Electronic Journal of Statistics
Vol. 4 (2010) 1324–1344
ISSN: 1935-7524
DOI: 10.1214/09-EJS381

Non standard functional limit laws for the increments of the compound empirical distribution function

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Abstract: Let $(Y_i, Z_i)_{i\geq 1}$ be a sequence of independent, identically distributed (i.i.d.) random vectors taking values in $\mathbb{R}^k \times \mathbb{R}^d$, for some integers k and d. Given $z \in \mathbb{R}^d$, we provide a nonstandard functional limit law for the sequence of functional increments of the compound empirical process, namely

$$\mathbf{\Delta}_{n,\mathfrak{c}}(h_n, z, \cdot) := \frac{1}{nh_n} \sum_{i=1}^{n} 1_{[0,\cdot)}\Big(\frac{Z_i - z}{{h_n}^{1/d}}\Big) Y_i.$$

Provided that $nh_n \sim c \log n$ as $n \to \infty$, we obtain, under some natural conditions on the conditional exponential moments of $Y \mid Z = z$, that

$$\mathbf{\Delta}_{n,\mathfrak{c}}(h_n, z, \cdot) \rightsquigarrow \Gamma \text{ almost surely},$$

where $\rightsquigarrow$ denotes the clustering process under the sup norm on $[0,1)^d$. Here, Γ is a compact set that is related to the large deviations of certain compound Poisson processes.

Keywords and phrases: Empirical processes, Poisson process, large deviations.

Received February 2009.

Contents

*This is an original survey paper

1. Introduction and statement of the results

Let $(Y_i, Z_i)_{i\geq 1}$ be a sequence of independent, identically distributed (i.i.d.) random vectors taking values in $\mathbb{R}^k \times \mathbb{R}^d$, for some integers k and d. Given $s, t \in \mathbb{R}^d$ with respective coordinates $s_1, \ldots, s_d$ and $t_1, \ldots, t_d$, we shall write $[s,t] := [s_1,t_1] \times \cdots \times [s_d,t_d]$, $[s,t) := [s_1,t_1) \times \cdots \times [s_d,t_d)$ and given $a \in \overline{\mathbb{R}}$ we set $[a,t] := [a,t_1] \times \cdots \times [a,t_d]$. For each integer $n \geq 1$, define the compound empirical distribution function as:

$$\mathbb{U}_{n,\mathfrak{c}}(s) := \frac{1}{n}\sum_{i=1}^{n} 1_{(-\infty,s]}(Z_i)Y_i,\ s \in \mathbb{R}^d. \tag{1.1}$$

Here the letter $\mathfrak{c}$ stands for "compound". In this paper, we are concerned with the asymptotic behaviour of the functional increments of $\mathbb{U}_{n,\mathfrak{c}}$, namely, for fixed $h > 0$ and $z \in \mathbb{R}^d$,

$$\boldsymbol{\Delta}_{n,\mathfrak{c}}(h,z,s) := \frac{1}{nh}\sum_{i=1}^{n} 1_{[0,s)}\Big(\frac{Z_i - z}{h^{1/d}}\Big)Y_i,\ s \in [0,1)^d. \tag{1.2}$$

Note that, in the particular case where $k = 1$ and $Y_1 \equiv 1$, the $\boldsymbol{\Delta}_{n,\mathfrak{c}}(h,z,\cdot)$ are no more than the functional increments of the empirical distribution function, which have been intensively investigated in the literature (see, e.g., [4, 5, 10, 11]). Among these investigations, Deheuvels and Mason ([4, 5]) have established *nonstandard* functional limit laws for the $\boldsymbol{\Delta}_{n,\mathfrak{c}}(h,z,\cdot)$ when $k = 1$, $d = 1$, $Y_1 \equiv 1$ and Z_1 is uniformly distributed on $[0,1)$. To cite their results, we need to introduce some further notations. We shall write

$$|| \ s \ ||_k := \max\{| \ s_1 \ |, \ldots, | \ s_k \ |\}$$

for $s \in \mathbb{R}^k$, and we define $B_k([0,1)^d)$ as the space of all mappings from $[0,1)^d$ to $\mathbb{R}^k$ that are bounded. We shall endow $B_k([0,1)^d)$ with the usual sup-norm, namely $| \ g \ |_k := \sup_{s\in[0,1)^d} \ | \ g(s) \ |_k$. Given a convex real function $\mathfrak{h}$ on $\mathbb{R}^k$, we define the following functional on $B_k([0,1)^d)$: whenever a function g satisfies $g(0) = 0$ and admits a derivative g' with respect to the Lebesgue measure, set

$$J_{\mathfrak{h}}(g) := \int_{[0,1)^d} \mathfrak{h}(g'(s))ds, \tag{1.3}$$

and set $J_{\mathfrak{h}}(g) := \infty$ if it is not the case. We also write, for any $c > 0$,

$$\Gamma_{\mathfrak{h}}(c) := \big\{g \in B_k([0,1)^d),\ J_{\mathfrak{h}}(g) \leq c\big\}. \tag{1.4}$$

Now define the following (Chernoff) function on $[0,\infty)$:

$$\mathfrak{h}_1(x) := \begin{cases} x\log x - x + 1, & \text{for } x > 0; \\ 1, & \text{for } x = 0; \\ \infty, & \text{for } x < 0. \end{cases} \tag{1.5}$$

A sequence (f_n) in a metric space (E, ρ), is said to be *relatively compact* with limit set equal to K when K is (non void) compact and the following assertions are true

$$\lim_{n\to\infty} \inf_{f\in K} d(f_n, f) = 0, \tag{1.6}$$

$$\forall\, f \in K,\ \liminf_{n\to\infty} d(f_n, f) = 0. \tag{1.7}$$

We shall write this property $x_n \rightsquigarrow K$.

Throughout this article, we shall consider a sequence of constants $(h_n)_{n\geq 1}$ satisfying the so called local nonstandard conditions, namely, as $n \to \infty$,

(HV) $\quad 0 < h_n < 1,\ h_n \downarrow 0,\ nh_n \uparrow \infty,\ nh_n/\log_2 n \to c \in (0, \infty).$

Here we have set $\log_2 n := \log(\log(n \vee 3))$, with the notation $a \vee b := \max\{a, b\}$. In a pioneering work, Deheuvels and Mason [4] established a *nonstandard* functional law of the iterated logarithm for a single functional increment of the empirical distribution function. With the notation of the present paper, their theorem can be stated as follows.

Fact 1.1 (Deheuvels and Mason, 1990)**.** *Let $(h_n)_{n\geq 1}$ be a sequence satisfying (HV) for some constant $c > 0$. Assume that $k = 1$, $d = 1$, $Y_1 \equiv 1$, and that Z_1 is uniformly distributed on $[0, 1)$. Then, given $z \in [0, 1)$, we have almost surely*

$$\boldsymbol{\Delta}_{n,\mathfrak{c}}(h_n, z, \cdot) \ \rightsquigarrow \Gamma_{\mathfrak{h}_1}(1/c).$$

Later, Deheuvels and Mason [6] extended the just mentioned result to a more general setting, where $d > 1$ and with fewer assumptions on the law of the Z_i, considering the $\boldsymbol{\Delta}_{n,\mathfrak{c}}(h_n, z, \cdot)$ as random measures indexed by a class of sets. The aim of the present paper is to extend the above mentioned results to the case where the random vectors Y_i are not constant, but do satisfy some assumptions on their conditional exponential moments given $Z = z$. From now on $< \cdot, \cdot >$ will always denote the Euclidian scalar product on $\mathbb{R}^k$ and λ stands for the Lebesgue measure. Define $\mathcal{C}$ as the class of each $C \subset \mathbb{R}^d$ which is the union of d hypercubes of $\mathbb{R}^d$, and with $\lambda(C) > 0$. The two key assumptions that we shall make upon the law of (Y_1, Z_1) are stated as follows.

(HL1) There exists a constant $f(z) > 0$ satisfying, for each $C \in \mathcal{C}$
$\lim_{h\to 0} h^{-1}\mathbb{P}(Z_1 \in z + h^{1/d}C) = \lambda(C)f(z).$

(HL2) There exist two mappings $\mathcal{L}_Y : \mathbb{R}^k \mapsto [0, \infty)$ and $\mathcal{L}_{|Y|_k} : \mathbb{R} \mapsto [0, \infty)$ such that, for each $t \in \mathbb{R}^k$ and $t' \in \mathbb{R}$ and $C \in \mathcal{C}$, we have
$\lim_{h\to 0} \mathbb{E}(\exp(< t, Y_1 >)|Z_1 \in z + h^{1/d}C) = \mathcal{L}_Y(t),$
$\lim_{h\to 0} \mathbb{E}(\exp(t' \mid Y_1 \mid_k)|Z_1 \in z + h^{1/d}C) = \mathcal{L}_{|Y|_k}(t').$

Remark 1.1. Assumptions (HL1) and (HL2) seem to be the weakest that we can afford in this context, in regard to the methods we make use of in this paper. Note that $(HL2)$ implies that $\mathcal{L}_Y$ is infinitely differentiable on $\mathbb{R}^k$. Some straightforward analysis shows that these assumptions are fulfilled when the Y_i

are bounded by a constant and when Z_i admit a (version of) density f which is continuous at z. Another interesting case where (HL1) and (HL2) are fulfilled is a general semi parametric setting which appears in the following proposition.

Proposition 1.1. *Assume that there exist a σ-finite measure v on $\mathbb{R}^k$ and an application f_Y from $\mathbb{R}^k \times \mathbb{R}^d$ to $[0, \infty)$, such that*

1. *There exists a neighborhood $\mathcal{V}$ of z such that, for each $z' \in \mathcal{V}$, the law of $Y \mid Z' = z$ is dominated by v, with density $f_Y(\cdot, z')$.*
2. *For v-almost all y, the function $z \to f_Y(y, z)$ is continuous on $\mathcal{V}$.*
3. *For each $z' \in \mathcal{V}$, and each $t \in \mathbb{R}^k$ we have*

$$\int_{\mathbb{R}^k} \exp\big(< t, u > \big) f_Y(u, z') dv(u) < \infty.$$

4. *Z has a version of density (with respect to the Lebesgue measure λ) which is continuous on $\mathcal{V}$.*

Then the random vector (Y, Z) fulfills $(HL1)$ and $(HL2)$.

Proof. The proof is a straightforward application of Schéffé's lemma. □

Remark 1.2. Roughly speaking, assumption (HL2) imposes that the Laplace transform of the law $Y \mid Z = z$ is finite on $\mathbb{R}^k$. One could argue that this assumption could be weakened. However, it seems that, when this assumption is dropped, Theorem 1 (see below) does not hold anymore under the strong norm $|| \cdot ||_k$. A close look at the works of Deheuvels [3] and Borovkov [2] on the functional increments of random walks leads to the conjecture that the appropriate topology when $\mathcal{L}$ is finite only on a neighborhood of 0 seems to be the usually called *weak star* topology (see, e.g. [3]). This topic is however beyond the scope of this article, and shall be investigated in future works.

Notice that $\mathcal{L}_Y$ and $\mathcal{L}_{|Y|_k}$ are positive convex functions when they exist. We now introduce $\mathfrak{h}_Y$ (resp. $\mathfrak{h}_{|Y|_k}$), which is defined as the Legendre transform of $\mathcal{L}_Y - 1$ (resp. $\mathcal{L}_{|Y|_k} - 1$), namely:

$$\mathfrak{h}_Y(u) := \sup_{t \in \mathbb{R}^k} < t, u > - \big(\mathcal{L}_Y(t) - 1\big),\ u \in \mathbb{R}^k, \tag{1.8}$$

$$\mathfrak{h}_{|Y|_k}(x) := \sup_{t' \in \mathbb{R}} t'x - \big(\mathcal{L}_{|Y|}(t) - 1\big),\ x \in \mathbb{R}. \tag{1.9}$$

Recall that the constant $c > 0$ appears in assumption (HV) and that $\Gamma_{\mathfrak{h}_Y}(1/c(z))$ has been defined by (1.4) and (1.8). Our result can be stated as follows.

Theorem 1. *Under assumptions (HV), (HL1) and (HL2), we have almost surely*

$$f(z)^{-1} \boldsymbol{\Delta}_{n,\mathfrak{c}}(z, h_n, \cdot) \ \rightsquigarrow\ \Gamma_{\mathfrak{h}_Y}(1/cf(z)).$$

A consequence of this result is the following unconsistency result, for which no proof has been yet provided to the best of our knowledge: let K be real function on $\mathbb{R}^d$ with *bounded variation and compact support*. The Nadaraya-Watson regression estimator of $r(z) := \mathbb{E}(Y \mid Z = z)$ is defined as:

$$r_n(z) := \sum_{i=1}^{n} \frac{K\Big(\frac{Z_i - z}{h_n^{1/d}}\Big)}{\sum_{j=1}^{n} K\Big(\frac{Z_j - z}{h_n^{1/d}}\Big)} Y_j,\ z \in \mathbb{R}^d.$$

Theorem 1 entails that, under (HV), (HL1) and (HL2), the pointwise strong consistency of r_n does not hold.

Corollary 1.1. *Under (HV), (HL1) and (HL2), and assuming that $Y \not\equiv 0$, we have almost surely*

$$\limsup_{n\to\infty} \mid r_n(z) - r(z) \mid_k > 0.$$

Proof. We may assume without loss of generality that K vanishes outside $[0,1)^d$. Consider the random vectors $\widetilde{Y}_i := (Y_i, 1)$, taking values in $\mathbb{R}^{k+1}$. Some straightforward computations show that $\widetilde{Y}, Z$ satisfy $(HL1)$ and $(HL2)$, and that, writing $m := \nabla \mathcal{L}_{\widetilde{Y}}(0)$, we have

$$\mathfrak{h}_{\widetilde{Y}}(m) = 0. \tag{1.10}$$

Moreover, assuming without loss of generality that Y has a second moment matrix which is strictly positive, we have $\nabla^2 \mathcal{L}_{\widetilde{Y}} > 0$ (strictly positive matrix) on $\mathbb{R}^{k+1}$, which ensures that

$$t_0 \mapsto \nabla \mathcal{L}_{\widetilde{Y}}(t)\Big|_{t=t_0}$$

is a C^1 diffeomorphism from $\mathbb{R}^{k+1}$ to an open set $O \ni m$. And hence admits an inverse that we write $\nabla \mathcal{L}_{\widetilde{Y}}^{-1}$. We deduce that

$$\mathfrak{h}_{\widetilde{Y}}(x) = < x, \nabla \mathcal{L}_{\widetilde{Y}}^{-1}(x) > - \Big(\mathcal{L}_{\widetilde{Y}}\big(\nabla \mathcal{L}_{\widetilde{Y}}^{-1}(x)\big) - 1 \Big)$$

is continuous in x, which implies, that, for $\epsilon > 0$ small enough we have

$$\sup_{\substack{x \in \mathbb{R}^k \\ ||x-m||_k < \epsilon / f(z)}} \mid \mathfrak{h}_{\widetilde{Y}}(x) \mid_1 < \frac{1}{cf(z)}. \tag{1.11}$$

For $g \in B_{k+1}([0,1)^d)$, we shall write $g = (g_k, g_{k+1})$, where g_{k+1} denotes the last coordinate of g and $g_k \in B_k([0,1)^d)$ is equal to g without its last coordinate. We shall also write, for a Borel set A and for $\ell = (\ell_1, \ldots, \ell_k) \in B_k([0,1)^d)$

$$\ell(A) := \Big(\int_{\mathbb{R}^d} 1_A d\ell_1, \ldots, \int_{\mathbb{R}^d} 1_A d\ell_k \Big). \tag{1.12}$$

which is well defined as soon as either 1_A or each g_i has bounded variations on $\mathbb{R}^d$. Consider the following mappings:

$$\begin{array}{rcl}\Psi: \Big(B_{k+1}([0,1)^d), ||\cdot||_k\Big) & \mapsto & \mathbb{R}^{k+1} \\ g & \to & \Big(\int_{s\in[0,1)^d} g_k([s,1])dK(s), \\ & & \int_{s\in[0,1)^d} g_{k+1}([s,1])dK(s)\Big), \end{array}$$

$$\begin{array}{rcl}\Psi': \Big(B_{k+1}([0,1)^d), ||\cdot||_k\Big) & \mapsto & \mathbb{R}^{k+1} \\ g & \to & \Big(\int_{s\in[0,1)^d} \int_{[s,1]} \dot{g}_k(u)d\lambda(u)dK(s), \\ & & \int_{s\in[0,1)^d} \int_{[s,1]} \dot{g}_{k+1}(u)d\lambda(u)dK(s)\Big), \end{array}$$

$$\begin{array}{rcl}T: \mathbb{R}^k \times (\mathbb{R} - \{0\}) & \mapsto & \mathbb{R}^k \\ (x_1,\dots,x_k,x_{k+1}) & \to & \frac{1}{x_{k+1}}(x_1,\dots,x_k). \end{array}$$

Also consider

$$\widetilde{\Gamma}_{\mathfrak{h}_{\widetilde{Y}}}(1/cf(z)) := \Big\{ g \in L^2([0,1)^d)^{k+1},\ \int_{[0,1)^d} \mathfrak{h}_{\widetilde{Y}}(g) \leq 1/c \Big\}.$$

We obviously have $\Psi(f(z)\Gamma_{\mathfrak{h}_{\widetilde{Y}}}(1/cf(z))) = \Psi'(f(z)\widetilde{\Gamma}_{\mathfrak{h}_{\widetilde{Y}}}(1/cf(z)))$. As K has bounded variations, Ψ is continuous and so is $T\circ\Psi$. Applying Theorem 1, we then deduce that, almost surely

$$\begin{aligned} r_n(z) = T\circ\Psi\Big(\mathbf{\Delta}_{\mathbf{n},\mathfrak{c}}\Big) &\rightsquigarrow T\circ\Psi\Big(f(z)\Gamma_{\mathfrak{h}_{\widetilde{Y}}}(1/cf(z))\Big) \\ &\supset T\circ\Psi'\Big(f(z)\widetilde{\Gamma}_{\mathfrak{h}_{\widetilde{Y}}}(1/cf(z)) \cap B_{k+1}([0,1)^d)\Big). \end{aligned}$$

It hence remains to show that $T\circ\Psi'(f(z)\widetilde{\Gamma}_{\mathfrak{h}_{\widetilde{Y}}}(1/cf(z))\cap B_{k+1}([0,1)^d))$ has non empty interior, which shall obviously imply that, almost surely, $r_n(z) \not\to r(z)$ as $n\to\infty$. It is well known that, as Ψ' is continuous, surjective and linear from the Banach space $(B_{k+1}([0,1)^d), ||\cdot||_k)$ to $\mathbb{R}^{k+1}$, $\Psi'(O)$ is open for every open set O. Hence, it is sufficient to show that $f(z)\widetilde{\Gamma}_{\mathfrak{h}_{\widetilde{Y}}}(1/cf(z))\cap B_{k+1}([0,1)^d)$ has nonempty interior in $(B_{k+1}([0,1)^d), ||\cdot||_k)$. Consider $\epsilon > 0$ that appears in (1.11). Writing $g_m :\equiv m \in B_{k+1}([0,1)^d)$, we have, by (1.11)

$$||\ f(z)g - f(z)g_m\ ||_k < \epsilon \ \Rightarrow\ \int_{[0,1)^d} \mathfrak{h}_{\widetilde{Y}}(g) \leq 1/c(z),$$

which concludes the proof. □

The remainder of our paper is organised as follows. In §2, we introduce an almost sure approximation of $\mathbf{\Delta}_{n,\mathfrak{c}}(z,h_n,\cdot)$ by a sum of compound Poisson processes. This approximation is largely inspired by a lemma of Deheuvels and Mason [6]. We then focus on these "poissonised" processes and provide some exponential inequalities on their modulus of continuity. In §3, we establish a Large Deviation Principle (LDP). Then §4 and §5 are devoted to proving points (1.6) and (1.7) of Theorem 1 respectively.

2. A Poisson approximation

Recall that $z \in \mathbb{R}^d$ is fixed once for all in our problem. For ease of notation we write

$$\Delta_{n,\mathfrak{c}}(z,h,s) := \frac{1}{nhf(z)} \sum_{i=1}^{n} 1_{[0,s)}\Big(\frac{Z_i - z}{h^{1/d}}\Big) Y_i, \;\; s \in [0,1)^d, \; h > 0. \tag{2.1}$$

Throughout this article, we shall refer to a generic stochastic process U, usually called *compound Poisson process*. It is defined as follows: consider an infinite i.i.d array $(\mathfrak{Y}_{ij}, \mathfrak{Z}_{ij})_{i\geq 1,\, j\geq 1}$ having the same law as (Y_1, Z_1), as well as a Poisson random variable with expectation equal to 1 fulfilling $\eta \perp\!\!\!\perp (\mathfrak{Y}_{ij}, \mathfrak{Z}_{ij})_{i\geq 1,\, j\geq 1}$ (here $\perp\!\!\!\perp$ denotes stochastic independence). Now define

$$U(s) := \sum_{j=1}^{\eta} 1_{[0,s)}\big(\mathfrak{Z}_{ij} - z\big)\mathfrak{Y}_{ij}. \tag{2.2}$$

Note that the law of U is entirely determined by the following property:

For each $p \geq 1$ and for each partition $A_1, \ldots, A_p$ of $[0,1)^d$ we have:

$$\mathbb{E}\Bigg(\exp\Big(\sum_{j=1}^{p} < t_j, U(A_j) > \Big)\Bigg)$$

$$= \exp\Big(\sum_{j=1}^{p} \mathbb{P}(Z - z \in A_j)\big(\mathcal{L}_{Y|A_j}(t_j) - 1\big)\Big), \;\; \big(t_1, \ldots, t_p\big) \in \big(\mathbb{R}^k\big)^p \tag{2.3}$$

where $\mathcal{L}_{Y|A_j}(t) := \mathbb{E}(\exp(< t, Y > | Z - z \in A_j),\; j = 1, \ldots, p,\; t \in \mathbb{R}^k$. Recall the expression $U(A)$ is understood according to (1.12). The following proposition enables to switch the study of the almost behaviour of the sequence $(\Delta_{n,\mathfrak{c}}(z, h_n, \cdot))_{n\geq 1}$ to that of a sequence with the following generic term

$$\Delta\Pi_{n,\mathfrak{c}}(h,s) := \frac{1}{nhf(z)} \sum_{i=1}^{n} U_i(h^{1/d}s), \; s \in [0,1)^d, \; n \geq 1, \tag{2.4}$$

where the U_i are suitably built independent copies of U. This result is in the spirit of Deheuvels and Mason (see [6], Lemma 2.1, or [4], Proposition 2.1).

Proposition 2.1. *On a probability space rich enough $(\Omega, \mathcal{A}, \mathbb{P})$ we can construct an i.i.d. sequence of processes $(U_i)_{i\geq 1}$ having the same law as U and an sequence $(Y_{i1}, Z_{i1})_{i\geq 1}$ having the same law as $(Y_i, Z_i)_{i\geq 1}$ such that, considering the $\Delta_{n,\mathfrak{c}}(z, h_n, \cdot)$ as built with the sequence $(Y_{i1}, Z_{i1})_{i\geq 1}$ we have almost surely*

$$\limsup_{n\to\infty} \, (nh_n) \;\; || \, \Delta_{n,\mathfrak{c}}(z, h_n, \cdot) - \Delta\Pi_{n,\mathfrak{c}}\big(h_n, \cdot\big) \, ||_k < \infty, \tag{2.5}$$

with $\Delta\Pi_{n,\mathfrak{c}}(\cdot,\cdot)$ defined in (2.4).

Proof. Denote by U a process having the same law as in (2.2). Set $\mathcal{V}_i := z + h_i^{1/d}[0,1)^d$, $p_i = \mathbb{P}(Z_1 \in \mathcal{V}_i)$, and let $(Y_{ij}^{(1)}, Z_{ij}^{(1)})_{i\geq 1,\, j\geq 1}$, $(Y_{ij}^{(2)}, Z_{ij}^{(2)})_{i\geq 1,\, j\geq 1}$, $(\mathbf{b}_i)_{i\geq 1}$, $(U_i^*(\cdot))_{i\geq 1}$ and $(v_i)_{i\geq 1}$ be families of random elements such that

(a) $\mathbb{P}((Y_{ij}^{(1)}, Z_{ij}^{(1)}) \in B) = \mathbb{P}((Y_1, Z_1) \in B | Z_1 \in \mathcal{V}_i)$, B Borel set, $i, j \geq 1$.
(b) $\mathbb{P}((Y_{ij}^{(2)}, Z_{ij}^{(2)}) \in B) = \mathbb{P}((Y_1, Z_1) \in B | Z_1 \notin \mathcal{V}_i)$, B Borel set, $i, j \geq 1$.
(c) For each $i \geq 1$ we have $\mathbb{P}(v_i = 0) = 1 - p_i^{-1}(1 - e^{-p_i})$ and $\mathbb{P}(v_i = k) = (k!)^{-1} p_i^{k-1} e^{-p_i}$, $k = 1, 2, \ldots$.
(d) $\mathbb{P}(\mathbf{b}_i = 1) = 1 - \mathbb{P}(\mathbf{b}_i = 0) = p_i$, $i \geq 1$.
(e) The U_i^* are independent copies of U defined in (2.2).
(f) The union of these five families of random elements is a stochastically independent family.

In (e), equality in law is understood as an equality with respect to the σ-algebra $\mathcal{T}_0$ of $(B_k([0,1)^d), ||\cdot||_k)$ spawned by the open balls. In (f), stochastic independence is understood with respect to a suitably chosen product σ-algebra where each factor is either $\mathcal{T}_0$, the Borel σ-algebra of $\mathbb{R}^k \times \mathbb{R}^d$, or the subsets of $\{0, 1, 2, \ldots\}$. First, notice that $\eta_i^* := v_i \mathbf{b}_i$ is a Poisson random variable with expectation p_i for each $i \geq 1$, and that

$$\forall i \geq 1,\ \mathbb{P}\big(\eta_i^* = \mathbf{b}_i\big) \geq 1 - p_i^2. \tag{2.6}$$

In fact, η_i^* and $\mathbf{b}_i$ are a coupling of a Poisson and Bernouilli random variables $(\eta, \mathbf{b})$ with expectation p_i such that the probability $\mathbb{P}(\eta = \mathbf{b})$ is maximal. Second, notice that the following random vectors

$$\big(Y_{ij}, Z_{ij}\big) := 1_{\mathbf{b}_i = 1}\big(Y_{ij}^{(1)}, Z_{ij}^{(1)}\big) + 1_{\mathbf{b}_i = 0}\big(Y_{ij}^{(2)}, Z_{ij}^{(2)}\big),\ i \geq 1,\ j \geq 1, \tag{2.7}$$

are i.i.d. with common law equal to (Y_1, Z_1). Moreover, the following assertions are true with probability one, for each $i \geq 1$:

$$\forall s \in [0,1)^d,\ 1_{[0,s)}\Big(\frac{Z_{ij} - z}{h_i^{1/d}}\Big) Y_{ij} = \sum_{j=1}^{\mathbf{b}_i} 1_{[0,s)}\Big(\frac{Z_{ij}^{(1)} - z}{h_i^{1/d}}\Big) Y_{ij}^{(1)}. \tag{2.8}$$

We now define, for each $i \geq 1$,

$$U_i(s) := U_i^*\Big([0,s) \cap \{\mathcal{V}_i - z\}^C\Big) + \sum_{j=1}^{\eta_i^*} 1_{[0,s)}\big(Z_{ij}^{(1)} - z\big) Y_{ij}^{(1)}. \tag{2.9}$$

Here, $\mathcal{V}^C$ denotes the complement of a given set $\mathcal{V} \subset \mathbb{R}^d$. Some usual computations on characteristic functions show that the processes $U_i(\cdot)$ fulfill (2.3), and hence are distributed like U. Moreover since $h_{i+p}^{1/d}[0,1)^d \subset h_i^{1/d}[0,1)^d$ for $i \geq 1,\ q \geq 0$, we have almost surely

$$U_i\big(h_{i+q}^{1/d} s\big) = \sum_{j=1}^{\eta_i^*} 1_{[0,s)}\Big(\frac{Z_{ij}^{(1)} - z}{h_{i+q}{}^{1/d}}\Big) Y_{ij}^{(1)},\ s \in [0,1)^d. \tag{2.10}$$

It follows from (2.6), (2.8) and (2.10) that, for each $i \geq 1$,

$$\mathbb{P}\Big(U_i\big(h_{i+q}^{1/d}\,\cdot\,\big) \equiv 1_{[0,\cdot)}\Big(\frac{Z_i - z}{h_{i+q}^{1/d}}\Big)Y_{i1} \text{ for each } q \geq 0\Big) \geq \mathbb{P}\big(\eta_i^* = \mathbf{b}_i\big) \geq 1 - p_i^2. \tag{2.11}$$

Since $p_n = f(z)h_n(1 + o(1))$ as $n \to \infty$, and by assumption (HV), we have $\sum p_i^2 < \infty$, which entails, by making use of the Borel-Cantelli lemma, that (2.5) is true with respect to our construction. □

By Proposition 2.1, proving Theorem 1 is equivalent to proving a version of Theorem 1 with the process $\Delta_{n,\mathfrak{c}}(z, h_n, \cdot)$ replaced by their Poisson approxiations $\Delta\Pi_{n,\mathfrak{c}}(h, \cdot)$. This will be the aim of §3, §4 and §5. In each of these three sections, we shall require the following exponential inequality for the *absolute oscillations* of $\Delta\Pi_{n,\mathfrak{c}}$, which are defined as the oscillations of the following process:

$$\overline{\Delta\Pi_{n,\mathfrak{c}}}(h, s) := \frac{1}{nhf(z)}\sum_{i=1}^{n}\sum_{j=1}^{\eta_i} 1_{[0,s)}\Big(\frac{Z_{ij} - z}{h^{1/d}}\Big) \mid Y_{ij} \mid_k,\ s \in [0,1)^d,\ n \geq 1. \tag{2.12}$$

Recall that $\mathfrak{h}_{|Y|_k}$ has been defined by (1.9).

Lemma 2.1. *Given $\delta \in (0, \sqrt{2} - 1]$ and $x \geq 0$, there exists $h_x > 0$ such that, for each $0 < h < h_x$ and for each $n \geq 1$, we have*

$$\begin{aligned}
&\mathbb{P}\Bigg(\sup_{\substack{s,s' \in [0,1)^d \\ |s'-s|_d \leq \delta}} \Big|\overline{\Delta\Pi_{n,\mathfrak{c}}}\big(h, s\big) - \overline{\Delta\Pi_{n,\mathfrak{c}}}\big(h, s'\big)\Big|_1 \geq 2d\delta x\Bigg) \\
&\leq \Big(\frac{10}{\delta}\Big)^d \exp\Big(-d\delta nhf(z)\mathfrak{h}_{|Y|_k}(x)\Big),
\end{aligned} \tag{2.13}$$

$$\mathbb{P}\Big(\overline{\Delta\Pi_{n,\mathfrak{c}}}(h, 1) \geq x\Big) \leq \exp\big(-nhf(z)\mathfrak{h}_{|Y|_k}(x)\big). \tag{2.14}$$

Proof. Given s and $s' \in [0,1)^d$, we write $s \prec s'$ whenever each coordinate of s is lesser than the corresponding coordinate of s'. Obviously, the $\overline{\Delta\Pi_{n,\mathfrak{c}}}(h, s)$ are almost surely increasing in each coordinate of s. First fix $\delta > 0$ and set

$$M := 1 + \Big[\frac{3}{(\sqrt{2} - 1)\delta}\Big]. \tag{2.15}$$

We then discretise $[0,1)^d$ into the following finite grid:

$$s_{\mathbf{i}} := \frac{1}{M}\mathbf{i},\ \mathbf{i} \in \{0, 1, \ldots, M - 1\}^d. \tag{2.16}$$

By construction, for each s and s' with $\mid s' - s \mid_d \leq \delta$, there exists $\mathbf{i}_s \in \{0, 1, \ldots, M - 1\}^d$ fulfilling $s_{\mathbf{i}_s} \prec s$ and $\mid s - s_{\mathbf{i}_s} \mid_d \leq 1/M$, which entails

$\mid s' - s_{\mathbf{i}_s} \mid \leq 1/M + \delta$. Hence we can write

$$\mathbb{P}\Bigg(\sup_{\substack{s,s' \in [0,1)^d \\ |s'-s|_d \leq \delta}} \Big| \overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s) - \overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s') \Big|_1 \geq 2d\delta x \Bigg)$$

$$\leq \mathbb{P}\Bigg(\bigcup_{\mathbf{i} \in \{0,1,\dots,M-1\}^d} \Big\{ \sup_{\substack{s_{\mathbf{i}} \prec s', \\ |s'-s_{\mathbf{i}}|_d \leq \delta + 1/M}} \Big| \overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s') - \overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s_{\mathbf{i}}) \Big|_1 \geq 2d\delta x \Big\} \Bigg)$$

$$\leq M^d \max_{\mathbf{i} \in \{0,1,\dots,M-1\}^d} \mathbb{P}\Bigg(\sup_{\substack{s_{\mathbf{i}} \prec s', \\ |s'-s_{\mathbf{i}}|_d \leq \delta + 1/M}} \Big| \overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s') - \overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s_{\mathbf{i}}) \Big|_k \geq 2d\delta x \Bigg).$$

Now notice that, for each $n \geq 1$, we have

$$\big(\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s)\big)_{s \in [0,1)^d} =_{\mathcal{L}} \left(\frac{1}{nhf(z)} \sum_{i=1}^{\eta_n} 1_{[0,s)}\Big(\frac{Z_i - z}{h^{1/d}}\Big) \mid Y_i \mid_k \right)_{s \in [0,1)^d}, \tag{2.17}$$

where η_n is a Poisson random variable with expectation n and independent of $(Y_i, Z_i)_{i \geq 1}$ (here $=_{\mathcal{L}}$ stands for the equality in law for processes). For a Borel set $B \subset [0,1)^d$, write

$$\begin{aligned} \overline{\Delta\Pi_{n,\mathfrak{c}}}(h,B) &:= \int_{[0,1)^d} 1_B(s) d\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s) \\ &= \frac{1}{nhf(z)} \sum_{i=1}^{\eta_n} 1_B\Big(\frac{Z_i - z}{h^{1/d}}\Big) \mid Y_i \mid_k . \end{aligned} \tag{2.18}$$

By the triangle inequality we have almost surely

$$\begin{aligned} &\sup_{\substack{s_{\mathbf{i}} \prec s', \\ |s'-s_{\mathbf{i}}|_d \leq \delta + 1/M}} \Big| \overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s') - \overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s_{\mathbf{i}}) \Big|_1 \\ &\leq \sup_{\substack{s_{\mathbf{i}} \prec s', \\ |s'-s_{\mathbf{i}}|_d \leq \delta + 1/M}} \frac{1}{nhf(z)} \sum_{i=1}^{\eta_n} \big(1_{[0,s')} - 1_{[0,s_{\mathbf{i}})}\big)\Big(\frac{Z_i - z}{h^{1/d}}\Big) \mid Y_i \mid_k \\ &\leq \frac{1}{nhf(z)} \sum_{i=1}^{\eta_n} \big(1_{[0,s_{\mathbf{i}}^+)} - 1_{[0,s_{\mathbf{i}})}\big)\Big(\frac{Z_i - z}{h^{1/d}}\Big) \mid Y_i \mid_k, \end{aligned} \tag{2.19}$$

where $s_{\mathbf{i}}^+$ is defined by adding $M^{-1}([M\delta]+2)$ to each coordinate of s_i. Line (2.19) is a consequence of the fact that, if $s_{\mathbf{i}} \prec s'$ and $\mid s' - s \mid_d \leq \delta + 1/M$, then $s_{\mathbf{i}} \prec s' \prec s_{\mathbf{i}^+}$. We shall now write $B_{\mathbf{i}} := [0, s_{\mathbf{i}}^+) - [0, s_{\mathbf{i}})$. Now choose $t = t(x)$ fulfilling

$$tx - \big(\mathcal{L}_{|Y|}(t) - 1\big) \geq \frac{1}{2}\mathfrak{h}_{|Y|_k}(x). \tag{2.20}$$

By Markov's inequality we have

$$\begin{aligned}
&\mathbb{P}\Big(\frac{1}{nhf(z)}\sum_{i=1}^{\eta_n}\big(1_{[0,s_{\mathbf{i}}^+)}-1_{[0,s_{\mathbf{i}})}\big)\Big(\frac{Z_i-z}{h^{1/d}}\Big)\mid Y_i\mid_k\geq 2d\delta x\Big)\\
&\leq\ \exp\big(-2d\delta nhf(z)tx\big)\mathbb{E}\Big[\exp\Big(t\sum_{i=1}^{\eta_n}1_{B_{\mathbf{i}}}\Big(\frac{Z_i-z}{h^{1/d}}\Big)\mid Y_i\mid_k\Big)\Big]\\
&\leq\ \exp\big(-2d\delta nhf(z)tx\big)\exp\Big(n\big(\mathcal{L}_{h,\mathbf{i}}(t)-1\big)\Big), \qquad (2.21)
\end{aligned}$$

with

$$\mathcal{L}_{h,\mathbf{i}}(t):=\mathbb{E}\Big[\exp\Big(t1_{B_{\mathbf{i}}}\Big(\frac{Z_1-z}{h^{1/d}}\Big)\mid Y_1\mid_k\Big)\Big].$$

Note that (2.21) has been obtained by conditioning with respect to η_n. Now, by conditioning with respect to $E_{\mathbf{i},h}:=\{h^{-1/d}(Z_1-z)\in B_{\mathbf{i}}\}$, and writing

$$\mathcal{L}'_{h,\mathbf{i}}(t):=\mathbb{E}\Big[\exp\big(t\mid Y_1\mid_k\big)\Big|E_{\mathbf{i},h}\Big], \qquad (2.22)$$

we obtain

$$\begin{aligned}
\mathcal{L}_{h,\mathbf{i}}(t)-1&=\mathbb{P}\big(E_{\mathbf{i},h}\big)\mathcal{L}'_{h,\mathbf{i}}(t)+\Big(1-\mathbb{P}\big(E_{\mathbf{i},h}\big)\Big)-1\\
&=\mathbb{P}(E_{\mathbf{i},h})\Big(\mathcal{L}'_{h,\mathbf{i}}(t)-1\Big). \qquad (2.23)
\end{aligned}$$

Note that assumptions (HL1) and (HL2) readily entail

$$\lim_{h\to 0}\max_{\mathbf{i}\in\{0,\dots,M-1\}^d}\Big|\frac{\mathbb{P}(E_{\mathbf{i},h})\big(\mathcal{L}_{2,h,\mathbf{i}}(t)-1\big)}{f(z)\lambda(B_{\mathbf{i}})h\big(\mathcal{L}_{|Y|_k}(t)-1\big)}-1\Big|=0. \qquad (2.24)$$

Choose $h_x>0$ small enough so that the quantity involved in (2.24) is lesser that $\sqrt{2}-1$ and notice that for each $\mathbf{i}$ we have $\lambda(B_{\mathbf{i}})\leq d(\delta+1/M)\leq\sqrt{2}d\delta$ by (2.15). By combining (2.19), (2.21) and (2.23), we conclude that, for all $0<h<h_x$,

$$\begin{aligned}
&\max_{\mathbf{i}\in\{0,1,\dots,M-1\}^d}\mathbb{P}\Bigg(\sup_{\substack{s_{\mathbf{i}}\prec s',\\ |s'-s_{\mathbf{i}}|_d\leq\delta+1/M}}\Big|\overline{\Delta\Pi_{n,\mathfrak{c}}}\big(h,s'\big)-\overline{\Delta\Pi_{n,\mathfrak{c}}}\big(h,s_{\mathbf{i}}\big)\Big|_k\geq 2d\delta x\Bigg)\\
&\leq\exp\Big(-2d\delta nhf(z)\big(tx-\mathcal{L}_{|Y|_k}(t)+1\big)\Big), \qquad (2.25)
\end{aligned}$$

whence, by (2.20) we get

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{\substack{s,s'\in[0,1)^d\\ |s'-s|_d\leq\delta}}\Big|\overline{\Delta\Pi_{n,\mathfrak{c}}}\big(h,s\big)-\overline{\Delta\Pi_{n,\mathfrak{c}}}\big(h,s'\big)\Big|_k\geq 2d\delta x\Big)\\
&\leq M^d\exp\Big(-d\delta nhf(z)\mathfrak{h}_{|Y|_k}(x)\Big)\\
&\leq\Big(1+\frac{3}{(\sqrt{2}-1)\delta}\Big)^d\exp\Big(-d\delta nhf(z)\mathfrak{h}_{|Y|_k}(x)\Big)\\
&\leq\Big(\frac{10}{\delta}\Big)^d\exp\Big(-d\delta nhf(z)\mathfrak{h}_{|Y|_k}(x)\Big).
\end{aligned}$$

This concludes the proof of Lemma 2.1. $\square$

3. Large deviations for $\Delta\Pi_{n,\mathfrak{c}}(h_n,\cdot)$

In this section, we establish a Large Deviation Principle (LDP) for the sequence of processes $\Delta\Pi_{n,\mathfrak{c}}(h_n,\cdot)$. For the definition of large deviations for sequences for bounded stochastic processes and of a (good) rate function, we refer to Arcones [1].

3.1. Some tools in large deviation theory

We begin this subsection with some well known properties (see, e.g., [3], Lemma 2.1, or Borovkov [2] just above the main Theorem) of $\mathfrak{h}_Y$ and $\mathfrak{h}_{|Y|_k}$ given in (1.8) and (1.9) respectively.

Fact 3.1. *The functions $\mathfrak{h}_Y$ and $\mathfrak{h}_{|Y|_k}$ are positive convex. Moreover, since $\mathcal{L}_Y$ is finite on $\mathbb{R}^k$, we have*

$$\lim_{|u|_k\to\infty}\frac{\mathfrak{h}_Y(u)}{\mid u\mid_k}=\infty,$$
$$\lim_{|x|\to\infty}\frac{\mathfrak{h}_{|Y_k|}(x)}{\mid x\mid}=\infty.$$

Arcones (see [1], Theorem 3.1) has established a very useful criterion to establish a LDP for processes in $B_k([0,1)^d)$ (actually only with $k=1$ but the extension of his results to $k>1$ is straightforward). We cannot make a direct use of his Theorem 3.1 and shall make use of a slight modification of it. To state this modification, we shall introduce some more notations. For each integer $p\geq 1$, consider a finite grid

$$\begin{aligned}S_p&=\Big\{s_{\mathbf{j},p},\mathbf{j}\in\{1,\dots,2^p\}^d\Big\}\\&:=\Big\{2^{-p}\mathbf{j},\ \mathbf{j}\in\{1,\dots,2^p\}^d\Big\}.\end{aligned}\tag{3.1}$$

and consider its associated partition of $[0,1)^d$ into hypercubes, namely

$$C_{\mathbf{j},p}:=\Big[2^{-p}(\mathbf{j}-1),2^{-p}\mathbf{j}\Big),\ \mathbf{j}\in\{1,\dots,2^p\}^d.\tag{3.2}$$

Here we have written $\mathbf{j}-1=(j_1-1,\dots,j_d-1)$. Now for each integer $p\geq 1$ and for each $g\in B_k([0,1)^d)$ write

$$g^{(p)}(s):=g(s_{\mathbf{j},p}),\ s\in C_{\mathbf{j},p},\ \mathbf{j}\in\{1,\dots,2^p\}^d.\tag{3.3}$$

The following proposition is a straightforward variation of Theorem 1 of Arcones [1], and is written according to the notation of that theorem (in particular, we refer to [1] for a definition of the outer probability $\mathbb{P}^*$).

Proposition 3.1. *Let $(X_n)_{n\geq 1}$ be a sequence of stochastic processes and let $(\epsilon_n)_{n\geq 1}$ be a sequence of constants fulfilling $\epsilon_n>0$, $n\geq 1$ and $\epsilon_n\to 0$ as $n\to\infty$. Assume that the following conditions are satisfied.*

1. *The sequence of stochastic processes* $(X_n^{(p)})_{n\geq 1}$ *satisfies the LDP for* $(\epsilon_n^{-1})_{n\geq 1}$ *and for a rate function* J_p *on* $(B_k([0,1)^d), ||\cdot||_k)$.
2. *For each* $\tau > 0$ *and* $M > 0$ *there exists an integer* $p \geq 1$ *satisfying*

$$\limsup_{n\to\infty} \epsilon_n \log\Big(\mathbb{P}^*\Big(\max_{\mathbf{j}\in\{1,\ldots,2^p\}^d} \sup_{s\in C_{\mathbf{j},p}} \mid X_n(s) - X_n(s_{\mathbf{j},p}) \mid_k \geq \tau\Big)\Big) \leq -M.$$

Then $(X_n)_{n\geq 1}$ *satisfies the LDP for* $(\epsilon_n^{-1})_{n\geq 1}$ *and for the following rate function.*

$$J(g) := \sup_{p\geq 1} J_p\Big(g^{(p)}\Big),\ g\in B_k([0,1)^d).$$

Proof. The proof follows the same lines as in the proof of Theorem 3.1 of Arcones [1]. We omit details for sake of briefness. □

For $g = (g_1, \ldots, g_k) \in B_k([0,1)^d)$ and A Borel set, we shall write

$$g(A) := \Big(\int_{[0,1)^d} 1_A(s)dg_1(s), \ldots, \int_{[0,1)^d} 1_A(s)dg_k(s)\Big), \tag{3.4}$$

which is valid as long as 1_A or each g_l have bounded variations. We shall now consider the following (rate) functions on $(B_k([0,1)^d), ||\cdot||_k)$ that will play the role of successive approximations of $J_{\mathfrak{h}_Y}$: given $p\geq 1$ and $g\in B_k([0,1)^d)$ we set

$$J_{\mathfrak{h}_Y}^{(p)}(g) := \sum_{\mathbf{j}\in\{1,\ldots,2^p\}^d} \lambda(C_{\mathbf{j},p})\mathfrak{h}_Y\Big(\lambda(C_{\mathbf{j},p})^{-1} g(C_{\mathbf{j},p})\Big). \tag{3.5}$$

The following fact is a straightforward extension to the multivariate case of Proposition 2.1 in [12]. Recall that $J_{\mathfrak{h}_Y}$ has been defined through (1.3) and (1.8).

Fact 3.2. *For any* $g\in B_k([0,1)^d)$ *we have*

$$J_{\mathfrak{h}_Y}(g) = \lim_{p\to\infty} J_{\mathfrak{h}_Y}^{(p)}(g). \tag{3.6}$$

As a consequence, $J_{\mathfrak{h}_Y}$ *is lower semicontinuous on* $B_k([0,1)^d)$.

Our next lemma states that the function $J_{\mathfrak{h}_Y}$ (recall (1.3)) is a "rate" function.

Lemma 3.1. *The sets* $\Gamma_{J_{\mathfrak{h}_Y}}(a)$, $a\geq 0$ *are compact subsets of* $(B_k([0,1)^d), ||\cdot||_k)$. *In other words,* $J_{\mathfrak{h}_Y}$ *is a rate function in* $(B_k([0,1)^d), ||\cdot||_k)$.

Proof. By Fact 3.1 we have $\mid x\mid_k \leq \mid x\mid_k 1_{|x|_k\leq M} \wedge J_{\mathfrak{h}_Y}$ for some $M>0$ and for each x. Hence, for any $g\in\Gamma_{J_{\mathfrak{h}_Y}}(a)$ we have (recall that λ stands for the Lebesgue measure)

$$\int_{[0,1)^d} \mid g'\mid_k d\lambda = \int_{|g'|_k\leq M} \mid g'\mid_k d\lambda + \int_{|g'|_k>M} J_{\mathfrak{h}_Y}\big(g'\big)d\lambda \tag{3.7}$$

$$\leq M + a, \tag{3.8}$$

from where we conclude that $\Gamma_{J_{\mathfrak{h}_Y}}$ is relatively compact in $B_k([0,1)^d)$. It is also closed in $B_k([0,1)^d)$ by a combination of Fact 3.2 and (3.8), which proves Lemma 3.1. □

3.2. A large deviation principle

In this subsection, we state and prove a large deviation principle that will play a crucial role in the sequel of our proof of Theorem 1. This LDP is stated as follows:

Proposition 3.2. *Under assumptions* (HV), $(HL1)-(HL2)$, *the sequence* $(\Delta\Pi_{n,\mathfrak{c}}(h_n,\cdot))_{n\geq 1}$ *satisfies the LDP in* $B_k([0,1)^d)$ *for* $(\epsilon_n^{-1})_{n\geq 1}=((nh_nf(z))^{-1})_{n\geq 1}$ *and for the rate function* $J_{\mathfrak{h}_Y}$.

Proof. As we shall make use of Proposition 3.1, we have to check conditions 1 and 2 of that proposition, which will be the aim of the following lemmas. Notice that, almost surely, we have

$$\Delta\Pi_{n,\mathfrak{c}}(h,C):=\frac{1}{nhf(z)}\sum_{i=1}^{n}\sum_{j=1}^{\eta_i}1_C\Big(\frac{Z_{i,j}-z}{h^{1/d}}\Big)Y_{i,j},\ C\text{ Borel },$$

with $\Delta\Pi_{n,\mathfrak{c}}(h,C)$ defined according to (1.12). Our proof is divided in two steps, where we shall respectively verify conditions 1 and 2 of Proposition 3.1.

Step 1: To check condition 2 of Proposition 3.1, we shall make use of Lemma 2.1, which readily entails, for fixed $p\geq 1$ and $\tau>0$, and for all $n\geq n(p,\tau)$:

$$\begin{aligned}&\mathbb{P}\Big(\max_{\mathbf{j}\in\{1,\dots,2^p\}^d}\sup_{s\in C_{\mathbf{j},p}}\mid\Delta\Pi_{n,\mathfrak{c}}(h_n,s)-\Delta\Pi_{n,\mathfrak{c}}(h_n,s_{\mathbf{j},p})\mid_k\geq\tau\Big)\\&\leq 10^d2^{pd}\exp\Big(-d2^{-p}nh_nf(z)\mathfrak{h}_{|Y|_k}\big(d^{-1}2^{p-1}\tau\big)\Big).\end{aligned}$$

Now fix $M>0$ and $\tau>0$. By Fact (3.1), we have, for all large p:

$$\frac{\mathfrak{h}_{|Y|_k}\big(d^{-1}2^{p-1}\tau\big)}{d^{-1}2^{p-1}\tau}>4M\tau,$$

which implies that condition 2 of Proposition 3.1 is verified.

Step 2: To check condition 1 of Proposition 3.1, we shall require the following preliminary lemma. □

Lemma 3.2. *For any sequence* $(h_n)_{n\geq 1}$ *fulfilling* $h_n\to 0$ *and* $nh_n\to\infty$, *and for any fixed* $p\geq 1$, *the sequence of random vectors of* $\mathbb{R}^{k^{2^p}}$

$$\Big(\Delta\Pi_{n,\mathfrak{c}}(h_n,C_{\mathbf{j}}),\ \mathbf{j}\in\{1,\dots,2^p\}^d\Big)_{n\geq 1}\tag{3.9}$$

satisfies the LDP for the sequence $(\epsilon_n^{-1})_{n\geq 1}:=((nh_nf(z))^{-1})_{n\geq 1}$ *and the following rate function*

$$\begin{aligned}J^{(p)}:(\mathbb{R}^k)^{2^p}\quad&\mapsto\quad\mathbb{R}\\ x\quad&\mapsto\quad\sum_{\mathbf{j}\in\{1,\dots,2^p\}^d}^{p}\lambda(C_{\mathbf{j},p})\mathfrak{h}_Y\Big(\lambda(C_{\mathbf{j},p})^{-1}x_{\mathbf{j}}\Big).\end{aligned}$$

Here we write $x=(x_{\mathbf{j}},\ \mathbf{j}\in\{1,\dots,2^p\}^d)$, *with* $x_{\mathbf{j}}\in\mathbb{R}^k$ *for each* $\mathbf{j}\in\{1,\dots,2^p\}^d$.

Proof. The proof of Lemma 3.2 is divided into three steps. The two first steps deal with a single component of the random vectors written in (3.9).

Step 1: In our *first step*, we make an additional assumption on $\mathcal{L}_Y$, which allows us to make a full use of the Gärtner-Ellis theorem (see, e.g., [7], p. 44).

$$(H_0): \quad \forall x \in \mathbb{R}^k \text{ fulfilling } \mathfrak{h}_Y(x) < \infty,\ \exists \eta \in \mathbb{R}^k,\ x = \nabla \mathcal{L}_Y(\eta).$$

Lemma 3.3. *Assume that* (H_0) *is true in addition to the assumptions of Theorem 1. Then, for each* $p \geq 1$ *and* $\mathbf{j} \in \{1, \dots, 2^p\}^d$*, the sequence*

$$\Big(\Delta\Pi_{n,\mathfrak{c}}(h_n, C_{\mathbf{j},p})\Big)_{n\geq 1}$$

satisfies the LDP for the sequence $(nh_nf(z))^{-1}$ *and the rate function* $\lambda(C_{\mathbf{j},p})\mathfrak{h}_Y(\lambda(C_{\mathbf{j},p})^{-1}\cdot)$.

Proof of Lemma 3.3. We shall first show that, for each $t \in \mathbb{R}^k$, we have

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{nh_nf(z)} \log\bigg(\mathbb{E}\Big(\exp < t, nh_nf(z)\Delta\Pi_{n,\mathfrak{c}}(C_{\mathbf{j},p}, h_n) > \Big)\bigg)\\ &= \lambda\big(C_{\mathbf{j},p}\big)\Big(\mathcal{L}_Y\Big(\lambda\big(C_{\mathbf{j},p}\big)^{-1}t\Big)\Big).\end{aligned} \tag{3.10}$$

To show this, we start from the equality (2.17) to obtain by convolution:

$$\begin{aligned}&\log\bigg(\mathbb{E}\Big(\exp < t, nh_nf(z)\Delta\Pi_{n,\mathfrak{c}}(C_{\mathbf{j},p}, h_n) > \Big)\bigg)\\ &= n\log\bigg(\mathbb{E}\Big(\exp < t, nh_nf(z)U\big(h_n^{1/d}C_{\mathbf{j},p}\big) > \Big)\bigg).\end{aligned}$$

Recall that U has been defined in (2.2). Next, we use the characterisation (2.3), which is applied to the simple partition $(h_n^{1/d}C_{\mathbf{j},p},\ [0,1)^d - h_n^{1/d}C_{\mathbf{j},p})$. Using that relation with $t_1 = t$ and $t_2 = 0$, we obtain

$$\begin{aligned}&\log\bigg(\mathbb{E}\Big(\exp < t, nh_nf(z)\Delta\Pi_{n,\mathfrak{c}}(C_{\mathbf{j},p}, h_n) > \Big)\bigg)\\ &= n\mathbb{P}\Big(Z - z \in h_n^{1/d}C_{\mathbf{j},p}\Big)\mathbb{E}\Big(\exp\big(< t, Y >\big)\Big| Z - z \in h_n \in z + h_n^{1/d}C_{\mathbf{j},p})\Big) - 1\Big).\end{aligned}$$

Hence (3.10) follows from assumptions $(HL1) - (HL2)$.

By Lemma 2.3.9 in [7], p 46, we know that (H_0) implies that the set of *exposed points* of $\mathfrak{h}_Y$ is equal to $\{x \in \mathbb{R}^k,\ \mathfrak{h}(x) < \infty\}$, from where the proof of Lemma 3.3 is concluded by an application of the Gärtner-Ellis theorem (see, e.g., [7], p. 44). □

Step 2: In our *second step*, we shall get rid of assumption (H_0), which is unfortunately not verified in all situations (for example, take $k = 1$, $Y \equiv 1$, which leads to $\mathcal{L}_Y(t) = \exp(t)$, $t \in \mathbb{R}$ and $\mathfrak{h}_Y(0) = 1$, but (H_0) is not satisfied for $x = 0$).

Lemma 3.4. *Lemma 3.3 is true without making assumption* (H_0).

Proof of Lemma 3.4. First notice that the "closed sets" part of the LDP stated in Lemma 3.3 can be proved by making use of the Gärtner-Ellis theorem, without making assumption (H_0). Only the "open sets" part of Lemma 3.3 needs assumption (H_0), since it implies that the set of *exposed points* of $\mathfrak{h}_Y$ is equal to $\{x \in \mathbb{R}^k,\ \mathfrak{h}_Y(x) < \infty\}$. We only need to prove that, without assumption (H_0), for any open set $O \subset \mathbb{R}^k$ with $\mathfrak{h}_Y(O) < \infty$ (nontrivial case), we have

$$\liminf_{n\to\infty} \frac{1}{nh_n f(z)} \log\left(\mathbb{P}\Big(\Delta\Pi_{n,\mathfrak{c}}\big(C_{\mathbf{j},p}, h_n\big) \in O\Big)\right) \geq -\mathfrak{h}_Y(O). \tag{3.11}$$

To achieve this goal, we shall slightly modify the $Y_{i,j}$ by adding small Gaussian random vectors. Fix $O \subset \mathbb{R}^k$ open, with $\mathfrak{h}_Y(O) < \infty$, and $\delta > 0$. There exists $x \in O$ and $\delta_1 \in (0, \delta)$ such that $B(x, 2\delta_1) \subset O$ and $\mathfrak{h}_Y(O) \leq \mathfrak{h}_Y(B(x, 2\delta_1)) \leq \mathfrak{h}_Y(x) \leq \mathfrak{h}_Y(O) + \delta < \delta_1^{-1}$. Here $B(x, \epsilon)$ denotes the open ball with centre x and radius ϵ. Now introduce an array $(\zeta_{ij})_{i,j\geq 1}$ of $\mathbb{R}^k$ valued standard random vectors, that are independent of the array $(Y_{i,j}, Z_{i,j})_{i,j\in\mathbb{N}}$. Also define

$$\begin{aligned}
\Delta\Pi'_{n,\mathfrak{c}}\big(C_{\mathbf{j},p}, h_n\big) &:= \frac{1}{nh_n f(z)} \sum_{i=1}^{n} \sum_{j=1}^{\eta_i} 1_{C_{\mathbf{j},p}}\Big(\frac{Z_{ij} - z}{h_n^{1/d}}\Big)\zeta_{ij}, \\
\Delta\Pi''_{n,\mathfrak{c}}\big(C_{\mathbf{j},p}, h_n\big) &:= \frac{1}{nh_n f(z)} \sum_{i=1}^{n} \sum_{j=1}^{\eta_i} 1_{C_{\mathbf{j},p}}\Big(\frac{Z_{ij} - z}{h_n^{1/d}}\Big)\big(Y_{ij} + \delta_1^2\zeta_{ij}\big) \\
&= \Delta\Pi_{n,\mathfrak{c}}\big(C_{\mathbf{j},p}, h_n\big) + \delta_1^2\Delta\Pi'_{n,\mathfrak{c}}\big(C_{\mathbf{j},p}, h_n\big).
\end{aligned}$$

We shall first show that the vector $Y + \zeta$ fulfills assumptions (H_0). To prove this first notice that $\mathcal{L}_{Y+\zeta} = \mathcal{L}_Y \mathcal{L}_\zeta$, which holds since Y and ζ are independent conditionally to Z. Obviously we have, since $\zeta \perp\!\!\!\perp Z$,

$$\mathcal{L}_\zeta(t) = \exp\Big(\frac{1}{2} \mid t \mid_k^2\Big),$$

which shows that ζ fulfills (H_0). Moreover, by Jensen's inequality we have

$$\mathcal{L}_Y(t) \geq \exp\big(< m_Y, t > \big),\ t \in \mathbb{R}^k,$$

where $m_Y = \mathbb{E}(Y|Z = z)$, which leads to

$$\mathcal{L}_{Y+\delta_1^2\zeta}(t) \geq \exp\Big(< m_Y, t > + \frac{\delta_1^4}{2} \mid t \mid_k^2 \Big). \tag{3.12}$$

Now consider $x \in \mathbb{R}^k$, and define the function $g(t) =< x, t > -(\mathcal{L}_{Y+\delta_1^2\zeta}(t) - 1)$. By (3.12) we have $g(t) \to -\infty$ as $\mid t \mid_k \to \infty$. Hence, the continuous and differentiable function g admits a maximum at some $\eta \in \mathbb{R}^k$ fulfilling $0 = \nabla g(\eta) = y - \nabla\mathcal{L}_{Y+\zeta}(x)$. This proves that the vector ζ fulfills (H_0) and hence, by Lemma 3.3 we have:

$$\liminf_{n\to\infty} \frac{1}{nh_nf(z)} \log\Big(\mathbb{P}\Big(\Delta\Pi''_{n,\mathfrak{c}}\big(C_{\mathbf{j},p},h_n\big)\in O\Big)\Big) \geq -\,\mathfrak{h}_{Y+\delta_1^2\zeta}(O), \quad (3.13)$$

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{nh_nf(z)} \log\Big(\mathbb{P}\Big(\Big|\Big|\Delta\Pi'_{n,\mathfrak{c}}\big(C_{\mathbf{j},p},h_n\big)\Big|\Big|_k \geq \delta_1^{-1}\Big)\Big) &\leq -\inf_{||x||_k\geq\delta_1^{-1}} \mathfrak{h}_\zeta(x)\\ &\leq -\,\delta_1^{-1}. \quad (3.14)\end{aligned}$$

The last inequality holds for $\delta_1>0$ small enough, by Fact 3.1, replacing Y by ζ. Hence, by the triangle inequality, we have for all large n:

$$\begin{aligned}&\mathbb{P}\Big(\Delta\Pi_{n,\mathfrak{c}}(C_{\mathbf{j},p},h_n)\in O\Big)\\ &\geq \mathbb{P}\Big(\Big|\Big|\Delta\Pi_{n,\mathfrak{c}}(C_{\mathbf{j},p},h_n)-x\Big|\Big|_k<2\delta_1\Big)\\ &\geq \mathbb{P}\Big(\Big(\Big|\Big|\Delta\Pi''_{n,\mathfrak{c}}(C_{\mathbf{j},p},h_n)-x\Big|\Big|_k<\delta_1\Big)-\mathbb{P}\Big(\delta_1^2\Big|\Big|\Delta\Pi'_{n,\mathfrak{c}}(C_{\mathbf{j},p},h_n)\Big|\Big|_k>\delta_1\Big)\\ &\geq \exp\Big(-nh_nf(z)\Big(\delta+\mathfrak{h}_{Y+\zeta}\big(B(x,\delta_1)\big)\Big)\Big)-\exp\Big(-nh_nf(z)\delta_1^{-1}\Big)\\ &\geq \exp\Big(-nh_nf(z)\Big(\delta+\mathfrak{h}_Y(x)\Big)\Big)-\exp\Big(-nh_nf(z)\delta_1^{-1}\Big) \quad (3.15)\\ &\geq \frac{1}{2}\exp\Big(-nh_nf(z)\Big(2\delta+\mathfrak{h}_Y(O)\Big)\Big). \quad (3.16)\end{aligned}$$

Note that (3.15) is a consequence $\mathfrak{h}_{Y+\zeta}\leq\mathfrak{h}_Y$, which follows directly from $\mathcal{L}_\zeta\geq 1$. Also, (3.16) is a consequence of $\mathfrak{h}_Y(x)\leq\mathfrak{h}_Y(O)+\delta$ together with $\delta_1^{-1}>\mathfrak{h}_Y(O)+2\delta$, which is true by the choice of δ_1. The proof of Lemma 3.4 is then concluded since O and δ are arbitrary. □

Step 3: The proof of Lemma 3.2 by a tensorisation argument brought by Lynch an Sethuraman. Since, for each n, the collection

$$\Delta\Pi_{n,\mathfrak{c}}(h_n,C_{\mathbf{j}}),\ \mathbf{j}\in\{1,\dots,2^p\}^d$$

is independent, and since each sequence $(\Delta\Pi_{n,\mathfrak{c}}(h_n,C_{\mathbf{j}}))_{n\geq1}$ satisfies the LDP with the rate function $\lambda(C_{\mathbf{j},p})\mathfrak{h}_Y(\lambda(C_{\mathbf{j},p})^{-1}\cdot)$. Then Lemma 3.2 is proved by applying Lemma 2.8 in [9]. □

A direct consequence of Lemma 3.2 is that condition 1 of Proposition 3.1 is satisfied, as shows our next lemma.

Lemma 3.5. *If $h_n\to0$ and $nh_n\to\infty$, then the sequence of processes*

$$\Big(\Delta\Pi^{(p)}_{n,\mathfrak{c}}(h_n,\cdot)\Big)_{n\geq1}$$

satisfies the LDP for $\epsilon_n:=(nh_nf(z))^{-1}$. and for the rate function $J^{(p)}_{\mathfrak{h}_Y}$.

Proof. The proof is a straightforward application of the contraction principle (see, e.g., [1], Theorem 2.1), considering, for fixed p, the following application, from $\mathbb{R}^{k^{2^{pd}}}$ to $(B_k([0,1)^d);||\cdot||_k)$ (here we write $x=(x_{\mathbf{j}},\ \mathbf{j}\in\{1,\dots,2^p\}^d)$, with each $x_{\mathbf{j}}$ belonging to $\mathbb{R}^k$)

$$\begin{aligned}\mathcal{R}_p(x):[0,1)^d &\mapsto [0,\infty)\\ s &\to \sum_{C_{\mathbf{j},p}\subset[0,s_1]\times\cdots\times[0,s_p]} x_{\mathbf{i}}.\end{aligned}\quad\square$$

We conclude the proof of Proposition 3.2 by combining Step 1 and Step 2 with Proposition 3.1. $\square$

4. Proof of point (1.6) of Theorem 1

We shall make use of some usual blocking arguments along the following subsequence:

$$n_k := \Big[\exp\Big(k\exp\big(-\sqrt{\log k}\big)\Big)\Big],\tag{4.1}$$

with associated blocks $N_k := \{n_{k-1}+1,\dots,n_k\}$. Here, $[u]$ denotes the only integer fulfilling $[u]\le u\le[u]+1$. We point out two key properties of $(n_k)_{k\ge1}$:

$$\lim_{k\to\infty}\frac{n_k}{n_{k-1}}=1,\ \ \lim_{k\to\infty}\frac{\log_2 n_k}{\log k}=1.\tag{4.2}$$

For any $\epsilon>0$ and $A\subset B_k([0,1)^d)$, we shall write:

$$A^\epsilon := \Big\{g\in B_k([0,1)^d),\ \inf_{g'\in A}||\,g-g'\,||_k<\epsilon\Big\}.\tag{4.3}$$

Now, recalling the definition of $\Delta\Pi_{n,\mathfrak{c}}$ in (2.4), we define the following normalised Poisson processes that will play a crucial role in our blocking arguments.

$$\mathcal{H}_n(s) := \frac{1}{n_k h_{n_k} f(z)}\sum_{i=1}^{n} U_i(h_{n_k}s),\ k\ge1,\ n\in N_k,\ s\in[0,1)^d.\tag{4.4}$$

Fix $\epsilon>0$. We shall proceed in two steps: first, we will prove that, we have almost surely, ultimately as $n\to\infty$,

$$\mathcal{H}_n\in\Gamma_{J_{\mathfrak{b}_Y}}(1/cf(z))^{2\epsilon},\tag{4.5}$$

then we shall show that almost surely:

$$\lim_{k\to\infty}\max_{n\in N_k}||\,\mathcal{H}_n(\cdot)-\Delta\Pi_{n_k,\mathfrak{c}}(h_{n_k},\cdot)\,||_k\le 3\epsilon.\tag{4.6}$$

Step 1: We first prove (4.5). In order to make use of usual blocking arguments along the blocks N_k we shall first show that

$$\lim_{k\to\infty}\max_{n\in N_k}\mathbb{P}\Big(\,||\,\mathcal{H}_n(\cdot)-\Delta\Pi_{n_k,\mathfrak{c}}(h_{n_k},\cdot)\,||_k>\epsilon\Big)=0.\tag{4.7}$$

To prove this, choose $k \geq 1$ and $n \in N_k$ arbitrarily. A rough upper bound gives (excluding the trivial case where $n = n_k$).

$$\begin{aligned}
\mathbb{P}_{n,1} &:= \mathbb{P}\Big(\mid\mid \mathcal{H}_n(\cdot) - \Delta\Pi_{n_k,\mathfrak{c}}(h_{n_k},\cdot) \mid\mid_k > \epsilon \Big) \\
&\leq \mathbb{P}\Big(\sum_{i=1}^{n_k-n} \sum_{j=1}^{\eta_i} \mid Y_{i,j} \mid_k > \epsilon \frac{n_k}{n_k - n}(n_k - n) h_{n_k} f(z) \Big) \\
&= \mathbb{P}\Big(\overline{\Delta\Pi}_{n_k-n,\mathfrak{c}}(h_{n_k},\cdot) > \epsilon \frac{n_k}{n_k - n} \Big) \qquad (4.8)
\end{aligned}$$

Now making use of point (2.14) of Proposition 2.1 with $x := \epsilon n_k/(n_k - n)$ we get, for all large k and for each $n \in N_k$ with $n \neq n_k$,

$$\mathbb{P}_n \leq \exp\Big(-\epsilon n_k h_{n_k} f(z) \frac{n_k - n}{\epsilon n_k} \mathfrak{h}_{|Y|_k}\big(\frac{\epsilon n_k}{n_k - n}\big)\Big). \qquad (4.9)$$

Now, as $n_k - n \geq n_k - n_{k-1}$, $n_k/(n_k - n_{k-1}) \to \infty$ and by Fact 3.1 we readily infer (4.7).

We are now able to make use of a well known maximal inequality (see, e.g., Deheuvels and Mason [5], Lemma 3.4) to conclude that, for all large k,

$$\begin{aligned}
\mathbb{P}_{k,2} &:= \mathbb{P}\Big(\bigcup_{n \in N_k} \mathcal{H}_n \notin \Gamma^{2\epsilon}_{J_{\mathfrak{h}_Y}} \Big) \\
&\leq 2\mathbb{P}\Big(\mathcal{H}_{n_k} \notin \Gamma^{\epsilon}_{J_{\mathfrak{h}_Y}} \Big) \\
&= 2\mathbb{P}\Big(\Delta\Pi_{n_k,\mathfrak{c}}(h_{n_k},\cdot) \notin \Gamma^{\epsilon}_{J_{\mathfrak{h}_Y}} \Big). \qquad (4.10)
\end{aligned}$$

Applying proposition 3.2 to the closed set $F := B_k([0,1)^d) - (\Gamma_{J_{\mathfrak{h}_Y}})^{\epsilon}$, which satisfies $J_{\mathfrak{h}_Y}(F) \geq (1 + 3\alpha)/cf(z)$ for some $\alpha > 0$ (by lower semi continuity of $J_{\mathfrak{h}_Y}$) we get, ultimately as $k \to \infty$,

$$\begin{aligned}
\mathbb{P}_{k,2} &\leq 2\exp\Big(-\frac{n_k h_{n_k}(1 + 2\alpha)}{cf(z)} \Big) \\
&\leq \exp\big(-(1+\alpha)\log\log n_k \big), \qquad (4.11)
\end{aligned}$$

where (4.11) is a consequence of assumption (HV). By (4.2), we conclude that $(\mathbb{P}_{k,2})_{k\geq 1}$ is summable, which proves (4.5) by making use of the Borel-Cantelli lemma.

Step 2: To prove (4.6) we shall make use of the following almost sure equality

$$\Delta\Pi_{n,\mathfrak{c}}(h_n, s) := \frac{n_k h_{n_k}}{n h_n} \mathcal{H}_n\big(\frac{h_{n_k}}{h_n} s\big). \qquad (4.12)$$

By (4.2) together with (HV) we straightforwardly infer that

$$\lim_{k\to\infty} \max_{n \in N_k} \Big| \frac{n_k h_{n_k}}{n h_n} - 1 \Big| = 0, \; \lim_{k\to\infty} \max_{n \in N_k} \frac{h_{n_k}}{h_n} = 1. \qquad (4.13)$$

Moreover, making use of (3.8), we infer that

$$\lim_{T\to 1,\ \rho\uparrow 1} \sup_{g\in \Gamma_{J_{\mathfrak{h}_Y}}} \quad || \, Tg(\rho\cdot) - g(\cdot) \, ||_k = 0. \tag{4.14}$$

Hence, (4.6) follows from a combination of (4.13), (4.14) and (4.5) together with the triangle inequality.

The proof of point (1.6) of Theorem 1 is concluded by combining (4.5) and (4.6) and recalling that $\epsilon > 0$ was arbitrary. □

5. Proof of point (1.7) of Theorem 1

We introduce the following subsequence

$$\overline{n}_k := k^{2k},\ k \geq 1.$$

Obviously, $\overline{n}_k$ satisfies the following properties:

$$\log_2 \overline{n}_k = \log k + \log_2 k + \log 2,\ \ \overline{n}_k/\overline{n}_{k-1} = e^{-2}k^{-2}(1+o(1)). \tag{5.1}$$

we also shall write $v_k := \overline{n}_k - \overline{n}_{k-1}$. Now define the sequence

$$\mathcal{H}'_k := \frac{1}{v_k h_{\overline{n}_k} f(z)} \sum_{\overline{n}_{k-1}+1}^{\overline{n}_k} 1_{[0,\cdot]}\Big(\frac{Z_i - z}{h_{\overline{n}_k}^{1/d}}\Big) Y_i.$$

Now choose $\epsilon > 0$ and $g \in \Gamma_{\mathfrak{h}_Y}(1/cf(z))$ arbitrarily. We shall prove that, with probability one

$$\limsup_{n\to\infty} \Big|\Big|\mathcal{H}'_k - g\Big|\Big| \leq 2\epsilon, \tag{5.2}$$

which would conclude the proof of point (1.7) of Theorem 1 by a classical compactness argument. Obviously g satisfies

$$\lim_{\rho\to 1} || \, g(\rho\cdot) - g(\cdot) \, ||_k = 0. \tag{5.3}$$

Some routine analysis also shows that, for some $\alpha > 0$ we have $J(g^\epsilon) < (1 - 2\alpha)/cf(z)$. By (5.1) we have $v_k h_{\overline{n}_k} \to \infty$ as $k \to \infty$. Hence, by Proposition (3.2), which we apply to the open ball g^ϵ we obtain, for all large k

$$\begin{aligned}\mathbb{P}\Big(\mathcal{H}'_k \in g^\epsilon\Big) &\geq \ \exp\Big(- \frac{v_k h_{\overline{n}_k} f(z)(1-2\alpha)}{cf(z)}\Big) \\ &\geq \ \exp\Big(- \log k + \log_2 k + \log 2\Big),\end{aligned}$$

where the last inequality is a consequence of (5.1). As the $(\mathcal{H}'_k)_{k\geq 1}$ are independent, the Borel-Cantelli lemma entails, almost surely,

$$|| \, \mathcal{H}'_k - g \, ||_k \leq \epsilon \text{ for all large } k. \tag{5.4}$$

To conclude the proof, notice that

$$\mathcal{H}'_k = \frac{v_k}{\overline{n}_k}\mathcal{H}'_k + \frac{1}{\overline{n}_k h_{\overline{n}_k} f(z)} \sum_{i=1}^{\overline{n}_{k-1}} 1_{[0,\cdot]}\Big(\frac{Z_i - z}{h_{\overline{n}_k}^{1/d}}\Big) Y_i$$
$$=: \frac{v_k}{\overline{n}_k}\mathcal{H}'_k + \zeta_k. \tag{5.5}$$

Hence, if we show that $\|\ \zeta_k\ \|_k \to 0$ almost surely, then (5.2) will follow by noticing that $v_k/\overline{n}_k \to 1$ and applying both (5.3) and (5.4). Noticing that

$$\|\ \zeta_k\ \|_k \leq \frac{\overline{n}_{k-1}}{\overline{n}_k} \overline{\Delta\Pi}_{\overline{n}_{k-1},\mathfrak{c}}(h_{\overline{n}_k}, 1),$$

we readily infer, by (5.1) and point (2.14) of Lemma 2.1, that $\mathbb{P}(\|\ \zeta_k\ \|_k > \delta) = O(k^{-2})$ for any $\delta > 0$. This concludes the proof of point (1.7) of Theorem 1.

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