Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2011, Vol. 47, No. 1, 9–19
DOI: 10.1214/09-AIHP348

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

On Wiener–Hopf factors for stable processes

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Received 30 July 2009; revised 26 October 2009; accepted 29 October 2009

Abstract. We give a series representation of the logarithm of the bivariate Laplace exponent κ of α-stable processes for almost all $\alpha \in (0, 2]$.

Résumé. Nous donnons un développement en série du logarithme de l'exposant de Laplace bivarié κ des processus α-stables pour presque tous $\alpha \in (0, 2]$.

MSC: 60G51; 60E10

Keywords: Stable process; Wiener–Hopf factorization

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1. Introduction

The fluctuation theory of Lévy processes is one of the domains of probability very actively developing in the last years, and with important applications in mathematical finance; cf. the recent monograph of Kyprianou [12] and papers [2,5,6,13]. The α-stable Lévy processes play a primordial role in this theory. We address in this article one of the key problems of the Wiener–Hopf factorization theory of the α-stable processes: the computation of the bivariate Laplace exponent $\kappa(\gamma, \beta)$.

The aim of this paper is to give a series representation of the integral

$$g(\beta) = \frac{\sin(\pi\rho)}{\pi} \int_0^\infty \frac{\beta \log(1 + x^\alpha)}{x^2 + 2x\beta\cos(\pi\rho) + \beta^2}\,\mathrm{d}x \tag{1}$$

for almost all $\alpha \in (0, 2]$ and $\rho \in [1 - 1/\alpha, 1/\alpha] \cap (0, 1)$. This integral plays an important role in the theory of stable processes. Using (1) one may express the bivariate Laplace exponent $\kappa(\gamma, \beta)$ of the ascending ladder process built from the α-stable process X_t with index of stability α and $\rho = \mathbb{P}(X_1 > 0)$ (see, e.g., [3,12]). Namely

$$\gamma^\rho \exp\bigl\{g\bigl(\beta\gamma^{-1/\alpha}\bigr)\bigr\} = \kappa(\gamma, \beta) = k \exp\left\{\int_0^\infty \int_{(0,\infty)} \frac{\mathrm{e}^{-t} - \mathrm{e}^{-\gamma t}\mathrm{e}^{-\beta x}}{t}\mathbb{P}(X_t \in \mathrm{d}x)\,\mathrm{d}t\right\}.$$

The integral (1) was introduced by Darling in [7] for $\rho = 1/2$ and calculated in the case $\alpha = 1$ and $\rho = 1/2$, which corresponds to the symmetric Cauchy process and later by Bingham [4] for spectrally negative stable processes ($1/\rho = \alpha \in (1, 2)$). Doney in [8] calculated it for the set of parameters (α, ρ) satisfying $\rho + k = l/\alpha$ for some $k \in \mathbb{N} \cup \{0\}$ and $l \in \mathbb{N}$. Although the function κ plays an important role in the theory of stable (in general Lévy) processes, the only known closed expression for it is due to Doney. In this note we expand the function g to a power series for almost all

α and ρ. We denote by $\mathcal{L}$ the set of Liouville numbers, which will be defined in Section 2. Let $\mathcal{A} = (0, 2] \setminus (\mathbb{Q} \cup \mathcal{L})$. We note that if $\alpha \in \mathcal{A}$ then by Lemma 2, $1/\alpha \in \mathcal{A}$. The main result of this paper is

Theorem 1. *Let $\alpha \in \mathcal{A}$, $\rho \in [1 - 1/\alpha, 1/\alpha] \cap (0, 1)$ and $0 < \beta < 1$. Then*

$$g(\beta) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}\beta^m \sin(\rho m\pi)}{m \sin(m\pi/\alpha)} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\beta^{\alpha k} \sin(\rho\alpha k\pi)}{k \sin(\alpha k\pi)}. \tag{2}$$

We note that in view of Lemma 5 it suffices to consider only $0 < \beta < 1$. The series unfortunately does not converge for irrational numbers $\alpha \in \mathcal{L} \cap (0, 2)$, but $\mathcal{L}$ has a Lebesgue measure 0 hence $\mathcal{A}$ contains almost all $\alpha \in (0, 2)$. We obtain also a formula for rational α (Proposition 10) but the expression is not so closed as in Theorem 1. If $\rho + k = l/\alpha$ for some integers k and l the formula (2) may be simplified, in particular one may obtain results achieved by Doney in [8] (see Remark 1).

The formula (2) opens a way to applications for the study of various functionals of an α-stable Lévy process, in particular of the long time behavior of the supremum process or the law of the first passage time, cf. the recent results of Bernyk, Dalang and Peskir [2] and Kuznetsov [11]. We also profit from the Theorem 1 in a forthcoming work [9], devoted to the first passage time of symmetric stable processes.

The paper is organized as follows. In Section 2 we define Liouville numbers and prove some auxiliary lemmas. In Section 3 we prove the main Theorem 1. In Section 4 we give some remarks, applications and examples.

2. Liouville numbers

A number $x \in \mathbb{R}$ is called a Liouville number if it may be well approximated by rational numbers. More precisely for any $n \in \mathbb{N}$ there exist infinitely many pairs of integers p, q such that (see, e.g., [1])

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

We denote by $\mathcal{L}$ the set of all Liouville numbers. First we note the following lemma.

Lemma 2. *$x \in \mathcal{L}$ if and only if $1/x \in \mathcal{L}$.*

The proof does not seem available in the literature. The following proof was proposed by Waldschmidt [14].

Proof of Lemma 2. Let $x \notin \mathcal{L}$. There are $c \in \mathbb{R}$ and $d \in \mathbb{N}$ such that for all $p \in \mathbb{Z}, q \in \mathbb{N}$,

$$\left| x - \frac{p}{q} \right| \geq \frac{c}{q^d}.$$

Let $p \in \mathbb{Z}, q \in \mathbb{N}$. We may and do suppose that $|1/x - p/q| < 1$. Hence $|p|/q < (|x| + 1)/|x|$ and

$$\left| \frac{1}{x} - \frac{p}{q} \right| = \frac{|p|}{q|x|} \left| x - \frac{q}{p} \right| \geq \frac{|p|}{q|x|} \frac{c}{|p|^d} \geq \frac{|x|^{d-2}}{(1 + |x|)^{d-1}} \frac{c}{q^d}.$$ □

Lemma 3. *For any $x \in \mathcal{A}$ and $\beta \in (0, 1)$ we have*

$$\sum_{m=1}^{\infty} \frac{\beta^m}{|\sin(mx\pi)|} < \infty.$$

Proof. Since $x \in \mathbb{R} \setminus (\mathbb{Q} \cup \mathcal{L})$, there is $N \in \mathbb{N}$ such that $|x - \frac{p}{q}| > \frac{1}{q^N}$ for all integers $p, q > 0$. Hence $|\sin(mx\pi)| > \frac{1}{2m^{N-1}}$ and the lemma follows. □

In the sequel we will need following formulas taken from [10] (formulas 1.445.7, 1.422.3 and 1.353.1)

$$\sum_{m=1}^{\infty}(-1)^{m+1}\frac{m\sin(mz)}{m^2-w^2}=\frac{\pi}{2}\frac{\sin(zw)}{\sin(w\pi)},\quad z\in(-\pi,\pi),\,w\in\mathbb{R}\setminus\mathbb{Z}, \tag{3}$$

$$\frac{\pi}{\sin(\pi z)}=\frac{1}{z}-\sum_{k=1}^{\infty}\frac{(-1)^k 2z}{k^2-z^2},\quad z\in\mathbb{R}\setminus\mathbb{Z}, \tag{4}$$

$$\sum_{k=1}^{n-1}p^k\sin(kx)=\frac{p\sin(x)-p^n\sin(nx)+p^{n+1}\sin((n-1)x)}{1-2p\cos(x)+p^2}. \tag{5}$$

Lemma 4. *Let $\alpha\in\mathcal{A}$ and $\rho\in[1-1/\alpha,1/\alpha]\cap(0,1)$. Then there are constants C and N such that for all $M,k\in\mathbb{N}$*

$$\sum_{m=1}^{M}\frac{(-1)^m\sin(m\rho\pi)m}{m^2-(\alpha k)^2}\le Ck^N.$$

Proof. Let K be the smallest integer larger then $\alpha k+1$. Like in the proof of Lemma 3 we take N such that $|\alpha-\frac{p}{q}|>\frac{1}{q^N}$ for all integers p,q. Then $|m^2-(\alpha k)^2|>mk^{-N+1}$ for all $m,k\in\mathbb{N}$ and we get

$$\left|\sum_{m=1}^{K-1}\frac{(-1)^m\sin(m\rho\pi)m}{m^2-(\alpha k)^2}\right|\le(\alpha k+1)k^{N-1}\le 3k^N.$$

Denote $a_m=(-1)^m\sin(m\rho\pi)$ and $b_m=\frac{m}{m^2-(\alpha k)^2}$. By (5) for any $M\ge 1$ we have

$$\left|\sum_{m=1}^{M}a_m\right|\le\frac{3}{2(1+\cos(\rho\pi))}=c.$$

Since b_m is decreasing for $m\ge K$ we get

$$\begin{aligned}\left|\sum_{m=K}^{M}a_m b_m\right|&=\left|\sum_{m=K}^{M-1}(b_m-b_{m+1})\sum_{n=K}^{m}a_n+b_M\sum_{n=K}^{M}a_n\right|\le\sum_{m=K}^{M-1}(b_m-b_{m+1})\left|\sum_{n=K}^{m}a_n\right|+b_M\left|\sum_{n=K}^{M}a_n\right|\\&\le 2cb_K\le 2c.\end{aligned}$$

□

3. Proof of Theorem 1

The following lemma justifies our restriction in Theorem 1 to $0<\beta<1$.

Lemma 5.

$$\kappa(1,\beta)=\kappa(1,1/\beta)\beta^{\alpha\rho}.$$

Proof. After substituting $x=1/y$ we get

$$\begin{aligned}g(\beta)&=\beta\frac{\sin(\pi\rho)}{\pi}\int_0^{\infty}\frac{\log(1+y^\alpha)-\log(y^\alpha)}{1+2y\beta\cos(\pi\rho)+y^2\beta^2}\,\mathrm{d}y=g(1/\beta)+\frac{\alpha\sin(\pi\rho)}{\pi}\int_0^{\infty}\frac{\log(\beta)-\log(z)}{1+2z\cos(\pi\rho)+z^2}\,\mathrm{d}z\\&=g(1/\beta)+\frac{\alpha\log(\beta)}{\pi}\int_0^{\infty}\frac{\sin(\pi\rho)}{1+2z\cos(\pi\rho)+z^2}\,\mathrm{d}z=g(1/\beta)+\alpha\rho\log(\beta)\end{aligned}$$

and the lemma follows. □

A derivative of the function g is equal to

$$g'(\beta) = \frac{\partial}{\partial\beta}\frac{\sin(\pi\rho)}{\pi}\int_0^\infty \frac{\log(1+\beta^\alpha x^\alpha)}{x^2+2x\cos(\pi\rho)+1}\,\mathrm{d}x$$
$$= \frac{\sin(\pi\rho)\alpha}{\pi}\int_0^\infty \frac{x^\alpha}{1+x^\alpha}\frac{1}{x^2+2x\beta\cos(\pi\rho)+\beta^2}\,\mathrm{d}x.$$

Our aim is to prove the following lemmas.

Lemma 6. *Let $\alpha \in \mathcal{A}$, $\rho \in [1-1/\alpha, 1/\alpha] \cap (0,1)$ and $0<\beta<1$. Then*

$$g'(\beta) = \sum_{m=1}^\infty \frac{(-1)^{m+1}\beta^{m-1}\sin(\rho m\pi)}{\sin(m\pi/\alpha)} + \alpha\sum_{k=1}^\infty \frac{(-1)^{k+1}\beta^{\alpha k-1}\sin(\rho\alpha k\pi)}{\sin(\alpha k\pi)}.$$

Lemma 7. *For any $p>0$ and $0<b<1$*

$$\int_0^b \frac{y^p}{1+y}\,\mathrm{d}y = \sum_{k=0}^\infty \frac{(-1)^k b^{k+1+p}}{k+1+p}.$$

Proof. By Fubini theorem,

$$\int_0^b \frac{y^p}{1+y}\,\mathrm{d}y = \int_0^b \sum_{k=0}^\infty (-1)^k y^{p+k}\,\mathrm{d}y = \sum_{k=0}^\infty \frac{(-1)^k b^{k+1+p}}{k+1+p}.$$

□

Lemma 8. *For any $0<b\le 1$ and $p\in(0,\infty)\setminus\mathbb{N}$ we have*

$$\int_b^\infty \frac{y^{-p}}{1+y}\,\mathrm{d}y = \frac{\pi}{\sin(p\pi)} + \sum_{k=0}^\infty \frac{(-1)^{k+1} b^{k+1-p}}{k+1-p}. \tag{6}$$

Proof. Since the derivatives in b of both sides of (6) are equal we have for $b\in(0,1)$

$$\int_b^\infty \frac{y^{-p}}{1+y}\,\mathrm{d}y = C + \sum_{k=0}^\infty \frac{(-1)^{k+1} b^{k+1-p}}{k+1-p}.$$

To calculate the constant C we take $b\to 1$ and by (4) we get

$$C = \int_1^\infty \frac{y^{-p}}{1+y}\,\mathrm{d}y - \sum_{k=0}^\infty \frac{(-1)^{k+1}}{k+1-p} = \int_0^1 \frac{x^{p-1}}{1+x}\,\mathrm{d}x - \sum_{k=0}^\infty \frac{(-1)^{k+1}}{k+1-p}$$
$$= \sum_{k=0}^\infty (-1)^k \int_0^1 x^{p+k-1}\,\mathrm{d}x - \sum_{k=0}^\infty \frac{(-1)^{k+1}}{k+1-p} = \sum_{k=0}^\infty \frac{(-1)^k}{k+p} + \sum_{k=0}^\infty \frac{(-1)^k}{k+1-p}$$
$$= \frac{1}{p} - \sum_{k=1}^\infty (-1)^k \frac{2p}{k^2-p^2} = \frac{\pi}{\sin(p\pi)}.$$

□

Since for any $n\in\mathbb{N}$

$$\lim_{p\to n}\left(\frac{\pi}{\sin(p\pi)} + \frac{(-1)^n b^{n-p}}{n-p}\right) = (-1)^n \ln b,$$

we get the following corollary.

Corollary 9. *For $p \in \mathbb{N}$ and $0 < b < 1$,*

$$\int_b^\infty \frac{y^{-p}}{1+y}\,\mathrm{d}y = (-1)^p \ln b + \sum_{k\in\mathbb{N},k\neq p-1} \frac{(-1)^{k+1} b^{k+1-p}}{k+1-p}.$$

Proof of Lemma 6. We note that (see [10], formula 1.447.1)

$$\sum_{m=0}^\infty (-1)^m x^m \sin\bigl((m+1)z\bigr) = \frac{\sin(z)}{x^2+2x\cos(z)+1}, \quad |x|<1. \tag{7}$$

First we will calculate $\int_0^\beta$. From (5) we deduce

$$\sum_{k=1}^{n-1} (-1)^k p^k \sin(kz) = \frac{-p\sin(z) - (-1)^n p^n (p\sin((n-1)z) + \sin(nz))}{1+2p\cos(z)+p^2}.$$

Thus for any $M \geq 0$, $z \in (0,\pi)$ and $x \in (0,\beta)$

$$\begin{aligned}
&\left|\sum_{m=0}^M (-1)^m \sin\bigl((m+1)z\bigr)\left(\frac{x}{\beta}\right)^m\right| \\
&\quad = \left|\beta^2 \frac{(x/\beta)^{M+1}(-1)^M((x/\beta)\sin(z(M+1)) + \sin(z(M+2))) + \sin(z)}{x^2+2x\beta\cos(z)+\beta^2}\right| \\
&\quad < \frac{3}{\sin(z)^2}.
\end{aligned}$$

Hence by dominated convergence theorem, we get

$$\begin{aligned}
&\alpha\sin\rho\pi \int_0^\beta \frac{x^\alpha}{1+x^\alpha}\frac{1}{\beta^2+2x\beta\cos(\rho\pi)+x^2}\,\mathrm{d}x \\
&\quad = \alpha\sin\rho\pi \int_0^\beta \frac{x^\alpha}{1+x^\alpha}\frac{1}{\beta^2((x/\beta)^2+2(x/\beta)\cos(\rho\pi)+1)}\,\mathrm{d}x \\
&\quad = \alpha \int_0^\beta \frac{x^\alpha}{1+x^\alpha}\sum_{m=0}^\infty \beta^{-2}(-1)^m \sin\bigl(\rho(m+1)\pi\bigr)\left(\frac{x}{\beta}\right)^m \mathrm{d}x \\
&\quad = \alpha \int_0^\beta \lim_{M\to\infty}\sum_{m=0}^M \frac{x^\alpha}{1+x^\alpha}\beta^{-2}(-1)^m \sin\bigl(\rho(m+1)\pi\bigr)\left(\frac{x}{\beta}\right)^m \mathrm{d}x \\
&\quad = \sum_{m=0}^\infty (-1)^m \beta^{-2-m}\sin\bigl(\rho(m+1)\pi\bigr)\int_0^\beta \frac{\alpha x^{\alpha+m}}{1+x^\alpha}\,\mathrm{d}x.
\end{aligned}$$

By Lemma 7,

$$\int_0^\beta \frac{\alpha x^{\alpha+m}}{1+x^\alpha}\,\mathrm{d}x = \int_0^{\beta^\alpha} \frac{y^{(m+1)/\alpha}}{1+y}\,\mathrm{d}y = \sum_{k=0}^\infty \frac{(-1)^k \beta^{\alpha(k+1)+m+1}}{k+1+(m+1)/\alpha}.$$

Consequently

$$\begin{aligned}
&\alpha \sin \rho\pi \int_0^{\beta} \frac{x^\alpha}{1+x^\alpha} \frac{1}{\beta^2 + 2x\beta\cos(\rho\pi) + x^2}\, \mathrm{d}x \\
&\quad = \alpha \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+m} \beta^{\alpha(k+1)-1} \sin(\rho(m+1)\pi)}{\alpha(k+1)+(m+1)}.
\end{aligned} \tag{8}$$

Now we calculate $\int_\beta^\infty$. Similarly by the dominated convergence theorem we get

$$\begin{aligned}
&\alpha \sin(\rho\pi) \int_\beta^{\infty} \frac{x^\alpha}{1+x^\alpha} \frac{1}{\beta^2 + 2x\beta\cos(\rho\pi) + x^2}\, \mathrm{d}x \\
&\quad = \alpha \sin(\rho\pi) \int_\beta^{\infty} \frac{x^\alpha}{1+x^\alpha} \frac{1}{x^2(1+2(\beta/x)\cos(\rho\pi) + (\beta/x)^2)}\, \mathrm{d}x \\
&\quad = \alpha \int_\beta^{\infty} \frac{x^{\alpha-2}}{1+x^\alpha} \sum_{m=0}^{\infty} (-1)^m \left(\frac{\beta}{x}\right)^m \sin\big(\rho(m+1)\pi\big)\, \mathrm{d}x \\
&\quad = \sum_{m=0}^{\infty} (-1)^m \beta^m \sin\big(\rho(m+1)\pi\big) \int_\beta^{\infty} \frac{\alpha x^{\alpha-2-m}}{1+x^\alpha}\, \mathrm{d}x \\
&\quad = \sum_{m=0}^{\infty} (-1)^m \beta^m \sin\big(\rho(m+1)\pi\big) \int_{\beta^\alpha}^{\infty} \frac{y^{-(1+m)/\alpha}}{1+y}\, \mathrm{d}y.
\end{aligned}$$

Since $\alpha \in \mathcal{A}$ by Lemma 8 we get

$$\int_{\beta^\alpha}^{\infty} \frac{y^{-(1+m)/\alpha}}{1+y}\, \mathrm{d}y = \frac{\pi}{\sin(((m+1)/\alpha)\pi)} - \sum_{k=0}^{\infty} \frac{(-1)^k \beta^{\alpha(k+1)-(m+1)}}{k+1-(m+1)/\alpha}.$$

Therefore by Lemma 3

$$\begin{aligned}
&\alpha \sin \rho\pi \int_\beta^{\infty} \frac{x^\alpha}{1+x^\alpha} \frac{1}{\beta^2 + 2x\beta\cos(\rho\pi) + x^2}\, \mathrm{d}x \\
&\quad = \pi \sum_{m=0}^{\infty} \frac{(-1)^m \beta^m \sin(\rho(m+1)\pi)}{\sin(((m+1)/\alpha)\pi)} - \alpha \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+m} \beta^{\alpha(k+1)-1} \sin(\rho(m+1)\pi)}{\alpha(k+1)-(m+1)}.
\end{aligned} \tag{9}$$

Hence by (8), (9) and (3) we get

$$\begin{aligned}
&\frac{1}{\pi} \int_0^{\infty} \frac{x^\alpha}{1+x^\alpha} \frac{\alpha \sin \rho\pi}{\beta^2 + 2x\beta\cos(\rho\pi) + x^2}\, \mathrm{d}x \\
&\quad = \sum_{m=0}^{\infty} \frac{(-1)^m \beta^m \sin(\rho(m+1)\pi)}{\sin(((m+1)/\alpha)\pi)} + \frac{2\alpha}{\pi} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+m} \beta^{\alpha(k+1)-1} (m+1) \sin(\rho(m+1)\pi)}{(m+1)^2 - (\alpha(k+1))^2} \\
&\quad = \sum_{m=0}^{\infty} \frac{(-1)^m \beta^m \sin(\rho(m+1)\pi)}{\sin(((m+1)/\alpha)\pi)} + \alpha \sum_{k=0}^{\infty} \frac{(-1)^k \beta^{\alpha(k+1)-1} \sin(\alpha\rho(k+1)\pi)}{\sin(\alpha(k+1)\pi)}.
\end{aligned}$$

The change of order of summation in the second line is justified by Lemma 4 and the Lebesgue theorem. $\square$

Now Theorem 1 follows easily from Lemma 6.

4. Remarks and applications

Remark 1. *Put*

$$g_k(a,x)=\sum_{m=1}^{\infty}\frac{x^m U_{k-1}(\cos(m\pi a))}{m}, \tag{10}$$

where $U_k(x)$ are the Chebyshev polynomials of the second type (we put $U_{-1}\equiv 0$). If $\rho+k=l/\alpha$ (like in [8]), $l\geq 0$ and $k\geq 1$ we obtain for all $\alpha\in(0,2]$

$$g(\beta)=g_k\big(\alpha,(-1)^{l+1}\beta^{\alpha}\big)-g_l\big(1/\alpha,(-1)^{k+1}\beta\big), \tag{11}$$

we note that sums above correspond to the function f_k defined in [8].

Proof. First suppose $\alpha\in\mathcal{A}$ and $\rho=l/\alpha-k$. Since $U_k(\cos(x))=\frac{\sin((k+1)x)}{\sin x}$ we get (11) for all $\alpha\in\mathcal{A}$. Now for $\alpha\in(0,2]\setminus\mathcal{A}$ we take $\mathcal{A}\ni\alpha_n\to\alpha$ and $\rho_n=l/\alpha_n-k$. Passing to the limit we get (11) for $\alpha\in(0,2]$. ☐

Using formulas [10], formulas 1.342.4, 1.342.2 and 1.448.2,

$$U_{k-1}\big(\cos(z)\big)=\begin{cases}2\sum_{n=0}^{m}\cos\big((2n+1)z\big) & \text{for } k=2m+2,\\ 1+2\sum_{n=1}^{m}\cos(2nz) & \text{for } k=2m+1,\end{cases}$$

$$2\sum_{m=1}^{\infty}\frac{x^m\cos(mz)}{m}=-\log\big(x^2-2x\cos(z)+1\big),$$

the functions $g_k(a,x)$ may be expressed by finite sums

$$-g_k(a,x)=\begin{cases}\sum_{n=0}^{k/2-1}\log\big(x^2-2x\cos\big((2n+1)a\pi\big)+1\big) & \text{for even } k,\\ \log(1-x)+\sum_{n=1}^{(k-1)/2}\log\big(x^2-2x\cos(2na\pi)+1\big) & \text{for odd } k.\end{cases}$$

Example 1. *Let $k=l=1$ then $\alpha\in(0,1)$ and*

$$g(\beta)=-\sum_{m=1}^{\infty}\beta^m/m+\sum_{m=1}^{\infty}\beta^{\alpha m}/m=-\log\big(1-\beta^{\alpha}\big)+\log(1-\beta).$$

Hence $\kappa(1,\beta)=\tilde{C}\frac{1-\beta}{1-\beta^{\alpha}}$.

A first application of Theorem 1 is to obtain new expressions for the functions $g'(\beta)$, $g(\beta)$ and consequently $\kappa(1,\beta)$ and $\kappa(\gamma,\beta)$ for the values of β not concerned by the results of [8].

Proposition 10. *Let $\alpha\in\mathbb{Q}\cap(0,2]$ and $\beta\in(0,1)$. Then*

$$\begin{aligned}g'(\beta)={}&\sum_{m=1,m/\alpha\notin\mathbb{N}}^{\infty}\frac{(-1)^{m+1}\beta^{m-1}\sin(\rho m\pi)}{\sin(m\pi/\alpha)}+\alpha\sum_{k=1,\alpha k\notin\mathbb{N}}^{\infty}\frac{(-1)^{k+1}\beta^{\alpha k-1}\sin(\rho\alpha k\pi)}{\sin(\alpha k\pi)}\\ &+\frac{\alpha\log(\beta)}{\pi}\sum_{m=1,m/\alpha\in\mathbb{N}}^{\infty}(-1)^{m+m/\alpha}\beta^{m-1}\sin(\rho m\pi)+\alpha\rho\sum_{k=1,\alpha k\in\mathbb{N}}^{\infty}(-1)^{k(\alpha+1)}\beta^{\alpha k-1}\cos(\alpha\rho k\pi).\end{aligned} \tag{12}$$

Proof. Let $\alpha = \frac{p}{q}$. Like in Remark 1 we take $\mathcal{A} \ni \alpha_j = \frac{p}{q} + \frac{\sqrt{2}}{j}$. We obtain result by passing to the limit $j \to \infty$ in the expression

$$\sum_{m=1, m/\alpha \notin \mathbb{N}}^{\infty} \frac{(-1)^{m+1}\beta^{m-1}\sin(\rho m\pi)}{\sin(m\pi/\alpha_j)} + \alpha_j \sum_{k=1,\alpha k \notin \mathbb{N}}^{\infty} \frac{(-1)^{k+1}\beta^{\alpha_j k-1}\sin(\rho\alpha_j k\pi)}{\sin(\alpha_j k\pi)}$$
$$+ \sum_{m=1, m/\alpha \in \mathbb{N}}^{\infty} \frac{(-1)^{m+1}\beta^{m-1}\sin(\rho m\pi)}{\sin(m\pi/\alpha_j)} + \alpha_j \sum_{k=1,\alpha k \in \mathbb{N}}^{\infty} \frac{(-1)^{k+1}\beta^{\alpha_j k-1}\sin(\rho\alpha_j k\pi)}{\sin(\alpha_j k\pi)}.$$

By Lemma 3 we pass with limit under sum signs. The first two terms obviously converge to the first two terms in (12). If we take $m = np$, $k = nq$, the second line is equal to

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^{np+1}\beta^{np-1}\sin(\rho np\pi)}{\sin(np\pi/\alpha_j)} + \alpha_j \frac{(-1)^{nq+1}\beta^{nq\alpha_j-1}\sin(\rho nq\alpha_j\pi)}{\sin(nq\alpha_j\pi)}\right)$$
$$\xrightarrow{j\to\infty} \sum_{n=1}^{\infty} \frac{(-1)^{nq+1}(-\beta)^{np-1}p(\pi\rho\cos(np\pi\rho) + \log(\beta)\sin(np\pi\rho))}{\pi q} \tag{13}$$

and the assertion of the proposition holds. For the detailed proof of (13) we refer to the Appendix. □

Remark 2. *In fact Proposition 10 holds for $\alpha \in \mathcal{A} \cup (\mathbb{Q} \cap (0, 2])$ and (12) may be treated as a generalization of Lemma 6.*

Example 2. *Let $\alpha = 1/2$. Then $p = 1$ and $q = 2$. We get*

$$g'(\beta) = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\beta^{k-1/2}\sin\left(\rho\left(k+\frac{1}{2}\right)\pi\right)$$
$$+ \frac{1}{2\pi}\sum_{n=1}^{\infty}(-1)^n\beta^{n-1}\big(\rho\pi\cos(n\rho\pi) + \log(\beta)\sin(n\rho\pi)\big)$$
$$= \frac{(1+\beta)\cos((\pi\rho)/2)/(2\sqrt{\beta}) - \rho(\beta+\cos(\pi\rho))/2 - \log(\beta)\sin(\pi\rho)/\pi}{\beta^2 + 2\beta\cos(\pi\rho) + 1}.$$

Analogous simple expressions can be given for other rational α not covered by the results of [8].

Further applications of formula (2) from Theorem 1 are planned in the forthcoming paper [9] where symmetric α-stable processes X_t in $\mathbb{R}$ are considered. The starting point is the formula (see [12])

$$\int_0^{\infty}\int_0^{\infty} \mathrm{e}^{-\eta t}\mathrm{e}^{-\theta x}\mathbb{E}_x\big(\mathrm{e}^{-\gamma X_t};\ \tau > t\big)\,\mathrm{d}t\,\mathrm{d}x = \frac{1}{(\theta+\gamma)\kappa(\eta,\gamma)\kappa(\eta,\theta)}, \tag{14}$$

where $\tau = \tau_{(0,\infty)}$ is the first exit time from $(0, \infty)$ of the process X_t.

A better knowledge of κ then permits to get from (14) more information about the law of τ.

Appendix

Here we give a detailed proof of (13).

Lemma 11. *Let* $\alpha_j = \frac{p}{q} + \frac{\sqrt{2}}{j}$. *We have*

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^{np+1}\beta^{np-1}\sin(\rho np\pi)}{\sin(np\pi/\alpha_j)} + \alpha_j\frac{(-1)^{nq+1}\beta^{nq\alpha_j-1}\sin(\rho nq\alpha_j\pi)}{\sin(nq\alpha_j\pi)}\right)$$

$$\xrightarrow{j\to\infty}\sum_{n=1}^{\infty}\frac{(-1)^{nq+1}(-\beta)^{np-1}p(\pi\rho\cos(np\pi\rho)+\log(\beta)\sin(np\pi\rho))}{\pi q}. \tag{15}$$

Proof. Let us call

$$F_1(n,j) = \frac{(-1)^{np+1}\sin(\rho np\pi)}{\sin(np\pi/\alpha_j)} + \alpha_j\frac{(-1)^{nq+1}\sin(\rho np\pi)}{\sin(nq\alpha_j\pi)},$$

$$F_2(n,j) = \alpha_j\frac{(-1)^{nq+1}(\sin(\rho nq\alpha_j\pi)-\sin(\rho np\pi))}{\sin(nq\alpha_j\pi)},$$

$$F_3(n,j) = \alpha_j\frac{(-1)^{nq+1}\sin(\rho nq\alpha_j\pi)}{\sin(nq\alpha_j\pi)}\left(\beta^{nq\alpha_j-np}-1\right).$$

The proof of Lemma 13 consists of two parts:

(1) *Term by term convergence:* We note that

$$\sin\left(\frac{np\pi}{\alpha_j}\right) = (-1)^{nq+1}\sin\left(\frac{nq^2\sqrt{2}\pi}{pj+\sqrt{2}}\right), \qquad \sin(nq\alpha_j\pi) = (-1)^{np}\sin\left(\frac{nq\sqrt{2}\pi}{j}\right).$$

Hence for fixed n and large j we have

$$\left|\frac{F_1(n,j)}{\sin(\rho np\pi)}\right| = \left|\frac{\sin(nq\sqrt{2}\pi/j)-((pj+\sqrt{2}q)/(qj))\sin(nq^2\sqrt{2}\pi/(pj+\sqrt{2}q))}{\sin(nq\sqrt{2}\pi/j)\sin(nq^2\sqrt{2}\pi/(pj+\sqrt{2}q))}\right|$$

$$\le \frac{\sum_{k=1}^{\infty}(1/(2k+1)!)((nq\sqrt{2}\pi/j)^{2k+1}+(2p/q)(nq^2\sqrt{2}\pi/(pj+\sqrt{2}q))^{2k+1})}{(nq\sqrt{2}\pi/(2j))(nq^2\sqrt{2}\pi/(2(pj+\sqrt{2}q)))} \le \frac{Kn}{j}\xrightarrow{j\to\infty}0,$$

where K is some constant independent of n and j. Therefore $\lim_{j\to\infty}F_1(n,j)=0$. Further

$$\lim_{j\to\infty}F_2(n,j) = \lim_{j\to\infty}2\alpha_j(-1)^{n(p+q)+1}\frac{\sin(\rho nq\sqrt{2}\pi/(2j))\cos(\rho n\pi(q\alpha_j+p)/2)}{\sin(nq\sqrt{2}\pi/j)}$$

$$= \frac{(-1)^{n(p+q)+1}p\rho\cos(\rho np\pi)}{q}.$$

Similarly

$$\lim_{j\to\infty}F_3(n,j) = \lim_{j\to\infty}\alpha_j(-1)^{n(p+q)+1}\sin(\rho nq\alpha_j\pi)\frac{(\beta^{nq\sqrt{2}/j}-1)}{\sin(nq\sqrt{2}\pi/j)}$$

$$= \frac{(-1)^{n(p+q)+1}p\log(\beta)\sin(np\pi\rho)}{\pi q}.$$

(2) *Uniform integrability with respect to the measure* $\mu=\sum_{n=1}^{\infty}\beta^{np-1}\delta_n$*:* We will show that for each $k=1,2,3$, we have

$$\sup_{j\in\mathbb{N}}\sum_{n=1}^{\infty}\left|F_k(n,j)\right|\beta^{np-1}<\infty,$$

and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{j\in\mathbb{N}} \sum_{n\in G} \left|F_k(n,j)\right|\beta^{np-1} < \varepsilon,$$

whenever $\mu(G) < \delta$.

From part (1) of the proof we see that for $n < j/(2q\sqrt{2})$, we have $F_k(n,j) < C$, where C does not depend on n, j and $k = 1, 2, 3$. Denote $G_j = \{m \in \mathbb{N}\colon\ m < j/(2q\sqrt{2})\}$. Let $k \in \mathbb{N}$ be the closest integer to $nq\sqrt{2}/j$. Then by Diophantine approximation

$$\begin{aligned}\left|\sin(nq\alpha_j\pi)\right| &= \left|\sin\big((k - nq\sqrt{2}/j)\pi\big)\right| \geq \frac{|k - nq\sqrt{2}/j|}{2}\\ &= \frac{nq}{2j}\left|\sqrt{2} - \frac{kj}{nq}\right| \geq \frac{nq}{j}\frac{c_1}{(nq)^2} = \frac{c_2}{nj}.\end{aligned}$$

Similarly, we show that

$$\left|\sin(np\pi/\alpha_j)\right| = \left|\sin\left(\frac{nq^2\sqrt{2}\pi}{pj + \sqrt{2}q}\right)\right| \geq \frac{c_3}{nj}.$$

Therefore

$$\sup_{j\in\mathbb{N}} \sum_{n=1}^{\infty} \left|F_1(n,j)\right|\beta^{np-1} \leq \sup_{j\in\mathbb{N}}\left(\sum_{n\in G_j} C\beta^{np-1} + c\sum_{n\in\mathbb{N}\setminus G_j} nj\beta^{np-1}\right) < \infty.$$

Now let $\varepsilon > 0$. First we note that

$$b_j = \sum_{n\in\mathbb{N}\setminus G_j} nj\beta^{np-1} \to 0 \quad \text{if } j \to \infty.$$

Hence there is $j_0 \in \mathbb{N}$ such that for $j > j_0$ we have $b_j < \varepsilon/3$. We take $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty} n\beta^{np-1} < \varepsilon/(3cj_0)$ and $\sum_{n=n_0}^{\infty} \beta^{np-1} < \varepsilon/(3C)$. Now let $\delta = \beta^{n_0p-1}$. If $\mu(G) < \delta$ then $G \subset \{n_0, n_0+1, \ldots\}$ and

$$\begin{aligned}\sup_{j\in\mathbb{N}} \sum_{n\in G} \left|F_1(n,j)\right|\beta^{np-1} &\leq \sup_{j\in\mathbb{N}} \sum_{n\in G\cap G_j} C\beta^{np-1} + \sup_{j\in\mathbb{N}} c \sum_{n\in G\setminus G_j} nj\beta^{np-1}\\ &\leq C\sum_{n\in G}\beta^{np-1} + c\sup_{j>j_0}\sum_{n\in\mathbb{N}\setminus G_j} nj\beta^{np-1} + c\sum_{n\in G} nj_0\beta^{np-1}\\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

In the same way we prove uniform integrability of $F_2(n,j)$ and $F_3(n,j)$ and we obtain the assertion of the lemma. □

Acknowledgements

We thank Zbigniew Palmowski for introducing us into the subject. We also thank Michel Waldschmidt for discussions about this paper. P. Graczyk was partially supported by Grants MNiSW N N201 373136 and ANR-09-BLAN-0084-01. T. Jakubowski was partially supported by Grants MNiSW N N201 397137, ANR-09-BLAN-0084-01 and the fellowship of CCRRDT Pays de la Loire.

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