begin{aligned}&(\mathbb{E}[|h'_t(z)|^d|h'_t(w)|^d\mathcal{I}_t(z, w; D)])^2 \\ &\quad \le \mathbb{E}[|h'_t(z)|^{2d}\mathcal{I}_t(z; D)]\mathbb{E}[|h'_t(w)|^{2d}\mathcal{I}_t(w; D)].\end{aligned}$$

If z and w are close [e.g., if w is in the disk of radius $\mathrm{Im}(z)/2$ about z], then the distortion theorem tells us that $|h'_t(w)| \asymp |h'_t(z)|$ and then the two sides of the inequality agree up to a multiplicative constant. However, if z, w are far apart (in the hyperbolic metric), the right-hand side can be much larger than the left-hand side. Improving this estimate for z, w far apart is the key for proving good second moment bounds.

The next lemma proves the $s = 0$ case of (5.5). A similar argument proves (5.5) for all $0 \le s \le 3(1+a)$, so in the next section we can restrict our consideration to $s \ge 3(1+a)$.

LEMMA 5.10. *If $\kappa < \kappa_0$, there is a $c < \infty$ such that for all $t \ge 1$,*

$$\int_{\mathbb{H}}\int_{\mathbb{H}} F(z, w, t^2)\, d\mu(z)\, d\mu(w) \le ct^{-\zeta}.$$

PROOF. Since $t \ge 1$, (5.30) gives

$$F(z, w, t^2) \le ct^{-\zeta}|z|^{\zeta/2}|w|^{\zeta/2}[\sin\theta_z]^{-3\kappa/32-3/4}[\sin\theta_w]^{-3\kappa/32-3/4}.$$

Hence by (5.2) it suffices to show that

$$\int_{\mathbb{H}} |z|^{\zeta/2}[\sin\theta_z]^{-3\kappa/32-3/4}[\sin\theta_z]^{\kappa/8+8/\kappa-2}|z|^{\kappa/8-1}e^{-\beta|z|^2}\,dA(z) < \infty.$$

This will be true provided that

$$-\frac{3\kappa}{32}-\frac{3}{4}+\frac{\kappa}{8}+\frac{8}{\kappa}-2 > -1,$$

which holds for $\kappa < \kappa_0$ [see (3.4)]. $\square$

5.6. *The correlation.* In this section, we state the hardest estimate and then show how it can be used to prove the main result. It will be useful to introduce some notation. For $s \geq 3(1+a)$, let

$$v(w,s) = v_m(w,s) = s^{2-d-\zeta/2}\sup[t^{\zeta}y_z^{-\zeta/2}[\sin\theta_z]^{1/4+2/\kappa}F_{tD}(w,z,t^2)],$$

where the supremum is over all $D \in \mathcal{D}_m$, $t \geq 2s$ and all $z \in \mathbb{H}$ with $|z| \geq 3(1+a)s$. In other words, if $t \geq |z| \geq 3(1+a)$,

$$\begin{aligned} F_{tD}(w,z,t^2) &\leq ct^{-\zeta}|z|^{d-2+\zeta/2}y_z^{\zeta/2}[\sin\theta_z]^{-1/4-2/\kappa}v(w,|z|) \\ &= ct^{-\zeta}[\sin\theta_z]^{\zeta/2-1/4-2/\kappa}|z|^{d-2+\zeta}v(w,|z|). \end{aligned} \tag{5.33}$$

The main estimate is the following. The hardest part, (5.34), will be proved in the next subsection.

PROPOSITION 5.11. *If $\kappa < \kappa_0$, there exists $u < \frac{8}{\kappa}$ such that for each m there exists $c < \infty$ such that for all $s \geq 3(a+1)$ and $w \in \mathbb{H}$ with $y_w \leq 2a$,*

$$v(w,s) \leq c[\sin\theta_w]^{2-d-u}|w|^{2-d} = c[\sin\theta_w]^{1-\kappa/8-u}|w|^{2-d}. \tag{5.34}$$

In particular,

$$\int v(w,s)\,d\mu(w) < c,$$

and hence if $D \in \mathcal{D}_m$ and $z \in \mathbb{H}$,

$$\int F_{tD}(w,z,t^2)\,d\mu(w) \leq ct^{-\zeta}[\sin\theta_z]^{\zeta/2-1/4-2/\kappa}|z|^{d-2+\zeta}. \tag{5.35}$$

PROOF. We delay the proof of (5.34) to Section 5.7, but we will show here how it implies the other two statements. Using (5.2), we have

$$\begin{aligned} &\int v(w,s)\,d\mu(w) \\ &\quad \leq c\int [\sin\theta_w]^{1-\kappa/8-u}|w|^{2-d}|w|^{d-2}[\sin\theta_w]^{\kappa/8+8/\kappa-2}e^{-\beta|w|^2}\,dA(w) < \infty. \end{aligned}$$

The last inequality uses $u < 8/\kappa$. The estimate (5.35) for $|z| \geq 3(1+a)$ follows immediately from (5.33); for other z it is proved as in Lemma 5.10. $\square$

COROLLARY 5.12. *If $\kappa < \kappa_0$, then for every m there is a c such that if $D \in \mathcal{D}_m$, $s^2 \geq 1, t^2 \geq 1$.*

$$\sum_{j=0}^{s^2-1} \int_{\mathbb{H}} \int_{\mathbb{H}} F_{tD}(z, w, j + t^2, t^2)\, d\mu(w)\, d\mu(z) \leq c(t/s)^{-\zeta} s^{2(d-1)}.$$

PROOF ASSUMING PROPOSITION 5.11. From Lemma 5.4 and (5.2), we know that

$$\sum_{j=0}^{s^2-1} \int_{\mathbb{H}} \int_{\mathbb{H}} F(z, w, j + t^2, t^2)\, d\mu(w)\, d\mu(z)$$
$$\leq c \int_{\mathbb{H}} \left[\int_{\mathbb{H}} F(z, w, t^2)\, d\mu(w) \right] [\sin \theta_z]^{\kappa/8 + 8/\kappa - 2} |z|^{d-2} e^{-\beta |z|^2/s^2}\, dA(z).$$

Using the previous lemma and estimating as in Lemma 5.10, we see that for $\kappa < \kappa_0$ this is bounded by a constant times

$$t^{-\zeta} \int_{\mathbb{H}} |z|^{\zeta + 2(d-2)} e^{-\beta |z|^2/s^2}\, dA(z) = c t^{-\zeta} s^{\zeta + 2d - 2}. \qquad \square$$

5.7. *Proof of* (5.34). It was first observed in [12] that when studying moments of $|h_t'(z)|$ for a fixed z it is useful to consider a paraametrization such that $Y_t(z)$ grows deterministically. The next lemma uses this reparametrization to get a result about fixed time. The idea is to have a stopping time in the new parametrization that corresponds to a bounded stopping time in the original parametrization. A version of this stopping time appears in [6] in the proof of the first moment estimate. If $\kappa < \kappa_0$, there exists u satisfying

$$\frac{7}{4} - \frac{\kappa}{32} < u < \frac{8}{\kappa}. \tag{5.36}$$

For convenience, we fix one such value of u. Let $N_t(w)$ be the martingale from (5.19) which we can write as

$$N_t(w) = |h_t'(w)|^d Y_t(w)^{2-d} [R_t(w)^2 + 1]^{1/2}, \qquad R_t(w) = X_t(w)/Y_t(w),$$

and recall $\hat{u}$ from (5.23).

LEMMA 5.13. *If $a > a_0$, there exists c such that the following is true. For each t and each $w = x + yi$ with $y \leq t$, there exists a stopping time τ such that*

$$\tau \leq t^2,$$

$$|U_s| \leq (a + 2)t, \qquad 0 \leq s \leq \tau, \tag{5.37}$$

$$\begin{aligned}
&\mathbb{E}[|h'_\tau(w)|^d Y_\tau^{\zeta/2}(R_\tau^2+1)^{1/8+1/\kappa}] \\
&\quad = \mathbb{E}[N_\tau Y_\tau^{\zeta/2+d-2}(R_\tau^2+1)^{a/2-3/8}] \\
&\quad \le c(t+1)^{d-2+\zeta/2}|w|^{2-d}[\sin\theta_w]^{2-d-u}.
\end{aligned}$$

Here $N_s = N_s(w)$, $Y_s = Y_s(w)$, $R_s = X_s(w)/Y_s(w)$.

Moreover, if $a > 1/4$, *there exists* c *such that*

$$\mathbb{E}[|h'_{t^2}(w)|^d] \le c[(x/y)^2+1]^{\hat{u}/2}\left(\frac{t}{y}\vee 1\right)^{d-2}. \tag{5.38}$$

PROOF. By scaling, it suffices to prove the lemma for $y = 1$, that is, $w = x + i$. Without loss of generality, we assume $x \ge 0$. If $t \le 1$, we can choose the trivial stopping time $\tau \equiv 0$ and (5.38) is easily derived from the Loewner equation. Hence we may assume $t \ge 1$. We write $t = e^{al}$, $x = e^{am}$. For notational ease we will assume that l, m are integers, but it is easy to adjust the proof for other l, m. We will define the stopping time for all $a > 1/4$; it will be used for proving (5.38).

We consider a parametrization in which the logarithm of the imaginary part grows linearly. Let

$$\sigma(s) = \inf\{u : Y_u = e^{as}\}, \qquad \hat{X}_s = X_{\sigma(s)}, \qquad K_s = R_{\sigma(s)} = e^{-as}\hat{X}_s,$$

and note that $\hat{Y}_s = Y_{\sigma(s)} = e^{as}$. Using the Loewner equation, we can see that

$$\partial_s\sigma(s) = \hat{X}_s^2 + \hat{Y}_s^2 = e^{2as}(K_s^2+1).$$

Let $\hat{N}_s = N_{\sigma(s)}(x+i)$,

$$\hat{N}_s = \hat{N}_s(x+i) = |h'_{\sigma(s)}(x+i)|^d e^{(2-d)as}(K_s^2+1)^{1/2}.$$

Since $\hat{N}_s$ is a time change of a martingale, it is easy to see that it is a martingale. Note that $K_0 = x = e^{am}$.

We first define our stopping time in terms of the new parametrization. Let ρ be the smallest r such that

$$\hat{X}_r^2 + \hat{Y}_r^2 \ge \frac{e^{2al}}{(l-r+1)^4},$$

that is,

$$\sqrt{K_r^2+1} \ge \frac{e^{a(l-r)}}{(l-r+1)^2}.$$

One can readily check that the following properties hold:

$$\begin{aligned}
&\rho \le l, \\
&\rho = 0 \qquad \text{if } m \ge l,
\end{aligned}$$

$$\hat{X}_s^2 \le \hat{X}_s^2 + \hat{Y}_s^2 \le \frac{e^{2al}}{(l-s+1)^4}, \qquad 0 \le s \le \rho,$$

$$\sigma(\rho) = \int_0^\rho e^{2as}[K_s^2 + 1]\,ds \le \int_0^l \frac{e^{2al}}{(l-s+1)^4}\,ds \le e^{2al},$$

$$\int_0^\rho |\hat{X}_r|\,dr \le \int_0^l \frac{e^{al}}{(l-s+1)^2}\,ds \le e^{al}.$$

We define

$$\tau = \sigma(\rho) \le e^{2al},$$

that is, τ is essentially the same stopping time as ρ except using the original parametrization. Note that

$$\int_0^\tau \frac{|X_t|}{X_t^2 + Y_t^2}\,dt = \int_0^\rho |\hat{X}_t|\,dt \le e^{al}.$$

Recall that

$$dX_s = \frac{aX_s}{X_s^2 + Y_s^2}\,ds - dU_s,$$

which implies

$$-U_s = (X_s - X_0) - \infty_0^s \frac{aX_r}{X_r^2 + Y_r^2}\,dr.$$

If $X_0 \ge e^{al}$, then $\tau = 0$ and (5.37) holds immediately. Otherwise,

$$|U_t| \le |X_t| + |X_0| + a\int_0^\rho \frac{|X_s|}{X_s^2 + Y_s^2}\,ds \le (2+a)e^{al}.$$

This gives (5.37).

Let A_j be the event

$$A_j = \{t - j < \rho \le t - j + 1\} = \{e^{a(t-j)} < Y_\tau \le e^{a(t-j+1)}\}.$$

On the event A_j, we have

$$Y_\tau \asymp e^{at}e^{-aj}, \qquad R_\tau^2 + 1 \asymp e^{2aj} j^{-4}.$$

The Girsanov theorem implies that

$$\mathbb{E}[N_\tau 1_{A_j}] = N_0 \mathbf{P}^*(A_j) = (x^2+1)^{1/2}\mathbf{P}^*(A_j) \asymp e^{am}\mathbf{P}^*(A_j),$$

where we use $\mathbf{P}^*$ to denote the probabilities given by weighting by the martingale N_t. We claim that there exist c, β such that

$$\mathbf{P}^*(A_j) \le c j^\beta e^{(4a-1)(m-j)a}. \tag{5.39}$$

To see this, one considers the process in the new parametrization and notes that after weighting by the martingale $\hat{N}$, K_t satisfies

$$dK_s = (1-2a)K_s\,ds + \sqrt{K_s^2+1}\,dW_s, \tag{5.40}$$

where W_s is a standard Brownian motion with $K_0 = x$. Equivalently, $K_s = \sinh J_s$ where J_s satisfies

$$dJ_s = -q\tanh J_s\,ds + dW_s, \tag{5.41}$$

with $J_0 = \sinh^{-1} x, q = \frac{1}{2} - 2a$. Standard techniques (see [6], Section 7) show that J_t is positive recurrent with invariant density proportional to $[\cosh x]^{-2q}$. If $0 < x < y$, then the probability starting at x of reaching y before 0 is bounded by $c[\cosh x/\cosh y]^{2q}$. Using these ideas, we get that for every k,

$$\mathbf{P}^x\{y \le J_t \le y+1 \text{ for some } k \le t \le k+1\} \le c\left(\frac{\cosh x}{\cosh y}\right)^{2q}.$$

On the event A_j, we know that $Y_{t^2} \ge Y_\tau \asymp e^{al}e^{-aj}$. The martingale property and (5.39) imply that

$$\mathbb{E}[N_{t^2}1_{A_j}] = \mathbb{E}[N_\tau 1_{A_j}] = (x^2+1)^{1/2}\mathbf{P}^*(A_j) \le ce^{am}\big[1 \wedge j^\beta e^{(4a-1)(m-j)a}\big].$$

Therefore,

$$\begin{aligned}
&e^{-am}e^{al(2-d)}\mathbb{E}[|h'_{t^2}(z)|^d 1_{A_j}]\\
&\qquad\le ce^{aj(2-d)}e^{-am}\mathbb{E}[N_{t^2}1_{A_j}]\\
&\qquad\le ce^{aj(2-d)}\big[1 \wedge j^\beta e^{(4a-1)(m-j)a}\big],\\
&e^{-am}e^{at(2-d)}\mathbb{E}[|h'_{t^2}(z)|^d]\\
&\qquad\le c\sum_{j=1}^{\infty} e^{aj(2-d)}\big[1 \wedge j^\beta e^{(4a-1)(m-j)a}\big]\\
&\qquad\le c\left[\sum_{j=1}^{m} e^{aj(2-d)} + e^{(4a-1)m}\sum_{j=m+1}^{\infty} j^\beta e^{aj[(2-d)+1-4a]}\right]\\
&\qquad\le c\left[e^{am(2-d)} + e^{(4a-1)m}\sum_{j=m+1}^{\infty} j^\beta e^{aj[(2-d)+1-4a]}\right]\\
&\qquad\le cm^\beta e^{am(2-d)}.
\end{aligned}$$

The last inequality requires

$$2-d+1-4a<0,$$

which is readily checked for $a > 1/4$. Therefore,

$$\begin{aligned}\mathbb{E}[|h'_{t^2}(z)|^d] &\le ct^{d-2}m^{\beta}e^{am(3-d)}\\ &\le ct^{d-2}[\log(x^2+2)]^{\beta}(x^2+1)^{1-\kappa/16}\\ &\le ct^{d-2}(x^2+1)^{\hat{u}/2}.\end{aligned}$$

This establishes (5.38).

Note that

$$\begin{aligned}&\mathbb{E}[N_\tau(R_\tau^2+1)^{a/2-3/8}Y_\tau^{\zeta/2+d-2}1_{A_j}]\\ &\qquad \asymp j^{-2a+3/2}e^{aj(a-3/4)}e^{(l-j)a(\zeta/2+d-2)}\mathbb{E}[N_\tau 1_{A_j}].\end{aligned}$$

Therefore,

$$\begin{aligned}&e^{-am}e^{-al(\zeta/2+d-2)}\mathbb{E}[N_\tau(R_\tau^2+1)^{a/2-3/8}Y_\tau^{\zeta/2+d-2}]\\ &\quad\le c\sum_{j=1}^{\infty} j^{-2a+3/2}e^{aj(a-3/4)}e^{-ja(\zeta/2+2-d)}\bigl[1\wedge j^{\beta}e^{(4a-1)(m-j)}\bigr]\\ &\quad\le cm^{\beta}e^{ma(-\zeta/2+d-2+a-3/4)}.\end{aligned}$$

The last inequality requires

$$-\frac{\zeta}{2}+2-d+a-\frac{3}{4}+1-4a<0.$$

Recalling that

$$\frac{\zeta}{2}=a-\frac{3}{16a}-\frac{1}{2},$$

this becomes

$$-4a-\frac{1}{16a}+\frac{7}{4}<0.$$

This is true if $\kappa < \kappa_0$ [see (3.4)]. $\square$

PROPOSITION 5.14. *If $a > a_0$, for every $\varepsilon > 0$ there is a c such that the following is true. Assume*

$$z = x + iy, \qquad w = \hat{x} + i\hat{y} \in \mathbb{H}$$

with $\hat{y} \le 2a+1$ and $|z| \ge 3(a+1)$. Then for $t \ge 2$, $s \ge 1$,

$$\begin{aligned}F_{stD}(sz, w, (st)^2) &\le ct^{-\zeta}|z|^{\zeta/2}[\sin\theta_z]^{\zeta/2-1/4-2/\kappa}\\ &\qquad\times[\sin\theta_w]^{2-d-u}(|w|/s)^{2-d}.\end{aligned}\tag{5.42}$$

REMARK. If $|z| \geq 3(a+1)s$ and $t \geq 2s$, we can write (5.42) as

$$F_{stD}(z, w, t^2) \leq c(t/s)^{-\zeta} s^{d-2} [\sin\theta_z]^{\zeta/2-1/4-2/\kappa} [\sin\theta_w]^{2-d-u} |w|^{2-d}.$$

Therefore this proposition completes the proof of (5.34).

PROOF OF (5.34). By scaling, $F_{stD}(sz, w, (st)^2) = F_{tD}(z, w/s, t^2)$; hence, without loss of generality we may assume $s = 1$. We assume that τ is a stopping time as in the previous lemma for w and time 1. In particular, $0 \leq \tau \leq 1$. We can find a domain D' such that for all $1/2 \leq r \leq 1$, $rD \subset D'$.

We write $Z_s(z) = h_s(z) - U_s$, $Z_s(w) = h_s(w) - U_s$, etc. for the images under the flow. By definition, $F_{tD}(z, w, t^2) = \mathbb{E}[\Lambda]$ where Λ denotes the random variable

$$\Lambda = \Lambda_D(z, w, t^2) := |h'_{t^2}(z)|^d |h'_{t^2}(w)|^d \mathcal{I}_{t^2}(z, w; D),$$

and note that

$$\mathbb{E}[\Lambda | \mathcal{G}_\tau] = |h'_\tau(z)|^d |h'_\tau(w)|^d F_{tD}\big(Z_\tau(z), Z_\tau(w), t^2 - \tau\big). \tag{5.43}$$

Since $|U_s| \leq 2 + a$ for $s \leq \tau$, it follows from (5.10) that

$$\partial_s |h_s(z)| \leq 1, \qquad |h_s(z) - z| \leq s, \qquad s \leq \hat{\tau},$$

where $\hat{\tau}$ denotes the minimum of τ and the first time that $|h_s(z)| \leq 2 + 2a$. Since $|z| \geq 3 + 3a$, this implies the following estimates for $0 \leq s \leq \tau$:

$$|h_s(z) - z| \leq 1, \qquad |h_s(z)| \geq 2 + 3a, \qquad |U_s - h_s(z)| \geq a,$$
$$|Z_s(z) - z| \leq |h_s(z) - z| + |U_s| \leq 3 + a.$$

In particular, since $|z| \geq 3(1+a)$, there exists c_1, c_2 such that

$$c_1(x^2+1) \leq X_s(z)^2 + 1 \leq c_2(x^2+1), \qquad 0 \leq s \leq \tau,$$
$$y_z \leq Y_s(z) \leq c_2 y_z, \qquad 0 \leq s \leq \tau.$$

We therefore get

$$F_{tD}\big(Z_\tau(z), Z_\tau(w), t^2 - \tau\big) \leq \sup F_{\tilde{t}D'}(\tilde{z}, Z_\tau(w), \tilde{t}^2),$$

where the supremum is over all $t^2 - 1 \leq \tilde{t}^2 \leq t^2$ and all $\tilde{z} = \tilde{x} + i\tilde{y}$ with

$$c_1(x^2+1) \leq \tilde{x}^2 + 1 \leq c_2(x^2+1), \qquad y \leq \tilde{y} \leq c_2 y.$$

Using (5.30), we get

$$F_{tD}\big(Z_\tau(z), Z_\tau(w), t^2 - \tau\big)$$
$$\leq ct^{-\zeta} |z|^{\zeta/2} [\sin\theta_z]^{\zeta/2-1/4-2/\kappa} \big(R_\tau^2(w) + 1\big)^{1/8+1/\kappa} Y_\tau(w)^{\zeta/2}.$$

By differentiating (5.10), we get

$$|\partial_s h'_s(z)| \leq a|h'_s(z)|, \qquad 0 \leq s \leq \tau, \qquad |h'_\tau(z)| \leq e^a.$$

Therefore, plugging into (5.43), we get

$$\mathbb{E}[\Lambda|\mathcal{G}_\tau] \le ct^{-\zeta}|z|^{\zeta/2}[\sin\theta_z]^{\zeta/2-1/4-2/\kappa}|h'_\tau(w)|^d\big(R^2_\tau(w)+1\big)^{1/8+1/\kappa}Y_t(w)^{\zeta/2}.$$

Taking expectations, and using the previous lemma, we get

$$F_{tD}(z,w,t^2) \le ct^{-\zeta}|z|^{\zeta/2}[\sin\theta_z]^{\zeta/2-1/4-2/\kappa}|w|^{2-d}[\sin\theta_w]^{2-d-u}. \qquad \square$$

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