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A MARKOVIAN SLOT MACHINE AND PARRONDO'S PARADOX

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The antique Mills Futurity slot machine has two unusual features. First, if a player loses 10 times in a row, the 10 lost coins are returned. Second, the payout distribution varies from coup to coup in a manner that is nonrandom and periodic with period 10. It follows that the machine is driven by a 100-state irreducible period-10 Markov chain. Here, we evaluate the stationary distribution of the Markov chain, and this leads to a strong law of large numbers and a central limit theorem for the sequence of payouts. Following a suggestion of Pyke [In *Mathematical Statistics and Applications: Festschrift for Constance van Eeden* (2003) 185–216 Institute of Mathematical Statistics], we address the question of whether there exists a two-armed version of this "one-armed bandit" that obeys Parrondo's paradox. More precisely, is there such a machine with the property that the casino can honestly advertise that both arms are fair, yet when players alternate arms in certain random or nonrandom ways, the casino makes money in the long run? The answer is a qualified yes. Although this "history-dependent" game is conceptually simpler than the original such games of Parrondo, Harmer and Abbott [*Phys. Rev. Lett.* (2000) **85** 5226–5229], it is nearly as complicated analytically, and open problems remain.

1. Introduction. The Futurity slot machine, a 1936 design of Mills Novelty Company of Chicago, has two unusual features, one readily apparent and the other less so. The readily apparent feature is that, if the player loses 10 times in a row, the 10 lost coins are returned. At the top of the machine is a pointer that indicates the number of consecutive losses incurred. It advances by 1 after each loss, and resets at 0 after a win or after 10 consecutive losses. The less apparent feature is that there are 20 symbols on each of the three reels but only the ones in even-numbered positions can appear on the payline if the machine is in mode E, while only the ones in odd-numbered positions can appear on the payline if the machine is in mode O. The mode is nonrandom and is determined by a cam that rotates through 10 positions, advancing one position with each coup and resulting in a specific mode pattern of length 10, EEEEEOEEEO, which is repeated ad infinitum. (Note that we could substitute any cyclic permutation of this mode pattern, such as EEEOEEEEEO, without effect.) When in mode E, the machine is extremely

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"tight" (i.e., the mean payout from a one-coin bet is much less than 1). When in mode O, it is extremely "loose."

There are several questions that might be asked. Does the sequence of payouts obey the strong law of large numbers and the central limit theorem, as it would for a traditional slot machine for which the sequence can be assumed independent and identically distributed? If so, what are the mean and variance parameters? What is the asymptotic probability of a nonzero payout? How frequently does the player lose 10 times in a row, thereby receiving the so-called Futurity award? Are there advantageous opportunities depending on the information available to the player about the state of the machine?

Notice that the machine is driven by a Markov chain with state space $\Sigma := \{0, 1, \dots, 9\} \times \{0, 1, \dots, 9\}$ interpreted as follows. The machine is in state (i, j) if the cam position is i and the pointer position is j. (If the cam position is 5 or 9, the machine is in mode O; if the cam position is 0–4 or 6–8, the machine is in mode E.) If we kept track of the mode (E or O) instead of the cam position (0–9), we would lose the Markov property. There is also a pointer position 10, but from that position the pointer instantly moves to position 0, so we can ignore pointer position 10. By evaluating the stationary distribution of this Markov chain, we can infer the long-term behavior of the slot machine. Specifically, we can establish a strong law of large numbers and a central limit theorem for the sequence of payouts.

The Futurity came to our attention via articles of Geddes (1980) and Geddes and Saul (1980) that appeared in *Loose Change*, a magazine for collectors of antique slot machines (published 1977–1998 and archived at the UNLV Lied Library). Geddes and Saul used Monte Carlo simulation to study the Futurity, claiming that an analytical solution "falls somewhere between formidable and monumental on a relative scale of mathematical difficulty." As we will see, the claim is untrue.

Parrondo's paradox can be regarded as the observation that there exist two fair games that can be combined, by either random mixture or nonrandom alternation, to create an unfair game. See the survey articles by Harmer and Abbott (2002), Parrondo and Dinís (2004), Epstein (2007) and Abbott (2009). To motivate his discussion of the paradox, Pyke (2003) raised the following question without providing an explicit answer.

> You are about to play a two-armed slot machine. The casino that owns this two-armed bandit advertises that both arms on their two-armed machines are "fair" in the sense that any player who plays either of the arms is assured that the average cost per play approaches zero as the number of plays increases. However, the casino does not constrain you to stay with one arm; you are allowed to use either arm on every play. [...] The question of interest in this context would be whether it is possible for the casino to still make money using only "fair" games.

Our aim here is to formulate a two-armed version of the Mills Futurity that answers Pyke's question affirmatively. The feature of the Futurity that permits Parrondian behavior is the Futurity award (the return of the 10 lost coins after 10 consecutive losses); the periodicity of the payout distribution is not important.

This "history-dependent" bonus feature makes our hypothetical two-armed slot machine not unlike the history-dependent games introduced by Parrondo, Harmer and Abbott (2000). In fact, it has some advantages over the original such games: It is conceptually simpler and less contrived. On the other hand, it is nearly as complicated analytically.

Actually, our answer to Pyke's question must be qualified. It is an unqualified yes for the random-mixture strategies. It is a qualified yes for the nonrandom-alternation strategies because certain assumptions are needed and our conclusions rely on an unproved conjecture. And the answer is simply no if the player's strategy is completely unrestricted because there exist strategies that actually give the player an advantage. In particular, our two-armed version of the Futurity is not ready for casino play.

We should clarify how it works. The player can pull either arm at each coup. After 10 consecutive losses, regardless of the order of play of the two arms, the 10 lost coins are returned to the player. On the other hand, each arm has its own cam mechanism, each with 10 positions, hence its own periodic pattern of payout distributions (though the payout distribution need not vary). The cam position for an arm advances only when that arm is pulled. Indeed, if this were not the case and both cam positions advanced when either arm was pulled, astute players would simply pull the arm with the higher mean payout, and the casino would be beaten at its own game. Of course, there is nothing special about the number 10 in this context, so we replace it throughout by the integer $J \geq 2$.

The question of whether Parrondo's paradox can appear in the casino setting was raised by Harmer and Abbott (2002), Section 2.3.3. Our example shows that the potential exists, even though it will not likely be realized. However, in our case the winning game created from two fair games is winning for the casino, not for the player. If it were the other way around, the casino would likely discontinue the game or change the rules.

In a previous paper [Ethier and Lee (2009)], the authors formulated a general version of Parrondo's games. The results of that paper do not immediately apply here because the present underlying irreducible Markov chain is periodic. Even if that issue could be overcome, the Markov chain here is rather complicated relative to the three- and four-state chains that were studied in the previous paper. It is therefore preferable to use a different approach here that avoids having to evaluate the fundamental matrix and spectral representation associated with the one-step transition matrix of the Markov chain.

2. The Markov chain at equilibrium.

We will analyze a generalized (one-armed) version of the Futurity, dependent on several parameters. In Section 5, we will substitute the actual numbers.

We assume that the cam controlling the payout distribution has I positions, denoted by $0, 1, \dots, I-1$. When in cam position i, the probability of a nonzero payout is p_i, the mean payout is μ_i and the variance of the payout is σ_i^2; none of these parameters takes the Futurity award into account. As for the Futurity award, we assume that, if the player loses J times in a row, the J lost coins are returned. A pointer that indicates the number of consecutive losses advances by 1 after each loss, and resets at 0 after a win or after J consecutive losses.

If we were interested solely in the Futurity, we would take $I = J$ and simplify matters considerably. However, in studying Parrondo's paradox for a two-armed version of the Futurity, it will be necessary to allow I in the generalized one-armed machine to be an integer multiple of J, say $I = dJ$ for a positive integer d. Of course, the case $d = 1$ is included and is in fact of primary interest.

The Markov chain $\{(X_n, Y_n)\}_{n\geq 0}$ that drives (or controls) the generalized (one-armed) Futurity has state space $\Sigma := \{0, 1, \ldots, I-1\} \times \{0, 1, \ldots, J-1\}$. It is in state (i, j) at time n if the cam position is i and the pointer position is j following the nth coup. The transition probabilities have a very simple form:

$$\begin{aligned} P((i,j),(k,l)) &:= \mathrm{P}\big((X_{n+1}, Y_{n+1}) = (k,l) \mid (X_n, Y_n) = (i,j)\big) \\ &= \begin{cases} p_i & \text{if } (k,l) = \big(i+1 \ (\mathrm{mod}\, I), 0\big) \text{ and } j \leq J-2, \\ q_i & \text{if } (k,l) = \big(i+1 \ (\mathrm{mod}\, I), j+1\big) \text{ and } j \leq J-2, \\ 1 & \text{if } (k,l) = \big(i+1 \ (\mathrm{mod}\, I), 0\big) \text{ and } j = J-1, \end{cases} \end{aligned}$$

where $0 < p_i < 1$ and $q_i := 1 - p_i$ for $i = 0, 1, \ldots, I-1$. We notice that the one-step transition matrix $\mathbf{P}$ is irreducible and periodic with period I.

THEOREM 1. *The unique stationary distribution $\boldsymbol{\pi}$ for the Markov chain in Σ with one-step transition matrix $\mathbf{P}$ is given recursively by*

$$\pi(i,0) = \frac{p_{i-1} + q_{i-1}\cdots q_{i-J}p_{i-J-1} + \cdots + q_{i-1}\cdots q_{i-(d-1)J}p_{i-(d-1)J-1}}{I(1-Q)} \tag{1}$$

for $i = 0, 1, \ldots, I-1$,

$$\pi(i,1) = q_{i-1}\pi(i-1,0), \qquad i = 0, 1, \ldots, I-1, \tag{2}$$

$$\pi(i,2) = q_{i-1}\pi(i-1,1), \qquad i = 0, 1, \ldots, I-1, \tag{3}$$

$$\vdots$$

$$\pi(i,J-1) = q_{i-1}\pi(i-1,J-2), \qquad i = 0, 1, \ldots, I-1, \tag{4}$$

where $Q := q_0q_1\cdots q_{I-1}$, $p_{-i} := p_{I-i}$ *and* $q_{-i} := q_{I-i}$ *for* $i = 1, 2, \ldots, I$, *and* $\pi(-1, j) := \pi(I-1, j)$ *for* $j = 0, 1, \ldots, J-1$. *Furthermore*,

$$\pi(i,0) + \pi(i,1) + \cdots + \pi(i,J-1) = \frac{1}{I}, \qquad i = 0, 1, \ldots, I-1. \tag{5}$$

REMARK. In the special case $I = J$ (i.e., $d = 1$), (1) and (4) simplify to

$$\pi(i,0) = \frac{p_{i-1}}{J(1-Q)}, \qquad \pi(i, J-1) = \frac{p_i Q}{q_i J(1-Q)}.$$

PROOF OF THEOREM 1. The stationary distribution is the unique probability (row) vector $\boldsymbol{\pi}$ satisfying

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}. \tag{6}$$

Equations (2)–(4) are immediate from this. This reduces the problem to a system of I linear equations in I variables, $\pi(i,0)$, $i = 0, 1, \dots, I-1$. The system is a rather complicated one, so we take a different approach, noticing that these probabilities can be obtained probabilistically.

If the Markov chain has the stationary distribution as its initial distribution, it is a stationary process, and we can extend its time parameter to the set of all integers. Intuitively, we can assume that the machine has been operating forever. What is the probability that, at a particular time, the Markov chain is in state $(i, 0)$? First the cam position must be i, the probability of which is $1/I$. Second, either the last coup resulted in a win (conditional probability p_{i-1}) or the last coup completed a string of J or $2J$ or $3J$ or ... consecutive losses, causing the pointer to reset at 0 and the Futurity award to be paid. Thus, the conditional probability that the pointer position is 0, given that the cam position is i, is

$$\begin{aligned}
&p_{i-1} + q_{i-1}\cdots q_{i-J} p_{i-J-1} + q_{i-1}\cdots q_{i-2J} p_{i-2J-1} + \cdots \\
&\quad + q_{i-1}\cdots q_{i-dJ} p_{i-dJ-1} + q_{i-1}\cdots q_{i-(d+1)J} p_{i-(d+1)J-1} + \cdots \\
&\qquad = p_{i-1}(1 + Q + Q^2 + \cdots) + q_{i-1}\cdots q_{i-J} p_{i-J-1}(1 + Q + Q^2 + \cdots) + \cdots \\
&\qquad\quad + q_{i-1}\cdots q_{i-(d-1)J} p_{i-(d-1)J-1}(1 + Q + Q^2 + \cdots) \\
&\qquad = \big(p_{i-1} + q_{i-1}\cdots q_{i-J} p_{i-J-1} + \cdots \\
&\qquad\quad + q_{i-1}\cdots q_{i-(d-1)J} p_{i-(d-1)J-1}\big)/(1-Q),
\end{aligned}$$

where $p_{i-mI} := p_i$ for all $i \in \{0, 1, \dots, I-1\}$ and $m \ge 1$, and similarly for complementary probabilities q_{i-mI}. This implies (1).

This argument is a bit heuristic [since we essentially assumed (5), one of the conclusions of the theorem], but now we can make it rigorous. First, we verify that $\boldsymbol{\pi}$, given by (1)–(4), is a probability vector by proving (5). Using (2)–(4) and then (1), the left-hand side of (5) is equal to

$$\begin{aligned}
&\pi(i,0) + q_{i-1}\pi(i-1,0) + q_{i-1}q_{i-2}\pi(i-2,0) + \cdots \\
&\quad + q_{i-1}\cdots q_{i-J+1}\pi(i-J+1,0) \\
&\qquad = \big[p_{i-1} + q_{i-1}\cdots q_{i-J} p_{i-J-1} + \cdots \\
&\qquad\quad + q_{i-1}\cdots q_{i-(d-1)J} p_{i-(d-1)J-1}
\end{aligned}$$

$$
\begin{aligned}
&+ q_{i-1}\big(p_{i-2} + q_{i-2}\cdots q_{i-J-1}p_{i-J-2} + \cdots \\
&\qquad + q_{i-2}\cdots q_{i-(d-1)J-1}p_{i-(d-1)J-2}\big) + \cdots \\
&+ q_{i-1}\cdots q_{i-J+1}\big(p_{i-J} + q_{i-J}\cdots q_{i-2J+1}p_{i-2J} + \cdots \\
&\qquad\qquad + q_{i-J}\cdots q_{i-dJ+1}p_{i-dJ}\big)\big]/[I(1-Q)] \\
&= \frac{p_{i-1} + q_{i-1}p_{i-2} + q_{i-1}q_{i-2}p_{i-3} + \cdots + q_{i-1}\cdots q_{i-dJ+1}p_{i-dJ}}{I(1-Q)} \\
&= \frac{1 - q_{i-1}\cdots q_{i-dJ}}{I(1-Q)} = \frac{1}{I},
\end{aligned}
$$

where the second equality amounts to a rearrangement of terms, and the third equality is an algebraic identity.

Next, for (6) it will suffice to show, for $i = 0, 1, \ldots, I-1$, that

$$\pi(i,0) = p_{i-1}[\pi(i-1,0) + \cdots + \pi(i-1,J-2)] + \pi(i-1,J-1).$$

This can be rewritten, using (5) and (2)–(4), as

$$
\begin{aligned}
\pi(i,0) &= p_{i-1}[\pi(i-1,0) + \cdots + \pi(i-1,J-1)] + q_{i-1}\pi(i-1,J-1) \\
&= \frac{p_{i-1}}{I} + q_{i-1}\cdots q_{i-J}\pi(i-J,0).
\end{aligned}
$$

Fix i and substitute (1). It is enough that

$$
\begin{aligned}
&p_{i-1} + q_{i-1}\cdots q_{i-J}p_{i-J-1} + \cdots + q_{i-1}\cdots q_{i-(d-1)J}p_{i-(d-1)J-1} \\
&\quad = (1-Q)p_{i-1} + q_{i-1}\cdots q_{i-J}\big(p_{i-J-1} + q_{i-J-1}\cdots q_{i-2J}p_{i-2J-1} + \cdots \\
&\qquad\qquad + q_{i-J-1}\cdots q_{i-dJ}p_{i-dJ-1}\big).
\end{aligned}
$$

Canceling like terms, this reduces to $p_{i-1} = (1-Q)p_{i-1} + Qp_{i-1}$, which proves that $\boldsymbol{\pi}$, defined by (1)–(4), is the stationary distribution for $\mathbf{P}$. □

At equilibrium, what is the probability p° that, at a particular coup, the player wins the J-coin Futurity award by losing for the Jth (or $2J$th or $3J$th or ...) consecutive time? This happens if and only if the Markov chain is in state $(i, J-1)$ for some $i \in \{0, 1, \ldots, I-1\}$ just before the specified coup *and* that coup results in a loss. Using (1)–(4), the probability is

$$p^\circ = \sum_{i=0}^{I-1} \pi(i, J-1)q_i = \frac{1}{I(1-Q)} \sum_{i=0}^{I-1}\sum_{k=1}^{d} q_i \cdots q_{i-kJ+1}p_{i-kJ}. \tag{7}$$

Notice that the last of the d terms in the inner sum is Qp_i.

Therefore the mean payout, at equilibrium, is

$$\mu^* := \frac{1}{I}\sum_{i=0}^{I-1}\mu_i + Jp^\circ. \tag{8}$$

Incidentally, in the special case $I = J$ (i.e., $d = 1$), (7) reduces to

$$p^\circ = \left(\frac{1}{J}\sum_{i=0}^{J-1} p_i\right)\frac{Q}{1-Q}. \tag{9}$$

3. Strong law of large numbers. Mean payout is the most important statistic of a slot machine. It can be interpreted as the long-term proportion of coins played that are paid out to the player. The justification of this interpretation is the strong law of large numbers, which is well known to hold for traditional machines, whose sequence of payouts is independent and identically distributed (i.i.d.). Does the same conclusion hold for the Futurity, even though the independence assumption and the identically distributed assumption fail?

We will show that the answer is affirmative.

Let $R_1, R_2, \ldots$ be the sequence of payouts of the slot machine *excluding* the Futurity awards, given that the initial state $(X_0, Y_0) = (i_0, j_0) \in \Sigma$ is specified. This sequence clearly satisfies the strong law of large numbers. Indeed, $R_1, R_2, \ldots$ are independent, uniformly bounded, nonnegative random variables, with $\{R_{n+mI}, m \geq 0\}$ identically distributed as the payout distribution in cam position i (which has mean μ_i), where $i_0 + n - 1 \equiv i \pmod I$. We conclude that, if n is a multiple of I, then

$$n^{-1}\mathrm{E}[R_1 + \cdots + R_n] = \frac{1}{I}\sum_{i=0}^{I-1}\mu_i =: \mu.$$

It follows from a version of the strong law of large numbers for independent, but not identically distributed, random variables that

$$n^{-1}(R_1 + \cdots + R_n) \to \mu \qquad \text{a.s.}$$

Now, how does this change when the Futurity awards are taken into account? Let $R_1^*, R_2^*, \ldots$ be the sequence of payouts of the slot machine *including* the Futurity awards, given that the initial state $(X_0, Y_0) = (i_0, j_0) \in \Sigma$ is specified. Notice that, for each $n \geq 1$, Y_n is a nonrandom function of (X_0, Y_0) and $1_{\{R_1=0\}}, \ldots, 1_{\{R_n=0\}}$; in particular, Y_{n-1} is independent of R_n. Clearly,

$$\begin{aligned} R_n^* &= R_n + J \cdot 1_{\{Y_{n-1}=J-1, R_n=0\}} \\ &= R_n + J\sum_{i=0}^{I-1} 1_{\{(X_{n-1},Y_{n-1})=(i,J-1), R_n=0\}}, \qquad n \geq 1. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{R_1^* + \cdots + R_n^*}{n} &= \frac{R_1 + \cdots + R_n}{n} + J\sum_{i=0}^{I-1}\frac{1}{n}\sum_{l=1}^{n} 1_{\{(X_{l-1},Y_{l-1})=(i,J-1), R_l=0\}} \\ &\to \mu + J\sum_{i=0}^{I-1}\pi(i, J-1)q_i = \mu + Jp^\circ = \mu^* \qquad \text{a.s.,} \end{aligned}$$

where μ^* is as in (8); here the limit assertion requires additional justification. Since the Markov chain is finite, irreducible and periodic,

$$\frac{1}{n}\sum_{l=1}^{n} 1_{\{(X_{l-1},Y_{l-1})=(i,J-1)\}} \to \pi(i, J-1) \qquad \text{a.s.}$$

for $i = 0, 1, \dots, I-1$, hence

$$\begin{aligned}
&\frac{1}{n}\sum_{l=1}^{n} 1_{\{(X_{l-1},Y_{l-1})=(i,J-1),R_l=0\}} \\
&\quad = \left(\frac{1}{n}\sum_{l=1}^{n} 1_{\{(X_{l-1},Y_{l-1})=(i,J-1)\}}\right)\left(\frac{\sum_{l=1}^{n} 1_{\{(X_{l-1},Y_{l-1})=(i,J-1),R_l=0\}}}{\sum_{l=1}^{n} 1_{\{(X_{l-1},Y_{l-1})=(i,J-1)\}}}\right) \qquad (10)\\
&\quad \to \pi(i, J-1)q_i \qquad \text{a.s.}
\end{aligned}$$

for $i = 0, 1, \dots, J-1$. We are using the fact that the ratio of sums in (10) represents the proportion of visits to $(i, J-1)$ (through time $n-1$) that result in a Futurity award. At each visit to $(i, J-1)$ the probability of such an award is q_i and the results are determined independently; hence the ratio tends to q_i a.s. by the strong law of large numbers.

We have established the following version of the strong law of large numbers.

THEOREM 2. *Let $R_1^*, R_2^*, \dots$ be the sequence of payouts of the generalized Futurity slot machine starting in an arbitrary initial state $(X_0, Y_0) = (i_0, j_0)$. Then*

$$n^{-1}(R_1^* + \cdots + R_n^*) \to \mu^* \qquad \textit{a.s.}$$

Observe that we can similarly obtain the asymptotic frequency of nonzero payouts, the so-called "hit frequency" (usually reported as a percentage):

$$\begin{aligned}
&n^{-1}\big(1_{\{R_1^*>0\}} + \cdots + 1_{\{R_n^*>0\}}\big) \\
&\quad = \frac{1_{\{R_1>0\}} + \cdots + 1_{\{R_n>0\}}}{n} \\
&\qquad + \frac{1_{\{Y_0=J-1,R_1=0\}} + \cdots + 1_{\{Y_{n-1}=J-1,R_n=0\}}}{n} \qquad (11)\\
&\quad \to \frac{1}{I}\sum_{i=0}^{I-1} p_i + p^\circ =: p^* \qquad \text{a.s.}
\end{aligned}$$

4. Central limit theorem. The second-most important statistic of a slot machine is the variance of the payout. (This is arguable. Some would say that the hit frequency p^* is more important.) The variance permits determination of the asymptotic distribution of the cumulative number of coins paid out by the machine, via the central limit theorem. The central limit theorem is well known to hold for traditional machines, whose sequence of payouts is i.i.d. Does the same conclusion hold for the Futurity, even though the independence assumption and the identically distributed assumption fail?

We will show in two steps that the answer is affirmative. First, we will apply the central limit theorem for stationary, strongly mixing sequences, and this will allow us to evaluate the variance parameter. Then, using a simple coupling argument, we will treat the general case in which the initial state is fixed but arbitrary.

It will be convenient to index time by $\mathbf{Z}$, the set of integers. So we let $\{R_n\}_{n\in\mathbf{Z}}$ be independent, uniformly bounded, nonnegative random variables, with $\{R_n : n-1 \equiv i \pmod{I}\}$ identically distributed as the payout distribution in cam position $i \in \{0, 1, \dots, I-1\}$. We interpret $\{R_n\}_{n\in\mathbf{Z}}$ as the sequence of payouts of the slot machine *excluding* the Futurity awards. Thus,

$$\mathrm{P}(R_n > 0) = p_{n-1}, \qquad \mathrm{E}[R_n] = \mu_{n-1}, \qquad \mathrm{Var}(R_n) = \sigma_{n-1}^2$$

for all $n \in \mathbf{Z}$, provided we extend these parameters periodically; for example, $p_{i+mI} := p_i$ for all $i \in \{0, 1, \dots, I-1\}$ and $m \in \mathbf{Z}$.

Notice that we can define the Markov chain $\{(X_n, Y_n)\}_{n\in\mathbf{Z}}$ as a nonrandom function of $\{R_n\}_{n\in\mathbf{Z}}$. Indeed, $X_n = i \in \{0, 1, \dots, I-1\}$ if $n \equiv i \pmod{I}$, so $\{X_n\}_{n\in\mathbf{Z}}$ is deterministic, and $Y_n = j \in \{0, 1, \dots, J-1\}$ if

$$R_{n-kJ-j} > 0, \quad R_{n-kJ-j+1} = \cdots = R_n = 0 \qquad \text{for some } k \geq 0.$$

To take the Futurity awards into account, we define $\{R_n^*\}_{n\in\mathbf{Z}}$ by

$$\begin{aligned} R_n^* &:= R_n + J \cdot 1_{\{Y_{n-1}=J-1, R_n=0\}} \\ &= R_n + J \sum_{k=1}^{\infty} 1_{\{R_{n-kJ}>0, R_{n-kJ+1}=\cdots=R_{n-1}=R_n=0\}} \qquad (12) \\ &= u(\dots, R_{n-2}, R_{n-1}, R_n), \qquad n \in \mathbf{Z}, \end{aligned}$$

for some nonrandom function u.

The sequence $\{R_n\}_{n\in\mathbf{Z}}$ is independent but not identically distributed, so we consider the sequence of random vectors

$$\mathbf{R}_k := \big(R_{kI+1}, \dots, R_{(k+1)I}\big), \qquad k \in \mathbf{Z},$$

which *is* i.i.d., hence by (12),

$$\mathbf{R}_k^* := \big(R_{kI+1}^*, \dots, R_{(k+1)I}^*\big), \qquad k \in \mathbf{Z},$$

is a stationary sequence. In particular, the sequence

$$S_k := R_{kI+1} + \cdots + R_{(k+1)I}, \qquad k \in \mathbf{Z},$$

is also i.i.d., and the sequence

$$S_k^* := R_{kI+1}^* + \cdots + R_{(k+1)I}^*, \qquad k \in \mathbf{Z},$$

is also stationary, despite the fact that the Markov chain $\{(X_n, Y_n)\}_{n\in\mathbf{Z}}$ is not stationary in this construction. S_k and S_k^* represent the total payout, excluding and including the Futurity awards, respectively, over the segment of I consecutive coups numbered $kI+1, \dots, (k+1)I$.

We claim that the stationary sequence $\{S_k^*\}_{k\in\mathbf{Z}}$ is strongly mixing, that is, the quantities

$$\alpha(m) := \sup_{A\in\sigma(S_k^*:k\le -m),\, B\in\sigma(S_k^*:k\ge 0)} |\mathrm{P}(A\cap B) - \mathrm{P}(A)\mathrm{P}(B)| \tag{13}$$

satisfy $\alpha(m) \to 0$ as $m \to \infty$. For $m \ge 2$, let $C_m := \{R_k > 0$ for some $k \in \{-(m-1)I+1, -(m-1)I+2, \dots, 0\}$. Then, with A and B as in (13), A is independent of $B\cap C_m$, so

$$\begin{aligned}
&|\mathrm{P}(A\cap B) - \mathrm{P}(A)\mathrm{P}(B)| \\
&\quad \le |\mathrm{P}(A\cap B\cap C_m) - \mathrm{P}(A)\mathrm{P}(B\cap C_m)| \\
&\qquad + |\mathrm{P}(A\cap B\cap C_m^c) - \mathrm{P}(A)\mathrm{P}(B\cap C_m^c)| \\
&\quad = |\mathrm{P}(A\cap B\cap C_m^c) - \mathrm{P}(A)\mathrm{P}(B\cap C_m^c)| \\
&\quad \le \mathrm{P}(C_m^c) \\
&\quad = \mathrm{P}\big(R_{-(m-1)I+1} = R_{-(m-1)I+2} = \cdots = R_0 = 0\big) \\
&\quad = Q^{m-1},
\end{aligned}$$

and this shows that $\alpha(m)$ converges to 0 geometrically fast.

Letting $\bar\mu := \mathrm{E}[S_0^*]$ and noting that the random variables of interest are uniformly bounded, the central limit theorem for stationary, strongly mixing sequences [e.g., Bradley (2007), Theorem 10.3] tells us that

$$\frac{S_0^* + \cdots + S_{m-1}^* - m\bar\mu}{\sqrt{m\bar\sigma^2}} \xrightarrow{d} N(0,1),$$

provided

$$\bar\sigma^2 := \mathrm{Var}(S_0^*) + 2\sum_{m=1}^{\infty} \mathrm{Cov}(S_0^*, S_m^*) > 0.$$

We now evaluate $\bar\sigma^2$.

First, we will frequently encounter

$$\begin{aligned}
&\mathrm{P}(Y_{i-1} = J-1, R_i = 0) \\
&\quad = \sum_{k=1}^{\infty} \mathrm{P}(R_{i-kJ} > 0, R_{i-kJ+1} = \cdots = R_i = 0) \\
&\quad = \frac{1}{1-Q} \sum_{k=1}^{d} q_{i-1}\cdots q_{i-kJ}\, p_{i-kJ-1} \\
&\quad =: P_{i-1}
\end{aligned}$$

for $i = 1, 2, \ldots, I$. For example,

$$\begin{aligned}
\bar{\mu} &:= \mathrm{E}[S_0^*] = \mathrm{E}[R_1^* + \cdots + R_I^*] \\
&= \sum_{i=1}^{I} \mathrm{E}[R_i^*] = \sum_{i=1}^{I} \mathrm{E}\bigl[R_i + J \cdot 1_{\{Y_{i-1}=J-1, R_i=0\}}\bigr] \\
&= \sum_{i=1}^{I} (\mu_{i-1} + J P_{i-1}) = \sum_{i=0}^{I-1} \mu_i + J \sum_{i=0}^{I-1} P_i.
\end{aligned}$$

Next, for $i = 1, 2, \ldots, I$,

$$\begin{aligned}
\mathrm{Var}(R_i^*) &= \mathrm{Var}\bigl(R_i + J \cdot 1_{\{Y_{i-1}=J-1, R_i=0\}}\bigr) \\
&= \mathrm{Var}(R_i) + 2J \,\mathrm{Cov}\bigl(R_i, 1_{\{Y_{i-1}=J-1, R_i=0\}}\bigr) \\
&\quad + J^2 \,\mathrm{Var}\bigl(1_{\{Y_{i-1}=J-1, R_i=0\}}\bigr) \\
&= \mathrm{Var}(R_i) - 2J\mathrm{E}[R_i]\mathrm{P}(Y_{i-1} = J-1, R_i = 0) \\
&\quad + J^2\mathrm{P}(Y_{i-1} = J-1, R_i = 0)\bigl(1 - \mathrm{P}(Y_{i-1} = J-1, R_i = 0)\bigr) \\
&= \sigma_{i-1}^2 - 2J\mu_{i-1}P_{i-1} + J^2 P_{i-1}(1 - P_{i-1}),
\end{aligned}$$

and, for $1 \le i < j \le I$,

$$\begin{aligned}
\mathrm{Cov}(R_i^*, R_j^*) &= \mathrm{Cov}\bigl(R_i + J \cdot 1_{\{Y_{i-1}=J-1, R_i=0\}}, R_j + J \cdot 1_{\{Y_{j-1}=J-1, R_j=0\}}\bigr) \\
&= J \,\mathrm{Cov}\bigl(R_i, 1_{\{Y_{j-1}=J-1, R_j=0\}}\bigr) \\
&\quad + J^2 \,\mathrm{Cov}\bigl(1_{\{Y_{i-1}=J-1, R_i=0\}}, 1_{\{Y_{j-1}=J-1, R_j=0\}}\bigr) \\
&= J\bigl\{\mathrm{E}\bigl[R_i 1_{\{Y_{j-1}=J-1, R_j=0\}}\bigr] - \mathrm{E}[R_i]\mathrm{P}(Y_{j-1} = J-1, R_j = 0)\bigr\} \\
&\quad + J^2[\mathrm{P}(Y_{i-1} = J-1, R_i = 0, Y_{j-1} = J-1, R_j = 0) \\
&\qquad - \mathrm{P}(Y_{i-1} = J-1, R_i = 0)\mathrm{P}(Y_{j-1} = J-1, R_j = 0)] \\
&= J\mu_{i-1}\biggl(\sum_{1 \le k < (j-i)/J} q_{j-1} \cdots q_{j-kJ} p_{j-kJ-1} \\
&\qquad\qquad + \Delta_{ij} q_{j-1} \cdots q_i - P_{j-1}\biggr) \\
&\quad + J^2\biggl(\sum_{1 \le k < (j-i)/J} q_{j-1} \cdots q_{j-kJ} p_{j-kJ-1} P_{i-1} \\
&\qquad + \Delta_{ij} \sum_{k > (j-i)/J} q_{j-1} \cdots q_{j-kJ} p_{j-kJ-1} - P_{i-1}P_{j-1}\biggr) \\
&=: A_{ij},
\end{aligned}$$

where $\Delta_{ij} := 1$ if $j - i \equiv 0 \pmod J$ and $:= 0$ otherwise, and the infinite series in the definition of A_{ij} can be expressed as the finite sum

$$\sum_{(j-i)/J<k\le d} q_{j-1}\cdots q_{j-kJ}p_{j-kJ-1} + QP_{j-1}$$

when $j - i \equiv 0 \pmod J$. We conclude that

$$\begin{aligned}\operatorname{Var}(S_0^*) &= \operatorname{Var}(R_1^* + \cdots + R_I^*) = \sum_{i=1}^{I}\operatorname{Var}(R_i^*) + 2\sum\sum_{1\le i<j\le I}\operatorname{Cov}(R_i^*, R_j^*)\\ &= \sum_{i=0}^{I-1}[\sigma_i^2 - 2J\mu_i P_i + J^2P_i(1-P_i)] + 2\sum\sum_{1\le i<j\le I} A_{ij}.\end{aligned}\tag{14}$$

Notice that this formula depends solely on the basic parameters (I, J, p_i, μ_i and σ_i^2).

Next, for $i, j = 1, 2, \ldots, I$ and $m \ge 1$,

$$\begin{aligned}&\operatorname{Cov}(R_i^*, R_{mI+j}^*)\\ &\quad= \operatorname{Cov}\big(R_i + J\cdot 1_{\{Y_{i-1}=J-1,R_i=0\}}, R_{mI+j} + J\cdot 1_{\{Y_{mI+j-1}=J-1,R_{mI+j}=0\}}\big)\\ &\quad= J\operatorname{Cov}\big(R_i, 1_{\{Y_{mI+j-1}=J-1,R_{mI+j}=0\}}\big)\\ &\qquad+ J^2\operatorname{Cov}\big(1_{\{Y_{i-1}=J-1,R_i=0\}}, 1_{\{Y_{mI+j-1}=J-1,R_{mI+j}=0\}}\big).\end{aligned}$$

Now

$$\begin{aligned}&\operatorname{Cov}\big(R_i, 1_{\{Y_{mI+j-1}=J-1,R_{mI+j}=0\}}\big)\\ &\quad= \sum_{1\le k\le md+(j-i)/J} \mathrm{E}\big[R_i 1_{\{R_{mI-kJ+j}>0,R_{mI-kJ+j+1}=\cdots=R_{mI+j}=0\}}\big]\\ &\qquad - \mu_{i-1}P_{j-1}\\ &\quad= \sum_{1\le k<md+(j-i)/J} q_{j-1}\cdots q_{j-kJ}p_{j-kJ-1}\mu_{i-1}\\ &\qquad + \Delta_{ij}q_{mI+j-1}\cdots q_i\mu_{i-1} - \mu_{i-1}P_{j-1}\\ &\quad= \mu_{i-1}\Bigg(\sum_{k=1}^{d} q_{j-1}\cdots q_{j-kJ}p_{j-kJ-1}(1 + Q + \cdots + Q^{m-2})\\ &\qquad\qquad + Q^{m-1}\sum_{1\le k<d+(j-i)/J} q_{j-1}\cdots q_{j-kJ}p_{j-kJ-1}\\ &\qquad\qquad\qquad + \Delta_{ij}q_{j-1}\cdots q_0 Q^{m-1}q_{I-1}\cdots q_i - P_{j-1}\Bigg)\end{aligned}$$

$$\begin{aligned}&= \mu_{i-1}\biggl(-P_{j-1} + \sum_{1\le k<d+(j-i)/J} q_{j-1}\cdots q_{j-kJ}p_{j-kJ-1}\\ &\qquad\qquad + \Delta_{ij} q_{I-1}\cdots q_i q_{j-1}\cdots q_0\biggr)Q^{m-1}\\ &=: B_{ij}Q^{m-1},\end{aligned}$$

where $q_{I-1}\cdots q_i := 1$ if $i = I$, and

$$\begin{aligned}&\operatorname{Cov}\bigl(1_{\{Y_{i-1}=J-1,R_i=0\}}, 1_{\{Y_{mI+j-1}=J-1,R_{mI+j}=0\}}\bigr)\\ &\quad = \mathrm{P}(Y_{i-1}=J-1, R_i=0, Y_{mI+j-1}=J-1, R_{mI+j}=0) - P_{i-1}P_{j-1}\\ &\quad = \sum_{1\le k<md+(j-i)/J} P_{i-1}\mathrm{P}(R_{mI-kJ+j}>0, R_{mI-kJ+j+1}=\cdots=R_{mI+j}=0)\\ &\qquad + \Delta_{ij}\sum_{k>md+(j-i)/J} \mathrm{P}(R_{mI-kJ+j}>0, R_{mI-kJ+j+1}=\cdots=R_{mI+j}=0)\\ &\qquad - P_{i-1}P_{j-1}\\ &\quad = \sum_{1\le k<md+(j-i)/J} q_{j-1}\cdots q_{j-kJ}p_{j-kJ-1}P_{i-1}\\ &\qquad + \Delta_{ij}\sum_{k>md+(j-i)/J} q_{j-1}\cdots q_{j-kJ}p_{j-kJ-1} - P_{i-1}P_{j-1}\\ &\quad = \sum_{k=1}^{d} q_{j-1}\cdots q_{j-kJ}p_{j-kJ-1}(1+Q+\cdots+Q^{m-2})P_{i-1}\\ &\qquad + Q^{m-1}\sum_{1\le k<d+(j-i)/J} q_{j-1}\cdots q_{j-kJ}p_{j-kJ-1}P_{i-1}\\ &\qquad + \Delta_{ij}Q^{m-1}\sum_{k>d+(j-i)/J} q_{j-1}\cdots q_{j-kJ}p_{j-kJ-1} - P_{i-1}P_{j-1}\\ &\quad = \biggl(-P_{i-1}P_{j-1} + \sum_{1\le k<d+(j-i)/J} q_{j-1}\cdots q_{j-kJ}p_{j-kJ-1}P_{i-1}\\ &\qquad\qquad + \Delta_{ij}\sum_{k>d+(j-i)/J} q_{j-1}\cdots q_{j-kJ}p_{j-kJ-1}\biggr)Q^{m-1}\\ &\quad =: C_{ij}Q^{m-1},\end{aligned}$$

and the infinite series in the definition of C_{ij} can be expressed as

$$Q\sum_{(j-i)/J<k\le d} q_{j-1}\cdots q_{j-kJ}p_{j-kJ-1} + Q^2P_{j-1}$$

when $j - i \equiv 0 \pmod{J}$ and $j \geq i$, and as

$$\sum_{d+(j-i)/J<k\leq d} q_{j-1}\cdots q_{j-kJ}p_{j-kJ-1} + QP_{j-1}$$

when $j - i \equiv 0 \pmod{J}$ and $j < i$. We conclude that

$$\begin{aligned}\mathrm{Cov}(S_0^*, S_m^*) &= \mathrm{Cov}\big(R_1^* + \cdots + R_I^*, R_{mI+1}^* + \cdots + R_{(m+1)I}^*\big)\\ &= \sum_{i=1}^{I}\sum_{j=1}^{I}\mathrm{Cov}(R_i^*, R_{mI+j}^*)\\ &= \sum_{i=1}^{I}\sum_{j=1}^{I}(JB_{ij}Q^{m-1} + J^2C_{ij}Q^{m-1}),\end{aligned}$$

and hence that

$$\sum_{m=1}^{\infty}\mathrm{Cov}(S_0^*, S_m^*) = \frac{J}{1-Q}\sum_{i=1}^{I}\sum_{j=1}^{I}(B_{ij} + JC_{ij}). \tag{15}$$

Again, this formula depends solely on the basic parameters. Summing (14) and twice (15), we obtain $\bar{\sigma}^2$.

Finally, we observe that the central limit theorem for the stationary sequence $\{S_k^*\}_{k\in\mathbf{Z}}$ yields a central limit theorem for $\{R_n^*\}_{n\in\mathbf{Z}}$ as well. Indeed, with

$$\mu^* = \bar{\mu}/I \quad \text{and} \quad (\sigma^*)^2 := \bar{\sigma}^2/I,$$

we find that

$$\frac{R_1^* + \cdots + R_n^* - n\mu^*}{\sqrt{n(\sigma^*)^2}} - \frac{S_0^* + \cdots + S_{\lfloor n/I\rfloor-1}^* - \lfloor n/I\rfloor\bar{\mu}}{\sqrt{(n/I)\bar{\sigma}^2}} \tag{16}$$

tends to 0 a.s. as $n \to \infty$ because the difference between the numerators, namely

$$R_{I\lfloor n/I\rfloor+1}^* + \cdots + R_n^* - (n - I\lfloor n/I\rfloor)\mu^*,$$

is uniformly bounded in n and the denominators are equal. Thus,

$$\frac{R_1^* + \cdots + R_n^* - n\mu^*}{\sqrt{n(\sigma^*)^2}} \xrightarrow{d} N(0,1).$$

We can go one step further and derive a central limit theorem for $\{\hat{R}_n^*\}_{n\geq 0}$ with $(\hat{X}_0, \hat{Y}_0) = (i_0, j_0)$ specified, where the hats on $\hat{R}_n^*$, $\hat{X}_0$ and $\hat{Y}_0$ distinguish them from the R_n^*, X_0 and Y_0 already defined. The idea of the proof is the same as in (16). We define $\hat{R}_n := R_{n+i_0}$ for $n \geq 1$, and we define $(\hat{X}_n, \hat{Y}_n)$ for $n \geq 1$ in terms of $(\hat{X}_0, \hat{Y}_0) = (i_0, j_0)$ and $1_{\{\hat{R}_1=0\}}, \ldots, 1_{\{\hat{R}_n=0\}}$ in the usual way. Then

$$\frac{\hat{R}_1^* + \cdots + \hat{R}_n^* - n\mu^*}{\sqrt{n(\sigma^*)^2}} - \frac{R_1^* + \cdots + R_n^* - n\mu^*}{\sqrt{n(\sigma^*)^2}}$$

tends to 0 a.s. as $n \to \infty$ because $\hat{R}_n^* = R_{n+i_0}^*$ unless $\hat{R}_1^* = \cdots = \hat{R}_{n-1}^* = 0$. In words, the sequences $\hat{R}_1^*, \ldots, \hat{R}_n^*$ and $R_1^*, \ldots, R_n^*$ differ only by a shift (of i_0 terms), once the $\hat{Y}$ process and the shifted Y process couple, which occurs after the first win. We have therefore established the following central limit theorem.

THEOREM 3. *Let $\hat{R}_1^*, \hat{R}_2^*, \ldots$ be the sequence of payouts of the generalized Futurity slot machine starting in an arbitrary initial state. Then*

$$\frac{\hat{R}_1^* + \cdots + \hat{R}_n^* - n\mu^*}{\sqrt{n(\sigma^*)^2}} \xrightarrow{d} N(0, 1).$$

5. Numerical results for the Futurity. The Futurity was in production from 1936 to 1941. (After December 7, 1941, Mills Novelty stopped producing slot machines and became a defense contractor for the duration of the war. When it resumed slot production in 1945, it did so with new designs.) In particular, there were minor variations in the payouts and reel strip labels used with the machine, but the fundamental properties, the Futurity award and the periodic mode changes, are common to every Mills Futurity. The precise version we consider here is the one described by Geddes (1980).

To simplify matters, we code the six symbols as lemon = 0, cherry = 1, orange = 2, plum = 3, bell = 4 and bar = 5. The pay table can then be described by the function $p : \{0, 1, 2, 3, 4, 5\}^3 \mapsto \mathbf{Z}_+$ given by $p(5, 5, 5) := 150$, $p(4, 4, 4) = p(4, 4, 5) := 18$, $p(3, 3, 3) = p(3, 3, 5) := 14$, $p(2, 2, 2) = p(2, 2, 5) := 10$, $p(1, 1, 0) = p(1, 1, 4) := 5$ and $p(1, 1, 2) = p(1, 1, 3) = p(1, 1, 5) := 3$; otherwise $p := 0$. The three reel strips can be described as follows, in which the symbols in odd-numbered positions are italicized for convenience:

$$\begin{aligned}
&\text{reel 1:} \quad \mathit{1}, 5, \mathit{1}, 2, \mathit{1}, 5, \mathit{1}, 5, \mathit{1}, 3, \mathit{1}, 2, \mathit{5}, 1, \mathit{4}, 3, \mathit{1}, 5, \mathit{1}, 2,\\
&\text{reel 2:} \quad \mathit{1}, 4, \mathit{1}, 3, \mathit{1}, 4, \mathit{1}, 2, \mathit{1}, 4, \mathit{1}, 4, \mathit{1}, 2, \mathit{1}, 2, \mathit{4}, 1, \mathit{5}, 4,\\
&\text{reel 3:} \quad \mathit{3}, 4, \mathit{2}, 0, \mathit{3}, 4, \mathit{2}, 0, \mathit{4}, 0, \mathit{2}, 3, \mathit{2}, 4, \mathit{2}, 4, \mathit{5}, 2, \mathit{3}, 5.
\end{aligned}$$

Table 1 summarizes the relevant information from these reel strips. Of course, the reels operate independently, and the 10 possible positions at which each reel can stop (given the mode) are assumed equally likely.

With $f_{\mathrm{E}}(i, j)$ denoting the frequency of symbol i on reel j in mode E (see Table 1), we find that the mean payout in mode E is

$$\mu_{\mathrm{E}} = \frac{1}{(10)^3} \sum_{i_1=0}^{5} \sum_{i_2=0}^{5} \sum_{i_3=0}^{5} f_{\mathrm{E}}(i_1, 1) f_{\mathrm{E}}(i_2, 2) f_{\mathrm{E}}(i_3, 3) p(i_1, i_2, i_3) = 0.28.$$

Similarly, the mean payout in mode O is $\mu_{\mathrm{O}} = 2.234$. Certainly, these numbers justify our descriptions of mode E as "tight" and mode O as "loose," as do the facts that the probability of a nonzero payout in mode E, other than a Futurity award,

TABLE 1
Reel strip inventories for the Futurity in both modes

	Mode E			Mode O		
Symbol	**Reel 1**	**Reel 2**	**Reel 3**	**Reel 1**	**Reel 2**	**Reel 3**
Lemon (= 0)	0	0	3	0	0	0
Cherry (= 1)	1	1	0	8	8	0
Orange (= 2)	3	3	1	0	0	5
Plum (= 3)	2	1	1	0	0	3
Bell (= 4)	0	5	4	1	1	1
Bar (= 5)	4	0	1	1	1	1
Total	10	10	10	10	10	10

is $p_E = 0.032$, and the corresponding probability in mode O is $p_O = 0.643$. See Table 2.

With the statistics of Table 2, we can define

$$(p_0, p_1, \dots, p_9) := (p_E, p_E, p_E, p_E, p_E, p_O, p_E, p_E, p_E, p_O),$$
$$(\mu_0, \mu_1, \dots, \mu_9) := (\mu_E, \mu_E, \mu_E, \mu_E, \mu_E, \mu_O, \mu_E, \mu_E, \mu_E, \mu_O),$$
$$(\sigma_0^2, \sigma_1^2, \dots, \sigma_9^2) := (\sigma_E^2, \sigma_E^2, \sigma_E^2, \sigma_E^2, \sigma_E^2, \sigma_O^2, \sigma_E^2, \sigma_E^2, \sigma_E^2, \sigma_O^2),$$

and $q_i := 1 - p_i$ for $i = 0, 1, \dots, 9$. With $I = J = 10$ (in particular, $d = 1$), we can apply Theorem 1 to obtain the stationary distribution for the driving Markov chain. Numerical values are shown in Table 3. Geddes and Saul (1980) obtained an approximate stationary distribution from their simulation, essentially accurate

TABLE 2
Payout frequencies and statistics for the Futurity in both modes, excluding Futurity awards. Results are exact (no rounding)

Payout	**Mode E**	**Mode O**
0	968	357
3	3	576
5	7	64
10	18	0
14	4	0
18	0	2
150	0	1
Total	1000	1000
Mean payout	$\mu_E = 0.28$	$\mu_O = 2.234$
Variance of payout	$\sigma_E^2 = 2.7076$	$\sigma_O^2 = 24.941244$
Probability of nonzero payout	$p_E = 0.032$	$p_O = 0.643$

TABLE 3
Stationary distribution of the Markov chain, rounded to six decimal places. Rows indicate cam position, and columns indicate pointer position. Entries greater than 1/100 *are shaded*

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**Sum**
0	0.071306	0.001267	0.001226	0.001187	0.023090	0.000410	0.000397	0.000384	0.000372	0.000360	1/10
1	0.003549	0.069024	0.001226	0.001187	0.001149	0.022351	0.000397	0.000384	0.000372	0.000360	1/10
2	0.003549	0.003435	0.066815	0.001187	0.001149	0.001112	0.021636	0.000384	0.000372	0.000360	1/10
3	0.003549	0.003435	0.003325	0.064677	0.001149	0.001112	0.001077	0.020943	0.000372	0.000360	1/10
4	0.003549	0.003435	0.003325	0.003219	0.062608	0.001112	0.001077	0.001042	0.020273	0.000360	1/10
5	0.003549	0.003435	0.003325	0.003219	0.003116	0.060604	0.001077	0.001042	0.001009	0.019624	1/10
6	0.071306	0.001267	0.001226	0.001187	0.001149	0.001112	0.021636	0.000384	0.000372	0.000360	1/10
7	0.003549	0.069024	0.001226	0.001187	0.001149	0.001112	0.001077	0.020943	0.000372	0.000360	1/10
8	0.003549	0.003435	0.066815	0.001187	0.001149	0.001112	0.001077	0.001042	0.020273	0.000360	1/10
9	0.003549	0.003435	0.003325	0.064677	0.001149	0.001112	0.001077	0.001042	0.001009	0.019624	1/10
Sum	0.171001	0.161193	0.151837	0.142915	0.096857	0.091152	0.050526	0.047593	0.044797	0.042130	

to three decimal places. One drawback of a simulation in this context is that it does not clearly show that, when the stationary distribution is expressed as a matrix, several entries in each column are equal.

We calculate from (9), (8) and (11) that

$$p^\circ \approx 0.0168011, \qquad \mu^* \approx 0.838811, \qquad p^* \approx 0.171001.$$

Based on their simulation of 1,000,000 coups, Geddes and Saul (1980) obtained the estimates 0.016638, 0.838995 and 0.171451, respectively. They did not attempt to estimate the variance parameter. Using (14) and (15), we find that

$$\mathrm{Var}(S_0^*) \approx 69.860263, \qquad \sum_{m=1}^{\infty} \mathrm{Cov}(S_0^*, S_m^*) \approx -0.951088,$$

hence

$$(\sigma^*)^2 \approx 6.795809.$$

All displayed numbers are exact except for rounding.

Geddes and Saul (1980) also proposed a very interesting betting strategy: Simply play the machine until, and only until, a payout occurs. Let $E(i, j)$ be the player's expected profit when starting from cam position i and pointer position j. Then

$$E(i, 9) = -1 + \mu_i + 10q_i, \qquad i = 0, 1, \dots, 9,$$

where of course the 10 is the Futurity award. Furthermore,

$$E(i, j) = -1 + \mu_i + q_i E\big(i + 1 \ (\mathrm{mod}\, 10), j + 1\big), \qquad i = 0, 1, \dots, 9,$$

for $j = 8, 7, \dots, 0$ (in that order). These expectations are evaluated numerically in Table 4. This result is due to Geddes and Saul.

We find that, if the pointer position is 3–9, a positive expectation is assured (regardless of the cam position). In fact, 90 of the 100 expectations are positive. Perhaps more surprising is the fact that

$$\sum_{(i,j)\in\Sigma} \pi(i, j)E(i, j) \approx 0.960501.$$

In other words, the "stop after the next payout" betting system has positive expectation when played at equilibrium. This observation, however, is less useful than it may first appear to be. For if the player has reached approximate equilibrium through extensive play, then the positive expected profit the system promises will not make up for the negative expected profit already incurred. And the player should not expect to find a machine at approximate equilibrium after extensive play by others. Indeed, a player quits not at a fixed time, such as after the 10,000th coup, but rather at a random stopping time, such as after the next win, or after running out of coins. Moreover, as we have seen, if the pointer position is 3–9,

TABLE 4
Expected player profit when playing until a payout occurs, as a function of initial cam position (row) and pointer position (column), rounded to six decimal places; columns 8 and 9 are exact

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	−1.640567	−0.210554	0.085131	0.390591	0.706148	5.122320	6.035454	6.978775	7.953280	8.960
1	−1.056559	−0.950999	0.526288	0.831747	1.147305	1.473294	6.035454	6.978775	7.953280	8.960
2	−0.453244	−0.347685	−0.238636	1.287487	1.603045	1.929034	2.265799	6.978775	7.953280	8.960
3	0.170015	0.275574	0.384623	0.497277	2.073850	2.399839	2.736605	3.084503	7.953280	8.960
4	0.813877	0.919437	1.028486	1.141140	1.257518	2.886209	3.222975	3.570873	3.930272	8.960
5	1.479024	1.584584	1.693633	1.806287	1.922665	2.042890	3.725423	4.073321	4.432720	4.804
6	−0.743671	0.686343	0.982028	1.287487	1.603045	1.929034	2.265799	6.978775	7.953280	8.960
7	−0.130013	−0.024453	1.452834	1.758293	2.073850	2.399839	2.736605	3.084503	7.953280	8.960
8	0.503931	0.609491	0.718540	2.244663	2.560220	2.886209	3.222975	3.570873	3.930272	8.960
9	1.158832	1.264392	1.373441	1.486095	3.062668	3.388657	3.725423	4.073321	4.432720	4.804

a player has positive equity and may not want to relinquish it by walking away. It seems likely that most players would notice this at least for pointer positions 7, 8 and 9, for in those cases a loss is impossible.

Geddes and Saul (1980) remarked that "the machine tends to leave the player at an unprofitable starting point most of the time after paying off." One way to confirm this is to evaluate the asymptotic distribution of the Markov chain's state after a payout. Arguing as in (11), we get

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\sum_{l=1}^n 1_{\{R_l^*>0,(X_l,Y_l)=(i,0)\}}}{\sum_{l=1}^n 1_{\{R_l^*>0\}}} \\
&\quad = \lim_{n\to\infty} \frac{(1/n)\sum_{l=1}^n (1_{\{X_{l-1}=i-1,R_l>0\}} + 1_{\{(X_{l-1},Y_{l-1})=(i-1,9),R_l=0\}})}{(1/n)\sum_{l=1}^n (1_{\{R_l>0\}} + 1_{\{Y_{l-1}=9,R_l=0\}})} \\
&\quad = \frac{(0.1)p_{i-1} + \pi(i-1,9)q_{i-1}}{(0.8)p_{\mathrm{E}} + (0.2)p_{\mathrm{O}} + p^\circ} =: \rho(i,0) \qquad \text{a.s.},
\end{aligned}$$

where $p_{-1} := p_9$, etc. We find that $\rho(0,0) = \rho(6,0) \approx 0.416991$ and $\rho(i,0) \approx 0.020752$ otherwise, and of course states $(0,0)$ and $(6,0)$ have negative entries in Table 4. Geddes and Saul obtained approximations from their simulation. Observe that states $(0,0)$ and $(6,0)$ account for about 0.833982 of the probability, which can be interpreted as the long-term proportion of payouts that occur when the machine is in mode O. This is the same as the long-term proportion of Futurity awards that occur when the machine is in mode O.

Finally, we observe that the previous mode (E or O) is clear at a glance. This depends on the fact that the machine's payout window displays not only the three symbols on the payline (from the last coup) but also the symbols on the line above and the line below the payline. If the previous mode was O, then exactly four coups are needed to determine the cam position with certainty; if the previous mode was E, then at least four and at most seven coups are needed. The player who is unwilling to play without a positive expectation should play with pointer position 3 or greater, but also pointer position 2 if the previous mode was O.

6. A two-armed slot machine. Motivated by Parrondo's paradox, here we consider a two-armed generalization of the Futurity slot machine, and we label the arms A and B. Excluding the Futurity award, the sequence of payouts from each arm is assumed nonnegative i.i.d., with arm A (resp., B) having probability p_A (resp., p_B) of a nonzero payout and mean payout μ_A (resp., μ_B) in all cam positions. The two arms are linked only by the Futurity award: After J consecutive losses, regardless of the order of play of the two arms, the J lost coins are returned to the player. We assume that $J \ge 2$, and we let $q_A := 1 - p_A$ and $q_B := 1 - p_B$.

The asymptotic mean payout per coup, including the Futurity award, from playing arm A only (resp., arm B only) is

$$\mu_A^* = \mu_A + J p_A^\circ \qquad \text{where } p_A^\circ := \frac{p_A q_A^J}{1 - q_A^J}$$

[resp., $\mu_B^* = \mu_B + Jp_B^\circ$, where $p_B^\circ := p_B q_B^J/(1-q_B^J)$]. If we play arm A with probability γ $(0 < \gamma < 1)$ and arm B otherwise, a strategy we denote by $C := \gamma A + (1-\gamma)B$, then this random mixture has probability $p_C := \gamma p_A + (1-\gamma)p_B$ of a nonzero payout and mean payout $\mu_C := \gamma\mu_A + (1-\gamma)\mu_B$ in all cam positions, excluding the Futurity award. Let $q_C := 1 - p_C$. Then the asymptotic mean payout per coup, including the Futurity award, from playing the random-mixture strategy with parameter γ is

$$\mu_C^* := \mu_C + Jp_C^\circ \qquad \text{where } p_C^\circ := \frac{p_C q_C^J}{1-q_C^J}.$$

We will say that the *Parrondo effect* is present for the random-mixture strategy with parameter γ if

$$\mu_C^* < \gamma\mu_A^* + (1-\gamma)\mu_B^*.$$

In words, the asymptotic mean payout per coup from playing the random-mixture strategy on the two-armed machine is less than the asymptotic mean payout per coup from playing the same random-mixture strategy on two one-armed machines, one of them equivalent to arm A and the other equivalent to arm B, each with its own Futurity award.

THEOREM 4. *If $p_A \neq p_B$, $J \geq 2$ and $0 < \gamma < 1$, then the Parrondo effect is present for the random-mixture strategy with parameter γ.*

REMARK. As Abbott (2009) remarked, "In its most general form, Parrondo's paradox can occur where there is a nonlinear interaction of random behavior with an asymmetry." Here $J \geq 2$ ensures the nonlinearity, while $p_A \neq p_B$ ensures the asymmetry.

In the scenario of Pyke (2003) described in Section 1, $\mu_A^* = \mu_B^* = 1$ (both arms are fair), hence $\mu_C^* < 1$ (the random mixture-strategy is losing for the player, hence winning for the casino).

PROOF OF THEOREM 4. The function $f(x) := (1-x)x^J/(1-x^J)$ is strictly convex on $(0, 1)$ for each $J \geq 2$ because

$$\begin{aligned} f''(x) &= \frac{Jx^{J-2}[J(1-x)(1+x^J) - (1+x)(1-x^J)]}{(1-x^J)^3} \\ &= \frac{J(1-x)x^{J-2}}{(1-x^J)^3}\sum_{j=1}^{J-1}(1-x^j)(1-x^{J-j}) \\ &> 0, \qquad 0 < x < 1. \end{aligned}$$

Therefore,

$$\begin{aligned}\mu_C^* &- [\gamma\mu_A^* + (1-\gamma)\mu_B^*]\\ &= J\{p_C^\circ - [\gamma p_A^\circ + (1-\gamma)p_B^\circ]\}\\ &= J\{f(\gamma q_A + (1-\gamma)q_B) - [\gamma f(q_A) + (1-\gamma)f(q_B)]\}\\ &< 0\end{aligned}$$

since $q_A \neq q_B$ and $0 < \gamma < 1$. □

In fact, the function f of the proof satisfies

$$1 - x + Jf(x) = (1-x)\frac{1 + (J-1)x^J}{1 - x^J} = \frac{1 + (J-1)x^J}{1 + x + \cdots + x^{J-1}} < 1$$

for $0 < x < 1$. In particular, $p_A + Jp_A^\circ < 1$ and $p_B + Jp_B^\circ < 1$. If we assume nonnegative integer payouts, then $\mu_A \geq p_A$ and $\mu_B \geq p_B$. It follows that μ_A and μ_B can be chosen in such a way that $\mu_A^* = \mu_B^* = 1$. Actually, fractional payouts per unit bet are commonplace on modern slot machines. (For example, a machine with five paylines might return three coins from a five-coin bet.) In such cases, a loss, for the purpose of the Futurity award, means a zero payout, not just a payout that is less than the amount bet.

Now we turn to strategies involving nonrandom patterns of the two arms. Let D denote a (finite) nonrandom pattern of As and Bs, with at least one A and at least one B, that is repeated ad infinitum. For example, D could be as simple as AB or ABB, or it could be more complicated, such as $ABBAB$. Let $r \geq 1$ and $s \geq 1$ be the numbers of As and Bs, respectively, in pattern D. Then the asymptotic mean payout per coup, including the Futurity award, from playing pattern D repeatedly is given by (8) with I equal to the least common multiple of $r+s$ and J. More precisely,

$$\mu_D^* := \frac{r\mu_A + s\mu_B}{r+s} + Jp_D^\circ,$$

where p_D° can be inferred from (7). The simplest case is that in which $r+s$ divides J because then (9) applies and we have

$$p_D^\circ := \frac{rp_A + sp_B}{r+s}\left(\frac{(q_A^r q_B^s)^{J/(r+s)}}{1 - (q_A^r q_B^s)^{J/(r+s)}}\right). \tag{17}$$

In this case, p_D° (and hence μ_D^*) depends on D only through r and s. For example, $p_{AABBB}^\circ = p_{ABBAB}^\circ$ as long as J is a multiple of 5.

We will say that the *Parrondo effect* is present for the nonrandom-pattern strategy with pattern D (with r As and s Bs) if

$$\mu_D^* < \frac{r\mu_A^* + s\mu_B^*}{r+s}.$$

The interpretation is analogous to that of the random-mixture strategy.

THEOREM 5. *If $p_A \neq p_B$, $J \geq 2$, and $r, s \geq 1$, and if $r + s$ divides J, then the Parrondo effect is present for the nonrandom-pattern strategy with pattern D.*

REMARK. While J is a characteristic of the machine, the pattern D (and hence r and s) is chosen by the player, so the assumption that $r + s$ divides J is too restrictive. We believe that this assumption can be weakened considerably (see below), but it cannot simply be omitted.

PROOF OF THEOREM 5. The presence of the Parrondo effect is equivalent to

$$p_D^\circ < \frac{rp_A^\circ + sp_B^\circ}{r+s}.$$

By the arithmetic mean-geometric mean inequality and $q_A \neq q_B$,

$$(q_A^r q_B^s)^{1/(r+s)} < \frac{rq_A + sq_B}{r+s}.$$

Since the function $g(x) := x^J/(1-x^J)$ is increasing on (0, 1), we have

$$\begin{aligned} p_D^\circ &= \frac{rp_A + sp_B}{r+s}\left(\frac{(q_A^r q_B^s)^{J/(r+s)}}{1-(q_A^r q_B^s)^{J/(r+s)}}\right) \\ &< \frac{rp_A + sp_B}{r+s}\left(\frac{[(rq_A + sq_B)/(r+s)]^J}{1-[(rq_A + sq_B)/(r+s)]^J}\right) \\ &= p_C^\circ < \frac{rp_A^\circ + sp_B^\circ}{r+s}, \end{aligned}$$

where p_C° is as in the proof of Theorem 4 with $\gamma := r/(r+s)$, and the second inequality uses Theorem 4. $\square$

Various attempts have been made at explaining why Parrondo's paradox holds in the nonrandom-pattern case; see, for example, Ethier and Lee (2009). When the assumptions of Theorem 5 are met, we have an especially simple explanation: the AM-GM inequality and convexity.

Let us generalize (17) to arbitrary D, r, s and J. Although we can minimize the number of terms by taking I to be the least common multiple of $r + s$ and J, we can equally well take I to be *any* multiple of $r + s$ and J, and the simplest choice is $I := (r+s)J$ (i.e., $d := r + s$). Define each of $p_1, p_2, \ldots, p_{r+s}$ to be p_A or p_B in accordance with the corresponding term in the pattern D. Extend this definition by $p_{i+r+s} = p_i$ for all $i \in \{1, 2, \ldots, r+s\}$, and define $q_i := 1 - p_i$ for $i = 1, 2, \ldots, 2(r+s)$. With this notation, we can write

$$p_D^\circ = \frac{1}{r+s}\sum_{k=1}^{r+s}\left(\sum_{j=1}^{r+s} p_j \prod_{i=j+1}^{j+kJ-(r+s)\lfloor kJ/(r+s)\rfloor} q_i\right)\frac{(q_A^r q_B^s)^{\lfloor kJ/(r+s)\rfloor}}{1-(q_A^r q_B^s)^J},$$

where empty products are 1. For example, if $r+s$ divides J, then all products are empty and this reduces algebraically to (17). For a less trivial example, consider $D = ABB$. Then, if $J = 3K + 1$ for a positive integer K,

$$p^\circ_{ABB} = (1/3)[(p_A q_B + p_B q_B + p_B q_A)(q_A q_B^2)^K + (p_A q_B^2 + p_B q_B q_A + p_B q_A q_B)(q_A q_B^2)^{2K} + (p_A + 2p_B)(q_A q_B^2)^J]/[1 - (q_A q_B^2)^J],$$

and, if $J = 3K + 2$ for a nonnegative integer K,

$$p^\circ_{ABB} = (1/3)[(p_A q_B^2 + p_B q_B q_A + p_B q_A q_B)(q_A q_B^2)^K + (p_A q_B + p_B q_B + p_B q_A)(q_A q_B^2)^{2K+1} + (p_A + 2p_B)(q_A q_B^2)^J]/[1 - (q_A q_B^2)^J].$$

Despite the impression that may be given by the proof of Theorem 5, it is not true in general that $p^\circ_D < p^\circ_C$ when $\gamma := r/(r+s)$, and it is easy to find counterexamples. It is also not true in general that p°_D depends on D only through r and s. For example, with $J = 6$, $p^\circ_{AABB} > p^\circ_{ABAB} = p^\circ_{AB}$ if $p_A \neq p_B$. However, extensive numerical computation suggests the following.

CONJECTURE. Under the assumptions of Theorem 5, the conclusion holds for patterns of the form $D := A^r B^s$ if we replace the assumption that $r+s$ divides J by any one of the following four assumptions:

(a) $J = 2$.
(b) $\min(r, s) = 1$.
(c) $r + s \le J$.
(d) $p_A + p_B > 1/3$.

We can confirm the sufficiency of condition (b) at least in the simplest case, $r = s = 1$. The case of even J is covered by Theorem 5, so we suppose that J is odd, say $J = 2K + 1$ for some positive integer K. Then, by algebra,

$$\begin{aligned} p^\circ_{AB} - \frac{1}{2}(p^\circ_A + p^\circ_B) &= \frac{(p_A q_B + p_B q_A)(q_A q_B)^K + (p_A + p_B)(q_A q_B)^J}{2[1 - (q_A q_B)^J]} \\ &\quad - \frac{1}{2}\left(\frac{p_A q_A^J}{1 - q_A^J} + \frac{p_B q_B^J}{1 - q_B^J}\right) \\ &= -\frac{h(q_A, q_B)}{2(1 - q_A^J)(1 - q_B^J)[1 - (q_A q_B)^J]} \\ &< 0, \end{aligned}$$

where

$$
\begin{aligned}
h(x, y) :=& [x^{K+1} - y^{K+1} + (xy)^{K+1}(x^K - y^K)] \\
& \cdot [x^K(1-x)(1-y^{2K+1}) - y^K(1-y)(1-x^{2K+1})] \\
=& (1-x)(1-y)[x^{K+1} - y^{K+1} + (xy)^{K+1}(x^K - y^K)] \\
& \cdot \sum_{k=0}^{K-1} (x^{K-k} - y^{K-k})[(xy)^k - (xy)^K] \\
>& 0, \qquad x, y \in (0, 1), x \neq y.
\end{aligned}
$$

We conclude this section by asking, at what rate can the casino make money with our two-armed machine, assuming that both arms are fair in the sense that $\mu_A^* = \mu_B^* = 1$? For simplicity, we suppose the player adopts the random-mixture strategy with $\gamma = \frac{1}{2}$. Then the casino's win rate is

$$
\begin{aligned}
& J\left[\frac{1}{2}(p_A^\circ + p_B^\circ) - p_C^\circ\right] \\
& \quad = J\left[\frac{1}{2}\left(\frac{p_A q_A^J}{1 - q_A^J} + \frac{p_B q_B^J}{1 - q_B^J}\right) - \frac{[(p_A + p_B)/2][(q_A + q_B)/2]^J}{1 - [(q_A + q_B)/2]^J}\right],
\end{aligned}
\tag{18}
$$

which for fixed $J \geq 2$ has supremum $\frac{1}{2}[1 - J2^{-J}/(1 - 2^{-J})]$, achieved as $p_A \to 0$ and $p_B \to 1$ (and vice versa). But this case is unrealistic.

Kilby, Fox and Lucas (2005), page 137, reported a simulation study of the effect of hit frequency on player longevity. They considered 10 slot machines with hit frequencies ranging from 6.7% to 29.6% and mean payouts being roughly equal. So we take $p_A = 3/10$ and $p_B = 1/15$ as being the extremes among hit frequencies considered typical in the industry (for single-payline machines). Notice that condition (d) of the conjecture is met. We find that (18) is increasing in J for $J \leq 20$ and decreasing in J for $J \geq 20$. At $J = 20$ its value is about 0.161553 (i.e., 16.2%), while at $J = 10$ its value is about 0.100383. Similar calculations can be done for other strategies. It would seem from the numerical evidence that there is a reasonable profit potential (for the casino) in a two-armed version of the Futurity with both arms fair and $J = 10$. However, it must be recognized that there are strategies other than those ordinarily associated with Parrondo's paradox, so our tentative conclusion about the viability of this machine on the casino floor is premature.

Consider a strategy for which the choice of arm depends on the Futurity pointer. Specifically, let K and L be positive integers such that $K + L = J$, and assume that, if the Futurity pointer shows j consecutive losses and $0 \leq j \leq K - 1$, then arm A is pulled, otherwise arm B is pulled. The driving Markov chain has state space $\Sigma_1 := \{0, 1, \ldots, J-1\}$ and one-step transition matrix $\mathbf{P}_1$ defined by

$$P_1(i, j) = \begin{cases} p_A & \text{if } 0 \le i \le K-1 \text{ and } j = 0, \\ q_A & \text{if } 0 \le i \le K-1 \text{ and } j = i+1, \\ p_B & \text{if } K \le i \le J-2 \text{ and } j = 0, \\ q_B & \text{if } K \le i \le J-2 \text{ and } j = i+1, \\ 1 & \text{if } i = J-1 \text{ and } j = 0. \end{cases}$$

This chain is irreducible and aperiodic, and its unique stationary distribution $\boldsymbol{\pi}_1$ is given by

$$\pi_1(j) = \begin{cases} c^{-1} q_A^j & \text{if } 0 \le j \le K-1, \\ c^{-1} q_A^K q_B^{j-K} & \text{if } K \le j \le J-1, \end{cases}$$

where

$$c := 1 + q_A + \cdots + q_A^{K-1} + q_A^K(1 + q_B + \cdots + q_B^{L-1}).$$

If the mean payouts of arms A and B are

$$1 = \mu_A^* = \mu_A + J p_A^\circ \quad \text{and} \quad 1 = \mu_B^* = \mu_B + J p_B^\circ,$$

then the mean payout at equilibrium under our strategy is

$$\begin{aligned} \mu^* &= \left(\sum_{j=0}^{K-1} \pi_1(j)\right)\mu_A + \left(\sum_{j=K}^{J-1} \pi_1(j)\right)\mu_B + J\pi_1(J-1)q_B \\ &= 1 - \left(\sum_{j=0}^{K-1} \pi_1(j)\right) J p_A^\circ - \left(\sum_{j=K}^{J-1} \pi_1(j)\right) J p_B^\circ + J\pi_1(J-1)q_B, \end{aligned}$$

and we find that the Parrondo effect (in favor of the casino) holds if and only if

$$\pi_1(J-1)q_B < \left(\sum_{j=0}^{K-1} \pi_1(j)\right) p_A^\circ + \left(\sum_{j=K}^{J-1} \pi_1(j)\right) p_B^\circ.$$

Now if we substitute the formula for the stationary distribution, the constant c is irrelevant, and the condition becomes

$$q_A^K q_B^L < (1 + q_A + \cdots + q_A^{K-1}) \frac{p_A q_A^J}{1 - q_A^J} + q_A^K(1 + q_B + \cdots + q_B^{L-1}) \frac{p_B q_B^J}{1 - q_B^J}$$

or

$$q_A^K q_B^L < \frac{(1 - q_A^K) q_A^J}{1 - q_A^J} + \frac{q_A^K (1 - q_B^L) q_B^J}{1 - q_B^J}.$$

This is equivalent to

$$q_A^K \left(\frac{1 - q_A^L}{1 - q_A^J} - \frac{1 - q_B^L}{1 - q_B^J} \right) < 0,$$

which holds if and only if $q_A > q_B$ or equivalently $p_A < p_B$. Here we are using the fact that the function $f_1(x) := (1-x^L)/(1-x^J)$ is decreasing on (0, 1), which follows from

$$\begin{aligned} f_1'(x) &= \frac{Jx^{J-1}(1-x^L) - Lx^{L-1}(1-x^J)}{(1-x^J)^2} \\ &= -\frac{(1-x)x^{L-1}}{(1-x^J)^2}\sum_{k=1}^{K}\sum_{l=1}^{L} x^{k-1}(1-x^{K-k+l}) \\ &< 0, \qquad 0 < x < 1. \end{aligned}$$

So we have the Parrondo effect if $p_A < p_B$. If, however, $p_A > p_B$, then the Parrondo effect fails and the *player* has the advantage.

Returning to our example in which $p_A = 3/10$ and $p_B = 1/15$, we suppose that $J = 10$ and consider the above strategy with $K = 4$. If $\mu_A^* = \mu_B^* = 1$, then we find that the player's win rate is about 0.145747 (i.e., 14.6%). We conclude that our machine is not ready for casino play.

Figure 1 compares several strategies in terms of the expected casino cumulative profit.

The fact that the player can achieve a substantial advantage by using the information available from the Futurity pointer will not come as a surprise to those familiar with the original history-dependent Parrondo games [Parrondo, Harmer

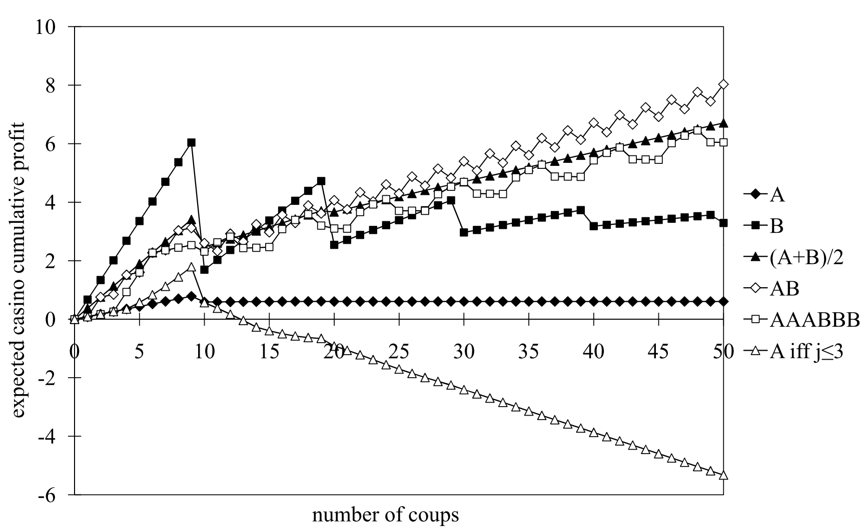

FIG. 1. *Expected casino cumulative profit for various player strategies. We assume a two-armed slot machine with hit frequencies $p_A = 3/10$ and $p_B = 1/15$; Futurity award paid after $J = 10$ consecutive losses, regardless of the order of play of the two arms; initial pointer position* 0; *and both arms fair when played exclusively* ($\mu_A^* = \mu_B^* = 1$). *j is the Futurity pointer position. Results are by direct calculation* (*not simulation*).

and Abbott (2000)]. Let us recall the assumptions: In game A the player tosses a $1/2$-coin (heads has probability $1/2$), whereas in game B, the player tosses a $9/10$-coin if his last two results are two losses, a $1/4$-coin if his last two results are a loss and a win in either order, and a $7/10$-coin if his last two results are two wins. In both games, the player wins one unit with heads and loses one unit with tails. If the player can use information about his two most recent results to choose which game to play, the optimal strategy is clear: Play game A if the last two results differ and game B otherwise. Most studies of Parrondo's paradox disregard this strategy and consider only "blind" strategies, those that do not rely on the player's past. In the casino setting, however, one cannot expect a player to disregard information that may prove to be profitable.

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