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Annealed upper tails for the energy of a charged polymer

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Abstract. We study the upper tails for the energy of a randomly charged symmetric and transient random walk. We assume that only charges on the same site interact pairwise. We consider *annealed* estimates, that is when we average over both randomness, in dimension three or more. We obtain a large deviation principle, and an explicit rate function for a large class of charge distributions.

Résumé. Nous étudions les grandes déviations pour l'énergie d'un polymère. L'espace est discret, et le polymère est une chaine linéaire de n monomères associés à des charges. Nous supposons que deux charges n'intéragissent que lorsqu'elles occupent le même site de $\mathbb{Z}^d$. Nous considérons le cas où les deux aléas, valeurs des charges et positions des monomères, sont moyennés, et où la dimension de l'espace est 3 ou plus. Nous obtenons un principe de grande déviations, et pour certaines distributions de charges, la fonctionnelle de taux est explicite.

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Keywords: Random polymer; Large deviations; Random walk in random scenery; Self-intersection local times

1. Introduction

We consider the following toy model for a charged polymer in dimension 3 or more. Time and space are discrete, and two independent sources of randomness enter into the model:

(i) A symmetric random walk, $\{S(n), n \in \mathbb{N}\}$, evolving on the sites of $\mathbb{Z}^d$ with $d \geq 3$. When the walk starts at $z \in \mathbb{Z}^d$, its law is denoted $\mathbb{P}_z$.
(ii) A random field of charges, $\{\eta(n), n \in \mathbb{N}\}$. The charges are centered i.i.d. with variance 1, and their law is denoted by Q. We assume that each charge variable satisfies Cramer's condition: for some $\lambda_0 > 0$, $E_Q[\exp(\lambda_0 \eta)] < \infty$.

For a large integer n, our polymer is a linear chain of n monomers each carrying a random charge, and sitting sequentially on $\{S(0), \dots, S(n-1)\}$. The monomers interact pairwise only when they occupy the same site on the lattice, and produce a local energy

$$H_n(z) = \sum_{0 \leq i \neq j < n} \eta(i)\eta(j)\mathbb{1}\{S(i) = S(j) = z\} \quad \forall z \in \mathbb{Z}^d. \tag{1.1}$$

The energy of the polymer, H_n, is the sum of $H_n(z)$ over $z \in \mathbb{Z}^d$.

In this paper, we study the annealed probability, i.e. when averaging over both randomness, that H_n is large. The study of the event $\{-H_n \text{ large}\}$, that is the lower tails, is tackled in a companion paper [1], since the phenomenology and the techniques are different.

A few words of motivation. Our toy-model comes from physics, where it is used to model proteins or DNA folding. For mathematical works on random polymers, we refer to the review paper [13], and the recent books [7,12,14].

Our interest actually stems from works of Chen [8] and Chen and Khoshnevisan [9], dealing with central limit theorems for H_n (in the transient, and in the more delicate recurrent case). The former paper shows some analogy between H_n and the l_2-norm of the local times of the walk, whereas the latter paper shows similarities between the typical fluctuations of H_n and of a *random walk in random scenery*, to be defined later. Chen [8] establishes the following annealed moderate deviation principle. First, some notations: for two positive sequences $\{a_n, b_n, n \in \mathbb{N}\}$, we say that $a_n \ll b_n$, when $\limsup \frac{\log(a_n)}{\log(b_n)} < 1$, we denote the annealed law with P, and we denote with η a generic charge variable.

Proposition 1.1 (Chen [8]). *Assume $d \geq 3$, and $E[\exp(\lambda\eta^2)] < \infty$, for some $\lambda > 0$. When ξ_n goes to infinity, with $n^{1/2} \ll \xi_n \ll n^{5/8}$, then*

$$\lim_{n\to\infty} \frac{n}{\xi_n^2} \log\bigl(P(\pm H_n \geq \xi_n)\bigr) = -\frac{1}{2c_d}, \quad \text{where } c_d = \sum_{n\geq 1} \mathbb{P}_0\bigl(S(n) = 0\bigr). \tag{1.2}$$

Features of the charge distribution do not enter into the moderate deviations estimates, but play a role in the large deviation regime, which starts when $\{|H_n| \geq n^{2/3}\}$. Thus, to present our results, we first distinguish tail-behaviors of the η-variables. For $\alpha > 0$, we say that $\mathcal{H}_\alpha$ holds, or simply that $\eta \in \mathcal{H}_\alpha$, when $|\eta|^\alpha$ satisfies Cramer's condition. Also, in order to write shorter proofs, we assume two non-essential but handy features: η is symmetric with a unimodal distribution (see [6]), and we consider the simplest aperiodic walk: the walk jumps to a nearest neighbor site or stays still with equal probability.

Finally, we rewrite the energy into a convenient form. For $z \in \mathbb{Z}^d$, and $n \in \mathbb{N}$, we call $l_n(z)$ the *local times* and $\check{q}_n(z)$ the *local charges*. That is

$$l_n(z) = \sum_{k=0}^{n-1} \mathbb{1}\bigl\{S(k) = z\bigr\} \quad \text{and} \quad \check{q}_n(z) = \sum_{k=0}^{n-1} \eta(k)\mathbb{1}\bigl\{S(k) = z\bigr\}.$$

Inspired by Eq. (1) of [10] and (1.18) of [8], we write $H_n(z) = \check{X}_n(z) + Y_n(z)$ with

$$\check{X}_n(z) = \check{q}_n^2(z) - l_n(z) \quad \text{and} \quad Y_n(z) = l_n(z) - \sum_{i=0}^{n-1} \eta(k)^2 \mathbb{1}\bigl\{S(k) = z\bigr\}.$$

Now,

$$Y_n = \sum_{z\in\mathbb{Z}^d} Y_n(z) = \sum_{i=0}^{n-1} \bigl(1 - \eta^2(i)\bigr), \tag{1.3}$$

is a sum of centered independent random variables, and its large deviation asymptotics are well known (see below Remark 1.7). Thus, we focus on $\check{X}_n = \sum_{\mathbb{Z}^d} \check{X}_n(z)$.

Theorem 1.2. *Assume $d \geq 3$, and $n^{2/3} \ll \xi_n \ll n^2$. If $\eta \in \mathcal{H}_\alpha$ with $\alpha > 1$, then there is a positive constant $\mathcal{Q}_2$,*

$$\lim_{n\to\infty} \frac{1}{\sqrt{\xi_n}} \log P(\check{X}_n \geq \xi_n) = -\mathcal{Q}_2. \tag{1.4}$$

Moreover, if we define for $A > 0$, $\mathcal{D}_n^*(A) = \{z \in \mathbb{Z}^d\colon A\sqrt{\xi_n} > l_n(z) > \sqrt{\xi_n}/A\}$, then

$$\limsup_{A\to\infty} \limsup_{n\to\infty} \frac{1}{\sqrt{\xi_n}} \log P\Biggl(\sum_{z\notin\mathcal{D}_n^*(A)} \check{X}_n(z) \geq \xi_n\Biggr) = -\infty. \tag{1.5}$$

In words, Theorem 1.2 says that realizing an excess energy, of order ξ_n, forces a transient polymer to pile of the order $\sqrt{\xi_n}$ ($\ll n$) monomers, on a finite number of sites (independent of n), where the *local charge* is of order

$\sqrt{\xi_n}$. Indeed, the fact that we can assume that $\mathcal{D}_n^*(A)$ is finite comes from the transience of the random walk: by inequality (1.14) below, we have easily that for any $C>0$,

$$\mathbb{P}_0\big(\big|\mathcal{D}_n^*(A)\big|>C\big)\le (2n)^{dC}\exp\bigg(-C^{1-2/d}\frac{\sqrt{\xi}_n}{A}\bigg).$$

Thus, as soon as $C^{1-2/d}>A\mathcal{Q}_2$, $\{|\mathcal{D}_n^*(A)|>C\}$ is negligible in our asymptotics (1.4). Note also that

$$\sum_{z\in\mathcal{D}_n^*(A)}\check{X}_n(z)=\sum_{z\in\mathcal{D}_n^*(A)}\check{q}_n^2(z)+\mathrm{O}\big(\sqrt{\xi_n}\big),$$

and we can write

$$\check{q}_n^2(z)=l_n(z)^2\bigg(\frac{\check{q}_n(z)}{l_n(z)}\bigg)^2.$$

This suggests that on the *piles* (i.e., sites of $\mathcal{D}_n^*(A)$), the average charge is of order unity, and the *local charges* perform large deviations.

Remark 1.3. *There is a gap between the regime studied in Chen [8] defined by $\xi_n\ll n^{5/8}$ (see Proposition 1.1 above), and our regime defined by $n^{2/3}\ll\xi_n$ in Theorem 1.2. When comparing the two asymptotics, and looking for $\min(\sqrt{\xi}_n,\xi_n^2/n)$, one expects that the boundary between the two regimes is $\xi_n=n^{2/3}$, so that (1.2) should hold for $\xi_n\le n^{2/3}$.*

Theorem 1.2 is based ultimately on a subadditive argument, and the rate $\mathcal{Q}_2$ is beyond the reach of this method. However, there is large family of charge distributions for which we can explicitly compute $\mathcal{Q}_2$. To formulate a more precise result, we need additional assumptions and notations.

We call the log-Laplace transform of the charge distribution $\Gamma(x)=\log E_Q[\exp(x\eta)]$. Since the charges satisfy Cramer's condition, their empirical measure obeys a Large Deviation Principle with rate function $\mathcal{I}$, the Legendre-transform of Γ:

$$\mathcal{I}(x)=\sup_{y\in\mathbb{R}}\big[yx-\Gamma(y)\big]. \tag{1.6}$$

Finally, we define χ_d (when $d\ge 3$) such that

$$\mathbb{P}_0(S_k=0,\ \text{for some } k>0)=\exp(-\chi_d).$$

Theorem 1.4. *Assume $d\ge 3$, and $n^{2/3}\ll\xi_n\ll n^2$. Assume Γ is twice differentiable and satisfies*

$$x\mapsto\Gamma\big(\sqrt{x}\big)\ \textit{is convex on}\ \mathbb{R}^+. \tag{1.7}$$

Then,

$$\lim_{n\to\infty}\frac{1}{\sqrt{\xi_n}}\log\big(P(\check{X}_n\ge\xi_n)\big)=-\Gamma^{-1}(\chi_d). \tag{1.8}$$

Example 1.5. *For any $\alpha\in\,]1,2]$, we give in Section 4.1 examples of charge distributions in $\mathcal{H}_\alpha$ satisfying (1.7).*

Remark 1.6. *Note that when η is a centered Gaussian variable of variance σ, then $\Gamma^{-1}(\chi_d)=\sqrt{2\chi_d/\sigma}$. In this case, our proof of Theorem 1.4 applies equally well to random walk in the following random scenery: let $\{\zeta(z),z\in\mathbb{Z}^d\}$ be i.i.d. with $\zeta(0)$ distributed as $Z^2-\sigma$, where σ is a positive real, and Z is a centered Gaussian variable with variance σ. Then, for $d\ge 3$ and $n^{2/3}\ll\xi_n\ll n^2$, we have*

$$\lim_{n\to\infty}\frac{1}{\sqrt{\xi_n}}\log\big(P\big(\langle l_n,\zeta\rangle\ge\xi_n\big)\big)=-\sqrt{\frac{2\chi_d}{\sigma}},\quad \textit{where}\ \langle l_n,\zeta\rangle=\sum_{z\in\mathbb{Z}^d}l_n(z)\zeta(z). \tag{1.9}$$

Note that $\zeta \in \mathcal{H}_1$. A large deviations principle for RWRS was obtained in [11] *in the case $\mathcal{H}_\alpha$ with $0 < \alpha < 1$ and $\xi_n \gg n^{(1+\alpha)/2}$, and in* [3] *with*

$$1 < \alpha < \frac{d}{2} \quad \text{and} \quad n^{1-1/(\alpha+2)} \ll \xi_n \ll n^{1+1/\alpha}.$$

To our knowledge, (1.9) *is the first result for a special case of the border-line regime $\mathcal{H}_1$.*

Remark 1.7. *Since $H_n = \check{X}_n + Y_n$, and Y_n, given in* (1.3), *is a sum of n independent variables bounded by* 1, *it is clear that* (1.4) *and* (1.5) *hold for H_n as well as for $\check{X}_n$.*

We wish now to present intuitively two ways of understanding Theorem 1.2. A first step in proving Theorem 1.2 is the following result.

Proposition 1.8. *Assume $d \geq 3$ and $n^{2/3} \ll \xi_n \ll n^2$. Consider η of type $\mathcal{H}_\alpha$ with $1 \leq \alpha \leq 2$. There are positive constants c_-, c_+ such that*

$$\mathrm{e}^{-c_- \sqrt{\xi_n}} \leq P(\check{X}_n \geq \xi_n) \leq \mathrm{e}^{-c_+ \sqrt{\xi_n}}. \tag{1.10}$$

When $\xi_n = \xi n^2$ and $\xi > 0$ with $Q(\eta > \sqrt{\xi}) > 0$, (1.10) *holds.*

Assume now that η is of type $\mathcal{H}_1$. Note that $\check{X}_n$ is the l^2-norm of an additive random fields $n \mapsto \{\check{q}_n(z), z \in \mathbb{Z}^d\}$. This is analogous to the self-intersection local times

$$\|l_n\|_2^2 := \sum_{z \in \mathbb{Z}^d} l_n^2(z).$$

Now, our (naive) approach in the study of the excess self-intersection local time in [4–6] is to slice the l_2-norm over the level sets of the local times. By analogy, we define here the level sets of the *local charges* $\{\check{q}_n(z), z \in \mathbb{Z}^d\}$. For a value $\xi > 0$,

$$\mathcal{E}_n(\xi) = \bigl\{z \in \mathbb{Z}^d \colon \check{q}_n(z) \sim \xi\bigr\}.$$

For simplicity, we choose $\xi_n = n^\beta$, and we focus on the contribution of $\mathcal{E}_n(n^x)$ for $0 < x \leq \beta/2$,

$$\biggl\{\sum_{z \in \mathcal{E}_n(n^x)} \check{q}_n^2(z) \geq n^\beta\biggr\} \subset \bigl\{\bigl|\mathcal{E}_n\bigl(n^x\bigr)\bigr| \geq n^{\beta-2x}\bigr\} \subset \bigl\{\exists \Lambda \subset [-n, n]^d, |\Lambda| \leq n^{\beta-2x} \text{ and } \check{q}_n(\Lambda) \geq |\Lambda| n^x\bigr\}, \tag{1.11}$$

where $\check{q}_n(\Lambda)$ is the charge collected in Λ by the random walk in a time n. Thus, (1.11) requires an estimate for $P(\check{q}_n(\Lambda) \geq t)$. Note that by standard estimates, if we denote by $|\Lambda|$ the number of sites of Λ,

$$E\bigl[\check{q}_n(\Lambda)^2\bigr] = \sum_{z \in \Lambda} E_Q\bigl[\eta^2\bigr] \mathbb{E}_0\bigl[l_n(z)\bigr] \leq \sum_{z \in \Lambda} \mathbb{E}_0\bigl[l_\infty(z)\bigr] \leq C|\Lambda|^{2/d}. \tag{1.12}$$

Equation (1.12) motivates the following simple concentration lemma.

Lemma 1.9. *Assume dimension $d \geq 3$. For some constant $\kappa_d > 0$, and any finite subset Λ of $\mathbb{Z}^d$, we have for any $t > 0$ and any integer n*

$$P\bigl(\check{q}_n(\Lambda) \geq t\bigr) \leq \exp\biggl(-\kappa_d \frac{t}{|\Lambda|^{1/d}}\biggr). \tag{1.13}$$

Note the fundamental difference with the total time spent in Λ (denoted $l_\infty(\Lambda)$): for some positive constants $\tilde\kappa_d$

$$\mathbb{P}_0\big(l_\infty(\Lambda)\ge t\big)\le \exp\left(-\tilde\kappa_d\frac{t}{|\Lambda|^{2/d}}\right). \tag{1.14}$$

We have established (1.14) in Lemma 1.2 of [6]. Thus, using (1.13) in (1.11), we obtain

$$P\left(\sum_{z\in\mathcal{E}_n(n^x)}\check q_n^2(z)\ge n^\beta\right)\le C_n(x)\exp\big(-\kappa_d n^{\zeta(x)}\big), \tag{1.15}$$

with

$$\zeta(x)=\beta\left(1-\frac{1}{d}\right)-\left(1-\frac{2}{d}\right)x \quad\text{and}\quad C_n(x)=(2n+1)^{dn^{\beta-2x}}. \tag{1.16}$$

Looking at $\zeta(x)$, we observe that the high level sets (of $\check q_n$) give the dominant contribution and $\zeta(\frac{\beta}{2})=\frac{\beta}{2}$ in dimension three or more. Note that (1.16) also suggests that $d=2$ is a critical dimension, even though Lemma 1.9 fails in $d=2$. If one is to pursue this approach rigorously, one has to tackle the contribution of $C_n(x)$. Nonetheless, these simple heuristics show that inequality (1.13) is essentially responsible for the upper bound (1.10). It is easy to see that (1.13) is wrong when η is of type $\mathcal{H}_\alpha$ with $0<\alpha<1$, and a different phenomenology occurs.

First, an observation of Chen [8] is that fixing a realization of the walk, $\{\check q_n(z), z\in\mathbb{Z}^d\}$ are Q-independent random variables, and

$$\big\{\check q_n(z), z\in\mathbb{Z}^d\big\}\overset{Q\text{-law}}{=}\big\{q_n(z), z\in\mathbb{Z}^d\big\},\quad \text{where } q_n(z)=\sum_{i=1}^{l_n(z)}\eta_z(i), \tag{1.17}$$

where we denote by $\{\eta_z(i), z\in\mathbb{Z}^d, i\in\mathbb{N}\}$ i.i.d. variables distributed as η. Also,

$$\big\{\check X_n(z), z\in\mathbb{Z}^d\big\}\overset{Q\text{-law}}{=}\big\{X_n(z), z\in\mathbb{Z}^d\big\},\quad \text{where } X_n(z)=q_n^2(z)-l_n(z). \tag{1.18}$$

Now, a convenient way of thinking about X_n is to first fix a realization of the random walk, and to rewrite (1.18) as

$$X_n(z)=l_n(z)\big(\zeta_z\big(l_n(z)\big)-1\big),\quad \text{where for any } n\ \zeta_z(n)=\left(\frac{1}{\sqrt n}\sum_{i=1}^n\eta_z(i)\right)^2. \tag{1.19}$$

Thus, $\check X_n$ is equal in Q-law to a scalar product $\langle l_n,\zeta-1\rangle$ known as *random walk in random scenery* (RWRS). However, in our case the *scenery* is a function of the local times. To make this latter remark more concrete, we recall that when $\mathcal{H}_\alpha$ holds with $1<\alpha<2$, we have some constants C,κ_0,κ_∞ (see Section 2 for more precise statements), such that if

$$\zeta(n)=\left(\frac{1}{\sqrt n}\sum_{i=1}^n\eta(i)\right)^2,\quad \text{then } Q\big(\zeta(n)>t\big)\le C\begin{cases}\exp(-\kappa_0 t), & \text{when } t\ll n,\\ \exp\big(-\kappa_\infty t^{\alpha/2}n^{1-\alpha/2}\big), & \text{when } n\ll t.\end{cases} \tag{1.20}$$

Thus, according to the size of t/n, $\zeta(n)$ is either in $\mathcal{H}_1$ or is a heavy-tail variable (corresponding to $\mathcal{H}_{\alpha/2}$).

Consider the RWRS $\langle l_n,\rho\rangle$, where $\{\rho(z), z\in\mathbb{Z}^d\}$ are centered independent variables in $\mathcal{H}_\alpha$ with $\alpha=1+\varepsilon$ with a small $\varepsilon>0$. This corresponds to a field of charges with lighter tails than $\{\zeta_z(l_n(z)), z\colon l_n(z)>0\}$. Nonetheless, $\{\langle l_n,\rho\rangle\ge n^\beta\}$ with $\beta>\frac{2}{3}$, corresponds to a regime (region II of [5]) where a few sites, say in a region $\mathcal{D}$, are visited of order $\sqrt{\xi_n}$. The phase-diagram of [5] suggests that $\{\langle l_n,\zeta(l_n)-1\rangle\ge n^\beta\}$ behaves similarly. Thus, there is a finite region $\mathcal{D}$, over which the sum of $\zeta_z(\sqrt{\xi_n})$ should be of order $\sqrt{\xi_n}$. According to (1.20), this should make $\{\zeta_z(l_n(z)), z\in\mathcal{D}\}$ of type $\mathcal{H}_1$. It is easy to check, that if the walk were to spend less time on its most visited sites $\mathcal{D}$, say a time of order n^γ with $\gamma < \frac{\beta}{2}$, then the $\{\zeta_z(l_n(z)), z \in \mathcal{D}\}$ would be of type $\mathcal{H}_{\alpha/2}$, and an easy computation using (1.20) shows that

$$Q\Big(\sum_{z\in\mathcal{D}} \zeta_z\big(n^\gamma\big) > n^{\beta-\gamma}\Big) \sim \exp\big(-\kappa_\infty n^{\gamma(1-\alpha/2)} n^{(\beta-\gamma)(\alpha/2)}\big) \ll \exp\big(-n^{\beta/2}\big). \tag{1.21}$$

This explains intuitively (1.5). Note that if $\alpha = 1$ in (1.21), then for any $\gamma \le \beta/2$, we would have $Q(\zeta(n^\gamma) > n^{\beta-\gamma}) \sim \exp(-n^{\beta/2})$.

Assume now that the dominant contribution to the deviation $\{X_n \ge \xi_n\}$ comes from the random set $\mathcal{D}_n^*(A)$, whose *volume is independent of* n. Our next step is to fix a realization of $\mathcal{D}_n^*(A)$, and integrate over the charges of the *monomers piled up in* $\mathcal{D}_n^*(A)$.

Theorem 1.4 is made possible by the following result.

Proposition 1.10. *Assume* Γ *is twice differentiable and satisfies* (1.7). *Then, for any finite subset* $\mathcal{D}$, *any* $\gamma > 0$, *and any positive sequence* $\{\lambda(z), z \in \mathcal{D}\}$, *we have*

$$\inf_{\kappa\ge 0}\Big[\sum_{z\in\mathcal{D}} \lambda(z)\mathcal{I}\big(\kappa(z)\big) \colon \sum_{z\in\mathcal{D}} \lambda^2(z)\kappa^2(z) \ge \gamma^2\Big] = \Big(\max_{\mathcal{D}} \lambda\Big)\mathcal{I}\Big(\frac{\gamma}{\max_{\mathcal{D}} \lambda}\Big). \tag{1.22}$$

Moreover, for any α, β *positive*,

$$\inf_{\lambda>0}\Big[\alpha\lambda + \lambda\mathcal{I}\Big(\frac{\beta}{\lambda}\Big)\Big] = \beta\Gamma^{-1}(\alpha). \tag{1.23}$$

Without the assumption of Proposition 1.10, (1.5) allows us to borrow a strategy developed in [3] to prove a large deviation principle for the self-intersection local times. Indeed, the approach of [3] relies on the fact that a finite number of piles are responsible for producing the excess energy.

The rest of the paper is organized as follows. In Section 2, we recall well-known bounds on sums of independent random variables, prove Lemma 1.9 in Section 2.2, and recall the large deviations for the q-norm of the local times. In Section 3, we prove Proposition 1.8, whose upper bound is divided in three cases: $\eta \in \mathcal{H}_2$, in $\mathcal{H}_\alpha$ for $\alpha \in]1, 2[$, and in $\mathcal{H}_1$. The lower bound in Proposition 1.8 is established in Section 3.4. In Section 4.2, we prove the large deviation principle of Theorem 1.4. We first discuss useful features of the rate functions, and then prove Proposition 1.10 in Section 4.2. The upper bound of the LDP is proved in Section 4.4, and the lower bound follows in Section 4.5. In Section 5, we prove Theorem 1.2. It is based on a subadditive argument, Lemma 5.1, which mimics Lemma 7.1 of [3]. Finally, an Appendix collects proofs which have been postponed because of their analogy with known arguments.

2. Preliminaries

2.1. *Sums of independent variables*

In this section, we collect well-known results scattered in the literature. Since we are not pursuing sharp asymptotics, we give bounds good enough for our purpose, and for the convenience of the reader we have given a proof of the non-referenced results in the Appendix.

We are concerned with the tail distribution of the ζ-variable, given in (1.20), with $\bar{\zeta}(n) = \zeta(n) - 1$. We recall the following result, which we prove in the Appendix to ease to reading.

Lemma 2.1. *There are positive constants* $\beta_0, \{C_\alpha, \kappa_\alpha, 1 \le \alpha \le 2\}$ (*depending on the distribution of* η), *such that the following holds*:

- *For type* $\mathcal{H}_1$, *we have*

$$Q\big(\zeta(n) > t\big) \le C_1 \begin{cases} \exp(-\kappa_1 t), & \text{when } t < \beta_0 n, \\ \exp\big(-\kappa_1\sqrt{\beta_0 t n}\big), & \text{when } t \ge \beta_0 n. \end{cases} \tag{2.1}$$

- *For type $\mathcal{H}_\alpha$, with $1<\alpha<2$, we have*

$$Q\big(\zeta(n)>t\big)\le C_\alpha\begin{cases}\exp(-\kappa_\alpha t), & \text{when } t<\beta_0 n,\\ \exp\big(-\kappa_\alpha t^{\alpha/2}(\beta_0 n)^{1-\alpha/2}\big), & \text{when } t\ge\beta_0 n.\end{cases}\tag{2.2}$$

- *For type $\mathcal{H}_2$, we have for any $t>0$,*

$$Q\big(\zeta(n)>t\big)\le C_2\exp(-\kappa_2 t).\tag{2.3}$$

Nagaev has considered in [16] a sequence $\{\bar{Y}_n, n\in\mathbb{N}\}$ of independent centered i.i.d. satisfying $\mathcal{H}_\alpha$ with $0<\alpha<1$, and has obtained the following upper bound (see also inequality (2.32) of Nagaev [17]).

Proposition 2.2 (of Nagaev). *Assume $E[\bar{Y}_i]=0$ and $E[(\bar{Y}_i)^2]\le 1$. There is a constant C_Y, such that for any integer n and any positive t,*

$$P(\bar{Y}_1+\cdots+\bar{Y}_n\ge t)\le C_Y\left(nP\left(\bar{Y}_1>\frac{t}{2}\right)+\exp\left(-\frac{t^2}{20n}\right)\right).\tag{2.4}$$

Remark 2.3. *Note that if $\eta\in\mathcal{H}_\alpha$ for $1<\alpha\le 2$, then $\eta^2\in\mathcal{H}_{\alpha/2}$. Thus, for $\bar{Y}_i=\eta(i)^2-1$, Proposition 2.2 yields*

$$P\left(\sum_{i=1}^n\big(\eta(i)^2-1\big)\ge\xi_n\right)\le C_Y\left(n\exp\big(-c_\alpha(\xi_n)^{\alpha/2}\big)+\exp\left(-\frac{\xi_n^2}{20n}\right)\right).\tag{2.5}$$

When we take $t\sim n^\beta$ in (2.4), we have the following asymptotical result due to Nagaev [16] (see also Theorem 2.1 and (2.22) of [17]).

Lemma 2.4. *Assume $\{\bar{Y}_n, n\in\mathbb{N}\}$ are centered independent variables in $\mathcal{H}_\alpha$ with $0<\alpha<1$. If $\xi_n\gg n^{1/(2-\alpha)}$, then for n large enough, we have*

$$P(\bar{Y}_1+\cdots+\bar{Y}_n\ge\xi_n)\le 2n\max_{k\le n}P(\bar{Y}_k>\xi_n).\tag{2.6}$$

Remark 2.5. *With the notations of Remark 2.3, Lemma 2.4 yields for $\xi_n\gg n^{1/(2-\alpha/2a)}$,*

$$P\left(\sum_{i=1}^n\big(\eta(i)^2-1\big)\ge\xi_n\right)\le 2nP\big(\eta^2-1\ge\xi_n\big)\le 2C_\alpha n\exp\big(-c_\alpha(\xi_n)^{\alpha/2}\big).\tag{2.7}$$

Note that $\alpha>1$ implies that $\frac{1}{2-\alpha/2}>\frac{2}{3}$.

For completeness, note that when Y_i-variables are in $\mathcal{H}_1$, we can use a special form of Lemma 5.1 of [4] to obtain an analogue of (2.4).

2.2. *On a concentration inequality: Lemma 1.9*

We assume that for some $\lambda_0>0$, we have $E_Q[\exp(\lambda_0\eta)]<\infty$.

Note that when $\lambda<\lambda_0/2$, there is a positive constant C such that $E_Q[\exp(\lambda\eta)]\le 1+C\lambda^2$. Now, note that

$$q_n(\Lambda)=\sum_{z\in\Lambda}\sum_{i=1}^{l_n(z)}\eta_z(i)\overset{\text{law}}{=}\sum_{i=1}^{l_n(\Lambda)}\eta_0(i).\tag{2.8}$$

We use Chebychev's inequality, for $\lambda>0$, and integrate only over the η-variables

$$Q\big(q_n(\Lambda)>t\big)\le e^{-\lambda t}\big(E_Q\big[e^{\lambda\eta}\big]\big)^{l_n(\Lambda)}\le e^{-\lambda t}\exp\big(C\lambda^2 l_n(\Lambda)\big).\tag{2.9}$$

Now, using (1.14), if we choose

$$C\lambda^2 \le \frac{\tilde{\kappa}_d}{2|\Lambda|^{2/d}}, \quad \text{then we have } \mathbb{E}_0\bigl[\exp\bigl(C\lambda^2 l_n(\Lambda)\bigr)\bigr] \le 2. \tag{2.10}$$

Thus, (1.13) follows at once.

3. Proof of Proposition 1.8

We have divided the proof of the upper bounds in Proposition 1.8 into the three cases $\mathcal{H}_2$, $\mathcal{H}_\alpha$ and $\mathcal{H}_1$. The lower bound in (1.10) is obtained in Section 3.4.

3.1. *The case* $\mathcal{H}_2$

We consider first annealed upper bounds for $\{X_n \ge \xi_n\}$. We choose a sequence $\{\xi_n\}$ such that $\xi_n \gg n^{2/3}$, and $\varepsilon > 0$ such that $(\sqrt{\xi}_n)^{1-(4/3)\varepsilon} \gg n^{1/3}$.

When averaging first with respect to the charges, and then with respect to the random walk, we write

$$P(X_n \ge \xi_n) = \mathbb{E}_0\biggl[Q\biggl(\sum_{z\in\mathbb{Z}^d} l_n(z)\bar{\zeta}_z\bigl(l_n(z)\bigr) \ge \xi_n\biggr)\biggr]. \tag{3.1}$$

Thus, we think of X_n as a weighted sum of independent centered variables $\bar{\zeta}_z(l_n(z))$. Thanks to the uniform bound (2.3), the dependence of ζ_z on the local time $l_n(z)$ vanishes. Indeed, since the $\{\zeta_z\}$ are independent and satisfy Cramer's condition, we can use Lemma 5.1 of [4] and obtain that for some $c_u > 0$, any $0 < \delta < 1$, any finite subset $\Lambda \subset \mathbb{Z}^d$, any $x > 0$, and any integer-sequence $\{k(z), z \in \mathbb{N}\}$,

$$Q\biggl(\sum_{z\in\Lambda} \bar{\zeta}_z\bigl(k(z)\bigr) \ge x\biggr) \le \exp\biggl(c_u|\Lambda|\delta^{2(1-\delta)} \max_n\bigl(E_Q\bigl[\zeta(n)^2\bigr]\bigr) - \frac{\lambda_1\delta}{2}x\biggr). \tag{3.2}$$

In order to use (3.2), we first need to uncouple the product $l_n(z) \times \bar{\zeta}_z$ in (3.1).

We fix a constant $A \ge 1$, $b_0 = 1/A$ and $b_{i+1} = 2b_i$. We define the following subdivision of $[1, \sqrt{\xi_n}/A]$. We define i_0 and N as follows, $b_{i_0} \le 1 < b_{i_0+1}$,

$$b_N \le \frac{\sqrt{\xi_n}}{A} < b_{N+1}, \quad \text{we set for } i = i_0, \dots, N,\ \mathcal{D}_i = \bigl\{z\colon\ b_i \le l_n(z) < b_{i+1}\bigr\}. \tag{3.3}$$

Furthermore, for δ_0 to be chosen small later (independent of A), we define

$$\delta_i = \delta_0\biggl(\frac{Ab_i}{\sqrt{\xi_n}}\biggr)^{(1-\varepsilon)} \quad \text{and} \quad p_i = p\biggl(\frac{Ab_i}{\sqrt{\xi_n}}\biggr)^{\varepsilon}, \quad \text{with } p \text{ such that } \sum_{i\ge 0} p_i = 1. \tag{3.4}$$

It is important to note that p is independent on A.

Now, we perform the decomposition of X_n in terms of level sets $\mathcal{D}_i$. Note that for any $A \ge 1$,

$$\begin{aligned}\mathbb{E}_0\biggl[Q\biggl(\sum_{z:l_n(z)\le b_N} l_n(z)\bar{\zeta}_z\bigl(l_n(z)\bigr) \ge \xi_n\biggr)\biggr] &\le \sum_{i=i_0}^{N-1} \mathbb{E}_0\biggl[Q\biggl(\sum_{z\in\mathcal{D}_i} l_n(z)\bar{\zeta}_z\bigl(l_n(z)\bigr) \ge p_i\xi_n\biggr)\biggr] \\ &\le \sum_{i=i_0}^{N-1} Q\biggl(\sum_{z\in\mathcal{D}_i} \frac{l_n(z)}{b_{i+1}}\bar{\zeta}_z\bigl(l_n(z)\bigr) \ge \frac{p_i\xi_n}{b_{i+1}}\biggr).\end{aligned} \tag{3.5}$$

Fix now a realization of the random walk, and let $z \in \mathcal{D}_i$. We have (recall that $\zeta_z \geq 0$)

$$Q\left(\frac{l_n(z)}{b_{i+1}}\zeta_z\big(l_n(z)\big) \geq t\right) \leq Q\big(\zeta_z\big(l_n(z)\big) \geq t\big) \leq \mathrm{e}^{-\lambda_1 t}. \tag{3.6}$$

We use now Lemma 5.1 of [4], that we have recalled in (3.2) with the choice of δ_i given in (3.4) (and whose dependence on n is omitted)

$$Q\left(\sum_{z\in\mathcal{D}_i}\frac{l_n(z)}{b_{i+1}}\bar{\zeta}_z\big(l_n(z)\big) \geq \frac{p_i\xi_n}{b_{i+1}}\right) \leq \exp\left(c_0|\mathcal{D}_i|\delta_i^{2(1-\delta_i)} - \frac{\delta_i p_i \xi_n}{4b_{i+1}}\right), \quad \text{with } c_0 = c_u \sup_k E_Q\big[\zeta^2(k)\big]. \tag{3.7}$$

If we denote $\kappa_0 = \delta_0^{-2\delta_0}$, then note that $\delta_i^{-2\delta_i(n)} \leq \kappa_0$. Then, the bound (3.7) is useful if the first term on the right-hand side is negligible, that is if

$$8\kappa_0 c_0|\mathcal{D}_i|\delta_i^2 \leq \delta_i p_i \frac{\xi_n}{b_{i+1}}. \tag{3.8}$$

Note that

$$\delta_i p_i \frac{\xi_n}{b_{i+1}} \geq p\frac{Ab_i}{\sqrt{\xi_n}}\frac{\xi_n}{b_{i+1}} = \frac{p}{2}A\sqrt{\xi_n}. \tag{3.9}$$

Assuming (3.8), the result follows right away by (3.9). The remaining point is to show that (3.8) holds. Note that (3.8) holds as long as $|\mathcal{D}_i|$ is not *large*. On the other hand, we express $\{|\mathcal{D}_i|\ \textit{large}\}$ as a large deviation event for the self-intersection local time. Thus, (3.8) holds when

$$8\kappa_0 c_0|\mathcal{D}_i| \leq \frac{1}{2\delta_i^2}pA\sqrt{\xi_n} = \left(\frac{\sqrt{\xi_n}}{Ab_i}\right)^{2-2\varepsilon}\frac{p}{2}A\sqrt{\xi_n} \leq \frac{(\sqrt{\xi_n})^{3-2\varepsilon}}{b_i^{2-2\varepsilon}}\frac{p}{2A^{1-2\varepsilon}}. \tag{3.10}$$

When (3.10) does not hold, we note for some constant c_1 (independent of A) and any $q > 1$,

$$\mathcal{E}_i = \left\{|\mathcal{D}_i| > \frac{c_1}{A^{1-2\varepsilon}}\frac{(\sqrt{\xi_n})^{3-2\varepsilon}}{b_i^{2-2\varepsilon}}\right\} \subset \left\{\|l_n\|_q^q \geq \frac{c_1}{A^{1-2\varepsilon}}\big(\sqrt{\xi_n}\big)^{3-2\varepsilon} b_i^{q-2+2\varepsilon}\right\}. \tag{3.11}$$

Note that the event on the right hand-side of (3.11) is a large deviation event since from our hypotheses on ξ_n and ε, we have, when $q \geq 2$ and in the worse case where $b_i = 1$, that

$$\mathbb{E}_0\big[\|l_n\|_q^q\big] \sim \kappa(q,d)n \ll \frac{c_1}{A^{1-2\varepsilon}}\big(\sqrt{\xi_n}\big)^{3-2\varepsilon}.$$

Case $d = 3$. We consider (3.11) with $d = 3$ and $q = 2 < q_c(3) = 3$, and use Theorem 1.1 and Remark 1.3 of [2]. This yields

$$\begin{aligned}\mathbb{P}(\mathcal{E}_i) &\leq \mathbb{P}\left(\|l_n\|_2^2 - E\big[\|l_n\|_2^2\big] \geq \frac{c_1}{2A^{1-2\varepsilon}}\big(\sqrt{\xi_n}\big)^{3-2\varepsilon} b_i^{2\varepsilon}\right) \\ &\leq C\exp\left(-c(2,3)\left(\frac{c_1}{2A^{1-2\varepsilon}}\big(\sqrt{\xi_n}\big)^{3-2\varepsilon} b_i^{2\varepsilon}\right)^{2/3} n^{-1/3}\right). \end{aligned} \tag{3.12}$$

Thus, the quantity $\sum_i \mathbb{P}(\mathcal{E}_i)$ is negligible if

$$\sqrt{\xi_n}n^{1/3} \ll \left(\frac{c_1}{2A^{1-2\varepsilon}}\big(\sqrt{\xi_n}\big)^{3-2\varepsilon}\right)^{2/3}. \tag{3.13}$$

Condition (3.13) is equivalent to $n^{2/3} \ll \xi_n$.

Case of $d \geq 4$. First, in dimension 5 or more, we have from Theorem 1.2 of [2], with $q = 2 > q_c(d)$,

$$\mathbb{P}(\mathcal{E}_i) \leq \exp\left(-c(2,d)\left(\frac{c_1}{2A^{1-2\varepsilon}}\left(\sqrt{\xi_n}\right)^{3-2\varepsilon} b_i^{2\varepsilon}\right)^{1/2}\right). \tag{3.14}$$

This term is negligible.

When $d = 4$, $q_c(4) = 2$, and we choose $q > 2$,

$$\mathbb{P}(\mathcal{E}_i) \leq \exp\left(-c(q,d)\left(\frac{c_1}{2A^{1-2\varepsilon}}\left(\sqrt{\xi_n}\right)^{3-2\varepsilon} b_i^{q-2+2\varepsilon}\right)^{1/q}\right). \tag{3.15}$$

This term is negligible if

$$\sqrt{\xi_n} \ll \left(\sqrt{\xi_n}\right)^{3/q} \quad \Longleftarrow \quad q < 3.$$

Thus, if we choose $2 < q < 3$, $P(\mathcal{E}_i) \ll \exp(-\sqrt{\xi_n})$.

In conclusion, we obtain that

$$\begin{aligned}
P\left(\sum_{l_n(z) < \sqrt{\xi_n}/A} X_n(z) \geq \xi_n\right) &\leq \sum_{i=i_0}^{N-1} \mathbb{E}_0 Q\left(\sum_{z \in \mathcal{D}_i} l_n(z)\bar{\zeta}_z\big(l_n(z)\big) \geq p_i \xi_n\right) \\
&\leq \sum_{i=i_0}^{N-1} \mathbb{E}_0 Q\left(\sum_{z \in \mathcal{D}_i} \bar{\zeta}_z\big(l_n(z)\big) \geq p_i \frac{\xi_n}{b_{i+1}}, |\mathcal{D}_i| < \frac{c_1}{A^{1-2\varepsilon}} \frac{(\sqrt{\xi_n})^{3-2\varepsilon}}{b_i^{2-2\varepsilon}}\right) \\
&\quad + \sum_{i=i_0}^{N-1} \mathbb{P}\left(|\mathcal{D}_i| \geq \frac{c_1}{A^{1-2\varepsilon}} \frac{(\sqrt{\xi_n})^{3-2\varepsilon}}{b_i^{2-2\varepsilon}}\right) \\
&\leq C \log(n) \times \exp\left(-\frac{p}{8} A \sqrt{\xi_n}\right) + \exp\left(-\xi_n^{1/2+\varepsilon}\right).
\end{aligned} \tag{3.16}$$

Thus, taking $A = 1$ in (3.16), we cover the levels $\{z\colon l_n(z) < \sqrt{\xi_n}\}$, whereas taking A larger than 1, we cover the levels $\{z\colon l_n(z) < \sqrt{\xi_n}/A\}$. Thus, combining these two regimes, we obtain the upper bound (1.10), whereas taking A to infinity, we obtain the asymptotic (1.5).

3.2. *The case* $\mathcal{H}_\alpha$ *with* $1 < \alpha < 2$

Charges in $\mathcal{H}_\alpha$ have a much fatter tails than in $\mathcal{H}_2$. Thus, we decompose ζ into its small and large values. For $z \in \mathbb{Z}^d$, define for all positive integer k,

$$\zeta_z'(k) = \zeta_z(k)\mathbb{1}\big\{\zeta_z(k) \leq \beta_0 k\big\} \quad \text{and} \quad \zeta_z''(k) = \zeta_z(k)\mathbb{1}\big\{\zeta_z(k) > \beta_0 k\big\}. \tag{3.17}$$

We add a bar on top of ζ', ζ'' to denote the centered variables, and we define

$$\bar{X}_n' = \sum_{z \in \mathbb{Z}^d} l_n(z)\bar{\zeta}_z'\big(l_n(z)\big) \quad \text{and} \quad \bar{X}_n'' = \sum_{z \in \mathbb{Z}^d} l_n(z)\bar{\zeta}_z''\big(l_n(z)\big).$$

Note that

$$\{\bar{X}_n \geq \xi_n\} \subset \left\{\bar{X}_n' \geq \frac{\xi_n}{2}\right\} \cup \left\{\bar{X}_n'' \geq \frac{\xi_n}{2}\right\}. \tag{3.18}$$

The $\{\zeta_z', z \in \mathbb{Z}^d\}$ look like coming from η in $\mathcal{H}_2$. Indeed, for any $t > 0$

$$\big\{\zeta_z'(k) > t\big\} = \big\{t < \zeta_z(k) \leq \beta_0 k\big\} \quad \Longrightarrow \quad Q\big(\zeta_z'(k) > t\big) \leq C_\alpha \exp(-\kappa_\alpha t). \tag{3.19}$$

Thus, the term $\{\bar{X}'_n \geq n^\beta \frac{\xi}{2}\}$ follows the same treatment as that of Section 3.1, with the upper bound (1.10), and the asymptotic (1.5).

We focus on the large values of ζ_z. Note that

$$\{\zeta''_z(k) > t\} = \{\zeta_z(k) \geq \max(t, \beta_0 k)\} \quad \Longrightarrow \quad \forall t > 0 \quad Q(\zeta''_z(k) \geq t) \leq C_\alpha \mathrm{e}^{-\kappa_\alpha t^{\alpha/2}(\beta_0 k)^{1-\alpha/2}}. \tag{3.20}$$

For convenience, set $\tilde{\alpha} = \frac{2}{\alpha} - 1$, with $0 < \tilde{\alpha} < 1$, and note that (3.20) implies that for $u > 0$

$$Q(k^{\tilde{\alpha}} \zeta''_z(k) \geq u) \leq C_\alpha \exp(-\kappa_\alpha \beta_0^{1-\alpha/2} u^{\alpha/2}). \tag{3.21}$$

We can therefore think of

$$Y_z := (l_n(z))^{\tilde{\alpha}} \zeta''_z(l_n(z)) \qquad (\bar{Y}_z := (l_n(z))^{\tilde{\alpha}} \bar{\zeta}''_z(l_n(z))),$$

as having a heavy-tail (of type $\mathcal{H}_{\alpha/2}$). Using the level decomposition of Section 3.1, we first fix a realization of the random walk and estimate

$$\mathcal{A}_i := Q\left(\sum_{z \in \mathcal{D}_i} \left(\frac{l_n(z)}{b_{i+1}}\right)^{1-\tilde{\alpha}} \bar{Y}_z \geq p_i \frac{\xi_n}{b_{i+1}^{1-\tilde{\alpha}}}\right). \tag{3.22}$$

Note that from (3.21), we have some constant C such that for $z \in \mathcal{D}_i$

$$Q\left(\left(\frac{l_n(z)}{b_{i+1}}\right)^{1-\tilde{\alpha}} Y_z \geq u\right) \leq Q(Y_z \geq u) \leq C_\alpha \exp(-C u^{\alpha/2}). \tag{3.23}$$

This implies that for some $\sigma_Y > 0$, we have $E[Y_z^2] \leq \sigma_Y^2$, and we can use Proposition 2.2,

$$\mathcal{A}_i \leq C_Y\left(|\mathcal{D}_i| Q\left(Y_1 \geq \frac{p_i \xi_n}{2(b_{i+1})^{1-\tilde{\alpha}}}\right) + \exp\left(-\frac{p_i^2}{20\sigma_Y^2 |\mathcal{D}_i|}\left(\frac{\xi_n}{b_{i+1}^{1-\tilde{\alpha}}}\right)^2\right)\right). \tag{3.24}$$

We show now that the first term of the right-hand side of (3.24) is the dominant term. Note that

$$Q\left(Y_1 \geq \frac{p_i \xi_n}{2(b_{i+1})^{1-\tilde{\alpha}}}\right) \leq C_\alpha \exp\left(-C\left(\frac{p_i \xi_n}{2(b_{i+1})^{1-\tilde{\alpha}}}\right)^{\alpha/2}\right). \tag{3.25}$$

Now, we estimate the exponent in the right-hand side of (3.25),

$$\frac{p_i \xi_n}{2(b_{i+1})^{1-\tilde{\alpha}}} = p\left(\frac{A b_{i+1}}{2\sqrt{\xi_n}}\right)^\varepsilon \frac{\xi_n}{2(b_{i+1})^{1-\tilde{\alpha}}}. \tag{3.26}$$

When $\tilde{\alpha} < 1$, we choose ε small enough so that $1 - \tilde{\alpha} > \varepsilon$, and then the least value of the last term in (3.26) is for i such that $b_{i+1} = \sqrt{\xi_n}/A$. Thus

$$\frac{p_i \xi_n}{2(b_{i+1})^{1-\tilde{\alpha}}} \geq \frac{p}{2^\varepsilon} \frac{2A^{1-\tilde{\alpha}}}{(\sqrt{\xi_n})^{1-\tilde{\alpha}}} \xi_n. \tag{3.27}$$

When taking a power $\alpha/2$ in (3.27), the power of ξ_n in the right-hand side of (3.27) satisfies

$$\frac{\alpha}{2}\left(1 - \frac{1-\tilde{\alpha}}{2}\right) = \frac{1}{2} \quad \Longrightarrow \quad \left(\frac{p_i \xi_n}{2(b_{i+1})^{1-\tilde{\alpha}}}\right)^{\alpha/2} \geq \left(\frac{p}{2^{1+\varepsilon}}\right)^{\alpha/2} A^{(\alpha/2)(1-\tilde{\alpha})} \sqrt{\xi_n}. \tag{3.28}$$

To deal with $|\mathcal{D}_i|$ in the second term on the right-hand side of (3.24) (the Gaussian bound), we can assume as in Section 3.1 that we restrict ourselves to $\mathcal{E}_i$ defined in (3.11). Thus, on $\mathcal{E}_i$

$$\frac{p_i^2}{|\mathcal{D}_i|}\left(\frac{\xi_n}{(b_{i+1})^{1-\tilde{\alpha}}}\right)^2 \geq \frac{p_i^2}{c_1} A^{1-2\varepsilon} \xi_n^{1/2+\varepsilon} b_{i+1}^{2(\tilde{\alpha}-\varepsilon)} = \frac{p^2 A}{c} b_{i+1}^{2\tilde{\alpha}} \sqrt{\xi_n} \geq \frac{p^2 A}{c} \sqrt{\xi_n}. \tag{3.29}$$

Equation (3.29) holds as soon as n is large enough. Thus, the Gaussian bound is negligible. The proof is concluded as in Section 3.1.

3.3. *The case* $\mathcal{H}_1$

In this case, no information on the dominant level set can be obtained. We perform the same decomposition as in Section 3.2. The term X'_n follows the case $\mathcal{H}_2$, and we focus on X''_n. Note that $\tilde{\alpha} = 1$, so that the analysis of Section 3.2 is not adequate (note that (3.26) would yield an upper bound of the form $\exp(-\xi_n^{1-\varepsilon})$). In our case,

$$Y_z = l_n(z)\zeta''_z\big(l_n(z)\big) \quad \text{and} \quad \bar{Y}_z = Y_z - E[Y_z]. \tag{3.30}$$

Note that (2.1) implies that on $\{l_n(z) > 0\}$ and for some constant c_1 we have $t > 0$,

$$\begin{aligned} Q(Y_z > t) &= Q\left(\zeta_z\big(l_n(z)\big) > \max\left(\beta_0 l_n(z), \frac{t}{l_n(z)}\right)\right) \\ &\le C_1 \exp\left(-\kappa_1\sqrt{\beta_0 l_n(z)\max\left(\beta_0 l_n(z), \frac{t}{l_n(z)}\right)}\right) \\ &\le C_1 \exp\big(-\kappa_1 \max\big(\beta_0 l_n(z), \sqrt{\beta_0 t}\big)\big) \le C_1 \mathrm{e}^{-c_1\sqrt{t}}. \end{aligned} \tag{3.31}$$

When fixing a realization of the random walk, we think of Y_z as in $\mathcal{H}_{1/2}$, and use Lemma 2.4 with $\xi_n \gg n^{2/3}$, and n large to obtain

$$\begin{aligned} P\big(X''_n \ge \xi_n\big) &= \mathbb{E}_0\left[Q\left(\sum_{z:l_n(z)>0} \bar{Y}_z \ge \xi_n\right)\right] \le 2\mathbb{E}_0\Big[n \max_{z:l_n(z)>0} Q(\bar{Y}_z \ge \xi_n)\Big] \\ &\le 2nC_1 \exp\big(-c_1\sqrt{\xi_n}\big). \end{aligned} \tag{3.32}$$

3.4. *Lower bound in* (1.10)

A scenario compatible with the cost in (1.10) is as follows. The walk is pinned at the origin a time t_n of order $\sqrt{\xi_n}$, building up an energy $t_n\bar{\zeta}(t_n)$ required to be of order ξ_n. The remaining time, the walk roams freely, and the total energy should be made of $t_n\bar{\zeta}(t_n)$, and a part close to zero due to the centering. Note that the relevant order for $\zeta(t_n)$ is $t_n \sim \sqrt{\xi_n}$, thus, we are in the *central limit regime*. This is why we can treat at once the three cases we have considered.

We first show a more general lemma.

Lemma 3.1. *Let $\{\mathcal{D}_n, n \in \mathbb{N}\}$ be a sequence of random subsets, with $\mathcal{D}_n \in]-n, n[^d$ and $\mathcal{D}_n$ measurable with respect to $\sigma(S(k), k < n)$. Let $\{m_n, M_n, n \in \mathbb{N}\}$ be positive sequences with $m_n \le n$. Assume either* (i) *$\xi_n = \xi n^2$ and $Q(\eta > \sqrt{\xi}) > 0$, or* (ii) *$n^{2/3} \ll \xi_n \ll n^2$. Then, for any $\varepsilon > 0$,*

$$\begin{aligned} P\big(\|q_n\|_2^2 - n \ge \xi_n\big) \ge{}& 2^{-2M_n-1} P\big(\|\mathbb{1}_{\mathcal{D}_n} q_{m_n}\|_2^2 \ge (1+\varepsilon)\xi_n, |\mathcal{D}_n| \le M_n\big) \\ &- \mathbb{P}_0\left(l_n(\mathcal{D}_n) \ge \frac{\varepsilon}{2}\xi_n, |\mathcal{D}_n| \le M_n\right) - \mathbb{P}_0\left(\|l_n\|_2 \ge \frac{\xi_n}{n^\varepsilon}\right). \end{aligned} \tag{3.33}$$

Based on Lemma 3.1, we distinguish two cases: (i) $\xi_n = \xi n^2$ with $\xi < 1$, and (ii) $\xi_n \ll n^2$.

First, we state a corollary of Theorems 1.1 and 1.2 (and Remark 1.3) of [2] whose immediate proof is omitted.

Corollary 3.2. *Assume $d \ge 3$ and $\xi_n \gg n^{2/3}$. For $\varepsilon > 0$ small enough, and n large enough*

$$\mathbb{P}_0\left(\|l_n\|_2 \ge \frac{\xi_n}{n^\varepsilon}\right) \le \exp\big(-\sqrt{\xi_n} n^\varepsilon\big). \tag{3.34}$$

In case (i), we choose ε small enough so that $Q(\eta > \sqrt{(1+\varepsilon)\xi}) > 0$. Our scenario is obtained as we choose $\mathcal{D}_n = \{0\}$, $M_n = 1$ and $m_n = n$ in Lemma 3.1. Inequality (3.33) reads

$$P\big(\|q_n\|_2^2 - n \ge \xi n^2\big) \ge \frac{1}{4} Q\left(\frac{1}{n}\sum_{i=1}^{n} \eta_0(i) \ge 1\right) - \mathbb{P}_0\left(l_n(0) \ge \frac{\varepsilon}{2}\xi n^2\right) - \mathbb{P}_0\big(\|l_n\|_2 \ge n^{2-\varepsilon}\big). \tag{3.35}$$

Now, using $\{\eta_0(i) \ge \sqrt{(1+\varepsilon)\xi}, \forall i \le n\} \subset \{\eta_0(1) + \cdots + \eta_0(n) \ge \sqrt{(1+\varepsilon)\xi} n\}$, as well as (1.14), and Corollary 3.2, we have

$$P\big(\|q_n\|_2^2 - n \ge \xi_n\big) \ge \frac{1}{8} Q\big(\eta > \sqrt{(1+\varepsilon)\xi}\big)^n. \tag{3.36}$$

In case (ii), we take any $\xi > 0$, and $m_n^2 = (1+\varepsilon)\xi_n$. Then, (3.33) reads

$$\begin{aligned} P\big(\|q_n\|_2^2 - n \ge \xi_n\big) &\ge \frac{1}{4} Q\left(\sum_{i=1}^{m_n} \eta_0(i) \ge m_n\right) - \mathbb{P}_0\left(l_n(0) \ge \frac{\varepsilon}{2}\xi_n\right) - \mathbb{P}_0\left(\|l_n\|_2 \ge \frac{\xi_n}{n^\varepsilon}\right) \\ &\ge \frac{1}{4} Q(\eta \ge 1)^{m_n} - \mathbb{P}_0\left(l_n(0) \ge \frac{\varepsilon}{2}\xi_n\right) - \mathbb{P}_0\left(\|l_n\|_2 \ge \frac{\xi_n}{n^\varepsilon}\right). \end{aligned} \tag{3.37}$$

To deal with the two last terms in (3.37), we use (1.14) and Corollary 3.2.

Proof of Lemma 3.1. To establish (3.33), we need (i) to control the process outside $\mathcal{D}_n$, and (ii) to control the process in the time period $[m_n, n[$. We start with (i), and introduce notations $\mathcal{S}_n = \{|\mathcal{D}_n| \le M_n\}$ and $\mathcal{B}_n = \{\|l_n\|_2 \le \xi_n n^{-\varepsilon}\}$. First,

$$\big\{\|q_n\|_2^2 - n \ge \xi_n\big\} \supset \left\{\|\mathbb{1}_{\mathcal{D}_n} q_n\|_2^2 - l_n(\mathcal{D}_n) \ge \left(1 + \frac{\varepsilon}{2}\right)\xi_n\right\} \cup \left\{\|\mathbb{1}_{\mathcal{D}_n^c} q_n\|_2^2 - l_n\big(\mathcal{D}_n^c\big) \ge -\frac{\varepsilon}{2}\xi_n\right\}. \tag{3.38}$$

Note that if we show that

$$\mathbb{1}_{\mathcal{B}_n} Q\left(\|\mathbb{1}_{\mathcal{D}_n^c} q_n\|_2^2 - l_n\big(\mathcal{D}_n^c\big) \le -\frac{\varepsilon}{2}\xi_n\right) \le \frac{1}{2}\mathbb{1}_{\mathcal{B}_n}, \tag{3.39}$$

then, using the independence of the charges on different regions, we would have that

$$2\mathbb{1}_{\mathcal{B}_n} Q\big(\|q_n\|_2^2 - n \ge \xi_n, \mathcal{S}_n\big) \ge \mathbb{1}_{\mathcal{B}_n} Q\left(\|\mathbb{1}_{\mathcal{D}_n} q_n\|_2^2 - l_n(\mathcal{D}_n) \ge \left(1 + \frac{\varepsilon}{2}\right)\xi_n, \mathcal{S}_n\right). \tag{3.40}$$

Then, upon integrating (3.40) over the random walk, we would reach

$$\begin{aligned} P\big(\|q_n\|_2^2 - n \ge \xi_n\big) &\ge P\big(\mathcal{B}_n \cap \mathcal{S}_n, \|q_n\|_2^2 - n \ge \xi_n\big) \\ &\ge \frac{1}{2} P\left(\mathcal{B}_n \cap \mathcal{S}_n, \|\mathbb{1}_{\mathcal{D}_n} q_n\|_2^2 - l_n(\mathcal{D}_n) \ge \left(1 + \frac{\varepsilon}{2}\right)\xi_n\right) \\ &\ge \frac{1}{2} P\left(\|\mathbb{1}_{\mathcal{D}_n} q_n\|_2^2 - l_n(\mathcal{D}_n) \ge \left(1 + \frac{\varepsilon}{2}\right)\xi_n, \mathcal{S}_n\right) - \frac{1}{2} P\big(\mathcal{B}_n^c\big). \end{aligned} \tag{3.41}$$

We now show (3.39). We expand q_n^2,

$$q_n^2(z) - l_n(z) = \left(\sum_{i \le l_n(z)} \eta_z(i)\right)^2 - l_n(z) = \sum_{i \le l_n(z)} \big(\eta_z^2(i) - 1\big) + 2 \sum_{1 \le i < j \le l_n(z)} \eta_z(i)\eta_z(j). \tag{3.42}$$

It is immediate to obtain, for $\chi_1 = E[\eta^4]+1$

$$E_Q\big[\big(q_n^2(z)-l_n(z)\big)^2\big] = l_n(z)\big(E_Q\big[\eta^4\big]-1\big) + 2\big(l_n^2(z)-l_n(z)\big) \le \chi_1 l_n^2(z). \tag{3.43}$$

By Markov's inequality

$$\begin{aligned}\mathbb{1}_{\mathcal{B}_n} Q\big(\|\mathbb{1}_{\mathcal{D}_n^c} q_n\|_2^2 - l_n\big(\mathcal{D}_n^c\big) \le -\varepsilon\xi_n\big) &\le \mathbb{1}_{\mathcal{B}_n} \frac{\sum_{z\notin\mathcal{D}_n} \operatorname{var}(q_n^2(z)-l_n(z))}{(\varepsilon\xi_n)^2} \\ &\le \mathbb{1}_{\mathcal{B}_n} \frac{\chi_1 \|\mathbb{1}_{\mathcal{D}_n^c} l_n\|_2^2}{(\varepsilon\xi_n)^2} \le \mathbb{1}_{\mathcal{B}_n} \frac{\chi_1}{\varepsilon^2} n^{-\varepsilon}.\end{aligned} \tag{3.44}$$

Thus, for any $\varepsilon > 0$, (3.39) holds for n large enough.

We now deal with (ii), and show that

$$\begin{aligned}&P\left(\|\mathbb{1}_{\mathcal{D}_n} q_n\|_2^2 - l_n(\mathcal{D}_n) \ge \left(1+\frac{\varepsilon}{2}\right)\xi_n, \mathcal{S}_n\right) + \mathbb{P}_0\left(l_n(\mathcal{D}_n) \ge \frac{\varepsilon}{2}\xi_n, \mathcal{S}_n\right) \\ &\quad \ge \left(\frac{1}{2}\right)^{2M_n} P\big(\|\mathbb{1}_{\mathcal{D}_n} q_{m_n}\|_2^2 \ge (1+\varepsilon)\xi_n, \mathcal{S}_n\big).\end{aligned} \tag{3.45}$$

We impose that the *local charge* on each $z \in \mathcal{D}_n$ during both time periods $[0, m_n[$ and $[m_n, n[$ be of a same sign. Indeed, this would have the effect that

$$\forall z \in \mathcal{D}_n \quad q_{[0,m_n[}^2(z) + q_{[m_n,n[}^2(z) \le \big(q_{[0,m_n[}(z) + q_{[m_n,n[}(z)\big)^2. \tag{3.46}$$

Thus, if we set

$$\tilde{\mathcal{S}}_n = \big\{\|\mathbb{1}_{\mathcal{D}_n} q_{m_n}\|_2^2 \ge (1+\varepsilon)\xi_n\big\} \cap \big\{\forall z \in \mathcal{D}_n, q_{[0,m_n[}(z) \ge 0\big\} \tag{3.47}$$

and

$$\mathcal{S}_n' = \big\{\forall z \in \mathcal{D}_n, q_{[m_n,n[}(z) \ge 0\big\}, \quad \text{then } \tilde{\mathcal{S}}_n \cap \mathcal{S}_n' \subset \big\{\|\mathbb{1}_{\mathcal{D}_n} q_n\|_2^2 \ge (1+\varepsilon)\xi_n\big\}. \tag{3.48}$$

Note also that when integrating over the charges, $\tilde{\mathcal{S}}_n$ and $\mathcal{S}_n'$ are independent, and by symmetry of the charges' distribution

$$Q(\tilde{\mathcal{S}}_n) \ge \left(\frac{1}{2}\right)^{|\mathcal{D}_n|} Q(\mathcal{S}_n) \quad \text{and} \quad Q\big(\mathcal{S}_n'\big) \ge \left(\frac{1}{2}\right)^{|\mathcal{D}_n|}. \tag{3.49}$$

Now, (3.48) and (3.49) imply that

$$P\big(\|\mathbb{1}_{\mathcal{D}_n} q_n\|_2^2 \ge (1+\varepsilon)\xi_n, \mathcal{S}_n\big) \ge \left(\frac{1}{2}\right)^{2M_n} P\big(\|\mathbb{1}_{\mathcal{D}_n} q_{m_n}\|_2^2 \ge (1+\varepsilon)\xi_n, \mathcal{S}_n\big). \tag{3.50}$$

Now we center $\|\mathbb{1}_{\mathcal{D}_n} q_n\|_2^2$. Note that

$$\big\{\|\mathbb{1}_{\mathcal{D}_n} q_n\|_2^2 \ge (1+\varepsilon)\xi_n\big\} \subset \left\{\|\mathbb{1}_{\mathcal{D}_n} q_n\|_2^2 - l_n(\mathcal{D}_n) \ge \left(1+\frac{\varepsilon}{2}\right)\xi_n\right\} \cup \left\{l_n(\mathcal{D}_n) \ge \frac{\varepsilon}{2}\xi_n\right\}. \tag{3.51}$$

Now (3.45) follows from (3.50) and (3.51).

In conclusion, (3.33) is obtained as we put together (3.41) and (3.45). □

4. Explicit rate functions

4.1. *Some examples*

For constants $0 < a, \beta$, and normalizing constant $c(a,\beta)$, consider the *even* density (on $\mathbb{R}$)

$$g(x) = c(a,\beta)\int_0^1 \exp\left(-\frac{a}{u^\beta} - ux^2\right) \mathrm{d}u. \tag{4.1}$$

When x is large, the value u^* where $\frac{a}{u^\beta} + ux^2$ reaches a minimum, arises as you equal $\frac{a\beta}{u^\beta} = ux^2$. Thus,

$$\lim_{x\to\infty} \frac{1}{x^\alpha}\log g(x) = -c_\alpha, \quad \text{with } \alpha = \frac{2\beta}{1+\beta} \text{ and } c_\alpha = \frac{(1+\beta)}{\beta^{\beta/(1+\beta)}} a^{1/(1+\beta)}. \tag{4.2}$$

Now, the log-Laplace transform is

$$\begin{aligned}
\Gamma(y) &= \log c(a,\beta)\int_0^1 \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}x \left(\exp\left(yx - \frac{a}{u^\beta} - ux^2\right)\right) \\
&= \log\left(c(a,\beta)\int_0^1 \mathrm{d}u \exp\left(\frac{y^2}{4u} - \frac{a}{u^\beta}\right) \times \int_{\mathbb{R}} \exp\left(-\left(x\sqrt{u} - y/\left(2\sqrt{u}\right)\right)^2\right)\mathrm{d}x\right) \\
&= \log \int_0^1 \mathrm{d}u \exp\left(\frac{y^2}{4u}\right)\phi(u), \quad \text{with } \phi(u) = c(a,\beta)\exp\left(-\frac{a}{u^\beta}\right)\frac{1}{\sqrt{u}}.
\end{aligned} \tag{4.3}$$

The last integral converges if $\beta > 1$, that is $\alpha \in]1,2]$. Note, that at infinity $\Gamma(y) \sim d_\alpha y^{2\beta/(\beta-1)}$, as expected by Tauberian theorems. Now, $\Gamma(\sqrt{y})$ is convex, using Hölder's inequality on the representation of the last time of (4.3).

4.2. *When $x \mapsto \mathcal{I}(\sqrt{x})$ is concave*

In this section, we review some useful property of the rate function, and prove Proposition 1.10. First, we state a simple observation.

Lemma 4.1. *Assume that Γ is twice differentiable, and $y \mapsto \Gamma(\sqrt{y})$ is convex for $y > 0$. Then, $x \mapsto \mathcal{I}(\sqrt{x})$ is concave for $x > 0$.*

Proof. Note first that Γ is strictly convex. Indeed, note first that $\Gamma(x) > 0$ for $x \neq 0$. This forces $\Gamma'(x) > 0$ for $x > 0$. Second, $y \mapsto \Gamma(\sqrt{y})$ convex, for $y > 0$, implies that $\Gamma'(x)/x$ is increasing for $x > 0$, which in turn says that $\Gamma'(x)$ is strictly increasing, which implies that Γ is strictly convex.

Also, $\mathcal{I}$ is differentiable, and $\mathcal{I}'$ is the inverse of Γ' on $\mathbb{R}^+$. Note that $\Gamma''(x) > 0$ for $x > 0$, so that $\mathcal{I}'$ is differentiable and $\mathcal{I}''(x)\Gamma''(\mathcal{I}'(x)) = 1$.

Now, $\Gamma'(x)/x$ is increasing for $x > 0$ is equivalent to

$$\forall y > 0 \quad \Gamma'(y) \le y\Gamma''(y) \quad \Longrightarrow \quad \forall x > 0 \quad \mathcal{I}'(x) \ge x\mathcal{I}''(x).$$

Thus, $x \mapsto \mathcal{I}(\sqrt{x})$ is concave for $x > 0$. □

Proof of Proposition 1.10. We show first two useful properties. First, note that for $0 \le p \le 1$ and $x > 0$,

$$p\mathcal{I}(x) \ge \mathcal{I}(px), \tag{4.4}$$

with equality if and only if $p = 1$. Indeed, using that $\mathcal{I}(0) = 0$, (4.4) is equivalent to

$$\frac{\mathcal{I}(x) - \mathcal{I}(0)}{x} \ge \frac{\mathcal{I}(px) - \mathcal{I}(0)}{px}, \quad \text{with } 0 \le xp \le x. \tag{4.5}$$

The strict convexity of $\mathcal{I}$ implies that (4.5) is true. Secondly, for any $x_1, \dots, x_n$ positive

$$\mathcal{I}\big(\sqrt{x_1}\big) + \cdots + \mathcal{I}\big(\sqrt{x_n}\big) \geq \mathcal{I}\big(\sqrt{x_1 + \cdots + x_n}\big). \tag{4.6}$$

It is easy to see that (4.6) is obtained by induction as a direct consequence of $x \mapsto \mathcal{I}(\sqrt{x})$ concave and $\mathcal{I}(0) = 0$.

Now, assume that $\sum_{z \in \mathcal{D}} \lambda^2(z)\kappa^2(z) \geq \gamma^2$, and let $z^* \in \mathcal{D}$ be such that $\lambda(z^*) = \max_{z \in \mathcal{D}} \lambda(z)$. By using that $\mathcal{I}$ is increasing in $\mathbb{R}^+$ ((4.4) is stronger than this latter property)

$$\sum_{z \in \mathcal{D}} \frac{\lambda^2(z)\kappa^2(z)}{\lambda^2(z^*)} \geq \frac{\gamma^2}{\lambda^2(z^*)} \quad \Longrightarrow \quad \mathcal{I}\left(\sqrt{\sum_{z \in \mathcal{D}} \frac{\lambda^2(z)\kappa^2(z)}{\lambda^2(z^*)}}\right) \geq \mathcal{I}\left(\frac{\gamma}{\lambda(z^*)}\right). \tag{4.7}$$

By (4.6), we have

$$\sum_{z \in \mathcal{D}} \mathcal{I}\left(\frac{\lambda(z)}{\lambda(z^*)}\kappa(z)\right) \geq \mathcal{I}\left(\frac{\gamma}{\lambda(z^*)}\right). \tag{4.8}$$

From (4.4), we deduce that

$$\sum_{z \in \mathcal{D}} \lambda(z)\mathcal{I}\big(\kappa(z)\big) \geq \lambda\big(z^*\big)\mathcal{I}\left(\frac{\gamma}{\lambda(z^*)}\right). \tag{4.9}$$

Note that the inequality in (4.9) is an equality if and only if $\kappa(z) = 0$ for all $z \neq z^*$, and $\lambda(z^*)\kappa(z^*) = \gamma$. Thus, (1.22) holds.

We prove now (1.23). First, note that since $\mathcal{I}$ is differentiable

$$\inf_{x>0}\left[\alpha x + x\mathcal{I}\left(\frac{\beta}{x}\right)\right] = \alpha x^* + x^*\mathcal{I}\left(\frac{\beta}{x^*}\right), \tag{4.10}$$

where x^* satisfies

$$\alpha = -\mathcal{I}\left(\frac{\beta}{x^*}\right) + \frac{\beta}{x^*}\mathcal{I}'\left(\frac{\beta}{x^*}\right) \qquad \left(\text{and } \alpha x^* + x^*\mathcal{I}\left(\frac{\beta}{x^*}\right) = \beta\mathcal{I}'\left(\frac{\beta}{x^*}\right)\right). \tag{4.11}$$

Recall now that for any x

$$-\mathcal{I}(x) + x\mathcal{I}'(x) = \Gamma\big(\mathcal{I}'(x)\big). \tag{4.12}$$

Thus, combining (4.11) and (4.12), we obtain

$$\alpha = \Gamma\left(\mathcal{I}'\left(\frac{\beta}{x^*}\right)\right). \tag{4.13}$$

The parenthesis in (4.11) and (4.13) imply (1.23), and x^* satisfies

$$\frac{\beta}{x^*} = \Gamma'\big(\Gamma^{-1}(\alpha)\big). \tag{4.14}$$

□

4.3. *A corollary of Proposition* 1.8

We first derive a corollary of Proposition 1.8.

Corollary 4.2. *For any $\varepsilon > 0$, there are positive constants A and M_A such that, for n large enough*

$$P(X_n \geq \xi_n) \leq 2P\big(\|\mathbb{1}_{\mathcal{D}_n^*(A)} q_n\|_2^2 \geq (1-\varepsilon)\xi_n, \big|\mathcal{D}_n^*(A)\big| \leq M_A\big) \tag{4.15}$$

and

$$\frac{1}{4^{M_A+1}}P\left(\|\mathbb{1}_{\mathcal{D}_n^*(A)}q_n\|_2^2\geq(1-\varepsilon)\xi_n,\left|\mathcal{D}_n^*(A)\right|\leq M_A\right)\leq P(X_n\geq\xi_n). \tag{4.16}$$

Since, ε is eventually taken to 0, Theorem 1.4 follows from Corollary 4.2, once we find the same upper and lower bound for

$$P\left(\|\mathbb{1}_{\mathcal{D}_n^*(A)}q_n\|_2^2\geq\xi_n,\left|\mathcal{D}_n^*(A)\right|\leq M_A\right). \tag{4.17}$$

Proof of Corollary 4.2. Fix $\varepsilon>0$ small enough, and A, M_A to be chosen later,

$$\begin{aligned}P(X_n\geq\xi_n)&\leq P\left(\sum_{z\in\mathcal{D}_n^*(A)}q_n^2(z)\geq(1-\varepsilon)\xi_n,\left|\mathcal{D}_n^*(A)\right|\leq M_A\right)+R_n\\ &\quad\text{with } R_n=P\left(\left|\mathcal{D}_n^*(A)\right|\geq M_A\right)+P\left(\sum_{z\notin\mathcal{D}_n^*(A)}X_n(z)\geq\varepsilon\xi_n\right).\end{aligned} \tag{4.18}$$

From (1.5), there is A, such that

$$P\left(\sum_{z\notin\mathcal{D}_n^*(A)}X_n(z)\geq\varepsilon\xi_n\right)\leq\frac{1}{4}\mathrm{e}^{-c_-\sqrt{\xi_n}}. \tag{4.19}$$

Also, an application of (1.14) stated as Lemma 2.2 of [5] shows that

$$P\left(\left|\mathcal{D}_n^*(A)\right|\geq M_A\right)\leq P\left(\left|\left\{z\colon l_n(z)\geq\frac{\sqrt{\xi_n}}{A}\right\}\right|\geq M_A\right)\leq\left|B(n)\right|^{M_A}\exp\left(-\tilde{\kappa}_d\frac{M_A^{1-2/d}\sqrt{\xi_n}}{A}\right). \tag{4.20}$$

Thus, there is M_A such that

$$P\left(\left|\mathcal{D}_n^*(A)\right|\geq M_A\right)\leq\frac{1}{4}\mathrm{e}^{-c_-\sqrt{\xi_n}}. \tag{4.21}$$

Now, for A large enough, and the corresponding M_A such that (4.21) holds, we have

$$R_n\leq\frac{1}{2}\exp\left(-c_-\sqrt{\xi_n}\right). \tag{4.22}$$

Now, from the lower bound in (1.5), we have (4.15).

We turn now to (4.16). We invoke Lemma 3.1 with $\mathcal{D}_n=\mathcal{D}_n^*(A)$ and $M_n=M_A$, and $m_n=n$. Note that from (1.14), we have

$$\mathbb{P}_0\left(l_n\left(B(r)\right)\geq\frac{\varepsilon}{2}\xi_n,\left|\mathcal{D}_n^*(A)\right|<M_A\right)\leq(2n)^{dM_A}\exp\left(-\frac{\tilde{\kappa}_d\varepsilon\xi_n}{2M_A^{2/d}}\right), \tag{4.23}$$

which is negligible, as well as the term $P(\|l_n\|_2\geq\xi_n n^{-\varepsilon})$ by Corollary 3.2. □

4.4. *Upper bound in Proposition* 1.10

Our first task is to approximate $\|\mathbb{1}_{\mathcal{D}_n^*(A)}q_n\|_2$ by a convenient discrete object.

Step 1: *On discretizing the local charge.* First, write for any integer n

$$\|\mathbb{1}_{\mathcal{D}_n^*(A)}q_n\|_2^2=\sum_{z\in\mathcal{D}_n^*(A)}l_n^2(z)\left(\frac{q_n(z)}{l_n(z)}\right)^2. \tag{4.24}$$

Note now that for any $\varepsilon > 0$, there is $\delta > 0$ such that on $\{|\mathcal{D}_n^*(A)| \le M_A\}$,

$$\left\{\|\mathbb{1}_{\mathcal{D}_n^*(A)} q_n\|_2^2 \ge \xi_n\right\} \subset \left\{\sum_{z\in\mathcal{D}_n^*(A)} l_n^2(z)\pi_\delta\left[\left(\frac{q_n(z)}{l_n(z)}\right)^2\right] \ge (1-\varepsilon)\xi_n\right\}. \tag{4.25}$$

Indeed,

$$\pi_\delta\left[\left(\frac{q_n(z)}{l_n(z)}\right)^2\right] \ge \left(\frac{q_n(z)}{l_n(z)}\right)^2 - \delta. \tag{4.26}$$

We sum (4.26) over $z \in \mathcal{D}_n^*(A)$, on the event $\{|\mathcal{D}_n^*(A)| \le M_A\}$, and choose δ small enough so that $\delta A^2 M_A \le \varepsilon$, and

$$\begin{aligned}\sum_{z\in\mathcal{D}_n^*(A)} l_n^2(z)\pi_\delta\left[\left(\frac{q_n(z)}{l_n(z)}\right)^2\right] &\ge \|\mathbb{1}_{\mathcal{D}_n^*(A)} q_n\|_2^2 - \delta\sum_{\mathcal{D}_n^*(A)} l_n^2(z)\\ &\ge \|\mathbb{1}_{\mathcal{D}_n^*(A)} q_n\|_2^2 - \delta A^2 M_A \xi_n\\ &\ge \left(1-\delta A^2 M_A\right)\xi_n \ge (1-\varepsilon)\xi_n.\end{aligned} \tag{4.27}$$

Step 2: *Integrating over the charges.* As usual, we integrate first with respect to the η-variables. We introduce the following random set (of volume independent of n since $|\mathcal{D}_n^*(A)| \le M_A$),

$$\mathcal{B}_n = \left\{\kappa = \left\{\kappa(z), z \in \mathcal{D}_n^*\right\}:\ \kappa^2(z) \in \delta\mathbb{N}, 0 \le \kappa(z) \le 2A^2\xi_n\right\}.$$

Now, for $z \in \mathcal{D}_n^*(A)$, $q_n(z)/l_n(z)$ satisfies a Large Deviation Principle with rate function $\mathcal{I}$: for $z \in \mathcal{D}_n^*(A)$ (recalling that this implies that $l_n(z) \ge \sqrt{\xi_n}/A$), $\varepsilon(n)$ vanishing as n goes to infinity, and $\kappa(z) > 0$,

$$Q\left(\pi_\delta\left[\left(\frac{q_n(z)}{l_n(z)}\right)^2\right] = \kappa^2(z)\right) \le 2Q\left(\frac{1}{l_n(z)}\sum_{i=1}^{l_n(z)}\eta_i \ge \kappa(z)\right) \le 2\exp\left(-l_n(z)\left(\mathcal{I}\left(\kappa(z)\right)+\varepsilon(n)\right)\right). \tag{4.28}$$

Also, if we denote

$$\mathcal{C}_n(l_n) = \left\{\boldsymbol{\kappa} \in \mathcal{B}_n:\ \sum_{z\in\mathcal{D}_n^*(A)} l_n^2(z)\kappa^2(z) \ge (1-\varepsilon)\xi_n\right\}, \tag{4.29}$$

we have, when integrating only over the charge variables, on the event $\{|\mathcal{D}_n^*(A)| \le M_A\}$,

$$\begin{aligned}&Q\left(\sum_{z\in\mathcal{D}_n^*(A)} l_n^2(z)\pi_\delta\left[\left(\frac{q_n(z)}{l_n(z)}\right)^2\right] \ge (1-\varepsilon)\xi_n\right)\\ &\quad\le \left(\frac{A\sqrt{\xi_n}}{\delta}\right)^{|D_n^*(A)|} \sup_{\boldsymbol{\kappa}\in\mathcal{C}_n(l_n)} \prod_{z\in\mathcal{D}_n^*(A)} Q\left(\pi_\delta\left[\left(\frac{q_n(z)}{l_n(z)}\right)^2\right] = \kappa^2(z)\right)\\ &\quad\le \left(\frac{A\sqrt{\xi_n}}{\delta}\right)^{M_A} \sup_{\boldsymbol{\kappa}\in\mathcal{C}_n(l_n)} \exp\left(-\sum_{z\in\mathcal{D}_n^*(A)} l_n(z)\left(\mathcal{I}\left(\kappa(z)\right)+\varepsilon(n)\right)\right)\\ &\quad\le \left(\frac{A\sqrt{\xi_n}}{\delta}\right)^{M_A} \mathrm{e}^{\varepsilon(n)\sqrt{\xi_n}} \exp\left(-\inf_{\boldsymbol{\kappa}\in\mathcal{C}_n(l_n)} \sum_{z\in\mathcal{D}_n^*(A)} l_n(z)\mathcal{I}\left(\kappa(z)\right)\right).\end{aligned} \tag{4.30}$$

Step 3: *On an explicit infimum.* We apply Proposition 1.10 to the infimum in (4.30), since we take actually the infimum over a smaller (discrete) set $\mathcal{C}_n(l_n)$.

$$\mathbb{1}_{\{|\mathcal{D}_n^*(A)|\le M_A\}} Q\biggl(\sum_{z\in\mathcal{D}_n^*(A)} q_n^2(z)\ge \xi_n\biggr)$$
$$\le \mathrm{e}^{\varepsilon(n)\sqrt{\xi_n}}\exp\biggl(-\max_{\mathcal{D}_n^*(A)} l_n\times\mathcal{I}\biggl(\frac{\sqrt{(1-\varepsilon)\xi_n}}{\max_{\mathcal{D}_n^*(A)} l_n}\biggr)\biggr) \tag{4.31}$$

We now integrate over the random walk (over the event $\{|\mathcal{D}_n^*(A)|\le M_A\}$)

$$E\bigl(\|\mathbb{1}_{\mathcal{D}_n^*(A)}q_n\|_2^2\ge\xi_n, \bigl|\mathcal{D}_n^*(A)\bigr|\le M_A\bigr)$$
$$\le \mathrm{e}^{\varepsilon(n)\sqrt{\xi_n}}\mathbb{E}_0\biggl[\exp\biggl(-\max_{\mathcal{D}_n^*(A)} l_n\times\mathcal{I}\biggl(\frac{\sqrt{(1-\varepsilon)\xi_n}}{\max_{\mathcal{D}_n^*(A)} l_n}\biggr)\biggr)\biggr]$$
$$\le \mathrm{e}^{\varepsilon(n)\sqrt{\xi_n}}\sum_{k\ge1}\mathbb{P}_0\Bigl(\max_{z\in\mathbb{Z}^d} l_n(z)=k\Bigr)\exp\biggl(-k\mathcal{I}\biggl(\frac{\sqrt{(1-\varepsilon)\xi_n}}{k}\biggr)\biggr)$$
$$\le \mathrm{e}^{\varepsilon(n)\sqrt{\xi_n}}\sum_{z\in\mathbb{Z}^d}\mathbb{P}_0(H_z\le n)\sum_{k\ge1}\mathbb{P}_0\bigl(l_\infty(z)=k\bigr)\exp\biggl(-k\mathcal{I}\biggl(\frac{\sqrt{(1-\varepsilon)\xi_n}}{k}\biggr)\biggr). \tag{4.32}$$

Now, a simple coupling argument shows that for any z, $\mathcal{P}_0(l_\infty(z)=k)\le\mathbb{P}_0(l_\infty(0)=k)=\exp(-\chi_d k)$. Now, for $\varepsilon>0$ arbitrarily small, we call $x=k/\sqrt{\xi_n}$, and we use (1.23) of Proposition 1.10, with $\alpha=(1-\varepsilon)\chi_d$ and $\beta=\sqrt{(1-\varepsilon)}$ for ε small. It is clear from (4.14) that we can choose A large enough so that $x^*\in[1/A,A]$. Thus,

$$\inf_{1/A\le x\le A}\biggl[(1-\varepsilon)\chi_d x + x\mathcal{I}\biggl(\frac{\sqrt{(1-\varepsilon)}}{x}\biggr)\biggr]=\sqrt{(1-\varepsilon)}\times\Gamma^{-1}\bigl((1-\varepsilon)\chi_d\bigr). \tag{4.33}$$

Since the function we optimize is continuous, it is irrelevant whether we take x real in $[1/A,A]$, or along a subdivision of mesh $1/\sqrt{\xi_n}$ as n goes to infinity.

4.5. *Lower bound in Proposition* 1.10

Recall Corollary 4.2 and (4.16). Note that

$$P\biggl(\sum_{z\in\mathcal{D}_n^*(A)} q_n^2(z)\ge\xi_n\biggr)\ge P\bigl(q_n^2(0)\ge\xi_n, \{0\}\in\mathcal{D}_n^*(A)\bigr).$$

When A is large enough (recall that $\beta<2$)

$$P\bigl(q_n^2(0)\ge\xi_n, \{0\}\in\mathcal{D}_n^*(A)\bigr)\ge\sup_{A\sqrt{\xi_n}\ge m_n\ge\sqrt{\xi_n}/A}\mathcal{P}_0\bigl(l_n(0)=m_n\bigr)2Q\biggl(\sum_{i=1}^{m_n}\eta_0(i)\ge\sqrt{\xi_n}\biggr). \tag{4.34}$$

We first need to compare $\mathbb{P}_0\{l_n(0)=m_n\}$ with $\mathbb{P}_0\{l_\infty(0)=m_n\}$, where $m_n=\lfloor x\sqrt{\xi_n}\rfloor$ (the integer part of $x\sqrt{\xi_n}$), and $x\in[1/A,A]$. We state the following lemma, which we prove at the end of the section.

Lemma 4.3. *Assume $d\ge3$. For any $\varepsilon>0$, there is $t(\varepsilon)>0$, such that for $n\ge t(\varepsilon)m_n$ (with any $m_n\le A\sqrt{\xi_n}$)*

$$\mathbb{P}_0\bigl(l_\infty(0)=m_n\bigr)\le\mathrm{e}^{\varepsilon m_n}\mathbb{P}_0\bigl(l_n(0)=m_n\bigr). \tag{4.35}$$

Now, recalling that for any integer k, $\mathbb{P}_0(l_\infty(0)=k)=\exp(-\chi_d k)$, $\beta<2$, and using Lemma 4.3, we have (with $\varepsilon(n)$ is a vanishing sequence, and the supremum over m_n in $[\sqrt{\xi_n}/A, A\sqrt{\xi_n}]$),

$$\begin{aligned}
&P\big(q_n^2(0)\ge \xi_n, \{0\}\in \mathcal{D}_n^*(A)\big)\\
&\quad\ge \mathrm{e}^{-\varepsilon m_n}\sup_{m_n}\exp\left(-\chi_d m_n - m_n\mathcal{I}\left(\frac{\sqrt{\xi_n}}{m_n}\right)-\varepsilon(n)m_n\right)\\
&\quad\ge \mathrm{e}^{-(\varepsilon+\varepsilon(n))m_n}\exp\left(-\sqrt{\xi_n}\inf_{A\ge x\ge 1/A}\left(\chi_d x + x\mathcal{I}\left(\frac{1}{x}\right)\right)\right)\\
&\quad\ge \mathrm{e}^{-(\varepsilon+\varepsilon(n))m_n}\exp\big(-\sqrt{\xi_n}\times\Gamma^{-1}(\chi_d)\big).
\end{aligned}\tag{4.36}$$

We used in the second line of (4.36) the continuity of the infimum. Also, we need to choose A large enough so that x^* which minimizes the infimum in (4.36) is in $[1/A, A]$. A lower bound identical to the upper bound follows from (4.36), as we send ε to zero.

Proof of Lemma 4.3. Recall that $m_n=\lfloor x\sqrt{\xi_n}\rfloor$ and $x\in[1/A, A]$. Let $\{\tau_i, i\ge 1\}$ be the successive return times to 0, and recall the classical bound, which holds in $d\ge 3$ for some constant c_d

$$\mathbb{P}(\tau_i>t|\tau_i<\infty)\le \frac{c_d}{t^{d/2-1}}.\tag{4.37}$$

Inequality (4.37) implies that for any ε, there is $t(\varepsilon)$ such that

$$\prod_{i\le m_n}\mathbb{P}\big(\tau_i\le t(\varepsilon)|\tau_i<\infty\big)\ge\left(1-\frac{c_d}{t(\varepsilon)^{d/2-1}}\right)^{m_n}\ge\exp(-\varepsilon m_n).\tag{4.38}$$

Also, note that

$$\big\{l_\infty(0)=m_n\big\}=\mathcal{A}_n\cap\{\tau_{m_n+1}=\infty\}\quad\text{with } \mathcal{A}_n=\{\tau_i<\infty, \forall i=1,\dots,m_n\}.\tag{4.39}$$

Now

$$P(\mathcal{A}_n)=P\left(\sum_{i=1}^{m_n}\tau_i<m_n t(\varepsilon)\big|\mathcal{A}_n\right)P(\mathcal{A}_n)+P\left(\sum_{i=1}^{m_n}\tau_i\ge m_n t(\varepsilon)\big|\mathcal{A}_n\right)P(\mathcal{A}_n).\tag{4.40}$$

We show that the first term on the right-hand side of (4.40) is large enough. Using (4.38)

$$P\left(\sum_{i=1}^{m_n}\tau_i<m_n t(\varepsilon)\big|\mathcal{A}_n\right)\ge\prod_{i\le m_n}P\big(\tau_i<t(\varepsilon)|\tau_i<\infty\big)\ge\exp(-\varepsilon m_n).\tag{4.41}$$

Thus, (4.40) and (4.41) yield

$$P(\mathcal{A}_n)\le P\left(\sum_{i=1}^{m_n}\tau_i<m_n t(\varepsilon)\big|\mathcal{A}_n\right)P(\mathcal{A}_n)+\big(1-\exp(-\varepsilon m_n)\big)P(\mathcal{A}_n),\tag{4.42}$$

and this implies that

$$P(\mathcal{A}_n)P(\tau_{m_n+1}=\infty)\le \mathrm{e}^{\varepsilon m_n}P\left(\sum_{i=1}^{m_n}\tau_i<m_n t(\varepsilon)\right)P(\tau_{m_n+1}=\infty).\tag{4.43}$$

Note that (4.43) is equivalent to (4.35) when $n\ge m_n t(\varepsilon)$. $\square$

5. A general large deviation principle

To prove Theorem 1.2, we follow the approach of [3]. We recall the main steps of the approach, and we detail how to treat the features which are different. In all of Section 5, we assume that dimension is 3 or more.

In [3], dimension is 5 or more. There is actually three occurrences in [3] where $d > 3$ is used, and we now review them:

- For self-intersection local times, $d = 4$ is the critical dimension, and only for $d > 4$, do we have that the excess self-intersection is made up on a finite number of sites. Here, the phenomenology is different, with Lemma 1.9 suggesting that $d = 2$ is critical, and Proposition 1.8 holds for $d \geq 3$.
- For normalizing time in Section 6 of [3], we used that, conditioned on returning to 0, the return time to 0 has finite expectation if $d > 4$. We bypass this constraint in Section 5.2 (see the arguments following (5.14)).
- Lemma 4.9 of [3] uses an estimate on the probability of exiting a sphere from a given domain in (10.29). This latter inequality is useful in $d > 3$. In $d = 3$, one can use instead the more sophisticated estimate of Lemma 5(b) of [15] which states that for $z \in B(r)$, and Σ a domain on the boundary of $B(r)$, the probability a random walk starting on z exits $B(r)$ in Σ is bounded by a constant times $|\Sigma|/|z-r|^{d-1}$ (rather than $|\Sigma|/|z-r|^{d-2}$). Thus, the denominator of (10.30) of [3] has a power $2d-3$ (rather than $2d-4$), and (10.31) holds also in $d = 3$ once L is chosen large enough, where L is related to the diameter of a ball containing the piles of monomers producing the excess energy (see $\tilde{\Lambda}$ in the paragraph following (5.9)).

5.1. *On a subadditive argument*

We recall that Lemma 7.1 of [3] establishes that for any radius r and $\xi > 0$, there is a positive constant $\mathcal{J}(\xi, r)$, and the following limit exists

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_0\big(\|\mathbb{1}_{B(r)} l_n\|_2 \geq \xi n, S(n) = 0\big) = -\mathcal{J}(\xi, r).$$

One important difference between local times and local charges, is that the distribution of the latter is continuous. Thus, we cannot find an optimal strategy by maximizing over a finite number of values, as for $\{\|\mathbb{1}_{B(r)} l_n\|_2 \geq \xi n\}$. The remedy is to first discretize $\|\mathbb{1}_{B(r)} q_m\|_2$.

For any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\big\{\|\mathbb{1}_{B(r)} q_m\|_2 \geq \xi m\big\} \subset \Bigg\{\sum_{z\in B(r)} l_m^2(z) \pi_\delta^2\bigg(\frac{q_m(z)}{l_m(z)}\bigg) \geq \big((1-\varepsilon)\xi m\big)^2\Bigg\}. \tag{5.1}$$

Indeed, if z is such that $q_m(z) \geq l_m(z)$, then

$$\pi_\delta\bigg(\frac{q_m(z)}{l_m(z)}\bigg) \geq \frac{q_m(z)}{l_m(z)} - \delta \geq (1-\delta)\frac{q_m(z)}{l_m(z)}. \tag{5.2}$$

Now, if $q_m(z) < l_m(z)$, then

$$\pi_\delta^2\bigg(\frac{q_m(z)}{l_m(z)}\bigg) \geq \bigg(\frac{q_m(z)}{l_m(z)}\bigg)^2 - 2\delta. \tag{5.3}$$

We use now (5.2) and (5.3) to form $\|q_m\|^2_{B(r)}$. When summing over $z \in B(r)$, we bound $\pi_\delta(q_m(z)/l_m(z))$ according to the worse scenario. For $\xi > 0$ fixed, choose δ small enough, and recall that $\|l_m\|_{B(r)} \leq m$,

$$\begin{aligned}\sum_{z\in B(r)} l_m^2(z)\bigg(\frac{q_m(z)}{l_m(z)}\bigg)^2 &\geq (1-\delta)\|q_m\|^2_{B(r)} - 2\delta \sum_{B(r)} l_m^2(z)\\ &\geq (1-\delta)\|q_m\|^2_{B(r)} - 2\delta m^2\\ &\geq \big(\xi^2(1-\delta) - 2\delta\big)m^2 \geq \big((1-\varepsilon)\xi m\big)^2. \end{aligned}\tag{5.4}$$

We can now state our subadditive result which we prove in the Appendix.

Lemma 5.1. *For $\xi > 0$ small enough, for any $r > 0$ and for any δ small enough, there is a constant $\mathcal{J}(\xi, r, \delta)$ such that*

$$\lim_{m\to\infty} \frac{1}{m} \log P\left(\left\| \mathbb{1}_{B(r)} l_m(\cdot) \pi_\delta\left(\frac{q_m(\cdot)}{l_m(\cdot)}\right)\right\|_2 \geq \xi m\right) = -\mathcal{J}(\xi, r, \delta). \tag{5.5}$$

5.2. *On the upper bound for the LDP*

We show the following upper bound.

Proposition 5.2. *Assume $n^{2/3} \ll \xi_n \ll n^2$ and $d \geq 3$. Then,*

$$\begin{aligned} &\forall \varepsilon > 0, \exists r_0 > 0, \exists \alpha_0 > 0, \exists \delta_0 > 0, \forall r > r_0, \forall \alpha > \alpha_0, \forall \delta < \delta_0 \\ &\quad \limsup_{n\to\infty} \frac{1}{\sqrt{\xi_n}} \log P(X_n \geq \xi_n) \leq -\alpha \mathcal{J}\left(\frac{\sqrt{(1-\varepsilon)}}{\alpha}, r, \delta\right) + \varepsilon. \end{aligned} \tag{5.6}$$

Proof. The random walk cannot escape $[-n, n]^d$ in a time n. Fix $\varepsilon > 0$. By Corollary 4.2, and at the expense of a polynomial term, there is a constant M_A, and a finite volume Λ_n with $|\Lambda_n| \leq M_A$ such that

$$P(X_n \geq \xi_n) \leq n^\gamma, \qquad P\left(\sum_{z\in\Lambda_n} q_n^2(z) \geq (1-\varepsilon)\xi_n\right). \tag{5.7}$$

Recalling that the walk is transient, it is convenient to pass to an infinite time-horizon. Thus, we define

$$l_\infty(z) = \sum_{i\in\mathbb{N}} \mathbb{1}\{S(i) = z\} \quad \text{and} \quad q_\infty(z) = \sum_{i=1}^{l_\infty(z)} \eta_z(i). \tag{5.8}$$

We use now our monotony of the square charges, Corollary A.2, to conclude

$$P(X_n \geq \xi_n) \leq n^\gamma, \qquad P\left(\sum_{z\in\Lambda_n} q_\infty^2(z) \geq (1-\varepsilon)\xi_n\right). \tag{5.9}$$

Asselah [3] establishes that Λ_n can be transferred into a domain of finite diameter $\tilde{\Lambda}$. Let $\mathcal{T}: \Lambda_n \to \tilde{\Lambda}$ be the *transfer map* (see Proposition 5.2 of [3]). The following more precise statement is established in [3]: for any $\varepsilon > 0$, there is $r_0 > 0$, such that $\tilde{\Lambda} \subset B(r_0)$ for any large integer n, and for any sequence of integers $\{k(z), z \in \Lambda_n\}$, with $k(z) \leq A\sqrt{\xi_n}$ for all $z \in \Lambda_n$, we have

$$\begin{aligned} &P\left(\sum_{z\in\Lambda_n} q_\infty^2(z) \geq (1-\varepsilon)\xi_n \; l_\infty(z) = k(z), \forall z \in \Lambda_n\right) \\ &\quad \leq \mathrm{e}^{\varepsilon\sqrt{\xi_n}} \mathbb{P}_0\big(l_\infty(\mathcal{T}z) \geq k(z), \forall z \in \Lambda_n\big) Q\left(\sum_{z\in\Lambda_n}\left(\sum_{i=1}^{k(z)} \eta_z(i)\right)^2 \geq (1-\varepsilon)\xi_n\right). \end{aligned} \tag{5.10}$$

In the sum of the $\eta_z(i)$ over $[1, k(z)]$, in the right-hand side of (5.10), we need to replace $k(z)$ by the larger value $l_\infty(\mathcal{T}z)$. Again, we require a monotony of the l_2-norm of the charges (i.e., Corollary A.2 in the Appendix), with the consequence that for a fixed realization of the walk with $\{l_\infty(\mathcal{T}(z)) \geq k(z), \forall z \in \Lambda_n\}$, and any $r > r_0$, and with two shorthand notations

$$A_n=(1-\varepsilon)\xi_n \quad \text{and} \quad q_z(n)=\sum_{i=1}^{n}\eta_z(i),$$

$$Q\Big(\sum_{z\in\Lambda_n} q_z^2\big(k(z)\big)\ge A_n\Big)\le Q\Big(\sum_{z\in\Lambda_n} q_z^2\big(l_\infty(\mathcal{T}z)\big)\ge A_n\Big)=Q\Big(\sum_{z\in\tilde\Lambda} q_z^2\big(l_\infty(z)\big)\ge A_n\Big)$$
$$\le Q\Big(\sum_{z\in B(r)} q_z^2\big(l_\infty(z)\big)\ge A_n\Big). \tag{5.11}$$

Thus, after averaging over the walk in (5.11), and summing over the $\{k(z), z\in\Lambda_n\}$ each term of (5.11), we have that for any $\varepsilon>0$, there is $r>0$ such that

$$P\big(\|\mathbb{1}_{\Lambda_n}q_\infty\|_2^2\ge A_n\big)\le \mathrm{e}^{\varepsilon\sqrt{\xi_n}}\sum_{\mathbf{k}}\mathbb{E}_0\Big[\mathbb{1}_{\{l_\infty(\mathcal{T}(z))\ge k(z),\forall z\in\Lambda_n\}}Q\Big(\sum_{z\in B(r)} q_z^2\big(l_\infty(z)\big)\ge A_n\Big)\Big]$$
$$\le \mathrm{e}^{\varepsilon\sqrt{\xi_n}}E\Big[\prod_{z\in\tilde\Lambda} l_\infty(z)Q\Big(\sum_{z\in B(r)} q_z^2\big(l_\infty(z)\big)\ge A_n\Big)\Big]$$
$$\le \mathrm{e}^{\varepsilon\sqrt{\xi_n}}\big(A\sqrt{\xi_n}\big)^{|B(r)|}P\Big(\sum_{z\in B(r)} q_z^2\big(l_\infty(z)\big)\ge A_n, \max_{B(r)} l_\infty\le A\sqrt{\xi_n}\Big)$$
$$+\,\mathrm{e}^{\varepsilon\sqrt{\xi_n}}\mathbb{E}_0\Big[\prod_{z\in\tilde\Lambda} l_\infty(z)\mathbb{1}_{\{\exists z\in\tilde\Lambda, l_\infty(z)>A\sqrt{\xi_n}\}}\Big]. \tag{5.12}$$

The second term in the right-hand side of (5.12) is bounded by a term $\exp(-\chi_d A\sqrt{\xi_n})$, whereas the first term is estimated as follows.

$$P\Big(\sum_{z\in B(r)} q_z^2\big(l_\infty(z)\big)\ge A_n, \max_{B(r)} l_\infty\le A\sqrt{\xi_n}\Big)$$
$$\le \big(A\sqrt{\xi_n}\big)^{|B(r)|}\sup_k Q\Big(\sum_{z\in B(r)} q_z^2\big(k(z)\big)\ge A_n\Big)\mathbb{P}_0\big(l_\infty(z)=k(z), \forall z\in B(r)\big), \tag{5.13}$$

where the supremum in (5.13) is over integer sequences such that $\max_{B(r)} k(z)\le A\sqrt{\xi_n}$.

Now, we proceed similarly as in Section 6 of [3]. We choose an integer sequence $\{k(z), z\in B(r)\}$ with $\max_{B(r)} k(z)\le A\sqrt{\xi_n}$, and define $|k|=\sum_{B(r)} k(z)$, and

$$\mathcal{E}(k)=\Big\{\mathbf{z}=\big(z(1),\dots,z(|k|)\big)\in\tilde\Lambda^{|k|}\colon \sum_{i=1}^{|k|}\mathbb{1}\big\{z(i)=x\big\}=k(x), \forall x\in B(r)\Big\}. \tag{5.14}$$

Now, for $\{k(z), z\in B(r)\}$ with $\max_{B(r)} k(z)\le A\sqrt{\xi_n}$, we have, if $T_x=\inf\{n\ge 1\colon S_n=x\}$ and $T=\min\{T(x), x\in B(r)\}$,

$$P_0\big(l_\infty(z)=k(z), \forall z\in B(r)\big)=\sum_{\mathbf{z}\in\mathcal{E}(k)}\prod_{i=0}^{|k|-1}P_{z(i)}\big(T\big(z(i+1)\big)=T<\infty\big)P_{z(|k|)}(T=\infty). \tag{5.15}$$

We show now that if $d\ge 3$, and any $\varepsilon>0$, there is $\alpha(r,\varepsilon)$ such that for all $x, y\in B(r)$,

$$P_x\big(T(y)=T<\infty\big)\le(1-\varepsilon)P_x\big(T(y)=T<\alpha(r,\varepsilon)\big). \tag{5.16}$$

This would imply that

$$
\begin{aligned}
&P_0\big(l_\infty(z)=k(z),\forall z\in B(r)\big)\\
&\quad\le(1-\varepsilon)^{|k|}\sum_{\mathbf{z}\in\mathcal{E}(k)}\prod_{i=0}^{|k|-1}P_{z(i)}\big(T\big(z(i+1)\big)=T<\alpha\big)P_{z(|k|)}(T=\infty)\\
&\quad\le P_0\big(l_{\alpha(r,\varepsilon)|k|}(z)\ge k(z),\forall z\in B(r)\big).
\end{aligned}
\tag{5.17}
$$

Using (5.13), (5.17) and Corollary A.2 in the Appendix, we would obtain for $\alpha\ge|B(r)|\alpha(r,\varepsilon)$,

$$
P\bigg(\sum_{z\in B(r)}q_z^2\big(l_\infty(z)\big)\ge A_n,\max_{B(r)}l_\infty\le A\sqrt{\xi_n}\bigg)\le\big(A\sqrt{\xi_n}\big)^{|B(r)|}P\bigg(\sum_{z\in B(r)}q_z^2\big(l_{\alpha\sqrt{\xi_n}}(z)\big)\ge A_n\bigg).
\tag{5.18}
$$

We now prove (5.16). A classical result yields that for aperiodic symmetric walk, and any positive integer k, we have $P_x(T_x=k)\le c_d/k^{d/2}$ for some positive constant c_d. Also, if we only consider pairs $x,y\in B(r)$ such that $P_x(T_y=T<\infty)>0$, then there is an integer l_r and a positive constant c_r, such that

$$
\inf_{x,y\in B(r)}P_x(T_y=T<l_r)=c_r.
\tag{5.19}
$$

Now, by conditioning

$$
\frac{c_d}{(k+l_r)^{d/2}}\ge P_x(T_x=k+l_r)\ge P_x(T=T_y=k)P_y(T_x=T=l_r).
\tag{5.20}
$$

Thus,

$$
P_x(T=T_y=k)\le\frac{c_d}{c_r}\frac{1}{(k+l_r)^{d/2}}.
\tag{5.21}
$$

Therefore, there is $C>0$ such that for any $x,y\in B(r)$ (with $P_x(T_y=T<\infty)>0$), and any integer k,

$$
P_x(k<T_y=T<\infty)=\sum_{i>k}P_x(T=T_y=i)\le\frac{C}{k^{d/2-1}}.
\tag{5.22}
$$

We conclude that for any ε, there is $k(r,\varepsilon)$ such that for any $x,y\in B(r)$

$$
P_x\big(k(r,\varepsilon)<T_y=T<\infty\big)\le\varepsilon.
\tag{5.23}
$$

Inequality (5.23) implies that there is $\alpha(r,\varepsilon)$ such that (5.16) holds.

The purpose of squeezing $\mathcal{D}_n^*(A)$ inside $B(r)$ is to renormalize time. Indeed, the walk typically visits $\sqrt{\xi_n}$-times sites of $\tilde{\Lambda}_n$ in a total time of order $\sqrt{\xi_n}$. Thus, Section VII of [3] establishes that there is $\alpha_0>0$ such that for $\alpha>\alpha_0$, calling m_n the integer part of $\alpha\sqrt{\xi_n}$, we have for some $\gamma>0$

$$
P\big(\|q_\infty\|_{B(r)}\ge\sqrt{(1-\varepsilon)\xi_n}\big)\le n^\gamma P\bigg(\|q_{m_n}\|_{B(r)}\ge m_n\frac{\sqrt{(1-\varepsilon)}}{\alpha},S(m_n)=0\bigg).
\tag{5.24}
$$

As in [3], formula (7.19), in order that the walk returns to the origin at time m_n, we needed to add an piece of path of arbitrary length n, satisfying $\{S(0)=S(n)=0\}$ whose probability is polynomial in n. Under our hypothesis of aperiodicity of the walk, this latter fact is true. Putting together (5.1), (5.7), (5.9), (5.24) and invoking Lemma 5.1, we obtain (5.6) and conclude our proof. □

5.3. *Lower bound in Theorem 1.2*

We first show a lower bound, similar to Proposition 5.2, and then in Section 5.4, we take a limit as α, r go to infinity.

Proposition 5.3. *Assume $\frac{2}{3} < \beta < 2$ and $d \geq 3$. Then,*

$$\forall \varepsilon > 0, \forall r, \forall \alpha, \forall \delta > 0 \quad \liminf_{n\to\infty} \frac{1}{\sqrt{\xi_n}} \log P(X_n \geq \xi_n) \geq -\alpha \mathcal{J}\left(\frac{\sqrt{\xi(1+\varepsilon)}}{\alpha}, r, \delta\right). \tag{5.25}$$

In order to use Lemma 5.1, we need to show that there is C which might depend on (r, α, δ) such that (recall that m_n is the integer part of $\alpha\sqrt{\xi_n}$)

$$P\big(\|q_n\|_{\mathbb{Z}^d}^2 - n \geq \xi_n\big) \geq C P\left(\left\|\mathbb{1}_{B(r)} l_{m_n} \pi_\delta\left(\frac{q_{m_n}}{l_{m_n}}\right)\right\|_2^2 \geq (1+\varepsilon)\xi_n, S(m_n) = 0\right). \tag{5.26}$$

Since $\pi_\delta(x) \leq x$ for $x \geq 0$ and $\delta > 0$, it is obvious that (5.26) follows from Lemma 3.1 with the following choice. For a fixed $r > 0$, we set $\mathcal{D}_n = B(r)$ and $M_n = |B(r)|$. We take m_n as the integer part of $\alpha\sqrt{\xi_n}$ for $\xi_n \ll n^2$, with α as large as we wish, and n going to infinity. Note that from (1.14), we have

$$\mathbb{P}_0\left(l_n\big(B(r)\big) \geq \frac{\varepsilon}{2}\xi_n\right) \leq \exp\left(-\frac{\tilde{\kappa}_d \varepsilon \xi_n}{2|B(r)|^{2/d}}\right),$$

which is negligible. The term $\mathbb{P}_0(\|l_n\|_2^2 \geq n^{2\beta-\varepsilon})$ is dealt with Corollary 3.2 since $\zeta_d(\beta) > \beta/2$.

5.4. *About the rate function*

Using Lemmas (5.2) and (5.3), we have

$$\begin{aligned} &\forall \varepsilon > 0, \exists r_0 > 0, \exists \alpha_0 > 0, \exists \delta_0 > 0, \forall r, r' > r_0, \forall \alpha, \alpha' > \alpha_0, \forall \delta, \delta' < \delta_0 \\ &\quad \alpha \mathcal{J}\left(\frac{\sqrt{\xi(1-\varepsilon)}}{\alpha}, r, \delta\right) - \varepsilon \leq \alpha' \mathcal{J}\left(\frac{\sqrt{\xi(1+\varepsilon)}}{\alpha'}, r', \delta'\right). \end{aligned} \tag{5.27}$$

If we set

$$\varphi(x, r, \delta) = \frac{\mathcal{J}(x, r, \delta)}{x},$$

we note that (1.10), (5.6) and (5.25) imply that $\varphi(x, r, \delta)$ is bounded as follows:

$$\frac{c_+}{\sqrt{1+\varepsilon}} \leq \varphi(x, r, \delta) \leq \frac{1}{\sqrt{1-\varepsilon}}\left[c_- + \frac{\varepsilon}{\sqrt{\xi}}\right]. \tag{5.28}$$

Now, (5.27) reads as

$$\begin{aligned} &\forall \varepsilon > 0, \exists r_0 > 0, \exists x_0 > 0, \exists \delta_0 > 0, \forall r, r' > r_0, \forall x, x' < x_0, \forall \delta, \delta' < \delta_0 \\ &\quad \varphi\big(x', r', \delta'\big) \geq \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \varphi(x, r, \delta) - \frac{\varepsilon}{\sqrt{\xi(1+\varepsilon)}}. \end{aligned} \tag{5.29}$$

As we consider subsequences when $x \to 0, r \to \infty, \delta \to 0$, we obtain for any ε small

$$\liminf \varphi \geq \sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \limsup \varphi - \frac{\varepsilon}{\sqrt{\xi(1+\varepsilon)}}. \tag{5.30}$$

As ε vanishes in (5.30), we conclude that $\varphi(x, r, \delta)$ (along any subsequences) converges to a constant $\mathcal{Q}_2 > 0$.

Appendix

A.1. *Proof of Lemma 2.1*

Note that since we assume that η has a symmetric law,

$$Q\big(\zeta(n) > t\big) = 2\mathrm{Q}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\eta(i) \geq \sqrt{t}\right).$$

We first treat the case $\eta \in \mathcal{H}_1$, and $\lambda_0 > 0$ is such that $E[\exp(\lambda_0\eta)] < \infty$. We use a Chebychev's exponential inequality. For $\lambda > 0$,

$$\begin{aligned} Q\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\eta(i) \geq t\right) &\leq \mathrm{e}^{-\lambda t}\left(E_Q\left[\exp\left(\frac{\lambda}{\sqrt{n}}\eta\right)\right]\right)^n \\ &\leq \mathrm{e}^{-\lambda t}\left(1 + \frac{\lambda^2}{2n} + \frac{\lambda^3}{3!n\sqrt{n}}E\big[|\eta|^3\mathrm{e}^{(\lambda/\sqrt{n})|\eta|}\big]\right)^n. \end{aligned} \tag{A.1}$$

First, choose $\lambda = \lambda_0\sqrt{n}/2$. There is a constant c_1 such that

$$\begin{aligned} Q\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\eta(i) \geq t\right) &\leq \exp\left(-\frac{\lambda_0\sqrt{n}}{2}t\right)\left(1 + \frac{\lambda_0^2}{8} + E\big[\lambda_0^3|\eta|^3\mathrm{e}^{\lambda_0|\eta|/2}\big]\right)^n \\ &\leq \exp\left(-\frac{\lambda_0\sqrt{n}}{2}t\right)\left(1 + \frac{\lambda_0^2}{8} + c_1E\big[\mathrm{e}^{\lambda_0\eta}\big]\right)^n \\ &\leq \exp\left(-\frac{\lambda_0\sqrt{n}}{2}t + \beta_1 n\right), \quad \text{with } \beta_1 = \frac{\lambda_0^2}{8} + c_1E\big[\mathrm{e}^{\lambda_0\eta}\big]. \end{aligned} \tag{A.2}$$

Let $\beta_0 = 4\beta_1/\lambda_0$ and note that for $t \geq \sqrt{\beta_0 n}$, we have

$$Q\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\eta(i) \geq t\right) \leq \exp\left(-\frac{\lambda_0}{4}\sqrt{n}t\right). \tag{A.3}$$

Now, we assume that $t \leq \sqrt{\beta_0 n}$, and we choose $\lambda = \gamma t$ for γ to be adjusted latter. Inequality (A.1) yields

$$Q\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\eta(i) \geq t\right) \leq \mathrm{e}^{-\gamma t^2}\left(1 + \frac{\gamma^2t^2}{2n} + \frac{\gamma^3t^3}{n\sqrt{n}}E\left[|\eta|^3\exp\left(\frac{\gamma t}{\sqrt{n}}|\eta|\right)\right]\right)^n. \tag{A.4}$$

γ needs to satisfy many constraints. First, in order for the exponential of $|\eta|$ in (A.4) to be finite, we need that $\gamma\sqrt{\beta_0} \leq \lambda_0/2$, in which case

$$\begin{aligned} Q\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\eta(i) \geq t\right) &\leq \mathrm{e}^{-\gamma t^2}\left(1 + \frac{\gamma^2t^2}{2n} + \frac{\gamma^3t^3}{\lambda_0^3n\sqrt{n}}c_1E\big[\mathrm{e}^{\lambda_0\eta}\big]\right)^n \\ &\leq \exp\left(-\gamma t^2 + \frac{\gamma^2t^2}{2} + \frac{\gamma^3t^3}{\lambda_0^3\sqrt{n}}c_1E\big[\mathrm{e}^{\lambda_0\eta}\big]\right). \end{aligned} \tag{A.5}$$

In the right-hand side of (A.5), the term in γ^2 is innocuous as soon as $\gamma \leq 1/2$. Also, since $t \leq \sqrt{\beta_0 n}$, the term in γ^3 is innocuous as soon as

$$\gamma \leq \frac{\lambda_0^3}{\sqrt{\beta_0}c_1E[\mathrm{e}^{\lambda_0\eta}]},$$

so that (A.5) yields

$$Q\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\eta(i)\geq t\right)\leq \exp\bigl(-\kappa_\infty t^2\bigr),\quad \text{with } \kappa_\infty=\min\left(\frac{\lambda_0^3}{\sqrt{\beta_0}c_1E[\mathrm{e}^{\lambda_0\eta}]},\frac{1}{2},\frac{\lambda_0}{2\sqrt{\beta_0}}\right). \tag{A.6}$$

Using (A.3) and (A.6), it is immediate to deduce (2.1).

Assume now $\eta\in\mathcal{H}_\alpha$ for $1<\alpha<2$. From Kasahara's Tauberian theorem, there is a constant κ_α and $\beta_0>0$ such that for $t\geq\sqrt{\beta_0 n}$, we have

$$Q\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\eta(i)\geq t\right)\leq \exp\left(-\kappa_\alpha\frac{(t\sqrt{n})^\alpha}{n^{\alpha-1}}\right). \tag{A.7}$$

Now, when $t\leq\sqrt{\beta_0 n}$, we use the argument of the previous case $\mathcal{H}_1$, to obtain (A.6).

Finally, when $\eta\in\mathcal{H}_2$, Chen has shown in [8] there is a constant C such that for any $k\in\mathbb{N}$,

$$E_Q\left[\left(\sum_{i=1}^{n}\eta(i)\right)^{2k}\right]\leq C^k k! n^k. \tag{A.8}$$

Inequality (A.8) implies that for some $\lambda_1>0$,

$$\sup_n E_Q\left[\exp\left(\lambda_1\left(\frac{\sum_{i=1}^{n}\eta(i)}{\sqrt{n}}\right)^2\right)\right]<\infty. \tag{A.9}$$

Thus, there is a constant C_1 such that for any $n\in\mathbb{N}$ and $t>0$,

$$P_Q\bigl(\zeta(n)>t\bigr)\leq C_1\mathrm{e}^{-\lambda_1 t}. \tag{A.10}$$

This concludes the proof of Lemma 2.1.

A.2. *Proof of Lemma* 5.1

We fix two integers k and n, with k to be taken first to infinity. Let m, s be integers such that $k=mn+s$, and $0\leq s<n$. The phenomenon behind the subadditive argument (to come) is that the rare event

$$\mathcal{A}_k(\xi,r,\delta)=\left\{\left\|l_k\pi_\delta\left(\frac{q_k}{l_k}\right)\right\|_{B(r)}\geq\xi k,\, S(k)=0\right\},$$

can be built by concatenating the *same* optimal scenario realizing $\mathcal{A}_n(\xi,r,\delta)$ on m consecutive periods of length n, and one last period of length s where the scenario is necessarily special and its cost innocuous. The crucial independence between the different period is obtained by forcing the walk to return to the origin at the end of each period.

Thus, our first step is to exhibit an optimal strategy realizing $\mathcal{A}_n(\xi,r,\delta)$. For this purpose, we show that a finite number of values of the discrete variables $\{l_n(z),\pi_\delta(q_n/l_n(z)), z\in B(r)\}$ are needed to estimate the probability of $\mathcal{A}_n(\xi,r,\delta)$.

First recall that ξ is as small as we wish. In particular, we take it such that $Q(\eta>\xi)>0$. Note that $\sum_{B(r)}l_n(z)\leq n$. Also, using Lemma 2.1, there is a constant C such that

$$P\bigl(\exists z\in B(r),\zeta_z\bigl(l_n(z)\bigr)\geq An\bigr)\leq\exp(-CAn).$$

Consequently, the same holds for $q_n(z)/l_n(z)=\sqrt{\zeta_z(l_n(z))/l_n(z)}$. Thus, if we denote by

$$\mathcal{E}(A)=\left\{\exists z\in B(r),\frac{q_n(z)}{l_n(z)}\geq\sqrt{An}\right\},\quad\text{then } P\bigl(\mathcal{E}(A)\bigr)\leq\exp(-CAn). \tag{A.11}$$

On the other hand, there is an obvious lower bound obtained by considering monomers making up one single pile:

$$\mathcal{S} = \big\{l_n(0) = n, \eta(i) > \xi, \forall i < n\big\} \quad \text{and} \quad P(\mathcal{S}) = \left(\frac{1}{2d+1}\right)^n Q(\eta > \xi)^n. \tag{A.12}$$

Thus, using (A.11) and (A.12), we have for A large enough

$$2P\big(\mathcal{E}(A)\big) \le P\big(\mathcal{A}_n(\xi, r, \delta)\big) \quad \Longrightarrow \quad P\big(\mathcal{A}_n(\xi, r, \delta)\big) \le 2P\big(\mathcal{A}_n(\xi, r, \delta), \mathcal{E}^c(A)\big). \tag{A.13}$$

We conclude that for $z \in B(r)$ there are

$$\lambda_n(z) \in [0, n] \cap \mathbb{N} \quad \text{and} \quad \kappa_n(z) \in \big[0, \sqrt{An}\big] \cap \delta\mathbb{N}, \quad \text{with } \|\lambda_n \kappa_n\|_{B(r)} \ge \xi n, \tag{A.14}$$

such that

$$P\big(\mathcal{A}_n(\xi, r, \delta)\big) \le \left(\frac{2n\sqrt{An}}{\delta}\right)^{|B(r)|} P\left(l_n|_{B(r)} = \lambda_n, \pi_\delta\left(\frac{q_n}{l_n}\right)\bigg|_{B(r)} = \kappa_n\right). \tag{A.15}$$

The factor 2 in the constant appearing in the right-hand side of (A.15) is to account for the choice of a positive total charge on all sites of $B(r)$, by using the symmetry of the charge's distribution. Let $z^* \in B(r)$ be a site where

$$\lambda_n\big(z^*\big)\kappa_n^2\big(z^*\big) = \max_{B(r)} \lambda_n \kappa_n^2 \quad \text{and note that} \quad \lambda_n\big(z^*\big)\kappa_n^2\big(z^*\big) \ge n\xi^2. \tag{A.16}$$

Indeed, one uses that $\sum_{B(r)} \lambda_n(z) \le n$ and

$$(n\xi)^2 \le \sum_{z \in B(r)} \big(\lambda_n(z)\kappa_n(z)\big)^2 \le \max_{B(r)}\big(\lambda_n \kappa_n^2\big) \sum_{z \in B(r)} \lambda_n(z) \le n\lambda_n\big(z^*\big)\kappa_n^2\big(z^*\big).$$

Also,

$$P\left(\pi_\delta\left(\frac{q_n(z^*)}{l_n\big(z^*\big)}\right) = \kappa_n\big(z^*\big)\right) > 0 \quad \Longleftarrow \quad P\big(\eta \ge \kappa_n\big(z^*\big)\big) > 0. \tag{A.17}$$

We set $\lambda_r(z^*) = r$ and $\lambda_r(z) = 0$ for $z \neq z^*$. We define the following symbols, for integers $i < j$,

$$l_{[i,j[}(z) = \sum_{t=i}^{j-1} \mathbb{1}\{S_t - S_{i-1} = z\} \quad \text{and} \quad q_{[i,j[}(z) = \sum_{t=i}^{j-1} \eta(t)\mathbb{1}\{S_t - S_{i-1} = z\}. \tag{A.18}$$

Note that on disjoints sets $I_k = [i_k, j_k[$, the variables $\{(l_{I_k}, q_{I_k}), k \in \mathbb{N}\}$ are independent. Finally, we define and the following sets, for $i = 1, \dots, m$,

$$\begin{aligned} \mathcal{A}_n^{(i)} &= \left\{ l_{[(i-1)n, in[}|_{B(r)} = \lambda_n, \pi_\delta\left(\frac{q_{[(i-1)n, in[}}{l_{[(i-1)n, in[}}\right)\bigg|_{B(r)} = \kappa_n, S_{in} = 0 \right\}, \\ \mathcal{A}_s &= \big\{ l_{[mn, k[} = s\delta_{z^*}, \eta(i) > \kappa_n\big(z^*\big), \forall i \in [mn, k[\big\}. \end{aligned} \tag{A.19}$$

On the event $\{S_{in} = 0, \forall i = 0, \dots, m\}$, the local charges, and local times in $[0, k[$ on site z are, respectively,

$$q_k(z) = \sum_{i=1}^m q_{[(i-1)n, in[}(z) + q_{[mn, k[}(z) \quad \text{and} \quad l_k(z) = \sum_{i=1}^m l_{[(i-1)n, in[}(z) + l_{[mn, k[}(z). \tag{A.20}$$

We need to show now that

$$\bigcap_{i=1}^m \mathcal{A}_n^{(i)} \cap \mathcal{A}_s \subset \mathcal{A}_k(\xi, r, \delta). \tag{A.21}$$

In other words, we need to see that under $\bigcap_i \mathcal{A}_n^{(i)} \cap \mathcal{A}_s$, we have

$$\sum_{z\in B(r)} \big(m\lambda_n(z)+\lambda_s(z)\big)^2 \pi_\delta^2\left(\frac{q_k(z)}{m\lambda_n(z)+\lambda_s(z)}\right) \geq (\xi k)^2. \tag{A.22}$$

Note that under $\bigcap_i \mathcal{A}_n^{(i)} \cap \mathcal{A}_s$, for any $z \in B(r)$

$$\frac{q_{[(i-1)n,in[}(z)}{l_{[(i-1)n,in[}(z)} \in \big[\kappa_n(z), \kappa_n(z)+\delta\big[\quad \Longrightarrow \quad \frac{\sum_{i=1}^m q_{[(i-1)n,in[}(z)}{ml_{[(i-1)n,in[}(z)} \in \big[\kappa_n(z), \kappa_n(z)+\delta\big[. \tag{A.23}$$

Thus, if $s=0$, (A.22) would hold trivially.

We assume for simplicity that $z^*=0$, and postpone to Remark A.1 the general case. Note that for $z=z^*=0$

$$\frac{q_{[mn,k[(0)}}{\lambda_s(0)} \geq \kappa_n(0) \quad \text{and (A.23) imply that} \quad \frac{\sum_{i=1}^m q_{[(i-1)n,in[}(0)+q_{[mn,k[(0)}}{m\lambda_n(0)+s} \geq \kappa_n(0), \tag{A.24}$$

whereas for $z\neq 0$, $q_k(z)=q_{mn}(z)$ and $l_k(z)=l_{mn}(z)$, so that checking (A.22) reduces to checking

$$\begin{aligned}&\big(m\lambda_n(0)+s\big)^2\pi_\delta^2\left(\frac{\sum_{i=1}^m q_{[(i-1)n,in[}(0)+q_{[mn,k[(0)}}{m\lambda_n(0)+s}\right)\\ &\quad -\big(m\lambda_n(0)\big)^2\pi_\delta^2\left(\frac{\sum_{i=1}^m q_{[(i-1)n,in[}(0)}{m\lambda_n(0)}\right) \leq \big(k^2-(mn)^2\big)\xi^2.\end{aligned} \tag{A.25}$$

Using (A.24), it is enough to check that

$$\big(2m\lambda_n(0)+s\big)\kappa_n(0)^2 \geq (2mn+s)\xi^2. \tag{A.26}$$

Recall that (A.16) yields $\lambda_n(z^*)\kappa_n^2(z^*) \geq n\xi^2$ so that when $z^*=0$,

$$\lambda_n(0)\kappa_n(0)^2 \geq n\xi^2 \quad \text{and} \quad \kappa_n(0)^2 \geq \xi^2, \tag{A.27}$$

which implies (A.26) right away.

Now, (A.21) implies that for c, c' depending on δ, r and A, we have

$$\begin{aligned} P\big(\mathcal{A}_n(\xi,r,\delta)\big)^m P(\mathcal{A}_s) &\leq (cn)^{c'm} P\big(\mathcal{A}_n^{(1)}\big)\cdots P\big(\mathcal{A}_n^{(m)}\big)P(\mathcal{A}_s)\\ &\leq (cn)^{c'm} P\left(\bigcap_{i\leq m}\mathcal{A}_n^{(i)}\cap\mathcal{A}_s\right)\\ &\leq (cn)^{c'm} P\big(\mathcal{A}_k(\xi,r,\delta)\big).\end{aligned} \tag{A.28}$$

We now take the logarithm on each side of (A.28),

$$\frac{nm}{nm+s}\frac{\log(P(\mathcal{A}_n(\xi,r,\delta)))}{n} + \frac{\log(P(\mathcal{A}_s))}{k} \leq \frac{c'm\log(cn)}{nm+s} + \frac{\log(P(\mathcal{A}_k(\xi,r,\delta)))}{k}. \tag{A.29}$$

We take now the limit $k\to\infty$ while n is kept fixed (e.g., $m\to\infty$) so that

$$\frac{\log(P(\mathcal{A}_n(\xi,r,\delta)))}{n} \leq \frac{c'\log(cn)}{n} + \liminf_{k\to\infty}\frac{\log(P(\mathcal{A}_k(\xi,r,\delta)))}{k}. \tag{A.30}$$

By taking the limit sup in (A.30) as $n\to\infty$, we conclude that the limit in (5.5) exists.

Remark A.1. *We treat here the case $z^* \neq 0$. Note that this is related to the strategy on a single period of length s. If we could have that monomers in a piece of length s pile up on site z^*, then (A.21) would hold since it only uses that $\lambda_n(z^*)\kappa_n^2(z^*) \geq n\xi^2$. However, the walk starts at the origin, and each period of length n sees the walk returning to the origin. The idea is to insert a period of length s into the first time-period of length n at the first time the walk hits z^*. Then, since we still use a scenario with a single pile at site z^* with charges exceeding ξ, we should have*

$$l_{n+s} = \lambda_n + s\delta_{z^*}, \qquad \pi_\delta\left(\frac{q_{n+s}}{l_{n+s}}\right)\bigg|_{B(r)} \geq \kappa_n \quad \text{and} \quad S_{n+s} = 0. \tag{A.31}$$

More precisely, let $\tau^* = \inf\{n \geq 0 : \ S_n = z^*\}$, and note that

$$P\big(\mathcal{A}_n^{(1)}\big) = \sum_{i=0}^{n-1} P\big(\tau^* = i, \mathcal{A}_n^{(1)}\big). \tag{A.32}$$

Let $i^* < n$ be such that

$$P\big(\tau^* = i^*, \mathcal{A}_n^{(1)}\big) = \max_{i<n} P\big(\tau^* = i, \mathcal{A}_n^{(1)}\big).$$

Then,

$$P\big(\mathcal{A}_n^{(1)}\big) \leq n P\big(\tau^* = i^*, \mathcal{A}_n^{(1)}\big), \tag{A.33}$$

and, adding a subscript to P to explicit the starting point of the walk

$$\begin{aligned} &P_0\big(\mathcal{A}_n^{(1)}\big) P_0\big(l_{[0,s[} = s\delta_0, \eta(i) > \kappa_n\big(z^*\big), \forall i \in [0,s[\big) \\ &\quad \leq n P_0\big(\tau^* = i^*, \mathcal{A}_n^{(1)}\big) P_{z^*}\big(l_{[0,s[} = s\delta_{z^*}, \eta(i) > \kappa_n\big(z^*\big)\big) \\ &\quad \leq P_0\left(l_{[0,s[} = s, \pi_\delta\left(\frac{q_{n+s}}{l_{n+s}}\right)\bigg|_{B(r)} \geq \kappa_n, S_{n+s} = 0\right). \end{aligned} \tag{A.34}$$

A.3. *On a monotony property*

We prove in this section the following result which is a corollary of Lemma 5.3 of [6].

Corollary A.2. *For any integer n, assume that $\{\eta_j(i), j = 1, \ldots, n, i \in \mathbb{N}\}$ are independent symmetric variables, and for any sequence $\{n_j, n'_j, j = 1, \ldots, n\}$ with $n'_j \geq n_j$, and any $\xi > 0$, we have*

$$P\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n_j} \eta_j(i)\right)^2 > \xi\right) \leq P\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n'_j} \eta_j(i)\right)^2 > \xi\right). \tag{A.35}$$

Proof. We prove the result by induction. First, for $n = 1$, we use first the symmetry of the distribution of the η's and then Lemma 5.3 of [6] to have for $n'_1 \geq n_1$,

$$\begin{aligned} P\left(\left(\sum_{i=1}^{n_1} \eta_1(i)\right)^2 > \xi\right) &= 2P\left(\sum_{i=1}^{n_1} \eta_1(i) > \sqrt{\xi}\right) \leq 2P\left(\sum_{i=1}^{n'_1} \eta_1(i) > \sqrt{\xi}\right) \\ &\leq P\left(\left(\sum_{i=1}^{n'_1} \eta_1(i)\right)^2 > \xi\right). \end{aligned} \tag{A.36}$$

Now, assume that (A.35) is true for $n-1$, and call $\Gamma_j = (\eta_j(1) + \cdots + \eta_j(n_j))^2$ and Γ'_j the sum of the η_j up to n'_j. We only write the proof in the case where Γ_j has a density, say g_{Γ_j}. The case of a discrete distribution is trivially adapted. Then

$$\begin{aligned} P\left(\sum_{j=1}^{n} \Gamma_j > \xi\right) &= P(\Gamma_1 > \xi) + \int_0^{\xi} g_{\Gamma_1}(z) P\left(\sum_{j=2}^{n} \Gamma_j > \xi - z\right) \mathrm{d}z \\ &\le P(\Gamma_1 > \xi) + \int_0^{\xi} g_{\Gamma_1}(z) P\left(\sum_{j=2}^{n} \Gamma'_j > \xi - z\right) \mathrm{d}z \\ &= P\left(\Gamma_1 + \sum_{j=2}^{n} \Gamma'_j > \xi\right). \end{aligned} \tag{A.37}$$

Then, we rewrite the sum on the right-hand side of (A.37) $\Gamma_1 + (\Gamma'_2 + \cdots + \Gamma'_n) = \Gamma'_2 + (\Gamma_1 + \cdots + \Gamma'_n)$, and single out Γ'_2 in the first step of (A.37) to conclude. □

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