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On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions

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Abstract. In [*Probab. Theory Related Fields* **141** (2008) 543–567], the authors proved the uniqueness among the solutions of quadratic BSDEs with convex generators and unbounded terminal conditions which admit every exponential moments. In this paper, we prove that uniqueness holds among solutions which admit some given exponential moments. These exponential moments are natural as they are given by the existence theorem. Thanks to this uniqueness result we can strengthen the nonlinear Feynman–Kac formula proved in [*Probab. Theory Related Fields* **141** (2008) 543–567].

Résumé. Les auteurs de l'article [*Probab. Theory Related Fields* **141** (2008) 543–567] ont prouvé un résultat d'unicité pour les solutions d'EDSRs quadratiques de générateur convexe et de condition terminale non bornée ayant tous leurs moments exponentiels finis. Dans ce papier, nous prouvons que ce résultat d'unicité reste vrai pour des solutions qui admettent uniquement certains moments exponentiels finis. Ces moments exponentiels sont reliés de manière naturelle à ceux présents dans le théorème d'existence. À l'aide de ce résultat d'unicité nous pouvons améliorer la formule de Feynman–Kac non linéaire prouvée dans [*Probab. Theory Related Fields* **141** (2008) 543–567].

MSC: 60H10

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1. Introduction

In this paper, we consider the following quadratic backward stochastic differential equation (BSDE in short for the remaining of the paper)

$$Y_t = \xi - \int_t^T g(s, Y_s, Z_s)\,\mathrm{d}s + \int_t^T Z_s\,\mathrm{d}W_s, \quad 0 \le t \le T, \tag{1.1}$$

where the generator $-g$ is a continuous real function that is concave and has a quadratic growth with respect to the variable z. Moreover, ξ is an unbounded random variable (see, e.g., [8] for the case of quadratic BSDEs with bounded terminal conditions). Let us recall that, in the previous equation, we are looking for a pair of processes (Y, Z) which is required to be adapted with respect to the filtration generated by the $\mathbb{R}^d$-valued Brownian motion W. In [4], the authors prove the uniqueness among the solutions which satisfy for any $p > 0$,

$$\mathbb{E}\big[\mathrm{e}^{p \sup_{0\le t\le T}|Y_t|}\big] < \infty.$$

The main contribution of this paper is to strengthen their uniqueness result. More precisely, we prove the uniqueness among the solutions satisfying:

$$\exists p > \bar{\gamma}, \exists \varepsilon > 0 \quad \mathbb{E}\big[\mathrm{e}^{p \sup_{0\le t\le T}(Y_t^- + \int_0^t \bar{\alpha}_s\,\mathrm{d}s)} + \mathrm{e}^{\varepsilon \sup_{0\le t\le T} Y_t^+}\big] < +\infty,$$

where $\bar{\gamma} > 0$ and $(\bar{\alpha}_t)_{t\in[0,T]}$ is a progressively measurable nonnegative stochastic process such that, $\mathbb{P}$-a.s.,

$$\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1\times d} \quad g(t, y, z) \le \bar{\alpha}_t + \bar{\beta}|y| + \frac{\bar{\gamma}}{2}|z|^2.$$

Our method is different from that in [4] where the authors apply the so-called θ-difference method, i.e. estimating $Y^1 - \theta Y^2$, for $\theta \in (0, 1)$, and then letting $\theta \to 1$. Whereas in this paper, we apply a verification method: first we define a stochastic control problem and then we prove that the first component of any solution of the BSDE is the optimal value of this associated control problem. Thus the uniqueness follows immediately. Moreover, using this representation, we are able to give a probabilistic representation of the following PDE:

$$\partial_t u(t, x) + \mathcal{L}u(t, x) - g\big(t, x, u(t, x), -\sigma^* \nabla_x u(t, x)\big) = 0, \qquad u(T, \cdot) = h,$$

where h and g have a "not too high" quadratic growth with respect to the variable x. We remark that the probabilistic representation is also given by [4] under the condition that h and g are subquadratic, i.e.:

$$\forall (t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1\times d} \quad \big|h(x)\big| + \big|g(t, x, y, z)\big| \le f(t, y, z) + C|x|^p$$

with $f \ge 0$, $C > 0$ and $p < 2$.

The paper is organized as follows. In Section 2, we prove an existence result in the spirit of [3] and [4]: here we work with generators $-g$ such that g^- has a linear growth with respect to variables y and z. As in part 5 of [3], this assumption allows us to reduce hypothesis of [4]. Section 3 is devoted to the optimal control problem from which we get as a byproduct a uniqueness result for quadratic BSDEs with unbounded terminal conditions. Finally, in the last section we derive the nonlinear Feynman–Kac formula in this framework.

Let us close this introduction by giving the notations that we will use in all the paper. For the remaining of the paper, let us fix a nonnegative real number $T > 0$. First of all, $(W_t)_{t\in[0,T]}$ is a standard Brownian motion with values in $\mathbb{R}^d$ defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $(\mathcal{F}_t)_{t\ge 0}$ is the natural filtration of the Brownian motion W augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$. The sigma-field of predictable subsets of $[0, T] \times \Omega$ is denoted by $\mathcal{P}$.

As mentioned before, we will deal only with real valued BSDEs which are equations of type (1.1). The function $-g$ is called the generator and ξ the terminal condition. Let us recall that a generator is a random function $[0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{1\times d} \to \mathbb{R}$ which is measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^{1\times d})$ and a terminal condition is simply a real $\mathcal{F}_T$-measurable random variable. By a solution to the BSDE (1.1) we mean a pair $(Y_t, Z_t)_{t\in[0,T]}$ of predictable processes with values in $\mathbb{R} \times \mathbb{R}^{1\times d}$ such that $\mathbb{P}$-a.s., $t \mapsto Y_t$ is continuous, $t \mapsto Z_t$ belongs to $L^2(0, T)$, $t \mapsto g(t, Y_t, Z_t)$ belongs to $L^1(0, T)$ and $\mathbb{P}$-a.s. (Y, Z) verifies (1.1). We will sometimes use the notation BSDE(ξ, f) to say that we consider the BSDE whose generator is f and whose terminal condition is ξ.

For any real $p \ge 1$, $\mathcal{S}^p$ denotes the set of real-valued, adapted and càdlàg processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|_{\mathcal{S}^p} := \mathbb{E}\Big[\sup_{0\le t\le T} |Y_t|^p\Big]^{1/p} < +\infty.$$

M^p denotes the set of (equivalent class of) predictable processes $(Z_t)_{t\in[0,T]}$ with values in $\mathbb{R}^{1\times d}$ such that

$$\|Z\|_{M^p} := \mathbb{E}\bigg[\bigg(\int_0^T |Z_s|^2\,\mathrm{d}s\bigg)^{p/2}\bigg]^{1/p} < +\infty.$$

Finally, we will use the notation $Y^* := \sup_{0\le t\le T} |Y_t|$ and we recall that Y belongs to the class (D) as soon as the family $\{Y_\tau\colon \tau \le T \text{ stopping time}\}$ is uniformly integrable.

2. An existence result

In this section, we prove a slight modification of the existence result for quadratic BSDEs obtained in [4] by using a method applied in Section 5 of [3]. We consider here the case where g^- has a linear growth with respect to variables y and z. Let us assume the following on the generator.

Assumption A.1. *There exist three constants $\beta \geq 0$, $\gamma > 0$ and $r \geq 0$ together with two progressively measurable nonnegative stochastic processes $(\bar{\alpha}_t)_{0\leq t\leq T}$, $(\underline{\alpha}_t)_{0\leq t\leq T}$ and a deterministic continuous nondecreasing function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ with $\phi(0)=0$ such that, $\mathbb{P}$-a.s.:*

(1) *for all $t\in[0,T]$, $(y,z)\mapsto g(t,y,z)$ is continuous*;
(2) *monotonicity in y*: $\forall(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^{1\times d}$,

$$y\big(g(t,0,z)-g(t,y,z)\big)\leq\beta|y|^2;$$

(3) *growth condition*: $\forall(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^{1\times d}$,

$$-\underline{\alpha}_t-r\big(|y|+|z|\big)\leq g(t,y,z)\leq\bar{\alpha}_t+\phi\big(|y|\big)+\frac{\gamma}{2}|z|^2.$$

Theorem 2.1. *Let A.1 hold. If there exists $p>1$ such that*

$$\mathbb{E}\bigg[\exp\bigg(\gamma e^{\beta T}\xi^-+\gamma\int_0^T\bar{\alpha}_t e^{\beta t}\,dt\bigg)+\big(\xi^+\big)^p+\bigg(\int_0^T\underline{\alpha}_t\,dt\bigg)^p\bigg]<+\infty$$

then the BSDE (1.1) has a solution (Y,Z) such that

$$-\frac{1}{\gamma}\log\mathbb{E}\bigg[\exp\bigg(\gamma e^{\beta(T-t)}\xi^-+\gamma\int_t^T\bar{\alpha}_r e^{\beta(r-t)}\,dr\bigg)\bigg|\mathcal{F}_t\bigg]\leq Y_t\leq Ce^{CT}\bigg(\mathbb{E}\bigg[\big(\xi^+\big)^p+\bigg(\int_t^T\underline{\alpha}_r\,dr\bigg)^p\bigg|\mathcal{F}_t\bigg]\bigg)^{1/p},$$

with C a constant that does not depend on T.

Proof. We will fit the proof of Proposition 4 in [3] to our situation. Without loss of generality, let us assume that r is an integer. For each integer $n\geq r$, let us consider the function

$$g_n(t,y,z):=\inf\big\{g(t,p,q)+n|p-y|+n|q-z|,(p,q)\in\mathbb{Q}^{1+d}\big\}.$$

g_n is well defined and it is globally Lipschitz continuous with constant n. Moreover, $(g_n)_{n\geq r}$ is increasing and converges pointwise to g. Dini's theorem implies that the convergence is also uniform on compact sets. We have also, for all $n\geq r$,

$$h(t,y,z):=-\underline{\alpha}_t-r\big(|y|+|z|\big)\leq g_n(t,y,z)\leq g(t,y,z).$$

Let (Y^n,Z^n) be the unique solution in $\mathcal{S}^p\times M^p$ to BSDE$(\xi,-g_n)$. It follows from the classical comparison theorem that

$$Y^{n+1}\leq Y^n\leq Y^r.$$

Let us prove that for each $n\geq r$

$$Y_t^n\geq-\frac{1}{\gamma}\log\mathbb{E}\bigg[\exp\bigg(\gamma e^{\beta(T-t)}\xi^-+\gamma\int_t^T\bar{\alpha}_r e^{\beta(r-t)}\,dr\bigg)\bigg|\mathcal{F}_t\bigg]:=X_t.$$

Let $(\tilde{Y}^n,\tilde{Z}^n)$ be the unique solution in $\mathcal{S}^p\times M^p$ to BSDE$(-\xi^-,-g_n^+)$. It follows from the classical comparison theorem that $\tilde{Y}^n\leq Y^n$ and $\tilde{Y}^n\leq 0$. Then, according to Proposition 3 in [4], we have $\tilde{Y}^n\geq X$ and so $Y^n\geq X$ for all

$n \geq r$. We set $Y = \inf_{n\geq r} Y^n$ and, arguing as in the proof of Proposition 3 in [4] or Theorem 2 in [3] with a localization argument, we construct a process Z such that (Y, Z) is a solution to BSDE$(\xi, -g)$. For the upper bound, let $(\bar{Y}, \bar{Z})$ be the unique solution in $\mathcal{S}^p \times M^p$ to BSDE$(\xi^+, -h)$. Then the classical comparison theorem gives us that $Y \leq Y^n \leq \bar{Y}$ and we apply a classical a priori estimate for L^p solutions of BSDEs in [2] to $\bar{Y}$. □

Corollary 2.2. *Let A.1 hold. We suppose that $\xi^- + \int_0^T \bar{\alpha}_t \,\mathrm{d}t$ has an exponential moment of order $\gamma \mathrm{e}^{\beta T}$ and there exists $p > 1$ such that $\xi^+ + \int_0^T \underline{\alpha}_t \,\mathrm{d}t \in L^p$.*

- *If $\xi^- + \int_0^T \bar{\alpha}_t \,\mathrm{d}t$ has an exponential moment of order $q\mathrm{e}^{\beta T}$ with $q > \gamma$ then the BSDE (1.1) has a solution (Y, Z) such that $\mathbb{E}[\mathrm{e}^{qA^*}] < +\infty$ with $A_t := Y_t^- + \int_0^t \bar{\alpha}_s \,\mathrm{d}s$.*
- *If $\xi^+ + \int_0^T \underline{\alpha}_t \,\mathrm{d}t$ has an exponential moment of order $\varepsilon C\mathrm{e}^{CT}$, with C given in Theorem 2.1, then the BSDE (1.1) has a solution (Y, Z) such that $\mathbb{E}[\mathrm{e}^{\varepsilon (Y^+)^*}] < +\infty$.*

Proof. Let us apply the existence result: BSDE (1.1) has a solution (Y, Z) and we have

$$A_t = Y_t^- + \int_0^t \bar{\alpha}_s \,\mathrm{d}s \leq \frac{1}{\gamma} \log \underbrace{\mathbb{E}\left[\exp\left(\gamma \mathrm{e}^{\beta T}\left(\xi^- + \int_0^T \bar{\alpha}_r \,\mathrm{d}r\right)\right)\Big|\mathcal{F}_t\right]}_{:=M_t}.$$

So $\mathrm{e}^{qA_t} \leq (M_t)^{q/\gamma}$ with $q/\gamma > 1$. Since $M^{q/\gamma}$ is a submartingale, we are able to apply the Doob's maximal inequality to obtain

$$\mathbb{E}\left[\mathrm{e}^{qA^*}\right] \leq C_q \mathbb{E}\left[\mathrm{e}^{q\mathrm{e}^{\beta T}(\xi^- + \int_0^T \bar{\alpha}_s \,\mathrm{d}s)}\right] < +\infty.$$

To prove the second part of the corollary, we define

$$N_t := \mathbb{E}\left[\left(\xi^+\right)^p + \left|\int_0^T \underline{\alpha}_s \,\mathrm{d}s\right|^p \Big|\mathcal{F}_t\right].$$

We set $q > 1$. There exists $C_{C,\varepsilon,p,q} \geq 0$ such that $x \mapsto \mathrm{e}^{C\mathrm{e}^{CT} x^{1/p}\varepsilon/q}$ is convex on $[C_{C,\varepsilon,p,q}, +\infty[$. We have $\mathrm{e}^{\varepsilon/q Y_t^+} \leq \mathrm{e}^{C\mathrm{e}^{CT}(C_{C,\varepsilon,p,q}+N_t)^{1/p}\varepsilon/q}$. Since $\mathrm{e}^{(C_{C,\varepsilon,p,q}+N)^{1/p}\varepsilon/q}$ is a submartingale, we are able to apply the Doob's maximal inequality to obtain

$$\mathbb{E}\left[\mathrm{e}^{\varepsilon (Y^+)^*}\right] \leq C\mathbb{E}\left[\mathrm{e}^{\varepsilon C\mathrm{e}^{CT}(C_{C,\varepsilon,p,q}+(\xi^+)^p + (\int_0^T \underline{\alpha}_s \,\mathrm{d}s)^p)^{1/p}}\right] \leq C\mathbb{E}\left[\mathrm{e}^{\varepsilon C\mathrm{e}^{CT}(\xi^+ + \int_0^T \underline{\alpha}_s \,\mathrm{d}s)}\right] < +\infty.$$ □

3. A uniqueness result

To prove our uniqueness result for the BSDE (1.1), we will introduce a stochastic control problem. For this purpose, we use the following assumption on g:

Assumption A.2. *There exist three constants $K_{g,y} \geq 0$, $\bar{\beta} \geq 0$ and $\bar{\gamma} > 0$ together with a progressively measurable nonnegative stochastic process $(\bar{\alpha}_t)_{t\in[0,T]}$ such that, $\mathbb{P}$-a.s.,*

- *for each $(t, z) \in [0, T] \times \mathbb{R}^{1\times d}$,*

$$\left|g(t, y, z) - g\left(t, y', z\right)\right| \leq K_{g,y}\left|y - y'\right| \quad \forall \left(y, y'\right) \in \mathbb{R}^2;$$

- *for each $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1\times d}$,*

$$g(t, y, z) \leq \bar{\alpha}_t + \bar{\beta}|y| + \frac{\bar{\gamma}}{2}|z|^2;$$

- *for each* $(t,y) \in [0,T] \times \mathbb{R}$, $z \mapsto g(t,y,z)$ *is a convex function*.

Since $g(t,y,\cdot)$ is a convex function we can define the Legendre–Fenchel transformation of g:

$$f(t,y,q) := \sup_{z\in\mathbb{R}^{1\times d}} \big(zq - g(t,y,z)\big) \quad \forall t \in [0,T], q \in \mathbb{R}^d, y \in \mathbb{R}.$$

f is a function with values in $\mathbb{R} \cup \{+\infty\}$ that verifies direct properties.

Proposition 3.1.

- $\forall (t,y,y',q) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$ *such that* $f(t,y,q) < +\infty$,

$$f\big(t,y',q\big) < +\infty \quad \textit{and} \quad \big|f(t,y,z) - f\big(t,y',z\big)\big| \le K_{g,y}\big|y-y'\big|;$$

- f *is a convex function in* q;
- $f(t,y,q) \ge -\bar{\alpha}_t - \bar{\beta}|y| + \frac{1}{2\bar{\gamma}}|q|^2$.

For $\varepsilon > 0$ and $p > \bar{\gamma}$ given we set $N \in \mathbb{N}^*$ such that

$$\frac{T}{N} < \left(\frac{1}{\bar{\gamma}} - \frac{1}{p}\right)\frac{1}{\bar{\beta}(1/p + 1/\varepsilon)}. \tag{3.1}$$

For $i \in \{0,\dots,N\}$ we define $t_i := \frac{iT}{N}$ and, for all real $\mathcal{F}_{t_{i+1}}$-measurable random variable η,

$$\begin{aligned}\mathcal{A}_{t_i,t_{i+1}}(\eta) := \Bigg\{ &(q_s)_{s\in[t_i,t_{i+1}]}\colon \int_{t_i}^{t_{i+1}} |q_s|^2\,\mathrm{d}s < +\infty\ \mathbb{P}\text{-a.s.}, \mathbb{E}^{\mathbb{Q}^i}\left[\int_{t_i}^{t_{i+1}} |q_s|^2\,\mathrm{d}s\right] < +\infty, \\ &\big(M_t^i\big)_{t\in[t_i,t_{i+1}]} \text{ is a martingale}, \mathbb{E}^{\mathbb{Q}^i}\left[|\eta| + \int_{t_i}^{t_{i+1}} \big|f(s,0,q_s)\big|\,\mathrm{d}s\right] < +\infty, \\ &\text{with } M_t^i := \exp\left(\int_{t_i}^t q_s\,\mathrm{d}W_s - \frac{1}{2}\int_{t_i}^t |q_s|^2\,\mathrm{d}s\right) \text{ and } \frac{\mathrm{d}\mathbb{Q}^i}{\mathrm{d}\mathbb{P}} := M_{t_{i+1}}^i \Bigg\}.\end{aligned}$$

Let q be in $\mathcal{A}_{t_i,t_{i+1}}(\eta)$, if this set is not empty. We define $\mathrm{d}W_t^q := \mathrm{d}W_t - q_t\,\mathrm{d}t$. Thanks to the Girsanov theorem, $(W_{t_i+h}^q - W_{t_i}^q)_{h\in[0,T/N]}$ is a Brownian motion under the probability $\mathbb{Q}^i$. So, we are able to apply Proposition 6.4 in [2] to obtain this existence result:

Proposition 3.2. *There exist two processes* $(Y^{\eta,q}, Z^{\eta,q})$ *such that* $(Y_t^{\eta,q})_{t\in[t_i,t_{i+1}]}$ *belongs to the class (D) under* $\mathbb{Q}^i$, $\int_{t_i}^{t_{i+1}} |Z_s^{\eta,q}|^2\,\mathrm{d}s < +\infty$ $\mathbb{P}$*-a.s.*, $\int_{t_i}^{t_{i+1}} |f(s,Y_s^{\eta,q},q_s)|\,\mathrm{d}s < +\infty$ $\mathbb{P}$*-a.s. and*

$$Y_t^{\eta,q} = \eta + \int_t^{t_{i+1}} f\big(s,Y_s^{\eta,q},q_s\big)\,\mathrm{d}s + \int_t^{t_{i+1}} Z_s^{\eta,q}\,\mathrm{d}W_s^q, \quad t_i \le t \le t_{i+1}.$$

We are now able to define the admissible control set:

$$\begin{aligned}\mathcal{A} := \Big\{ &(q_s)_{s\in[0,T]}\colon q_{|[t_{N-1},T]} \in \mathcal{A}_{t_{N-1},T}(\xi), \forall i \in \{N-2,\dots,0\}, q_{|[t_i,t_{i+1}]} \in \mathcal{A}_{t_i,t_{i+1}}\big(Y_{t_{i+1}}^q\big) \\ &\text{with } Y_{t_{i+1}}^q := Y_{t_{i+1}}^{Y_{t_{i+2}}^q, q_{|[t_{i+1},t_{i+2}]}} \text{ and } Y_T^q := \xi \Big\}.\end{aligned}$$

$\mathcal{A}$ is well defined by a decreasing recursion on $i \in \{0,\dots,N-1\}$. For $q \in \mathcal{A}$ we can define our cost functional Y^q on $[0,T]$ by

$$\forall i \in \{N-1,\dots,0\}, \forall t \in [t_i,t_{i+1}] \quad Y_t^q := Y_t^{Y_{t_{i+1}}^q, q_{|[t_i,t_{i+1}]}},$$

and, similarly, we define the process Z^q associated to Y^q by

$$\forall i \in \{N-1,\dots,0\}, \forall t \in (t_i, t_{i+1}) \quad Z_t^q := Z_t^{Y^q_{t_{i+1}}, q|[t_i,t_{i+1}]}.$$

(Y^q, Z^q) is also well defined by a decreasing recursion on $i \in \{0,\dots,N-1\}$. Finally, the stochastic control problem consists in minimizing Y^q among all the admissible controls $q \in \mathcal{A}$. Our strategy to prove the uniqueness is to prove that given a solution (Y, Z), the first component is the optimal value.

Theorem 3.3. *We suppose that there exists a solution (Y, Z) of the BSDE (1.1) verifying*

$$\exists p > \bar{\gamma}, \exists \varepsilon > 0 \quad \mathbb{E}\bigl[\exp\bigl(pA^*\bigr) + \exp\bigl(\varepsilon\bigl(Y^+\bigr)^*\bigr)\bigr] < +\infty,$$

with $A_t := Y_t^- + \int_0^t \bar{\alpha}_s \,\mathrm{d}s$. Then we have $Y = \operatorname{ess\,inf}_{q\in\mathcal{A}} Y^q$, and there exists $q^* \in \mathcal{A}$ such that $Y = Y^{q^*}$. Moreover, this implies that the solution (Y, Z) is unique among solutions verifying such condition.

Proof. Let us first prove that for any q admissible, we have $Y \le Y^q$. To do this, we will show that $Y_{|[t_i,t_{i+1}]} \le Y^q_{|[t_i,t_{i+1}]}$ by decreasing recurrence on $i \in \{0,\dots,N-1\}$. Firstly, we have $Y_T = Y_T^q = \xi$. Then we suppose that $Y_t \le Y_t^q$, $\forall t \in [t_{i+1}, T]$. We set $t \in [t_i, t_{i+1}]$ and we define

$$\tau_n^i := \inf\left\{s \ge t\colon \sup\left\{\int_t^s |Z_u|^2 \,\mathrm{d}u, \int_t^s \bigl|Z_u^q\bigr|^2 \,\mathrm{d}u, \int_t^s |q_u|^2 \,\mathrm{d}u\right\} > n\right\} \wedge t_{i+1},$$

$h(s, y, z) := -g(s, y, z) + zq_s$, and

$$h_s := \begin{cases} \frac{h(s,Y_s^q,Z_s)-h(s,Y_s,Z_s)}{Y_s^q - Y_s} & \text{if } Y_s^q - Y_s \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We observe that $|h_s| \le K_{g,y}$. Then, by applying Itô formula to the process $(Y_s^q - Y_s)\mathrm{e}^{\int_t^s h_u \,\mathrm{d}u}$ we obtain

$$Y_t^q - Y_t = \mathrm{e}^{\int_t^{\tau_n^i} h_s \,\mathrm{d}s}\bigl[Y^q_{\tau_n^i} - Y_{\tau_n^i}\bigr] + \int_t^{\tau_n^i} \mathrm{e}^{\int_t^s h_u \,\mathrm{d}u}\bigl[f\bigl(s, Y_s^q, q_s\bigr) - h\bigl(s, Y_s^q, Z_s\bigr)\bigr]\,\mathrm{d}s + \int_t^{\tau_n^i} \mathrm{e}^{\int_t^s h_u \,\mathrm{d}u}\bigl[Z_s^q - Z_s\bigr]\,\mathrm{d}W_s^q.$$

By definition, $f(s, Y_s^q, q_s) - h(s, Y_s^q, Z_s) \ge 0$, so

$$Y_t^q - Y_t \ge \mathbb{E}^{\mathbb{Q}^i}\bigl[\mathrm{e}^{\int_t^{\tau_n^i} h_s \,\mathrm{d}s}\bigl[Y^q_{\tau_n^i} - Y_{\tau_n^i}\bigr]\big|\mathcal{F}_t\bigr].$$

Since $(Y^q_{\tau_n^i}\mathrm{e}^{\int_t^{\tau_n^i} h_s \,\mathrm{d}s})_n$ tends to $Y^q_{t_{i+1}}\mathrm{e}^{\int_t^{t_{i+1}} h_s \,\mathrm{d}s}$ almost surely and is uniformly integrable under $\mathbb{Q}^i$, we have

$$\lim_{n\to+\infty} \mathbb{E}^{\mathbb{Q}^i}\bigl[\mathrm{e}^{\int_t^{\tau_n^i} h_s \,\mathrm{d}s} Y^q_{\tau_n^i}\big|\mathcal{F}_t\bigr] = \mathbb{E}^{\mathbb{Q}^i}\bigl[\mathrm{e}^{\int_t^{t_{i+1}} h_s \,\mathrm{d}s} Y^q_{t_{i+1}}\big|\mathcal{F}_t\bigr].$$

Moreover, $|Y_{\tau_n^i}\mathrm{e}^{\int_t^{\tau_n^i} h_s \,\mathrm{d}s}| \le (Y^+)^*\mathrm{e}^{TK_{g,y}} + (Y^-)^*\mathrm{e}^{TK_{g,y}}$. Let us recall a useful inequality: from

$$xy \le \exp(x) + y\bigl(\log(y) - 1\bigr) \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R}^+,$$

we deduce

$$xy = px\frac{y}{p} \le \exp(px) + \frac{y}{p}(\log y - \log p - 1). \tag{3.2}$$

Thus

$$\mathbb{E}^{\mathbb{Q}^i}\left[\left(Y^-\right)^*\right]=\mathbb{E}\left[M^i_{t_{i+1}}\left(Y^-\right)^*\right]\le\mathbb{E}\left[\exp\left(p\left(Y^-\right)^*\right)\right]+\frac{1}{p}\mathbb{E}\left[M^i_{t_{i+1}}\left(\log M^i_{t_{i+1}}-\log p-1\right)\right]$$
$$\le C_p+\frac{1}{2p}\mathbb{E}^{\mathbb{Q}^i}\left[\int_{t_i}^{t_{i+1}}|q_s|^2\,\mathrm{d}s\right]$$
$$<+\infty,$$

and, in the same manner,

$$\mathbb{E}^{\mathbb{Q}^i}\left[\left(Y^+\right)^*\right]\le C_\varepsilon+\frac{1}{2\varepsilon}\mathbb{E}^{\mathbb{Q}^i}\left[\int_{t_i}^{t_{i+1}}|q_s|^2\,\mathrm{d}s\right]<+\infty.$$

So, by applying the dominated convergence theorem we obtain

$$\lim_{n\to+\infty}\mathbb{E}^{\mathbb{Q}^i}\left[\mathrm{e}^{\int_t^{\tau_n^i}h_s\,\mathrm{d}s}Y_{\tau_n^i}\middle|\mathcal{F}_t\right]=\mathbb{E}^{\mathbb{Q}^i}\left[\mathrm{e}^{\int_t^{t_{i+1}}h_s\,\mathrm{d}s}Y_{t_{i+1}}\middle|\mathcal{F}_t\right].$$

Finally,

$$Y_t^q-Y_t\ge\lim_{n\to+\infty}\mathbb{E}^{\mathbb{Q}^i}\left[\mathrm{e}^{\int_t^{\tau_n^i}h_s\,\mathrm{d}s}\left[Y^q_{\tau_n^i}-Y_{\tau_n^i}\right]\middle|\mathcal{F}_t\right]=\mathbb{E}^{\mathbb{Q}^i}\left[\mathrm{e}^{\int_t^{t_{i+1}}h_s\,\mathrm{d}s}\left(Y^q_{t_{i+1}}-Y_{t_{i+1}}\right)\middle|\mathcal{F}_t\right]\ge 0,$$

because $Y^q_{t_{i+1}}\ge Y_{t_{i+1}}$ by the recurrence's hypothesis.

Now we set ${}^tq_s^*\in\partial_z g(s,Y_s,Z_s)$ with $\partial_z g(s,Y_s,Z_s)$ the subdifferential of $z\mapsto g(s,Y_s,z)$ at Z_s. We recall that for a convex function $l:\mathbb{R}^{1\times d}\to\mathbb{R}$, the subdifferential of l at x_0 is the nonempty convex compact set of $u\in\mathbb{R}^{1\times d}$ such that

$$l(x)-l(x_0)\ge u^t(x-x_0)\quad\forall x\in\mathbb{R}^{1\times d}.$$

We have $f(s,Y_s,q_s^*)=Z_sq_s^*-g(s,Y_s,Z_s)$ for all $s\in[0,T]$, so

$$g(s,Y_s,Z_s)\le Z_sq_s^*-\frac{1}{2\bar\gamma}\left|q_s^*\right|^2+\bar\beta|Y_s|+\bar\alpha_s$$
$$\le\frac{1}{2}\left(2\bar\gamma|Z_s|^2+\frac{|q_s^*|^2}{2\bar\gamma}\right)-\frac{1}{2\bar\gamma}\left|q_s^*\right|^2+\bar\beta|Y_s|+\bar\alpha_s,$$
$$\frac{|q_s^*|^2}{4\bar\gamma}\le -g(s,Y_s,Z_s)+\bar\gamma|Z_s|^2+\bar\beta|Y_s|+\bar\alpha_s,$$

and finally, $\int_0^T|q_s^*|^2\,\mathrm{d}s<+\infty$, $\mathbb{P}$-a.s. Moreover, $\forall t,t'\in[0,T]$,

$$Y_t=Y_{t'}+\int_t^{t'}f\left(s,Y_s,q_s^*\right)\mathrm{d}s+\int_t^{t'}Z_s\left(\mathrm{d}W_s-q_s^*\,\mathrm{d}s\right).$$

Thus, we just have to show that q^* is admissible to prove that q^* is optimal, i.e. $Y=Y^{q^*}$. For this, we must prove that $(q_s^*)_{s\in[t_i,t_{i+1}]}\in\mathcal{A}_{t_i,t_{i+1}}(Y_{t_{i+1}})$ for $i\in\{0,\dots,N-1\}$. We define

$$M_t^i:=\exp\left(\int_{t_i}^t q_s^*\,\mathrm{d}W_s-\frac{1}{2}\int_{t_i}^t\left|q_s^*\right|^2\mathrm{d}s\right),\qquad\frac{\mathrm{d}\mathbb{Q}^{*,i}}{\mathrm{d}\mathbb{P}}:=M^i_{t_{i+1}},$$
$$\tau_n^i=\inf\left\{t\in[t_i,t_{i+1}]:\ \sup\left(\int_{t_i}^t\left|q_s^*\right|^2\mathrm{d}s,\int_{t_i}^t|Z_s|^2\,\mathrm{d}s\right)>n\right\}\wedge t_{i+1},\qquad\frac{\mathrm{d}\mathbb{Q}_n^{*,i}}{\mathrm{d}\mathbb{P}}:=M^i_{\tau_n^i}.$$

Let us show the following lemma:

Lemma 3.4. *$(M^i_{\tau^i_n})_n$ is uniformly integrable.*

Proof. We apply inequality (3.2) to obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^{*,i}_n}\big[A^*\big] = \mathbb{E}\big[M^i_{\tau^i_n}A^*\big] &\leq \mathbb{E}\big[\exp\big(pA^*\big)\big] + \frac{1}{p}\mathbb{E}\big[M^i_{\tau^i_n}\big(\log M^i_{\tau^i_n} - \log p - 1\big)\big] \\ &\leq C_p + \frac{1}{2p}\mathbb{E}^{\mathbb{Q}^{*,i}_n}\left[\int_{t_i}^{\tau^i_n}\left|q^*_s\right|^2 \mathrm{d}s\right],\end{aligned}$$

and, in the same manner,

$$\mathbb{E}^{\mathbb{Q}^{*,i}_n}\big[\big(Y^+\big)^*\big] \leq C_\varepsilon + \frac{1}{2\varepsilon}\mathbb{E}^{\mathbb{Q}^{*,i}_n}\left[\int_{t_i}^{\tau^i_n}\left|q^*_s\right|^2 \mathrm{d}s\right].$$

Since $g(s, Y_s, Z_s) = Z_s q^*_s - f(s, Y_s, q^*_s)$ and $(M^i_{t\wedge\tau^i_n})_{t\in[t_i,t_{i+1}]}$ is a martingale, we can apply the Girsanov theorem and we obtain

$$\mathbb{E}^{\mathbb{Q}^{*,i}_n}\left[Y_{\tau^i_n} + \int_{t_i}^{\tau^i_n} f\big(s, Y_s, q^*_s\big)\,\mathrm{d}s\right] = \mathbb{E}^{\mathbb{Q}^{*,i}_n}[Y_{t_i}] = \mathbb{E}\big[M^i_{\tau^i_n}Y_{t_i}\big] = \mathbb{E}[Y_{t_i}].$$

Moreover, $f(t, y, q) \geq \frac{1}{2\bar{\gamma}}|q|^2 - \bar{\beta}|y| - \bar{\alpha}_t$ and $Y_{\tau^i_n} \geq -Y^-_{\tau^i_n}$, so

$$\begin{aligned}\mathbb{E}[Y_{t_i}] &\geq -\mathbb{E}^{\mathbb{Q}^{*,i}_n}\big[Y^-_{\tau^i_n}\big] - \mathbb{E}^{\mathbb{Q}^{*,i}_n}\left[\int_{t_i}^{\tau^i_n}\bar{\alpha}_s\,\mathrm{d}s\right] + \frac{1}{2\bar{\gamma}}\mathbb{E}^{\mathbb{Q}^{*,i}_n}\left[\int_{t_i}^{\tau^i_n}\left|q^*_s\right|^2\mathrm{d}s\right] - \bar{\beta}\mathbb{E}^{\mathbb{Q}^{*,i}_n}\left[\int_{t_i}^{\tau^i_n}|Y_s|\,\mathrm{d}s\right] \\ &\geq C - \mathbb{E}^{\mathbb{Q}^{*,i}_n}\big[A^*\big] + \frac{1}{2\bar{\gamma}}\mathbb{E}^{\mathbb{Q}^{*,i}_n}\left[\int_{t_i}^{\tau^i_n}\left|q^*_s\right|^2\mathrm{d}s\right] - \frac{T}{N}\big(\bar{\beta}\mathbb{E}^{\mathbb{Q}^{*,i}_n}\big[\big(Y^-\big)^* + \big(Y^+\big)^*\big]\big) \\ &\geq C_{p,\varepsilon} + \frac{1}{2}\underbrace{\left(\frac{1}{\bar{\gamma}} - \frac{1}{p} - \frac{T}{N}\left(\frac{\bar{\beta}}{p} + \frac{\bar{\beta}}{\varepsilon}\right)\right)}_{>0}\mathbb{E}^{\mathbb{Q}^{*,i}_n}\left[\int_{t_i}^{\tau^i_n}\left|q^*_s\right|^2\mathrm{d}s\right].\end{aligned}$$

This inequality explains why we take N verifying (3.1). Finally we get that

$$\begin{aligned}2\mathbb{E}\big[M^i_{\tau^i_n}\log M^i_{\tau^i_n}\big] &= \mathbb{E}^{\mathbb{Q}^{*,i}_n}\left[\int_{t_i}^{\tau^i_n}\left|q^*_s\right|^2\mathrm{d}s\right] \\ &\leq C_{p,\varepsilon}.\end{aligned} \tag{3.3}$$

Then we conclude the proof of the lemma by using the de La Vallée Poussin lemma. □

Thanks to this lemma, we have that $\mathbb{E}[M^i_{t_{i+1}}] = 1$ and so $(M^i_t)_{t\in[t_i,t_{i+1}]}$ is a martingale. Moreover, applying Fatou's lemma and inequality (3.3), we obtain

$$\begin{aligned}2\mathbb{E}\big[M^i_{t_{i+1}}\log M^i_{t_{i+1}}\big] &= \mathbb{E}^{\mathbb{Q}^{*,i}}\left[\int_{t_i}^{t_{i+1}}\left|q^*_s\right|^2\mathrm{d}s\right] \\ &\leq \liminf_n \mathbb{E}^{\mathbb{Q}^{*,i}_n}\left[\int_{t_i}^{\tau^i_n}\left|q^*_s\right|^2\mathrm{d}s\right] < +\infty.\end{aligned} \tag{3.4}$$

So, by using this result and inequality (3.2) we easily show that $\mathbb{E}^{\mathbb{Q}^{*,i}}[(Y^+)^* + (Y^-)^*] < +\infty$. To conclude we have to prove that $\mathbb{E}^{\mathbb{Q}^{*,i}}[\int_{t_i}^{t_{i+1}} |f(s,0,q_s^*)|\,\mathrm{d}s] < +\infty$:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^{*,i}}\left[\int_{t_i}^{t_{i+1}}\left|f\left(s,0,q_s^*\right)\right|\mathrm{d}s\right] &\le \mathbb{E}^{\mathbb{Q}^{*,i}}\left[\int_{t_i}^{t_{i+1}}\left|f\left(s,Y_s,q_s^*\right)\right| + K_{g,y}|Y_s|\,\mathrm{d}s\right]\\ &\le \mathbb{E}^{\mathbb{Q}^{*,i}}\left[\int_{t_i}^{t_{i+1}}\left|f\left(s,Y_s,q_s^*\right)\right|\mathrm{d}s + K_{g,y}T\left(\left(Y^+\right)^* + \left(Y^-\right)^*\right)\right]\\ &\le C + \mathbb{E}^{\mathbb{Q}^{*,i}}\left[\int_{t_i}^{t_{i+1}} f^+\left(s,Y_s,q_s^*\right) + f^-\left(s,Y_s,q_s^*\right)\mathrm{d}s\right].\end{aligned}$$

Firstly,

$$\mathbb{E}^{\mathbb{Q}^{*,i}}\left[\int_{t_i}^{t_{i+1}} f^-\left(s,Y_s,q_s^*\right)\mathrm{d}s\right] \le \mathbb{E}^{\mathbb{Q}^{*,i}}\left[\int_{t_i}^{t_{i+1}} \bar{\alpha}_s + \bar{\beta}|Y_s|\,\mathrm{d}s\right] < +\infty.$$

Moreover, thanks to the Girsanov theorem we have

$$\mathbb{E}^{\mathbb{Q}^{*,i}}[Y_{t_i}] = \mathbb{E}^{\mathbb{Q}^{*,i}}\left[Y_{\tau_n^i} + \int_{t_i}^{\tau_n^i} f\left(s,Y_s,q_s^*\right)\mathrm{d}s\right],$$

so

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^{*,i}}\left[\int_{t_i}^{\tau_n^i} f^+\left(s,Y_s,q_s^*\right)\mathrm{d}s\right] &\le \mathbb{E}^{\mathbb{Q}^{*,i}}[Y_{t_i} - Y_{\tau_n^i}] + \mathbb{E}^{\mathbb{Q}^{*,i}}\left[\int_{t_i}^{\tau_n^i} f^-\left(s,Y_s,q_s^*\right)\mathrm{d}s\right]\\ &\le C + \mathbb{E}^{\mathbb{Q}^{*,i}}\left[\int_{t_i}^{t_{i+1}} f^-\left(s,Y_s,q_s^*\right)\mathrm{d}s\right] \le C.\end{aligned}$$

Finally, $\mathbb{E}^{\mathbb{Q}^{*,i}}[\int_{t_i}^{t_{i+1}} f^+(s,Y_s,q_s^*)\,\mathrm{d}s] < +\infty$ and $\mathbb{E}^{\mathbb{Q}^{*,i}}[\int_{t_i}^{t_{i+1}} |f(s,0,q_s^*)|\,\mathrm{d}s] < +\infty$. Thus, we prove that q^* is optimal, i.e. $Y^{q^*} = Y$.

The uniqueness of Y is a direct consequence of the fact that $Y = Y^{q^*} = \operatorname{ess\,inf}_{q\in\mathcal{A}} Y^q$. The uniqueness of Z follows immediately. ☐

Remark 3.5. *If we have $g(t,y,z) \le g(t,0,z)$, then $f(t,y,q) \ge f(t,0,q) \ge \frac{1}{2\gamma}|q|^2 - \bar{\alpha}_t$ and we do not have to introduce N in the proof of Lemma* 3.4. *So we have a simpler representation theorem*:

$$Y_t = \operatorname*{ess\,inf}_{q\in\mathcal{A}_{0,T}(\xi)} Y_t^q \quad \forall t\in[0,T].$$

For example, *when g is independent of y*, *we obtain*

$$Y_t = \operatorname*{ess\,inf}_{q\in\mathcal{A}_{0,T}(\xi)} \mathbb{E}^{\mathbb{Q}}\left[\xi + \int_t^T f(s,q_s)\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right] \quad \forall t\in[0,T].$$

4. Application to quadratic PDEs

In this section we give an application of our results concerning BSDEs to PDEs which are quadratic with respect to the gradient of the solution. Let us consider the following semilinear PDE

$$\partial_t u(t,x) + \mathcal{L}u(t,x) - g\big(t,x,u(t,x),-\sigma^*\nabla_x u(t,x)\big) = 0 \qquad u(T,\cdot) = h, \tag{4.1}$$

where $\mathcal{L}$ is the infinitesimal generator of the diffusion $X^{t,x}$ solution to the SDE

$$X_s^{t,x} = x + \int_t^s b\big(r, X_r^{t,x}\big)\,\mathrm{d}r + \int_t^s \sigma(r)\,\mathrm{d}W_r, \quad t \le s \le T, \quad \text{and} \quad X_s^{t,x} = x, \quad s \le t. \tag{4.2}$$

The nonlinear Feynman–Kac formula consists in proving that the function defined by the formula

$$\forall (t,x) \in [0,T] \times \mathbb{R}^d \quad u(t,x) := Y_t^{t,x}, \tag{4.3}$$

where, for each $(t_0, x_0) \in [0,T] \times \mathbb{R}^d$, $(Y^{t_0,x_0}, Z^{t_0,x_0})$ stands for the solution to the following BSDE

$$Y_t = h\big(X_T^{t_0,x_0}\big) - \int_t^T g\big(s, X_s^{t_0,x_0}, Y_s, Z_s\big)\,\mathrm{d}s + \int_t^T Z_s\,\mathrm{d}W_s, \quad 0 \le t \le T, \tag{4.4}$$

is a solution, at least a viscosity solution, to the PDE (4.1).

Assumption A.3. *Let $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \to \mathbb{R}^{d\times d}$ be continuous functions and let us assume that there exists $K \ge 0$ such that*:

(1) $\forall t \in [0,T]$, $|b(t,0)| \le K$;
(2) $\forall t \in [0,T]$, $\forall (x,x') \in \mathbb{R}^d \times \mathbb{R}^d$, $|b(t,x) - b(t,x')| \le K|x - x'|$.

Lemma 4.1.

$$\forall \lambda \in \left[0, \frac{1}{2\mathrm{e}^{2KT}\|\sigma\|_\infty^2 T}\right[, \exists C_T \ge 0, \exists C \ge 0 \quad \mathbb{E}\Big[\sup_{0\le t\le T} \mathrm{e}^{\lambda|X_t^{t_0,x_0}|^2}\Big] \le C_T \mathrm{e}^{C|x_0|^2},$$

with $T \mapsto C_T$ nondecreasing.

Proof. As in [4] we easily show that, for all $\varepsilon > 0$, we have

$$\sup_{0\le t\le T} \big|X_t^{t_0,x_0}\big| \le \left(|x_0| + KT + \sup_{0\le t\le T}\left|\int_0^t \mathbb{1}_{s\ge t_0}\sigma(s)\,\mathrm{d}W_s\right|\right)\mathrm{e}^{KT},$$

$$\sup_{0\le t\le T} \big|X_t^{t_0,x_0}\big|^2 \le C_\varepsilon\big(T^2 + |x_0|^2\big) + (1+\varepsilon)\mathrm{e}^{2KT} \sup_{0\le t\le T}\left|\int_0^t \mathbb{1}_{s\ge t_0}\sigma(s)\,\mathrm{d}W_s\right|^2.$$

We define $\tilde{\lambda} := \lambda(1+\varepsilon)\mathrm{e}^{2KT}$. It follows from the Dambis–Dubins–Schwarz representation theorem and the Doob's maximal inequality that

$$\mathbb{E}\Big[\sup_{0\le t\le T} \exp\Big(\tilde{\lambda}\Big|\int_0^t \mathbb{1}_{s\ge t_0}\sigma(s)\,\mathrm{d}W_s\Big|^2\Big)\Big] \le \mathbb{E}\Big[\sup_{0\le t\le \|\sigma\|_\infty^2 T} \mathrm{e}^{\tilde{\lambda}|W_t|^2}\Big] \le 4\mathbb{E}\big[\mathrm{e}^{\tilde{\lambda}\|\sigma\|_\infty^2 T|W_1|^2}\big],$$

which is a finite constant if $\tilde{\lambda}\|\sigma\|_\infty^2 T < 1/2$. □

With this observation in hands, we can give our assumptions on the nonlinear term of the PDE and the terminal condition.

Assumption A.4. *Let $g : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $h : \mathbb{R}^d \to \mathbb{R}$ be continuous and let us assume moreover that there exist five constants $r \ge 0$, $\beta \ge 0$, $\gamma \ge 0$, $\alpha \ge 0$ and $\alpha' \ge 0$ such that*:

(1) *for each $(t,x,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^{1\times d}$,*

$$\forall \big(y, y'\big) \in \mathbb{R}^2 \quad \big|g(t,x,y,z) - g\big(t,x,y',z\big)\big| \le \beta\big|y - y'\big|;$$

(2) *for each $(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$, $z \mapsto g(t,x,y,z)$ is convex on $\mathbb{R}^{1\times d}$*;

(3) *for each* $(t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1\times d}$,

$$-r\bigl(1+|x|^2+|y|+|z|\bigr) \le g(t,x,y,z) \le r+\alpha|x|^2+\beta|y|+\frac{\gamma}{2}|z|^2,$$
$$-r-\alpha'|x|^2 \le h(x) \le r\bigl(1+|x|^2\bigr);$$

(4) *for each* $(t, x, x', y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1\times d}$,

$$\bigl|g(t,x,y,z)-g\bigl(t,x',y,z\bigr)\bigr| \le r\bigl(1+|x|+\bigl|x'\bigr|\bigr)\bigl|x-x'\bigr|,$$
$$\bigl|h(x)-h\bigl(x'\bigr)\bigr| \le r\bigl(1+|x|+\bigl|x'\bigr|\bigr)\bigl|x-x'\bigr|;$$

(5) $$\alpha'+T\alpha < \frac{1}{2\gamma e^{(2K+\beta)T}\|\sigma\|_\infty^2 T}.$$

Thanks to Lemma 4.1, we see that there exist $q > \gamma e^{\beta T}$ and $\varepsilon > 0$ such that $h^-(X_T^{t_0,x_0}) + \int_0^T (r+\alpha|X_t^{t_0,x_0}|^2)\,dt$ has an exponential moment of order q and $h^+(X_T^{t_0,x_0}) + \int_0^T (r+r|X_t^{t_0,x_0}|^2)\,dt$ has an exponential moment of order ε. So we are able to apply Corollary 2.2 and Theorem 3.3 to obtain a unique solution $(Y^{t_0,x_0}, Z^{t_0,x_0})$ to the BSDE (4.4). Let us recall the definition of a viscosity solution and then, prove that u is a viscosity solution to the PDE (4.1).

Definition 4.2. *A continuous function u on $[0,T]\times\mathbb{R}^d$ such that $u(T,\cdot)=h$ is said to be a viscosity subsolution* (*respectively supersolution*) *to* (4.1) *if*

$$\partial_t\varphi(t_0,x_0)+\mathcal{L}\varphi(t_0,x_0)-g\bigl(t_0,x_0,u(t_0,x_0),-\sigma^*\nabla_x\varphi(t_0,x_0)\bigr) \ge 0 \quad (\textit{respectively} \le 0)$$

as soon as $u-\varphi$ has a local maximum (*respectively minimum*) *at $(t_0,x_0)\in(0,T)\times\mathbb{R}^d$ where φ is a smooth function. A viscosity solution is both a viscosity subsolution and a viscosity supersolution.*

Proposition 4.3. *Let Assumptions A.3 and A.4 hold. The function u defined by* (4.3) *is continuous on $[0,T]\times\mathbb{R}^d$ and satisfies*

$$\forall (t,x)\in[0,T]\times\mathbb{R}^d \quad \bigl|u(t,x)\bigr| \le C\bigl(1+|x|^2\bigr).$$

Moreover, u is a viscosity solution to the PDE (4.1).

Before giving a proof of this result, we will show some auxiliary results about admissible control sets. We have already notice in Remark 3.5 that we have a simpler representation theorem when T is small enough to take $N=1$ in (3.1). So we define a constant $T_1>0$ such that for all $T\in[0,T_1]$ we are allowed to set $N=1$. We will reuse notations of Section 3. For all $T\in[0,T_1]$, $t\in[0,T]$, $x\in\mathbb{R}^d$, we define the admissible control set

$$\begin{aligned}\mathcal{A}_{0,T}(t,x) := \Biggl\{&(q_s)_{s\in[0,T]}\colon \int_0^T |q_s|^2\,ds < +\infty\ \mathbb{P}\text{-a.s.}, \mathbb{E}^{\mathbb{Q}}\biggl[\int_0^T |q_s|^2\,ds\biggr] < +\infty,\\ &(M_t)_{t\in[0,T]} \text{ is a martingale}, \mathbb{E}^{\mathbb{Q}}\biggl[\bigl|h\bigl(X_T^{t,x}\bigr)\bigr| + \int_0^T \bigl|f\bigl(s,X_s^{t,x},0,q_s\bigr)\bigr|\,ds\biggr] < +\infty,\\ &\text{with } M_t := \exp\biggl(\int_0^t q_s\,dW_s - \frac{1}{2}\int_0^t |q_s|^2\,ds\biggr) \text{ and } \frac{d\mathbb{Q}}{d\mathbb{P}} := M_T\Biggr\}.\end{aligned}$$

We will prove a first lemma and then we will use it to show that this admissible control set does not depend on t and x.

Lemma 4.4. $\exists C>0$ *such that* $\forall T\in[0,T_1]$, $\forall t\in[0,T]$, $\forall x\in\mathbb{R}^d$, $\forall q\in\mathcal{A}_{0,T}(t,x)$, $\forall t'\in[0,T]$, $\forall x'\in\mathbb{R}^d$, $\forall s\in[t',T]$,

$$\mathbb{E}^{\mathbb{Q}}\bigl[\bigl|X_s^{t',x'}\bigr|^2\bigr] \le C\biggl(1+\bigl|x'\bigr|^2+T\int_{t'}^s \mathbb{E}^{\mathbb{Q}}\bigl[|q_u|^2\bigr]\,du\biggr).$$

Remark 4.5. *In the second part of the lemma q and $\mathbb{Q}$ depend on x and t but we do not write it to simplify notations.*

Proof of Lemma 4.4. For all $s \in [t', T]$ we have an obvious inequality

$$\left|X_s^{t',x'}\right|^2 \le C\left(1+\left|x'\right|^2+\left(\int_{t'}^s \left|X_u^{t',x'}\right| \mathrm{d}u\right)^2+\sup_{t'\le r\le T}\left|\int_{t'}^r \sigma(u)\,\mathrm{d}W_u^q\right|^2+\left(\int_{t'}^s |q_u|\,\mathrm{d}u\right)^2\right).$$

Then, by applying Cauchy–Schwarz's inequality and Doob's maximal inequality, we obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\left|X_s^{t',x'}\right|^2\right] \le C\Bigg(&1+\left|x'\right|^2+T\int_{t'}^s \mathbb{E}^{\mathbb{Q}}\left[\left|X_u^{t',x'}\right|^2\right]\mathrm{d}u+\mathbb{E}^{\mathbb{Q}}\left[\left|\int_{t'}^T \sigma(u)\,\mathrm{d}W_u^q\right|^2\right]\\ &+T\mathbb{E}^{\mathbb{Q}}\left[\int_{t'}^s |q_u|^2\,\mathrm{d}u\right]\Bigg).\end{aligned}$$

Finally, Gronwall's lemma gives us the result. □

Proposition 4.6. *$\mathcal{A}_{0,T}(t,x)$ is independent of t and x. We will write it $\mathcal{A}_{0,T}$.*

Proof. Let $x, x' \in \mathbb{R}^d$, $t, t' \in [0,T]$ and $q \in \mathcal{A}_{0,T}(t,x)$. We will show that $q \in \mathcal{A}_{0,T}(t',x')$. Firstly, thanks to Lemma 4.4 we have

$$\mathbb{E}^{\mathbb{Q}}\left[\left|h\left(X_T^{t',x'}\right)\right|\right] \le C\left(1+\mathbb{E}^{\mathbb{Q}}\left[\left|X_T^{t',x'}\right|^2\right]\right) \le C\left(1+\int_{t'}^T \mathbb{E}^{\mathbb{Q}}\left[|q_u|^2\right]\mathrm{d}u\right) < +\infty.$$

Moreover,

$$-C\left(1+\left|X_s^{t',x'}\right|^2\right) \le \frac{1}{2\gamma}|q_s|^2 - C\left(1+\left|X_s^{t',x'}\right|^2\right) \le f\left(s, X_s^{t',x'}, 0, q_s\right),$$

and

$$f\left(s, X_s^{t',x'}, 0, q_s\right) \le f\left(s, X_s^{t,x}, 0, q_s\right) + C\left(1+\left|X_s^{t,x}\right|^2+\left|X_s^{t',x'}\right|^2\right).$$

So, $|f(s, X_s^{t',x'}, 0, q_s)| \le |f(s, X_s^{t,x}, 0, q_s)| + C(1+|X_s^{t,x}|^2+|X_s^{t',x'}|^2)$ and finally

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \left|f\left(s, X_s^{t',x'}, 0, q_s\right)\right|\mathrm{d}s\right] < +\infty.$$

□

Now we will do a new restriction of the admissible control set.

Proposition 4.7. $\exists T_2 \in]0, T_1]$, $\exists \tilde{C} > 0$, *such that*, $\forall T \in [0, T_2]$, $\forall t \in [0,T]$, $\forall s \in [0,t]$, $\forall x \in \mathbb{R}^d$,

$$\left|Y_s^{t,x}\right| \le \tilde{C}\left(1+|x|^2\right) \quad \textit{and} \quad \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T \left|q_u^*\right|^2 \mathrm{d}u\right] \le \tilde{C}\left(1+|x|^2\right).$$

Proof. Firstly, we suppose that $s \le t$, so $Y_s^{t,x}$ is a deterministic variable. We are able to use estimates of the existence Theorem 2.1 and Lemma 4.1:

$$\begin{aligned}&-C\log\mathbb{E}\left[\sup_{0\le s\le T}\exp\left(C+\gamma \mathrm{e}^{\beta T}\left(\alpha'+T\alpha\right)\left|X_s^{t,x}\right|^2\right)\right] \le Y_s^{t,x} \le C\left(1+\mathbb{E}\left[\sup_{0\le s\le T}\left|X_s^{t,x}\right|^4\right]\right)^{1/2},\\ &-\tilde{C}\left(1+|x|^2\right) \le Y_s^{t,x} \le \tilde{C}\left(1+|x|^2\right).\end{aligned}$$

Then, according to the representation theorem, we have

$$\begin{aligned}
Y_0^{t,x} &= \mathbb{E}^{\mathbb{Q}^*}\left[h\left(X_T^{t,x}\right)+\int_0^T f\left(s, X_s^{t,x}, Y_s^{t,x}, q_s^*\right)\mathrm{d}s\right]\\
&\geq -C-\alpha'\mathbb{E}^{\mathbb{Q}^*}\left[\left|X_T^{t,x}\right|^2\right]+\frac{1}{2\gamma}\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T \left|q_u^*\right|^2\mathrm{d}u\right]-\alpha\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T \left|X_s^{t,x}\right|^2\mathrm{d}s\right]-\beta\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T \left|Y_s^{t,x}\right|\mathrm{d}s\right].
\end{aligned}$$

But, thanks to the uniqueness, we have $Y_s^{t,x}=Y_s^{s,X_s^{t,x}}$ for $s\geq t$, so $\mathbb{E}^{\mathbb{Q}^*}[|Y_s^{t,x}|]\leq C(1+\mathbb{E}^{\mathbb{Q}^*}[|X_s^{t,x}|^2])$. Moreover, we are allowed to use Lemma 4.4,

$$\begin{aligned}
Y_0^{t,x} &\geq -C\left(1+|x|^2\right)-C\left(\alpha'+T\alpha+\beta C\right)\left(1+|x|^2+T\int_t^T \mathbb{E}^{\mathbb{Q}^*}\left[\left|q_u^*\right|^2\right]\mathrm{d}u\right)+\frac{1}{2\gamma}\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T \left|q_u^*\right|^2\mathrm{d}u\right]\\
&\geq -C\left(1+|x|^2\right)+\left(\frac{1}{2\gamma}-CT\right)\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T \left|q_u^*\right|^2\mathrm{d}u\right].
\end{aligned}$$

We set $0<T_2\leq T_1$ such that $\frac{1}{2\gamma}-CT>0$ for all $T\in[0,T_2]$. Finally,

$$\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T \left|q_u^*\right|^2\mathrm{d}u\right]\leq C\left(1+|x|^2+Y_0^{t,x}\right)\leq \tilde{C}\left(1+|x|^2\right).$$

□

According to the Proposition 4.7 we know that $\mathbb{E}^{\mathbb{Q}^*}[\int_0^T |q_u^*|^2\,\mathrm{d}u]\leq\tilde{C}(1+|x|^2)$ so we are allowed to restrict $\mathcal{A}_{0,T}$: for all $R\geq 0$ we define

$$\mathcal{A}_{0,T}^R=\mathcal{A}_{0,T}\cap\left\{(q_s)_{s\in[0,T]}\colon\ \mathbb{E}^{\mathbb{Q}}\left[\int_0^T |q_u|^2\,\mathrm{d}u\right]\leq\tilde{C}\left(1+R^2\right)\right\}. \tag{4.5}$$

With this new admissible control set we will prove a last inequality:

Proposition 4.8. $\exists C\geq 0$, $\forall T\in[0,T_2]$, $\forall t,t'\in[0,T]$, $\forall x,x'\in\mathbb{R}^d$, $\forall q\in\mathcal{A}_{0,T}^{|x|\vee|x'|}$, $\forall s\in[0,T]$,

$$\mathbb{E}^{\mathbb{Q}}\left[\left|X_s^{t,x}-X_s^{t',x'}\right|^2\right]\leq C\left(\left|x-x'\right|^2+\left(1+|x|^2+\left|x'\right|^2\right)\left|t-t'\right|\right).$$

Proof.

$$\mathbb{E}^{\mathbb{Q}}\left[\left|X_s^{t,x}-X_s^{t',x'}\right|^2\right]\leq 2\mathbb{E}^{\mathbb{Q}}\left[\left|X_s^{t,x}-X_s^{t,x'}\right|^2\right]+2\mathbb{E}^{\mathbb{Q}}\left[\left|X_s^{t,x'}-X_s^{t',x'}\right|^2\right].$$

We have, for $s\geq t$,

$$X_s^{t,x}-X_s^{t,x'}=x-x'+\int_t^s \left(b\left(u,X_u^{t,x}\right)-b\left(u,X_u^{t,x'}\right)\right)\mathrm{d}u.$$

So,

$$\mathbb{E}^{\mathbb{Q}}\left[\left|X_s^{t,x}-X_s^{t,x'}\right|^2\right]\leq C\left(\left|x-x'\right|^2+\int_t^s \mathbb{E}^{\mathbb{Q}}\left[\left|X_u^{t,x}-X_u^{t,x'}\right|^2\right]\mathrm{d}u\right).$$

We apply Gronwall's lemma to obtain that

$$\mathbb{E}^{\mathbb{Q}}\left[\left|X_s^{t,x}-X_s^{t,x'}\right|^2\right]\leq C\left|x-x'\right|^2.$$

Now we deal with the second term. Let us assume that $t\leq t'$. For $s\leq t$, $X_s^{t,x'}-X_s^{t',x'}=0$. When $t\leq s\leq t'$, we have

$$X_s^{t,x'}-X_s^{t',x'}=\int_t^s b\left(u,X_u^{t,x'}\right)\mathrm{d}u+\int_t^s \sigma(u)\,\mathrm{d}W_u^q+\int_t^s \sigma(u)q_u\,\mathrm{d}u.$$

So,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\big[\big|X_s^{t,x'}-X_s^{t',x'}\big|^2\big] &\le C\bigg(\mathbb{E}^{\mathbb{Q}}\bigg[\bigg(\int_t^{t'}\big|b\big(u,X_u^{t,x'}\big)\big|\,\mathrm{d}u\bigg)^2\bigg]+\int_t^{t'}\big|\sigma(u)\big|^2\,\mathrm{d}u+\mathbb{E}^{\mathbb{Q}}\bigg[\bigg(\int_t^{t'}\big|\sigma(u)q_u\big|\,\mathrm{d}u\bigg)^2\bigg]\bigg)\\
&\le C\bigg(\big|t'-t\big|+\big|t'-t\big|\int_t^{t'}\mathbb{E}^{\mathbb{Q}}\big[\big|X_u^{t,x'}\big|^2\big]\,\mathrm{d}u+\big|t'-t\big|\int_t^{t'}\mathbb{E}^{\mathbb{Q}}\big[|q_u|^2\big]\,\mathrm{d}u\bigg)\\
&\le C\big|t'-t\big|\bigg(1+\big|x'\big|^2+\int_0^T\mathbb{E}^{\mathbb{Q}}\big[|q_u|^2\big]\,\mathrm{d}u\bigg)\\
&\le C\big(1+|x|^2+\big|x'\big|^2\big)\big|t'-t\big|.
\end{aligned}
$$

Lastly, when $t' \le s$,

$$
X_s^{t,x'}-X_s^{t',x'}=X_{t'}^{t,x'}-X_{t'}^{t',x'}+\int_{t'}^{s}\big(b\big(u,X_u^{t,x'}\big)-b\big(u,X_u^{t',x'}\big)\big)\,\mathrm{d}u.
$$

So,

$$
\mathbb{E}^{\mathbb{Q}}\big[\big|X_s^{t,x'}-X_s^{t,x'}\big|^2\big]\le C\big(1+|x|^2+\big|x'\big|^2\big)\big|t'-t\big|+C\int_{t'}^{s}\mathbb{E}^{\mathbb{Q}}\big[\big|X_u^{t,x'}-X_u^{t',x'}\big|^2\big]\,\mathrm{d}u,
$$

and according to Gronwall's lemma,

$$
\mathbb{E}^{\mathbb{Q}}\big[\big|X_s^{t,x'}-X_s^{t,x'}\big|^2\big]\le C\big(1+|x|^2+\big|x'\big|^2\big)\big|t'-t\big|. \qquad \square
$$

Proof of Proposition 4.3. First of all, let us assume that $T < T_2$. With this condition, we are allowed to use all previous propositions. Firstly, the quadratic increase of u is already proved in Proposition 4.7. Then, we will show continuity of u in $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$. We have

$$
\forall (t,x)\in[0,T]\times\mathbb{R}^d \quad \big|u(t,x)-u(t_0,x_0)\big|\le\big|u(t,x)-u(t,x_0)\big|+\big|u(t,x_0)-u(t_0,x_0)\big|.
$$

Let us begin with the fist term. We define $R := |x| \vee |x_0|$. Thanks to the representation theorem, we have

$$
Y_t^{t,x}=\operatorname*{ess\,inf}_{q\in\mathcal{A}_{0,T}^R}Y_t^{q,t,x} \quad \text{and} \quad Y_t^{t,x_0}=\operatorname*{ess\,inf}_{q\in\mathcal{A}_{0,T}^R}Y_t^{q,t,x_0}.
$$

So,

$$
\big|Y_t^{t,x}-Y_t^{t,x_0}\big|\le\operatorname*{ess\,sup}_{q\in\mathcal{A}_{0,T}^R}\big|Y_t^{q,t,x}-Y_t^{q,t,x_0}\big|.
$$

But, for $t \le s \le T$,

$$
\begin{aligned}
\big|Y_s^{q,t,x}-Y_s^{q,t,x_0}\big| &= \bigg|\mathbb{E}^{\mathbb{Q}}\bigg[h\big(X_T^{t,x}\big)-h\big(X_T^{t,x_0}\big)+\int_s^T\big(f\big(u,X_u^{t,x},Y_u^{q,t,x},q_u\big)-f\big(u,X_u^{t,x_0},Y_u^{q,t,x_0},q_u\big)\big)\,\mathrm{d}u\Big|\mathcal{F}_s\bigg]\bigg|,\\
\mathbb{E}^{\mathbb{Q}}\big[\big|Y_s^{q,t,x}-Y_s^{q,t,x_0}\big|\big] &\le \mathbb{E}^{\mathbb{Q}}\big[C\big(1+\big|X_T^{t,x}\big|^2+\big|X_T^{t,x_0}\big|^2\big)\big]^{1/2}\mathbb{E}^{\mathbb{Q}}\big[\big|X_T^{t,x}-X_T^{t,x_0}\big|^2\big]^{1/2}\\
&\quad+\int_s^T\mathbb{E}^{\mathbb{Q}}\big[C\big(1+\big|X_u^{t,x}\big|^2+\big|X_u^{t,x_0}\big|^2\big)\big]^{1/2}\mathbb{E}^{\mathbb{Q}}\big[\big|X_u^{t,x}-X_u^{t,x_0}\big|^2\big]^{1/2}\,\mathrm{d}u\\
&\quad+C\int_s^T\mathbb{E}^{\mathbb{Q}}\big[\big|Y_u^{q,t,x}-Y_u^{q,t,x_0}\big|\big]\,\mathrm{d}u,
\end{aligned}
$$

thanks to Assumption A.4 and Hölder's inequality. According to Lemma 4.4, the definition of $\mathcal{A}^R_{0,T}$ and Proposition 4.8, we obtain

$$\mathbb{E}^{\mathbb{Q}}\big[\big|Y_s^{q,t,x}-Y_s^{q,t,x_0}\big|\big]\le C\big(1+|x|^2+|x_0|^2\big)^{1/2}|x-x_0|+C\int_s^T\mathbb{E}^{\mathbb{Q}}\big[\big|Y_u^{q,t,x}-Y_u^{q,t,x_0}\big|\big]\,\mathrm{d}u.$$

Then, Gronwall's lemma gives us $|Y_t^{q,t,x}-Y_t^{q,t,x_0}|\le C(1+|x|+|x_0|)|x-x_0|$. Since this bound is independent of q, we finally obtain that

$$\big|Y_t^{t,x}-Y_t^{t,x_0}\big|\le C\big(1+|x|+|x_0|\big)|x-x_0|.$$

Now, we will study the second term. Without loss of generality, let us assume that $t<t_0$.

$$\begin{aligned}\big|Y_t^{t,x_0}-Y_{t_0}^{t_0,x_0}\big|&\le\big|Y_t^{t,x_0}-Y_t^{t_0,x_0}\big|+\int_t^{t_0}\big|g\big(s,x_0,Y_s^{t_0,x_0},0\big)\big|\,\mathrm{d}s\\&\le\big|Y_t^{t,x_0}-Y_t^{t_0,x_0}\big|+\int_t^{t_0}C\big(1+|x_0|^2+\big|Y_s^{t_0,x_0}\big|\big)\,\mathrm{d}s.\end{aligned}$$

We apply Proposition 4.7 to obtain

$$\big|Y_t^{t,x_0}-Y_{t_0}^{t_0,x_0}\big|\le\big|Y_t^{t,x_0}-Y_t^{t_0,x_0}\big|+C\big(1+|x_0|^2\big)(t_0-t).$$

We still have

$$\big|Y_t^{t,x_0}-Y_t^{t_0,x_0}\big|\le\operatorname*{ess\,sup}_{q\in\mathcal{A}^R_{0,T}}\big|Y_t^{q,t,x_0}-Y_t^{q,t_0,x_0}\big|.$$

Moreover, exactly as the bound estimation for $\mathbb{E}^{\mathbb{Q}}|Y_s^{q,t,x}-Y_s^{q,t,x_0}|$, we have, for $t\le s\le T$,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\big[\big|Y_s^{q,t,x_0}-Y_s^{q,t_0,x_0}\big|\big]&\le\mathbb{E}^{\mathbb{Q}}\big[C\big(1+\big|X_T^{t,x_0}\big|^2+\big|X_T^{t_0,x_0}\big|^2\big)\big]^{1/2}\mathbb{E}^{\mathbb{Q}}\big[\big|X_T^{t,x_0}-X_T^{t_0,x_0}\big|^2\big]^{1/2}\\&\quad+\int_s^T\mathbb{E}^{\mathbb{Q}}\big[C\big(1+\big|X_u^{t,x_0}\big|^2+\big|X_u^{t_0,x_0}\big|^2\big)\big]^{1/2}\mathbb{E}^{\mathbb{Q}}\big[\big|X_u^{t,x_0}-X_u^{t_0,x_0}\big|^2\big]^{1/2}\,\mathrm{d}u\\&\quad+C\int_s^T\mathbb{E}^{\mathbb{Q}}\big[\big|Y_u^{q,t,x_0}-Y_u^{q,t_0,x_0}\big|\big]\,\mathrm{d}u.\end{aligned}$$

According to Lemma 4.4, the definition of $\mathcal{A}^R_{0,T}$, Proposition 4.8 and Gronwall's lemma, we obtain $|Y_t^{q,t,x_0}-Y_t^{q,t_0,x_0}|\le C(1+|x|^2+|x_0|^2)|t-t_0|^{1/2}$. Since this bound is independent of q, we finally obtain that

$$\big|Y_t^{t,x_0}-Y_t^{t_0,x_0}\big|\le C\big(1+|x|^2+|x_0|^2\big)|t-t_0|^{1/2}.$$

So,

$$\big|u(t,x)-u(t_0,x_0)\big|\le C\big(1+|x|+|x_0|\big)|x-x_0|+C\big(1+|x|^2+|x_0|^2\big)|t-t_0|^{1/2}.$$

We now return to the general case (for T): we set $N\in\mathbb{N}$ such that $T/N<T_2$ and, for $i\in\{0,\dots,N\}$, we define $t_i:=iT/N$. According to the beginning of the proof, u is continuous on $[t_{N-1},T]\times\mathbb{R}^d$. We define $h_{N-1}(x):=Y_{t_{N-1}}^{t_{N-1},x}$. Since $|h_{N-1}(x)-h_{N-1}(x')|\le C(1+|x|+|x'|)|x-x'|$, we are allowed to reuse previous results to show the continuity of u on $[t_{N-2},t_{N-1}]\times\mathbb{R}^d$. Thus, we can iterate this argument to show the continuity of u on $[0,T]\times\mathbb{R}^d$. Moreover, the quadratic increase of u with respect to the variable x results from the quadratic increase of u on each interval.

Finally, we will use a stability result to show that u is a viscosity solution to the PDE (4.1). As in the proof of Theorem 2.1, let us consider the function

$$g_n(t,x,y,z):=\inf\big\{g(t,x,p,q)+n|p-y|+n|q-z|,(p,q)\in\mathbb{Q}^{1+d}\big\}.$$

We have already seen that $(g_n)_{n\geq\lceil r\rceil}$ is increasing and converges uniformly on compact sets to g. Let $(Y^{n,t,x}, Z^{n,t,x})$ be the unique solution in $\mathcal{S}^2\times M^2$ to BSDE$(h(X_T^{t,x}), -g_n(\cdot, X_\cdot^{t,x}, \cdot, \cdot))$. We define $u_n(t,x) := Y_t^{n,t,x}$. Then by a classical theorem (see, e.g., [7,9]), u_n is a viscosity solution to the PDE

$$\partial_t u(t,x) + \mathcal{L}u(t,x) - g_n\bigl(t,x,u(t,x), -\sigma^*\nabla_x u(t,x)\bigr) = 0, \qquad u(T,\cdot) = h.$$

Moreover, it follows from the classical comparison theorem that $(u_n)_{n\geq\lceil r\rceil}$ is decreasing and, by construction, converges pointwise to u. Since u is continuous, Dini's theorem implies that the convergence is also uniform on compacts sets. Then, we apply a stability result (see, e.g., Theorem 1.7 of Chapter 5 in [1]) to prove that u is a viscosity solution to the PDE (4.1). □

Remark. *The uniqueness of viscosity solution to PDE is considered by Da Lio and Ley in* [6] *and* [5].

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